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Higher-dimensional open quantum walk in environment of quantum Bernoulli noises

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Open quantum walks (OQWs) (also known as open quantum random walks) are quantum analogs of classical Markov chains in probability theory, and have potential application in quantum information and quantum computation. Quantum Bernoulli noises (QBNs) are annihilation and creation operators acting on Bernoulli functionals, and can be used as the environment of an open quantum system. In this paper, by using QBNs as the environment, we introduce an OQW on a general higher-dimensional integer lattice. We obtain a quantum channel representation of the walk, which shows that the walk is indeed an OQW. We prove that all the states of the walk are separable provided its initial state is separable. We also prove that, for some initial states, the walk has a limit probability distribution of higher-dimensional Gauss type. Finally, we show links between the walk and a unitary quantum walk recently introduced in terms of QBNs.

Keywords: Open quantum walk; quantum Bernoulli noises; quantum probability.

AMS Subject Classification: 81S25; 81S22

1. Introduction

Random walks have been widely used as a fundamental mathematical tool for modeling various physical processes and for the development of stochastic algorithm [10]. Recent years have seen great attention paid to quantum walks [7, 16], which are also known as quantum random walks [1, 6]. As quantum extensions of random walks, quantum walks have found wide application in quantum information and quantum computation [12]. Due to the quantum interference effects, quantum walks greatly outperform random walks at certain computational tasks, and moreover it has turned out that quantum walks constitute universal models of quantum computation [12].

From a perspective of mathematical physics, quantum walks can be divided into two categories: unitary quantum walks and nonunitary quantum walks. Unitary quantum walks do not interact with the environments and their time evolutions are described mainly by unitary operators. The past two decades have witnessed a considerable amount of research on unitary quantum walks and their application (see, e.g., [6, 7, 12, 16] and references therein). In 2012, Attal *et al.* introduced a class of nonunitary quantum walks, called open quantum walks (OQWs) [4].

OQWs are also known as open quantum random walks [3], which can be viewed as quantum analogs of classical Markov chains in probability theory. As a new type of quantum walks, OQWs are finding application in the generalizations of the theory of quantum probability, and have potential application in quantum state engineering, dissipative quantum computation and transport in mesoscopic systems [15].

Much attention has been paid to OQWs. Attal *et al.* [2] established the central limit theorem (CLT) for a class of homogeneous OQWs with a unique invariant state. Konno and Yoo [8] applied the CLT to the study of limit probability distributions for various OQWs. Sadowski and Pawela [13] considered a generalization of the CLT for the case of nonhomogeneous OQWs. Carbone and Pautrat [5], from a perspective of classical Markov chain, introduced notions of irreducibility, period, communicating classes for OQWs. Lardizabal [9] defined a notion of hitting time for OQWs and obtained some useful formulas for certain cases. There are other researches on OQWs (see [15] and references therein).

From a physical point of view, OQWs interact with their environments. More specifically, for an OQW, the transitions between the sites (or vertices) are driven by the interaction with the environment. Thus, the effects of environment can play an important role in the time evolution of OQWs. On the other hand, being annihilation and creation operators acting on Bernoulli functionals, quantum Bernoulli noises (QBNs) have turned out to be an alternative approach to the environment of an open quantum system (see, e.g., [17, 19]). It is then natural to apply QBNs to the study of OQWs.

In 2018, by using QBNs, Wang *et al.* [21] introduced a model of OQW on the 1-dimensional integer lattice $\mathbb{Z}$, which we call *the* 1*-dimensional open QBN walk* below. In this paper, we would like to extend the 1-dimensional open QBN walk to a higher-dimensional case. More specifically, with QBNs as the environment, we will introduce a model of OQW on the d-dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 2$, and examine its dynamical behavior from a perspective of probability distribution. Our main work is as follows.

- We introduce a notion of d-dimensional nucleus on the space $\mathcal{H}$ of square integrable Bernoulli functionals.
- With QBNs as the fundamental tool, we construct a sequence of transforms $\mathfrak{J}_n^{(d)}$, $n \geq 0$, on the d-dimensional nucleuses, and then we use these transforms to establish our model of OQW on $\mathbb{Z}^d$, which we call *the d-dimensional open QBN walk*.

- We obtain a quantum channel representation of the d-dimensional open QBN walk, which shows that the d-dimensional open QBN walk is indeed an OQW.
- We find that all the states of the d-dimensional open QBN walk are separable provided its initial state is separable, and moreover we prove that, for some initial states, the d-dimensional open QBN walk has a limit probability distribution of d-dimensional Gauss type.
- We show links between the d-dimensional open QBN walk and a unitary quantum walk recently introduced in [20].

Some other results are also proven of the d-dimensional open QBN walk.

The paper is organized as follows. In Sec. 2.1, we briefly recall some necessary notions and facts about QBNs. Sections 2.2–2.4 are one part of our main work, which includes the technical theorems we prove, the definition of our model (namely the d-dimensional open QBN walk) and the quantum channel representation of the walk. The other part of our main work lies in Secs. 3 and 4, where we examine the separability of the walk's states, calculate its limit probability distribution and show its links with a unitary quantum walk recently introduced in [20].

Throughout this paper, $\mathbb{Z}$ always denotes the set of all integers, while $\mathbb{N}$ means the set of all nonnegative integers. We denote by Γ the finite power set of $\mathbb{N}$, namely

$$\Gamma = \{\sigma \,|\, \sigma \subset \mathbb{N} \text{ and } \#\,\sigma < \infty\}, \tag{1.1}$$

where $\#\sigma$ means the cardinality of σ. Unless otherwise stated, letters like j, k and n stand for nonnegative integers, namely elements of $\mathbb{N}$. If $\mathcal{X}$ is a Hilbert space, then $\mathfrak{B}(\mathcal{X})$ denotes the set of all bounded linear operators on $\mathcal{X}$. As usual, a density operator on a Hilbert space means a positive operator of trace class with unit trace on that space. By convention, $\mathrm{Tr}A$ denotes the trace of an operator A of trace class.

2. Definition of Walk and Its Basic Properties

In this section, we mainly introduce our model of OQW, which we will call the d-dimensional open QBN walk, and examine its basic properties.

2.1. *Quantum Bernoulli noises*

We first briefly recall some necessary notions and facts about QBNs. We refer to [17] for details about QBNs.

Let Ω be the set of all functions $f\colon \mathbb{N} \mapsto \{-1, 1\}$, and $(\zeta_n)_{n\geq 0}$ the sequence of canonical projections on Ω given by

$$\zeta_n(f) = f(n), \quad f \in \Omega. \tag{2.1}$$

Let $\mathscr{F}$ be the σ-field on Ω generated by the sequence $(\zeta_n)_{n\geq 0}$, and $(p_n)_{n\geq 0}$ a given sequence of positive numbers with the property that $0 < p_n < 1$ for all $n \geq 0$. Then there exists a unique probability measure $\mathbb{P}$ on $\mathscr{F}$ such that

$$\mathbb{P}\circ(\zeta_{n_1},\zeta_{n_2},\ldots,\zeta_{n_k})^{-1}\big\{(\epsilon_1,\epsilon_2,\ldots,\epsilon_k)\big\}=\prod_{j=1}^{k}p_{n_j}^{\frac{1+\epsilon_j}{2}}(1-p_{n_j})^{\frac{1-\epsilon_j}{2}} \tag{2.2}$$

for $n_j\in\mathbb{N}$, $\epsilon_j\in\{-1,1\}$ $(1\le j\le k)$ with $n_i\neq n_j$ when $i\neq j$ and $k\in\mathbb{N}$ with $k\ge 1$. Thus one has a probability measure space $(\Omega,\mathscr{F},\mathbb{P})$, which is referred to as the Bernoulli space and random variables on it are known as Bernoulli functionals.

Let $Z=(Z_n)_{n\ge 0}$ be the sequence of Bernoulli functionals generated by sequence $(\zeta_n)_{n\ge 0}$, namely

$$Z_n=\frac{\zeta_n+q_n-p_n}{2\sqrt{p_nq_n}},\quad n\ge 0, \tag{2.3}$$

where $q_n=1-p_n$. Clearly, $Z=(Z_n)_{n\ge 0}$ is an independent sequence of random variables on the probability measure space $(\Omega,\mathscr{F},\mathbb{P})$. Let $\mathcal{H}$ be the space of square integrable complex-valued Bernoulli functionals, namely

$$\mathcal{H}=L^2(\Omega,\mathscr{F},\mathbb{P}). \tag{2.4}$$

We denote by $\langle\cdot,\cdot\rangle$ the usual inner product of the space $\mathcal{H}$, and by $\|\cdot\|$ the corresponding norm. It is known that Z has the chaotic representation property. Thus, $\mathfrak{Z}=\{Z_\sigma\mid\sigma\in\Gamma\}$ form an orthonormal basis (ONB) of $\mathcal{H}$, which is known as the canonical ONB of $\mathcal{H}$. Here, $Z_\emptyset=1$ and

$$Z_\sigma=\prod_{j\in\sigma}Z_j,\quad \sigma\in\Gamma,\ \ \sigma\neq\emptyset. \tag{2.5}$$

Clearly $\mathcal{H}$ is infinite-dimensional as a complex Hilbert space.

It can be shown that [17], for each $k\in\mathbb{N}$, there exists a bounded operator ∂_k on $\mathcal{H}$ such that

$$\partial_k Z_\sigma=\mathbf{1}_\sigma(k)Z_{\sigma\setminus k},\quad \partial_k^* Z_\sigma=[1-\mathbf{1}_\sigma(k)]Z_{\sigma\cup k}\quad \sigma\in\Gamma, \tag{2.6}$$

where ∂_k^* denotes the adjoint of ∂_k, $\sigma\setminus k=\sigma\setminus\{k\}$, $\sigma\cup k=\sigma\cup\{k\}$ and $\mathbf{1}_\sigma(k)$ the indicator of σ as a subset of $\mathbb{N}$.

The operators ∂_k and ∂_k^* are usually known as the annihilation and creation operators acting on Bernoulli functionals, respectively. And the family $\{\partial_k,\partial_k^*\}_{k\ge 0}$ is referred to as *QBNs*.

A typical property of QBNs is that they satisfy the canonical anti-commutation relations (CAR) in equal-time [17]. More specifically, for k, $l\in\mathbb{N}$, it holds true that

$$\partial_k\partial_l=\partial_l\partial_k,\quad \partial_k^*\partial_l^*=\partial_l^*\partial_k^*,\quad \partial_k^*\partial_l=\partial_l\partial_k^*\quad (k\neq l) \tag{2.7}$$

and

$$\partial_k\partial_k=\partial_k^*\partial_k^*=0,\quad \partial_k\partial_k^*+\partial_k^*\partial_k=I, \tag{2.8}$$

where I is the identity operator on $\mathcal{H}$.

For a nonnegative integer $n \geq 0$, one can use QBNs to define two operators L_n and R_n on $\mathcal{H}$ in the following manner:

$$L_n = \frac{1}{2}\big(\partial_n^* + \partial_n - I\big), \quad R_n = \frac{1}{2}\big(\partial_n^* + \partial_n + I\big). \tag{2.9}$$

It then follows from the properties of QBNs that the operators L_n, R_n, $n \geq 0$, form a commutative family, namely

$$L_k L_l = L_l L_k, \quad R_k L_l = L_l R_k, \quad R_k R_l = R_l R_k, \quad k,\, l \geq 0. \tag{2.10}$$

Lemma 2.1 ([21]). *For all $n \geq 0$, operators L_n and R_n admit the following operational properties*:

$$L_n^2 = -L_n, \quad L_n R_n = R_n L_n = 0, \quad R_n^2 = R_n, \quad L_n^2 + R_n^2 = I. \tag{2.11}$$

2.2. *Technical theorems*

In this subsection, we prove some technical theorems, which will be used in defining our model of OQW and examining its properties.

In what follows, we always assume that $d \geq 2$ is a given positive integer and $\Lambda = \{-1, +1\}$. We denote by Λ^d the d-fold Cartesian product of Λ, and by $\mathcal{H}^{\otimes d}$ the d-fold tensor product space of $\mathcal{H}$. In addition, we assume that $\mathsf{K}\colon \mathcal{H}^{\otimes d} \to \mathcal{H}$ is a fixed unitary isomorphism. Such a unitary isomorphism does exist because $\mathcal{H}$ is infinite-dimensional and separable.

To facilitate our discussions, we further write $\mathfrak{S}(\mathcal{H})$ for the space of all operators of trace class on $\mathcal{H}$ with the trace norm and $\mathfrak{S}_+(\mathcal{H})$ for the cone of all positive elements of $\mathfrak{S}(\mathcal{H})$. Similarly, we use symbol $\mathfrak{S}\big(\mathcal{H}^{\otimes d}\big)$ and $\mathfrak{S}_+\big(\mathcal{H}^{\otimes d}\big)$.

Definition 2.1. For $n \geq 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \in \Lambda^d$, we define

$$C_n^{(\varepsilon)} = \mathsf{K}\left(\bigotimes_{j=1}^{d} B_n^{(\varepsilon_j)}\right)\mathsf{K}^{-1}, \tag{2.12}$$

where K^{-1} is the inverse of the unitary isomorphism $\mathsf{K}\colon \mathcal{H}^{\otimes d} \to \mathcal{H}$, and $B_n^{(\varepsilon_j)}$ is given by

$$B_n^{(\varepsilon_j)} = \begin{cases} L_n, & \varepsilon_j = -1, \\ R_n, & \varepsilon_j = +1, \end{cases} \tag{2.13}$$

for $i = 1, 2, \ldots, d$.

It can be shown that, for all $n \geq 0$, $\big\{C_n^{(\varepsilon)} \mid \varepsilon \in \Lambda^d\big\}$ are self-adjoint operators on $\mathcal{H}$. And moreover, they admit the following useful properties: $C_n^{(\varepsilon)} C_n^{(\varepsilon')} = 0$ for ε, $\varepsilon' \in \Lambda^d$ with $\varepsilon \neq \varepsilon'$; and their sum $\sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)}$ is a unitary operator on $\mathcal{H}$ (see [20] for details).

Theorem 2.1. *Let $n \geq 0$. Then $\sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)} C_n^{(\varepsilon)} = I$, where I denotes the identity operator on $\mathcal{H}$.*

Proof. By the definition of B_n^{-1} and $B_n^{(+1)}$ and Lemma 2.1, we have

$$\sum_{\varepsilon_j \in \Lambda} B_n^{(\varepsilon_j)} B_n^{(\varepsilon_j)} = B_n^{(-1)} B_n^{(-1)} + B_n^{(+1)} B_n^{(+1)} = L_n^2 + R_n^2 = I$$

for $j = 1, 2, \ldots, d$, where I denotes the identity operator on $\mathcal{H}$. Making tensor products gives

$$\sum_{\varepsilon \in \Lambda^d} \left(\bigotimes_{j=1}^{d} B_n^{(\varepsilon_j)} \right) \left(\bigotimes_{j=1}^{d} B_n^{(\varepsilon_j)} \right)$$

$$= \sum_{\varepsilon \in \Lambda^d} \left(\bigotimes_{j=1}^{d} B_n^{(\varepsilon_j)} B_n^{(\varepsilon_j)} \right) = \bigotimes_{j=1}^{d} \sum_{\varepsilon_j \in \Lambda} B_n^{(\varepsilon_j)} B_n^{(\varepsilon_j)} = \bigotimes_{j=1}^{d} I.$$

This, together with the definition of $C_n^{(\varepsilon)}$ as well as properties of the unitary isomorphism K, yields

$$\sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} C_n^{(\varepsilon)} = \mathsf{K} \left[\sum_{\varepsilon \in \Lambda^d} \left(\bigotimes_{j=1}^{d} B_n^{(\varepsilon_j)} \right) \left(\bigotimes_{j=1}^{d} B_n^{(\varepsilon_j)} \right) \right] \mathsf{K}^{-1} = \mathsf{K} \left[\bigotimes_{j=1}^{d} I \right] \mathsf{K}^{-1} = I.$$

Here, we note that $\bigotimes_{j=1}^{d} I$ is just the identity operator on $\mathcal{H}^{\otimes d}$. □

Theorem 2.2. *For all $\varrho \in \mathfrak{S}_+(\mathcal{H})$ and $n \geq 0$, the sum operator $\sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} \varrho C_n^{(\varepsilon)}$ belongs to $\mathfrak{S}_+(\mathcal{H})$, and moreover it holds true that*

$$\mathrm{Tr} \left[\sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} \varrho C_n^{(\varepsilon)} \right] = \mathrm{Tr}\, \varrho. \tag{2.14}$$

Proof. For each $\varepsilon \in \Lambda^d$, $C_n^{(\varepsilon)} \varrho C_n^{(\varepsilon)}$ is a positive operator of trace class on $\mathcal{H}$ since ϱ is such an operator and $C_n^{(\varepsilon)}$ is self-adjoint. Thus, $\sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} \varrho C_n^{(\varepsilon)}$ is also a positive operator of trace class on $\mathcal{H}$, namely it belongs to $\mathfrak{S}_+(\mathcal{H})$. Now, by using Theorem 2.1, we find

$$\mathrm{Tr} \left[\sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} \varrho C_n^{(\varepsilon)} \right] = \sum_{\varepsilon \in \Lambda^d} \mathrm{Tr}\big[\varrho C_n^{(\varepsilon)} C_n^{(\varepsilon)}\big] = \mathrm{Tr} \left[\varrho \sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} C_n^{(\varepsilon)} \right] = \mathrm{Tr}\, \varrho.$$

This completes the proof. □

Definition 2.2. A d-dimensional nucleus ω on $\mathcal{H}$ is a mapping $\omega\colon \mathbb{Z}^d \to \mathfrak{S}_+(\mathcal{H})$ satisfying that

$$\sum_{\mathrm{x} \in \mathbb{Z}^d} \mathrm{Tr}[\omega(\mathrm{x})] = 1. \tag{2.15}$$

The set of all d-dimensional nucleuses on $\mathcal{H}$ is denoted as $\mathscr{T}^{(d)}(\mathcal{H})$.

It can be seen that, for each d-dimensional nucleus $\omega \in \mathscr{T}^{(d)}(\mathcal{H})$, the corresponding function $\mathrm{x} \mapsto \mathrm{Tr}[\omega(\mathrm{x})]$ defines a probability distribution on $\mathbb{Z}^d$.

Theorem 2.3. *For each $n \geq 0$, there exists a mapping $\mathfrak{J}_n^{(d)} \colon \mathscr{T}^{(d)}(\mathcal{H}) \to \mathscr{T}^{(d)}(\mathcal{H})$ such that*

$$\big[\mathfrak{J}_n^{(d)}\omega\big](\mathrm{x}) = \sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)}\omega(\mathrm{x}-\varepsilon)C_n^{(\varepsilon)}, \quad \mathrm{x} \in \mathbb{Z}^d,\ \omega \in \mathscr{T}^{(d)}(\mathcal{H}), \tag{2.16}$$

where $\sum_{\varepsilon\in\Lambda^d}$ means to sum over Λ^d.

Proof. Let $n \geq 0$. For each $\omega \in \mathscr{T}^{(d)}(\mathcal{H})$, there is naturally a mapping ω' on $\mathbb{Z}^d$ associated with ω in the following way:

$$\omega'(\mathrm{x}) = \sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)}\omega(\mathrm{x}-\varepsilon)C_n^{(\varepsilon)}, \quad \mathrm{x} \in \mathbb{Z}^d.$$

We observe that $C_n^{(\varepsilon)}\omega(\mathrm{x}-\varepsilon)C_n^{(\varepsilon)} \in \mathfrak{S}_+(\mathcal{H})$ for all $\mathrm{x} \in \mathbb{Z}^d$ and all $\varepsilon \in \Lambda^d$, which implies that $\omega'(\mathrm{x}) \in \mathfrak{S}_+(\mathcal{H})$ for all $\mathrm{x} \in \mathbb{Z}^d$, hence ω' is a mapping from $\mathbb{Z}^d$ to $\mathfrak{S}_+(\mathcal{H})$.

Next, we show that $\sum_{\mathrm{x}\in\mathbb{Z}^d} \mathrm{Tr}[\omega'(\mathrm{x})] = 1$. In fact, for each $\mathrm{x} \in \mathbb{Z}^d$, by Theorem 2.2 we have

$$\sum_{\varepsilon\in\Lambda^d} \mathrm{Tr}\big[C_n^{(\varepsilon)}\omega(\mathrm{x})C_n^{(\varepsilon)}\big] = \mathrm{Tr}[\omega(\mathrm{x})].$$

Thus, in view of the fact that all the series involved have positive terms, we get

$$\begin{aligned}\sum_{\mathrm{x}\in\mathbb{Z}^d} \mathrm{Tr}[\omega'(\mathrm{x})] &= \sum_{\mathrm{x}\in\mathbb{Z}^d}\sum_{\varepsilon\in\Lambda^d} \mathrm{Tr}\big[C_n^{(\varepsilon)}\omega(\mathrm{x}-\varepsilon)C_n^{(\varepsilon)}\big] \\ &= \sum_{\mathrm{x}\in\mathbb{Z}^d}\sum_{\varepsilon\in\Lambda^d} \mathrm{Tr}\big[C_n^{(\varepsilon)}\omega(\mathrm{x})C_n^{(\varepsilon)}\big] \\ &= \sum_{\mathrm{x}\in\mathbb{Z}^d} \mathrm{Tr}[\omega(\mathrm{x})],\end{aligned}$$

which together with $\omega \in \mathscr{T}^{(d)}(\mathcal{H})$ implies that $\sum_{\mathrm{x}\in\mathbb{Z}^d} \mathrm{Tr}[\omega'(\mathrm{x})] = 1$. Now, according to Definition 2.2, we know that $\omega' \in \mathscr{T}^{(d)}(\mathcal{H})$. Finally, we define a mapping $\mathfrak{J}_n^{(d)} \colon \mathscr{T}^{(d)}(\mathcal{H}) \to \mathscr{T}^{(d)}(\mathcal{H})$ as

$$\mathfrak{J}_n^{(d)}\omega = \omega', \quad \omega \in \mathscr{T}^{(d)}(\mathcal{H}).$$

Then $\mathfrak{J}_n^{(d)}$ is the desired. □

2.3. *Definition of walk*

This subsection gives the definition of our model of OQW on the d-dimensional integer lattice $\mathbb{Z}^d$.

As mentioned above, we call a positive operator of trace class (on a Hilbert space) a density operator if it has unit trace. Recall that $d \geq 2$ is a given positive integer. We denote by $l^2(\mathbb{Z}^d)$ the space of square summable complex-valued functions on the d-dimensional integer lattice $\mathbb{Z}^d$. As a Hilbert space, $l^2(\mathbb{Z}^d)$ has a countable ONB $\{\delta_{\mathrm{x}} \,|\, \mathrm{x} \in \mathbb{Z}^d\}$, which is known as the canonical ONB of $l^2(\mathbb{Z}^d)$, where δ_{x} is the function on $\mathbb{Z}^d$ given by

$$\delta_{\mathrm{x}}(\mathrm{z}) = \begin{cases} 1, & \mathrm{z} = \mathrm{x},\ \mathrm{z} \in \mathbb{Z}^d, \\ 0, & \mathrm{z} \neq \mathrm{x},\ \mathrm{z} \in \mathbb{Z}^d. \end{cases}$$

By convention, we use $|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}|$ to mean the Dirac operator associated with δ_{x}, which is a density operator on $l^2(\mathbb{Z}^d)$.

By the general theory of trace class operators on a Hilbert space [14], one can easily come to the next lemma, which provides a way to construct a density operator on the tensor space $l^2(\mathbb{Z}^d) \otimes \mathcal{H}$ from the canonical ONB of $l^2(\mathbb{Z}^d)$ and elements of $\mathscr{T}^{(d)}(\mathcal{H})$.

Lemma 2.2. *Let* $\omega \in \mathscr{T}^{(d)}(\mathcal{H})$. *Then, for each* $\mathrm{x} \in \mathbb{Z}^d$, $|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \omega(\mathrm{x})$ *is a positive operator of trace class on* $l^2(\mathbb{Z}^d) \otimes \mathcal{H}$. *Moreover, the operator series*

$$\sum_{\mathrm{x} \in \mathbb{Z}^d} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \omega(\mathrm{x}) \tag{2.17}$$

is convergent in the trace operator norm and its sum operator is a density operator on $l^2(\mathbb{Z}^d) \otimes \mathcal{H}$.

With help of this lemma, we are now ready to introduce our model of OQW as follows.

Definition 2.3. The d-dimensional open QBN walk is an OQW on the d-dimensional integer lattice $\mathbb{Z}^d$ that admits the following features:

- Its states are represented by density operators on the tensor space $l^2(\mathbb{Z}^d) \otimes \mathcal{H}$.
- Let $\widetilde{\omega^{(n)}}$ be the state of the walk at time $n \geq 0$. Then $\widetilde{\omega^{(n)}}$ takes the form

$$\widetilde{\omega^{(n)}} = \sum_{\mathrm{x} \in \mathbb{Z}^d} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \omega^{(n)}(\mathrm{x}), \tag{2.18}$$

  where $\omega^{(n)} \in \mathscr{T}^{(d)}(\mathcal{H})$, which is called the nucleus of the state $\widetilde{\omega^{(n)}}$.
- The time evolution of the walk is governed by equation

$$\omega^{(n+1)} = \mathfrak{J}_n^{(d)} \omega^{(n)}, \quad n \geq 0, \tag{2.19}$$

  where $\omega^{(n+1)}$ and $\omega^{(n)}$ are the nucleuses of the states $\widetilde{\omega^{(n+1)}}$ and $\widetilde{\omega^{(n)}}$, respectively, and $\mathfrak{J}_n^{(d)}$ is the mapping described in Theorem 2.3.

In that case, the function $\mathrm{x} \mapsto \mathrm{Tr}[\omega^{(n)}(\mathrm{x})]$ on $\mathbb{Z}^d$ is called the probability distribution of the walk at time $n \geq 0$, while the quantity $\mathrm{Tr}[\omega^{(n)}(\mathrm{x})]$ is the probability to find out the walker at position $\mathrm{x} \in \mathbb{Z}^d$ and time $n \geq 0$. By convention, the state $\widetilde{\omega^{(0)}}$ of the walk at time $n = 0$ is usually known as its initial state.

Physically, $l^2(\mathbb{Z}^d)$ describes the position of the walk, while $\mathcal{H}$ describes the internal degrees of freedom of the walk. As shown above, $\mathcal{H}$ is infinitely dimensional, which means that the d-dimensional open QBN walk has infinitely many internal degrees of freedom.

Remark 2.1. It is not hard to see that the d-dimensional open QBN walk is completely determined by the nucleus sequence of its states. Let $\left(\omega^{(n)}\right)_{n\geq 0}$ be the nucleus sequence of states of the d-dimensional open QBN walk. Then, by Theorem 2.3, one has the following evolution relations:

$$\omega^{(n+1)}(\mathrm{x}) = \sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)}\omega^{(n)}(\mathrm{x}-\varepsilon)C_n^{(\varepsilon)}, \quad \mathrm{x}\in\mathbb{Z}^d,\ n\geq 0, \tag{2.20}$$

which actually give an alternative description of the evolution of the d-dimensional open QBN walk.

2.4. *Quantum channel representation*

In this subsection, we establish a quantum channel representation of the d-dimensional open QBN walk, which shows that the d-dimensional open QBN walk is indeed an OQW.

For $n \geq 0$ and x, y $\in \mathbb{Z}^d$, we define an operator $M^{(n)}(\mathrm{x},\mathrm{y})$ on the tensor space $l^2(\mathbb{Z}^d)\otimes\mathcal{H}$ as

$$M_{\mathrm{x},\mathrm{y}}^{(n)} = \begin{cases} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}|\otimes C_n^{(\mathrm{x}-\mathrm{y})}, & \mathrm{x}-\mathrm{y}\in\Lambda^d; \\ 0, & \mathrm{x}-\mathrm{y}\notin\Lambda^d. \end{cases} \tag{2.21}$$

Clearly, for each $n \geq 0$, the operator family $\left\{M_{\mathrm{x},\mathrm{y}}^{(n)} \mid \mathrm{x},\mathrm{y} \in \mathbb{Z}^d\right\}$ is infinite and countable.

Theorem 2.4. *For each $n \geq 0$, the countable family $\left\{M_{\mathrm{x},\mathrm{y}}^{(n)} \mid \mathrm{x},\mathrm{y}\in\mathbb{Z}^d\right\}$ of operators satisfies the following relation*:

$$\sum_{\mathrm{x},\mathrm{y}\in\mathbb{Z}^d} M_{\mathrm{x},\mathrm{y}}^{(n)^*} M_{\mathrm{x},\mathrm{y}}^{(n)} = I, \tag{2.22}$$

where $M_{\mathrm{x},\mathrm{y}}^{(n)^*}$ means the adjoint of $M_{\mathrm{x},\mathrm{y}}^{(n)}$, I denotes the identity operator on $l^2(\mathbb{Z}^d)\otimes\mathcal{H}$ and the operator series converges strongly.

Proof. First, for each y $\in \mathbb{Z}^d$, we find that there are at most finitely many x $\in \mathbb{Z}^d$ such that $M_{\mathrm{x},\mathrm{y}}^{(n)} \neq 0$, which implies that the series $\sum_{\mathrm{x}\in\mathbb{Z}^d} M_{\mathrm{x},\mathrm{y}}^{(n)^*} M_{\mathrm{x},\mathrm{y}}^{(n)}$ is actually a sum of finitely many summands. On the other hand, for each y $\in \mathbb{Z}^d$, by a direct calculation we have

$$\begin{aligned}\sum_{\mathrm{x}\in\mathbb{Z}^d} M_{\mathrm{x},\mathrm{y}}^{(n)^*} M_{\mathrm{x},\mathrm{y}}^{(n)} &= \sum_{\mathrm{x}-\mathrm{y}\in\Lambda^d} \left(|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}|\otimes C_n^{(\mathrm{x}-\mathrm{y})}\right)^*\left(|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}|\otimes C_n^{(\mathrm{x}-\mathrm{y})}\right) \\ &= \sum_{\mathrm{x}-\mathrm{y}\in\Lambda^d} \left(|\delta_{\mathrm{y}}\rangle\langle\delta_{\mathrm{x}}|\otimes C_n^{(\mathrm{x}-\mathrm{y})}\right)\left(|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}|\otimes C_n^{(\mathrm{x}-\mathrm{y})}\right)\end{aligned}$$

$$= \sum_{x-y\in\Lambda^d} |\delta_y\rangle\langle\delta_y| \otimes \big(C_n^{(x-y)*}C_n^{(x-y)}\big)$$

$$= |\delta_y\rangle\langle\delta_y| \otimes \sum_{x-y\in\Lambda^d} C_n^{(x-y)*}C_n^{(x-y)}$$

$$= |\delta_y\rangle\langle\delta_y| \otimes I_{\mathcal{H}}.$$

Therefore, using the fact that the series $\sum_{y\in\mathbb{Z}^d} |\delta_y\rangle\langle\delta_y|$ strongly converges to $I_{l^2(\mathbb{Z}^d)}$, we finally come to

$$\sum_{x,y\in\mathbb{Z}^d} M_{x,y}^{(n)*} M_{x,y}^{(n)}$$

$$= \sum_{y\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d} M_{x,y}^{(n)*} M_{x,y}^{(n)} = \sum_{y\in\mathbb{Z}^d} |\delta_y\rangle\langle\delta_y| \otimes I_{\mathcal{H}} = I_{l^2(\mathbb{Z}^d)} \otimes I_{\mathcal{H}} = I.$$

Here, $I_{l^2(\mathbb{Z}^d)}$ and $I_{\mathcal{H}}$ mean the identity operators on $l^2(\mathbb{Z}^d)$ and $\mathcal{H}$, respectively. □

As above, we denote by $\mathfrak{S}\big(l^2(\mathbb{Z}^d)\otimes\mathcal{H}\big)$ the space of trace class operators on $l^2(\mathbb{Z}^d)\otimes\mathcal{H}$. Then, by the general theory of trace class operators [14], for each $n \geq 0$ and each $\widetilde{\omega} \in \mathfrak{S}\big(l^2(\mathbb{Z}^d)\otimes\mathcal{H}\big)$, the operator series

$$\sum_{x,y\in\mathbb{Z}^d} M_{x,y}^{(n)}\,\widetilde{\omega} M_{x,y}^{(n)*}$$

converges in the trace norm, and moreover its sum still belongs to $\mathfrak{S}\big(l^2(\mathbb{Z}^d)\otimes\mathcal{H}\big)$.

Definition 2.4. For $n \geq 0$, we define a mapping $\mathsf{M}^{(n)}\colon \mathfrak{S}\big(l^2(\mathbb{Z}^d)\otimes\mathcal{H}\big) \to \mathfrak{S}\big(l^2(\mathbb{Z}^d)\otimes\mathcal{H}\big)$ as

$$\mathsf{M}^{(n)}(\widetilde{\omega}) = \sum_{x,y\in\mathbb{Z}^d} M_{x,y}^{(n)}\,\widetilde{\omega} M_{x,y}^{(n)*}, \quad \widetilde{\omega} \in \mathfrak{S}\big(l^2(\mathbb{Z}^d)\otimes\mathcal{H}\big), \tag{2.23}$$

where $\sum_{x,y\in\mathbb{Z}^d}$ means to sum for all x, y $\in \mathbb{Z}^d$.

From a point of quantum information theory [11], the mapping $\mathsf{M}^{(n)}$ is actually a quantum channel of Krauss type. The next result then gives a quantum channel representation of the d-dimensional open QBN walk.

Theorem 2.5. *Let $\big(\widetilde{\omega^{(n)}}\big)_{n\geq 0}$ be the state sequence of the d-dimensional open QBN walk. Then it satisfies the following evolution equation*:

$$\widetilde{\omega^{(n+1)}} = \mathsf{M}^{(n)}\big(\widetilde{\omega^{(n)}}\big), \quad n \geq 0. \tag{2.24}$$

Proof. Let $n \geq 0$. By Definition 2.3, $\widetilde{\omega^{(n)}}$ has a representation of the following form:

$$\widetilde{\omega^{(n)}} = \sum_{z\in\mathbb{Z}^d} |\delta_z\rangle\langle\delta_z| \otimes \omega^{(n)}(z),$$

where the series converges strongly. For x, y $\in \mathbb{Z}^d$ with x $-$ y $\in \Lambda^d$, in view of the fact

$$|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}|\,|\delta_{\mathrm{z}}\rangle\langle\delta_{\mathrm{z}}| = \begin{cases} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}|, & \mathrm{z}=\mathrm{y}; \\ 0, & \mathrm{z}=\mathrm{y},\ \ \mathrm{z}\in\mathbb{Z}^d, \end{cases}$$

we have

$$\begin{aligned} M_{\mathrm{x,y}}^{(n)}\,\widetilde{\omega^{(n)}}\,M_{\mathrm{x,y}}^{(n)^*} &= \left[|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}| \otimes C_n^{(\mathrm{x-y})}\right]\left[\sum_{\mathrm{z}\in\mathbb{Z}^d} |\delta_{\mathrm{z}}\rangle\langle\delta_{\mathrm{z}}| \otimes \omega^{(n)}(\mathrm{z})\right]\left[|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}| \otimes C_n^{(\mathrm{x-y})}\right]^* \\ &= \left[|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}| \otimes \left(C_n^{(\mathrm{x-y})}\omega^{(n)}(\mathrm{y})\right)\right]\left[|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}| \otimes C_n^{(\mathrm{x-y})}\right]^* \\ &= \left[|\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{y}}| \otimes \left(C_n^{(\mathrm{x-y})}\omega^{(n)}(\mathrm{y})\right)\right]\left[|\delta_{\mathrm{y}}\rangle\langle\delta_{\mathrm{x}}| \otimes C_n^{(\mathrm{x-y})}\right] \\ &= |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \left(C_n^{(\mathrm{x-y})}\omega^{(n)}(\mathrm{y})C_n^{(\mathrm{x-y})}\right). \end{aligned}$$

For $\mathrm{x},\mathrm{y} \in \mathbb{Z}^d$ with x $-$ y $\in \Lambda^d$, in view of $M_{\mathrm{x,y}}^{(n)} = 0$, we simply have $M_{\mathrm{x,y}}^{(n)}\,\widetilde{\omega^{(n)}}\,M_{\mathrm{x,y}}^{(n)^*} = 0$. Thus, for each y $\in \mathbb{Z}^d$, as a sum actually with a finite number of summands

$$\begin{aligned} \sum_{\mathrm{x}\in\mathbb{Z}^d} M_{\mathrm{x,y}}^{(n)}\,\widetilde{\omega^{(n)}}\,M_{\mathrm{x,y}}^{(n)^*} &= \sum_{\mathrm{x-y}\in\Lambda^d} M_{\mathrm{x,y}}^{(n)}\,\widetilde{\omega^{(n)}}\,M_{\mathrm{x,y}}^{(n)^*} \\ &= \sum_{\mathrm{x-y}\in\Lambda^d} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \left(C_n^{(\mathrm{x-y})}\omega^{(n)}(\mathrm{y})C_n^{(\mathrm{x-y})}\right) \\ &= \sum_{\varepsilon\in\Lambda^d} |\delta_{\mathrm{y}+\varepsilon}\rangle\langle\delta_{\mathrm{y}+\varepsilon}| \otimes \left(C_n^{(\varepsilon)}\omega^{(n)}(\mathrm{y})C_n^{(\varepsilon)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathsf{M}^{(n)}\left(\widetilde{\omega^{(n)}}\right) &= \sum_{\mathrm{y}\in\mathbb{Z}^d}\sum_{\mathrm{x}\in\mathbb{Z}^d} M_{\mathrm{x,y}}^{(n)}\,\widetilde{\omega^{(n)}}\,M_{\mathrm{x,y}}^{(n)^*} \\ &= \sum_{\mathrm{y}\in\mathbb{Z}^d}\sum_{\varepsilon\in\Lambda^d} |\delta_{\mathrm{y}+\varepsilon}\rangle\langle\delta_{\mathrm{y}+\varepsilon}| \otimes \left(C_n^{(\varepsilon)}\omega^{(n)}(\mathrm{y})C_n^{(\varepsilon)}\right) \\ &= \sum_{\mathrm{x}\in\mathbb{Z}^d} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)}\omega^{(n)}(\mathrm{x}-\varepsilon)C_n^{(\varepsilon)} \\ &= \sum_{\mathrm{x}\in\mathbb{Z}^d} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes \omega^{(n+1)}(\mathrm{x}) \\ &= \widetilde{\omega^{(n+1)}}. \end{aligned}$$

Here, we make use of the strong convergence of the involved series. $\square$

3. Probability Distribution of Walk

In this section, we consider the d-dimensional open QBN walk from a perspective of probability distribution. We first show that all the states of the walk are separable provided its initial state is separable, and then, based on this property, we calculate explicitly the limit probability distribution of the walk.

3.1. *Separability of state*

Let $\mathscr{T}(\mathcal{H})$ be the 1-dimensional counterpart of the d-dimensional nucleus set $\mathscr{T}^{(d)}(\mathcal{H})$, namely

$$\mathscr{T}(\mathcal{H}) = \left\{ \rho\colon \mathbb{Z} \to \mathfrak{S}_+(\mathcal{H}) \,\middle|\, \sum_{x=-\infty}^{\infty} \mathrm{Tr}[\rho(x)] = 1 \right\}. \tag{3.1}$$

Elements of $\mathscr{T}(\mathcal{H})$ are called 1-dimensional nucleuses on $\mathcal{H}$.

Lemma 3.1 ([21]). *For each $n \geq 0$, there exists a mapping $\mathfrak{J}_n\colon \mathscr{T}(\mathcal{H}) \to \mathscr{T}(\mathcal{H})$ such that*

$$[\mathfrak{J}_n\rho](x) = L_n\rho(x+1)L_n + R_n\rho(x-1)R_n, \quad x \in \mathbb{Z}, \;\; \rho \in \mathscr{T}(\mathcal{H}). \tag{3.2}$$

By this lemma, we find that $\left(\prod_{k=0}^{n}\mathfrak{J}_k\right)\rho \in \mathscr{T}(\mathcal{H})$ for all $n \geq 0$ whenever $\rho \in \mathscr{T}(\mathcal{H})$, where $\prod_{k=0}^{n}\mathfrak{J}_k$ means the composition of mappings $\mathfrak{J}_0, \mathfrak{J}_1, \ldots, \mathfrak{J}_n$.

Theorem 3.1. *Let $\rho_1, \rho_2, \ldots, \rho_d \in \mathscr{T}(\mathcal{H})$ be 1-dimensional nucleuses on $\mathcal{H}$. Define*

$$\omega(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d}\rho_j(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d. \tag{3.3}$$

Then $\omega \in \mathscr{T}^{(d)}(\mathcal{H})$, namely ω is a d-dimensional nucleus on $\mathcal{H}$.

Proof.

$$\sum_{\mathrm{x}\in\mathbb{Z}^d} \mathrm{Tr}[\omega(\mathrm{x})] = \sum_{\mathrm{x}\in\mathbb{Z}^d} \mathrm{Tr}\left[\bigotimes_{j=1}^{d}\rho_j(x_j)\right] = \sum_{\mathrm{x}\in\mathbb{Z}^d}\prod_{j=1}^{d}\mathrm{Tr}[\rho_j(x_j)]$$

$$= \prod_{j=1}^{d}\sum_{x_j\in\mathbb{Z}}\mathrm{Tr}[\rho_j(x_j)] = 1.$$

□

Definition 3.1. A d-dimensional nucleus $\omega \in \mathscr{T}^{(d)}(\mathcal{H})$ is said to to be separable if there exist 1-dimensional nucleuses $\rho_1, \rho_2, \ldots, \rho_d \in \mathscr{T}(\mathcal{H})$ such that

$$\omega(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d}\rho_j(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d. \tag{3.4}$$

A state $\widetilde{\omega^{(n)}}$ of the d-dimensional open QBN walk is said to be separable if its nucleus $\omega^{(n)}$ is separable.

The next theorem actually shows that all the states of the d-dimensional open QBN walk are separable provided its initial state is separable. In other words, the d-dimensional open QBN walk has the "separability-preserving" property.

Theorem 3.2. *Let* $\left(\omega^{(n)}\right)_{n\geq 0}$ *be the nucleus sequence of the states of the d-dimensional open QBN walk. Suppose that*

$$\omega^{(0)}(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d} \rho_j^{(0)}(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \tag{3.5}$$

where $\rho_1^{(0)}, \rho_2^{(0)}, \ldots, \rho_d^{(0)} \in \mathscr{T}(\mathcal{H})$. *Then, for all* $n \geq 1$, $\omega^{(n)}$ *has a representation of the following form:*

$$\omega^{(n)}(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d} \rho_j^{(n)}(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \tag{3.6}$$

where

$$\rho_j^{(n)} = \left(\prod_{k=0}^{n-1} \mathfrak{J}_k\right)\rho_j^{(0)} \tag{3.7}$$

for $j = 1, 2, \ldots, d$.

Proof. According to Lemma 3.1, $\rho_j^{(n)} \in \mathscr{T}(\mathcal{H})$ for each j with $1 \leq j \leq d$ and each $n \geq 0$. Thus, by Theorem 3.1, there exists a sequence $\left(\omega'^{(n)}\right)_{n\geq 0}$ in $\mathscr{T}^{(d)}(\mathcal{H})$ such that

$$\omega'^{(n)}(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d} \rho_j^{(n)}(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \ \ n \geq 0. \tag{3.8}$$

In particular, we have

$$\omega'^{(0)}(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d} \rho_j^{(0)}(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d,$$

which, together with the assumption given in (3.5), implies that $\omega^{(0)} = \omega'^{(0)}$.

On the other hand, for $n \geq 1$, by (3.7) we have $\rho_j^{(n)} = \mathfrak{J}_{n-1}\rho_j^{(n-1)}$, $1 \leq j \leq d$, which together with Lemma 3.1 implies that

$$\rho_j^{(n)}(x_j) = \sum_{\varepsilon_j \in \Lambda} B_{n-1}^{(\varepsilon_j)} \rho_j^{(n-1)}(x_j - \varepsilon_j) B_{n-1}^{(\varepsilon_j)}, \quad x_j \in \mathbb{Z}, \ \ 1 \leq j \leq d.$$

Here, $B_{n-1}^{(-1)} = L_{n-1}$ and $B_{n-1}^{(+1)} = R_{n-1}$ as indicated in Definition 2.1. Taking tensor product gives

$$\bigotimes_{j=1}^{d} \rho_j^{(n)}(x_j) = \sum_{\varepsilon \in \Lambda^d} \left(\bigotimes_{j=1}^{d} B_{n-1}^{(\varepsilon_j)}\right)\left(\bigotimes_{j=1}^{d} \rho_j^{(n-1)}(x_j - \varepsilon_j)\right)\left(\bigotimes_{j=1}^{d} B_{n-1}^{(\varepsilon_j)}\right),$$

$\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, $n \geq 1$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d)$. This, together with (3.8) and Definition 2.1, yields

$$\omega'^{(n)}(\mathrm{x}) = \sum_{\varepsilon \in \Lambda^d} C_{n-1}^{(\varepsilon)} \omega'^{(n-1)}(\mathrm{x} - \varepsilon) C_{n-1}^{(\varepsilon)}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \ \ n \geq 1,$$

namely

$$\omega'^{(n)} = \mathfrak{J}_{n-1}^{(d)}\omega'^{(n-1)}, \quad n \geq 1,$$

which together with (2.19) and $\omega^{(0)} = \omega'^{(0)}$ implies that $\omega^{(n)} = \omega'^{(n)}$ for all $n \geq 0$, which together with (3.8) gives (3.6). □

As an immediate consequence of the previous theorem, we have the following corollary, which gives a formula for calculating the probability distributions of the d-dimensional open QBN walk.

Corollary 3.1. *Let the nucleus $\omega^{(0)}$ of the initial state of the d-dimensional open QBN walk take the following form*:

$$\omega^{(0)}(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d} \rho_j^{(0)}(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \tag{3.9}$$

where $\rho_1^{(0)}, \rho_2^{(0)}, \ldots, \rho_d^{(0)} \in \mathscr{T}(\mathcal{H})$. Then, at time $n \geq 1$, the walk has a probability distribution of the following form:

$$\mathrm{Tr}\big[\omega^{(n)}(\mathrm{x})\big] = \prod_{j=1}^{d} \mathrm{Tr}\big[\rho_j^{(n)}(x_j)\big], \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \tag{3.10}$$

where

$$\rho_j^{(n)} = \left(\prod_{k=0}^{n-1} \mathfrak{J}_k\right)\rho_j^{(0)} \tag{3.11}$$

for $j = 1, 2, \ldots, d$.

3.2. *Limit probability distribution*

Let ρ be a 1-dimensional nucleus on $\mathcal{H}$, namely $\rho \in \mathscr{T}(\mathcal{H})$. A sequence $\big(\rho^{(n)}\big)_{n\geq 0}$ of 1-dimensional nucleuses on $\mathcal{H}$ is said to be generated by ρ and $\big(\mathfrak{J}_n\big)_{n\geq 0}$ if $\rho^{(0)} = \rho$ and

$$\rho^{(n+1)} = \mathfrak{J}_n\rho^{(n)}, \quad n \geq 0.$$

We note that if a sequence $\big(\rho^{(n)}\big)_{n\geq 0}$ is generated by ρ and $\big(\mathfrak{J}_n\big)_{n\geq 0}$, then $\rho^{(0)} = \rho$ and

$$\rho^{(n)} = \left(\prod_{k=0}^{n-1} \mathfrak{J}_k\right)\rho^{(0)}, \quad n \geq 1,$$

where $\prod_{k=0}^{n-1} \mathfrak{J}_k$ means the composition of mappings $\{\mathfrak{J}_k \,|\, 0 \leq k \leq n-1\}$.

Definition 3.2. A 1-dimensional nucleus ρ on $\mathcal{H}$ is said to be regular if the sequence $\big(\rho^{(n)}\big)_{n\geq 0}$ generated by ρ and $\big(\mathfrak{J}_n\big)_{n\geq 0}$ satisfies that

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}} \mathrm{e}^{\frac{\mathrm{i}tx}{\sqrt{n}}} \mathrm{Tr}\big[\rho^{(n)}(x)\big] = \mathrm{e}^{-\frac{t^2}{2}}, \quad t \in \mathbb{R}.$$

The next example shows that there exist infinitely many 1-dimensional nucleuses on $\mathcal{H}$ that are regular.

Example 3.1. Let $\sigma \in \Gamma$ and Z_σ the corresponding basis vector of the canonical ONB of $\mathcal{H}$. Define

$$\rho_\sigma(x) = \begin{cases} |Z_\sigma\rangle\langle Z_\sigma|, & x = 0; \\ 0, & x \neq 0, \ \ x \in \mathbb{Z}, \end{cases} \tag{3.12}$$

where $|Z_\sigma\rangle\langle Z_\sigma|$ is the Dirac operator associated with the basis vector Z_σ. Then ρ_σ is a regular 1-dimensional nucleus on $\mathcal{H}$.

Proof. Clearly, ρ_σ is a 1-dimensional nucleus on $\mathcal{H}$. Next, we show that it is also regular. To this end, we consider the space $l^2(\mathbb{Z}, \mathcal{H})$ of square summable $\mathcal{H}$-valued functions defined on $\mathbb{Z}$, which is endowed the usual inner product and norm. It can be verified (see [18] and references therein) that for each $n \geq 0$, there exists a unitary operator $\mathcal{U}_n$ on $l^2(\mathbb{Z}, \mathcal{H})$ such that

$$(\mathcal{U}_n\Phi)(x) = R_n\Phi(x-1) + L_n\Phi(x+1), \quad x \in \mathbb{Z}, \ \ \Phi \in l^2(\mathbb{Z}, \mathcal{H}). \tag{3.13}$$

Now, define $\Phi_n = \big(\prod_{k=0}^{n-1}\mathcal{U}_k\big)\Phi_0$, $n \geq 1$, where $\Phi_0 \in l^2(\mathbb{Z}, \mathcal{H})$ is taken as

$$\Phi_0(x) = \begin{cases} Z_\sigma, & x = 0; \\ 0, & x \neq 0, \ \ x \in \mathbb{Z}. \end{cases}$$

Then, by a result recently proven by Wang *et al.* (see [18] for details), we have

$$\|\Phi_n(x)\|^2 = \begin{cases} \dfrac{1}{2^n}\dbinom{n}{j}, & x = n-2j, \ \ 0 \leq j \leq n; \\ 0, & \text{otherwise}, \end{cases} \tag{3.14}$$

for all $n \geq 1$. On the other hand, let $\big(\rho^{(n)}\big)_{n\geq 0}$ be the 1-dimensional nucleus sequence on $\mathcal{H}$ generated by ρ_σ and $\big(\mathfrak{I}_n\big)_{n\geq 0}$. Then, by a careful check, we find that

$$\rho^{(0)}(x) = \rho_\sigma(x) = |\Phi_0(x)\rangle\langle\Phi_0(x)|, \quad x \in \mathbb{Z}.$$

This, together with [21, Theorem 4.2], as well as (3.14), implies that

$$\mathrm{Tr}[\rho^{(n)}(x)] = \begin{cases} \dfrac{1}{2^n}\dbinom{n}{j}, & x = n-2j, \ \ 0 \leq j \leq n; \\ 0, & \text{otherwise}. \end{cases} \tag{3.15}$$

Consequently, by a careful calculation, we finally come to

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}} \mathrm{e}^{\frac{itx}{\sqrt{n}}}\mathrm{Tr}\big[\rho^{(n)}(x)\big] = \lim_{n\to\infty}\cos^n\frac{t}{\sqrt{n}} = \mathrm{e}^{-\frac{t^2}{2}}, \quad t \in \mathbb{R},$$

which means that ρ_σ is regular. $\square$

The next theorem shows that, for a wide range of choices of its initial state, the d-dimensional open QBN walk has a limit probability distribution of d-dimensional Gauss type.

Theorem 3.3. *Let* $\big(\widetilde{\omega^{(n)}}\big)_{n\geq 0}$ *be the state sequence of the* d*-dimensional open QBN walk and* $\omega^{(n)}$ *the nucleus of* $\widetilde{\omega^{(n)}}$, $n \geq 0$. *Suppose that the nucleus* $\omega^{(0)}$ *of the initial state* $\widetilde{\omega^{(0)}}$ *takes the following form*:

$$\omega^{(0)}(\mathrm{x}) = \mathsf{K}\left(\bigotimes_{j=1}^{d} \rho_j^{(0)}(x_j)\right)\mathsf{K}^{-1}, \quad \mathrm{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d, \tag{3.16}$$

where $\big\{\rho_j^{(0)} \,|\, 1 \leq j \leq d\big\}$ *are regular* 1*-dimensional nucleuses on* $\mathcal{H}$. *For* $n \geq 0$, *let* X_n *be a* d*-dimensional random vector with probability distribution*

$$P\{X_n = \mathrm{x}\} = \mathrm{Tr}\big[\omega^{(n)}(\mathrm{x})\big], \quad \mathrm{x} \in \mathbb{Z}^d. \tag{3.17}$$

Then

$$\frac{X_n}{\sqrt{n}} \Rightarrow N(\mathbf{0}, I_{d\times d}),$$

namely $\frac{X_n}{\sqrt{n}}$ *converges in law to the* d*-dimensional standard Gauss distribution as* $n \to \infty$.

Proof. Let $n \geq 1$. Consider the characteristic function $C_{\frac{X_n}{\sqrt{n}}}(\mathbf{t})$ of the random vector $\frac{X_n}{\sqrt{n}}$. By definition, we have

$$C_{\frac{X_n}{\sqrt{n}}}(\mathbf{t}) = \sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{\frac{\mathrm{i}}{\sqrt{n}}\sum_{j=1}^{d} t_j x_j}\,\mathrm{Tr}\big[\omega^{(n)}(\mathrm{x})\big], \quad \mathbf{t} = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d, \tag{3.18}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$. Using Corollary 3.1 gives

$$C_{\frac{X_n}{\sqrt{n}}}(\mathbf{t}) = \prod_{j=1}^{d}\left(\sum_{x_j\in\mathbb{Z}} \mathrm{e}^{\frac{\mathrm{i}t_j x_j}{\sqrt{n}}}\,\mathrm{Tr}\big[\rho_j^{(n)}(x_j)\big]\right), \quad \mathbf{t} = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d,$$

where

$$\rho_j^{(n)} = \left(\prod_{k=0}^{n-1}\mathfrak{J}_k\right)\rho_j^{(0)}, \quad 1 \leq j \leq d.$$

Now, consider $\lim_{n\to\infty} C_{\frac{X_n}{\sqrt{n}}}(\mathbf{t})$. For each j, since $\rho_j^{(0)}$ is regular and $\big(\rho_j^{(n)}\big)_{n\geq 0}$ is generated by $\rho_j^{(0)}$ and $(\mathfrak{J}_n)_{n\geq 0}$, we have

$$\lim_{n\to\infty}\sum_{x_j\in\mathbb{Z}} \mathrm{e}^{\frac{\mathrm{i}t_j x_j}{\sqrt{n}}}\,\mathrm{Tr}\big[\rho_j^{(n)}(x_j)\big] = e^{-\frac{t_j^2}{2}}, \quad t_j \in \mathbb{R},$$

which implies that

$$\lim_{n\to\infty} C_{\frac{X_n}{\sqrt{n}}}(\mathbf{t}) = \prod_{j=1}^{d}\left(\lim_{n\to\infty}\sum_{x_j\in\mathbb{Z}} \mathrm{e}^{\frac{\mathrm{i}t_j x_j}{\sqrt{n}}}\,\mathrm{Tr}\big[\rho_j^{(n)}(x_j)\big]\right) = \prod_{j=1}^{d} e^{-\frac{t_j^2}{2}} = e^{-\frac{1}{2}\sum_{j=1}^{d} t_j^2},$$

$\mathbf{t} = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d$. Thus, $\frac{X_n}{\sqrt{n}}$ converges in law to the d-dimensional standard Gauss distribution as $n \to \infty$. □

4. Links with Unitary Quantum Walk

In this section, we show links between the d-dimensional open QBN walk and a unitary quantum walk recently introduced in [20].

As before, $d \geq 2$ is a given positive integer. We denote by $l^2(\mathbb{Z}^d, \mathcal{H})$ the space of square summable functions defined on $\mathbb{Z}^d$ and valued in $\mathcal{H}$, namely

$$l^2(\mathbb{Z}^d, \mathcal{H}) = \left\{ W : \mathbb{Z}^d \to \mathcal{H} \,\middle|\, \sum_{\mathrm{x} \in \mathbb{Z}^d} \|W(\mathrm{x})\|^2 < \infty \right\}, \tag{4.1}$$

where $\|\cdot\|$ is the norm in $\mathcal{H}$. It is known that $l^2(\mathbb{Z}^d, \mathcal{H})$ forms a separable Hilbert space with the inner product induced by the that in $\mathcal{H}$.

Recall that, for each $n \geq 0$, $\{C_n^{(\varepsilon)} \mid \varepsilon \in \Lambda^d\}$ are self-adjoint operators on $\mathcal{H}$ with properties that: $C_n^{(\varepsilon)} C_n^{(\varepsilon')} = 0$ for ε, $\varepsilon' \in \Lambda^d$ with $\varepsilon \neq \varepsilon'$; and their sum $\sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)}$ is a unitary operator on $\mathcal{H}$ (see Sec. 2.2 for details). Using these facts, we can prove that there exists a sequence of unitary operators $(\mathcal{U}_n^{(d)})_{n \geq 0}$ on $l^2(\mathbb{Z}^d, \mathcal{H})$ such that

$$(\mathcal{U}_n^{(d)} W)(\mathbf{x}) = \sum_{\varepsilon \in \Lambda^d} C_n^{(\varepsilon)} W(\mathbf{x} - \varepsilon), \quad \mathbf{x} \in \mathbb{Z}^d, \; W \in l^2(\mathbb{Z}^d, \mathcal{H}), \; n \geq 0. \tag{4.2}$$

With these unitary operators as the evolution operators, the authors of [20] introduced a unitary quantum walk on the d-dimensional integer lattice $\mathbb{Z}^d$ in the following manner.

Definition 4.1 ([20]). The d-dimensional QBN walk is a discrete-time unitary quantum walk on the d-dimensional integer lattice $\mathbb{Z}^d$ that satisfies the following requirements:

- Its states are represented by unit vectors in space $l^2(\mathbb{Z}^d, \mathcal{H})$.
- The time evolution of the walk is governed by equation

$$W_{n+1} = \mathcal{U}_n^{(d)} W_n, \quad n \geq 0, \tag{4.3}$$

where $W_n \in l^2(\mathbb{Z}^d, \mathcal{H})$ denotes the state of the walk at time $n \geq 0$, in particular W_0 is the initial state and $\mathcal{U}_n^{(d)}$ is the unitary operator indicated in (4.2).

In that case, the function $\mathbf{x} \mapsto \|W_n(\mathbf{x})\|^2$ on $\mathbb{Z}^d$ is called the probability distribution of the walk at time $n \geq 0$, while the quantity $\|W_n(\mathbf{x})\|^2$ is the probability to find out the walker at position $\mathbf{x} \in \mathbb{Z}^d$ and time $n \geq 0$.

Remark 4.1. According to (4.2), which describes the definition of unitary operators $\mathcal{U}_n^{(d)}$, the evolution equation of the d-dimensional QBN walk can also be represented as

$$W_{n+1}(\mathrm{x}) = \sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)} W_n(\mathrm{x}-\varepsilon), \quad \mathrm{x}\in\mathbb{Z}^d,\ \ n\geq 0, \tag{4.4}$$

which is more convenient to use.

As is seen, the d-dimensional QBN walk is driven by the sequence $\big(\mathcal{U}_n^{(d)}\big)_{n\geq 0}$ of unitary operators. Hence, it belongs to the category of unitary quantum walks. In other words, it is indeed a unitary quantum walk. The next result shows links between the d-dimensional open QBN walk and the d-dimensional QBN walk.

Theorem 4.1. *Let* $\big(\widetilde{\omega^{(n)}}\big)_{n\geq 0}$ *be the state sequence of the* d*-dimensional open QBN walk, where* $\omega^{(n)}$ *is the nucleus of* $\widetilde{\omega^{(n)}}$. *Let* $\big(W_n\big)_{n\geq 0}$ *be the state sequence of the* d*-dimensional QBN walk. Suppose that*

$$\omega^{(0)}(\mathrm{x}) = |W_0(\mathrm{x})\rangle\langle W_0(\mathrm{x})|, \quad \mathrm{x}\in\mathbb{Z}^d, \tag{4.5}$$

where $|W_0(\mathrm{x})\rangle\langle W_0(\mathrm{x})|$ *is the Dirac operator associated with* $W_0(\mathrm{x})$. *Then, for all* $n\geq 0$, *it holds that*

$$\mathrm{Tr}\big[\omega^{(n)}(\mathrm{x})\big] = \|W_n(\mathrm{x})\|^2, \quad \mathrm{x}\in\mathbb{Z}^d. \tag{4.6}$$

Proof. We first recall some algebraic and analytical properties of operators $C_n^{(\varepsilon)}$. As is indicated above, for each $n\geq 0$, $\big\{C_n^{(\varepsilon)}\,|\,\varepsilon\in\Lambda^d\big\}$ are self-adjoint operators on $\mathcal{H}$ with the property that $C_n^{(\varepsilon)}C_n^{(\varepsilon')}=0$ for ε, $\varepsilon'\in\Lambda^d$ with $\varepsilon\neq\varepsilon'$, which implies that

$$\Bigg\|\sum_{\varepsilon\in\Lambda^d} C_n^{(\varepsilon)}\xi_\varepsilon\Bigg\|^2 = \sum_{\varepsilon\in\Lambda^d}\big\|C_n^{(\varepsilon)}\xi_\varepsilon\big\|^2 \tag{4.7}$$

whenever $\big\{\xi_\varepsilon\,|\,\varepsilon\in\Lambda^d\big\}\subset\mathcal{H}$. And moreover, it follows from (2.10) and Definition 2.1 that all the operators $\big\{C_n^{(\varepsilon)}\,|\,\varepsilon\in\Lambda^d, n\geq 0\big\}$ form a commutative family.

Now, we consider (4.6). Clearly, it holds for $n=0$. In the following, we let $n\geq 1$ and $\mathrm{x}\in\mathbb{Z}^d$. Using (4.4) and (4.7) gives

$$\|W_n(\mathrm{x})\|^2 = \sum_{\varepsilon^{(n-1)}\in\Lambda^d}\big\|C_{n-1}^{(\varepsilon^{(n-1)})}W_{n-1}\big(\mathrm{x}-\varepsilon^{(n-1)}\big)\big\|^2,$$

where, by using the commutativity of the family $\big\{C_k^{(\varepsilon)}\,|\,\varepsilon\in\Lambda^d, k\geq 0\big\}$ and (4.7), we have

$$\begin{aligned}&\big\|C_{n-1}^{(\varepsilon^{(n-1)})}W_{n-1}\big(\mathrm{x}-\varepsilon^{(n-1)}\big)\big\|^2\\ &\quad= \sum_{\varepsilon^{(n-2)}\in\Lambda^d}\big\|C_{n-1}^{(\varepsilon^{(n-1)})}C_{n-2}^{(\varepsilon^{(n-2)})}W_{n-2}\big(\mathrm{x}-\varepsilon^{(n-1)}-\varepsilon^{(n-2)}\big)\big\|^2.\end{aligned}$$

Thus

$$\|W_n(\mathrm{x})\|^2 = \sum_{\varepsilon^{(n-1)}\in\Lambda^d}\ \sum_{\varepsilon^{(n-2)}\in\Lambda^d}\big\|C_{n-1}^{(\varepsilon^{(n-1)})}C_{n-2}^{(\varepsilon^{(n-2)})}W_{n-2}\big(\mathrm{x}-\varepsilon^{(n-1)}-\varepsilon^{(n-2)}\big)\big\|^2.$$

It then follows by the induction that

$$\|W_n(\mathrm{x})\|^2 = \sum_{\varepsilon^{(n-1)}\in\Lambda^d} \cdots \sum_{\varepsilon^{(0)}\in\Lambda^d} \times \left\|C_{n-1}^{(\varepsilon^{(n-1)})} \cdots C_0^{(\varepsilon^{(0)})} W_0\big(\mathrm{x} - \varepsilon^{(n-1)} - \cdots - \varepsilon^{(0)}\big)\right\|^2. \qquad (4.8)$$

Similarly, repeatedly using the evolution relation (2.20) of the d-dimensional open QBN walk yields that

$$\omega^{(n)}(\mathrm{x}) = \sum_{\varepsilon^{(n-1)}\in\Lambda^d} \cdots \sum_{\varepsilon^{(0)}\in\Lambda^d} \times \prod_{j=0}^{n-1} C_{n-1-j}^{(\varepsilon^{(n-1-j)})} \omega^{(0)}\big(\mathrm{x} - \varepsilon^{(n-1)} - \cdots - \varepsilon^{(0)}\big) \prod_{j=0}^{n-1} C_j^{(\varepsilon^{(j)})}.$$

Taking the trace gives

$$\mathrm{Tr}\big[\omega^{(n)}(\mathrm{x})\big] = \sum_{\varepsilon^{(n-1)}\in\Lambda^d} \cdots \sum_{\varepsilon^{(0)}\in\Lambda^d} \times \mathrm{Tr}\left[\prod_{j=0}^{n-1} C_{n-1-j}^{(\varepsilon^{(n-1-j)})} \omega^{(0)}\big(\mathrm{x} - \varepsilon^{(n-1)} - \cdots - \varepsilon^{(0)}\big) \prod_{j=0}^{n-1} C_j^{(\varepsilon^{(j)})}\right]. \qquad (4.9)$$

On the other hand, for $\varepsilon^{(n-1)} \in \Lambda^d, \ldots, \varepsilon^{(0)} \in \Lambda^d$, by using the assumption (4.5) we find

$$\left\|C_{n-1}^{(\varepsilon^{(n-1)})} \cdots C_0^{(\varepsilon^{(0)})} W_0\big(\mathrm{x} - \varepsilon^{(n-1)} - \cdots - \varepsilon^{(0)}\big)\right\|^2 = \mathrm{Tr}\left[\prod_{j=0}^{n-1} C_{n-1-j}^{(\varepsilon^{(n-1-j)})} \omega^{(0)}\big(\mathrm{x} - \varepsilon^{(n-1)} - \cdots - \varepsilon^{(0)}\big) \prod_{j=0}^{n-1} C_j^{(\varepsilon^{(j)})}\right],$$

which, together with (4.8) and (4.9), implies that $\mathrm{Tr}\big[\omega^{(n)}(\mathrm{x})\big] = \|W_n(\mathrm{x})\|^2$. □

Remark 4.2. It should be mentioned that under the conditions given in Theorem 4.1, one has, in general, that

$$\omega^{(n)}(\mathrm{x}) \neq |W_n(\mathrm{x})\rangle\langle W_n(\mathrm{x})|, \quad \mathrm{x} \in \mathbb{Z}^d, \; n \geq 1. \qquad (4.10)$$

This suggests that the d-dimensional open QBN walk is mathematically different from the d-dimensional QBN walk.

However, from a perspective of physical realization, transition may happen between these two walks. As an immediate consequence of Theorem 4.1, the next corollary describes the limit case when such transition happens.

Corollary 4.1. *Let the initial state $\widetilde{\omega^{(0)}}$ of the d-dimensional open QBN walk be such that*

$$\widetilde{\omega^{(0)}} = \sum_{\mathrm{x}\in\mathbb{Z}^d} |\delta_{\mathrm{x}}\rangle\langle\delta_{\mathrm{x}}| \otimes |W_0(\mathrm{x})\rangle\langle W_0(\mathrm{x})|, \qquad (4.11)$$

where W_0 is the initial state of the d-dimensional QBN walk. Then, the d-dimensional open QBN walk has a limit probability distribution if and only if the d-dimensional QBN walk has a limit probability distribution. In that case, their limit probability distributions are identical.

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