Calculus of Variations

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Regularity of harmonic maps from polyhedra to CAT(1) spaces

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Abstract We determine regularity results for energy minimizing maps from an n-dimensional Riemannian polyhedral complex X into a CAT(1) space. Provided that the metric on X is Lipschitz regular, we prove Hölder regularity with Hölder constant and expo-

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nent dependent on the total energy of the map and the metric on the domain. Moreover, at points away from the $(n-2)$-skeleton, we improve the regularity to locally Lipschitz. Finally, for points $x \in X^{(k)}$ with $k \le n - 2$, we demonstrate that the Hölder exponent depends on geometric and combinatorial data of the link of $x \in X$.

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1 Introduction

A natural notion of energy for a map between geometric spaces is defined by measuring the total stretch of the map at each point of the domain and then integrating it over the domain. Harmonic maps are critical points of the energy functional. They can be seen as both a generalization of harmonic functions in complex analysis and a higher dimensional analogue of parameterized geodesics in Riemannian geometry. In the absence of a totally geodesic map, a harmonic map is perhaps the most natural way to map one given geometric space into another.

The celebrated work of Eells and Sampson [9] initiated a wide interest in the study of harmonic maps between Riemannian manifolds, and harmonic maps have proven to be a useful tool in geometry. A more recent development is the harmonic map theory for non-smooth spaces. The seminal works of Gromov–Schoen [13] and Korevaar–Schoen [15] consider harmonic maps from a Riemannian domain into a non-Riemannian target. Further exploration of harmonic map theory in the singular setting includes works of Jost [14], Chen [3], Eells–Fuglede [8] and Daskalopoulos–Mese [5]. The above mentioned works all assume non-positivity of curvature (NPC) in the target space. In this paper, the goal is to investigate the regularity issues of harmonic maps in the case when the target curvature is bounded above by a constant that is not necessarily 0. In this direction, we mention earlier works of Serbinowski [20] for harmonic maps from Riemannian manifold domains and Fuglede [11,12] for polyhedral domains.

By understanding the regularity of harmonic maps, we can realize the potential applications of harmonic map theory. The key issue is to prove regularity theorems strong enough to be able to apply differential geometric methods. Applications of harmonic maps already in the literature include those in rigidity problems (for example, [4,13,21]) and in Teichmüller theory (for example, [6,7,22]) amongst others. Our goal is to apply harmonic map theory in a more general setting (namely for CAT(1) targets) than the NPC targets considered in the above mentioned applications. Indeed, in the follow-up of this paper [1], we prove a generalization to the metric space setting of Sacks and Uhlenbeck's celebrated work [19] on the bubbling phenomena for harmonic maps. The generalization of Sacks and Uhlenbeck's work has important connections to the non-smooth uniformization problem (cf. [2] and references therein) which in turn is related to the Cannon conjecture and the asymptotic geometry of negatively curved spaces. Details of these connections are provided in the introduction of [1]. We now state our main theorems.

Theorem 1.1 (Hölder regularity) *Let $B(r)$ be a ball of radius r around a point x in an admissible complex X endowed with a Lipschitz Riemannian metric g and let (Y, d) be a CAT(1) space and $\varrho \in (0, 1)$. If $f : (B(r), g) \to \mathcal{B}_\tau(P) \subset Y$ is an energy minimizing map, where $0 < \tau < \frac{\pi}{4}$, then there exist $C > 0$, $\gamma > 0$ such that*

$$d(f(x), f(y)) \le C|x - y|^\gamma \quad \text{for all } x, y \in B(\varrho r).$$

The constants C, γ depend only on the total energy E_g^f of the map, $(B(r), g)$ and ϱ.

Remark 1.2 Note that in the statement of the main theorems, the radius r of the ball $B(r)$ is measured with respect to the Euclidean metric δ_{ij} on each cell.

Fuglede proves a similar result in [11,12]. The main improvements of our results are the following: First, the metric on the domain space is more general; more specifically, the metrics considered in this paper are only assumed to be Lipschitz continuous while Fuglede considers simplex-wise smooth metrics (cf. p. 380, subsection "Maps into metric spaces" in [11]). We hope that this will lead to wider applications for the theory of harmonic maps from polyhedral domains. Second, and more importantly, we explicitly give the dependence of the Hölder constant and exponent on the total energy of the map. This statement in the special case of NPC targets has been crucial in the applications of harmonic map theory. In particular, the explicit dependence leads to a compactness result for a family of harmonic maps with uniformly bounded energy (see [1, Lemma 2.3]). Moreover, we can deduce the existence of tangent maps associated to harmonic maps (see Proposition 7.5).

We further remark that our proof uses very different techniques from those in [11,12]. Specifically, we take advantage of the work done by Daskalopoulos and Mese for NPC targets in [5], using the order function and a Campanato type theorem to prove the Hölder regularity. One of the advantages of this method is that, on high dimensional faces, we can improve the regularity to gain Lipschitz control, as given in Theorem 1.3. Moreover, as in [5], for points in the lower dimensional skeleta, we provide a lower bound on the Hölder exponent of the minimizing map in terms of the first eigenvalue of the link of the normal strata of the skeleton, λ_1^N.

Theorem 1.3 (Lipschitz regularity) *Let $B(r)$ be a ball of radius r around a point x in an admissible complex X endowed with a Lipschitz Riemannian metric g and let (Y, d) be a CAT(1) space. Suppose that $f : (B(r), g) \to \mathcal{B}_\tau(P) \subset Y$ is an energy minimizing map where $0 < \tau < \frac{\pi}{4}$.*

(1) *For $x \in X - X^{(n-2)}$, let $\bar{d}$ denote the distance of x to $X^{(n-2)}$. Then for $\varrho \in (0, 1)$ and $d' \leq \min\{\varrho r, \varrho \bar{d}\}$, f is Lipschitz continuous in $B(d')$ with Lipschitz constant depending on the total energy E_g^f of the map f, $(B(r), g)$, and d'.*

(2) *For $x \in X^{(k)} - X^{(k-1)}$ and $k = 0, 1, \ldots, n-2$, let $\bar{d}$ denote the distance of x to $X^{(k-1)}$. Then for $\varrho \in (0, 1)$ and $d' \leq \min\{\varrho r, \varrho \bar{d}\}$, f is Hölder continuous in $B(d')$ with Hölder exponent and constant depending on the total energy E_g^f of the map f, $(B(r), g)$, and d'. More precisely, the Hölder exponent α has lower bound given by the following: If $\lambda_1^N \geq \beta \ (> \beta)$ then $\alpha \ (\alpha + n - k - 2) \geq \beta \ (> \beta)$. In particular, if $\lambda_1^N \geq n - k - 1$, then f is Lipschitz continuous in a neighborhood of x.*

To understand the second item, for $x \in X^{(k)} - X^{(k-1)}$, let $N = N(x)$ denote the link of $X^{(k)}$ at x with metric induced by the Lipschitz Riemannian metric on X. Set

$$\lambda_1^N := \inf_{Q \in Y} \lambda_1(N, T_Q Y),$$

where $\lambda_1(N, T_Q Y)$ denotes the first eigenvalue of the Laplacian of N with values in the tangent cone of Y at Q. For more details, see Sect. 8.

Serbinowski [20], in an unpublished thesis, proves Lipschitz regularity from a Riemannian domain. Again, our proof is quite different from his.

The paper is organized as follows. In Sect. 2, we define the domains and targets of interest and prove a few key estimates on CAT(1) spaces. Section 3 includes background and necessary references for defining the energy and minimizing maps into metric spaces. This section also includes the definition of the cone over Y and important distance relations. In Sect. 4, we

prove a monotonicity formula for minimizing maps into CAT(1) spaces. In Sect. 5, we use the monotonicity formula to prove Theorem 1.1. Section 6 uses Theorem 1.1 to improve the monotonicity result which in turn allows us to improve the Hölder regularity so that the Hölder exponent is given by the order of the map. In Sect. 7, we determine a tangent map construction using the cone over Y, where the existence of a tangent map is given by the Hölder regularity. Finally, in Sect. 8, we use the tangent map construction and the improved Hölder regularity to prove Theorem 1.3.

2 Domain and target spaces

2.1 Admissible cell complexes and local models

Throughout the paper, X will denote an admissible n-dimensional cell complex (i.e. a dimensionally homogeneously, locally $(n-1)$-chainable convex cell complex) with a Lipschitz continuous Riemannian metric defined on each cell. We refer to [5, Section 2.2] for more details. In particular, since the regularity theorems we prove are local, we will study harmonic maps from a "local model" that represents a neighborhood of a point of X. We refer the reader to [5, Section 2.1] for the precise formulation of a local model, but will briefly describe this here. To do so, we inductively define a k-dimensional *conical cell*. First, a 1-dimensional conical cell is either the interval $[0, \infty)$ or the interval $(-\infty, 0]$. Having defined $(k-1)$-dimensional conical cells, we define a k-dimensional conical cell C as a subset of $\mathbb{R}^k$ with the following properties:

(i) The set C is non-empty and closed.
(ii) The set C is conical; i.e. if $x \in C$, then $tx \in C$ for $t \geq 0$.
(iii) The intersection of C with the unit sphere $\mathbb{S}^{k-1} \subset \mathbb{R}^k$ is geodesically convex (with respect the standard metric on $\mathbb{S}^{k-1}$).
(iv) The boundary ∂C of C is a finite union of $\{c_i\}$ where each c_i is a subset of a $(k-1)$-dimensional hyperplane H_i of $\mathbb{R}^k$ containing the origin such that if we identify $H_i \subset \mathbb{R}^k$ with $\mathbb{R}^{k-1} = \{(x^1, \ldots, x^{k-1}, 0)\} \subset \mathbb{R}^k$, (via an orthogonal transformation which takes H_i to $\mathbb{R}^{k-1}$), then c_i is a $(k-1)$-dimensional conical cell. We will say that c_i is a $(k-1)$-*dimensional boundary cell of C*.

An l-*dimensional boundary cell* of a k-dimensional conical cell C is $H \cap C$, where H is again a hyperplane of $\mathbb{R}^k$ containing the origin, such that there exists an orthogonal transformation of $\mathbb{R}^k$ which takes $H \cap C$ into $\mathbb{R}^l = \{(x^1, \ldots, x^l, 0, \ldots, 0)\} \subset \mathbb{R}^k$ but there exists no orthogonal transformation which takes $H \cap C$ into $\mathbb{R}^{l-1} = \{(x^1, \ldots, x^{l-1}, 0, \ldots, 0)\} \subset \mathbb{R}^k$. The union of l-dimensional boundary cells is called the l-*skeleton of C*.

Note that since ∂C bounds a conical cell, the hyperplanes H_i containing c_i are linearly independent in the sense that the set of normal vectors defining the hyperplanes are all linearly independent. Indeed, one may consider C as the intersection of appropriately oriented half-spaces, each with boundary one of the H_i. A k-dimensional conical cell C is said to have *codimension* ν if $\partial C = \cup_{i=1}^{\nu} c_i$. In that case, there exists a hyperplane H of $\mathbb{R}^k$ containing the origin and an orthogonal transformation T of $\mathbb{R}^k$ such that $T(H \cap \partial C)$ is equal to $\mathbb{R}^{k-\nu} = \{(x^1, \ldots, x^{k-\nu}, 0, \ldots, 0)\} \subset \mathbb{R}^k$. We let $D := T^{-1}(\mathbb{R}^{k-\nu})$.

A *dimension-n*, *codimension-ν local model* (of a neighborhood of a point in an n-dimensional cell complex) is $\mathbf{B} := \bigsqcup W/ \sim$, i.e. a disjoint union of a finite number $\mathcal{W} = \{W\}$ of n-dimensional conical cells of codimension ν modulo an equivalence relation $\sim$. We refer to $W \in \mathcal{W}$ as a *wedge*. The equivalence relation $\sim$ is defined by a finite set of isometries

$\{\varphi\}$ where each φ maps a boundary cell of one wedge to a boundary cell of another wedge. Note that the equivalence relation implies that we may consider a single D as belonging to the local model **B**.

We assume **B** is *admissible*, i.e. whenever $W \in \mathcal{W}$ and S is a $(n - 2)$-skeleton of W, $\bigsqcup W \backslash S / \sim$ is connected.

Each wedge W of **B** is a subset of $\mathbb{R}^n$ and therefore **B** comes equipped with the Euclidean metric (because each W inherits the Euclidean metric from $\mathbb{R}^n$). Let $\mathbf{B}(r)$ denote the ball of radius r, with respect to the Euclidean metric, centered at the origin of **B**. Throughout the rest of the paper $B_x(\sigma)$ will denote a Euclidean ball in **B**, centered at x and of radius σ. Furthermore, using the coordinates inherited from $\mathbb{R}^n$, we can define a Riemannian metric g on **B** by defining component functions (g_{ij}) on each wedge W. We say g is a *Lipschitz Riemannian metric* on **B** if on each W

$$|g_{ij}(x) - g_{ij}(\bar{x})| \le c|x - \bar{x}|, \quad \forall x, \bar{x} \in W.$$

As explained in [5, Proposition 2.1], we can and will often assume that the Lipschitz metrics are *normalized*, i.e.

$$|g_{ij}(x) - \delta_{ij}| \le c\sigma \quad \text{for } |x| \le \sigma. \tag{2.1}$$

Thus, for a *normalized* Lipschitz metric,

$$g_{ij}(0) = \delta_{ij}.$$

Lastly, we say $\lambda \in (0, 1]$ is an *ellipticity constant* for g if for each wedge W and for $x \in W$,

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n g_{ij}(x)\xi^i\xi^j \le \lambda^{-1}|\xi|^2.$$

2.2 CAT(1) spaces

Given a complete metric space (Y, d), Y is called a *geodesic space* if for each $P, Q \in Y$, there exists a curve γ_{PQ} such that the length of γ_{PQ} is exactly $d(P, Q)$. We call γ_{PQ} a *geodesic* between P and Q.

Remark 2.1 For ease of notation, we will often denote $d(P, Q)$ by d_{PQ}.

We determine a weak notion of an upper sectional curvature bound on Y by using comparison triangles. Given any three points $P, Q, R \in Y$ such that $d_{PQ} + d_{QR} + d_{RS} < 2\pi$, the *geodesic triangle* $\triangle PQR$ is the triangle in Y with sides given by the geodesics $\gamma_{PQ}, \gamma_{QR}, \gamma_{RS}$.

Let $\triangle \tilde{P}\tilde{Q}\tilde{R}$ denote a geodesic triangle on the standard sphere $\mathbb{S}^2$ such that $d_{PQ} = d_{\tilde{P}\tilde{Q}}$, $d_{QR} = d_{\tilde{Q}\tilde{R}}$ and $d_{RP} = d_{\tilde{R}\tilde{P}}$. We call $\triangle \tilde{P}\tilde{Q}\tilde{R}$ a *comparison triangle* for the geodesic triangle $\triangle PQR$. Note that a comparison triangle is convex since the perimeter of the geodesic triangle is less than 2π.

Definition 2.2 Given a geodesic space (Y, d) and a geodesic γ_{PQ} with $d_{PQ} < \pi$, for $\tau \in [0, 1]$ let $(1 - \tau)P + \tau Q$ denote the point on γ_{PQ} at distance τd_{PQ} from P. That is,

$$d((1 - \tau)P + \tau Q, P) = \tau d_{PQ}.$$

Definition 2.3 Let (Y, d) be a complete geodesic space. Then Y is a CAT(1) space if:

Given any geodesic triangle $\triangle PQR$ (with perimeter less than 2π) and a comparison triangle $\triangle \tilde{P}\tilde{Q}\tilde{R}$ in $\mathbb{S}^2$,

$$d_{P_t R_s} \leq d_{\tilde{P}_t \tilde{R}_s} \tag{2.2}$$

where

$$P_t = (1-t)P + tQ, \quad R_s = (1-s)R + sQ,$$
$$\tilde{P}_t = (1-t)\tilde{P} + t\tilde{Q}, \quad \tilde{R}_s = (1-s)\tilde{R} + s\tilde{Q}.$$

We conclude this section with a few key estimates that we will use later in the paper. The first estimate appeared in the thesis of [20, Estimate II] without proof. See [1, Lemma A.4] for a proof.

Lemma 2.4 *Let $\triangle PQS$ be a geodesic triangle in a CAT(1) space (Y, d). For a pair of numbers $0 \leq \eta, \eta' \leq 1$ define*

$$P_{\eta'} = (1-\eta')P + \eta'Q$$
$$S_\eta = (1-\eta)S + \eta Q.$$

Then

$$
\begin{aligned}
d^2(P_{\eta'}, S_\eta) \leq{} & \frac{\sin^2((1-\eta)d_{QS})}{\sin^2 d_{QS}} \left(d_{PS}^2 - (d_{QS} - d_{QP})^2\right) \\
& + \left((1-\eta)(d_{QS} - d_{QP}) + (\eta' - \eta)d_{QS}\right)^2 \\
& + \mathrm{Cub}\left(d_{PS}, d_{QS} - d_{QP}, \eta - \eta'\right).
\end{aligned}
\tag{2.3}
$$

As an immediate consequence, we have the following lemma.

Lemma 2.5 *Let $\triangle PQS$ be a geodesic triangle in a CAT(1) space (Y, d). For $0 \leq \eta, \eta' \leq 1$ and $P_{\eta'}, S_\eta$ as above,*

$$
\begin{aligned}
d^2(P_{\eta'}, S_\eta) \leq{} & \left(1 - 2\eta + \eta d_{QS}^2\right)d_{PS}^2 - 2(\eta - \eta')(d_{QS} - d_{QP})d_{QS} + (\eta' - \eta)^2 d_{QS}^2 \\
& + \mathrm{Quad}(\eta, \eta')\mathrm{Quad}(d_{PS}, d_{QS} - d_{QP}) + \mathrm{Cub}\left(d_{PS}, d_{QS} - d_{QP}, \eta - \eta'\right).
\end{aligned}
$$

Proof By Taylor expansion, $\sin((1-\eta)d_{QS}) = \sin d_{QS} - \eta d_{QS} \cos d_{QS} + O(\eta^2)$. Since $\frac{a}{\sin a} \geq 1$ and $\cos a \geq 1 - \frac{a^2}{2}$ for $0 \leq a < \pi$,

$$\frac{\sin^2((1-\eta)d_{QS})}{\sin^2 d_{QS}} = \left(1 - \eta \frac{d_{QS}}{\sin d_{QS}} \cos d_{QS} + O(\eta^2)\right)^2 \leq 1 - 2\eta + \eta d_{QS}^2 + O(\eta^2).$$

Substituting into (2.3) implies that

$$
\begin{aligned}
d^2(P_{\eta'}, S_\eta) \leq{} & \left(1 - 2\eta + \eta d_{QS}^2\right)\left(d_{PS}^2 - (d_{QS} - d_{QP})^2\right) \\
& + \left((1-\eta)(d_{QS} - d_{QP}) + (\eta' - \eta)d_{QS}\right)^2 \\
& + \eta^2 \mathrm{Quad}(d_{PS}, d_{QS} - d_{QP}) + \mathrm{Cub}\left(d_{PS}, d_{QS} - d_{QP}, \eta - \eta'\right).
\end{aligned}
$$

Expanding the quadratic term and collecting the remaining like terms implies the result. □

We conclude this section with a convexity bound.

Lemma 2.6 *Let* $\triangle PQR$ *be a geodesic triangle in a CAT(1) space* (Y, d). *If* $d_{PQ}, d_{PR} < \frac{\pi}{2}$, *then*

$$\frac{1}{8} \cos \left(d_{Q_{\frac{1}{2}} P} \right) d_{QR}^2 \leq \frac{1}{2} \left(d_{RP}^2 + d_{QP}^2 \right) - d_{Q_{\frac{1}{2}} P}^2 \tag{2.4}$$

where $Q_{\frac{1}{2}}$ *denotes the midpoint between* Q *and* R.

Proof By the triangle comparison, it suffices to prove inequality (2.4) assuming that $\triangle PQR$ is a geodesic triangle on the unit sphere. Let $\gamma(s)$ be an arclength parameterized geodesic on the sphere. Let

$$d_\gamma(s) := d(\gamma(s), P)$$

and assume that for all s, $d_\gamma(s) < \frac{\pi}{2}$. The function $d_\gamma(s)$ satisfies

$$(\cos d_\gamma(s))'' = -\cos d_\gamma(s).$$

Direct computation shows that

$$(d_\gamma(s))'' = \frac{1}{\tan d_\gamma(s)} \left(1 - (d_\gamma'(s))^2 \right)$$

$$\left(d_\gamma^2(s) \right)'' = 2 \frac{d_\gamma(s)}{\tan d_\gamma(s)} \left(1 - (d_\gamma'(s))^2 \right) + 2 \left(d_\gamma'(s) \right)^2.$$

Thus,

$$\left(d_\gamma^2(s) + 2 \cos d_\gamma(s) \right)'' = 2 \left(\frac{d_\gamma(s)}{\tan d_\gamma(s)} - \cos d_\gamma(s) \right) + 2 \left(1 - \frac{d_\gamma(s)}{\tan d_\gamma(s)} \right) \left(d_\gamma'(s) \right)^2.$$

Now let $\sigma(t)$ be a constant speed parameterization of the geodesic with $\sigma(0) = Q$ and $\sigma(1) = R$. Thus, $\gamma(s) := \sigma(s/\delta)$, where $\delta = d_{QR}$, is an arclength parameterized geodesic. With

$$d(t) := d(\sigma(t), P),$$

the chain rule implies that

$$(d^2(t) + 2 \cos d(t))'' = 2\delta^2 \left(\frac{d(t)}{\tan d(t)} - \cos d(t) \right) + 2\delta^2 \left(1 - \frac{d(t)}{\tan d(t)} \right) (d'(t))^2.$$

Since $0 \leq d(t) < \frac{\pi}{2}$,

$$\cos d(t) \leq \frac{d(t)}{\tan d(t)} \leq 1.$$

Therefore

$$(d^2(t) + 2 \cos d(t))'' \geq 0.$$

The convexity of $t \mapsto d^2(t) + 2 \cos d(t)$ implies that

$$d_{Q_{\frac{1}{2}} P}^2 + 2 \cos d_{Q_{\frac{1}{2}} P} \leq \frac{1}{2} \left(d_{QP}^2 + 2 \cos d_{QP} + d_{RP}^2 + 2 \cos d_{RP} \right). \tag{2.5}$$

Using the identity on the sphere and a double angle formula,

$$\cos d_{Q_{\frac{1}{2}} P} = \frac{\sin \frac{d_{QR}}{2}}{\sin d_{QR}} \cos d_{QP} + \frac{\sin \frac{d_{QR}}{2}}{\sin d_{QR}} \cos d_{RP} = \frac{1}{2 \cos \frac{d_{QR}}{2}} \left(\cos d_{QP} + \cos d_{RP} \right).$$

Since $\cos a \leq 1 - \frac{a^2}{4}$ for $0 \leq a < \frac{\pi}{4}$,

$$\cos d_{QP} + \cos d_{RP} = 2\cos d_{Q_{\frac{1}{2}}P}\cos\frac{d_{QR}}{2} \leq 2\cos\left(d_{Q_{\frac{1}{2}}P}\right) - \frac{1}{8}\cos\left(d_{Q_{\frac{1}{2}}P}\right)d_{QR}^2.$$

The desired inequality follows from inserting the above into (2.5). □

3 Sobolev space and the energy density

In the seminal work of Korevaar–Schoen (cf. [15, Chapter 1]) the authors define the energy density and directional energies for maps from Riemannian manifolds into metric spaces. Using [5, Proposition 2.1], these definitions immediately extend to include maps from an admissible complex X (cf. [5, Section 2]). Following the usual convention, we say $f \in W^{1,2}(\Omega, Y)$ if $f \in L^2(\Omega)$ and the energy is finite. We then write $|\nabla f|_g^2(x)$ in place of the energy density function and let

$$E_g^f = \int_{\mathbf{B}(r)} |\nabla f|_g^2 d\mu_g.$$

For a set $S \subset \mathbf{B}(r)$, let

$$E_g^f[S] = \int_S |\nabla f|_g^2 d\mu_g.$$

To study energy minimizing maps, we use the notion of the trace of f, for $f \in W^{1,2}(\Omega, Y)$, as defined in [8,15]. We denote the space of admissible maps $W_f^{1,2}(\Omega, \mathcal{B}) := \{h \in W^{1,2}(\Omega, \mathcal{B}) : d(f, h) \in W_0^{1,2}(\Omega)\}$.

Definition 3.1 Let Ω be a compact domain in an admissible complex with Lipschitz Riemannian metric g and (Y, d) be a CAT(1) space. A finite energy map $f : \Omega \to \mathcal{B}_\tau(P) \subset Y$ is *energy minimizing* if f minimizes energy amongst maps in $W_f^{1,2}(\Omega, \overline{\mathcal{B}_\tau(P)})$.

The existence and uniqueness of energy minimizers from Riemannian domains appeared in the thesis [20] and the same result from Riemannian complexes into small balls in a CAT(1) space was established in [10]. We verify the existence and uniqueness in the Riemannian case in the appendix of [1].

Remark 3.2 Note that unlike the definition in [20], the comparison maps in Definition 3.1 not only have the same trace as f but also map into the same ball. The reason that we define energy minimizing maps in this way is that, unlike in the NPC setting, the projection map onto convex domains in a CAT(1) space is not globally distance decreasing. Therefore, one cannot guarantee that a minimizer in the class $W_f^{1,2}(\Omega, Y)$ maps into the closure of $\mathcal{B}_\tau(P)$ without some extra hypotheses. For simplicity, we define a minimizer by considering only competitors in the smaller class of maps.

3.1 The pullback metric

The directional energies are defined in a fashion similar to the energy density function. See [15] for the definition of the directional energy and of the pull-back inner product π when Y is an NPC space.

We use the triangle comparison in CAT(1) spaces to demonstrate that directional energies and the pull-back inner product are well defined for finite energy maps into CAT(1) spaces.

The next lemma appeared in [17, Lemma 3.6] and is a consequence of (2.2). We include the proof here both for completeness and because we have simplified the proof.

Lemma 3.3 *Let Y be a CAT(1) space. For every $\epsilon_0 > 0$, there exists $\delta_0 > 0$ such that if $P, Q, R, S \in Y$ with $\max\{d_{PQ}, d_{QR}, d_{RS}, d_{PS}\} \le \delta_0$, then*

$$d_{PR}^2 + d_{QS}^2 \le d_{PQ}^2 + d_{QR}^2 + d_{RS}^2 + d_{PS}^2 + \epsilon_0 \delta_0^2.$$

Proof By comparing a geodesic quadrilateral $\square PQRS$ in Y to a comparison quadrilateral $\square \tilde{P}\tilde{Q}\tilde{R}\tilde{S}$ in $\mathbb{S}^2$ (and noting [18] which says that the pairwise distance of points on $\square PQRS$ is bounded by the distance of the corresponding pair in $\square \tilde{P}\tilde{Q}\tilde{R}\tilde{S}$), it is sufficient to prove the assertion when $Y = \mathbb{S}^2$. Suppose the assertion is not true on $\mathbb{S}^2$. Then there exists $\epsilon_0 > 0$ and a sequence P_i, Q_i, R_i, S_i with $\max\{d_{P_i Q_i}, d_{Q_i R_i}, d_{R_i S_i}, d_{P_i S_i}\} \le \delta_i \to 0$ such that

$$d_{P_i R_i}^2 + d_{Q_i S_i}^2 > d_{P_i Q_i}^2 + d_{Q_i R_i}^2 + d_{R_i S_i}^2 + d_{P_i S_i}^2 + \epsilon_0 \delta_i^2.$$

For each i, denote by $\frac{1}{\delta_i}\mathbb{S}^2$ the rescaling of the unit sphere $\mathbb{S}^2$ by a factor of $\frac{1}{\delta_i}$ and let $P_i', Q_i', R_i', S_i' \in \frac{1}{\delta_i}\mathbb{S}^2$ be the corresponding points to $P_i, Q_i, R_i, S_i \in \mathbb{S}^2$ respectively. Thus,

$$d_{P_i' R_i'}^2 + d_{Q_i' S_i'}^2 > d_{P_i' Q_i'}^2 + d_{Q_i' R_i'}^2 + d_{R_i' S_i'}^2 + d_{P_i' S_i'}^2 + \epsilon_0.$$

and $\max\{d_{P_i' Q_i'}, d_{Q_i' R_i'}, d_{R_i' S_i'}, d_{P_i' S_i'}\} = 1$. This is a contradiction since the Gauss curvature of the sphere $\frac{1}{\delta_i}\mathbb{S}^2$ goes to 0 as $\delta_i \to 0$ and

$$d_{\bar{P}\bar{R}}^2 + d_{\bar{Q}\bar{S}}^2 \le d_{\bar{P}\bar{Q}}^2 + d_{\bar{Q}\bar{R}}^2 + d_{\bar{R}\bar{S}}^2 + d_{\bar{S}\bar{P}}^2$$

for every $\bar{P}, \bar{Q}, \bar{R}, \bar{S} \in \mathbb{R}^2$. $\qquad\square$

Lemma 3.4 *Let $f : (\mathbf{B}(r), g) \to Y$ be a finite energy map and (Y, d) a CAT(1) space. Then letting $|f_*(Z)|^2$ denote the directional energy density function in the direction of Z (cf. [15, Section 1.8]), the parallelogram identity*

$$|f_*(Z + W)|_g^2 + |f_*(Z - W)|_g^2 = 2|f_*(Z)|_g^2 + 2|f_*(W)|_g^2$$

holds for a.e. $x \in \mathbf{B}(r)$ and any pair of Lipschitz vector fields Z, W on $\mathbf{B}(r)$.

Proof Fix $\epsilon_0 > 0$. Let $\epsilon \mapsto x_1(x, \epsilon)$, $\epsilon \mapsto x_2(x, \epsilon)$ and $\epsilon \mapsto x_3(x, \epsilon)$ be the flow induced by the vector fields Z, $Z + W$ and W respectively with $x_1(x, 0) = x_2(x, 0) = x_3(x, 0) = x$. By [15, Lemma 1.9.2], $\epsilon \mapsto f(x_i(x, \epsilon))$ is continuous at $\epsilon = 0$, $i = 1, 2, 3$ for a.e. $x \in \Omega$. For such x, apply Lemma 3.3 with $P = f(x)$, $Q = f(x_1(x, \epsilon))$, $R = f(x_2(x, \epsilon))$, $S = f(x_3(x, \epsilon))$, divide by ϵ^2 and multiply the resulting inequality by $\phi \in C_c^\infty(\Omega)$. Now following the argument of [15, Lemma 2.3.1], we conclude that

$$|f_*(Z + W)|_g^2(x) + |f_*(Z - W)|_g^2(x) \le 2|f_*(Z)|_g^2(x) + 2|f_*(W)|_g^2(x) + \epsilon_0 \Delta$$

where

$$\Delta \ge \max\left\{|f_*(Z + W)|_g^2(x), |f_*(Z - W)|_g^2(x), |f_*(Z)|_g^2(x), |f_*(W)|_g^2(x)\right\}.$$

Since $\epsilon_0 > 0$ is arbitrary, we conclude that

$$|f_*(Z + W)|_g^2 + |f_*(Z - W)|_g^2 \le 2|f_*(Z)|_g^2 + 2|f_*(W)|_g^2.$$

Repeat using $Z + W$ and $Z - W$ in place of Z and W to get the opposite inequality. $\qquad\square$

Lemma 3.5 *Let* $f : (\mathbf{B}(r), g) \to Y$ *be a finite energy map and let* Z, W *be Lipschitz vector fields on* $\mathbf{B}(r)$*. The operator* ${}^g\pi^f$ *defined by*

$$
{}^g\pi^f(Z, W) := \frac{1}{4}|f_*(Z + W)|_g^2 - \frac{1}{4}|f_*(Z - W)|_g^2
$$

is symmetric, bilinear, non-negative and tensorial.

Proof Using Lemma 3.4, we can follow the proof of [15, Theorem 2.3.2]. □

Notation 3.6 *Let* $\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}$ *be the standard Euclidean basis defined on each wedge inherited from* $\mathbb{R}^n$ *and* δ *the standard Euclidean metric. Set*

$$
\frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j} = {}^\delta\pi^f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \quad and \quad \left|\frac{\partial f}{\partial x^i}\right|^2 = \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^i}.
$$

Similarly for the standard Euclidean polar coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ *on each wedge we denote*

$$
\frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} = {}^\delta\pi^f\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial r}\right), \quad \left|\frac{\partial f}{\partial r}\right|^2 = \frac{\partial f}{\partial r} \cdot \frac{\partial f}{\partial r} = {}^\delta\pi^f\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)
$$

and

$$
\frac{\partial f}{\partial \theta_i} \cdot \frac{\partial f}{\partial \theta_j} = {}^\delta\pi^f\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right).
$$

Note that the energy density with respect to the metric g is given by

$$
|\nabla f|_g^2 = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j},
$$

whereas the energy density with respect to the Euclidean metric is given by

$$
|\nabla f|^2 = |\nabla f|_\delta^2 = \sum_i \left|\frac{\partial f}{\partial x^i}\right|^2.
$$

3.2 The cone over Y and energy comparisons

We denote by $\mathcal{C}Y$ the metric cone over Y. Topologically, $\mathcal{C}Y$ is defined by

$$
\mathcal{C}Y = Y \times [0, \infty)/Y \times \{0\}.
$$

A point in $\mathcal{C}Y$ is a pair $[P, t]$ for $P \in Y$ and $t \in [0, \infty)$, with $[P, 0]$ and $[Q, 0]$ representing the same point in $\mathcal{C}Y$ for all $P, Q \in Y$. We endow $\mathcal{C}Y$ with a distance function D defined by $D^2([P, t], [Q, s]) = t^2 + s^2 - 2ts \cos \min(d_{PQ}, \pi)$. It is well known that when Y is a CAT(1) space, the metric space $(\mathcal{C}Y, D)$ is an NPC space.

For $P, Q \in Y$ with $d_{PQ} < \pi/2$,

$$
\frac{1}{2} \leq \frac{D^2([P, 1], [Q, 1])}{d_{PQ}^2} \leq 1, \tag{3.1}
$$

$$
\lim_{P \to Q} \frac{D^2([P, 1], [Q, 1])}{d_{PQ}^2} = \lim_{P \to Q} \frac{2(1 - \cos(d_{PQ}))}{d_{PQ}^2} = 1, \tag{3.2}
$$

i.e., Y is isometrically embedded into $Y \times \{1\}$ in an infinitesimal sense. Moreover, for d_{PQ} small,

$$d_{PQ}^2(1 - d_{PQ}^2) \leq D^2([P, 1], [Q, 1]). \tag{3.3}$$

Definition 3.7 For any map $w : \Omega \to Y$, we let $\overline{w} : \Omega \to Y \times \{1\}$ be given by $\overline{w}(x) = [w(x), 1]$. We call $\overline{w}$ the *lifted map* of w.

If $w \in W^{1,2}(\Omega, Y)$ then $\overline{w} \in W^{1,2}(\Omega, \mathcal{C}Y)$ and the definition of energy implies that

$$^d E_g^w[\Omega] = {}^D E_g^{\overline{w}}[\Omega]. \tag{3.4}$$

We let $\Pi : \mathcal{C}Y \to Y \times \{1\}$ denote the projection map $\Pi([P, t]) = [P, 1]$. Then for $d_{PQ} < \pi$,

$$\begin{aligned}
D^2([P, t], [Q, s]) &= t^2 + s^2 - 2st \cos(d_{PQ}) \\
&= (t - s)^2 + 2st(1 - \cos(d_{PQ})) \\
&\geq 2st(1 - \cos(d_{PQ})) \\
&= st\, D^2(\Pi([P, t]), \Pi([Q, s])).
\end{aligned} \tag{3.5}$$

4 A monotonicity formula

The goal of this section is to prove a proposition analogous to [5, Proposition 3.1]. The reader would benefit from familiarity with Sect. 3, up through Lemma 3.5, of that paper.

Let **B** be a local model. In each wedge W, we use Euclidean coordinates $(x^1, \ldots, x^n)$. For $x, y \in \mathbf{B}$, denote the induced Euclidean distance by $|x - y|$. Thus, if $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ are on the same wedge of **B**, then $|x - y|^2 = \sum_{i=1}^n (x^i - y^i)^2$. Let $(r, \theta_1, \ldots, \theta_{n-1})$ denote polar coordinates, so r represents radial distance from the origin and $\theta = (\theta_1, \ldots, \theta_{n-1})$ are the standard coordinates on the $(n - 1)$-sphere.

Presume, unless otherwise stated, that g is a normalized Lipschitz metric defined on $\mathbf{B}(r)$. For $\sigma \in (0, r)$, set

$$E_g^f(\sigma) = \int_{\mathbf{B}(\sigma)} |\nabla f|_g^2 d\mu_g \tag{4.1}$$

and

$$I_g^f(\sigma, Q) = \int_{\partial \mathbf{B}(\sigma)} d^2(f, Q) d\Sigma_g \tag{4.2}$$

for $Q \in Y$. Here $d\Sigma_g$ is the measure on $\partial \mathbf{B}(\sigma)$ induced by g.

Notation 4.1 *For simplicity, in the rest of this section we will use the notation*

$$E(\sigma) = E_g^f(\sigma) \ \text{and} \ I(\sigma) = I(\sigma, Q) = I_g^f(\sigma, Q),$$

if Q is a generic point. Furthermore in all statements we assume that the metric g is normalized.

We begin with a technical lemma which provides a unique center of mass for energy minimizers into sufficiently small balls. See [15, Lemma 2.5.1] for the analogous statement for L^2 maps with NPC targets.

Lemma 4.2 *Let (Y, d) be a CAT(1) space and $0 < \tau < \frac{\pi}{4}$. If $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ is an L^2 map, then for each $0 < \sigma < r$ there exists a unique $Q_\sigma \in Y$ such that*

$$I(\sigma, Q_\sigma) = \inf_{Q \in Y} I(\sigma, Q).$$

Proof Note that it is enough to consider points Q in $\mathcal{B}_\tau(P)$ since the projection function is distance decreasing on balls of radius $\frac{\pi}{4}$ in CAT(1) spaces. By Lemma 2.6, for $x \in \mathbf{B}(r)$ and $Q, R \in \mathcal{B}_\tau(P)$, (2.4) implies that

$$\frac{1}{8} \cos\left(d_{Q_{\frac{1}{2}} f(x)}\right) d_{QR}^2 \leq \frac{1}{2}(d^2(R, f(x)) + d^2(Q, f(x))) - d^2(Q_{\frac{1}{2}}, f(x))$$

where $Q_{\frac{1}{2}}$ is the midpoint between Q, R. Note that $\cos\left(d_{Q_{\frac{1}{2}} f(x)}\right) \geq \cos(2\tau) > 0$. Thus, integrating over $(\mathbf{B}(\sigma), g)$ implies that

$$d_{QR}^2 \leq \frac{C}{\cos(2\tau)} \left(\frac{1}{2}\left(I(\sigma, R) + I(\sigma, Q)\right) - I(\sigma, Q_{\frac{1}{2}})\right).$$

It follows that any minimizing sequence for $I(\sigma, Q)$ is Cauchy and therefore there is a unique minimum. $\qquad\square$

We next prove a type of subharmonicity result for the d^2 function. See [13, Proposition 2.2.], [5, Lemma 3.3] for a similar result when Y is NPC. Note that in the NPC setting the integral of $d^2(f, Q)|\nabla f|_g^2$ does not appear in (4.3).

Lemma 4.3 *Let $0 < \tau < \frac{\pi}{2}$ and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ be an energy minimizing map and (Y, d) a CAT(1) space. Presume that $Q \in \mathcal{B}_\tau(P)$. Then for all $0 < \sigma \leq r$*

$$2E(\sigma) - \int_{\mathbf{B}(\sigma)} d^2(f, Q)|\nabla f|_g^2 d\mu_g \leq \int_{\partial \mathbf{B}(\sigma)} \langle \nabla |x|, \nabla d^2(f, Q)\rangle_g d\Sigma_g. \qquad (4.3)$$

Proof Define $f_\eta : (\mathbf{B}(r), g) \to Y$ by setting

$$f_\eta(x) = (1 - \eta(x))f(x) + \eta(x)Q$$

for $\eta \in C_c^\infty(\mathbf{B}(r))$. Letting $S = f(x)$, $P = f(y)$, $\eta' = \eta(y)$, we use the estimate of Lemma 2.5 to observe that for $\hat{d}(x) := d(Q, f(x))$,

$$\begin{aligned}
d^2(f_\eta(y), f_\eta(x)) &\leq (1 - 2\eta(x) + \eta(x)\hat{d}^2(x))d^2(f(x), f(y)) \\
&\quad - 2(\eta(x) - \eta(y))(\hat{d}(x) - \hat{d}(y))\hat{d}(x) \\
&\quad + (\eta(y) - \eta(x))^2\hat{d}^2(x) + \eta^2(x)\text{Quad}(d(f(x), f(y)), \hat{d}(x) - \hat{d}(y)) \\
&\quad + \text{Cub}\left(d(f(x), f(y)), \hat{d}(x) - \hat{d}(y), \eta(x) - \eta(y)\right).
\end{aligned}$$

Divide by ϵ^{n+1} and fix $x \in \mathbf{B}(r)_\epsilon$ where

$$\mathbf{B}(r)_\epsilon = \{x \in \mathbf{B}(r) : d(x, \partial \mathbf{B}(r)) > \epsilon\}.$$

Let $S(x, \epsilon)$ denote the ϵ-sphere centered at x. By integrating over all $y \in S(x, \epsilon)$ with respect to the induced measure on $S(x, \epsilon)$, integrating over all $x \in \mathbf{B}(r)_\epsilon$, and letting $\epsilon \to 0$, we obtain

$$\begin{aligned}
\int_{\mathbf{B}(r)} |\nabla f_\eta|_g^2 d\mu_g &\leq \int_{\mathbf{B}(r)} |\nabla f|_g^2 d\mu_g - 2\int_{\mathbf{B}(r)} \eta|\nabla f|_g^2 d\mu_g + \int_{\mathbf{B}(r)} \eta d^2(f, Q)|\nabla f|_g^2 d\mu_g \\
&\quad - \int_{\mathbf{B}(r)} \langle \nabla \eta, \nabla d^2(f, Q)\rangle_g d\mu_g + O(\eta^2, |\nabla \eta|_g^2).
\end{aligned}$$

Note that the cubic error terms either vanish as $\epsilon \to 0$ or can be absorbed into the remaining error.

Now note that the energy of f is bounded from above by the energy of f_η. Thus,

$$2 \int_{\mathbf{B}(r)} \eta |\nabla f|_g^2 d\mu_g - \int_{\mathbf{B}(r)} \eta d^2(f, Q) |\nabla f|_g^2 d\mu_g$$

$$\leq - \int_{\mathbf{B}(r)} \langle \nabla \eta, \nabla d^2(f, Q) \rangle_g d\mu_g + O(\eta^2, |\nabla \eta|_g^2).$$

Replace η by $\alpha \eta$, divide by α and let $\alpha \to 0$ to cancel out the $O(\eta^2, |\nabla \eta|_g^2)$ term. Letting η approximate the characteristic function on $\mathbf{B}(\sigma)$ implies (4.3). $\square$

Lemma 4.4 *Let $0 < \tau < 1$ and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ be an energy minimizing map, (Y, d) a CAT(1) space, and g a normalized Lipschitz metric. Then, for all $0 < \sigma < r$,*

$$\frac{1}{2} E(\sigma) \leq I(\sigma)^{1/2} \left(\left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^{1/2} + c\sigma (E'(\sigma))^{1/2} \right), \qquad (4.4)$$

where c depends on $\mathbf{B}(r)$, and the Lipschitz bound and ellipticity constant of g.

Proof By (4.3) and the Lipschitz bound $|g^{ij} - \delta^{ij}| \leq c\sigma$, for $Q \in \mathcal{B}_\tau(P)$,

$$(2 - 4\tau^2) E(\sigma) \leq \int_{\partial \mathbf{B}(\sigma)} \langle \nabla |x|, \nabla d^2(f, Q) \rangle_g d\Sigma_g$$

$$= \int_{\partial \mathbf{B}(\sigma)} g^{ij} \frac{x_j}{|x|} \frac{\partial}{\partial x^i} d^2(f, Q) \, d\Sigma_g$$

$$\leq \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g + c\sigma \int_{\partial \mathbf{B}(\sigma)} \sum_i \left| \frac{\partial}{\partial x^i} d^2(f, Q) \right| d\Sigma_g$$

$$= 2 \int_{\partial \mathbf{B}(\sigma)} d(f, Q) \frac{\partial}{\partial r} d(f, Q) d\Sigma_g$$

$$+ 2c\sigma \int_{\partial \mathbf{B}(\sigma)} d(f, Q) \sum_i \left| \frac{\partial}{\partial x^i} d(f, Q) \right| d\Sigma_g$$

$$\leq 2I(\sigma)^{1/2} \left(\left(\int_{\partial \mathbf{B}(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^{1/2} + c\sigma (E'(\sigma))^{1/2} \right). \qquad (4.5)$$

In the final inequality we use Hölder's inequality and the fact that

$$\left| \frac{\partial}{\partial r} d(f, Q) \right|^2 \leq \left| \frac{\partial f}{\partial r} \right|^2, \quad \left| \frac{\partial}{\partial x^i} d(f, Q) \right|^2 \leq \left| \frac{\partial f}{\partial x^i} \right|^2, \quad |\nabla f|^2 \leq \frac{1}{\lambda^2} |\nabla f|_g^2,$$

where λ is the ellipticity constant of g. $\square$

Lemma 4.5 *Let $0 < \tau < \frac{\pi}{4}$ and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ be an energy minimizing map into a CAT(1) space (Y, d). There exist $\sigma_0 > 0$ and $\gamma > 0$ depending on $\mathbf{B}(r)$, and the Lipschitz bound and the ellipticity constant of g so that*

$$\sigma \mapsto \frac{E(\sigma)}{\sigma^{n-2+2\gamma}}, \quad \sigma \in (0, \sigma_0)$$

is non-decreasing.

Proof Let $Q_\sigma \in Y$ such that

$$I(\sigma, Q_\sigma) = \inf_{Q \in Y} I(\sigma, Q)$$

where the existence is guaranteed by Lemma 4.2. We now follow the exact argument of [5, Lemma 3.5]. Note that their invocation of [5, (3.12)] is replaced by (4.4) here. All other inequalities they reference arise from appropriate domain variations and are therefore true for maps into Y. □

As in [5], all of the previous results extend from the setting of normalized metrics to admissible complexes with Lipschitz Riemannian metrics. See [5, p. 289-290] to understand how the properties of the map L_x in [5, Proposition 2.1] affect the energy of f and the domain over which Lemma 4.5 can be applied.

Let **B** be a dimension-n, codimension-$(n-k)$ local model and g a Lipschitz metric on **B**(r) with ellipticity constant $\lambda \in (0, 1]$. For $x \in$ **B**(r), let $R(x)$ denote the radius of the largest homogeneous ball centered at x contained in **B**(r). The value $\sigma_0 > 0$ was defined above as the upper bound for which the monotonicity formula of Lemma 4.5 holds for any energy minimizing map from a local model with a normalized metric. Therefore, the monotonicity formula for $f \circ L_x$ is valid for balls **B**$'(\sigma)$ contained in **B**$'(r_0(x))$ where

$$r_0(x) := \min\{\sigma_0, \lambda R(x)\}. \tag{4.6}$$

Recalling that $B_x(\sigma)$ is the Euclidean ball about x of radius σ, we define $E_x(\sigma)$ for σ sufficiently small by setting

$$E_x(\sigma) = \int_{B_x(\sigma)} |\nabla f|^2 d\mu_g.$$

Proposition 4.6 *Let* **B** *be a dimension-n, codimension-ν local model, g a Lipschitz Riemannian metric defined on* **B**(r) *with ellipticity constant $\lambda \in (0, 1]$, (Y, d) a CAT(1) space and $f : ($**B**$(r), g) \rightarrow \mathcal{B}_\tau(P) \subset Y$ an energy minimizing map. If $0 < \tau < \frac{\pi}{4}$, then there exist constants $\gamma > 0$ and $C \geq 1$ depending on* **B**(r), *the Lipschitz bound and the ellipticity constant of g so that for every $x \in$* **B**(r),*

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\gamma}} \leq C \frac{E_x(\rho)}{\rho^{n-2+2\gamma}}, \quad 0 < \sigma < \rho \leq r(x) \tag{4.7}$$

where

$$r(x) := \lambda r_0(x) = \min\{\lambda \sigma_0, \lambda^2 R(x)\}. \tag{4.8}$$

Here, $R(x)$ is defined as above and $\sigma_0 > 0$ is as in Lemma 4.5.

Proof The proof proceeds exactly as in the proof of [5, Proposition 3.1], using Lemma 4.5. □

5 Hölder regularity

The goal of this section is to prove Theorem 1.1. The proof is modeled on the proof of Hölder regularity in [5] for minimizing maps into an NPC space. The reader would benefit from a familiarity with Section 4 of that paper. The method of proof is classical, as the regularity result will follow from a Campanato theorem and the monotonicity given by (4.7). Many of the technical aspects of this argument in [5] are related to the singular nature of the domain

and thus can be immediately applied for CAT(1) targets. We will highlight the key places where the target curvature plays a role and provide suitable adaptations of the arguments involved.

Before proceeding to the main argument, we prove a technical lemma.

Lemma 5.1 *Let Ω be a Euclidean domain, (Y, d) a CAT(1) space, and $0 < \tau < \frac{\pi}{4}$. If $f : \Omega \to \mathcal{B}_\tau(P) \subset Y$ is an L^2 map then, for all $\epsilon > 0$, there exists $h_\epsilon : \Omega_\epsilon \to Y$ Lipschitz such that*

$$\int_{\Omega_\epsilon} d^2(f, h_\epsilon) d\mu_g < \epsilon,$$

where $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$.

Proof For $f : \Omega \to Y$, recall $\overline{f} : \Omega \to \mathcal{C}Y$ is defined by setting

$$\overline{f}(x) = [f(x), 1].$$

(See Sect. 3.2 for further relevant definitions and energy comparisons.)

We use the notation $\mathcal{B}_r^{\mathcal{C}Y}(\cdot)$ to denote a ball of radius r in $\mathcal{C}Y$. By (3.1),

$$\overline{f}(x) \in \mathcal{B}_\tau^{\mathcal{C}Y}([P, 1]), \quad \forall x \in \Omega.$$

and $\overline{f}$ is an L^2-map into $\mathcal{C}Y$. Since $\mathcal{C}Y$ is NPC, we can apply the mollification procedure of [16, Section 1.5] to produce a Lipschitz map $g_\epsilon : \Omega_\epsilon \to \mathcal{B}_\tau^{\mathcal{C}Y}([P, 1]) \subset \mathcal{C}Y$ such that

$$\int_{\Omega_\epsilon} D^2(\overline{f}, g_\epsilon) dx < \frac{\epsilon}{4}.$$

Write $g_\epsilon(x) = [\varphi(x), t(x)]$. The map $g_\epsilon(x)$ is constructed as the center of mass of the map $\overline{f}$ with respect to a probability measure $\eta_\epsilon(x - y)dy$ where η_ϵ can be chosen to be a function with compact support in a small ball centered at 0. Therefore, since Image$(\overline{f}) \subset Y \times \{1\}$, we can assume that $D(g_\epsilon(x), Y \times \{1\})$ satisfies

$$|1 - t(x)| < (\text{volume}(\Omega_\epsilon))^{-1/2} \sqrt{\frac{\epsilon}{4}}.$$

Thus,

$$\int_{\Omega_\epsilon} D^2(\overline{f}, \Pi \circ g_\epsilon) dx \leq \int_{\Omega_\epsilon} D^2(\overline{f}, g_\epsilon) dx + \int_{\Omega_\epsilon} D^2(g_\epsilon, \Pi \circ g_\epsilon) dx < \frac{\epsilon}{2}$$

where $\Pi : \mathcal{C}Y \to Y \times \{1\}$ is the projection map as in Sect. 3.2. Since $|1 - t(x)|$ is bounded for all $x \in \Omega$ and g_ϵ is Lipschitz, (3.5) implies that $\Pi \circ g_\epsilon$ is Lipschitz on Ω_ϵ. Define

$$h_\epsilon : \Omega_\epsilon \to Y, \quad h_\epsilon = \Pi \circ g_\epsilon$$

by identifying Y with $Y \times \{1\} \subset \mathcal{C}Y$. Then by (3.1)

$$\int_{\Omega_\epsilon} d^2(f, h_\epsilon) dx \leq 2 \int_{\Omega_\epsilon} D^2(\overline{f}, \Pi \circ g_\epsilon) dx < \epsilon.$$

$\square$

We now prove a Campanato type lemma. In [5, Lemma 4.1], the authors prove a similar result for any L^2 map into an NPC space.

Lemma 5.2 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a Lipschitz Riemannian metric on $\mathbf{B}(r)$, (Y, d) a CAT(1) space, $0 < \tau < \frac{\pi}{4}$ and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ an L^2 map. Fix $\varrho \in (0, 1)$. If there exist $K > 0$, $R \in (0, (1 - \varrho)r)$ and $\beta \in (0, 1]$ such that*

$$\inf_{Q \in Y} \sigma^{-n} \int_{B_x(\sigma)} d^2(f, Q) \, d\mu_g \leq K^2 \sigma^{2\beta}, \quad \forall x \in \mathbf{B}(\varrho r) \text{ and } \sigma \in (0, R), \qquad (5.1)$$

then there exists $C > 0$ and a representative in the L^2-equivalence class of f, which we still denote by f, such that

$$d(f(x), f(y)) \leq C|x - y|^\beta, \quad \forall x, y \in \mathbf{B}(\varrho r)$$

with C depending on K, r, R, β, ϱ and $\mathbf{B}(r)$.

Proof The lemma will follow from the Campanato lemma [5, Lemma 4.1], provided that each aspect of the proof that relied on the non-positive curvature of the target still holds if the target is CAT(1) and f has small image. The NPC hypothesis gave the existence and uniqueness of $Q_{x,\sigma}$ for each $x \in \mathbf{B}(\varrho r)$ and $\sigma \in (0, R)$. Lemma 4.2 above provides this for our setting. The NPC condition also provided the existence of Lipschitz maps L^2 close to f. For a CAT(1) space Y, we appeal to Lemma 5.1 above, since, by hypothesis, f has small image. All other aspects of the proof are related to properties of the domain, and thus carry through with no trouble. □

Recall the following proposition [5, Proposition 4.3], which converts the monotonicity information of (4.7) into a uniform estimate on the decay of the scale invariant energy for all x.

Proposition 5.3 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a Lipschitz Riemannian metric defined on $\mathbf{B}(r)$, (Y, d) a metric space and $f : (\mathbf{B}(r), g) \to Y$ a finite energy map. Fix $\varrho \in (0, 1)$ and suppose that for $x \in \mathbf{B}(\varrho r)$ there exist $\beta > 0$ and $\hat{C} \geq 1$ so that*

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\beta}} \leq \hat{C} \frac{E_x(\rho)}{\rho^{n-2+2\beta}}, \quad 0 < \sigma \leq \rho \leq r(x) \qquad (5.2)$$

where $r(x)$ is as defined in (4.8). Then there exist K and $R > 0$ depending only on the total energy of f, E^f, the ellipticity constant and Lipschitz bound of g, $\mathbf{B}(r)$ and ϱ so that

$$E_x(\sigma) \leq K^2 \sigma^{n-2+2\beta}, \quad \forall x \in \mathbf{B}(\varrho r), \sigma < R.$$

This immediately implies the Hölder regularity for a local model.

Theorem 5.4 *Let $\mathbf{B}$ be a local model, g a Lipschitz Riemannian metric defined on $\mathbf{B}(r)$, (Y, d) a CAT(1) space and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ an energy minimizing map where $0 < \tau < \frac{\pi}{4}$. For $\varrho \in (0, 1)$, there exist $C_H > 0$ and $\gamma > 0$ depending only on the Lipschitz bound and ellipticity constant of g, E^f, $\mathbf{B}(r)$ and ϱ such that*

$$d(f(x), f(y)) \leq C_H |x - y|^\gamma, \quad \forall x, y \in \mathbf{B}(\varrho r).$$

Proof The result follows immediately from (4.7), Proposition 5.3, the Poincaré inequality of [5, Theorem 2.7], and Lemma 5.2. □

By using [5, Proposition 2.1] we obtain Theorem 1.1.

6 Improved Hölder regularity

To extend the regularity from Hölder to Lipschitz requires a better result than Theorem 5.4 provides. The objective of this section is twofold. First, we prove that the order function $\mathrm{ord}^f(x)$ is well-defined (see Proposition 6.5 and Definition 6.9). Second, we use monotonicity to demonstrate that the Hölder regularity on a ball can be improved to have Hölder exponent equal to $\alpha \le \mathrm{ord}^f(x)$ for $x \in \mathbf{B}(r)$. The Lipschitz regularity will then immediately hold in any neighborhood with $\alpha \ge 1$.

In [5], the authors proved the stronger Hölder regularity in parallel with the weaker version. In the CAT(1) setting, however, we rely in a fundamental way on the weaker Hölder result. We use the weak Hölder result in (4.3) to improve the inequality from (4.5). This improvement allows us adapt the techniques of [5] to our setting. Following their ideas, we demonstrate that the order function is well-defined. We then demonstrate that $\frac{E(\sigma)}{\sigma^{n-2+\alpha}}$ is monotone, which immediately implies the improved regularity.

Throughout this section, unless explicitly stated otherwise, presume that g is a normalized Lipschitz metric on $\mathbf{B}(r)$.

6.1 The order function

The goal of this subsection is to prove that the order $\alpha := \lim_{\sigma \to 0^+} \frac{\sigma E(\sigma)}{I(\sigma)}$ exists. In the Euclidean setting, the existence of the limit follows from proving the differential inequality $\frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} \ge 0$, which implies that the function $\sigma \mapsto \frac{\sigma E(\sigma)}{I(\sigma)}$ is monotone. Under the current hypotheses, we cannot hope to prove a differential inequality of exactly the desired type. The inequality we determine includes additional terms. Nevertheless, we still show that the limit α exists.

We begin by recalling two essential inequalities derived in [5, (3.9),(3.17)] for energy minimizing maps from a local model into a metric space target. These calculations use only domain variations and the Lipschitz assumption on the domain metric and thus immediately extend to our setting.

Lemma 6.1 *Let $f : (\mathbf{B}(r), g) \to Y$ be an energy minimizing map, Y a metric space, and g a normalized Lipschitz metric. Then there exist constants $c_1, c_2 > 0$ and $\sigma_0 > 0$ small, all depending only on $\mathbf{B}(r)$ and the Lipschitz bounds of g, such that for all $0 < \sigma \le \sigma_0$ and $Q \in Y$,*

$$(1 + c\sigma)\frac{E'(\sigma)}{E(\sigma)} \ge \frac{n-2}{\sigma} + \frac{2}{E(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g - c_1,$$

and

$$\left| \frac{I'(\sigma)}{I(\sigma)} - \frac{n-1}{\sigma} - \frac{1}{I(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g \right| \le c_2. \tag{6.1}$$

Therefore, for $c_3 = c_1 + c_2$,

$$(1 + c\sigma)\frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3$$

$$\ge \frac{2}{E(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g - \frac{1}{I(\sigma)} \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g$$

$$= \frac{1}{E(\sigma)I(\sigma)} \left(2I(\sigma) \int_{\partial \mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g - E(\sigma) \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g \right). \tag{6.2}$$

We now determine a lower bound for the right hand side of (6.2), modifying the differential inequality to one more conducive to the proof of monotonicity.

Lemma 6.2 *Suppose that* **B** *is a local model,* (Y, d) *is a CAT(1) space, and* g *is a normalized Lipschitz metric. If* $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ *is an energy minimizing map with* $0 < \tau < \frac{\pi}{4}$, *then for* $\mathbf{B}(\sigma) \subset \mathbf{B}(r/2)$ *and any* $Q \in \overline{\mathcal{B}_{C_H \sigma^\gamma}(f(0))}$,

$$2\left(1 - c'\sigma^{2\gamma}\right)E(\sigma) \leq \int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f, Q)d\Sigma_g + I(\sigma) + k\sigma^2 E'(\sigma) \tag{6.3}$$

and

$$E(\sigma)\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f, Q)d\Sigma_g \leq 2I(\sigma)\int_{\partial\mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g + \left(c\sigma + cc'\sigma^{2\gamma}\right)I(\sigma)E'(\sigma) \tag{6.4}$$

where c, k *depend on* $\mathbf{B}(r)$, *the Lipschitz bound and the ellipticity constant of* g *and* c', γ *depend on* E^f, $\mathbf{B}(r)$, *and the Lipschitz bound and ellipticity constant of* g.

Proof First observe that by Theorem 5.4, $f(\partial\mathbf{B}(\sigma)) \subset \overline{\mathcal{B}_{C_H \sigma^\gamma}(f(0))}$. By (4.3) and the Lipschitz bound $|g^{ij}(x) - \delta^{ij}| \leq c\sigma$, for $|x| \leq \sigma$, we can improve the estimate in (4.5) to

$$\left(2 - C_H^2\sigma^{2\gamma}\right)E(\sigma) \leq \int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f, Q)\, d\Sigma_g + c\sigma \int_{\partial\mathbf{B}(\sigma)} \sum_i \left|\frac{\partial}{\partial x_i}d^2(f, Q)\right| d\Sigma_g \tag{6.5}$$

where C_H is the Hölder constant, and γ is the Hölder exponent.

Then, for $c' = C_H^2/2$, (6.3) follows by applying the following elementary inequality to the last term in (6.5)

$$2c\sigma d(f, Q) \sum_{i=1}^n \left|\frac{\partial}{\partial x_i}d(f, Q)\right| \leq d^2(f, Q) + c^2\sigma^2 \sum_{i=1}^n \left|\frac{\partial}{\partial x_i}d(f, Q)\right|^2$$

$$\leq d^2(f, Q) + c^2\sigma^2 \sum_{i=1}^n \left|\frac{\partial f}{\partial x_i}\right|^2.$$

To prove (6.4), first note that if $\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f, Q)d\Sigma_g \leq 0$, the result holds simply because the right hand side of the inequality is non-negative. So suppose that $\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f, Q)d\Sigma_g \geq 0$. Recall the estimates determined in (4.5):

$$\int_{\partial\mathbf{B}(\sigma)} \frac{\partial}{\partial r}d^2(f, Q)d\Sigma_g \leq 2\left(I(\sigma)\int_{\partial\mathbf{B}(\sigma)} \left|\frac{\partial f}{\partial r}\right|^2 d\Sigma_g\right)^{1/2} \leq 2cI(\sigma)^{1/2}E'(\sigma)^{1/2}, \tag{6.6}$$

$$\int_{\partial\mathbf{B}(\sigma)} \sum_i \left|\frac{\partial}{\partial x_i}d^2(f, Q)\right|d\Sigma_g \leq 2cI(\sigma)^{1/2}E'(\sigma)^{1/2}. \tag{6.7}$$

Note that in the above equations, c depends on the ellipticity constant of g. In what follows, c may increase from one line to the next, but its dependence will always be only on $\mathbf{B}(r)$, the Lipschitz bound and the ellipticity constant of g.

Using (6.5), (6.6), (6.7) we observe that

$$
2\big(1 - c'\sigma^{2\gamma}\big)E(\sigma) \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g
$$

$$
\leq \Big(\int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g + c\sigma \int_{\partial \mathbf{B}(\sigma)} \sum_i \Big| \frac{\partial}{\partial x_i} d^2(f, Q) \Big| d\Sigma_g \Big) \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g
$$

$$
\leq c(1 + \sigma) I(\sigma) E'(\sigma).
$$

(6.8)

Thus, for sufficiently small $\sigma > 0$,

$$
E(\sigma) \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g \leq c I(\sigma) E'(\sigma).
$$

Now, using the middle inequality in (6.6) and substituting the above inequality into (6.8) implies that

$$
E(\sigma) \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g
$$

$$
\leq 2I(\sigma) \int_{\partial \mathbf{B}(\sigma)} \Big| \frac{\partial f}{\partial r} \Big|^2 d\Sigma_g + c\sigma I(\sigma) E'(\sigma) + 2c'\sigma^{2\gamma} E(\sigma) \int_{\partial \mathbf{B}(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g
$$

$$
\leq 2I(\sigma) \int_{\partial \mathbf{B}(\sigma)} \Big| \frac{\partial f}{\partial r} \Big|^2 d\Sigma_g + c\sigma I(\sigma) E'(\sigma) + cc'\sigma^{2\gamma} I(\sigma) E'(\sigma).
$$

$\square$

Combining (6.2) and (6.4), we conclude that for sufficiently small σ,

$$
\big(1 + c\sigma + cc'\sigma^{2\gamma}\big) \frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0.
$$

Note that if $\gamma \geq \frac{1}{2}$, we may appeal directly to the work of [5] since the term $c\sigma$ dominates. Therefore, we presume that $\gamma < 1/2$. In what follows, for notational simplicity, we rescale the domain metric g so that $c \leq 1$, since c depends only on the domain metric. If we assume that $\sigma_0 = 1$ and let $C \geq 1 + 2c'$ then

$$
(1 + C\sigma^{2\gamma}) \frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0.
$$

(6.9)

For the analogous inequality in the NPC setting see [5, (3.20)].

We remark that due to the extra term $\sigma^{2\gamma}$, the original monotonicity in [13] no longer works. Following the ideas of [5], we introduce a modified energy. Let

$$
J(\sigma) = \max_{s \in (0, \sigma]} I(s),
$$

and

$$
A := \Big\{ \sigma : \frac{E'(\sigma)}{E(\sigma)} - \frac{J'(\sigma)}{J(\sigma)} + \frac{1}{\sigma} + c_3 \leq 0 \Big\}.
$$

Note that A is exactly the set on which a standard monotonicity formula fails. For $\sigma \in (0, 1)$, we define the modified energy

$$
F(\sigma) = E(\sigma) \exp\big(\varphi(\sigma)\big),
$$

(6.10)

where

$$\varphi(\sigma) = -\int_{A\cap(\sigma,1)} Cs^{2\gamma} \frac{E'(s)}{E(s)} ds$$

and C is as in (6.9). Then exactly as in the proof of [5, Lemma 3.7],

$$\frac{F'(\sigma)}{F(\sigma)} = \begin{cases} \frac{E'(\sigma)}{E(\sigma)} & \text{if } \sigma \notin A \\ (1 + C\sigma^{2\gamma})\frac{E'(\sigma)}{E(\sigma)} & \text{if } \sigma \in A. \end{cases} \tag{6.11}$$

By (6.9) and (6.11), we observe that

$$\sigma \mapsto e^{c_3\sigma} \frac{\sigma F(\sigma)}{I(\sigma)} \tag{6.12}$$

is monotone nondecreasing for any $Q \in \overline{\mathcal{B}_{C_H\sigma^\gamma}(f(0))}$. For $\sigma > 0$ sufficiently small, $C_H\sigma^\gamma < \tau < \frac{\pi}{4}$ and thus the projection map $\pi_\sigma : \mathcal{B}_\tau(P) \to \overline{\mathcal{B}_{C_H\sigma^\gamma}(f(0))}$ is distance decreasing. It follows that for every $\sigma > 0$ sufficiently small, $Q_\sigma \in \overline{\mathcal{B}_{C_H\sigma^\gamma}(f(0))}$. Thus applying (6.12) for $\sigma_1 < \sigma_2$ sufficiently small, and noting that by the definition of $I(\sigma, Q_\sigma)$, $I(\sigma_2, Q_{\sigma_1}) \geq I(\sigma_2, Q_{\sigma_2})$, we observe that

$$e^{c_3\sigma_1} \frac{\sigma_1 F(\sigma_1)}{I(\sigma_1, Q_{\sigma_1})} \leq e^{c_3\sigma_2} \frac{\sigma_2 F(\sigma_2)}{I(\sigma_2, Q_{\sigma_1})} \leq e^{c_3\sigma_2} \frac{\sigma_2 F(\sigma_2)}{I(\sigma_2, Q_{\sigma_2})}.$$

Therefore, $\sigma \mapsto e^{c_3\sigma} \frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)}$ is monotone and $\lim_{\sigma\to0^+} e^{c_3\sigma} \frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)}$ exists. To show that $\lim_{\sigma\to0^+} e^{c_3\sigma} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)}$ exists, it is therefore enough to consider

$$\lim_{\sigma\to0^+} \frac{E(\sigma)}{F(\sigma)} = \lim_{\sigma\to0^+} \exp\left(-\varphi(\sigma)\right).$$

Lemma 6.3 $\lim_{\sigma\to0^+} -\varphi(\sigma) = \lim_{\sigma\to0^+} \int_{A\cap(\sigma,1)} Cs^{2\gamma} \frac{E'(s)}{E(s)} ds < \infty.$

Proof The proof follows from straightforward modifications of the argument in [5, Lemma 3.8].

By the definition of A, for all $s \in A$

$$\frac{E'(s)}{E(s)} \leq \frac{J'(s)}{J(s)}. \tag{6.13}$$

So it suffices to show that

$$\lim_{\sigma\to0^+} \int_{A\cap(\sigma,1)} Cs^{2\gamma} \frac{J'(s)}{J(s)} ds < \infty.$$

Following [5, Proof of Lemma 3.8], there exists a sufficiently large constant C' depending on the domain such that for any $\epsilon > 0$ and $0 < \theta_1 < \theta_2 \leq 1$,

$$\left[1 - \left(\frac{1}{\epsilon} + \frac{C'}{\theta_1}\right)(\theta_2 - \theta_1)\right] J(\theta_2) - \epsilon M E(\theta_2) \leq J(\theta_1) \tag{6.14}$$

where

$$\frac{1}{E(\sigma)} \int_{\mathbf{B}(\sigma)} |\nabla f|^2 d\mu_g \leq M.$$

Let

$$\phi(\theta, n, j) = \frac{1}{2} - \left(2K\theta^{-jp(\theta,n)} + \frac{C'}{\theta}\right)(1 - \theta),$$

where

$$p(\theta, n) = C\frac{1 - \theta^{2\gamma n}}{1 - \theta^{2\gamma}}, \qquad K = Me^{c_3}\frac{F(1)}{J(1)}.$$

Note that $p(\theta, 0) = 0$ and

$$\lim_{\theta \to 1^-} \phi(\theta, n, j) = \frac{1}{2} \quad \text{uniformly in } j, n.$$

Therefore, there exists $\theta_0 < 1$ sufficiently close to 1, such that $\phi(\theta_0, n, j) > \frac{1}{4}$. We also choose j such that $\theta_0^j < \frac{1}{4}$. Then for all n, we have that

$$\theta_0^j < \phi(\theta_0, n, j) = \frac{1}{2} - \left(2K\theta_0^{-jp(\theta_0,n)} + \frac{C'}{\theta_0}\right)(1 - \theta_0). \tag{6.15}$$

Now, for the chosen θ_0 and j as above, we let

$$\epsilon = \frac{1}{2K}\theta_0^{jp(\theta_0,n)+n}.$$

Then by (6.14),

$$\left[1 - \left(2K\theta_0^{-jp(\theta_0,n)-n} + \frac{C'}{\theta_1}\right)(\theta_2 - \theta_1)\right]J(\theta_2) - \frac{E(\theta_2)J(1)}{2e^{c_3}F(1)}\theta_0^{jp(\theta_0,n)+n} \leq J(\theta_1). \tag{6.16}$$

$\square$

Claim 6.4 *For any n,*

$$\theta_0^j J(\theta_0^n) < J(\theta_0^{n+1}), \tag{6.17}$$

and

$$\int_{\theta_0^n}^{1} Cs^{2\gamma}\frac{J'(s)}{J(s)}ds \leq \log\theta_0^{-jp(\theta_0,n)} \leq C(\theta_0, \gamma, j). \tag{6.18}$$

Note that the proof of the lemma will follow once we prove the claim since

$$\lim_{\sigma \to 0^+}\int_{A\cap(\sigma,1)}Cs^{2\gamma}\frac{E'(s)}{E(s)}ds \leq \lim_{\sigma \to 0^+}\int_{A\cap(\sigma,1)}Cs^{2\gamma}\frac{J'(s)}{J(s)}ds \leq \lim_{n\to\infty}\int_{\theta_0^n}^{1}Cs^{2\gamma}\frac{J'(s)}{J(s)}ds$$

and by the claim, the right hand side is bounded independent of n.

$\square$

Proof of claim We proceed by induction on the powers of θ_0. First, take $n = 0$, $\theta_1 = \theta_0$, $\theta_2 = 1$, and notice that $F(1) = E(1)$ by the definition of $F(\sigma)$, and thus $\frac{E(1)J(1)}{e^{c_3}F(1)} = \frac{J(1)}{e^{c_3}} \leq J(1)$. Then by (6.15) and (6.16),

$$\theta_0^j J(1) < \phi(\theta_0, 0, j)J(1) = \left[\frac{1}{2} - \left(2K + \frac{C'}{\theta_0}\right)(1 - \theta_0)\right]J(1) \leq J(\theta_0).$$

Next, we assume $\theta_0^j J(\theta_0^k) < J(\theta_0^{k+1})$ for all $k = 0, 1, 2, \ldots, n - 1$.

By the definition of $F(\sigma)$ and (6.13),

$$
\begin{aligned}
\log \frac{E(\theta_0^n)}{F(\theta_0^n)} = -\varphi(\theta_0^n) &= \int_{A \cap (\theta_0^n, 1)} Cs^{2\gamma} \frac{E'(s)}{E(s)} ds \\
&\leq \int_{\theta_0^n}^1 Cs^{2\gamma} \frac{J'(s)}{J(s)} ds.
\end{aligned}
\tag{6.19}
$$

We estimate

$$
\begin{aligned}
\int_{\theta_0^n}^1 Cs^{2\gamma} \frac{J'(s)}{J(s)} ds &= \sum_{k=0}^{n-1} \int_{\theta_0^{k+1}}^{\theta_0^k} Cs^{2\gamma} \frac{d}{ds} \log J(s) ds \\
&\leq \sum_{k=0}^{n-1} C\theta_0^{2\gamma k} \log \frac{J(\theta_0^k)}{J(\theta_0^{k+1})} \\
&\leq \sum_{k=0}^{n-1} C\theta_0^{2\gamma k} \log \theta_0^{-j} \quad \text{(by the induction hypothesis)} \\
&= \log \theta_0^{-jC \frac{1-\theta_0^{2\gamma n}}{1-\theta_0^{2\gamma}}} = \log \theta_0^{-jp(\theta_0, n)}.
\end{aligned}
$$

So since

$$
\frac{\sigma F(\sigma)}{J(\sigma)} \leq e^{c_3\sigma} \frac{\sigma F(\sigma)}{J(\sigma)} \leq e^{c_3} \frac{F(1)}{J(1)}
$$

(6.19) and the integral estimate imply that

$$
\theta_0^{jp(\theta_0, n)} E(\theta_0^n) \leq F(\theta_0^n) \leq \frac{e^{c_3} F(1) J(\theta_0^n)}{J(1)\theta_0^n}
$$

That is

$$
\theta_0^{jp(\theta_0, n)+n} \frac{E(\theta_0^n)J(1)}{e^{c_3} F(1)} \leq J(\theta_0^n).
$$

We now take $\theta_1 = \theta_0^{n+1}, \theta_2 = \theta_0^n$ in (6.16), and together with (6.15) and the above inequality we conclude that

$$
\theta_0^j J(\theta_0^n) < \phi(\theta_0, n, j) J(\theta_0^n) = \left[\frac{1}{2} - \left(2K\theta_0^{-jp(\theta_0, n)} + \frac{C'}{\theta_0} \right)(1-\theta_0) \right] J(\theta_0^n) \leq J(\theta_0^{n+1}).
$$

This implies that (6.17) is true for all n. Therefore, we may make the substitution in the integral estimate to conclude that for all n

$$
\int_{\theta_0^n}^1 Cs^{2\gamma} \frac{J'(s)}{J(s)} ds \leq \log \theta_0^{-jC \frac{1-\theta_0^{2\gamma n}}{1-\theta_0^{2\gamma}}} = \log \theta_0^{-jp(\theta_0, n)}.
\tag{6.20}
$$

Since $p(\theta_0, n)$ is increasing in n and $\lim_{n\to\infty} p(\theta_0, n) = C \frac{1}{1-\theta_0^{2\gamma}} = C(\theta_0, \gamma)$, we prove the integral estimate. $\qquad \square$

Proposition 6.5 *Suppose that* **B** *is a local model,* (Y, d) *is a CAT(1) space, and g is a normalized Lipschitz metric. If* $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ *is an energy minimizing map*

where $0 < \tau < \frac{\pi}{4}$, then the order

$$\alpha := \lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} < \infty$$

is well defined.

Proof The monotonicity of $\sigma \mapsto \frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)}$ together with Lemma 6.3 implies that

$$\alpha := \lim_{\sigma \to 0+} \frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} = \lim_{\sigma \to 0+} \left(\frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)} \frac{E(\sigma)}{F(\sigma)} \right)$$

$$= \lim_{\sigma \to 0+} \frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)} \lim_{\sigma \mapsto 0+} \frac{E(\sigma)}{F(\sigma)} < \infty.$$

$\square$

Definition 6.6 The value α is the *order of f at 0* and denoted by $\alpha = \mathrm{ord}^f(0)$.

6.2 Improved monotonicity

Using the Hölder regularity of Theorem 5.4 and the definition of α given by Proposition 6.5, we improve the Hölder result to have exponent corresponding to the order function. Such a result allows us to immediately conclude Lipschitz regularity whenever $\alpha \geq 1$.

Lemma 6.7 *Let **B** be a local model, g a normalized Lipschitz metric defined on **B**(r), (Y, d) a CAT(1) space and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ an energy minimizing map where $0 < \tau < \frac{\pi}{4}$. Let $\alpha = \mathrm{ord}^f(0)$ and $\gamma > 0$ be the Hölder exponent of Theorem 5.4. There exist constants c_0, c_0' and σ_0 depending only on **B**(r), E^f, and the Lipschitz bound and the ellipticity constant of g so that if C is as in (6.9) and*

$$\tilde{E}(\sigma) := E(\sigma) \exp \left(c_0 \int_{A \cap (0,\sigma)} C s^{2\gamma} \frac{E'(s)}{E(s)} ds \right),$$

then

$$\sigma \mapsto e^{c_0\sigma + c_0'\sigma^{2\gamma}} \frac{\tilde{E}(\sigma)}{\sigma^{2\alpha+n-2}}$$

is non-decreasing for $\sigma \in (0, \sigma_0)$.

Proof Let

$$G(\sigma) = E(\sigma) \exp \left(\int_{A \cap (0,\sigma)} C s^{2\gamma} \frac{E'(s)}{E(s)} ds \right).$$

Since (6.13), (6.20) imply that $\exp \left(\int_{A \cap (0,1)} C s^{2\gamma} \frac{E'(s)}{E(s)} ds \right)$ is finite and by definition

$$G(\sigma) = F(\sigma) \exp \left(\int_{A \cap (0,1)} C s^{2\gamma} \frac{E'(s)}{E(s)} ds \right),$$

the monotonicity of (6.12) implies that

$$\sigma \mapsto e^{c_3\sigma} \frac{\sigma G(\sigma)}{I(\sigma)}$$

is non-decreasing. Since $\int_{A\cap(0,\sigma)} Cs^{2\gamma}\frac{E'(s)}{E(s)}ds \geq 0$, $E(\sigma) \leq G(\sigma)$. Therefore,

$$\alpha = \lim_{\sigma\to 0}\frac{\sigma E(\sigma)}{I(\sigma,Q_\sigma)} \leq \lim_{\sigma\to 0}\frac{\sigma G(\sigma)}{I(\sigma,Q_\sigma)} = \lim_{\sigma\to 0}e^{c_3\sigma}\frac{\sigma G(\sigma)}{I(\sigma,Q_\sigma)}. \tag{6.21}$$

Thus for σ sufficiently small,

$$\alpha \leq e^{c_3\sigma}\frac{\sigma G(\sigma)}{I(\sigma,Q_\sigma)} = e^{c_3\sigma}\frac{\sigma E(\sigma)}{I(\sigma,Q_\sigma)}\cdot\exp\left(\int_{A\cap(0,\sigma)}Cs^{2\gamma}\frac{E'(s)}{E(s)}ds\right) \tag{6.22}$$

and

$$\frac{\sigma E(\sigma)}{I(\sigma,Q_\sigma)} \leq e^{c_3\sigma}\frac{\sigma G(\sigma)}{I(\sigma,Q_\sigma)} \leq e^{c_3}\frac{G(1)}{I(1)} =: K. \tag{6.23}$$

We now use arguments from the proof of Lemma 6.3. By (6.17), for n such that $\theta_0^{n+1} \leq \sigma < \theta_0^n$,

$$\int_{A\cap(0,\sigma)}Cs^{2\gamma}\frac{E'(s)}{E(s)}ds \leq \int_0^\sigma Cs^{2\gamma}\frac{J'(s)}{J(s)}ds$$

$$\leq \sum_{k=n}^\infty \int_{\theta_0^{k+1}}^{\theta_0^k}Cs^{2\gamma}\frac{J'(s)}{J(s)}ds$$

$$\leq \left(\sum_{k=n}^\infty C\theta_0^{2\gamma k}\right)\log\theta_0^{-j}$$

$$= \left(\frac{C\theta_0^{2\gamma n}}{1-\theta_0^{2\gamma}}\right)\log\theta_0^{-j}$$

$$\leq c_4\theta_0^{2\gamma n}$$

$$\leq \frac{c_4}{\theta_0^{2\gamma}}\sigma^{2\gamma} =: c_5\sigma^{2\gamma}. \tag{6.24}$$

By (6.22) and (6.24)

$$\alpha \leq e^{c_3\sigma+c_5\sigma^{2\gamma}}\frac{\sigma E(\sigma)}{I(\sigma,Q_\sigma)} \leq e^{c_6\sigma^{2\gamma}}\frac{\sigma E(\sigma)}{I(\sigma,Q_\sigma)}. \tag{6.25}$$

By (6.1) and (6.3),

$$2\left(1-c'\sigma^{2\gamma}\right)E(\sigma) \leq \int_{\partial\mathbf{B}(\sigma)}\frac{\partial}{\partial r}d^2(f,f(0))d\Sigma_g + I(\sigma) + k\sigma^2 E'(\sigma)$$

$$\leq I'(\sigma) - \frac{n-1}{\sigma}I(\sigma) + (1+c_2)I(\sigma) + k\sigma^2 E'(\sigma).$$

Here k, c_2 depend on the Lipschitz bound of g. Therefore

$$\frac{\frac{2(1-c'\sigma^{2\gamma})\sigma E(\sigma)}{I(\sigma)} + n - 1 - O(\sigma)}{\sigma} \leq \frac{I'(\sigma)}{I(\sigma)} + k\sigma^2\frac{E'(\sigma)}{I(\sigma)}$$

$$\leq \frac{I'(\sigma)}{I(\sigma)} + kK\sigma\frac{E'(\sigma)}{E(\sigma)} \quad \text{by (6.23)}$$

$$\leq \frac{G'(\sigma)}{G(\sigma)} + \frac{1}{\sigma} + kK\sigma\frac{E'(\sigma)}{E(\sigma)} + c_3.$$

For the last inequality we use (6.9) and (6.11) to show that

$$\frac{G'(\sigma)}{G(\sigma)} = \frac{F'(\sigma)}{F(\sigma)} \geq \frac{I'(\sigma)}{I(\sigma)} - \frac{1}{\sigma} - c_3.$$

Note further that by applying (6.25) and absorbing the higher order terms of the exponential into $O(\sigma)$, there exists c_0' such that

$$\frac{G'(\sigma)}{G(\sigma)} + kK\sigma\frac{E'(\sigma)}{E(\sigma)} \geq \frac{2\alpha - c_0'\sigma^{2\gamma}}{\sigma} + \frac{n - 2 - O(\sigma)}{\sigma} - c_3.$$

From this point forward, we presume that $\sigma \in A$. Indeed, if $\sigma \notin A$ then the appropriate differential inequality is satisfied which immediately proves the monotonicity. By definition

$$\frac{\tilde{E}'(\sigma)}{\tilde{E}(\sigma)} \geq \frac{E'(\sigma)}{E(\sigma)}\left(1 + c_0C\sigma^{2\gamma}\right).$$

Choose c_0 sufficiently large so that $c_0C \geq kK + C$. Then, since $\sigma \in A$,

$$\frac{\tilde{E}'(\sigma)}{\tilde{E}(\sigma)} \geq \frac{E'(\sigma)}{E(\sigma)}\left(1 + (kK + C)\sigma^{2\gamma}\right) \geq \frac{G'(\sigma)}{G(\sigma)} + kK\sigma\frac{E'(\sigma)}{E(\sigma)}.$$

Then, we may increase c_0 if necessary to determine that

$$\frac{\tilde{E}'(\sigma)}{\tilde{E}(\sigma)} \geq \frac{n - 2 + 2\alpha}{\sigma} - c_0'\sigma^{2\gamma-1} - c_0.$$

It follows that

$$\frac{d}{d\sigma}\log\left(\frac{e^{c_0\sigma + c_0'\sigma^{2\gamma}}\tilde{E}(\sigma)}{\sigma^{n-2+2\alpha}}\right) \geq 0.$$

□

Corollary 6.8 *Let* **B** *be a dimension-n, codimension-v local model, g a normalized Lipschitz metric defined on* $\mathbf{B}(r)$, *and* (Y, d) *a CAT(1) space and* $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ *an energy minimizing map where* $0 < \tau < \frac{\pi}{4}$. *Let* $\alpha = \mathrm{ord}^f(0)$ *and* $\gamma > 0$ *be the Hölder exponent of Theorem* 5.4. *Then there exist constants* $c, k > 0$, $\sigma_0 < 1$ *depending on* $\mathbf{B}(r)$, E^f, *and the Lipschitz bound and the ellipticity constant of g so that*

$$\frac{\sigma E(\sigma)}{I(\sigma, Q_\sigma)} \leq ce^{k\rho}\frac{\rho E(\rho)}{I(\rho, Q_\rho)} \quad for\ 0 < \sigma \leq \rho \leq \sigma_0, \tag{6.26}$$

and

$$\frac{E(\sigma)}{\sigma^{n-2+2\alpha}} \leq e^{k\rho^{2\gamma}}\frac{E(\rho)}{\rho^{n-2+2\alpha}} \quad for\ 0 < \sigma \leq \rho \leq \sigma_0. \tag{6.27}$$

Proof For $F(\sigma)$ defined as in (6.10), (6.13) and (6.20) imply that

$$\frac{1}{c}E(\sigma) \leq F(\sigma) \leq E(\sigma)$$

for some $c > 1$ depending on the Hölder constant from Theorem 5.4. Using this uniform bound and the fact that $e^{c_3\sigma}\frac{\sigma F(\sigma)}{I(\sigma, Q_\sigma)}$ is monotone by (6.12) implies (6.26).

Lemma 6.7, the definition of $\tilde{E}$, and the fact that (6.24) gives the bound

$$E(\sigma) \leq \tilde{E}(\sigma) \leq e^{c_0c_5\sigma^{2\gamma}}E(\sigma).$$

together imply (6.27). □

As in the conclusion of Sect. 4, we now consider monotonicity for metrics g that are not necessarily normalized. Recall that if g is a Lipschitz metric and $h := L_x^* g$, where L_x is the map given by [5, Proposition 2.1], then h is normalized. Moreover, when f is minimizing with respect to the metric g, then $f \circ L_x$ is minimizing with respect to h.

Definition 6.9 For f minimizing with respect to a Lipschitz metric g, we define the *order of f at x* as

$$\alpha_x = \mathrm{ord}^f(x) := \mathrm{ord}^{f \circ L_x}(0).$$

Recalling that

$$E_x(\sigma) := \int_{B_x(\sigma)} |\nabla f|^2 d\mu_g,$$

and $B_x(\sigma)$ denotes the Euclidean ball about x of radius σ, we prove the monotonicity of Proposition 4.6 with exponent $n - 2 + 2\alpha$.

Proposition 6.10 *Let $\mathbf{B}$ be a dimension-n, codimension-ν local model, g a Lipschitz metric defined on $\mathbf{B}(r)$ with ellipticity constant $\lambda \in (0, 1]$, (Y, d) a CAT(1) space and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ an energy minimizing map where $0 < \tau < \frac{\pi}{4}$. Then there exists $C \geq 1$ depending on $\mathbf{B}(r)$, the Lipschitz bound and the ellipticity constant of g so that for every $x \in \mathbf{B}(r)$,*

$$\frac{E_x(\sigma)}{\sigma^{n-2+2\alpha_x}} \leq C \frac{E_x(\rho)}{\rho^{n-2+2\alpha_x}}, \quad 0 < \sigma < \rho \leq r(x) \tag{6.28}$$

where $r(x)$ is defined as in (4.8).

Proof Following the proof of Proposition 4.6, the result follows from [5, Proposition 2.1] and Corollary 6.8. □

Using (6.28) and the techniques of Sect. 5, we immediately determine Hölder regularity for f with exponent depending on the order function.

Theorem 6.11 *Let $\mathbf{B}$ be a local model, g a Lipschitz Riemannian metric defined on $\mathbf{B}(r)$, (Y, d) a CAT(1) space and $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ an energy minimizing map where $0 < \tau < \frac{\pi}{4}$. If $0 < \alpha \leq \alpha_x$ for all $x \in \mathbf{B}(\varrho r)$ where $\varrho \in (0, 1)$, then there exists C depending only on the Lipchitz bound and ellipticity constant of g, E^f, $\mathbf{B}(r)$ and ϱ such that*

$$d(f(x), f(y)) \leq C|x - y|^\alpha, \quad \forall x, y \in \mathbf{B}(\varrho r).$$

Proof The result follows immediately from (6.28), Proposition 5.3, the Poincaré inequality of [5, Theorem 2.7], and Lemma 5.2. □

7 Tangent map construction

Given a domain Ω, NPC spaces (Y_k, d_k), and maps $f_k : \Omega \to Y_k$, Korevaar and Schoen [16, Section 3] develop the notion of convergence of maps in the pullback sense. This allowed [5] to define a tangent map of $f : \mathbf{B}(r) \to Y$ when Y is NPC. They then related the homogeneity of a tangent map to the order of f and used this to get the Lipschitz regularity.

Rather than reconstruct the entire argument when Y is CAT(1), we will consider the tangent map of f that is determined by the tangent map construction in [5] for the lifted map $\overline{f} : \mathbf{B}(r) \to \mathcal{C}Y$. Since $\mathcal{C}Y$ is NPC, we do not need to reconstruct the theory. Instead, we use the minimizing property of f to prove that the proposed tangent map exists.

7.1 Limit maps in the pullback sense

We first recall the construction in [16, Section 3] and its extension to local models in [5, Section 5].

Let $f_k : \mathbf{B}(r) \to Y_k$ where each (Y_k, d_k) is an NPC space. Since each f_k maps to a different metric space, convergence cannot be understood in a pointwise sense without further work. If one considers the closed convex hull of each set $f_k(\mathbf{B}(r))$ and corresponding pseudodistances $d_{k,\infty}$, convergence can be well understood by considering convergence of the pseudodistances. The construction proceeds as follows.

Let $f : \mathbf{B}(r) \to Y$ and denote $\Omega_0 = \mathbf{B}(r)$, $f_0 = f$, and let $d_0 : \Omega_0 \times \Omega_0 \to \mathbb{R}^+ \cup \{0\}$ be the pseudodistance function $d_0(x, y) := d(f_0(x), f_0(y))$. Inductively define $\Omega_{i+1} = \Omega_i \times \Omega_i \times [0, 1]$ and identify $\Omega_i \subset \Omega_{i+1}$ via the inclusion $x \mapsto (x, x, 0)$. Define $f_{i+1} : \Omega_{i+1} \to Y$ by

$$f_{i+1}(x, y, t) = (1 - t)f_i(x) + tf_i(y).$$

Let

$$d_{i+1}(x, y) := d(f_{i+1}(x), f_{i+1}(y)).$$

Then

$$\begin{aligned}
d_{i+1}((x, x, 0), (y, y, 0)) &= d_i(x, y), \\
d_{i+1}((x, y, s), (x, y, t)) &= |s - t|d_i(x, y), \\
d_{i+1}^2(z, (x, y, s)) &\leq (1 - s)d_{i+1}^2(z, (x, x, 0)) + sd_{i+1}^2(z, (y, y, 0)) \\
&\quad - s(1 - s)d_{i+1}^2((x, x, 0), (y, y, 0)).
\end{aligned}$$

Set $\Omega_\infty = \cup_i \Omega_i$ and define $f_\infty : \Omega_\infty \to (Y, d)$ such that $f_\infty|_{\Omega_i} = f_i$. Define a pseudodistance function $d_\infty(x, y) := d(f_\infty(x), f_\infty(y))$. Define the metric space (Y_*, d_*) as the completion of the quotient metric space constructed from $(\Omega_\infty, d_\infty)$, where the quotient space is defined via the equivalence relation of zero pseudodistance. Then by construction, in particular the properties of d_i, (Y_*, d_*) is an NPC space. Moreover, (Y_*, d_*) is isometric to $\mathrm{Cvx}(f(\mathbf{B}(r)))$, the closed convex hull of $f(\mathbf{B}(r))$, with isometry given by the unique extension of f_∞ to Y_*.

Definition 7.1 Let $v_k : \mathbf{B}(r) \to (Y_k, d_k)$ be a sequence of maps to NPC spaces. We say v_k *converges to v_* in the pullback sense* if the corresponding pullback pseudodistances $d_{k,\infty}$ converge pointwise to a limit pseudodistance d_∞ on $\Omega_\infty \times \Omega_\infty$, and $v_* = \pi \circ \iota$ where $\iota : \mathbf{B}(r) \to \Omega_\infty$ is the inclusion map and π is the natural projection map of Ω_∞ onto the metric completion (Y_*, d_*) of the quotient space constructed from $(\Omega_\infty, d_\infty)$.

Given v_* as above, we can replace f in the outlined construction by v_*. Then $d_{*,i}$ denotes the corresponding pullback pseudodistance function of $v_{*,i}$ and $d_{*,\infty}$ denotes the corresponding pullback pseudodistance function of $v_{*,\infty}$. In this case, $(d_*)_* = d_*$.

Definition 7.2 Suppose v_k converge to v_* in the pullback sense. Let $d_{k,i}$ (resp. $d_{\infty,i}$) be the corresponding pullback pseudodistance function to $v_{k,i} : \Omega_i \to (Y_k, d_k)$ (resp. $v_{*,i} : \Omega_i \to (Y_*, d_*)$). We say that the convergence is *locally uniform* if the convergence of $d_{k,i}$ to the limit $d_{*,i}$ is uniform on each compact subset of $\Omega_i \times \Omega_i$. In this case, we also say $v_k \to v_*$ *locally uniformly in the pullback sense*.

Proposition 7.3 [16, Proposition 3.7], [5, Proposition 5.1] *Let $v_k : \mathbf{B}(r) \to (Y_k, d_k)$ be a sequence of maps to NPC spaces Y_k for which there is uniform modulus of continuity control. That is, assume that for each $x \in \mathbf{B}(r)$ and $R > 0$ there exists a positive function $\omega(x, R)$ which is monotone in R satisfying*

$$\lim_{R \to 0} \omega(x, R) = 0,$$

and so that for each $k \in \mathbb{Z}$

$$\max_{y \in B(x,R)} d_k(v_k(x), v_k(y)) \le \omega(x, R).$$

Then there is a subsequence of v_k which converges locally uniformly in the pullback sense to a limit map $v_* : \mathbf{B}(r) \to (Y_*, d_*)$. Moreover, v_* satisfies the same modulus of continuity estimates.

7.2 The tangent map construction

Let $\mathbf{B}$ be a dimension-n, codimension-ν local model and g a normalized Lipschitz metric on $\mathbf{B}(1)$. For $r \in (0, 1)$ and a map $f : \mathbf{B}(r) \to Y$ we will consider the λ-*blow up map* $f_\lambda : \mathbf{B}(r/\lambda) \to (Y, d_\lambda)$ and the *lifted λ-blow up map* $\overline{f}_\lambda : \mathbf{B}(r/\lambda) \to (CY, D_\lambda)$ where

$$g_\lambda(x) := g(\lambda x)$$
$$\mu_\lambda^d := (\lambda^{1-n} \, {}^d I(\lambda))^{1/2}$$
$$\mu_\lambda^D := (\lambda^{1-n} \, {}^D I(\lambda))^{1/2}$$
$$d_\lambda(P, Q) := (\mu_\lambda^d)^{-1} d(P, Q)$$
$$D_\lambda(P, Q) := (\mu_\lambda^D)^{-1} D(P, Q)$$
$$f_\lambda(x) := f(\lambda x) \in Y$$
$$\overline{f}_\lambda(x) := [f(\lambda x), 1] \in Y \times \{1\} \subset CY.$$

Above we have denoted

$$^d I(\lambda) := \inf_{Q \in Y} \int_{\partial \mathbf{B}(\lambda)} d^2(f, Q) d\Sigma_g$$
$$^D I(\lambda) := \inf_{Q \in CY} \int_{\partial \mathbf{B}(\lambda)} D^2([f, 1], Q) d\Sigma_g.$$

Definition 7.4 If there exists an NPC space (Y_*, d_*) and a sequence $\lambda_k \to 0$ such that f_{λ_k} converges locally uniformly in the pullback sense to $\overline{f}_* : \mathbf{B} \to (Y_*, d_*)$ then $\overline{f}_*$ is called a *tangent map of f*.

Proposition 7.5 *For $\mathbf{B}$ a dimension-n, codimension-ν local model, g a normalized Lipschitz metric defined on $\mathbf{B}(r)$, and (Y, d) a CAT(1) space, let $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ be an energy minimizing map where $0 < \tau < \frac{\pi}{4}$. Then f has a tangent map $\overline{f}_* : (\mathbf{B}, g) \to (Y_*, d_*)$. Moreover, $\overline{f}_* : \mathbf{B}(1) \to (Y_*, d_*)$ is a non-constant, energy minimizing map.*

Proof We first determine uniform modulus of continuity control on the maps $\overline{f}_\lambda$. Following [5, Lemma 6.1], for the maps $f_\lambda : \mathbf{B}(r) \to (Y, d_\lambda)$, for sufficiently small $\lambda > 0$

$$^{d_\lambda} E_{g_\lambda}^{f_\lambda}[\mathbf{B}(1)] = \int_{\mathbf{B}(1)} |\nabla f_\lambda|^2 d\mu_{g_\lambda} \le 2\alpha$$

where α is the order of f at zero. Given the uniform Lipschitz bounds on the metrics g_λ, we appeal to Theorem 5.4 and note that

$$d_\lambda(f_\lambda(x), f_\lambda(y)) \leq C|x - y|^\gamma \text{ for all } x, y \in \mathbf{B}(r), \tag{7.1}$$

where C, γ are independent of λ. This immediately implies uniform modulus of continuity control on the maps f_λ but not on their lifted maps $\overline{f}_\lambda$. To determine the necessary control for the lifted maps, we will consider the relation between d_λ and D_λ as $\lambda \to 0$.

Since f is energy minimizing into Y, Theorem 5.4 implies that $f(\partial \mathbf{B}(\lambda)) \subset \mathcal{B}_{C\lambda^\gamma}(f(0)) \subset Y$ where C depends only on the Lipschitz bound and ellipticity constant of g, E^f, and $\mathbf{B}(r)$. By (3.1), given $Q \in \mathcal{B}_{C\lambda^\gamma}(f(0))$, for all $x \in \partial \mathbf{B}(\lambda)$,

$$D^2(\overline{f}(x), [Q, 1]) \leq d^2(f(x), Q). \tag{7.2}$$

It follows that for Q_λ^d such that $^d I(\lambda, Q_\lambda^d) = \inf {}^d I(\lambda, Q)$,

$$^D I(\lambda) \leq {}^D I(\lambda, [Q_\lambda^d, 1]) \leq {}^d I(\lambda).$$

Let $Q_\lambda^D \in \mathcal{C}Y$ such that $^D I(\lambda, Q_\lambda^D) = \inf {}^D I(\lambda, Q)$. Then $Q_\lambda^D \in \mathcal{B}_{C\lambda^\gamma}([f(0), 1]) \subset \mathcal{C}Y$ and by (3.3),

$$^d I(\lambda) \leq {}^d I\left(\lambda, \pi_1(Q_\lambda^D)\right) \leq (1 + C\lambda^{2\gamma}) {}^D I(\lambda),$$

where $\pi_1 : \mathcal{C}Y \to Y$ is the projection map onto the first component of $Y \times [0, \infty)$. Therefore, for $\mu_\lambda^d, \mu_\lambda^D$ the rescalings of d, D respectively, and $\lambda > 0$ sufficiently small,

$$\frac{1}{2} \leq (1 + C\lambda^{2\gamma})^{-1/2} \leq \frac{\mu_\lambda^D}{\mu_\lambda^d} \leq 1. \tag{7.3}$$

It follows by (3.1), (7.2), and (7.3) that for sufficiently small $\lambda > 0$ and all $x, y \in \mathbf{B}(r)$,

$$\begin{aligned} D_\lambda(\overline{f}_\lambda(x), \overline{f}_\lambda(y)) &= (\mu_\lambda^D)^{-1} D([f(\lambda x), 1], [f(\lambda y), 1]) \\ &\leq 2(\mu_\lambda^d)^{-1} D([f(\lambda x), 1], [f(\lambda y), 1]) \\ &\leq 2(\mu_\lambda^d)^{-1} d(f(\lambda x), f(\lambda y)) \\ &= 2d_\lambda(f_\lambda(x), f_\lambda(y)). \end{aligned}$$

By (7.1), the maps $\overline{f}_\lambda$ into $(\mathcal{C}Y, D_\lambda)$ possess uniform modulus of continuity control. Therefore, by Proposition 7.3, there exists a sequence $\lambda_k \to 0$ and an NPC space $((\mathcal{C}Y)_*, D_*)$ such that $\overline{f}_{\lambda_k}$ converge locally uniformly in the pullback sense to a limit map $\overline{f}_* : \mathbf{B}(1) \to ((\mathcal{C}Y)_*, D_*)$.

Claim 7.6 $\overline{f}_*$ *is a tangent map of* f.

Proof We need to show that $d_{\lambda,n} \to d_{*,n}$ uniformly on $\Omega_n \times \Omega_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since the uniform convergence for $D_{\lambda,n} \to d_{*,n}$ is already established, and since $\mu_\lambda^d/\mu_\lambda^D \to 1$ uniformly by (7.3), it is enough to show that for all $n \in \mathbb{N} \cup \{0\}$,

$$\frac{d(f_{\lambda,n}(x), f_{\lambda,n}(y))}{D(\overline{f}_{\lambda,n}(x), \overline{f}_{\lambda,n}(y))} \to 1 \tag{7.4}$$

uniformly for all $x \neq y \in \Omega_n$. Proceeding by induction requires that we also demonstrate that

$$D([f_{\lambda,n}(x), 1], \overline{f}_{\lambda,n}(x)) \to 0 \tag{7.5}$$

uniformly for $x \in \Omega_n$.

Observe that by (3.2),

$$\frac{d(f_\lambda(x), f_\lambda(y))}{D(\overline{f}_\lambda(x), \overline{f}_\lambda(y))} \to 1$$

uniformly for all $x \neq y \in \mathbf{B}(1) = \Omega_0$ so (7.4) holds easily for $n = 0$. Moreover, (7.5) is trivial for $n = 0$ since $\overline{f}_{\lambda,0}(x) = [f_{\lambda,0}(x), 1] \in \mathcal{C}Y$.

Now suppose that

$$\frac{d(f_{\lambda,i-1}(x), f_{\lambda,i-1}(y))}{D(\overline{f}_{\lambda,i-1}(x), \overline{f}_{\lambda,i-1}(y))} \to 1$$

uniformly for $x \neq y \in \Omega_{i-1}$ and that

$$D([f_{\lambda,i-1}(x), 1], \overline{f}_{\lambda,i-1}(x)) \to 0 \tag{7.6}$$

uniformly for $x \in \Omega_{i-1}$. We claim that together these imply that (7.4) and (7.5) hold for $n = i$ and $\mathbf{x} \neq \mathbf{y} \in \Omega_i$.

Consider $\mathbf{x}, \mathbf{y} \in \Omega_i$ with $\mathbf{x} = (x_1, x_2, s)$ and $\mathbf{x} \neq \mathbf{y}$. Since, by Theorem 5.4, $f_{\lambda,i}(\mathbf{B}(1)) \subset \mathcal{B}_{C\lambda^\gamma}(f(0))$, (3.2) implies that

$$\frac{d(f_{\lambda,i}(\mathbf{x}), f_{\lambda,i}(\mathbf{y}))}{D([f_{\lambda,i}(\mathbf{x}), 1], [f_{\lambda,i}(\mathbf{y}), 1])} \to 1 \text{ uniformly for } \mathbf{x} \neq \mathbf{y} \in \Omega_i.$$

Thus, it is enough to show that

$$\frac{D([f_{\lambda,i}(\mathbf{x}), 1], [f_{\lambda,i}(\mathbf{y}), 1])}{D(\overline{f}_{\lambda,i}(\mathbf{x}), \overline{f}_{\lambda,i}(\mathbf{y}))} \to 1 \text{ uniformly for } \mathbf{x} \neq \mathbf{y} \in \Omega_i.$$

Note that if $x_1 = x_2$ then $f_{\lambda,i}(\mathbf{x}) = f_{\lambda,i-1}(x_1)$ and $\overline{f}_{\lambda,i}(\mathbf{x}) = \overline{f}_{\lambda,i-1}(x_1)$. Thus $D([f_{\lambda,i}(\mathbf{x}), 1], \overline{f}_{\lambda,i}(\mathbf{x})) \to 0$ uniformly by (7.6). Now suppose that $x_1 \neq x_2$. By hypothesis, with $\gamma_\lambda \subset Y$ the geodesic connecting $f_{\lambda,i-1}(x_1)$ to $f_{\lambda,i-1}(x_2)$ and $\overline{\gamma}_\lambda \subset \mathcal{C}Y$ the geodesic connecting $\overline{f}_{\lambda,i-1}(x_1)$ to $\overline{f}_{\lambda,i-1}(x_2)$,

$$\frac{\ell(\gamma_\lambda)}{\ell(\overline{\gamma}_\lambda)} = \frac{d(f_{\lambda,i-1}(x_1), f_{\lambda,i-1}(x_2))}{D(\overline{f}_{\lambda,i-1}(x_1), \overline{f}_{\lambda,i-1}(x_2))} \to 1 \tag{7.7}$$

uniformly. For $j = 1, 2$, let $\overline{\gamma}_\lambda^j \subset \mathcal{C}Y$ be the geodesic connecting $\overline{f}_{\lambda,i-1}(x_j)$ to $[f_{\lambda,i}(\mathbf{x}), 1]$. Then by the triangle inequality,

$$\ell(\overline{\gamma}_\lambda) \leq \ell(\overline{\gamma}_\lambda^1) + \ell(\overline{\gamma}_\lambda^2) \leq \ell(\gamma_\lambda) + \sum_{j=1,2} D(\overline{f}_{\lambda,i-1}(x_j), [f_{\lambda,i-1}(x_j), 1]).$$

Thus by (7.6) and (7.7),

$$\frac{\ell(\overline{\gamma}_\lambda^1) + \ell(\overline{\gamma}_\lambda^2)}{\ell(\overline{\gamma}_\lambda)} \to 1 \text{ uniformly.}$$

We consider the geodesic triangle in $\mathcal{C}Y$ with endpoints $\overline{f}_{\lambda,i-1}(x_1), \overline{f}_{\lambda,i-1}(x_2), [f_{\lambda,i}(\mathbf{x}), 1]$. Using a comparison triangle in $\mathbb{R}^2$, the side length relation implies that

$$D([f_{\lambda,i}(\mathbf{x}), 1], \overline{f}_{\lambda,i}(\mathbf{x})) \to 0 \text{ uniformly for all } \mathbf{x} \in \Omega_i.$$

Therefore (7.5) holds for $n = i$. By the triangle inequality,

$$D([f_{\lambda,1}(\mathbf{x}), 1], [f_{\lambda,1}(\mathbf{y}), 1]) - D(\overline{f}_{\lambda,1}(\mathbf{x}), \overline{f}_{\lambda,1}(\mathbf{y})) \to 0 \text{ uniformly} \qquad (7.8)$$

and thus (7.4) holds for $n = i$ and $\mathbf{x} \neq \mathbf{y}$.

Therefore f_{λ_k} converges uniformly locally in a pullback sense to $\overline{f}_*$ and it is reasonable to consider the target using the notation (Y_*, d_*). □

Finally, we prove that $\overline{f}_*$ is minimizing. Let $v_\lambda =^{Dir} \overline{f}_\lambda : \mathbf{B}(1) \to (\mathcal{C}Y, D_\lambda)$ denote the Dirichlet solution for $\overline{f}_\lambda$. As before, we note that $f_\lambda(\mathbf{B}(1)) \subset \mathcal{B}_{C\lambda^\gamma}(f(0))$, and so by (3.1), it follows that $\overline{f}_\lambda(\mathbf{B}(1)) \subset \mathcal{B}^{\mathcal{C}Y}_{C\lambda^\gamma}([f(0), 1])$. Now, since $\mathcal{C}Y$ is an NPC space, projection onto convex domains decreases energy. Therefore, since v_λ is an energy minimizer and $v_\lambda|_{\partial\mathbf{B}(1)} = \overline{f}_\lambda|_{\partial\mathbf{B}(1)} \subset \mathcal{B}^{\mathcal{C}Y}_{C\lambda^\gamma}([f(0), 1])$, it follows that $v_\lambda(\mathbf{B}(1)) \subset \mathcal{B}^{\mathcal{C}Y}_{C\lambda^\gamma}([f(0), 1])$. Thus, $\pi_2(v_\lambda) \geq 1 - C\lambda^\gamma$, where $\pi_2 : \mathcal{C}Y \to [0, \infty)$ is the projection onto the second component of the cone over Y. For λ sufficiently small, we apply (3.5) and observe that

$$D^2(v_\lambda(x), v_\lambda(y)) \geq (1 - C\lambda^\gamma)^2 D^2(\Pi(v_\lambda(x)), \Pi(v_\lambda(y))),$$

and thus $^D E^{\Pi(v_\lambda)}[\mathbf{B}(1)] \leq (1 - C\lambda^\gamma)^{-2} {}^D E^{v_\lambda}[\mathbf{B}(1)]$. By (3.4), if $\pi_1 : \mathcal{C}Y \to Y$ denotes the projection onto the first component of the cone over Y, then $^d E^{\pi_1(v_\lambda)}[\mathbf{B}(1)] = {}^D E^{\Pi(v_\lambda)}[\mathbf{B}(1)]$, and thus

$$^d E^{\pi_1(v_\lambda)}[\mathbf{B}(1)] \leq (1 - C\lambda^\gamma)^{-2} {}^D E^{v_\lambda}[\mathbf{B}(1)]. \qquad (7.9)$$

Again using (3.4), and noting that f_λ is energy minimizing with respect to d_λ,

$$^D E^{\overline{f}_\lambda}[\mathbf{B}(1)] = {}^d E^{f_\lambda}[\mathbf{B}(1)]$$
$$= (\mu_\lambda^d)^2 {}^{d_\lambda} E^{f_\lambda}[\mathbf{B}(1)] \leq (\mu_\lambda^d)^2 {}^{d_\lambda} E^{\pi_1(v_\lambda)}[\mathbf{B}(1)] = {}^d E^{\pi_1(v_\lambda)}[\mathbf{B}(1)]. \qquad (7.10)$$

Combining (7.9) and (7.10) we observe that

$$^D E^{\overline{f}_\lambda}[\mathbf{B}(1)] \leq (1 - C\lambda^\gamma)^{-2} {}^D E^{v_\lambda}[\mathbf{B}(1)]$$

and therefore

$$^{D_\lambda} E^{\overline{f}_\lambda}[\mathbf{B}(1)] = (\mu_\lambda^D)^{-2} {}^D E^{\overline{f}_\lambda}[\mathbf{B}(1)] \leq (\mu_\lambda^D)^{-2}(1 - C\lambda^\gamma)^{-2} {}^D E^{v_\lambda}[\mathbf{B}(1)]$$
$$= (1 - C\lambda^\gamma)^{-2} {}^{D_\lambda} E^{v_\lambda}[\mathbf{B}(1)].$$

Finally, since v_λ is energy minimizing with respect to D_λ, we have that

$$^{D_\lambda} E^{v_\lambda}[\mathbf{B}(1)] \leq {}^{D_\lambda} E^{\overline{f}_\lambda}[\mathbf{B}(1)] \leq (1 - C\lambda^\gamma)^{-2} {}^{D_\lambda} E^{v_\lambda}[\mathbf{B}(1)],$$

and so it follows from [16, Theorem 3.11] that $\overline{f}_*$ is minimizing. The non-constancy of $\overline{f}_*$ follows exactly as in the proof of [13, Proposition 3.3]. □

8 Higher regularity results

8.1 Lipschitz regularity

The Lipschitz regularity of f at points in $X - X^{(n-2)}$ will follow from regularity results for minimizing maps into an NPC space, once we show that the order of f is bounded below by the order of its tangent map $\overline{f}_*$.

Lemma 8.1 *Let* **B** *be a dimension-n, codimension-v local model, g a normalized Lipschitz metric defined on* **B**(r) *and* (Y, d) *a CAT(1) space. If* $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ *is an energy minimizing map with* $0 < \tau < \frac{\pi}{4}$, *let* $\overline{f}_* : \mathbf{B} \to (Y_*, d_*)$ *denote a tangent map of f, constructed as in Proposition* 7.5. *Then*

$$\mathrm{ord}^{\overline{f}_*}(0) \leq \mathrm{ord}^f(0), \quad i.e., \ \alpha_* \leq \alpha.$$

Proof Note that $\overline{f}_*$ is a minimizing map into an NPC space and thus by [5, Corollary 3.1],

$$\alpha_* := \lim_{\sigma \to 0} \frac{\sigma \, {}^{d_*}E^{\overline{f}_*}(\sigma)}{{}^{d_*}I^{\overline{f}_*}(\sigma)} < \infty.$$

Let λ_k denote the sequence defining the lifted tangent map $\overline{f}$. Define $f_k := f_{\lambda_k}$, $\mu_k := \mu_{\lambda_k}^d$, $d_k := d_{\lambda_k}$, and $g_k := g_{\lambda_k}$. Then,

$$\lim_{k \to \infty} \frac{\sigma \, {}^{d_k}E_{g_k}^{f_k}(\sigma)}{{}^{d_k}I_{g_k}^{f_k}(\sigma)} = \lim_{k \to \infty} \frac{\sigma \mu_k^{-2} \lambda_k^{2-n} \, {}^{d}E_g^f(\lambda_k \sigma)}{\mu_k^{-2} \lambda_k^{1-n} \, {}^{d}I_g^f(\lambda_k \sigma)}$$

$$= \lim_{k \to \infty} \frac{\lambda_k \sigma \, {}^{d}E_g^f(\lambda_k \sigma)}{{}^{d}I_g^f(\lambda_k \sigma)}$$

$$= \alpha.$$

Therefore, it is enough to show that

$$\frac{\sigma \, {}^{d_*}E^{\overline{f}_*}(\sigma)}{{}^{d_*}I^{\overline{f}_*}(\sigma)} \leq \lim_{k \to \infty} \frac{\sigma \, {}^{d_k}E_{g_k}^{f_k}(\sigma)}{{}^{d_k}I_{g_k}^{f_k}(\sigma)}.$$

By the pointwise convergence of $(\mu_{\lambda_k}^D)^{-1}D := D_k \to D_*$ locally, uniformly on compact sets,

$$\!{}^{D_k}I_{g_k}^{\overline{f}_k}(\sigma) \to \!{}^{d_*}I^{\overline{f}_*}(\sigma).$$

By a change of variables and properties of the λ_k-blow up maps,

$$\!{}^{d_k}I_{g_k}^{f_k}(\sigma) = \left(\mu_k^d\right)^{-2} \lambda_k^{1-n} ({}^{d}I)_g^f(\lambda_k \sigma),$$

$$\!{}^{D_k}I_{g_k}^{\overline{f}_k}(\sigma) = \left(\mu_k^D\right)^{-2} \lambda_k^{1-n} ({}^{D}I)_g^{\overline{f}}(\lambda_k \sigma).$$

The proof of Proposition 7.5 immediately implies that

$$\!{}^{d_k}I_{g_k}^{f_k}(\sigma) - \!{}^{D_k}I_{g_k}^{\overline{f}_k}(\sigma) \to 0$$

as $k \to \infty$ and thus

$$\!{}^{d_k}I_{g_k}^{f_k}(\sigma) \to \!{}^{d_*}I^{\overline{f}_*}(\sigma) \text{ as } k \to \infty.$$

By [16, Lemma 8.1, Lemma 8.7], $\liminf_{k \to \infty} \!{}^{D_k}E_{g_k}^{\overline{f}_k}(\sigma) \geq \!{}^{d_*}E^{\overline{f}_*}(\sigma)$. By (3.2), ${}^{d}E_{g_k}^{f_k}(\sigma) = {}^{D}E_{g_k}^{\overline{f}_k}(\sigma)$ for all k. Moreover, since $\mu_k^d - \mu_k^D \to 0$ as $k \to \infty$,

$$\!{}^{d_*}E^{\overline{f}_*}(\sigma) \leq \liminf_{k \to \infty} \!{}^{d_k}E_{g_k}^{f_k}(\sigma).$$

This implies the result. □

Proposition 8.2 *Let* **B** *be a dimension-n, codimension-ν local model with $\nu \in \{0, 1\}$, g a normalized Lipschitz metric defined on* **B**(r), (Y, d) *a CAT(1) space and $f : ($**B**$(r), g) \to \mathcal{B}_\tau(P) \subset Y$ an energy minimizing map with $0 < \tau < \frac{\pi}{4}$. Then f is Lipschitz continuous in* **B**(ϱr) *with Lipschitz constant depending on $\varrho \in (0, 1)$, $($**B**$(r), g)$, and the total energy of the map f.*

Proof For each $x \in$ **B**(ϱr), consider the normalized map $f_x := f \circ L_x$, minimizing with respect to the normalized metric $h = L_x^* g$. Here the map L_x is as in [5, Proposition 2.1] but it plays a slightly different role. The metric on **B** is normalized at the origin of **B**. To determine a lower bound for the order of f at x, we want to consider its tangent map about x, which requires that the metric be normalized about x. L_x does this for us, and thus we may consider the tangent map of f_x at 0 (or f at x).

By Proposition 7.5, the tangent map $(f_x)_* :$ **B**$(1) \to Y_*$ is minimizing into the NPC space Y_*. Therefore, by [5, Lemma 8.1, Lemma 8.7], $\alpha_* \geq 1$. Lemma 8.1 implies that ord$^{f_x}(0) \geq 1$ and thus ord$^f(x) \geq 1$. The result now follows from Theorem 6.11. □

The Lipschitz regularity, Theorem 1.3 item (1), follows immediately from [5, Proposition 2.1].

8.2 Regularity at a higher codimension singular point

Given a Riemannian complex X and an NPC space T, we define the center of mass of a map $u \in L^2(X, T)$ to be the unique point $\overline{u} \in T$ (with existence and uniqueness given by [15, Proposition 2.5.4]) such that

$$\int_X d_T^2(u, \overline{u}) \, d\mu_g := \inf_{P \in T} \int_X d_T^2(u, P) \, d\mu_g.$$

We define the first eigenvalue of X with values in T by the Rayleigh quotient

$$\lambda_1(X, T) := \inf_{u \in W^{1,2}(X,T)} \frac{\int_X |\nabla u|^2 \, d\mu_g}{\int_X d^2(u, \overline{u}) \, d\mu_g}.$$

In application, T will be the tangent cone of the CAT(1) space Y at a point $Q \in Y$. While Y is not NPC, $T_Q Y$ is always NPC by construction.

To prove item (2) in Theorem 1.3, we first note that all of the results of [5, section 6] can be immediately applied. In particular, set $f_k := f_{\lambda_k}$ and define $h_k : ($**B**$(1), \delta) \to (\mathcal{C}Y, D_{\lambda_k})$ to be the minimizer into $\mathcal{C}Y$ with $h_k|_{\partial \mathbf{B}(1)} = \overline{f}_k|_{\partial \mathbf{B}(1)}$. Then h_k converges uniformly locally in the pull-back sense to $\overline{f}_*$ and in fact the pseudodistance functions d_*^f, d_*^h, determined by the maps f_k and h_k, are equal.

The results of [5, section 8.3] rely on the fact that the sequence f_k satisfies the monotonicity formula and that the tangent map f_* is homogeneous. We have already established monotonicity for the sequence f_k. To prove homogeneity for the tangent map $\overline{f}_*$, we proceed exactly as in [5].

Lemma 8.3 *For* **B** *a dimension-n, codimension-ν local model, $\nu \geq 2$, and g a normalized Lipschitz metric defined on* **B**(r), (Y, d) *a CAT(1) space, let $f : ($**B**$(r), g) \to \mathcal{B}_\tau(P) \subset Y$ be an energy minimizing map where $0 < \tau < \frac{\pi}{4}$. Let $f_k, h_k, \overline{f}_*$ be as above. Then the directional energies of the sequences h_k, f_k converge to the directional energies of $\overline{f}_*$.*

Proof Since the h_k are Hölder continuous and satisfy a monotonicity formula, the proof of [5, Lemma 8.8] can be followed verbatim. To prove the directional energies converge relies

only on estimates relating the energies of h_k, f_k, and lower semi-continuity of the energy. Since the energy comparisons follow from the comments above, the result is immediate. □

Now, following the proof of [5, Lemma 6.3], since the directional energies of h_k converge to those of $\overline{f}_*$, $\overline{f}_*$ is homogeneous of order α, i.e.

$$d_*(\overline{f}_*(x), \overline{f}_*(0)) = |x|^\alpha d_* \left(\overline{f}_* \left(\frac{x}{|x|} \right), \overline{f}_*(0) \right).$$

Note that in the proof in [5], the right hand side of the equation for $(E^{f_*}(\sigma))'$ should include the term $\frac{n-2}{\sigma} E^{f_*}(\sigma)$ and the first term in the parenthesis in (6.10) should be the product of boundary integrals.

With the homogeneity in hand, we can now follow the proofs of [5] to conclude the necessary results. For **B** a dimension-n, codimension-ν local model, recall that D is isomorphic to $\mathbb{R}^{n-\nu}$. For each $x \in D$, let $N(x)$ be the ν-plane orthogonal to D at x. Then, for $|x| < 1$, $\partial\mathbf{B}(1) \cap N(x)$ is a spherical $(\nu - 1)$-complex. The key proposition, which follows exactly the proof of [5, Theorem 8.4], is as follows.

Proposition 8.4 *Let* **B** *be a dimension-n, codimension-ν local model, $\nu \geq 2$, and g a normalized Lipschitz metric defined on* **B**(r) *and* (Y, d) *a CAT(1) space, let* $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ *be an energy minimizing map where* $0 < \tau < \frac{\pi}{4}$. *If* $\lambda_1(\partial\mathbf{B}(1) \cap N(0), T_Q Y) \geq \beta(> \beta)$ *for all* $Q \in Y$ *and* $\alpha < 1$, *then the order* α *of f at 0 satisfies* $\alpha(\alpha + \nu - 2) \geq \beta(> \beta)$.

For a local model **B** and any $x \in \mathbf{B}(r)$, recall that $f \circ L_x : \mathbf{B}'_x(r(x)) \to \mathcal{B}_\tau P$ where L_x is given by [5, Proposition 2.1] and $\mathbf{B}'_x(r(x))$ is a local model centered at x. Define

$$\lambda_1^N := \inf_{x \in \mathbf{B}(r) \cap N(0), Q \in Y} \lambda_1(\partial\mathbf{B}'_x(1), T_Q Y).$$

As an immediately corollary of the previous proposition, we observe that:

Corollary 8.5 *For* **B** *a dimension-n, codimension-ν local model, $\nu \geq 2$, g a Lipschitz Riemannian metric defined on* **B**(r), *and* (Y, d) *a CAT(1) space, let* $f : (\mathbf{B}(r), g) \to \mathcal{B}_\tau(P) \subset Y$ *be an energy minimizing map where* $0 < \tau < \frac{\pi}{4}$. *If* $\lambda_1^N \geq \nu - 1$ *then f is Lipschitz continuous in* **B**(ϱr) *for* $\varrho \in (0, 1)$.

Theorem 1.3, item (2), immediately follow from the above results following the observation that $\nu = n - k$.

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