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Coupled Sasaki-Ricci solitons

In Memory of Professor Zhengguo Bai (1916–2015)

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Abstract Motivated by the study of coupled Kähler-Einstein metrics by Hultgren and Witt Nyström (2018) and coupled Kähler-Ricci solitons by Hultgren (2017), we study in this paper coupled Sasaki-Einstein metrics and coupled Sasaki-Ricci solitons. We first show an isomorphism between the Lie algebra of all transverse holomorphic vector fields and certain space of coupled basic functions related to coupled twisted Laplacians for basic functions, and obtain extensions of the well-known obstructions to the existence of Kähler-Einstein metrics to this coupled case. These results are reduced to coupled Kähler-Einstein metrics when the Sasaki structure is regular. Secondly we show the existence of toric coupled Sasaki-Einstein metrics when the basic first Chern class is positive extending the work of Hultgren (2017).

Keywords coupled Kähler-Einstein metrics, coupled Sasaki-Einstein metrics, coupled Sasaki-Ricci solitons

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1 Introduction

Motivated by the proposed study of coupled Kähler-Einstein metrics by Hultgren and Witt Nyström [17] and coupled Kähler-Ricci solitons by Hultgren [16] we study in this paper coupled Sasaki-Einstein metrics and coupled Sasaki-Ricci solitons. Our work started in trying to understand their works from the viewpoint of the former studies [6–10] of Kähler-Einstein metrics, and hopefully would serve as a supplement to their papers since our results reduce to the case of coupled Kähler-Einstein metrics when the Sasaki structure is regular.

Let M be a Fano manifold and choose the Kähler class to be the anti-canonical class K_M^{-1} or equivalently the first Chern class $c_1(M)$. A decomposition of $c_1(M)$ is a sum

$$c_1(M) = (\gamma_1 + \cdots + \gamma_k)/2\pi \tag{1.1}$$

with Kähler classes $\gamma_1, \ldots, \gamma_k$. If we choose a Kähler form ω_α representing γ_α for each α, since both the Ricci form $\mathrm{Ric}(\omega_\alpha)$ and $\sum_{\beta=1}^k \omega_\beta$ represent $2\pi c_1(M)$, there exists a smooth function f_α such that

$$\mathrm{Ric}(\omega_\alpha) - \sqrt{-1}\partial\overline{\partial} f_\alpha = \sum_{\beta=1}^k \omega_\beta, \quad \alpha = 1, \ldots, k. \tag{1.2}$$

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Coupled Kähler-Ricci solitons are defined in [16] to be Kähler metrics with Kähler forms ω_α representing γ_α such that, for each α, f_α is a Hamiltonian Killing potential with respect to ω_α so that $J\mathrm{grad}_\alpha f_\alpha$ is a Hamiltonian Killing vector field where grad_α denotes the gradient with respect to ω_α. Coupled Kähler-Einstein metrics defined in [17] are the case where f_α's are all constant so that

$$\mathrm{Ric}(\omega_1) = \cdots = \mathrm{Ric}(\omega_k) = \sum_{\beta=1}^{k} \omega_\beta. \tag{1.3}$$

Now we consider Sasakian analogues of the above. Let S be a compact Sasaki manifold with the positive basic first Chern class $c_1^B(S)$, which means $c_1^B(S)$ is represented by a positive basic $(1,1)$-form. We assume that the real dimension of S is $2m+1$. We define similarly a decomposition of $c_1^B(S)$ to be a sum

$$c_1^B(S) = (\gamma_1 + \cdots + \gamma_k)/2\pi \tag{1.4}$$

of positive basic $(1,1)$ classes γ_α. If we choose basic Kähler forms ω_α representing γ_α, there exist smooth basic functions f_α such that

$$\mathrm{Ric}^{\mathrm{T}}(\omega_\alpha) - \sqrt{-1}\partial_B\overline{\partial}_B f_\alpha = \sum_{\beta=1}^{k} \omega_\beta, \quad \alpha = 1, \ldots, k, \tag{1.5}$$

where $\mathrm{Ric}^{\mathrm{T}}$ denotes the transverse Ricci form (see (2.2) below). We say ω_α's are coupled Sasaki-Ricci solitons if for each α, f_α is a Hamiltonian Killing potential for ω_α and coupled Sasaki-Einstein metrics if f_α is constant so that coupled Sasaki-Einstein metrics satisfy

$$\mathrm{Ric}^{\mathrm{T}}(\omega_1) = \cdots = \mathrm{Ric}^{\mathrm{T}}(\omega_k) = \sum_{\beta=1}^{k} \omega_\beta. \tag{1.6}$$

Remark 1.1. When $k=1$, the definition does not coincide with the usual definition of the transverse Kähler-Einstein metric of a Sasaki-Einstein metric which is known to be equivalent to saying the transverse Kähler metric is Einstein. This is because the Riemannian metric of a Sasaki manifold naturally determines a transverse Kähler form written in the form $\frac{1}{2}d\eta$ for the contact 1-form with respect to the given Reeb vector field but the basic Chern class $c_1^B(S)$ need not be represented by $\frac{1}{2}d\eta$. However if we assume $c_1(D) = 0$ as a de Rham class for the contact distribution D, we have $c_1^B(S) = [\frac{1}{2}d\eta]$ and the definition coincides with the transverse Kähler-Einstein form of a Sasaki-Einstein metric (see [2, Corollary 7.5.26]).

To study the coupled equations in Sasakian situation as described above we wish to extend the results for the Fano manifold as in [9]. Suppose we are given a compact Sasaki manifold (S,g) with the positive basic first Chern class and a decomposition of $c_1^B(S)$ as (1.4). We also choose ω_α in γ_α and then have f_α satisfying (1.5). We may normalize f_α so that

$$\mathrm{e}^{f_1}\omega_1^m = \cdots = \mathrm{e}^{f_k}\omega_k^m. \tag{1.7}$$

We define the twisted basic Laplacian Δ_{α,f_α} acting on basic smooth functions u by

$$\Delta_{\alpha,f_\alpha} u = \Delta_\alpha u + (\mathrm{grad}_\alpha u) f_\alpha, \tag{1.8}$$

where $\Delta_\alpha = -\overline{\partial}_{B,\alpha}^{*}\overline{\partial}_B$ is the basic Beltrami-Laplacian with respect to ω_α and $\mathrm{grad}_\alpha u = g_\alpha^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$, where $(x, z^1, \ldots, z^m)$ is the local foliation chart (see Section 2 below).

Recall that a compact Sasaki manifold (S,g) is characterized by its Riemannian cone manifold $C(S)$ being a Kähler manifold. A transverse holomorphic vector field is a holomorphic vector field X on the Kähler cone $C(S)$ which commutes with the extended Reeb vector field on $C(S)$ (see Section 2 for more details). It descends to a vector field on S, and also descends to a holomorphic vector field on local orbit spaces of the Reeb flow which we denote by the same letter X. Since $c_1^B(S)$ is positive the basic first cohomology is zero. Thus, for each basic Kähler form ω_α, there is a basic complex-valued Hamiltonian function u_α such that $X = \mathrm{grad}_\alpha u_\alpha$. Let $\mathfrak{h}^{\mathrm{T}}(S)$ be the complex Lie algebra of all transverse holomorphic vector fields.

Theorem 1.2. *Let (S, g) be a compact Sasaki manifold with the positive basic first Chern class $c_1^B(S)$ with the decomposition satisfying* (1.4). *We choose basic Kähler forms and associated potential functions satisfying* (1.5) *and* (1.7).

(1) *If non-constant complex valued basic functions $u_1, \dots, u_k$ satisfy*

(a) $\mathrm{grad}_\alpha u_\alpha = \mathrm{grad}_\beta u_\beta$, $\alpha, \beta = 1, 2, \dots, k$;

(b) $-\Delta_{\alpha, f_\alpha} u_\alpha = \lambda \sum_{\beta=1}^k u_\beta$, *for* $\alpha = 1, 2, \dots, k$,

then $\lambda \geqslant 1$. Moreover if $\lambda = 1$, the complex vector field $V = \mathrm{grad}_\alpha u_\alpha = \mathrm{grad}_\beta u_\beta$ is a transverse holomorphic vector field.

(2) *The Lie algebra $\mathfrak{h}^{\mathrm{T}}(S)$ of all transverse holomorphic vector fields is isomorphic to the set of all k-tuples of complex valued smooth functions $(u_1, \dots, u_k)$ satisfying* (a) *and* (b) *with $\lambda = 1$ endowed with the Lie algebra structure with respect to the Poisson bracket.*

This theorem is a generalization of [9, Theorem 2.4.3] (see also [8]). The case of $k = 1$ of (1) is an eigenvalue estimate of the twisted basic Laplacian. If $f_1, \dots, f_k$ are all constant then $\Delta_{\alpha, f_\alpha} = \Delta_\alpha$ is a real operator. It follows that the gradients of both the real part and the imaginary part of $u_1, \dots, u_k$ give a holomorphic vector field. Furthermore, since if a (real) Hamiltonian vector field is holomorphic it is necessarily Killing and the group of all isometries is compact, we obtain the following extension of a theorem of Matsushima [22].

Corollary 1.3. *If a compact Sasaki manifold S admits coupled Kähler-Einstein metrics, then the Lie algebra $\mathfrak{h}^{\mathrm{T}}(S)$ of transverse holomorphic vector fields is reductive.*

This result was stated in [17] for coupled Kähler-Einstein metrics, but our proof would be more elementary.

As in other non-linear problems in Kähler geometry (see, e.g., [12, 13, 18, 19]), a Lie algebra character obstruction as in [6] appears in pairs with the reductiveness result of Theorem 1.2. Using the isomorphism in Theorem 1.2(2), we define

$$\begin{aligned} \mathrm{Fut} : \mathfrak{h}^{\mathrm{T}}(S) &\to \mathbf{C} \\ V &\mapsto \sum_{\alpha=1}^{k} \frac{\int_S u_\alpha \, \omega_\alpha^m \wedge \eta}{\int_S \omega_\alpha^m \wedge \eta}. \end{aligned} \tag{1.9}$$

We will see in Section 4 that this definition is independent of the choice of $u_1, \dots, u_k$ satisfying (a) and (b) with $\lambda = 1$ above.

Theorem 1.4. *Suppose the basic Kähler classes γ_α give a basic decomposition $c_1^B(S) = \sum_{\alpha=1}^k \gamma_\alpha / 2\pi$. Then* $\mathrm{Fut}(V)$ *is independent of the choice of basic Kähler forms $\omega_\alpha \in \gamma_\alpha$, $\alpha = 1, \dots, k$. Furthermore, if S admits coupled Sasaki-Einstein metrics for the decomposition $c_1^B(S) = \sum_{\alpha=1}^k \gamma_\alpha / 2\pi$ then* Fut *identically vanishes.*

The lifting of the Lie algebra character in [6] to a group character was obtained in [7]. This lifting is expressed in terms of Ricci forms, and if we replace them by Kähler forms it becomes the form of the Monge-Ampère energy or Aubin's J-functional [1]. In [4], another form of lifting was obtained in the sense that it satisfies the cocycle conditions, and it is now called Ding's functional. The definition of Fut in this paper uses the relationship of [4, 7]. This seems to be already implicitly used in [16, 17].

We also extend the existence results of Wang and Zhu [25] and Hultgren [16] to toric coupled Sasaki-Einstein metrics as follows.

Theorem 1.5. *Let S be a compact toric Sasaki manifold with the positive basic first Chern class with the decomposition satisfying* (1.4). *Then S admits coupled Sasaki-Einstein metrics for the decomposition* (1.4) *if and only if* Fut *identically vanishes.*

Here, a Sasaki manifold is said to be toric if the cone $C(S)$ is toric, i.e., if $C(S)$ admits an effective $(\mathbf{C}^*)^{m+1}$-action. Theorem 1.5 follows from an existence result of toric coupled Sasaki-Ricci solitons (see Theorem 5.1 in Section 5). In the case of toric Sasaki-Einstein metrics one can deform the Reeb vector field so that Fut vanishes, and proving the existence of Sasaki-Ricci solitons one can conclude that there always exists a Sasaki-Einstein metric under the condition of $c_1(D) = 0$ (see [11, 14, 21]). It is not clear whether such a volume minimization argument applies in the coupled case.

The rest of this paper is organized as follows. In Section 2 we review the transverse Kähler structure and the notions of transverse holomorphic vector fields and Hamiltonian holomorphic vector fields. In Section 3, the proof of Theorem 1.2 is given. We also prove in Theorem 3.2 an identification of the Lie algebra of all transverse holomorphic vector fields with the Lie algebra expressed in terms of Hamiltonian functions satisfying a normalization condition. In Section 4, the proof of Theorem 1.4 is given. We also give an obstruction to the existence of coupled Sasaki-Ricci solitons. In Section 5, we first show in Theorem 5.2 that the normalization in Theorem 3.2 is equivalent to a Minkowski sum condition of the moment map image. Using this we reduce the proof of Theorem 1.5 to the same type of real Monge-Ampère equations as considered by Hultgren [16]. In Appendix A, we supplement the proof of Theorem 1.5 by making the standard moment map for the ample anti-canonical class explicit in terms of the first non-zero eigenfunctions of the twisted Laplacian (5.3).

2 The transverse Kähler structure on a Sasaki manifold

A compact Riemannian manifold (S, g) of dimension $2m+1$ is called a Sasaki manifold if its Riemannian cone $(C(S), dr^2 + r^2 g)$ is a Kähler manifold. S is identified with the submanifold $\{r = 1\}$. By using the convention $d^c = \sqrt{-1}(\overline{\partial} - \partial)$, the restriction η of $d^c \log r$ to $S = \{r = 1\}$ is a contact form. If we denote by J the complex structure of $C(S)$, the restriction ξ of the vector field $Jr\frac{\partial}{\partial r}$ to S is the Reeb vector field of the contact form η so that $i_\xi \eta = 1$ and $i_\xi d\eta = 0$.

Since the Sasaki structure is characterized by the Kähler structure of the cone $C(S)$ the geometry of Sasaki manifold S is often described in terms of the Kähler geometry of $C(S)$. Therefore it is convenient to extend the Reeb vector field ξ and the contact form η on S to $C(S)$, and we use the same letters to denote them. Thus on $C(S)$ we have

$$\xi = Jr\frac{\partial}{\partial r} \quad \text{and} \quad \eta = d^c \log r. \tag{2.1}$$

As this shows, when the holomorphic structure of the cone $C(S)$ is fixed, the radial function r has all the information about the Sasaki structure on S and the Kähler structure on $C(S)$, and in fact the Kähler form on $C(S)$ is given by $\frac{\sqrt{-1}}{2}\partial\overline{\partial}r^2$. The complex vector field $\xi - \sqrt{-1}J\xi$ is a holomorphic vector field on $C(S)$. It generates a $\mathbf{C}^*$ action on $C(S)$. The local orbit of this action defines a transversely holomorphic foliation on S, given by one-dimensional leaves generated by ξ. Let $\bigcup_\lambda U_\lambda$ be an open covering of S with $\pi_\lambda : U_\lambda \to V_\lambda \subset \mathbf{C}^m$ a submersion to the local orbit spaces. Then, when $U_\lambda \cap U_\mu \neq \emptyset$,

$$\pi_\lambda \circ \pi_\mu^{-1} : \pi_\mu(U_\lambda \cap U_\mu) \to \pi_\lambda(U_\lambda \cap U_\mu)$$

is a biholomorphic transformation. We have the transverse Kähler structure in the following sense. On each V_λ, we can give a Kähler structure as follows. Let $D = \ker \eta \subset TS$, i.e.,

$$D = \{X \in TS \,|\, \eta(X) = 0\}.$$

There is a canonical isomorphism

$$d\pi_\lambda : D_p \to T_{\pi_\lambda(p)}V_\lambda, \quad p \in U_\lambda.$$

Then there is a Kähler form γ_λ on V_λ such that $\pi_\lambda^*\gamma_\lambda = \frac{1}{2}d\eta$.

Let $\bigcup_\lambda U_\lambda$ with $U_\lambda = \{(x_\lambda, z_\lambda^1, \dots, z_\lambda^m)\}$ be the foliation chart on S. If $U_\lambda \cap U_\mu \neq \emptyset$, and $(x_\lambda, z_\lambda^1, \dots, z_\lambda^m)$ and $(x_\mu, z_\mu^1, \dots, z_\mu^m)$ are local foliation coordinates on U_λ and U_μ, respectively, where $\frac{\partial}{\partial x_\lambda} = \xi\,|_{U_\lambda}$, then

$$\frac{\partial z_\mu^i}{\partial x_\lambda} = 0, \quad \frac{\partial z_\mu^i}{\partial \overline{z}_\lambda^j} = 0.$$

Consequently, a (p,q)-form $\alpha = \alpha_{i_1,\dots,i_p,\overline{j}_1,\dots,\overline{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}$ is well-defined on S.

Definition 2.1. A p-form α on S is called basic if

$$i_\xi \alpha = 0, \quad L_\xi \alpha = 0.$$

Let Λ_B^p be the sheaf of germs of basic p-forms, and Ω_B^p be the set of all global sections of Λ_B^p.

The lifted Kähler form $\pi_\lambda^* \gamma_\lambda = \frac{1}{2} d\eta$ is a basic $(1,1)$-form. In the local holomorphic foliation coordinate $(x, z^1, \dots, z^m)$, we write

$$\frac{1}{2} d\eta = \sqrt{-1} g_{i\overline{j}}^T dz^i \wedge d\overline{z}^j.$$

For basic forms, we have the following two lemmas which can be proved by using Stokes theorem and the fact that $d\eta$ is basic.

Lemma 2.2. *If α is a basic $(2m-1)$-form, then*

$$\int_S d_B \alpha \wedge \eta = 0.$$

Lemma 2.3. *If α and β are basic forms with* $\deg \alpha + \deg \beta = 2m - 1$, *then*

$$\int_S d_B \alpha \wedge \beta \wedge \eta = (-1)^{\deg \alpha} \int_S \alpha \wedge d_B \beta \wedge \eta.$$

We also have well-defined operators

$$\partial_B : \Lambda_B^{p,q} \to \Lambda_B^{p+1,q}, \quad \overline{\partial}_B : \Lambda_B^{p,q} \to \Lambda_B^{p,q+1}.$$

Put

$$d_B = \partial_B + \overline{\partial}_B, \quad d_B^c = \sqrt{-1}(\overline{\partial}_B - \partial_B).$$

Then

$$d_B = d\,|_{\Omega_B^p}, \quad d_B d_B^c = 2\sqrt{-1}\partial_B \overline{\partial}_B, \quad d_B^2 = (d_B^c)^2 = 0.$$

The basic p-th de Rham cohomology group is

$$H_B^p(S) = \frac{\ker\{d_B : \Omega_B^p \to \Omega_B^{p+1}\}}{\operatorname{Im}\{d_B : \Omega_B^{p-1} \to \Omega_B^p\}},$$

and the basic (p,q)-Dolbeault cohomology group is

$$H_B^{p,q}(S) = \frac{\ker\{\overline{\partial}_B : \Omega_B^{p,q} \to \Omega_B^{p,q+1}\}}{\operatorname{Im}\{\overline{\partial}_B : \Omega_B^{p,q-1} \to \Omega_B^{p,q}\}}$$

on S. With respect to the volume form $(\frac{1}{2} d\eta)^m \wedge \eta$, we define the adjoint operator of $d_B^* : \Omega_B^{p+1} \to \Omega_B^p$ by

$$\langle d_B^* \alpha, \beta \rangle = \langle \alpha, d_B \beta \rangle.$$

Similarly, the adjoint operator of $\overline{\partial}_B^* : \Omega_B^{p,q+1} \to \Omega_B^{p,q}$ is defined by

$$\langle \overline{\partial}_B^* \alpha, \beta \rangle = \langle \alpha, \overline{\partial}_B \beta \rangle.$$

The corresponding basic Laplacian operators are defined by

$$\square_{d_B} = d_B d_B^* + d_B^* d_B, \quad \square_{\overline{\partial}_B} = \overline{\partial}_B \overline{\partial}_B^* + \overline{\partial}_B^* \overline{\partial}_B.$$

It is known that by the transverse Kähler structures, $\square_{d_B} = 2\square_{\overline{\partial}_B}$.

The transverse Ricci form $\mathrm{Ric}^{\mathrm{T}}$ is defined as

$$\mathrm{Ric}^{\mathrm{T}} = -\sqrt{-1}\partial_B\overline{\partial}_B \log\det(g^{\mathrm{T}}). \tag{2.2}$$

$\mathrm{Ric}^{\mathrm{T}}$ is a d_B-closed form and defines the basic cohomology class of type $(1,1)$. The basic cohomology class $c_1^B(S) = [\mathrm{Ric}^{\mathrm{T}}]/2\pi$ is called the basic first Chern class of S. We say the basic first Chern class $c_1^B(S)$ is positive, if $c_1^B(S)$ is represented by a positive basic $(1,1)$-form.

Remark 2.4. Note that if two real closed basic $(1,1)$-forms σ_1 and σ_2 represent the same basic cohomology class there is a basic smooth function φ such that $\sigma_1 - \sigma_2 = \sqrt{-1}\partial_B\overline{\partial}_B\varphi$. If we fix the complex structure of the Kähler cone $C(S)$ and the Reeb vector field ξ, the Sasaki structure can be deformed by the change of the radial function r by $\tilde{r} = \mathrm{e}^{\varphi}r$ for a basic function φ. Then the contact form is deformed from $\eta = d^c\log r$ to $\tilde{\eta} = \eta + d^c\log\varphi$ and thus the transverse Kähler form is deformed from $\frac{1}{2}d\eta$ to $\frac{1}{2}d\eta + \sqrt{-1}\partial\overline{\partial}\varphi$. The basic Chern class is independent of the choice of such contact form η.

Definition 2.5. Let S be a compact Sasaki manifold, ξ be the Reeb vector field and $\mathfrak{h}(S)$ be the Lie algebra of all holomorphic vector fields on the cone $C(S)$. We define

$$\mathfrak{h}^{\mathrm{T}}(S) := \{X \in \mathfrak{h}(S) \,|\, [\xi, X] = 0\}$$

to be the Lie algebra of transverse holomorphic vector fields.

We remark that for $X \in \mathfrak{h}^{\mathrm{T}}(S)$, we also have $[\xi - \sqrt{-1}J\xi, X] = 0$. It follows that X descends to a holomorphic vector field on S, and also that X descends to local orbit spaces of the Reeb flow. By abuse of notation, we use the same letter X to denote the corresponding vector fields on S and local orbit spaces of the Reeb flow.

Definition 2.6. A complex vector field X on a Sasaki manifold is called a Hamiltonian holomorphic vector field if

(1) $d\pi_\alpha(X)$ is a holomorphic vector field on V_α;

(2) the complex valued function $u_X = \sqrt{-1}\eta(X)$ satisfies

$$\overline{\partial}_B u_X = -\sqrt{-1}i_X\left(\frac{1}{2}d\eta\right).$$

Suppose that the basic first Chern class is positive. Then for any other basic Kähler class there is a basic Kähler form of positive transverse Ricci form and thus the basic first cohomology vanishes by the standard Bochner technique. From this we have the following lemma.

Lemma 2.7 (See [3]). *Let S be a compact Sasaki manifold of positive basic first Chern class. Then the Lie algebra of the automorphism group of transverse holomorphic structure is the Lie algebra of all Hamiltonian holomorphic vector fields.*

Let γ be a basic $(1,1)$-class which contains a positive $(1,1)$-form ω. Then we say that ω is a basic Kähler form and that γ is a basic Kähler class. For example the transverse Kähler form $\frac{1}{2}d\eta$ is a basic Kähler form and its basic cohomology class $[\frac{1}{2}d\eta]$ is a basic Kähler class. If we assume the basic first Chern class $c_1^B(S)$ is positive then it is a basic Kähler class. For each basic Kähler form ω we can define the transverse Ricci form $\mathrm{Ric}^{\mathrm{T}}(\omega)$ as in (2.2), and it represents $c_1^B(S)$. Definition 2.6 and Lemma 2.7 also apply even if we replace $\frac{1}{2}d\eta$ by ω.

3 Transverse holomorphic vector fields and transverse elliptic operators

Let (S, g) be a compact Sasaki manifold. We assume the basic first Chern class $c_1^B(S) > 0$, and it admits a basic decomposition $c_1^B(S) = \sum_{\alpha=1}^k \gamma_\alpha/2\pi$, where each γ_α is a basic Kähler class. We fix a basic Kähler form ω_α in each basic Kähler class γ_α. In this section, we prove Theorem 1.2 stated in Section 1 which is the relationship between the Lie algebra of transverse holomorphic vector fields and the twisted basic Laplacians on S.

For each basic Kähler form ω_α, since both $\mathrm{Ric}^{\mathrm{T}}(\omega_\alpha)$ and $\sum_{\alpha=1}^k \omega_\alpha$ are in the basic first Chern class $c_1^B(S)$, by the transverse $\partial_B\overline{\partial}_B$ lemma [5], there is a basic function f_α satisfying (1.5). We normalize f_α by (1.7). This means $\mathrm{e}^{f_\alpha}\omega_\alpha^m$ is independent of α. If we put

$$dV := \mathrm{e}^{f_\alpha}\omega_\alpha^m \wedge \eta, \tag{3.1}$$

then dV defines a volume form on S. We may normalize it so that $\int_S dV = 1$.

For a transverse holomorphic vector field $V \in \mathfrak{h}^{\mathrm{T}}(S)$, by Lemma 2.7, let the basic function u_α be the Hamiltonian function of V with respect to the basic Kähler form ω_α. In the local holomorphic foliation coordinate $(x, z^1, \dots, z^m)$, $\omega_\alpha = \sqrt{-1}g^\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j$. Since for each α from 1 to k, $\overline{\partial}_B u_\alpha = \sqrt{-1}i_V\omega_\alpha$, we have the coordinate components of V are given by

$$V = \mathrm{grad}_\alpha u_\alpha = \mathrm{grad}_\beta u_\beta. \tag{3.2}$$

Define Δ_{α,f_α} by (1.8).

Proposition 3.1. *For a transverse holomorphic vector field V, the basic functions $u_1, \dots, u_k$ satisfying* (3.2) *after suitable modifications by addition of constants satisfy*

$$\Delta_{1,f_1}u_1 = \Delta_{2,f_2}u_2 = \cdots = \Delta_{k,f_k}u_k = -\sum_{\beta=1}^k u_\beta. \tag{3.3}$$

Proof. First, we show that for $\alpha \neq \beta$,

$$\Delta_{\alpha,f_\alpha}u_\alpha = \Delta_{\beta,f_\beta}u_\beta.$$

Note that (3.2) implies

$$\partial_{\overline{p}}u_\alpha = g^\alpha_{i\overline{p}}g^{i\overline{j}}_\beta\partial_{\overline{j}}u_\beta.$$

Thus

$$\partial_k\partial_{\overline{p}}u_\alpha = \partial_k(g^\alpha_{i\overline{p}}g^{i\overline{j}}_\beta\partial_{\overline{j}}u_\beta) = g^\alpha_{i\overline{p}}g^{i\overline{j}}_\beta\partial_k\partial_{\overline{j}}u_\beta + g^{i\overline{j}}_\beta\partial_{\overline{j}}u_\beta\partial_k g^\alpha_{i\overline{p}} + g^\alpha_{i\overline{p}}\partial_k g^{i\overline{j}}_\beta\partial_{\overline{j}}u_\beta.$$

By taking trace with respect to $g^{k\overline{p}}_\alpha$ and using the Kähler condition $\partial_i g^\alpha_{j\overline{k}} = \partial_j g^\alpha_{i\overline{k}}$,

$$\begin{aligned}
\Delta_\alpha u_\alpha &= \Delta_\beta u_\beta + (g^{k\overline{p}}_\alpha\partial_k g^\alpha_{i\overline{p}})g^{i\overline{j}}_\beta\partial_{\overline{j}}u_\beta + \partial_i g^{i\overline{j}}_\beta\partial_{\overline{j}}u_\beta\\
&= \Delta_\beta u_\beta + g^{i\overline{j}}_\beta(\partial_i\log\omega_\alpha^m)\partial_{\overline{j}}u_\beta - g^{i\overline{j}}_\beta(\partial_i\log\omega_\beta^m)\partial_{\overline{j}}u_\beta\\
&= \Delta_\beta u_\beta + g^{i\overline{j}}_\beta\partial_i\log\frac{\omega_\alpha^m}{\omega_\beta^m}\partial_{\overline{j}}u_\beta\\
&= \Delta_\beta u_\beta + g^{i\overline{j}}_\beta\partial_i f_\beta\partial_{\overline{j}}u_\beta - g^{i\overline{j}}_\beta\partial_i f_\alpha\partial_{\overline{j}}u_\beta && (\star 1)\\
&= \Delta_\beta u_\beta + g^{i\overline{j}}_\beta\partial_i f_\beta\partial_{\overline{j}}u_\beta - g^{i\overline{j}}_\alpha\partial_i f_\alpha\partial_{\overline{j}}u_\alpha, && (\star 2)
\end{aligned}$$

where $(\star 1)$ is due to $\mathrm{e}^{f_\alpha}\omega_\alpha^m = \mathrm{e}^{f_\beta}\omega_\beta^m$, and $(\star 2)$ is due to $\mathrm{grad}_\alpha u_\alpha = \mathrm{grad}_\beta u_\beta$. Hence

$$\Delta_{\alpha,f_\alpha}u_\alpha = \Delta_{\beta,f_\beta}u_\beta.$$

Next, we show for any $\alpha = 1, \dots, k$, after a normalization,

$$\Delta_{\alpha,f_\alpha}u_\alpha + \sum_{\beta=1}^k u_\beta = 0.$$

We denote by ∇_α the covariant derivative with respect to ω_α. Then for smooth functions $\nabla''_\alpha = \overline{\partial}_B$ for any α, and for this reason the derivative with respect to the i-th coordinate will be written as $\nabla_{\alpha,\bar{i}} = \nabla_{B,\bar{i}}$. Using the fact that for all α, $\nabla_{\alpha,\bar{i}}\nabla_{\alpha,\bar{j}}u_\alpha = 0$, and $\nabla^i_\alpha u_\alpha = \nabla^i_\beta u_\beta$, we have

$$\nabla_{B,\overline{k}}(\Delta_{\alpha,f_\alpha}u_\alpha) = \nabla_{\alpha,\overline{k}}\nabla_{\alpha,i}\nabla^i_\alpha u_\alpha + \nabla^i_\alpha u_\alpha\nabla_{\alpha,\overline{k}}\nabla_{\alpha,i}f_\alpha$$

$$
\begin{aligned}
&= -g_\alpha^{i\bar{j}} R_{\alpha,i\bar{k}} \nabla_{\alpha,\bar{j}} u_\alpha + g_\alpha^{i\bar{j}} \nabla_{\alpha,\bar{k}} \nabla_{\alpha,i} f_\alpha \nabla_{\alpha,\bar{j}} u_\alpha \\
&= \bigg(-\sum_{\beta=1}^{k} g_{i\bar{k}}^{\beta} \bigg) \nabla_\alpha^i u_\alpha = -\sum_{\beta=1}^{k} g_{i\bar{k}}^{\beta} \nabla_\beta^i u_\beta \\
&= -\nabla_{B,\bar{k}} \bigg(\sum_{\beta=1}^{k} u_\beta \bigg).
\end{aligned}
$$

From the assumptions that u_α are basic functions we obtain

$$
\Delta_{\alpha,f_\alpha} u_\alpha + \sum_{\beta=1}^{k} u_\beta = c_\alpha
$$

for constants c_α. In particular, $c_\alpha = c_\beta = c$. Furthermore, we choose $v_\alpha = u_\alpha - \frac{c}{k}$. Then v_α satisfies

$$
\Delta_{\alpha,f_\alpha} v_\alpha + \sum_{\beta=1}^{k} v_\beta = 0.
$$

This completes the proof. □

Theorem 3.2. *Let S be a compact Sasaki manifold with $c_1^B(S) > 0$, the decomposition* (1.4) *and the choice of basic Kähler forms ω_α and basic smooth functions f_α satisfying* (1.5)*. Then the Lie algebra $\mathfrak{h}^{\mathrm{T}}(S)$ of all transverse holomorphic vector fields is isomorphic to*

$$
\mathfrak{h}' := \bigg\{ u_1 + \cdots + u_k \,\bigg|\, \mathrm{grad}_\alpha u_\alpha = \mathrm{grad}_\beta u_\beta \in \mathfrak{h}^{\mathrm{T}}(S),\ \alpha,\ \beta = 1, \ldots, k,\ \int_S (u_1 + \cdots + u_k) dV = 0 \bigg\},
$$

where the Lie algebra structure of the latter is given by the Poisson bracket of each u_α. Here, the isomorphism is given by $V \mapsto u_1 + \cdots + u_k$ with $V = \mathrm{grad}_\alpha u_\alpha$.

Proof. There is a natural injection from $\mathfrak{h}'$ to $\mathfrak{h}^{\mathrm{T}}(S)$ sending $u_1 + \cdots + u_k$ to $V = \mathrm{grad}_\alpha u_\alpha$. This map is also surjective by Proposition 3.1. □

Now we turn to the proof of Theorem 1.2. First, we prove it in the regular Sasaki case, namely the Kähler case. The statement in this case should be helpful to understand the product configuration of the definition of K-stability in the coupled Kähler case.

Theorem 3.3. *Let M be a Fano Kähler manifold of complex dimension n. Suppose that for Kähler forms $\omega_1, \ldots, \omega_k$, there exist real smooth functions $f_1, \ldots, f_k \in C^\infty(M)$, such that*

$$
\begin{cases}
\mathrm{Ric}(\omega_\alpha) = \displaystyle\sum_{\beta=1}^{k} \omega_\beta + \sqrt{-1}\partial\bar{\partial} f_\alpha, \\
\mathrm{e}^{f_1}\omega_1^m = \mathrm{e}^{f_2}\omega_2^m = \cdots = \mathrm{e}^{f_k}\omega_k^m.
\end{cases}
$$

Suppose also that non-constant complex valued smooth functions $u_1, \ldots, u_k \in C^\infty(X) \otimes \mathbf{C}$ satisfy

(1) $\nabla_\alpha^i u_\alpha = \nabla_\beta^i u_\beta$ *for $i = 1, 2, \ldots, n$ and $\alpha, \beta = 1, 2, \ldots, k$.*

(2) $-\Delta_{\alpha,f_\alpha} u_\alpha = \lambda \sum_{\beta=1}^{k} u_\beta$ *for $\alpha = 1, 2, \ldots, k$, where $\Delta_{\alpha,f_\alpha} u_\alpha = \Delta_\alpha u_\alpha + \nabla_\alpha^i u_\alpha \nabla_i f_\alpha$, and $\Delta_\alpha = -\bar{\partial}_\alpha^* \bar{\partial}$ is the Beltrami-Laplacian with respect to the Kähler form ω_α.*

Then $\lambda \geqslant 1$. Moreover, if $\lambda = 1$, the vector field $V = \mathrm{grad}_\alpha u_\alpha = \mathrm{grad}_\beta u_\beta$ is a holomorphic vector field.

Proof. We compute

$$
\begin{aligned}
&\int_M |\nabla_\alpha'' \nabla_\alpha'' u_\alpha|_{\omega_\alpha}^2 \ \mathrm{e}^{f_\alpha} \omega_\alpha^m \\
&\quad = \int_M \nabla_\alpha^i \nabla_\alpha^j u_\alpha \nabla_{\alpha,i} \nabla_{\alpha,j} \bar{u}_\alpha \ \mathrm{e}^{f_\alpha} \omega_\alpha^m \\
&\quad = -\int_M \nabla_{\alpha,j} (\nabla_\alpha^i \nabla_\alpha^j u_\alpha \mathrm{e}^{f_\alpha}) \nabla_{\alpha,i} \bar{u}_\alpha \ \omega_\alpha^m
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_M (\nabla^i_\alpha(\Delta_\alpha u_\alpha) + R_{\alpha,j}{}^i \nabla^j_\alpha u_\alpha + \nabla^i_\alpha \nabla^j_\alpha u_\alpha \nabla^\alpha_j f_\alpha)\nabla_{\alpha,i}\overline{u}_\alpha \ \mathrm{e}^{f_\alpha}\omega_\alpha^m \\
&= -\int_M (\nabla^i_\alpha(\Delta_\alpha u_\alpha) + \nabla_{\alpha,j}\nabla^i_\alpha f_\alpha \nabla^j_\alpha u_\alpha + \nabla^i_\alpha \nabla^j_\alpha u_\alpha \nabla_{\alpha,j} f_\alpha)\nabla_{\alpha,i}\overline{u}_\alpha \ \mathrm{e}^{f_\alpha}\omega_\alpha^m \\
&\quad - \int_M \bigg(\sum_{\beta=1}^k g_{\beta,i\overline{j}}\bigg)\nabla^i_\alpha u_\alpha \overline{\nabla^j_\alpha u_\alpha} \ \mathrm{e}^{f_\alpha}\omega_\alpha^m \\
&= -\int_M \nabla^i_\alpha(\Delta_{\alpha,f_\alpha} u_\alpha)\nabla_{\alpha,i}\overline{u}_\alpha \ \mathrm{e}^{f_\alpha}\omega_\alpha^m - \int_M \sum_{\beta=1}^k g_{\beta,i\overline{j}} \nabla^i_\beta u_\beta \overline{\nabla^j_\beta u_\beta} \ \mathrm{e}^{f_\beta}\omega_\beta^m \\
&= \int_M |\Delta_{\alpha,f_\alpha} u_\alpha|^2 \ \mathrm{e}^{f_\alpha}\omega_\alpha^m + \sum_{\beta=1}^k \int_M (\Delta_{\beta,f_\beta} u_\beta)\overline{u}_\beta \ \mathrm{e}^{f_\beta}\omega_\beta^m \\
&= \int_M \lambda^2 \bigg|\sum_{\beta=1}^k u_\beta\bigg|^2 \ \mathrm{e}^{f_\alpha}\omega_\alpha^m - \lambda \sum_{\beta=1}^k \int_M \bigg(\sum_{\gamma=1}^k u_\gamma\bigg)\overline{u}_\beta \ \mathrm{e}^{f_\beta}\omega_\beta^m \\
&= \lambda(\lambda-1)\int_M \bigg|\sum_{\beta=1}^k u_\beta\bigg|^2 \ \mathrm{e}^{f_\beta}\omega_\beta^m, \qquad (3.4)
\end{aligned}
$$

where we have used $\mathrm{e}^{f_\alpha}\omega_\alpha^m = \mathrm{e}^{f_\beta}\omega_\beta^m$ and $\nabla^i_\alpha u_\alpha = \nabla^i_\beta u_\beta$. Taking the L^2-inner product with u_γ on both sides of $-\Delta_{\alpha,f_\alpha} u_\alpha = \lambda \sum_{\beta=1}^k u_\beta$ and taking sum over $\gamma = 1,\ldots,k$, we see that $\lambda > 0$ since we assumed u_α are non-constant. Then from the computations (3.4) we conclude $\lambda \geqslant 1$. Moreover, if $\lambda = 1$, then $V = \mathrm{grad}_\alpha u_\alpha$ is a holomorphic vector field. □

Proof of Theorem 1.2. In the proof of Theorem 3.3 we replace the volume form $\mathrm{e}^{f_1}\omega_1^m = \cdots = \mathrm{e}^{f_k}\omega_k^m$ by $dV = \mathrm{e}^{f_1}\omega_1^m \wedge \eta = \cdots = \mathrm{e}^{f_k}\omega_k^m \wedge \eta$. Then by Lemmas 2.2 and 2.3 the same computations of Theorem 3.3 prove Theorem 1.2(1). The part (2) follows from Lemma 2.7, Proposition 3.1 and Theorem 3.3. □

4 The invariant for the coupled Sasaki-Einstein manifold

Let S be a compact Sasaki manifold, ξ be the Reeb vector field and η be the contact form. We assume that the basic first Chern class $c_1^B(S)$ is positive and that it admits a basic decomposition $c_1^B(S) = (\sum_{\alpha=1}^k \gamma_\alpha)/2\pi$, where $\gamma_1,\ldots,\gamma_k$ are basic Kähler classes. In this section, we define an invariant on S which gives an obstruction to the existence of coupled Sasaki-Einstein metrics. This invariant extends the obstruction to the existence of Kähler-Einstein metrics obtained in [6], but it is expressed in the form obtained in [10, Proposition 2.3].

As in Section 1, we let ω_α be the basic Kähler forms in γ_α and f_α be the basic smooth functions such that $\mathrm{Ric}^{\mathrm{T}}(\omega_\alpha) = \sum_{\beta=1}^k \omega_\beta + \sqrt{-1}\partial_B\overline{\partial}_B f_\alpha$, and the basic functions $f_1,\ldots,f_k$ are normalized by (1.7), i.e.,

$$\mathrm{e}^{f_1}\omega_1^m = \cdots = \mathrm{e}^{f_k}\omega_k^m.$$

Take a transverse holomorphic vector field $V \in \mathfrak{h}^{\mathrm{T}}(S)$. Let complex-valued basic functions $u_1,\ldots,u_k$ be the Hamiltonian functions of V with respect to basic Kähler forms $\omega_1,\ldots,\omega_k$.

In Section 1, we defined $\mathrm{Fut} : \mathfrak{h}^{\mathrm{T}}(S) \to \mathbf{C}$ by (1.9) using Theorem 1.2(2). But since the condition (3.3) is equivalent to the normalization $\int_S (u_1 + \cdots + u_k)dV = 0$ we may re-define Fut using the latter normalization. Then this definition is independent of the choice $u_1,\ldots,u_k$ satisfying the normalization $\int_S (u_1 + \cdots + u_k)\, dV = 0$ since if $\widetilde{u}_1,\ldots,\widetilde{u}_k$ are another choice to represent the same element in $\mathfrak{h}'$ then $\widetilde{u}_\alpha = u_\alpha + c_\alpha$ for some constants $c_1,\ldots,c_k$ with $c_1 + \cdots + c_k = 0$.

Proof of Theorem 1.4. If we take basic Kähler forms $\tilde{\omega}_\alpha = \omega_\alpha + \sqrt{-1}\partial_B\overline{\partial}_B\varphi_\alpha$, where $\varphi_1,\ldots,\varphi_k$ are basic functions, then $\sum_{\alpha=1}^k \tilde{\omega}_\alpha = \sum_{\alpha=1}^k \omega_\alpha + \sqrt{-1}\partial_B\overline{\partial}_B(\sum_{\alpha=1}^k \varphi_\alpha)$. Let $\tilde{f}_\alpha$ be the basic functions satisfying

$$\begin{cases} \mathrm{Ric}^{\mathrm{T}}(\tilde{\omega}_\alpha) = \displaystyle\sum_{\beta=1}^{k} \tilde{\omega}_\beta + \sqrt{-1}\partial_B\overline{\partial}_B \tilde{f}_\alpha, \\ \mathrm{e}^{\tilde{f}_1}\tilde{\omega}_1^m = \cdots = \mathrm{e}^{\tilde{f}_k}\tilde{\omega}_k^m. \end{cases}$$

Then

$$\mathrm{e}^{\sum_{\beta=1}^{k}\varphi_\beta}\mathrm{e}^{\tilde{f}_\alpha}\tilde{\omega}_\alpha^m = \mathrm{e}^{f_\alpha}\omega_\alpha^m. \tag{4.1}$$

For the transverse holomorphic vector field V, we take the Hamiltonian functions u_α with respect to ω_α satisfying the normalization condition $\int_S (u_1 + \cdots + u_k)\, dV = 0$ or equivalently the condition (3.3). We then take $\tilde{u}_\alpha = u_\alpha + V(\varphi_\alpha)$ to be the Hamiltonian function of V with respect to $\tilde{\omega}_\alpha$. We first show that $\tilde{u}_\alpha$'s satisfy the normalization condition $\int_S(\tilde{u}_1 + \cdots + \tilde{u}_k)\, d\tilde{V} = 0$ where $d\tilde{V}$ denotes the corresponding volume form $\mathrm{e}^{\tilde{f}_\alpha}\tilde{\omega}_\alpha^m \wedge \eta$. We have

$$\sum_{\alpha=1}^{k}\tilde{u}_\alpha = \sum_{\alpha=1}^{k} u_\alpha + V\bigg(\sum_{\alpha=1}^{k}\varphi_\alpha\bigg).$$

We compute using the condition (3.3) that

$$\begin{aligned} \int_S (\tilde{u}_1 + \cdots + \tilde{u}_k)\, d\tilde{V} &= \int_S \bigg(\sum_{\beta=1}^{k}\tilde{u}_\beta\bigg)\, \mathrm{e}^{\tilde{f}_\alpha}\tilde{\omega}_\alpha^m \wedge \eta \\ &= \int_S \bigg(\sum_{\alpha=1}^{k} u_\alpha + V\bigg(\sum_{\beta=1}^{k}\varphi_\beta\bigg)\bigg)\mathrm{e}^{-\sum_{\beta=1}^{k}\varphi_\beta}\mathrm{e}^{f_\alpha}\omega_\alpha^m \wedge \eta \\ &= -\int_S \Delta_{\alpha,f_\alpha} u_\alpha\ \mathrm{e}^{-\sum_{\beta=1}^{k}\varphi_\beta}\mathrm{e}^{f_\alpha}\ \omega_\alpha^m \wedge \eta - \int_S V(\mathrm{e}^{-\sum_{\beta=1}^{k}\varphi_\beta})\mathrm{e}^{f_\alpha}\omega_\alpha^m \wedge \eta \\ &= 0. \end{aligned}$$

Next, we show that Fut is independent of the choice of the basic Kähler forms ω_α in γ_α. To show this, consider the deformation $\omega_\alpha(t) = \omega_\alpha + \sqrt{-1}\partial_B\overline{\partial}_B(t\varphi_\alpha)$ for basic functions φ_α. Let u_α and $u_\alpha(t)$ be the Hamiltonian functions of V with respect to ω_α and $\omega_\alpha(t)$, where $u_\alpha(t) = u_\alpha + tV(\varphi_\alpha)$. Then it is enough to show the integrals $\int_S u_\alpha(t)(\omega_\alpha(t))^m \wedge \eta$ and $\int_S (\omega_\alpha(t))^m \wedge \eta$ are independent of t. But this follows from

$$\begin{aligned} &\frac{d}{dt}\int_S (\omega_\alpha(t))^m \wedge \eta = \int_S (\Delta_{\omega_\alpha(t)}\varphi_\alpha)\ (\omega_\alpha(t))^m \wedge \eta = 0, \\ &\frac{d}{dt}\int_S u_\alpha(t)\ (\omega_\alpha(t))^m \wedge \eta = \int_S V(\varphi_\alpha)(\omega_\alpha(t))^m \wedge \eta + \int_S u_\alpha(t)\Delta_{\omega_\alpha(t)}\varphi_\alpha\ (\omega_\alpha(t))^m \wedge \eta = 0. \end{aligned}$$

If S admits the coupled Sasaki-Einstein metrics $\omega_1, \dots, \omega_k$, then we can take all f_α's to be zeros. Thus the normalization $\int_S (u_1 + \cdots + u_k)\, dV = 0$ implies $\mathrm{Fut}(V) = 0$. This completes the proof. □

Let ω_α be a basic Kähler form in the basic Kähler class γ_α. For $V, W_1, \dots, W_k \in \mathfrak{h}^{\mathrm{T}}(S)$ we put

$$\mathrm{Fut}_{W_1,\dots,W_k}(V) = \sum_{\alpha=1}^{k} \frac{\int_S u_{\alpha,V}\ \mathrm{e}^{u_{W_\alpha}}\omega_\alpha^m \wedge \eta}{\int_S \mathrm{e}^{u_{W_\alpha}}\omega_\alpha^m \wedge \eta}, \tag{4.2}$$

where $\mathrm{grad}_\alpha u_{\alpha,V} = V$ and $\mathrm{grad}_\alpha u_{W_\alpha} = W_\alpha$ and we assume $u_{\alpha,V}$'s satisfy (3.3) or equivalently the normalization

$$\int_S (u_{1,V} + \cdots + u_{k,V})\, dV = 0.$$

By a similar proof to that of Theorem 1.4, one can show that this is also independent of the choice of ω_α's in γ_α's. When $W_1 = \cdots = W_k = 0$, $\mathrm{Fut}_{W_1,\dots,W_k}(V)$ coincides with $\mathrm{Fut}(V)$ in (1.9).

Theorem 4.1. *If there exists the coupled Sasaki-Ricci solitons* (1.5) *for the decomposition* $c_1^B(S) = \sum_{\alpha=1}^k \gamma_\alpha/2\pi$ *then* $\mathrm{Fut}_{W_1,\dots,W_k}$ *identically vanishes for* $W_\alpha = \mathrm{grad}_\alpha f_\alpha$*,* $\alpha = 1,\dots,k$*.*

Proof. Suppose we have a solution of coupled Sasaki-Ricci solitons (1.5). We may normalize f_α so that (1.7) is satisfied. Then since $W_\alpha = \mathrm{grad}_\alpha f_\alpha$ and thus $u_{W_\alpha} = f_\alpha$ in (4.2), we obtain using (3.1),

$$\mathrm{Fut}_{W_1,\dots,W_k}(V) = \left(\sum_{\alpha=1}^k \int_S u_{\alpha,V}\, dV\right) \Big/ \int_S dV, \tag{4.3}$$

which vanishes by Theorem 3.2. This completes the proof of Theorem 4.1. □

The above theorem is an extension of [6, 24].

5 Toric Sasaki-Ricci solitons

Let T be a real torus of dimension $m+1$ acting effectively on S as isometries and $\mathfrak{t}$ its Lie algebra. Naturally, T acts on the Kähler cone $C(S)$ as holomorphic isometries. Note that in this case we have an effective $(\mathbf{C}^*)^{m+1}$-action on $C(S)$, and $m+1$ is the maximal dimension of the torus action because of the dimension reason. In such a case we say $C(S)$ is a toric Kähler cone and S is a toric Sasaki manifold (see [20] for a concise description of toric Sasaki geometry).

We identify an element $X \in \mathfrak{t}$ with a vector field on $C(S)$ and denote it by the same letter X. Note that the Reeb vector field ξ lies in $\mathfrak{t}$. Since the Kähler form on $C(S)$ is given by $\frac{\sqrt{-1}}{2}\partial\overline{\partial} r^2$ the moment map $\mu : C(S) \to \mathfrak{t}^*$ on the Kähler cone $C(S)$ for the action of T is given for $X \in \mathfrak{t}$ by

$$\langle \mu, X\rangle = r^2\eta(X),$$

where we recall that the contact form η is extended on $C(S)$ by (2.1). It is well known that the image of μ is a rational convex polyhedral cone which we denote by $\mathcal{C}$. The Sasaki manifold $S \cong \{r = 1\}$ is characterized as $\langle \mu, \xi\rangle = 1$, and its moment map image is $\{p \in \mathcal{C} \,|\, \langle p, \xi\rangle = 1\}$. We denote by $\mathcal{P}_\xi$ the image of $\{p \in \mathcal{C} \,|\, \langle p, \xi\rangle = 1\}$ under the projection $\pi_\xi : \mathfrak{t}^* \to (\mathfrak{t}/\mathbf{R}\xi)^*$. $\mathcal{P}_\xi$ is the image of $\mu_\xi := \pi_\xi \circ \mu\,|_S : S \to (\mathfrak{t}/\mathbf{R}\xi)^*$, which we call the transverse moment map for S. $\mathcal{P}_\xi$ is a rational convex polytope when ξ defines a quasi-regular Sasaki structure, but otherwise it is not rational. The inverse image by μ of each facet of $\mathcal{C}$ is the fixed point set in $C(S)$ of a one-dimensional torus. If $\lambda \in \mathfrak{t}$ is its infinitesimal generator then λ is normal to the facet. The inverse image by μ_ξ of the corresponding facet of $\mathcal{P}_\xi$ is the fixed point set in S of the same one-dimensional torus, and $\lambda/\mathbf{R}\xi \in \mathfrak{t}/\mathbf{R}\xi$ is normal to the corresponding facet of $\mathcal{P}_\xi$. (Note that there is no meaning of rationality in $\mathfrak{t}/\mathbf{R}\xi$ if ξ is not rational.)

Suppose that the basic first Chern class is positive. For any basic Kähler form ω invariant under T we have a transverse moment map $\mu_\omega : S \to (\mathfrak{t}/\mathbf{R}\xi)^*$ defined up to translation by the same reason as in the paragraph after Lemma 2.7. Note that the Hamiltonian functions in Lemma 2.7 can be taken to be real functions since T acts by isometries. Then the image of the transverse moment map is a convex polytope, which we denote by $\mathcal{P}_\omega$, and the facets have the same description as above. In particular, the facets of $\mathcal{P}_\xi$ and $\mathcal{P}_\omega$ are parallel to each other.

Let $c_1^B(S) = (\sum_{\alpha=1}^k \gamma_\alpha)/2\pi$ be a basic decomposition where $\gamma_1,\dots,\gamma_k$ are basic Kähler classes, and choose basic Kähler forms $\omega_\alpha \in \gamma_\alpha$. We put $\mathcal{P}_\alpha := \mathcal{P}_{\omega_\alpha}$ which is independent of $\omega_\alpha \in \gamma_\alpha$ up to translation. Choose $W_1,\dots,W_k \in \mathfrak{t}$. Then $\mathrm{Fut}_{W_1,\dots,W_k}$ in (4.2), which is independent of the choice of $\omega_\alpha \in \gamma_\alpha$, is expressed as

$$\mathrm{Fut}_{W_1,\dots,W_k} = \sum_{\alpha=1}^k \mathcal{A}_{\mathcal{P}_\alpha}(W_\alpha), \tag{5.1}$$

where

$$\mathcal{A}_{\mathcal{P}_\alpha}(W_\alpha) = \frac{\int_{\mathcal{P}_\alpha} p\ \mathrm{e}^{\langle W_\alpha, p\rangle}\, dp}{\int_{\mathcal{P}_\alpha} \mathrm{e}^{\langle W_\alpha, p\rangle}\, dp}.$$

Theorem 5.1. *Let S be a compact toric Sasaki manifold with the basic first Chern class $c_1^B(S)$ positive, and $c_1^B(S) = (\sum_{\alpha=1}^k \gamma_\alpha)/2\pi$ be a basic decomposition. Let $\mathcal{P}_\alpha$ be the convex polytope which is the image of the transverse moment map for γ_α with normalization $\sum_{\alpha=1}^k \int_{\mathcal{P}_\alpha} pdp = 0$. Then there exist coupled Kähler-Ricci solitons satisfying* (1.5) *for $W_\alpha = \mathrm{grad}_\alpha f_\alpha \in \mathfrak{t}$ if and only if*

$$\sum_{\alpha=1}^k \mathcal{A}_{\mathcal{P}_\alpha}(W_\alpha) = 0.$$

Note that the condition $\sum_{\alpha=1}^k \int_{\mathcal{P}_\alpha} pdp = 0$ implies that the barycenter of the Minkowski sum $\sum_{\alpha=1}^k \mathcal{P}_\alpha$ lies at the origin. This condition is equivalent to the normalization

$$\int_S (u_{1,V} + \cdots + u_{k,V})\ dV = 0$$

in Theorem 3.2. Let K_S denote the complex line bundle over S consisting of basic $(m,0)$-forms, and call K_S the transverse canonical line bundle and K_S^{-1} the transverse anti-canonical line bundle. We put $\omega := \omega_1 + \cdots + \omega_k$, which is a basic Kähler form in $c_1^B(S) = c_1(K_S^{-1})$. Let F be a basic smooth function such that

$$\mathrm{Ric}^{\mathrm{T}}(\omega) - \omega = \sqrt{-1}\partial_B\overline{\partial}_B F. \tag{5.2}$$

We put $\Delta_F u := \Delta u + (\mathrm{grad}_\omega u)F$. Then Theorems 1.2 and 3.2 assert that when $k = 1$ we have the isomorphisms of the Lie algebra $\mathfrak{h}^{\mathrm{T}}(S)$ of all transverse holomorphic vector fields

$$\begin{aligned} \mathfrak{h}^{\mathrm{T}}(S) &\cong \{u \in C^\infty(S)_B \otimes \mathbf{C} \mid -\Delta_F u = u\} \\ &= \left\{u \in C^\infty(S)_B \otimes \mathbf{C} \,\middle|\, \int_S u\ \mathrm{e}^F \omega^m \wedge \eta = 0\right\}. \end{aligned} \tag{5.3}$$

We will call the moment map for the class $c_1(K_S^{-1})$ defined by the Hamiltonian functions u in (5.3) the standard moment map, and denote its moment polytope by $\mathcal{P}_{-K_S}$. In Appendix A, we will show that this moment map is indeed standard.

Theorem 5.2. *The normalization condition $\int_S (u_{1,V} + \cdots + u_{k,V})dV = 0$ is equivalent to*

$$\sum_{\alpha=1}^k \mathcal{P}_\alpha = \mathcal{P}_{-K_S},$$

where the left-hand side is the Minkowski sum of the polytopes $\mathcal{P}_\alpha$'s.

Proof. By [5,26], there exists a unique basic Kähler form ω_0 in $c_1^B(S)$ such that $\mathrm{Ric}^{\mathrm{T}}(\omega_0) = \omega$. Then using (5.2) we have

$$\mathrm{e}^F \omega^m = \omega_0^m$$

by adding a constant to F. On the other hand by (1.5) with the normalization (1.7) we also have for any α,

$$\mathrm{e}^{f_\alpha} \omega_\alpha^m = \omega_0^m.$$

Thus, using (3.1), we have for any α,

$$\mathrm{e}^F \omega^m \wedge \eta = \mathrm{e}^{f_\alpha} \omega_\alpha^m \wedge \eta = dV.$$

Hence

$$\int_S (u_{1,V} + \cdots + u_{k,V})dV = 0$$

implies that $u = u_{1,V} + \cdots + u_{k,V}$ satisfies the condition in (5.3). This implies the Minkowski sum $\sum_{\alpha=1}^k \mathcal{P}_\alpha$ coincides with $\mathcal{P}_{-K_S}$. This completes the proof. □

When S is the unit circle bundle of K_X^{-1} of a Fano manifold X the Minkowski sum condition in Theorem 5.2 is equivalent to $\sum_{\alpha=1}^k \mathcal{P}_\alpha = \mathcal{P}_{-K_X}$ in Hultgren's paper [16].

Proof of Theorem 5.1. We choose basic Kähler forms ω_α in γ_α, and consider for $t \in [0,1]$ the family of Monge-Ampère equations

$$e^{g_1+W_1(\varphi_1)}(\omega_1+\sqrt{-1}\partial_B\overline{\partial}_B\varphi_1)^m=\cdots=e^{g_k+W_k(\varphi_k)}(\omega_k+\sqrt{-1}\partial_B\overline{\partial}_B\varphi_k)^m=e^{-t\sum_\alpha\varphi_\alpha}\omega_0^m \tag{5.4}$$

in terms of basic functions φ_α, where ω_0 is the unique basic Kähler form in a basic Kähler class such that $\mathrm{Ric}^{\mathrm{T}}(\omega_0)=\omega_1+\cdots+\omega_k$ and g_α is the potential function of W_α with respect to ω_α, i.e., $\mathrm{grad}_\alpha g_\alpha = W_\alpha$. If we have a solution for $t=1$ the Kähler forms $\omega_\alpha+\sqrt{-1}\partial\overline{\partial}\varphi_\alpha$ give the desired coupled Sasaki-Ricci solitons. By the same argument as in [16, 23], we only need to show the C^0-estimates. To do so, we wish to reduce the equation to holomorphic logarithmic coordinates, further to the real Monge-Ampère equation with respect to the real coordinates, and to show the same arguments in [16] apply in our Sasakian situation.

As in [14, Section 7], we take any subtorus $H\subset T$ of codimension 1 such that its Lie algebra $\mathfrak{h}$ does not contain ξ. Let $H^c\cong(\mathbf{C}^*)^m$ denote the complexification of H. Take any point $p\in\mu^{-1}(\mathrm{Int}\,\mathcal{C})$ and consider the orbit $\mathrm{Orb}_{C(S)}(H^c,p)$ of the H^c-action on $C(S)$ through p. Since H^c-action preserves $-J\xi=r\partial/\partial r$, it descends to an action on the set $S\cong\{r=1\}\subset C(S)$. More precisely this action is described as follows. Let $\gamma: H^c\times C(S)\to C(S)$ denote the H^c-action on $C(S)$. Let $\overline{p}$ and $\overline{\gamma(g,p)}$ respectively be the points on $\{r=1\}$ at which the flow lines through p and $\gamma(g,p)$ generated by $r\partial/\partial r$ respectively meet $\{r=1\}$. Then the H^c-action on $S\cong\{r=1\}$ is given by $\overline{\gamma}: H^c\times\{r=1\}\to\{r=1\}$ where

$$\overline{\gamma}(g,\overline{p})=\overline{\gamma(g,p)}.$$

Let $\mathrm{Orb}_S(H^c,\overline{p})$ be the orbit of the induced action of H^c on $S\cong\{r=1\}$. Then as in [14, Proposition 7.2], the transverse Kähler structure of the Sasaki manifold S is completely determined by the restriction of $\frac{1}{2}d\eta$ to $\mathrm{Orb}_{C(S)}(H^c,p)$, and also to $\mathrm{Orb}_S(H^c,\overline{p})$. For other basic Kähler forms ω on S we may restrict ω to these two orbits, and consider the transverse Kähler geometry as the Kähler geometry on $\mathrm{Orb}_{C(S)}(H^c,p)$ and $\mathrm{Orb}_S(H^c,\overline{p})$. The two Kähler manifolds $(\mathrm{Orb}_{C(S)}(H^c,p),\omega_{\mathrm{Orb}_{C(S)}(H^c,p)})$ and $(\mathrm{Orb}_S(H^c,\overline{p}),\omega_{\mathrm{Orb}_S(H^c,\overline{p})})$ thus obtained are essentially the same in that if we give them the holomorphic structures induced from the holomorphic structure of H^c then they are isometric Kähler manifolds. The difference between them is that $\mathrm{Orb}_{C(S)}(H^c,p)$ is a complex submanifold of the complex manifold $C(S)$ while $\mathrm{Orb}_S(H^c,\overline{p})$ is a complex submanifold in the real Sasaki manifold S. Furthermore, since the Reeb vector field can be approximated by quasi-regular ones, we may assume that the closure of $(\mathrm{Orb}_S(H^c,\overline{p}),\omega_{\mathrm{Orb}_S(H^c,\overline{p})})$ is a toric Kähler orbifold.

For any generic point $q'\in S$ the trajectory through q' generated by the Reeb vector field ξ meets $\mathrm{Orb}_S(H^c,\overline{p})$ and ξ generates a one-parameter subgroup of isometries. So, the transverse Kähler geometry at any q' is determined by the transverse Kähler geometry along the points on $\mathrm{Orb}_S(H^c,\overline{p})$. This trajectory may meet $\mathrm{Orb}_S(H^c,\overline{p})$ infinitely many times when the Sasaki structure is irregular. But the transverse structures at all of them define the same Kähler structure because ξ generates a subtorus in T^{m+1} and we assumed that T^{m+1} preserves the Sasaki structure.

Now we will express the Kähler potentials of ω_α in terms of real affine coordinates on H^c. On

$$\mathrm{Orb}_{C(S)}(H^c,p)\cong(\mathbf{C}^*)^m,$$

we use the affine logarithmic coordinates

$$(w^1,w^2,\ldots,w^m)=(x^1+\sqrt{-1}\theta^1,x^2+\sqrt{-1}\theta^2,\ldots,x^m+\sqrt{-1}\theta^m)$$

for a point

$$(e^{x^1+\sqrt{-1}\theta^1},e^{x^2+\sqrt{-1}\theta^2},\ldots,e^{x^m+\sqrt{-1}\theta^m})\in(\mathbf{C}^*)^m\cong H^c.$$

By [15, Subsection A.2.3], there is a real smooth function h_α unique up to an affine linear function such that

$$\omega_\alpha=\sqrt{-1}\frac{\partial^2 h_\alpha}{\partial x^i\partial x^j}dw^i\wedge dw^{\overline{j}}.$$

We call h_α the Kähler potential of ω_α. However, we have a fixed moment map image $\mathcal{P}_\alpha$ so that h_α is determined only up to a constant. Since $\omega = \sum_\alpha \omega_\alpha$ and we have the equality of Minkowski sum $\mathcal{P}_{-K_S} = \sum_\alpha \mathcal{P}_\alpha$ the Kähler potential of ω is equal to $\sum_\alpha \varphi_\alpha$ up to a constant. This implies on $\mathrm{Orb}_{C(S)}(H^c, p)$,

$$\omega_0^m = \mathrm{e}^{-\sum_\alpha h_\alpha}(\sqrt{-1}\,dw^1 \wedge dw^{\overline{1}}) \wedge \cdots \wedge (\sqrt{-1}\,dw^m \wedge dw^{\overline{m}}). \tag{5.5}$$

If we set

$$f_\alpha = h_\alpha + \varphi_\alpha,$$

then

$$\omega_\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\alpha = \sqrt{-1}\frac{\partial^2 f_\alpha}{\partial x^i \partial x^j} dw^i \wedge dw^{\overline{j}}. \tag{5.6}$$

By (5.5) and (5.6) we obtain

$$\mathrm{e}^{-t\sum_\alpha \varphi_\alpha}\omega_0^m = \mathrm{e}^{t\sum_\alpha f_\alpha - (1-t)\sum_\alpha h_\alpha}(\sqrt{-1}\,dw^1 \wedge dw^{\overline{1}}) \wedge \cdots \wedge (\sqrt{-1}\,dw^m \wedge dw^{\overline{m}}) \tag{5.7}$$

and

$$(\omega_\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\alpha)^m = \det\left(\frac{\partial^2 f_\alpha}{\partial x^i \partial x^j}\right)(\sqrt{-1}\,dw^1 \wedge dw^{\overline{1}}) \wedge \cdots \wedge (\sqrt{-1}\,dw^m \wedge dw^{\overline{m}}). \tag{5.8}$$

Since both $W_\alpha(f_\alpha)$ and $g_\alpha + W_\alpha(\varphi_\alpha)$ are Hamiltonian functions of W_α with respect to $\omega_\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\alpha$ we have

$$W_\alpha(f_\alpha) + C_\alpha = g_\alpha + W_\alpha(\varphi_\alpha)$$

for some constant C_α. Then normalizing g_α so that

$$\int_{\mathrm{Orb}_S(H^c,\overline{p})} \mathrm{e}^{g_\alpha}\omega_\alpha^m = 1$$

we obtain

$$\int_{\mathrm{Orb}_S(H^c,\overline{p})} \mathrm{e}^{g_\alpha + W_\alpha(\varphi_\alpha)}(\omega_\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\alpha)^m = 1$$

and

$$\begin{aligned}\int_{\mathrm{Orb}_S(H^c,\overline{p})} \mathrm{e}^{W_\alpha(f_\alpha)+C_\alpha}(\omega_\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\alpha)^m &= \int_{\mathbb{R}^m} \mathrm{e}^{W_\alpha(f_\alpha)+C_\alpha}\det\left(\frac{\partial^2 f_\alpha}{\partial x^i \partial x^j}\right)dx \\ &= \mathrm{e}^{C_\alpha}\int_{\mathcal{P}_\alpha} \mathrm{e}^{\langle W_\alpha, p\rangle}dp.\end{aligned}$$

Thus we obtain $C_\alpha = -\log \mathrm{Vol}_{W_\alpha}(\mathcal{P}_\alpha)$, and therefore

$$\frac{\mathrm{e}^{W_\alpha(f_\alpha)}}{\mathrm{Vol}_{W_\alpha}(\mathcal{P}_\alpha)} = \mathrm{e}^{g_\alpha + W_\alpha(\varphi_\alpha)}, \tag{5.9}$$

where

$$\mathrm{Vol}_{W_\alpha}(\mathcal{P}_\alpha) = \int_{\mathcal{P}_\alpha} \mathrm{e}^{\langle W_\alpha, p\rangle}dp.$$

From (5.7)–(5.9), the Monge-Ampère equation (5.4) reduces to the real Monge-Ampère equation

$$\frac{\mathrm{e}^{W_1(f_1)}}{\mathrm{Vol}_{W_1}(\mathcal{P}_1)}\det\left(\frac{\partial^2 f_1}{\partial x^i \partial x^j}\right) = \cdots = \frac{\mathrm{e}^{W_k(f_k)}}{\mathrm{Vol}_{W_k}(\mathcal{P}_k)}\det\left(\frac{\partial^2 f_k}{\partial x^i \partial x^j}\right) = \mathrm{e}^{t\sum_\alpha f_\alpha - (1-t)\sum_\alpha h_\alpha}.$$

For the rest of the proof the same arguments as in [16] apply. This completes the proof of Theorem 5.1. □

In [24,25], it is shown that for a toric Fano manifold Fut_W is the derivative of a proper convex function at $W \in \mathfrak{k}$ and that there exists a unique soliton vector field W, i.e., $\mathrm{Fut}_W = 0$. The same arguments apply in our coupled Sasaki Ricci-soliton case to find $W \in \mathfrak{k}$ with $\mathrm{Fut}_{W,\ldots,W} = 0$. Thus we obtain the following corollary to Theorem 5.1.

Corollary 5.3. *Let S be a compact toric Sasaki manifold with the basic first Chern class $c_1^B(S)$ positive, and $c_1^B(S) = (\sum_{\alpha=1}^k \gamma_\alpha)/2\pi$ be a basic decomposition. Then there exists a Killing potential W such that for $W_1 = \cdots = W_k = W$ we have coupled Sasaki Ricci-solitons.*

The general uniqueness result modulo automorphisms was established for coupled Kähler-Einstein metrics in [17]. The case for Sasaki-Ricci solitons will necessitate the pluripotential theory for Sasaki manifolds, and is beyond the scope of this paper.

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Appendix A

Let $L \to X$ be an ample line bundle over a compact complex manifold X. We choose a Hermitian metric h of L such that its curvature form ω is a positive form. Suppose that we have a Hamiltonian action of a torus T on X. The moment map for the torus action is defined up to a translation. This ambiguity depends on the choice of lifting of the action on X to L. However for the anti-canonical line bundle $L = K_X^{-1}$ we have the standard lifting, namely the action induced by the push-forward. We call the moment map of K_X^{-1} for the Fano manifold corresponding to the push-forward the *standard moment map*. Similarly we have the standard moment map for K_S^{-1} for a compact Sasaki manifold S with $c_1^B(S) > 0$.

Theorem A.1. *Let S (resp. X) be a toric Sasaki manifold with $c_1^B(S) > 0$ (resp. toric Fano manifold). The moment map given by the Hamiltonian functions u in (5.3) is the standard one for K_S^{-1} (resp. K_X^{-1}).*

Proof. We first consider the case of the toric Fano manifold X. Choose a Kähler form

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$$

with the positive Ricci form $\rho := \mathrm{Ric}(\omega)$. Then $\det g$ gives a Hermitian metric on K_X^{-1} and the connection form θ on the associated principal $\mathbf{C}^*$-bundle is given by $\theta = \zeta^{-1} d\zeta + (\det g)^{-1} \partial \det g$, where ζ is the fiber coordinate of the $\mathbf{C}^*$-bundle, and its curvature is $\rho = \sqrt{-1}\bar{\partial}\theta$. The Hamiltonian function in terms of ρ for an element $V \in \mathfrak{t}$ is given by $-\theta(V) = -\mathrm{div}_g V$, i.e., minus the divergence of V with respect to g. Thus the barycenter of the moment polytope with respect to the Kähler form ρ is given by $-\int_X \mathrm{div}_g V \rho_g^m$. By [10, Proposition 2.3], this is equal to $-\int_X V(F)\omega^m$. The Hamiltonian function u_V in terms of ω satisfies $\Delta_F u_V = -u_V$. Thus

$$-\int_X V(F)\omega^m = \int_X u_V \omega^m.$$

This shows the standard moment map has the same barycenter as the one given by the Hamiltonian functions in (5.3). The case of Sasaki manifolds is similar. This completes the proof. □