ZERO-VISCOSITY LIMIT FOR BOUSSINESQ EQUATIONS WITH VERTICAL VISCOSITY AND NAVIER BOUNDARY IN THE HALF PLANE

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Abstract. In this paper we study the zero-viscosity limit of 2-D Boussinesq equations with vertical viscosity and zero diffusivity, which is a nonlinear system with partial dissipation arising in atmospheric sciences and oceanic circulation. The domain is taken as $\mathbb{R}^2_+$ with Navier-type boundary. We prove the nonlinear stability of the approximate solution constructed by boundary layer expansion in conormal Sobolev space. The optimal expansion order and convergence rates for the inviscid limit are also identified in this paper. Our paper extends the partial zero-dissipation limit results of Boussinesq system with full dissipation by Chae D. [*Adv. Math. 203 (2006), no. 2, 497–513*] in the whole space to the case with partial dissipation and Navier boundary in the half plane.

Keywords: Boussinesq equations; Anisotropic dissipation; Zero-viscosity limit; Navier boundary.

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Contents

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1. Introduction

We consider the following 2-D Boussinesq equations with only vertical viscosity and zero diffusivity in $\mathbb{R}^2_+$:

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \partial_y^2 \mathbf{u}^\varepsilon = \theta^\varepsilon e_2, \\ \partial_t \theta^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \theta^\varepsilon = 0, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0, \\ \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0^\varepsilon(x,y), \quad \theta^\varepsilon|_{t=0} = \theta_0^\varepsilon(x,y), \end{cases} \tag{1.1}$$

where $t \geq 0$ and $(x,y) \in \mathbb{R} \times \mathbb{R}_+$ are time and space variables. Here $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ and p^ε are, respectively, the velocity and pressure of the fluid. The scalar function θ^ε denotes the temperature or the density. We use $e_2 = (0,1)$ to represent the unit vector in the vertical direction, and ε^2 to denote the kinematic viscosity. The Boussinesq equations with anisotropic (full or partial) dissipation play an important role in the study of atmospheric and oceanographic flows and Raleigh–Bernard convection, mathematically and physically. For more backgrounds of the Boussinesq equations, one can refer to [13, 45, 48, 59].

In this paper, the Navier-type (slip) boundary condition of the system (1.1) is given by

$$u_2^\varepsilon = 0, \quad \partial_y u_1^\varepsilon = \alpha u_1^\varepsilon \quad \text{on} \quad \{y = 0\}, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is used to characterize the tendency of the fluid to slip on the boundary. Here the initial data given in (1.1) and in the sequel should satisfy the compatibility conditions on the boundary and the divergence free condition.

Our purpose is to survey the zero-viscosity limit behaviour of the Boussinesq system (1.1) with partial viscosity and zero diffusivity in the half plane satisfying the Navier boundary condition (1.2). It is extremely challenging to deal with the loss of partial dissipation and the boundary layer effects, especially, for the bounded / unbounded domain with non-slip boundary. In this paper we set our problem in the half plane with the Navier boundary condition (1.2) for the first step. To our best knowledge, this paper is the first one to consider the strong zero-viscosity limit of Boussinesq equations with partial viscosity and boundaries, which is from physical consideration and involves layer effects.

Formally, letting $\varepsilon \to 0$, the partially viscous Boussinesq system (1.1) is then reduced to the following zero dissipation system in $\mathbb{R}^2_+$:

$$\begin{cases} \partial_t \mathbf{u}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0 + \nabla p^0 = \theta^0 e_2, \\ \partial_t \theta^0 + \mathbf{u}^0 \cdot \nabla \theta^0 = 0, \\ \nabla \cdot \mathbf{u}^0 = 0, \\ \mathbf{u}^0|_{t=0} = \mathbf{u}^0_0(x,y), \quad \theta^0|_{t=0} = \theta^0_0(x,y). \end{cases} \tag{1.3}$$

Also, the boundary condition for (1.3) turns into

$$u^0_2|_{y=0} = 0. \tag{1.4}$$

It is suggested that there are some strong analogues between the 2-D zero-dissipation system (1.3) and the 3-D Euler equations (see [14, 44]). Therefore, to understand the local existence theory of the 2-D Boussinesq system with no dissipation (1.3), one can refer to the local existence theory and finite time blow-up criteria for 3-D incompressible Euler equations [3, 9–11, 23, 24, 34, 42, 53]. Setting the zero-dissipation system (1.3) into the whole space $\mathbb{R}^2$ or a domain with boundary, the local in time existence theory of (1.3) has been studied in [6, 8] or [7, 9, 25, 28], respectively. In particular, the local in time existence of (1.3) in $\mathbb{R}^2_+$ holds for the C^∞-smooth initial data [25] and for the $C^{1,\alpha}_c(\mathbb{R}^2_+)$ initial data [9] with $\alpha < \alpha_0$ for some $\alpha_0 < 1$, both developing singularities in finite time. We also note that the initial data $(\mathbf{u}^0_0, \theta^0_0)$ satisfy the compatibility condition

$$\nabla \cdot \mathbf{u}^0_0 = 0, \quad \mathbf{u}^0_0 \cdot \mathbf{n} = 0, \quad \text{with} \quad \mathbf{n} = (0, -1).$$

The system (1.1) is a particular case of the 2-D Boussinesq system with anisotropic dissipation, which reads

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \nu_1 \partial_x^2 \mathbf{u}^\varepsilon + \nu_2 \partial_y^2 \mathbf{u}^\varepsilon + \theta^\varepsilon e_2, \\ \partial_t \theta^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \theta^\varepsilon = \kappa_1 \partial_x^2 \theta^\varepsilon + \kappa_2 \partial_y^2 \theta^\varepsilon, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0, \end{cases} \tag{1.5}$$

where ν_1, ν_2 and κ_1, κ_2 are nonnegative constants to characterize viscosity and diffusivity, respectively. Recently, extensive progress has been made on the global existence theory of the Boussinesq system (1.5) with full dissipation or partial dissipation in $\mathbb{R}^2$ (cf. [1, 2, 4, 5, 17, 18, 20, 22, 29, 30, 37, 39, 47, 61] and the references therein), or in an appropriate domain with boundary (cf. [21, 31, 36, 37, 52, 62] and the references therein). However, the global existence theory for the Boussinesq system with $\nu_2 > 0$ and $\nu_1 = \kappa_1 = \kappa_2 = 0$ (i.e. (1.1)), or $\nu_1 = \nu_2 = \kappa_1 = \kappa_2 = 0$ (i.e. (1.3)), are still challenging open problems even in the whole space $\mathbb{R}^2$. For the coupled advective scalar equation of θ in

(1.5) and the related quasigeostrophic equations, they are also attractive research topics in PDE theory [12–14, 16, 33, 49, 60].

In addition, the zero-dissipation limit behaviour of Boussinesq equations with boundary is a meaningful physical problem, which is associated with the boundary layer theory. Recently, Jiang et al [32] have studied the zero-diffusivity limit of the 2-D Boussinesq equations with full dissipation in the half plane. They have justified the zero-diffusivity limit of the velocity $\mathbf{u}$ and the temperature θ, respectively, in H^1 and L^2 norm uniformly on $[0, T]$ with convergence rate. In [56], the authors have investigated the L^2 zero-dissipation limit of Boussinesq equations in a bounded domain with Navier type boundary for the velocity and Neumann boundary for the temperature using boundary layer expansion. One can also consult [57] about the well-posedness of the boundary layer equation for a geophysical model or [26, 38] on the zero-diffusivity limit for the Boussinesq system in the weak sense.

In this paper we concentrate on investigating the strong L^∞ zero-viscosity limit of the Boussinesq equations (1.1) with Navier boundary in $\mathbb{R}^2_+$. To begin with, we construct an approximate solution with outer layer profile away from the boundary and inner layer profile near the boundary, using boundary layer expansion method. Due to the structure of the Navier boundary condition, we can derive an approximate solution with leading profile

$$(\mathbf{u}^0, \theta^0) + \varepsilon(U, \Theta)(t, x, \frac{y}{\varepsilon}) + \cdots$$

to characterize the singularity near the boundary as ε goes to zero. The study of zero-viscosity limit is then reduced to the stability analysis of the approximate solution and derive the uniform estimates on $[0, T]$ with T independent of ε. If the expansion order is lager than or equal to two, we can prove the linear stability in the conormal Sobolev setting by taking advantage of the anisotropic Sobolev embedding inequality, since the first order derivatives of the inner layer profile are bounded. In the usual nonlinear stability analysis, we need the a priori assumption of $\|(\mathbf{u}, \theta)\|_{W^{1,\infty}}$ to estimate the nonlinear terms and the pressure in higher-order energy estimates. To handle the strong singularity appearing in the higher-order energy estimates, instead of deriving the uniform H^3 energy estimates in Sobolev space to close the a priori assumption, we introduce the energy involving $\|(\eta, \partial_y\theta)\|_{1,\infty}$-norm inspired by the extraordinary work of Masmoudi and Rousset [46]. Moreover, we deal with the pressure estimates and the L^∞ estimates of the velocity $\mathbf{u}$ in spirit of the remarkable pressure estimates and L^∞ estimates for incompressible Navier-Stokes equations in [46]. As for the L^∞ estimates of the density (or temperature) θ, we apply the maximum principle for advective scalar equations [49]. Here we shall notice that in [46] the authors have proved the zero-viscosity limit of incompressible Navier-Stokes equations with Navier boundary using compactness argument, in which the convergence rate can not be deduced. Owing to the boundary layer expansion and stability analysis, we can deduce the convergence rate $O(\varepsilon)$ and identify the optimal expansion order $K=2$ for the boundary layer expansion of the partially dissipative Boussinesq system in this paper.

Now we state our main result in the following theorem.

Theorem 1.1. *Let $(\mathbf{u}^0,\theta^0)$ be a solution to (1.3) (1.4) such that*

$$(\mathbf{u}^0,\theta^0)\in C^0([0,T],C^\infty(\mathbb{R}^2_+)).$$

Then there is ε_0 such that for any $\varepsilon\in(0,\varepsilon_0]$, there exists $(\mathbf{u}^\varepsilon,\theta^\varepsilon)$ a solution to (1.1) (1.2) defined on $[0,T]$ with T independent of ε such that

$$\sup_{[0,T]}(\|\mathbf{u}^\varepsilon-\mathbf{u}^0\|_{L^2(\mathbb{R}^2_+)}+\|\theta^\varepsilon-\theta^0\|_{L^2(\mathbb{R}^2_+)})\to 0, \tag{1.6}$$

$$\sup_{[0,T]}(\|\mathbf{u}^\varepsilon-\mathbf{u}^0\|_{L^\infty(\mathbb{R}^2_+)}+\|\theta^\varepsilon-\theta^0\|_{L^\infty(\mathbb{R}^2_+)})\to 0, \tag{1.7}$$

as ε goes to 0.

Furthermore, the rate of convergence is $O(\varepsilon)$.

Remark 1.2. *The convergence results in Theorem 1.1 also hold for Boussinesq system (1.5) in $\mathbb{R}^2_+$ with either one of the following two conditions:*

(1) $\nu_2=\varepsilon^2,\nu_1=0$ or $\varepsilon^2,\ \kappa_1=0$ or $\varepsilon^2,\kappa_2=0$ and Navier boundary condition (1.2);

(2) $\nu_1=\varepsilon^2,\kappa_1=0$ or $\varepsilon^2,\ \nu_2=\kappa_2=0$ and boundary condition $\mathbf{u}\cdot\mathbf{n}=0$ on $\{y=0\}$.

The arrangement of the remaining sections is as following. We devote Section 2 to constructing an approximate solution using boundary layer expansion. We will give the stability analysis of the linearized system around the approximate solution in conormal Sobolev space in Section 3. In Section 4, we will focus on dealing with the nonlinear terms and deriving the L^∞ estimates to close the uniform estimates.

2. Boundary Layer Expansion

We construct an approximate solution of system (1.1) in the following form

$$(\mathbf{u}_a,\theta_a,p_a)=\sum_{i=0}^K\varepsilon^i(\mathbf{u}^i,\theta^i,p^i)(t,x,y)+\sum_{i=0}^K\varepsilon^i\left(U^i,\Theta^i,P^i\right)\left(t,x,\frac{y}{\varepsilon}\right), \tag{2.1}$$

where K is an arbitrarily large integer, and $U^i := (U_1^i, U_2^i)$. We use $(\mathbf{u}^i, \theta^i, p^i)(t,x,y)$ in the first sum to approximate the outer layer, meanwhile, $\left(U^i, \Theta^i, P^i\right)\left(t, x, \frac{y}{\varepsilon}\right)$ to characterize the inner layer behaviour near the boundary in the second sum. In the following presentation, we denote the fast variable as $z = y/\varepsilon$ for simplicity.

We shall impose the fast decay condition as following

$$\left(U^i, \Theta^i, P^i\right)(t,x,z) \to 0, \quad \text{as} \quad z \to +\infty. \tag{2.2}$$

In addition, the matched boundary conditions for Navier boundary (1.2) read

$$\begin{aligned}
&u_2^i(t,x,0) + U_2^i(t,x,0) = 0, \text{ for all } i \geq 0, \\
&\partial_z U_1^0(t,x,0) = 0, \\
&\partial_y u_1^i(t,x,0) + \partial_z U_1^{i+1}(t,x,0) = \alpha(u_1^i + U_1^i)(t,x,0), \ i \geq 0.
\end{aligned} \tag{2.3}$$

In the sequel, we denote Γf as

$$\Gamma f = f(t,x,y)|_{y=0}.$$

We expect the leading order of the outer layer in the approximation (2.1) to be the solution of the zero viscosity Boussinesq system (1.3) (1.4), whose local well-posedness theory can be deduced, respectively, in [25] for C^∞-smooth data, and in [9] for $C^{1,\alpha}(\mathbb{R}_+^2)$ data with $\alpha < \alpha_0 < 1$ for some α_0. In fact, substituting the approximate scheme (2.1) into the viscous Boussinesq system (1.1) and collecting the $O(1)$ order terms, consequently we can get the leading order terms for the outer layer satisfying (1.3) by taking $z \to +\infty$.

In the inner zone, by collecting the $O(\varepsilon^{-1})$ terms, one has

$$\begin{aligned}
&(\Gamma u_2^0 + U_2^0)\partial_z U_1^0 = 0, \\
&(\Gamma u_2^0 + U_2^0)\partial_z U_2^0 + \partial_z P^0 = 0, \\
&\partial_z U_2^0 = 0, \\
&(\Gamma u_2^0 + U_2^0)\partial_z \Theta^0 = 0.
\end{aligned}$$

Then we can deduce from (2.2) and $(2.3)_1$ that

$$U_2^0(t,x,z) = 0, \quad P^0(t,x,z) = 0. \tag{2.4}$$

In turn, $\Gamma u_2^0 = 0$ holds.

Now, we collect the $O(1)$ terms for the first equation of velocity in the inner zone to obtain

$$\partial_t(\Gamma u_1^0 + U_1^0) + (\Gamma u_1^0 + U_1^0)\partial_x(\Gamma u_1^0 + U_1^0)$$

$$+ (\Gamma u_2^1 + U_2^1 + z\Gamma\partial_y u_2^0)\partial_z(\Gamma u_1^0 + U_1^0) + \partial_x(\Gamma p^0) - \partial_{zz}(\Gamma u_1^0 + U_1^0) = 0, \tag{2.5}$$

Combining with the $O(1)$ terms from the divergence free condition in the inner layer,

$$\partial_x(\Gamma u_1^0 + U_1^0) + \partial_z(\Gamma u_2^1 + U_2^1 + z\Gamma\partial_y u_2^0) = 0, \tag{2.6}$$

one has a closed system for $\Gamma u_1^0 + U_1^0$ and $u_2^1 + U_2^1 + z\Gamma\partial_y u_2^0$, together with the boundary condition

$$\partial_z U_1^0(t, x, 0) = 0. \tag{2.7}$$

We notice that the equations (2.5) (2.6) and (2.7) have a trivial solution

$$U_1^0 = 0.$$

In the following analysis, we shall assume that the approximate solutions (2.1) satisfy $U_1^0 = 0$.

Clearly, from the $O(1)$ order terms of the second velocity equation and the transport equation, respectively,

$$\begin{aligned}
&\partial_t(\Gamma u_2^0 + U_2^0) + (\Gamma u_1^0 + U_1^0)\partial_x(\Gamma u_2^0 + U_2^0) \\
&+ (\Gamma u_2^0 + U_2^0)\partial_z(\Gamma u_2^1 + U_2^1 + z\Gamma\partial_y u_2^0) + \Gamma\partial_y p^0 + \partial_z P^1 = \Gamma\theta^0 + \Theta^0, \\
&\partial_t(\Gamma\theta^0 + \Theta^0) + (\Gamma u_1^0 + U_1^0)\partial_x(\Gamma\theta^0 + \Theta^0) + (\Gamma u_2^1 + U_2^1 + z\Gamma\partial_y u_2^0)\partial_z\Theta^0 = 0,
\end{aligned}$$

the leading order for the temperature (or the density) in the inner zone has a trivial solution

$$\Theta^0 = 0.$$

Consequently, we also have $P^1 = 0$.

Remark 2.1. *In the study of zero-viscosity limit for incompressible Navier-Stokes equations with non-slip boundary, one derive the system* (2.5) (2.6) *with boundary condition*

$$\Gamma u_1^0 + U_1^0(t, x, 0) = 0, \tag{2.8}$$

which just forms the famous Prandtl system. For the local well-posedness theory of Prandtl system(2.5) (2.6) (2.8) *or the inviscid limit of Navier-Stokes equations with non-slip boundary, one may refer to [19, 35, 40, 43, 50, 51, 55] and the references therein.*

Till now, we get the leading order for both inner zone and outer zone for the approximate solution. i.e. $(\mathbf{u}^0, p^0, \theta^0), (U^0, P^0, \Theta^0)$ are solved. Following the standard method on constructing approximate solution by boundary layer expansion (see [27, 56, 58] and the references therein for example), one can obtain the terms $(\mathbf{u}^i, p^i, \theta^i)$ and (U^i, P^i, Θ^i) order by order. Then we conclude that the approximate solutions $(\mathbf{u}_a, p_a, \theta_a)$ for (1.1) can be expressed in the form

$$(\mathbf{u}^0, p^0, \theta^0)(t,x,y) + \varepsilon\Big((\mathbf{u}^1, p^1, \theta^1)(t,x,y) + (U^1, P^1, \Theta^1)(t, x, \frac{y}{\varepsilon})\Big) + \cdots .$$

Let us denote

$$\omega^0 = \partial_y u_1^0 - \partial_x u_2^0, \quad \text{and} \quad \omega_0^0 = \omega^0(t=0,x,y).$$

Then we can gather our results on boundary layer approximations of the solutions for the system (1.1) and (1.2) in the following theorem.

Theorem 2.2. *Let $K \in \mathbb{N}_+$. For any initial data (θ_0^0, u_0^0) satisfying some compatibility conditions on $\{y=0\}$ and such that $(\mathbf{u}_0^0, \theta_0^0) \in C_c^\infty(\mathbb{R}_+^2)$, then there exists $T>0$ and a smooth approximate solution $(\mathbf{u}_a, \theta_a)$ of (1.1) under the form (2.1) with order K such that*

i) *we have $(\mathbf{u}^0, \theta^0) \in C^0([0,T], C^\infty(\mathbb{R}_+^2))$ as a solution of the inviscid system (1.3) with initial data $(\mathbf{u}_0^0, \theta_0^0)$;*

ii) *for all $1 \le i \le K$, $(\mathbf{u}^i, \theta^i) \in C^0([0,T], C^\infty(\mathbb{R}_+^2))$;*

iii) *for all $0 \le i \le K$, (U^i, P^i, Θ^i) are solved at least locally and satisfy the fast decay property (2.2) with respect to the last variable.*

iv) *we consider $(\mathbf{u}^\varepsilon, \theta^\varepsilon)$ a solution to (1.1), and denote the error terms $(\mathbf{u}, p, \theta)$ as following*

$$\mathbf{u} = (u_1, u_2) = \mathbf{u}^\varepsilon - \mathbf{u}_a, \quad p = p^\varepsilon - p_a, \quad \theta = \theta^\varepsilon - \theta_a,$$

where $\mathbf{u}_a$ naturally satisfies

$$\nabla \cdot \mathbf{u}_a = 0, \quad \mathbf{u}_a \cdot \mathbf{n}|_{y=0} = 0,$$

with $\mathbf{n} = (0,-1)$.

Then $(\mathbf{u}, \theta)$ satisfies the system of equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla(\mathbf{u}+\mathbf{u}_a) + \mathbf{u}_a\cdot\nabla\mathbf{u} + \nabla p - \varepsilon^2\partial_y^2\mathbf{u} = \theta e_2 + \varepsilon^K R_{\mathbf{u}}, \tag{2.9a}$$

$$\partial_t\theta + \mathbf{u}\cdot\nabla(\theta+\theta_a) + \mathbf{u}_a\cdot\nabla\theta = \varepsilon^K R_\theta, \tag{2.9b}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.9c}$$

$$\partial_y u_1 = \alpha u_1, \quad u_2 = 0, \quad \text{on} \quad \{y=0\}, \tag{2.9d}$$

where $R_{\mathbf{u}} := (R_{u_1}, R_{u_2})$ and R_θ are remainders satisfying

$$\sup_{[0,T]} \|\nabla^\beta R_{\mathbf{u},\theta}\| \le C_a \varepsilon^{-\beta_2}, \quad \forall \beta = (\beta_1,\beta_2) \in \mathbb{N}^2, \tag{2.10}$$

with $C_a > 0$ independent of ε.

3. Linear Stability Estimates

This section is devoted to the stability analysis of the approximate solution constructed by boundary layer expansion in Theorem 2.2. Due to the essential challenge in nonlinear energy estimates caused by the boundary layer effects and the loss of dissipation in x-direction, we first consider the linear stability of the boundary layer approximation. Recall the *Trace Theorem* (cf. [41, 54]) for a C^1 bounded domain Ω :

$$\|\mathbf{u}\|^2_{L^2(\partial\Omega)} \le C(\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\|^2_{L^2(\Omega)}), \tag{3.1}$$

$$\|\mathbf{u}\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \le C\|\mathbf{u}\|_{H^s(\Omega)}, \quad \text{for} \quad s > 1/2, \tag{3.2}$$

where $C > 0$ is some constant. We know that the Navier boundary condition (1.2) contains a constant α, which is used to characterize the slip length. If $\alpha < 0$, the boundary term is not good in the energy estimates. On the other hand, for a general smooth bounded domain Ω, the loss of dissipation in tangential direction is extremely challenging in energy estimates in Sobolev space, especially when we deal with the boundary term involving $\alpha \in \mathbb{R}$ by trace theorem, due to the boundary is not flat. From now on, Ω is taken as $\mathbb{R}^2_+$ for simplicity in our paper. As for the C^1 bounded domain in $\mathbb{R}^2$, we leave this case in our further study concerning the zero-viscosity limit of Boussinesq equations with partial dissipation. Therefore, in this section, we will present the uniform estimates for $\alpha \in \mathbb{R}$ in the half plane. In the following writing, we use the notation

$$\Omega := \mathbb{R}^2_+.$$

To begin with, we linearize the system (2.9) around the approximate solution to obtain that, in $\mathbb{R}^2_+$,

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_a + \mathbf{u}_a\cdot\nabla\mathbf{u} + \nabla p - \varepsilon^2\partial_y^2\mathbf{u} = \theta e_2 + \varepsilon^K R_{\mathbf{u}}, \tag{3.3a}$$

$$\partial_t\theta + \mathbf{u}\cdot\nabla\theta_a + \mathbf{u}_a\cdot\nabla\theta = \varepsilon^K R_\theta, \tag{3.3b}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3.3c}$$

$$\partial_y u_1 = \alpha u_1, \quad u_2 = 0, \quad \text{on} \quad \{y = 0\}, \tag{3.3d}$$

where $R_{\mathbf{u}} := (R_{u_1}, R_{u_2})$ and R_θ are remainders satisfying

$$\sup_{[0,T]}\|\nabla^\beta R_{\mathbf{u},\theta}\| \le C_a\varepsilon^{-\beta_2}, \quad \forall\beta = (\beta_1, \beta_2) \in \mathbb{N}^2, \tag{3.4}$$

with $C_a > 0$ independent of ε. The initial data of the system (2.9) and (3.3) are given by

$$\mathbf{u}|_{t=0} = \varepsilon^{K+1}\mathbf{u}_0, \quad \theta|_{t=0} = \varepsilon^{K+1}\theta_0. \tag{3.5}$$

In the remaining parts of this section, we will give the stability analysis of the linear system (3.3) for $\alpha \in \mathbb{R}$.

3.1. **Preliminaries.** Before we process the uniform estimates, some basic notations and useful inequalities will be given in this subsection. We will use $\lesssim$ to denote $\leq C(\cdot)$ or $\leq C_a(\cdot)$ for a generic constant C or a constant C_a depending on the approximate solution, but both independent of ε. The standard Sobolev norm is denoted by $\|\cdot\|_s$ for $\|\cdot\|_{H^s}$ with $s \geq 0$. In particular, $\|\cdot\|$ is for $\|\cdot\|_{L^2}$.

We introduce conormal operator $Z^\beta = Z_1^{\beta_1} Z_2^{\beta_2}$ with $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, where

$$Z_1 = \partial_x, \qquad Z_2 = \varphi(y)\partial_y, \tag{3.6}$$

and $\varphi(y)$ is a smooth function satisfying $\varphi(0) = 0$ and $\varphi'(0) > 0$, such as

$$\varphi(y) = \frac{y}{y+1}. \tag{3.7}$$

In the sequel, we denote

$$Z^k = Z^\beta, \quad \text{with} \ \ |\beta| = k, \ k \in \mathbb{N}.$$

The conormal Sobolev space H^s_{co} for $s \in \mathbb{N}$ is defined by

$$H^s_{co} := \left\{ u \in L^2_{x,y}(\mathbb{R}^2_+) : \|u\|_{H^s_{co}} = \sum_{|\beta| \leq s} \|Z_1^{\beta_1} Z_2^{\beta_2} u\|_{L^2_{x,y}(\mathbb{R}^2_+)} < \infty \right\}. \tag{3.8}$$

In spirit of this setting, we denote the H^s_{co}-inner product by $\langle \cdot, \cdot \rangle_{H^s_{co}}$. Set

$$\|u\|_{k,\infty} = \sum_{|\beta| \leq k} \|Z^\beta u\|_{L^\infty},$$

and we say that $u \in W^{k,\infty}_{co}$ if $\|u\|_{k,\infty}$ is finite.

Now we state the anisotropic Sobolev embedding inequality and some useful estimates (cf. [46]) in the following lemmas.

Lemma 3.1. *For $m_0 > 1, m_0 \in \mathbb{N}$, and conormal Sobolev norm defined in (3.8), we have*

$$\|u\|^2_{L^\infty} \lesssim \|\partial_y u\|_{H^{m_0}_{co}} \|u\|_{H^{m_0}_{co}} + \|u\|^2_{H^{m_0}_{co}}. \tag{3.9}$$

Let us denote the vorticity ω as

$$\omega = \operatorname{curl} \mathbf{u} = \partial_y u_1 - \partial_x u_2,$$

and set

$$\eta = \omega - \alpha u_1. \tag{3.10}$$

Lemma 3.2 (Proposition 12, [46]). *For $m_0 > 1$, we have*

$$\|\mathbf{u}\|_{W^{1,\infty}} \lesssim \|\mathbf{u}\|_{H_{co}^{m_0+2}} + \|\eta\|_{H_{co}^{m_0+1}} + \|\eta\|_{L^\infty}, \tag{3.11}$$

$$\|\mathbf{u}\|_{2,\infty} \lesssim \|\mathbf{u}\|_{H_{co}^{m_0+3}} + \|\eta\|_{H_{co}^{m_0+2}}, \tag{3.12}$$

$$\|\nabla\mathbf{u}\|_{1,\infty} \lesssim \|\mathbf{u}\|_{H_{co}^{m_0+3}} + \|\eta\|_{H_{co}^{m_0+3}} + \|\eta\|_{1,\infty}. \tag{3.13}$$

3.2. **Uniform Estimates for $\alpha \in \mathbb{R}$.** In this subsection, we will derive the uniform estimates in conormal Sobolev space. To begin with, we state our main results for the linear system with partial viscosity.

Theorem 3.3. *Let $m \geq 2$, and $(\theta_0^0, \mathbf{u}_0^0) \in C^\infty(\mathbb{R}_+^2)$ be initial data for (1.3) satisfying $\nabla \cdot \mathbf{u}_0^0$ and some compatibility conditions on $\{y = 0\}$. Let $K \in \mathbb{N}_+, K \geq 2$ and $(\mathbf{u}_a, p_a, \theta_a)$ an approximate solution at order K given by Theorem 2.2. Then, for every $\varepsilon \in (0, \varepsilon_0]$, there exists a $T_0 > 0$ such that the solution of (3.3) -(3.5) defined on $[0, T_0]$ satisfies the estimate*

$$\|(\mathbf{u}, \theta)\|_{H_{co}^m} + \|\partial_y(\mathbf{u}, \theta)\|_{H_{co}^{m-1}} \lesssim \varepsilon^{K-1}. \tag{3.14}$$

Therefore, it holds that

$$\|(\mathbf{u}, \theta)\|_{L^\infty(\mathbb{R}_+^2)} \lesssim \varepsilon^{K-1}. \tag{3.15}$$

The proof of the Theorem 3.3 can be deduced from the following L^2-estimates, conormal energy estimates, normal derivative estimates and pressure estimates.

L^2-estimates.

Lemma 3.4. *Let $(\mathbf{u}_a, \theta_a)$ be the approximate solution in Theorem 2.2. Given $\alpha \in \mathbb{R}$, if $(\mathbf{u}, \theta)$ is the solution of the linear system (3.3) defined on $[0, T]$, then it holds that*

$$\|(\mathbf{u}, \theta)(t, x, y)\|^2 + c_0\varepsilon^2 \int_0^T \|\partial_y u(\tau, x, y)\|^2 d\tau \lesssim \varepsilon^{2K} \tag{3.16}$$

for some positive constant c_0.

Remark 3.5. *The L^2 estimates (3.16) also holds for the nonlinear system (2.9), due to $\nabla \cdot \mathbf{u} = 0$, and $\mathbf{u} \cdot \mathbf{n}|_{y=0} = 0$.*

Remark 3.6. *If Ω is a bounded C^1 domain in $\mathbb{R}^2$, the trace theorem gives*

$$\int_{\partial\Omega} \mathbf{u}^2 \leq C\|\nabla\mathbf{u}\|\|\mathbf{u}\| + \|\mathbf{u}\|^2,$$

where C depends on Ω. The energy estimates for the partially viscous system (2.9) or (3.3) will meet another challenge in dealing with the boundary terms for $\alpha < 0$, due the loss of dissipation in x-direction. In this paper, we set the domain as a half plane.

Proof. We multiply the equation (3.3a) and (3.3b), respectively, by $\mathbf{u}$ and θ. Then, adding the resulting equations together and integrating over Ω, together with integration by parts, we can obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(\mathbf{u}^2+\theta^2)+\varepsilon^2\int_\Omega|\partial_y\mathbf{u}|^2+\alpha\varepsilon^2\int_{\partial\Omega}u_1^2\\
&=-\int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u}_a\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\theta_a\theta)+\varepsilon^K\int_\Omega(R_{\mathbf{u}}\mathbf{u}+R_\theta\theta),
\end{aligned}\tag{3.17}$$

since

$$\begin{aligned}
&\int_\Omega\nabla p\mathbf{u}=-\int_\Omega\nabla\cdot\mathbf{u}p+\int_{\partial\Omega}p\mathbf{u}\cdot\mathbf{n}=0,\\
&\int_\Omega\mathbf{u}_a\cdot\nabla\mathbf{u}\mathbf{u}=\int_\Omega\mathbf{u}_a\cdot\nabla|\mathbf{u}|^2=-\int_\Omega\nabla\cdot\mathbf{u}_a|\mathbf{u}|^2+\int_{\partial\Omega}\mathbf{u}_a\cdot\mathbf{n}|\mathbf{u}|^2=0,\\
&\int_\Omega\partial_y^2\mathbf{u}\mathbf{u}=-\int_\Omega|\partial_y\mathbf{u}|^2-\int_{\partial\Omega}\partial_yu_1u_1=-\int_\Omega|\partial_y\mathbf{u}|^2-\alpha\int_{\partial\Omega}u_1^2,\\
&\int_\Omega\mathbf{u}_a\cdot\nabla\theta\theta=-\int_\Omega\nabla\cdot\mathbf{u}_a\theta^2+\int_{\partial\Omega}\mathbf{u}_a\cdot\mathbf{n}\theta^2=0.
\end{aligned}$$

Due to

$$\int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u}_a\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\theta_a\theta)\le\|\nabla\mathbf{u}_a\|_{L^\infty}\int_\Omega|\mathbf{u}|^2+\|\nabla\theta_a\|_{L^\infty}\int_\Omega|\mathbf{u}||\theta|\le C_a\int_\Omega(|\mathbf{u}|^2+|\mathbf{u}||\theta|),$$

then we have, for $\alpha\ge 0$,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(\mathbf{u}^2+\theta^2)+\varepsilon^2\int_\Omega|\partial_y\mathbf{u}|^2+\alpha\varepsilon^2\int_{\partial\Omega}u_1^2\\
&\lesssim\int_\Omega(\mathbf{u}^2+\theta^2)+\varepsilon^{2K},
\end{aligned}\tag{3.18}$$

where the Cauchy inequality has been used. However, for $\alpha<0$, the boundary term is not good in the energy estimates. Using trace theorem in $\mathbb{R}^2_+$, we have

$$|\alpha|\varepsilon^2\int_{\partial\Omega}u_1^2\le|\alpha|\varepsilon^2\|\partial_yu_1\|\|u_1\|.$$

Then, together with the Young inequality, one has

$$\frac{d}{dt}\int_\Omega(\mathbf{u}^2+\theta^2)+c_0\varepsilon^2\int_\Omega|\partial_y\mathbf{u}|^2\lesssim\int_\Omega(\mathbf{u}^2+\theta^2)+\varepsilon^{2K}.\tag{3.19}$$

Hence (3.16) follows from (3.19) and the Gronwall's inequality. □

Conormal energy estimates.

Lemma 3.7. *Let $(\mathbf{u}_a, \theta_a)$ be the approximate solution in Theorem 2.2. Given $\alpha \in \mathbb{R}$, assume that $(\mathbf{u}, \theta)$ is a solution of the linear system (3.3) defined on $[0, T]$, then, for $m \geq 1$, we have*

$$\begin{aligned}
&\frac{d}{dt}(\|(\mathbf{u}, \theta)\|^2_{H^m_{co}}) + c_0\varepsilon^2\|\partial_y \mathbf{u}\|^2_{H^m_{co}} \\
&\lesssim \|\partial_y \mathbf{u}\|^2_{H^{m-1}_{co}} + \|\theta\|^2_{H^m_{co}} + \|\mathbf{u}\|^2_{H^m_{co}} + \|\nabla p\|_{H^{m-1}_{co}}\|\mathbf{u}\|_{H^m_{co}} + \varepsilon^{2K},
\end{aligned} \tag{3.20}$$

where c_0 is some positive constant independent of ε.

Proof. Acting Z^β with $|\beta| \leq m$ on the equations (3.3a) and (3.3b), we obtain that

$$\partial_t Z^\beta \mathbf{u} + \mathbf{u} \cdot \nabla Z^\beta \mathbf{u}_a + \mathbf{u}_a \cdot \nabla Z^\beta \mathbf{u} + \nabla Z^\beta p - \varepsilon^2 \partial_y^2 Z^\beta \mathbf{u} = Z^\beta \theta e_2 + \mathcal{C}_{\mathbf{u}} + \varepsilon^K Z^\beta R_{\mathbf{u}}, \tag{3.21}$$

$$\partial_t Z^\beta \theta + \mathbf{u} \cdot \nabla Z^\beta \theta_a + \mathbf{u}_a \cdot \nabla Z^\beta \theta = \mathcal{C}_\theta + \varepsilon^K Z^\beta R_\theta, \tag{3.22}$$

where the commutators $\mathcal{C}_{\mathbf{u}}, \mathcal{C}_\theta$ are defined as following

$$\begin{aligned}
&\mathcal{C}_{\mathbf{u}} = -[Z^\beta, \mathbf{u} \cdot \nabla]\mathbf{u}_a - [Z^\beta, \mathbf{u}_a \cdot \nabla]\mathbf{u} - [Z^\beta, \nabla]p + \varepsilon^2[Z^\beta, \partial_y^2] =: \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4, \\
&\mathcal{C}_\theta = -[Z^\beta, \mathbf{u} \cdot \nabla]\theta_a - [Z^\beta, \mathbf{u}_a \cdot \nabla]\theta := \mathcal{C}_5 + \mathcal{C}_6.
\end{aligned}$$

Now we calculate (3.21) $\times Z^\beta \mathbf{u}$ + (3.22) $\times Z^\beta \theta$ and then integrate the resulting equality over Ω to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|Z^\beta \mathbf{u}|^2 + |Z^\beta \theta|^2) + \varepsilon^2 \int_\Omega |\partial_y Z^\beta \mathbf{u}|^2 + \varepsilon^2 \int_{\partial\Omega} \partial_y Z^\beta u_1 Z^\beta u_1 \\
&= -\int_\Omega (\mathbf{u} \cdot \nabla Z^\beta \mathbf{u}_a Z^\beta \mathbf{u} + \mathbf{u} \cdot \nabla Z^\beta \theta_a Z^\beta \theta) - \int_\Omega \nabla Z^\beta p Z^\beta \mathbf{u} + \int_\Omega Z^\beta \theta e_2 Z^\beta u \\
&\quad + \int_\Omega \mathcal{C}_{\mathbf{u}} Z^\beta \mathbf{u} + \int_\Omega \mathcal{C}_\theta Z^\beta \theta + \varepsilon^K \int_\Omega (Z^\beta R_{\mathbf{u}} Z^\beta \mathbf{u} + Z^\beta R_\theta Z^\beta \theta) := \sum_{j=1}^{6} I_i.
\end{aligned}$$

We deduce from Navier boundary condition (1.2) that

$$\begin{aligned}
&\int_{\partial\Omega} \partial_y Z^\beta u_1 Z^\beta u_1 = \int_{\partial\Omega} Z^\beta \partial_y u_1 Z^\beta u_1 + \int_{\partial\Omega} [\partial_y, Z^\beta] u_1 Z^\beta u_1 \\
&= \alpha \int_{\partial\Omega} |Z^\beta u_1|^2 = \alpha \int_{\partial\Omega} |\partial_x^{|\beta|} u_1|^2 = \alpha \|u_1\|^2_{H^m_{co}(\partial\Omega)},
\end{aligned} \tag{3.23}$$

due to

$$\begin{aligned}
&[\partial_y, Z_1] = 0 \\
&Z^\beta u_1|_{\partial\Omega} = 0, \text{ if } \beta_2 > 0.
\end{aligned}$$

Here we note that if the boundary $\partial\Omega$ is not flat, $Z_2\mathbf{u}|_{\partial\Omega} = 0$ does not hold. Hence, by using trace theorem, we can derive that

$$|\alpha|\int_{\partial\Omega}|Z^\beta u_1|^2 \le C|\alpha|\|\partial_y u_1\|_{H^m_{co}}\|u_1\|_{H^m_{co}}. \tag{3.24}$$

Then we estimate I_i $(1\le i\le 6)$ one by one. For I_1,

$$\begin{aligned}|I_1| &\lesssim |\nabla Z^\beta \mathbf{u}_a|_{L^\infty}\int_\Omega \mathbf{u}Z^\beta\mathbf{u} \lesssim (|Z^{\beta-1}\partial_y\mathbf{u}_a|_{L^\infty} + |Z^\beta\partial_y\mathbf{u}_a|_{L^\infty})\int_\Omega(|\mathbf{u}|^2+|Z^\beta\mathbf{u}|^2)\\ &\lesssim \int_\Omega(|\mathbf{u}|^2+|Z^\beta\mathbf{u}|^2)\lesssim \|\mathbf{u}\|^2_{H^m_{co}}. \end{aligned}\tag{3.25}$$

Note that

$$I_2 = -\int_\Omega \nabla Z^\beta p Z^\beta\mathbf{u} = \int_\Omega Z^\beta p\nabla\cdot Z^\beta\mathbf{u} - \int_{\partial\Omega} Z^\beta p Z^\beta \mathbf{u}\cdot\mathbf{n} = -\int_\Omega Z^\beta p([Z^\beta,\nabla\cdot]\mathbf{u}),$$

and

$$\begin{aligned}&[Z_2,\nabla\cdot]\mathbf{u} = -\varphi'\partial_y u_2 = \varphi'\partial_x u_1,\\ &[Z_1,\nabla\cdot]\mathbf{u} = 0.\end{aligned}$$

Hence we obtain

$$|I_2| \lesssim \int_\Omega(|Z^\beta\mathbf{u}||Z^\beta p|)\lesssim \|\mathbf{u}\|_{H^m_{co}}\|\nabla p\|_{H^m_{co}}. \tag{3.26}$$

By virtue of the Cauchy inequality, we have

$$|I_3| \lesssim \int_\Omega(|Z^\beta\theta|^2+|Z^\beta\mathbf{u}|^2)\lesssim \|\mathbf{u}\|^2_{H^m_{co}}+\|\theta\|^2_{H^m_{co}}, \tag{3.27}$$

$$|I_6| \lesssim \int_\Omega(|Z^\beta\theta|^2+|Z^\beta\mathbf{u}|^2)+\varepsilon^{2K}\lesssim \|\mathbf{u}\|^2_{H^m_{co}}+\|\theta\|^2_{H^m_{co}}+\varepsilon^{2K}. \tag{3.28}$$

As for the term I_4 containing commutators $\mathcal{C}_i$ $(1\le i\le 4)$, we have

$$\begin{aligned}\mathcal{C}_1 &= \sum_{\gamma+\zeta=\beta,\gamma\neq 0} c^1_{\gamma,\zeta}Z^\gamma\mathbf{u}\cdot Z^\zeta\nabla\mathbf{u}_a + \mathbf{u}\cdot[Z^\beta,\nabla]\mathbf{u}_a,\\ \mathcal{C}_2 &= \sum_{\gamma+\zeta=\beta,\gamma\neq 0} c^2_{\gamma,\zeta}Z^\gamma\mathbf{u}_a\cdot Z^\zeta\nabla\mathbf{u} + \mathbf{u}_a\cdot[Z^\beta,\nabla]\mathbf{u},\end{aligned}$$

and

$$\int_\Omega \mathcal{C}_1^2 \lesssim \int_\Omega |Z^\beta\mathbf{u}|^2,\quad \int_\Omega \mathcal{C}_2^2\lesssim\int_\Omega|Z^\beta\mathbf{u}|^2,$$

due to

$$\begin{aligned}&|\nabla\mathbf{u}_a|_{L^\infty}\le C_a,\\ &u_{a2}=0,\quad\text{and}\quad \left\|u_{a2}\partial_y Z^{|\beta|-1}\mathbf{u}\right\| = \left\|\frac{u_{a2}}{\varphi(y)}\varphi(y)\partial_y Z^{|\beta|-1}\mathbf{u}\right\|\le |u_{a2}|_{W^{1,\infty}}\|Z^{|\beta|}\mathbf{u}\|.\end{aligned}$$

One can easily get

$$\|\mathcal{C}_3\| \lesssim \|Z^{|\beta|-1}\nabla p\| \lesssim \|\nabla p\|_{H^{m-1}_{co}}. \tag{3.29}$$

Together with

$$[Z_1, \partial_y^2]\mathbf{u} = 0, \quad [Z_2, \partial_y^2]\mathbf{u} = -\varphi''\partial_y \mathbf{u} - 2\varphi\partial_y^2\mathbf{u}, \quad Z_2\mathbf{u}|_{\partial\Omega} = 0,$$

we can derive the estimates for $\mathcal{C}_4$ by integration by parts:

$$\int_\Omega \mathcal{C}_4 Z^\beta \mathbf{u} \lesssim \varepsilon^2(\|\partial_y \mathbf{u}\|_{H^m_{co}} + \|\mathbf{u}\|_{H^m_{co}})\|\mathbf{u}\|_{H^m_{co}}. \tag{3.30}$$

Hence we can obtain the conormal estimates for I_4, using the Cauchy inequality,

$$|I_4| \lesssim \varepsilon^2(\|\partial_y \mathbf{u}\|_{H^m_{co}} + \|\mathbf{u}\|_{H^m_{co}})\|\mathbf{u}\|_{H^m_{co}} + \|\mathbf{u}\|^2_{H^m_{co}} + \|\nabla p\|_{H^{m-1}_{co}}\|\mathbf{u}\|_{H^m_{co}}. \tag{3.31}$$

Similarly, one can obtain that

$$|I_5| \lesssim \|\theta\|^2_{H^m_{co}} + \|\mathbf{u}\|^2_{H^m_{co}}. \tag{3.32}$$

Now one can deduce (3.20) from (3.23), (3.25), (3.26), (3.27), (3.28), (3.31), (3.32) and the Young inequality. The proof of Lemma 3.7 is completed. □

Normal derivatives estimates

In view of the conormal energy estimates, we shall give the estimates of the normal derivatives $\|\partial_y \mathbf{u}\|_{H^{m-1}_{co}}$. Due to the divergence free condition, we obtain that

$$\|\partial_y u_2\|_{H^{m-1}_{co}} \lesssim \|u_1\|_{H^m_{co}} \lesssim \|\mathbf{u}\|_{H^m_{co}}.$$

Hence it remains to estimate the normal derivatives of the tangential velocity $\|\partial_y u_1\|_{H^{m-1}_{co}}$.

Recall that the vorticity ω is defined as

$$\omega = \text{curl } \mathbf{u} = \partial_y u_1 - \partial_x u_2.$$

For the approximate solution, we denote ω_a as

$$\omega_a = \text{curl } \mathbf{u}_a = \partial_y u_{a1} - \partial_x u_{a2}.$$

Then we can derive from the velocity equation in (3.3) that

$$\partial_t \omega + \mathbf{u}\cdot\nabla\omega_a + \mathbf{u}_a\cdot\nabla\omega - \varepsilon^2\partial_y^2\omega = \partial_x\theta + \varepsilon^K \text{curl } R_{\mathbf{u}}, \tag{3.33}$$

where curl $R_{\mathbf{u}} = (\partial_y R_{u_1} - \partial_x R_{u_2})$.

For the new variables

$$\eta = \omega - \alpha u_1, \quad \eta_a = \omega_a - \alpha u_{a1},$$

we have the advantages on the boundary that

$$\eta|_{\partial\Omega} = 0, \quad \eta_a|_{\partial\Omega} = 0. \tag{3.34}$$

Moreover, it holds that

$$\|\partial_y u_1\|_{H^{m-1}_{co}} \lesssim \|\eta\|_{H^{m-1}_{co}} + \|\mathbf{u}\|_{H^m_{co}}.$$

Combining with the first equation of the velocity in (3.3), we rewrite the vorticity equation (3.33) into the following form

$$\partial_t\eta + \mathbf{u}\cdot\nabla\eta_a + \mathbf{u}_a\cdot\nabla\eta - \varepsilon^2\partial_y^2\eta = \partial_x p + \partial_x\theta - \varepsilon^K R_{u_1} + \varepsilon^K \mathrm{curl} R_{\mathbf{u}}. \tag{3.35}$$

Taking normal derivatives on (3.3b), one has

$$\partial_t\partial_y\theta + \mathbf{u}\cdot\nabla\partial_y\theta_a + \mathbf{u}_a\cdot\nabla\partial_y\theta = \mathcal{C}' + \varepsilon^K\partial_y R_\theta, \tag{3.36}$$

where $\mathcal{C}' = -[\partial_y, \mathbf{u}\cdot\nabla]\theta_a - [\partial_y, \mathbf{u}_a\cdot\nabla]\theta$.

Then we have the following normal derivatives estimates.

Lemma 3.8. *For every $m \geq 1$ and every smooth solution of the system (3.3) with partial viscocity, we have*

$$\begin{aligned} &\frac{d}{dt}(\|\eta\|^2_{H^{m-1}_{co}} + \|\partial_y\theta\|^2_{H^{m-1}_{co}}) + c_0\varepsilon^2\|\partial_y\eta\|^2_{H^{m-1}_{co}} \\ &\lesssim \|\nabla p\|_{H^{m-1}_{co}}\|\eta\|_{H^{m-1}_{co}} + \|\eta\|^2_{H^{m-1}_{co}} + \|\mathbf{u}\|^2_{H^m_{co}} + \|\partial_y\theta\|^2_{H^{m-1}_{co}} + \varepsilon^{2K-2} \end{aligned} \tag{3.37}$$

for some $c_0 > 0$.

Proof. We take L^2 inner product on equation (3.35) with η to get

$$\frac{d}{dt}\int_\Omega \eta^2 + \varepsilon^2\int_\Omega |\partial_y\eta|^2 = \int_\Omega \left[-\mathbf{u}\cdot\nabla\eta_a\eta + \partial_x p\eta + \partial_x\theta\eta - \varepsilon^K R_{u_1}\eta + \varepsilon^K \mathrm{curl} R_{\mathbf{u}}\eta\right] := RHS,$$

where RHS satisfies

$$\begin{aligned} RHS &\lesssim \int_\Omega \left[|u_1\partial_x\eta_a\eta| + |\frac{u_2}{\varphi(y)}\varphi(y)\partial_y\eta_a\eta| + |\partial_x p\eta| + |\partial_x\theta\eta| + \varepsilon^K|R_{u_1}\eta| + |\mathrm{curl} R_{\mathbf{u}}\eta|\right] \\ &\lesssim \|\mathbf{u}\|^2 + \|\nabla p\|\|\eta\| + \|\eta\|^2 + \|Z(\mathbf{u},\theta)\|^2 + \varepsilon^{2K-2}, \end{aligned}$$

with the aid of the Cauchy inequality and Hardy inequality. Then we have

$$\frac{d}{dt}\|\eta\|^2 + \varepsilon^2\|\partial_y\eta\|^2 \lesssim \|\mathbf{u}\|^2 + \|\nabla p\|\|\eta\| + \|\eta\|^2 + \|Z\theta\|^2 + \varepsilon^{2K-2}. \tag{3.38}$$

Similarly, we process L^2 inner product on the equation (3.36) with $\partial_y\theta$ to get

$$\frac{d}{dt}\|\partial_y\theta\|^2 \lesssim \|(\eta, \partial_y\theta)\|^2 + \|Zu\|^2 + \varepsilon^{2K-2}, \tag{3.39}$$

using the Cauchy inequality and the Hardy inequality.

We take H_{co}^{k-1}−inner product on the equations (3.35) (3.36) with η and $\partial_y\theta$, respectively. Then the desired results (3.37) can be verified by induction on $k \geq 1$. Here we omit the detail for simplicity. □

Pressure estimates.

In this part, we give the conormal estimates of the pressure. In view of the velocity equations in the partially viscous system (3.3a), we first consider the following system

$$\partial_t \mathbf{u} - \varepsilon^2 \partial_y^2 \mathbf{u} + \nabla p = F, \qquad y > 0, \tag{3.40}$$

$$\nabla \cdot \mathbf{u} = 0, \qquad y > 0, \tag{3.41}$$

$$u_2 = 0, \quad \partial_y u_1 = \alpha u_1, \qquad y = 0. \tag{3.42}$$

where F represents the given source term. Here F reads

$$F = -(\mathbf{u}_a \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_a) + \theta e_2 + \varepsilon^K R_{\mathbf{u}}$$

in the linear system.

By taking divergence on the equations (3.40), using divergence free condition, we obtain the following elliptic equation of pressure

$$\Delta p = \nabla \cdot F, \qquad y > 0. \tag{3.43}$$

One can deduce from (3.41) (3.42) that, the boundary condition for (3.43),

$$\begin{aligned}\partial_y p(x,0) &= \varepsilon^2 \partial_y^2 u_2(x,0) - \partial_t u_2(x,0) + F_2(x,0)\\ &= -\varepsilon^2 \partial_y \partial_x u_1(x,0) + F_2(x,0)\\ &= -\varepsilon^2 \alpha \partial_x u_1(x,0) + F_2(x,0).\end{aligned}$$

We can express p as following

$$p = p_1 + p_2,$$

where p_1 satisfies

$$\Delta p_1 = \nabla \cdot F, \quad y > 0, \quad \partial_y p(x,0) = F_2(x,0), \tag{3.44}$$

and p_2 solves

$$\Delta p_2 = 0, \qquad y > 0, \quad \partial_y p_2(x,0) = -\alpha \varepsilon^2 \partial_x u_1(x,0). \tag{3.45}$$

The desired estimates for the pressure p can be achieved from the estimates of p_1 and p_2 by standard elliptic theory. One can refer to the Theorem 11 in [46] by Masmoudi and Rousset for the details of proof. Here we only state the desired results for simplicity.

Proposition 3.9. *Considering the partially viscous system (3.40)-(3.42), for every $m \geq 2$, there exists $C > 0$ such that for every $t \geq 0$, we have the estimates*

$$\|\nabla p\|_{H^{m-1}_{co}} \leq C(\|F\|_{H^{m-1}_{co}} + \|\nabla \cdot F\|_{H^{m-2}_{co}} + \varepsilon^2 \|\nabla \mathbf{u}\|_{H^{m-1}_{co}} + \|\mathbf{u}\|_{H^{m-1}_{co}}). \tag{3.46}$$

Taking the source term F in (3.46) as

$$F = (\mathbf{u}_a \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_a) + \theta e_2 + \varepsilon^K R_{\mathbf{u}},$$

we can reach the desired estimates of the pressure p in the following lemma.

Lemma 3.10. *For every $m \geq 2$, and every $\varepsilon \in (0, 1]$, assume $(\mathbf{u}, \theta)$ be a smooth solution of (3.3) on $[0, T]$. Then it holds that*

$$\|\nabla p\|_{H^{m-1}_{co}} \lesssim (1 + \varepsilon^2)\|\mathbf{u}\|_{H^m_{co}} + \|(\mathbf{u}, \theta)\|_{H^{m-1}_{co}} + \|\partial_y \mathbf{u}\|_{H^{m-1}_{co}} + \|\partial_y \theta\|_{H^{m-2}_{co}} + \varepsilon^{K-1}. \tag{3.47}$$

Proof of the Theorem 3.3. Combining the estimates(3.18) (3.20) (3.37) (3.47), we can obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\|(\mathbf{u}, \theta)\|^2 + \|(\eta, \partial_y \theta)\|^2_{H^{m-1}_{co}}\Big) + c_0 \varepsilon^2 \|(\partial_y \mathbf{u}, \partial_y \eta)\|^2_{H^{m-1}_{co}} \\
&\lesssim \|(\mathbf{u}, \theta)\|^2_{H^m_{co}} + \|(\eta, \partial_y \theta)\|^2_{H^{m-1}_{co}} + \varepsilon^{2K-2}.
\end{aligned} \tag{3.48}$$

Then (3.14) follows from (3.48) and the Gronwall inequality. Consequently, we have (3.15) by using (3.14) and the Lemma 3.1. The proof the Theorem 3.3 is completed. □

4. Nonlinear Stability Estimates

In this section, we prove the nonlinear stability of the approximate solutions constructed in the Section 2 under the form (2.1). The zero-viscosity limit from the partially viscous system (1.1) to the inviscid system (1.3) will be verified with the detailed convergence rates.

In contrast with the linear stability estimates in the Section 3, we shall deal with the nonlinear terms

$$\sum_{\gamma+\zeta=\beta, \gamma\neq 0} c_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \mathbf{u} + \mathbf{u} \cdot [Z^\beta, \nabla]\mathbf{u}, \qquad \sum_{\gamma+\zeta=\beta, \gamma\neq 0} d_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \theta + \mathbf{u} \cdot [Z^\beta, \nabla]\theta$$

in the conormal energy estimates with $|\beta| \leq m$, and

$$\sum_{\gamma+\zeta=\beta, \gamma\neq 0} \tilde{c}_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \eta + \mathbf{u} \cdot [Z^\beta, \nabla]\eta, \qquad \sum_{\gamma+\zeta=\beta, \gamma\neq 0} \tilde{d}_{\gamma,\zeta} Z^\gamma \mathbf{u} \cdot Z^\zeta \nabla \partial_y \theta + \mathbf{u} \cdot [Z^\beta, \nabla]\partial_y \theta$$

in the normal derivatives estimates with $|\beta| \le m-1$, respectively. Here $c_{\gamma,\zeta}, d_{\gamma,\zeta}, \tilde{c}_{\gamma,\zeta}, \tilde{d}_{\gamma,\zeta}$ are positive constants depending on γ and ζ. The pressure estimates should involve the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$ in the source term F. In the nonlinear analysis, the estimates of $\|\eta\|_{1,\infty}$ and $\|\partial_y\theta\|_{1,\infty}$ are crucial to the closure of energy estimates. We deal with the term $\|\eta\|_{1,\infty}$ in the spirit of the methods in [46] (using Maximum principle for transport-diffusion equation and precise estimates for the Green's function of nonlinear operator). As for $\|\partial_y\theta\|_{1,\infty}$, we take advantage of a maximum principle for the generalized quasi-geostrophic equations (cf. [16, 49]).

Let us define

$$E_m(t) = \|(\mathbf{u},\theta)\|^2_{H^m_{co}} + \|(\eta,\partial_y\theta)\|^2_{H^{m-1}_{co}} + \|(\eta,\partial_y\theta)\|^2_{1,\infty}.$$

The proof of the Theorem 1.1 can be reduced to the justification of the following proposition, due to the Lemma 3.1.

Proposition 4.1. *Assume $(\mathbf{u},\theta)$ be a solution of (2.9) (3.5) defined on $[0,T]$ with T independent of ε. For $m>6$, $K\ge 2$, we have the a priori estimate*

$$E_m(t) \lesssim E_m(0) + (1+t)\int_0^t (E_m^2(s)+E_m(s))ds + \varepsilon^{2K-2}. \tag{4.1}$$

Note that

$$\|\mathbf{u}\cdot[Z^\beta,\nabla]\mathbf{u}\| \lesssim \sum_{|\gamma|\le m-1}\|u_2\partial_y Z^\gamma\mathbf{u}\| = \sum_{|\gamma|\le m-1}\left\|\frac{u_2}{\varphi(y)}\varphi(y)\partial_y Z^\gamma\mathbf{u}\right\| \lesssim \|u_2\|_{W^{1,\infty}}\|\mathbf{u}\|_{H^m_{co}}$$

for $|\beta|\le m$, due to $u_2|_{y=0}=0$ and Hardy inequality. Similarly, we have

$$\|\mathbf{u}\cdot[Z^\beta,\nabla]\theta\| \lesssim \|u_2\|_{W^{1,\infty}}\|\theta\|_{H^m_{co}}.$$

By virtue of the fact that

$$\|Z^{\beta_1}uZ^{\beta_2}v\| \lesssim \|u\|_{L^\infty}\|v\|_{H^k_{co}} + \|v\|_{L^\infty}\|u\|_{H^k_{co}}, \quad |\beta_1|+|\beta_2|=k, \tag{4.2}$$

one can deduce that

$$\|c_{\gamma,\zeta}Z^\gamma\mathbf{u}\cdot Z^\zeta\nabla\mathbf{u}\| \lesssim \|\nabla\mathbf{u}\|_{L^\infty}(\|\mathbf{u}\|_{H^m_{co}} + \|\partial_y\mathbf{u}\|_{H^{m-1}_{co}}), \tag{4.3}$$

$$\|d_{\gamma,\zeta}Z^\gamma\mathbf{u}\cdot Z^\zeta\nabla\theta\| \lesssim \|\nabla\mathbf{u}\|_{L^\infty}(\|\theta\|_{H^m_{co}} + \|\partial_y\theta\|_{H^{m-1}_{co}}) + \|\nabla\theta\|_{L^\infty}\|\mathbf{u}\|_{H^m_{co}}. \tag{4.4}$$

Hence we can obtain the conormal energy estimates for the nonlinear system

$$\begin{aligned}
&\frac{d}{dt}(\|(\mathbf{u},\theta)\|^2_{H^m_{co}}) + c_0\varepsilon^2\|\partial_y\mathbf{u}\|^2_{H^m_{co}} \\
&\lesssim (1+\|\mathbf{u}\|_{W^{1,\infty}})\left(\|(\theta,\mathbf{u})\|^2_{H^m_{co}} + \|\partial_y(\mathbf{u},\theta)\|^2_{H^{m-1}_{co}}\right) + \|\nabla p\|_{H^{m-1}_{co}}\|\mathbf{u}\|_{H^m_{co}} \\
&\quad + \|\nabla\theta\|_{L^\infty}\|\mathbf{u}\|^2_{H^m_{co}} + \varepsilon^{2K}.
\end{aligned} \tag{4.5}$$

In the estimates of normal derivatives $\|(\eta, \partial_y\theta)\|_{H^{m-1}_{co}}$, to avoid the appearance of the terms like $\|\partial_y\eta\|_{L^\infty}$ and $\|\partial_y\eta\|_{H^m_{co}}$, we write, for $\zeta+\gamma=\beta$, $\gamma\neq 0$, $|\beta|\leq m-1$,

$$\begin{aligned}
&\|\tilde{c}_{\gamma,\zeta} Z^\gamma \mathbf{u}\cdot Z^\zeta \nabla\eta\| \\
&\lesssim \|Z^\gamma u_1 Z^\zeta \partial_x \eta\| + \|Z^\gamma u_2 Z^\zeta \partial_y\eta\| \\
&\lesssim \|\nabla u_1\|_{L^\infty}\|\eta\|_{H^{m-1}_{co}} + \|\eta\|_{L^\infty}\|Zu_1\|_{H^{m-1}_{co}} + \left\|\frac{1}{\varphi(y)} Z^\gamma u_2 \varphi(y) Z^\zeta \partial_y \eta\right\| \\
&\lesssim \|\mathbf{u}\|_{W^{1,\infty}}(\|\eta\|_{H^{m-1}_{co}} + \|\mathbf{u}\|_{H^m_{co}}) + (\|\mathbf{u}\|_{W^{1,\infty}} + \|\mathbf{u}\|_{2,\infty} + \|Z\eta\|_{L^\infty})(\|\eta\|_{H^{m-1}_{co}} + \|\mathbf{u}\|_{H^m_{co}}), \\
&\lesssim (\|\mathbf{u}\|_{W^{1,\infty}} + \|\mathbf{u}\|_{2,\infty} + \|Z\eta\|_{L^\infty})(\|\eta\|_{H^{m-1}_{co}} + \|\mathbf{u}\|_{H^m_{co}}),
\end{aligned}$$

with the aid of (4.2), divergence free condition and the expansion of the form

$$\frac{1}{\varphi(y)} Z^\gamma u_2 \varphi(y) Z^\zeta \partial_y \eta = c_{\tilde{\gamma},\tilde{\zeta}} Z^{\tilde{\gamma}}\left(\frac{1}{\varphi(y)}u_2\right) Z^{\tilde{\zeta}}(\varphi\partial_y\eta), \tag{4.6}$$

where $|\tilde{\gamma}+\tilde{\zeta}|\leq m-1$, $|\tilde{\gamma}|\neq m-1$ and $c_{\tilde{\gamma},\tilde{\zeta}}$ is some smooth bounded coefficient. For the justification of this expansion (4.6), one can refer to [46] for more details.

Similarly, we can obtain that

$$\|\tilde{d}_{\gamma,\zeta} Z^\gamma \mathbf{u} Z^\zeta \nabla \partial_y\theta\| \lesssim (\|\mathbf{u}\|_{W^{1,\infty}} + \|\mathbf{u}\|_{2,\infty})\,\|\partial_y\theta\|_{H^{m-1}_{co}} + \|\partial_y\theta\|_{1,\infty}\|\mathbf{u}\|_{H^m_{co}}. \tag{4.7}$$

Together with

$$\|\mathbf{u}\cdot[Z^\beta,\nabla]\eta\| \lesssim \|\mathbf{u}\|_{W^{1,\infty}}\|\eta\|_{H^{m-1}_{co}}, \quad \|\mathbf{u}\cdot[Z^\beta,\nabla]\partial_y\theta\| \lesssim \|\mathbf{u}\|_{W^{1,\infty}}\|\partial_y\theta\|_{H^{m-1}_{co}}$$

for $|\beta|\leq m-1$, it follows that

$$\begin{aligned}
&\frac{d}{dt}(\|\eta\|^2_{H^{m-1}_{co}} + \|\partial_y\theta\|^2_{H^{m-1}_{co}}) + c_0\varepsilon^2\|\partial_y\eta\|^2_{H^{m-1}_{co}} \\
&\lesssim \|\nabla p\|_{H^{m-1}_{co}}\|\eta\|_{H^{m-1}_{co}} + (1+\|\mathbf{u}\|_{W^{1,\infty}} + \|\mathbf{u}\|_{2,\infty} + \|Z\eta\|_{L^\infty})(\|\eta\|^2_{H^{m-1}_{co}} + \|\mathbf{u}\|^2_{H^m_{co}}) \\
&\quad + \|\partial_y\theta\|_{1,\infty}\|\mathbf{u}\|^2_{H^m_{co}} + (1+\|\mathbf{u}\|_{W^{1,\infty}} + \|\mathbf{u}\|_{2,\infty})\|\partial_y\theta\|^2_{H^{m-1}_{co}} + \varepsilon^{2K-2}.
\end{aligned} \tag{4.8}$$

In the pressure estimates for the nonlinear system (2.9), we consider the source term as

$$F = (\mathbf{u}_a\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_a) + \mathbf{u}\cdot\nabla\mathbf{u} + \theta e_2 + \varepsilon^K R_{\mathbf{u}}.$$

Then it follows from (3.46) that

$$\|\nabla p\|_{H^{m-1}_{co}} \lesssim (1+\|\mathbf{u}\|_{W^{1,\infty}})\left(\|\mathbf{u}\|_{H^m_{co}} + \|\partial_y\mathbf{u}\|_{H^{m-1}_{co}}\right) + \|(\mathbf{u},\theta)\|_{H^{m-1}_{co}} + \|\partial_y\theta\|_{H^{m-2}_{co}} + \varepsilon^{K-1}, \tag{4.9}$$

due to

$$\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^{m-1}_{co}} \lesssim \|\mathbf{u}\|_{W^{1,\infty}}\left(\|\mathbf{u}\|_{H^{m-1}_{co}} + \|\nabla\mathbf{u}\|_{H^{m-1}_{co}}\right),$$

$$\|\nabla \mathbf{u}\cdot\nabla \mathbf{u}\|_{H^{m-2}_{co}} \lesssim \|\nabla \mathbf{u}\|_{L^\infty}\|\nabla \mathbf{u}\|_{H^{m-2}_{co}}.$$

Hence we can deduce from (4.5) (4.8) (4.9) that

$$\begin{aligned}
&\frac{d}{dt}\left(\|(\mathbf{u},\theta)\|^2_{H^m_{co}} + \|(\eta,\partial_y\theta)\|^2_{H^{m-1}_{co}}\right) + c_0\varepsilon^2\left(\|\partial_y \mathbf{u}\|^2_{H^m_{co}} + \|\partial_y\eta\|^2_{H^{m-1}_{co}}\right)\\
&\lesssim (1+\|\mathbf{u}\|_{W^{1,\infty}} + \|\mathbf{u}\|_{2,\infty} + \|Z\eta\|_{L^\infty})\left(\|(\theta,\mathbf{u})\|^2_{H^m_{co}} + \|(\eta,\partial_y\theta)\|^2_{H^{m-1}_{co}}\right)\\
&\quad + (\|\nabla\theta\|_{L^\infty} + \|\partial_y\theta\|_{1,\infty})\|\mathbf{u}\|^2_{H^m_{co}} + \varepsilon^{2K-2}. \qquad (4.10)
\end{aligned}$$

L^∞ estimates.

It follows from (3.11) (3.12) (3.13) in the Lemma 3.2 and (3.9) that

$$\|\mathbf{u}\|_{W^{1,\infty}} \lesssim E_m^{1/2}(t),\quad \|\mathbf{u}\|_{2,\infty} \lesssim E_m^{1/2}(t),\quad \|\nabla \mathbf{u}\|_{1,\infty} \lesssim E_m^{1/2}(t),\quad m\geq m_0+3, \qquad (4.11)$$

$$\|\theta\|_{1,\infty} \lesssim \|\theta\|_m + \|\partial_y\theta\|_{m-1} \lesssim E_m^{1/2}(t),\quad \|\theta\|_{2,\infty} \lesssim E_m^{1/2}(t),\quad m\geq m_0+3, \qquad (4.12)$$

for $m_0 > 1$.

Combining the inequalities (4.10) (4.11) and (4.12), we still need to estimate

$$\|\eta\|_{1,\infty},\quad \|\partial_y\theta\|_{1,\infty}$$

to close the energy estimates. As for the estimates of $\|\eta\|_{1,\infty}$, we can derive the following results as in [46]. One can refer to the proof of Proposition 13 in [46] for details.

Lemma 4.2. *For $m > 6$, we have the estimate*

$$\|\eta\|^2_{1,\infty} \lesssim E_m(0) + (1+t)\int_0^t (E_m^2(s) + E_m(s))ds + \varepsilon^{2K-2}. \qquad (4.13)$$

Therefore, it remains to estimate $\|\partial_y\theta\|_{1,\infty}$, which can be derived from the maximum principle for an advective scalar equation [16, 49].

Lemma 4.3. *For $m > 6$, we have the estimate*

$$\|\partial_y\theta\|^2_{1,\infty} \lesssim E_m(0) + \int_0^t E_m(s)ds + \varepsilon^{2K-2}. \qquad (4.14)$$

Proof. We rewrite (2.9b) in the following form

$$\partial_t\theta + (\mathbf{u}+\mathbf{u}_a)\cdot\nabla\theta = f(t,x,y) := -\mathbf{u}\cdot\nabla\theta_a + \varepsilon^K R_\theta,\quad y>0, \qquad (4.15)$$

where $\mathbf{u}, \mathbf{u}_a$ satisfy

$$\nabla\cdot\mathbf{u} = 0,\quad \nabla\cdot\mathbf{u}_a = 0,\quad u_2(t,x,0) = 0,\quad u_{a2}(t,x,0) = 0.$$

We transform the problem into the whole space by defining $\tilde{\theta}, \tilde{\theta}_a, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}_a, \tilde{f}$:

$$(\tilde{\theta}, \tilde{\theta}_a, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}_a) = (\theta, \theta_a, \mathbf{u}, \mathbf{u}_a)(t, x, y),\ y > 0; \quad (\tilde{\theta}, \tilde{\theta}_a, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}_a) = -(\theta, \theta_a, \mathbf{u}, \mathbf{u}_a)(t, x, -y),\ y < 0;$$
$$\tilde{f} = -\tilde{\mathbf{u}} \cdot \nabla\tilde{\theta}_a + \varepsilon^K \tilde{R}_\theta.$$

Then one has

$$\partial_t\tilde{\theta} + (\tilde{\mathbf{u}} + \tilde{\mathbf{u}}_a) \cdot \nabla\tilde{\theta} = \tilde{f}, \quad \tilde{\theta}_0 = \tilde{\theta}|_{t=0} \tag{4.16}$$

with $\nabla \cdot \tilde{\mathbf{u}} = 0$ and $\nabla \cdot \tilde{\mathbf{u}}_a = 0$. Applying maximum principle for advective scalar equations (see [49], 3.2 A priori Bounds. and [12, 16]), we obtain that, for $t \in [0, T]$,

$$\begin{aligned} \|\tilde{\theta}\|_{L^\infty} &\lesssim \|\tilde{\theta}|_{t=0}\|_{L^\infty} + \int_0^t \|\tilde{f}(\tau)\|_{L^\infty} d\tau \\ &\lesssim \varepsilon^{K+1}\|\theta_0\|_{L^\infty} + \int_0^t \|\mathbf{u}(\tau)\|_{L^\infty} d\tau + \varepsilon^K. \end{aligned} \tag{4.17}$$

Similarly, we can derive the L^∞-estimates of $\nabla\tilde{\theta}$ and $Z\partial_y\tilde{\theta}$,

$$\|\nabla\tilde{\theta}\|_{L^\infty} \lesssim \varepsilon^{K+1}\|\nabla\theta_0\|_{L^\infty} + \int_0^t (\|\mathbf{u}(\tau)\|_{L^\infty} + \|\nabla\mathbf{u}(\tau)\|_{L^\infty}) d\tau + \varepsilon^{K-1}, \tag{4.18}$$

$$\|Z\partial_y\tilde{\theta}\|_{L^\infty} \lesssim \varepsilon^{K+1}\|Z\partial_y\theta_0\|_{L^\infty} + \int_0^t (\|\mathbf{u}(\tau)\|_{W^{1,\infty}} + \|\mathbf{u}\|_{1,\infty} + \|Z\partial_y\mathbf{u}(\tau)\|_{L^\infty}) d\tau + \varepsilon^{K-1}, \tag{4.19}$$

due to

$$\|u_2\partial_y\partial_y\theta_a\|_{L^\infty} \le \left\| \frac{u_2}{\varphi(y)}\varphi(y)\partial_y\partial_y\theta_a \right\|_{L^\infty} \lesssim \|u_2\|_{W^{1,\infty}}. \tag{4.20}$$

Hence the proof of (4.14) is completed with the aid of Lemma 3.2. □

Therefore, the proof of Proposition 4.1 can be obtained by combining (4.10) (4.13) (4.14) and the standard continuous argument.

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