

ENERGY BOUNDS CONDITION FOR INTERTWINING OPERATORS OF TYPES B , C , AND G_2 UNITARY AFFINE VERTEX OPERATOR ALGEBRAS

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ABSTRACT. The energy bounds condition for intertwining operators of unitary rational vertex operator algebras (VOAs) was studied, first by A. Wassermann for type A affine VOAs, and later by T. Loke for $c < 1$ Virasoro VOAs, and by V. Toledano-Laredo for type D affine VOAs. In this paper, we extend their results to affine VOAs of types B , C , and G_2 . As a consequence, the modular tensor categories of these unitary vertex operator algebras are unitary.

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INTRODUCTION

In algebraic quantum field theory, the energy bounds condition is a very natural functional analytic property satisfied by many examples. This property guarantees not only the preclosedness of smeared field operators but also the strong commutativity of causally disjoint ones when the energy bounds are small. In 2d conformal field theory (CFT), the energy bounds condition has been extensively studied for chiral fields (vertex operators), from which one can construct conformal nets, i.e., nets of von Neumann algebras on the unit circle S^1 satisfying important properties such as Haag duality. See for instance [GW84, BSM90, CKLW18].

It is quite natural to think about extending the analysis of the energy bounds condition to the *intertwining operators* of a unitary rational vertex operator algebra (VOA), since intertwining operators are the building blocks (the chiral halves) of full CFT field operators [MS88, HK07], the energy bounds of which are clearly expected. Such analysis was initiated in the seminal work of A. Wassermann [Was98], followed by the works of T. Loke [Loke94] and V. Toledano-Laredo [TL04]. In these works,

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certain energy bounds were established for a generating set of intertwining operators of unitary affine type A and type D VOAs, as well as $c < 1$ unitary Virasoro VOAs. These results are crucial for constructing finite index subfactors [Jon83, Lon89], unitary fusion categories, unitary modular tensor categories [Xu00, KLM01, KL04], and hence unitary 3d topological quantum field theories [Tur94] under the settings of algebraic quantum field theories and operator algebras. Moreover, these functional analytic properties also have important applications in VOAs, as can be seen in our recent works [Gui19, Gui17], which used these properties to prove the unitarity of the representation tensor categories of many unitary rational VOAs.

Our goal in this article is to prove the energy bounds condition for the intertwining operators of unitary affine VOAs of types B , C , and the exceptional type G_2 . As a consequence, we show the unitarity of the modular tensor categories for these VOAs (Theorem 6.1). Further applications to conformal nets and operator algebras will be given in future work [Gui18]. Roughly speaking, if W_i, W_j, W_k are unitary representations of a unitary VOA V , \mathcal{Y} is a type $\binom{W_k}{W_i W_j} = \binom{k}{i j}$ intertwining operator V , and $w^{(i)}$ is a homogeneous vector of W_i , we say that \mathcal{Y} satisfies r -th order energy bounds ($r \geq 0$) if the smeared intertwining operator

$$\mathcal{Y}(w^{(i)}, f) = \oint_{S^1} \mathcal{Y}(w^{(i)}, z) f(z) \frac{dz}{2i\pi}$$

is bounded by L_0^r and is also continuous with respect to the smooth function f under some Sobolev norm on $C^\infty(S^1)$. First order energy bounds are especially important, as they imply the strong commutativity of two unbounded operators commuting adjointly on a common invariant core. So, when possible, we also determine those intertwining operators satisfying first order energy bounds. Although the concrete analysis of each type is different from another, the main idea, which is called the compression principle in our paper, is the same as those in [Was98] and also in [Loke94, TL04], which we now describe.

Compression principle. In general, energy bounds are easier to establish for vertex operators than for intertwining operators. For example, for free fermion vertex superalgebras, affine VOAs, and Virasoro VOAs, one can use the Lie (super)algebra relations to deduce the 0-th or first order energy bounds for primary fields (cf. [BSM90, section 2]), and use normal ordering (or more generally, the Jacobi identity in the form (1.10)) to prove the energy bounds condition for any vertex operators. The energy bounds condition is also preserved when passing to vertex subalgebras. So one can indeed establish energy bounds for many examples of vertex operators. Unfortunately, these methods seem to not work for intertwining operators, as intertwining operators don't in general form an algebra.¹

However, since we already know that many vertex operators are energy-bounded, we can try to embed our VOA V in a larger unitary vertex (super)algebra \tilde{V} whose vertex operators are energy-bounded, choose irreducible (unitary) V -submodules W_i, W_j, W_k of the action $V \curvearrowright \tilde{V}$, and restrict the vertex operator $\tilde{\mathcal{Y}}$ of \tilde{V} onto W_i, W_j, W_k . Then, up to a multiplication by a monomial of the formal variable x , this restricted vertex operator, which is clearly energy-bounded, is an intertwining operator V . If one can show that any irreducible intertwining operator \mathcal{Y} of V

¹Perhaps the only exceptions are the intertwining operators of a Heisenberg VOA, which form a so-called generalized vertex algebra [DL93].

can be realized as such a compression of an energy-bounded vertex operator or, more generally, as a compression of an intertwining operator \mathfrak{Y} of a larger \tilde{V} , the energy bounds condition which has already been established, then we can prove the energy bounds condition for \mathcal{Y} . The precise statement of this compression principle, at least for affine VOAs, can be found in Theorem 2.12.

Examples. So far, the compression principle is the only essential way to obtain the energy bounds condition for intertwining operators. Let us now explain how this principle can be applied to all known examples. First, for the intertwining operators of unitary $c < 1$ Virasoro VOAs, one can deduce the energy bounds condition from that of the intertwining operators of affine \mathfrak{sl}_2 VOAs, since the coset construction of Goddard-Kent-Olive [GKO86] realizes any $c < 1$ Virasoro VOA as a unitary vertex subalgebra of a tensor product of two affine \mathfrak{sl}_2 VOAs [Loke94].

Unitary affine VOAs are the examples that we are especially interested in in this paper. Let \mathfrak{g} be a unitary complex simple Lie algebra, and let $l = 1, 2, \dots$. Then it is well-known that the level l affine \mathfrak{g} -VOA $V_{\mathfrak{g}}^l$ can be embedded as a unitary vertex subalgebra of $(V_{\mathfrak{g}}^1)^{\otimes l}$ (Proposition 2.7). Moreover, when \mathfrak{g} is a classical Lie algebra or of type G_2 , it can be shown that any irreducible intertwining operator of $V_{\mathfrak{g}}^l$ is a compression of one of $(V_{\mathfrak{g}}^1)^{\otimes l}$. So for such \mathfrak{g} , it suffices to prove the energy bounds condition for the intertwining operators of the level 1 affine VOA $V_{\mathfrak{g}}^1$.² And when the level is 1, all classical Lie types can be reduced to type D due to the diagonal embeddings $\mathfrak{sl}_n \subset \mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n} \subset \mathfrak{so}_{4n}$, and the obvious embedding $\mathfrak{so}_{2n+1} \subset \mathfrak{so}_{2n+2}$, whose levels (i.e., Dynkin indexes; see section 2.1) are all 1. For the type G_2 simple Lie algebra \mathfrak{g}_2 , one should consider the embedding $\mathfrak{g}_2 \subset \mathfrak{e}_8$,³ whose Dynkin index is also 1, and realize the intertwining operators of $V_{\mathfrak{g}_2}^1$ as the compressions of the vertex operator (but not more generally intertwining operators) of $V_{\mathfrak{e}_8}^1$, the energy bounds condition of which is clearly known, as mentioned previously.

It seems that the only possible way of solving the type D level 1 case is through Heisenberg and lattice VOAs, as was carried out in [TL04, chapters V, VI]. As every classical Lie type can (or should) be reduced to this case, we feel it necessary to explain the key ideas of the proof. This method indeed works for any simply laced Lie algebra. So let us assume that \mathfrak{g} is of type A , D , or E . Then by Frenkel-Kac construction [FK80] (see also [Kac98, section 5.6]), the corresponding level 1 affine VOA is unitarily equivalent to the even lattice VOA whose lattice is the root lattice of \mathfrak{g} . So the problem is turned to show the energy bounds condition for (the vertex operators and) the intertwining operators of even lattice VOAs.

It is impossible to discuss lattice VOAs without a full understanding of their ‘‘Cartan subalgebras’’, the Heisenberg VOAs. Let \mathfrak{h} be a unitary (finite-dimensional) abelian Lie algebra. One can define, for any $l > 0$, a level l Heisenberg VOA $V_{\mathfrak{h}}^l$ in a similar way as affine VOAs are defined. The most striking differences between Heisenberg and affine VOAs are: (1) $V_{\mathfrak{h}}^l$ is irrelevant to the level l . So we can always choose $l = 1$. (2) The level adds no constraints on the irreducible representations of $V_{\mathfrak{h}}^1$. So the equivalence classes of the unitary irreducible representations of $V_{\mathfrak{h}}^1$ are in one-to-one correspondence with the elements of the dual space $(i\mathfrak{h}_{\mathbb{R}})^*$ of the

²In fact, we expect that this statement holds for all simple Lie algebras except type E_8 . See our discussion in section 6.

³The author would like to thank Marcel Bischoff for suggesting thinking about the embedding $\mathfrak{g}_2 \subset \mathfrak{e}_8$. All the embeddings mentioned in this paragraph were also suggested in [Was90].

self-adjoint (real) subspace $i\mathfrak{h}_{\mathbb{R}}$ of \mathfrak{h} . So the Heisenberg VOA $V_{\mathfrak{h}}^1$ is not a rational VOA. However, we can still study the “tensor category” of $V_{\mathfrak{h}}^1$ in a broader sense, as the fusion rules are always finite and the intertwining operators of $V_{\mathfrak{h}}^1$ actually satisfy some very simple fusion and (abelian) braid relations (cf. [DL93]; see also Appendix A). With the help of this tensor categorical structure, one can “rationalize” $V_{\mathfrak{h}}^1$ by extending it to a larger VOA using (irreducible) intertwining operators of $V_{\mathfrak{h}}^1$ whose charge spaces, source spaces, and target spaces all correspond to elements inside a non-degenerate even lattice Λ in $i\mathfrak{h}_{\mathbb{R}} \simeq (i\mathfrak{h}_{\mathbb{R}})^*$. This VOA V_{Λ} is the lattice VOA for Λ . The explicit form of an intertwining operator \mathfrak{Y} of V_{Λ} can be determined, which shows that \mathfrak{Y} must be a (globally infinite but locally finite) “homogeneous” sum of intertwining operators of the Heisenberg VOA $V_{\mathfrak{h}}^1$, so that the analytic behaviors of \mathfrak{Y} depend homogeneously on those of the intertwining operators of $V_{\mathfrak{h}}^1$ (Theorem A.5). Therefore, to study the energy bounds condition for the intertwining operators of lattice VOAs, it suffices to do this for any irreducible intertwining operator \mathcal{Y} of $V_{\mathfrak{h}}^1$. We assume that \mathcal{Y} is of type $\begin{pmatrix} \nu \\ \lambda \mu \end{pmatrix}$, where $\lambda, \mu, \nu \in (i\mathfrak{h}_{\mathbb{R}})^*$. Then, when $(\lambda|\lambda) = 1$, \mathcal{Y} can be regarded as a compression of the vertex operator of the lattice vertex superalgebra $V_{\mathbb{Z}^n}$ for the integral lattice \mathbb{Z}^n . Since, by Boson-Fermion correspondence, $V_{\mathbb{Z}^n}$ is equivalent to the vertex superalgebra for the n -dimensional free fermion, one can show that \mathcal{Y} satisfies 0-th order energy bounds. When $(\lambda|\lambda)$ takes general values, a tensor product argument due to [TL04] can be applied to prove the energy bounds condition for \mathcal{Y} . Thus the energy bounds conditions for Heisenberg VOAs and lattice VOAs are established. See Appendix A for more details.

Outline of this paper. Section 1 aims to fix the notation and provide the necessary background for establishing and understanding the compression principle and for analyzing the energy bounds condition for intertwining operators. In particular, we state the basic theories of unitary VOAs, affine Lie algebras, and finite-dimensional simple Lie algebras with a unifying language emphasizing their unitary properties. So, for example, the invariant *inner products* of complex simple Lie algebras are introduced at the very beginning of our discussion of Lie algebras, and the real structures of root systems and weight spaces become very natural from this point of view. On the other hand, the notion of invariant *bilinear forms* is completely avoided in our paper, as the author believes that this concept is, at some point, inconvenient for one to understand the unitary structures. In some sense our philosophy is similar to that of [Was10].

Section 2 gives a detailed account of the compression principle. In section 2.1 we discuss how a given unitary affine VOA $V_{\mathfrak{g}}^l$ can be realized as a unitary vertex subalgebra of a larger unitary VOA V . In section 2.2 we state the compression principle and show how this principle can be used to reduce higher level problems to level 1 problems.

After the presentation of general theories, in sections 3, 4, and 5 we apply the compression principle to establish the energy bounds condition for the intertwining operators of types B , C , and G_2 affine Lie algebras. Conclusions and future perspectives are given in section 6.

As mentioned earlier, the results on all classical Lie type affine VOAs depend on those of type D , which in turn depend on those of the intertwining operators of Heisenberg and lattice VOAs. Although the latter is implicitly contained in [TL04],

we feel it necessary to give a brief account of this theory in our paper, since, after all, the results in [TL04] are explicitly stated only for the lattice VOAs V_Λ where Λ is the root lattice of a simple Lie algebra. Moreover, [TL04] uses the notion of “primary fields” rather than intertwining operators, and the conditions that primary fields need to satisfy are, at first glance, much weaker than those on intertwining operators.⁴ The reason that we avoid primary fields in our paper is clear: it is hard to generalize them onto VOAs beyond affine and Virasoro ones. Therefore, as a general theory of even lattice VOAs, the account in [TL04, chapters V, VI] is incomplete and should be accompanied by the results in [DL93]. We will discuss this in Appendix A.

1. BACKGROUND

1.1. Unitary VOAs.

Unitary VOAs and unitary representations. We assume the reader is familiar with the basic concepts and computations in VOA, such as those in [FHL93], [Kac98], or [LL04].

Let (V, Y, Ω, ν) (or simply V) be a VOA, where Ω is the vacuum vector and ν is the conformal vector. For any vector $v \in V$ and a formal variable x , we write $Y(v, x) = \sum_{n \in \mathbb{Z}} Y(v, n)x^{-n-1}$, where each $Y(v, n) \in \text{End}(V)$ is a mode of the field operator $Y(v, x)$. Let $\{L_n = Y(v, n+1)\}$ be the Virasoro operators of V . The vector space V has grading $V = \bigoplus_{n \in \mathbb{Z}} V(n)$, where $L_0|_{V(n)} = n \cdot \text{id}_{V(n)}$. We assume that V is of CFT type; i.e., $V(0) = \mathbb{C}\Omega$ and $V(n) = 0$ when $n < 0$.

Let Θ be an antilinear automorphism of V . This means that $\Theta : V \rightarrow V$ is antilinear, that Ω and ν are fixed by Θ , and that $\Theta Y(v, x) = Y(\Theta v, x)\Theta$ for any $v \in V$. We say that $(V, Y, \Omega, \nu, \Theta)$ (or simply V) is a **unitary VOA** if there exists an inner product $\langle \cdot | \cdot \rangle$ on V , antilinear on the second variable, such that for any $v, v_1, v_2 \in V$,

$$(1.1) \quad \langle Y(v, x)v_1 | v_2 \rangle = \langle v_1 | Y(e^{xL_1}(-x^{-2})^{L_0}\Theta v, x^{-1})v_2 \rangle.$$

The above relation can be simply written as

$$Y(v, x)^\dagger = Y(e^{xL_1}(-x^{-2})^{L_0}\Theta v, x^{-1}),$$

where $Y(v, x)^\dagger$ means the formal adjoint of $Y(v, x)$. In particular, if we let $v = \nu$ and note that $L_1\nu = 0$, we have

$$L_n^\dagger = L_{-n} \quad (n \in \mathbb{Z}).$$

Note that $\Omega \in V$ is a cyclic vector under the action of vertex operators and that $V(0)$ is spanned by Ω . Therefore, the inner product $\langle \cdot | \cdot \rangle$ is uniquely determined by the positive value $\langle \Omega | \Omega \rangle$. If we **normalize** $\langle \cdot | \cdot \rangle$ so that $\langle \Omega | \Omega \rangle = 1$, then $\langle \cdot | \cdot \rangle$ is unique, or, more precisely, $\langle \cdot | \cdot \rangle$ is uniquely determined by Θ and ν . We call Θ the **PCT operator** of V . It is not hard to show that Θ is an anti-unitary map (cf. [CKLW18, Proposition 5.1]). One can also check easily that any unitary CFT type VOA is simple.

The invariant inner product on V also determines the conformal vector ν and the PCT operator Θ .

⁴They are indeed equivalent at least for affine VOAs, but the proofs are non-trivial. See [Gui17, section 8.2] for an explanation of this issue and the reference therein.

Proposition 1.1. *Suppose that $(V, Y, \Omega, \nu, \Theta)$ and $(V, Y, \Omega, \tilde{\nu}, \tilde{\Theta})$ are unitary VOAs of CFT type having the same normalized invariant inner product $\langle \cdot | \cdot \rangle$. Then $\nu = \tilde{\nu}$ and $\Theta = \tilde{\Theta}$.*

Proof (Cf. [CKLW18, Proposition 4.8]). Let $Y(\tilde{\nu}, x) = \sum_{n \in \mathbb{Z}} \tilde{L}_n x^{-n-2}$. If we can show that $L_{-2} = \tilde{L}_{-2}$, then $\nu = Y(\nu, -1)\Omega = L_{-2}\Omega = \tilde{L}_{-2}\Omega = Y(\tilde{\nu}, -1)\Omega = \tilde{\nu}$. So $L_n = \tilde{L}_n$ for any $n \in \mathbb{Z}$. Now for any $v, v_1, v_2 \in V$,

$$\begin{aligned} \langle Y(\Theta^{-1}v, x)v_1 | v_2 \rangle &= \langle v_1 | Y(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})v_2 \rangle \\ &= \langle v_1 | Y(e^{x\tilde{L}_1}(-x^{-2})^{\tilde{L}_0}v, x^{-1})v_2 \rangle = \langle Y(\tilde{\Theta}^{-1}v, x)v_1 | v_2 \rangle, \end{aligned}$$

showing that $\Theta = \tilde{\Theta}$.

We now show that $L_{-2} = \tilde{L}_{-2}$. By the translation property, for any $v \in V$ we have

$$Y(L_{-1}v, x) = \frac{d}{dx}Y(v, x) = Y(\tilde{L}_{-1}v, x).$$

The state-field correspondence now gives $L_1v = \tilde{L}_{-1}v$. Hence $L_{-1} = \tilde{L}_{-1}$. Taking the adjoint of both sides under the same inner product $\langle \cdot | \cdot \rangle$ corresponding to $(V, Y, \Omega, \nu, \Theta)$ and $(V, Y, \Omega, \tilde{\nu}, \tilde{\Theta})$ gives us $L_1 = \tilde{L}_1$. Hence, by the Virasoro relation, $2L_0 = [L_1, L_{-1}] = [\tilde{L}_1, \tilde{L}_{-1}] = 2\tilde{L}_0$. We conclude that $L_n = \tilde{L}_n$ when $n = -1, 0, 1$.

Using the Jacobi identity (e.g. (1.11)), we have

$$(1.2) \quad [L_{-2}, \tilde{L}_0] = [Y(\nu, -1), Y(\tilde{\nu}, 1)] = \sum_{l \geq 0} \binom{-1}{l} Y(Y(\nu, l)\tilde{\nu}, -l) = \sum_{n \geq 0} (-1)^l Y(L_{l-1}\tilde{\nu}, -l).$$

We now compute each summand on the right hand side. By the translation property, $Y(L_{-1}\tilde{\nu}, 0) = 0$. Also, $Y(L_0\tilde{\nu}, -1) = Y(\tilde{L}_0\tilde{\nu}, -1) = 2Y(\tilde{\nu}, -1) = 2\tilde{L}_{-2}$. Since $L_1\tilde{\nu} = \tilde{L}_1\tilde{\nu} = 0$, we have $Y(L_1\tilde{\nu}, -2) = 0$. For any $l \in \mathbb{Z}$, $L_0L_l\tilde{\nu} = [L_0, L_l]\tilde{\nu} + L_lL_0\tilde{\nu} = -lL_l\tilde{\nu} + L_l\tilde{L}_0\tilde{\nu} = (2-l)L_l\tilde{\nu}$. So $L_l\tilde{\nu} \in V(2-l)$. In particular, $L_2\tilde{\nu} \in V(0) = \text{Span}_{\mathbb{C}}(\Omega)$, and hence $Y(L_2\tilde{\nu}, 3)$ is proportional to $Y(\Omega, 3)$, which is 0. When $l > 2$, $V(2-l) = 0$. So $L_l\tilde{\nu} = 0$. Hence we conclude that the right hand side of equation (1.2) equals $-2\tilde{L}_{-2}$. On the other hand, the Virasoro relation implies that $[L_{-2}, \tilde{L}_0] = [L_{-2}, L_0] = -2L_{-2}$. Therefore $L_{-2} = \tilde{L}_{-2}$, which finishes our proof. \square

Corollary 1.2. *If $(V, Y, \Omega, \nu, \Theta)$ is a unitary VOA of CFT type, then Θ is an involution, i.e., $\Theta^2 = \text{id}_V$.*

Proof. Using the commuting relation for L_0 and L_1 , it is not hard to show that (1.1) also holds when Θ is replaced by Θ^{-1} . So $(V, Y, \Omega, \nu, \Theta^{-1})$ is also a unitary VOA of CFT type, and it has the same normalized invariant inner product as that of $(V, Y, \Omega, \nu, \Theta)$. So by our last proposition, $\Theta = \Theta^{-1}$. \square

Let V and \tilde{V} be VOAs with vacuum vectors $\Omega, \tilde{\Omega}$ and conformal vectors $\nu, \tilde{\nu}$, respectively. We also let Y, \tilde{Y} be their vertex operators. We say that V is a **vertex subalgebra** of \tilde{V} if there exists an injective linear map $\varphi : V \rightarrow \tilde{V}$ such that:

- (a) $\varphi\Omega = \tilde{\Omega}$.
- (b) $\varphi Y(v, x) = \tilde{Y}(\varphi v, x)\varphi$ for any $v \in V$.

(c) $\varphi(V)$ is a *graded* subspace of \tilde{V} .

In this case we may identify V with $\varphi(V)$ and regard V itself as a graded subspace of \tilde{V} . Note that we do *not* require that $\varphi\nu = \tilde{\nu}$. If V and \tilde{V} are unitary VOAs with PCT operators $\Theta, \tilde{\Theta}$, we say that V is a **unitary vertex subalgebra** of \tilde{V} if besides conditions (a), (b), (c) we also have:

(d) $\varphi\Theta = \tilde{\Theta}\varphi$.

(e) φ is an isometry; i.e., $\langle u|v \rangle = \langle \varphi u|\varphi v \rangle$ for any $u, v \in V$.

Remark 1.3. Assume that V is a unitary vertex subalgebra of \tilde{V} . We regard V as a subspace of \tilde{V} . So $Y(v, x) = \tilde{Y}(v, x)$ for any $v \in V$. If we let $\{L_n\}$ and $\{\tilde{L}_n\}$ be the Virasoro operators of V and \tilde{V} , respectively, then, as an easy consequence of the translation property and creation property, we must have

$$(1.3) \quad L_{-1}v = \tilde{L}_{-1}v \quad (\forall v \in V).$$

Convention 1.4. In this article, we always assume, unless otherwise stated, that V is a unitary VOA of CFT type.

Representations (modules) of V will be denoted by W_i, W_j, W_k , etc. If W_i is a V module, we let Y_i be the corresponding vertex operator; i.e., $Y_i(v, x) = \sum_{n \in \mathbb{Z}} Y_i(v, n)x^{-n-1}$ describes the action of V on W_i . Assume that W_i is equipped with an inner product $\langle \cdot | \cdot \rangle$. We say that $(W_i, \langle \cdot | \cdot \rangle)$ (or simply W_i) is a **unitary representation** of V if for any $v \in V$,

$$(1.4) \quad Y_i(v, x)^\dagger = Y_i(e^{xL_1}(-x^{-2})^{L_0}\Theta v, x^{-1}).$$

A vector $w^{(i)} \in W_i$ is called **homogeneous** if it is an eigenvector of L_0 , and we call its eigenvalue $\Delta_{w^{(i)}}$ the **conformal weight** of $w^{(i)}$. A homogeneous vector $w^{(i)} \in W_i$ is called **quasi-primary** if $L_1w^{(i)} = 0$. If $v \in V$ is quasi-primary, then equation (1.4) is simplified to

$$(1.5) \quad Y_i(v, x)^\dagger = (-1)^{\Delta_v}x^{-2\Delta_v}Y_i(\Theta v, x^{-1}).$$

Taking $v = \nu$, we obtain

$$(1.6) \quad L_n^\dagger = L_{-n}$$

when acting on W_i . So the action of V on W_i restricts to a unitary representation of Virasoro algebra. In particular, *the eigenvalues of $L_0 \curvearrowright W_i$ are non-negative* (cf., for example, [Gui19, Proposition 1.7]). So L_0 induces grading $W_i = \bigoplus_{s \geq 0} W_i(s)$.

If W_i is a unitary V -module, we let the vector space $W_{\bar{i}}$ be the complex conjugate of W_i and let $C_i : W_i \rightarrow W_{\bar{i}}$ be the antilinear isomorphism. We equip $W_{\bar{i}}$ with an inner product under which C_i becomes anti-unitary. The action of V on W_i is described by the vertex operator

$$Y_{\bar{i}}(v, x) = C_i Y_i(\Theta v, x) C_i^{-1}.$$

Clearly $W_{\bar{i}}$ is also a unitary representation of V . We call $W_{\bar{i}}$ the **contragredient representation (module)** of W_i .⁵

Note that the action of V on V is also a unitary representation, called the vacuum representation of V , which is also denoted by W_0 . We also identify $W_{\bar{0}}$ with V through the isomorphism $C_0\Theta : V = W_0 \rightarrow W_{\bar{0}}$. So the vacuum representation is self-dual.

⁵Our definition of contragredient modules is equivalent to the one given in [FHL93]. See [Gui19, equation (1.19)].

Remark 1.5. Suppose that V, \tilde{V} are unitary VOAs of CFT type, V is a unitary vertex subalgebra of \tilde{V} , and $(W_{\tilde{V}}, Y_{\tilde{V}})$ is a unitary representation of \tilde{V} . Then $(W_{\tilde{V}}, Y_{\tilde{V}})$ is also a unitary representation of V , for the Jacobi identity clearly holds, and the translation property follows from relation (1.3).

Intertwining operators. For any complex vector space U we set

$$(1.7) \quad U((x)) = \left\{ \sum_{n \in \mathbb{Z}} u_n x^n : u_n \in U, u_n = 0 \text{ for sufficiently small } n \right\},$$

$$(1.8) \quad U\{x\} = \left\{ \sum_{s \in \mathbb{R}} u_s x^s : u_s \in U \right\}.$$

Now let W_i, W_j, W_k be unitary V -modules. A type $\binom{W_k}{W_i W_j}$ (or type $\binom{k}{i j}$) **intertwining operator** \mathcal{Y}_α is a linear map

$$\begin{aligned} W_i &\rightarrow (\text{Hom}(W_j, W_k))\{x\}, \\ w^{(i)} &\mapsto \mathcal{Y}_\alpha(w^{(i)}, x) = \sum_{s \in \mathbb{R}} \mathcal{Y}_\alpha(w^{(i)}, s) x^{-s-1} \\ &\quad (\text{where } \mathcal{Y}_\alpha(w^{(i)}, s) \in \text{Hom}(W_j, W_k)), \end{aligned}$$

such that:

- (a) (Lower truncation) For any $w^{(j)} \in W_j$, $\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)} = 0$ for s sufficiently large.
- (b) (Jacobi identity) For any $u \in V, w^{(i)} \in W_i, m, n \in \mathbb{Z}, s \in \mathbb{R}$, we have

$$(1.9) \quad \begin{aligned} &\sum_{l \in \mathbb{Z}_{\geq 0}} \binom{m}{l} \mathcal{Y}_\alpha(Y_i(u, n+l)w^{(i)}, m+s-l) \\ &= \sum_{l \in \mathbb{Z}_{\geq 0}} (-1)^l \binom{n}{l} Y_k(u, m+n-l) \mathcal{Y}_\alpha(w^{(i)}, s+l) \\ &\quad - \sum_{l \in \mathbb{Z}_{\geq 0}} (-1)^{l+n} \binom{n}{l} \mathcal{Y}_\alpha(w^{(i)}, n+s-l) Y_j(u, m+l). \end{aligned}$$

- (c) (Translation property) $\frac{d}{dx} \mathcal{Y}_\alpha(w^{(i)}, x) = \mathcal{Y}_\alpha(L_{-1}w^{(i)}, x)$.

W_i, W_j, W_k are called the **charge space**, the **source space**, and the **target space** of \mathcal{Y}_α , respectively. If all three of these V -modules are irreducible, then \mathcal{Y}_α is called **irreducible**.

By setting $m = 0$ in the Jacobi identity, we obtain

$$(1.10) \quad \begin{aligned} \mathcal{Y}_\alpha(Y_i(u, n)w^{(i)}, s) &= \sum_{l \in \mathbb{Z}_{\geq 0}} (-1)^l \binom{n}{l} Y_k(u, n-l) \mathcal{Y}_\alpha(w^{(i)}, s+l) \\ &\quad - \sum_{l \in \mathbb{Z}_{\geq 0}} (-1)^{l+n} \binom{n}{l} \mathcal{Y}_\alpha(w^{(i)}, n+s-l) Y_j(u, l). \end{aligned}$$

If we let $n = 0$ instead, then the Jacobi identity becomes

$$(1.11) \quad Y_k(u, m) \mathcal{Y}_\alpha(w^{(i)}, s) - \mathcal{Y}_\alpha(w^{(i)}, s) Y_j(u, m) = \sum_{l \in \mathbb{Z}_{\geq 0}} \binom{m}{l} \mathcal{Y}_\alpha(Y_i(u, l)w^{(i)}, m+s-l)$$

or, equivalently,

(1.12)

$$Y_k(u, m)\mathcal{Y}_\alpha(w^{(i)}, x) - \mathcal{Y}_\alpha(w^{(i)}, x)Y_j(u, m) = \sum_{l \in \mathbb{Z}_{\geq 0}} \binom{m}{l} \mathcal{Y}_\alpha(Y_i(u, l)w^{(i)}, x)x^{m-l}.$$

Choose $u = \nu$ (the conformal vector of V) and $m = 1$. Then we have $[L_0, \mathcal{Y}_\alpha(w^{(i)}, x)] = \mathcal{Y}_\alpha(L_0w^{(i)}, x) + x\mathcal{Y}_\alpha(L_{-1}w^{(i)}, x)$. Therefore, the translation property is equivalent to

$$(1.13) \quad [L_0, \mathcal{Y}_\alpha(w^{(i)}, x)] = \mathcal{Y}_\alpha(L_0w^{(i)}, x) + x \frac{d}{dx} \mathcal{Y}_\alpha(w^{(i)}, x)$$

or, written in terms of modes,

$$(1.14) \quad [L_0, \mathcal{Y}_\alpha(w^{(i)}, s)] = \mathcal{Y}_\alpha(L_0w^{(i)}, s) - (s + 1)\mathcal{Y}_\alpha(w^{(i)}, s) \quad (s \in \mathbb{R}).$$

So, when $w^{(i)} \in W_i$ is homogeneous with conformal weight $\Delta_{w^{(i)}}$, the linear operator $\mathcal{Y}_\alpha(w^{(i)}, s) : W_j \rightarrow W_k$ raises the conformal weights by $\Delta_{w^{(i)}} - s - 1$. From this we conclude that (1.14) itself implies the lower truncation property. Therefore:

Proposition 1.6. *A linear map $\mathcal{Y}_\alpha : W_i \rightarrow (\text{Hom}(W_j, W_k))\{x\}$ is an intertwining operator of V if and only if relation (1.13) and the Jacobi identity (1.9) (both sides of which automatically converge) hold.*

Relation (1.13) will also be called the translation property.

Let $\mathcal{V} \binom{k}{i j}$ be the vector space of type $\binom{k}{i j}$ intertwining operators. Then, clearly, the vertex operator Y_i for W_i should be inside $\mathcal{V} \binom{i}{0 i}$. In particular, $Y \in \mathcal{V} \binom{0}{0 0}$, where Y is the vertex operator describing the action of V on V . Define the **fusion rule**

$$N_{ij}^k = \dim \mathcal{V} \binom{k}{i j}.$$

Then clearly $N_{0i}^i = 1$ when W_i is irreducible. In particular, $N_{00}^0 = 1$.

In this article, we are interested in those VOAs having finite fusion rules and finitely many equivalence classes of irreducible representations. For any such VOA V , it will also be useful to find a finite set \mathcal{F} of irreducible representations which tensor-generates the tensor category of V . More precisely:

Definition 1.7. Let $\{W_i : i \in \mathcal{F}\}$ be a finite set of irreducible V -modules with \mathcal{F} the set of indexes. With abuse of notation we also let \mathcal{F} denote this set. Let $\overline{\mathcal{F}} = \{W_{\bar{i}} : i \in \mathcal{F}\}$ be the set of irreducible V -modules contragredient to those of \mathcal{F} . We say that \mathcal{F} is **generating** if for any irreducible V -module W_k , there exist $j_1, i_1, i_2, \dots, i_n \in \mathcal{F} \cup \overline{\mathcal{F}}$ and irreducible V -modules W_{j_2}, \dots, W_{j_n} such that the vector spaces $\mathcal{V} \binom{j_2}{i_1 j_1}, \mathcal{V} \binom{j_3}{i_2 j_2}, \dots, \mathcal{V} \binom{j_n}{i_{n-1} j_{n-1}}, \mathcal{V} \binom{k}{i_n j_n}$ are non-trivial.

For any $\mathcal{Y}_\alpha \in \mathcal{V} \binom{k}{i j}$, we define its **adjoint intertwining operator** $(\mathcal{Y}_\alpha)^\dagger \equiv \mathcal{Y}_{\alpha^*} \in \mathcal{V} \binom{j}{\bar{i} k}$ such that for any $w^{(i)} \in W_i$ (recall that $C_i w^{(i)} \in W_{\bar{i}}$)

$$(1.15) \quad \mathcal{Y}_{\alpha^*}(C_i w^{(i)}, x) = \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi x^{-2}})^{L_0} w^{(i)}, x^{-1})^\dagger.$$

More precisely, for any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(k)} \in W_k$,

$$(1.16) \quad \langle \mathcal{Y}_{\alpha^*}(C_i w^{(i)}, x)w^{(k)} | w^{(j)} \rangle = \langle w^{(k)} | \mathcal{Y}_\alpha(e^{xL_1}(e^{-i\pi x^{-2}})^{L_0} w^{(i)}, x^{-1})w^{(j)} \rangle.$$

The notation $(e^{-i\pi}x^{-2})^{L_0}$ is understood as follows. If $w^{(i)}$ is homogeneous with conformal weight $\Delta_{w^{(i)}}$, then we set $(e^{-i\pi}x^{-2})^{L_0}w^{(i)} = e^{-i\pi\Delta_{w^{(i)}}}x^{-2\Delta_{w^{(i)}}}w^{(i)}$. The general case is defined using linearity. To prove that (1.15) actually defines an intertwining operator of V , one can use the argument in [FHL93, section 5.2].

Note that the unitarity of W_i implies that Y_i equals its adjoint intertwining operator.

One can also define the **braided intertwining operators** ([FHL93, chapter 5]) $\mathcal{Y}_{B_+\alpha}, \mathcal{Y}_{B_-\alpha} \in \mathcal{V}\left(\begin{smallmatrix} k \\ j \ i \end{smallmatrix}\right)$ of \mathcal{Y}_α using the formula

$$(1.17) \quad \mathcal{Y}_{B_\pm\alpha}(w^{(j)}, x)w^{(i)} = e^{xL_{-1}}\mathcal{Y}_\alpha(w^{(i)}, e^{\pm i\pi}x)w^{(j)} \quad (\forall w^{(i)} \in W_i, w^{(j)} \in W_j).$$

The two braided intertwining operators of the vertex operator $Y_i \in \mathcal{V}\left(\begin{smallmatrix} i \\ 0 \ i \end{smallmatrix}\right)$ are equal. We denote it by $\mathcal{Y}_{\kappa(i)}$ and call it the **creation operator** of W_i . Clearly $\mathcal{Y}_{\kappa(i)}$ is of type $\left(\begin{smallmatrix} i \\ i \ 0 \end{smallmatrix}\right)$.

It is obvious that the map $\mathcal{Y}_{\bar{\alpha}} : W_{\bar{i}} \otimes W_{\bar{j}} \rightarrow W_{\bar{k}}\{x\}$ defined by

$$\mathcal{Y}_{\bar{\alpha}}(C_i w^{(i)}, x)C_j w^{(j)} = C_k \mathcal{Y}_\alpha(w^{(i)}, x)w^{(j)} \quad (\forall w^{(i)} \in W_i, w^{(j)} \in W_j)$$

is a type $\left(\begin{smallmatrix} \bar{k} \\ \bar{i} \ \bar{j} \end{smallmatrix}\right)$ intertwining operator of V , called the **conjugate intertwining operator** of \mathcal{Y}_α .

Now we have linear maps

$$B_\pm : \mathcal{V}\left(\begin{smallmatrix} k \\ i \ j \end{smallmatrix}\right) \mapsto \mathcal{V}\left(\begin{smallmatrix} k \\ j \ i \end{smallmatrix}\right), \quad \mathcal{Y}_\alpha \mapsto \mathcal{Y}_{B_\pm\alpha},$$

and antilinear maps

$$\begin{aligned} \text{Ad} : \mathcal{V}\left(\begin{smallmatrix} k \\ i \ j \end{smallmatrix}\right) &\rightarrow \mathcal{V}\left(\begin{smallmatrix} j \\ \bar{i} \ k \end{smallmatrix}\right), & \mathcal{Y}_\alpha &\mapsto \mathcal{Y}_{\alpha^*}, \\ \text{Conj} : \mathcal{V}\left(\begin{smallmatrix} k \\ i \ j \end{smallmatrix}\right) &\rightarrow \mathcal{V}\left(\begin{smallmatrix} \bar{k} \\ \bar{i} \ \bar{j} \end{smallmatrix}\right), & \mathcal{Y}_\alpha &\mapsto \mathcal{Y}_{\bar{\alpha}}. \end{aligned}$$

One can show that B_+ is the inverse of B_- and $\alpha^{**} = \alpha, \bar{\bar{\alpha}} = \alpha$. (See also [FZ92].) So these maps are bijective. We conclude that

$$(1.18) \quad N_{ij}^k = N_{ji}^k = N_{\bar{i}\bar{k}}^{\bar{j}} = N_{\bar{i}\bar{j}}^{\bar{k}}.$$

Tensor products of unitary VOAs. Let V^1 and V^2 be two unitary VOAs of CFT type. For $i = 1, 2$, we let $Y^i, \Omega^i, \nu^i, \Theta^i$ be the vertex operator, the vacuum vector, the conformal vector, and the PCT operator of V^i . Consider $V^1 \otimes V^2$ as the tensor product of the two pre-Hilbert spaces V^1 and V^2 . We can define a vertex operator Y for $V^1 \otimes V^2$ satisfying that

$$(1.19) \quad Y(v^1 \otimes v^2, x) = Y^1(v^1, x) \otimes Y^2(v^2, x)$$

for any $v^1 \in V^1, v^2 \in V^2$. Let

$$(1.20) \quad \Omega = \Omega^1 \otimes \Omega^2,$$

$$(1.21) \quad \nu = \nu^1 \otimes \Omega^2 + \Omega^1 \otimes \nu^2,$$

$$(1.22) \quad \Theta = \Theta^1 \otimes \Theta^2$$

be the vacuum vector, the conformal vector, and the PCT operator of $V^1 \otimes V^2$, respectively. Then $V^1 \otimes V^2$ becomes a unitary VOA of CFT type. Note that (1.19) and (1.21) imply that

$$L_0 = L_0 \otimes \text{id}_{V^2} + \text{id}_{V^1} \otimes L_0,$$

which determines the grading of $V^1 \otimes V^2$.

Now let W_{i_1}, W_{i_2} be unitary representations of V^1, V^2 with vertex operators Y_{i_1}, Y_{i_2} , respectively. We define a vertex operator $Y_{i_1} \otimes Y_{i_2} \equiv Y_{i_1 \otimes i_2}$ of $W_i \otimes W_{i_2}$ to satisfy

$$(1.23) \quad (Y_{i_1} \otimes Y_{i_2})(w^{(i_1)} \otimes w^{(i_2)}, x) = Y_{i_1}(w^{(i_1)}, x) \otimes Y_{i_2}(w^{(i_2)}, x).$$

Then $Y_{i_1} \otimes Y_{i_2}$ satisfies Jacobi identity and the translation property, making $W_{i_1} \otimes W_{i_2}$ a representation of $V^1 \otimes V^2$. (The translation property is easy to check. The Jacobi identity can be proved by the method mentioned in [FHL93, section 4.6].) It is easy to check that this representation is unitary.

Let $W_{i_1}, W_{j_1}, W_{k_1}$ (resp., $W_{i_2}, W_{j_2}, W_{k_2}$) be unitary semisimple representations of V^1 (resp., V^2). If $\mathcal{Y}_{\alpha_1} \in \mathcal{V} \binom{k_1}{i_1 \ j_1}$ is an intertwining operator of V^1 and $\mathcal{Y}_{\alpha_2} \in \mathcal{V} \binom{k_2}{i_2 \ j_2}$ is an intertwining operator of V^2 , we can define a type $\binom{k_1 \otimes k_2}{i_1 \otimes i_2 \ j_1 \otimes j_2}$ intertwining operator of $V^1 \otimes V^2$, such that for any $w^{(i_1)} \in W_{i_1}, w^{(i_2)} \in W_{i_2}$,

$$(1.24) \quad (\mathcal{Y}_{\alpha_1} \otimes \mathcal{Y}_{\alpha_2})(w^{(i_1)} \otimes w^{(i_2)}, x) = \mathcal{Y}_{\alpha_1}(w^{(i_1)}, x) \otimes \mathcal{Y}_{\alpha_2}(w^{(i_2)}, x).$$

The right hand side of the above relation clearly converges because this is true when all the representations involved are irreducible.

The above discussion can be easily generalized to tensor products of more than two VOAs.

Energy bounds condition. Let W_i, W_j, W_k be unitary V -modules, $\mathcal{Y}_\alpha \in \mathcal{V} \binom{k}{i \ j}$. Given a homogeneous $w^{(i)} \in W_i, r \geq 0$, we say that $\mathcal{Y}_\alpha(w^{(i)}, x)$ satisfies **r -th order energy bounds** if there exist $M, t \geq 0$, such that for any $s \in \mathbb{R}, w^{(j)} \in W_j$, the mode $\mathcal{Y}_\alpha(w^{(i)}, s)$ satisfies the inequality

$$(1.25) \quad \|\mathcal{Y}_\alpha(w^{(i)}, s)w^{(j)}\| \leq M(1 + |s|)^t \|(1 + L_0)^r w^{(j)}\|.$$

In the case when the exact value of r is not very important, we just say that $\mathcal{Y}_\alpha(w^{(i)}, x)$ is **energy-bounded**.

We say that an intertwining operator $\mathcal{Y}_\alpha \in \mathcal{V} \binom{k}{i \ j}$ is **energy-bounded** if for any $w^{(i)} \in W_i, \mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded. A unitary V -module W_i is **energy-bounded** if $Y_i \in \mathcal{V} \binom{i}{0 \ i}$ is energy-bounded.

We now collect some useful criteria for the energy bounds conditions of intertwining operators. Since they are already proved in [Gui19, section 3.1], we will not prove them again here.

Proposition 1.8. *If $\mathcal{Y}_\alpha(w^{(i)}, x)$ satisfies r -th order energy bounds, so do $\mathcal{Y}_{\alpha^*}(C_i w^{(i)}, x)$ and $\mathcal{Y}_{\bar{\alpha}}(C_i w^{(i)}, x)$.*

Proposition 1.9. *Let \mathcal{Y}_{α_1} be a type $\binom{k_1}{i_1 \ j_1}$ intertwining operator of V^1 , and let \mathcal{Y}_{α_2} be a type $\binom{k_2}{i_2 \ j_2}$ intertwining operator of V^2 . If $w^{(i_1)} \in W_{i_1}, w^{(i_2)} \in W_{i_2}$ are*

homogeneous and $\mathcal{Y}_{\alpha_1}(w^{(i_1)}, x)$ and $\mathcal{Y}_{\alpha_2}(w^{(i_2)}, x)$ are energy-bounded, then

$$(\mathcal{Y}_{\alpha_1} \otimes \mathcal{Y}_{\alpha_2})(w^{(i_1)} \otimes w^{(i_2)}, x)$$

is also energy-bounded.

Proof. Although this proposition is not given in [Gui19], its proof is not hard. One can use, for instance, the same argument as in the proof of [Gui19, Proposition 3.5]. □

Given a set E of vectors of V , we say that V is **generated by** E (or V is **generating**) if the vector space V is spanned by vectors of the form $Y(v_1, n_1) \cdots Y(v_k, n_k)v_{k+1}$ ($k \in \mathbb{Z}_{\geq 0}, n_1, \dots, n_k \in \mathbb{Z}, v_1, \dots, v_{k+1} \in E$).

Proposition 1.10. *If W_i is a unitary V -module, E is a generating set of homogeneous vectors in V , and for any $v \in E$, $Y_i(v, x)$ is energy-bounded, then W_i is energy-bounded.*

Proposition 1.11. *If $\mathcal{Y}_\alpha \in \mathcal{V}(\binom{k}{i \ j})$, W_i is irreducible, W_j and W_k are energy-bounded, and there exists a non-zero homogeneous vector $w^{(i)} \in W_i$, such that $\mathcal{Y}_\alpha(w^{(i)}, x)$ is energy-bounded, then \mathcal{Y}_α is energy-bounded.*

1.2. Unitary Lie algebras. Let \mathfrak{g} be a finite-dimensional complex simple (non-abelian) Lie algebra. We give a $*$ -structure on \mathfrak{g} . Let $\mathfrak{g}_{\mathbb{R}}$ be a compact real form of \mathfrak{g} . So $\mathfrak{g}_{\mathbb{R}}$ is a real simple Lie algebra with a negative-definite (symmetric) bilinear form $(\cdot|\cdot)$ (unique up to scalar multiplication), such that

$$([X, Y]|Z) = -(Y|[X, Z]) \quad (X, Y, Z \in \mathfrak{g}_{\mathbb{R}}),$$

and $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus_{\mathbb{R}} i\mathfrak{g}_{\mathbb{R}}$. Define an antilinear isomorphism $*$: $\mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$(X + iY)^* = -X + iY \quad (X, Y \in \mathfrak{g}_{\mathbb{R}}).$$

Then $*$ is an involution, i.e., $** = \text{id}_{\mathfrak{g}}$. We also have the relation

$$[X, Y]^* = [Y^*, X^*] \quad (X, Y \in \mathfrak{g}).$$

The $(\cdot|\cdot)$ on $\mathfrak{g}_{\mathbb{R}}$ extends to an **invariant inner product** on \mathfrak{g} , also denoted by $(\cdot|\cdot)$. The word “invariant” means that

$$(1.26) \quad ([X, Y]|Z) = (Y|[X^*, Z]) \quad (X, Y, Z \in \mathfrak{g}).$$

In other words, under this inner product the adjoint representation of \mathfrak{g} is unitary. Due to the simplicity of \mathfrak{g} , the invariant inner products on \mathfrak{g} are unique up to multiplication by scalars. If an invariant inner product is chosen, one can easily check that $*$: $\mathfrak{g} \rightarrow \mathfrak{g}$ is an anti-unitary map under this inner product, i.e.,

$$(1.27) \quad (X|Y) = (Y^*|X^*) \quad (X, Y \in \mathfrak{g}).$$

Let U be a (complex) representation of \mathfrak{g} . If U has an inner product $\langle \cdot | \cdot \rangle$, we say that the representation is **unitary** if for any $X \in \mathfrak{g}, u_1, u_2 \in U$,

$$\langle Xu_1 | u_2 \rangle = \langle u_1 | X^*u_2 \rangle.$$

Remark 1.12. In general, let \mathfrak{p} be a (not necessarily simple) finite-dimensional complex Lie algebra with a $*$ -structure. So, by definition, $*$ satisfies $** = \text{id}_{\mathfrak{p}}$ and $[X, Y]^* = [Y^*, X^*]$ for any $X, Y \in \mathfrak{p}$. We say that \mathfrak{p} is **unitary** if there exists an invariant inner product $(\cdot|\cdot)$ on \mathfrak{p} , under which $*$ is anti-unitary; i.e., $*$: $\mathfrak{p} \rightarrow \mathfrak{p}$ is an antilinear isomorphism, and $(X^*|Y^*) = (Y|X)$ for any $X, Y \in \mathfrak{p}$. The anti-unitary

condition automatically holds when \mathfrak{p} is semisimple, i.e., when the center \mathfrak{z} of \mathfrak{p} is trivial.

A finite-dimensional complex simple Lie algebra \mathfrak{g} , with the $*$ -structure defined above, is a unitary Lie algebra. An abelian finite-dimensional Lie algebra \mathfrak{h}_0 with an arbitrary involution $*$ and an arbitrary inner product $(\cdot|\cdot)$ is also a unitary Lie algebra. In general, it is not hard to show that any finite-dimensional unitary Lie algebra \mathfrak{p} can be written as an orthogonal direct sum of unitary Lie subalgebras

$$(1.28) \quad \mathfrak{p} = \mathfrak{z} \oplus^\perp \mathfrak{g}_1 \oplus^\perp \cdots \oplus^\perp \mathfrak{g}_n,$$

where \mathfrak{z} , being the center of \mathfrak{p} , is an abelian unitary Lie algebra and $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ are simple unitary Lie algebras. Such decomposition can be obtained by considering the irreducible decomposition of the adjoint representation of $\mathfrak{p} \ominus^\perp \mathfrak{z}$ on itself. See [Was10, section II.1].

Let \mathfrak{h} be a maximal abelian unitary Lie subalgebra of \mathfrak{g} , i.e., a Cartan subalgebra of \mathfrak{g} . (Unitarity of the Lie subalgebra \mathfrak{h} requires, by definition, that $H^* \in \mathfrak{h}$ whenever $H \in \mathfrak{h}$.) Let \mathfrak{h}^* be the dual vector space of \mathfrak{h} . (The symbol $*$ in \mathfrak{h}^* has nothing to do with the $*$ -structure of \mathfrak{g} .) Then the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus^\perp \bigoplus_{\alpha \in \Phi}^\perp \mathfrak{g}_\alpha$$

gives the irreducible decomposition of the adjoint representation of \mathfrak{h} on \mathfrak{g} . Here $\Phi \subset \mathfrak{h}^*$ is the root system of \mathfrak{g} (in fact Φ is in the (real) dual space $(i\mathfrak{h}_\mathbb{R})^*$ of $i\mathfrak{h}_\mathbb{R}$, where $\mathfrak{h}_\mathbb{R} = \{X \in \mathfrak{h} : X^* = -X\}$), and \mathfrak{g}_α is the (1-dimensional) root space of the root α ; i.e., for any $H \in \mathfrak{h}, X \in \mathfrak{g}_\alpha$, we have $[H, X_\alpha] = \langle \alpha, H \rangle X$.

If we choose an invariant inner product $(\cdot|\cdot)$ on \mathfrak{g} , then the restricted inner product on \mathfrak{h} induces naturally an antilinear isomorphism $\mathfrak{h}^* \rightarrow \mathfrak{h}, \lambda \mapsto h_\lambda$ such that $\lambda(H) \equiv \langle H, \lambda \rangle = (H|h_\lambda)$. We can therefore define an inner product on \mathfrak{h}^* , also denoted by $(\cdot|\cdot)$, such that $(\lambda|\mu) = (h_\mu|h_\lambda)$ for any $\lambda, \mu \in \mathfrak{h}^*$. Now, for any $\alpha \in \Phi$, we choose a non-zero $X_\alpha \in \mathfrak{g}_\alpha$. Set

$$(1.29) \quad E_\alpha = \frac{\sqrt{2}}{\|X_\alpha\| \|\alpha\|} X_\alpha, \quad F_\alpha = \frac{\sqrt{2}}{\|X_\alpha\| \|\alpha\|} X_\alpha^*, \quad H_\alpha = \frac{2}{\|\alpha\|^2} h_\alpha.$$

Then $E_\alpha \in \mathfrak{g}_\alpha, F_\alpha \in \mathfrak{g}_{-\alpha}, H_\alpha \in \mathfrak{h}$, and it is not hard to check the following **unitary \mathfrak{sl}_2 relations**:

$$(1.30) \quad [E_\alpha, F_\alpha] = H_\alpha, \quad [H_\alpha, E_\alpha] = 2E_\alpha, \quad [H_\alpha, F_\alpha] = -2F_\alpha;$$

$$(1.31) \quad E_\alpha^* = F_\alpha, \quad H_\alpha^* = H_\alpha.$$

If there also exist non-zero $\tilde{E}_\alpha \in \mathfrak{g}_\alpha$ and $\tilde{F}_\alpha, \tilde{H}_\alpha \in \mathfrak{g}$ satisfying the unitary \mathfrak{sl}_2 relation, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that

$$(1.32) \quad \tilde{E}_\alpha = cE_\alpha, \quad \tilde{F}_\alpha = \bar{c}F_\alpha, \quad \tilde{H}_\alpha = H_\alpha.$$

In particular, the element H_α can be defined independently of the invariant inner product of \mathfrak{g} .

Convention 1.13. Unless otherwise stated, in this article, we always let $(\cdot|\cdot)$ be the **normalized** invariant inner product of \mathfrak{g} , i.e., the one under which the longest roots of \mathfrak{g} have length $\sqrt{2}$. The identification $\mathfrak{h}^* \rightarrow \mathfrak{h}, \lambda \mapsto h_\lambda$ are also defined with respect to this $(\cdot|\cdot)$.

Unitary representations of \mathfrak{g} . It is well-known that irreducible finite-dimensional representations of \mathfrak{g} are highest-weight representations, and their equivalence classes are characterized by their highest weights $\lambda \in \mathfrak{h}^*$. Let $\alpha_1, \dots, \alpha_n$ be the simple roots of \mathfrak{g} . Then the fundamental weights $\lambda_1, \dots, \lambda_n \in \mathfrak{h}^*$ are defined to satisfy that $\frac{2(\lambda_i|\alpha_j)}{\|\alpha_j\|^2} = \delta_{ij}$. Let

$$P_+(\mathfrak{g}) = \{k_1\lambda_1 + \dots + k_n\lambda_n : k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}\}$$

be the set of dominant integral weights of \mathfrak{g} . Then a weight $\lambda \in \mathfrak{h}^*$ is the highest weight of a finite-dimensional irreducible representation of \mathfrak{g} if and only if $\lambda \in P_+(\mathfrak{g})$.

Finite-dimensional irreducible representations of \mathfrak{g} are unitarizable (see, for example, [Was10, section II.14]). It is easy to show that unitary highest-weight representations are automatically irreducible, and the highest weights are also inside $P_+(\mathfrak{g})$ (since this is true for \mathfrak{sl}_2 , see [Was10, section II.4]). Therefore, *irreducible finite-dimensional \mathfrak{g} -modules and unitarizable highest-weight \mathfrak{g} -modules are the same thing.*

For any $\lambda \in P_+(\mathfrak{g})$, we choose a standard unitary highest-weight \mathfrak{g} -module $L_{\mathfrak{g}}(\lambda)$ whose highest weight is λ . Note that $L_{\mathfrak{g}}(0) = \mathbb{C}$, the 1-dimensional trivial representation of \mathfrak{g} .

Dual representations. We now discuss dual representations of \mathfrak{g} from the unitary point of view. Dual representations of Lie algebras are similar to contragredient representations of VOAs. Let U be a representation of \mathfrak{g} . We let U^* be the dual vector space of U . Define an action of \mathfrak{g} on U^* using the formula

$$\langle Xv', u \rangle = -\langle v', Xu \rangle \quad (\forall u \in U, v' \in U^*).$$

Then this action makes U^* a representation of \mathfrak{g} , called the **dual representation** of U .

Now we assume that U is a unitary representation. Then the inner product on U induces an antilinear isomorphism $C_U : U \rightarrow U^*$ satisfying

$$\langle u, C_U v \rangle = \langle u|v \rangle \quad (\forall u, v \in U).$$

We thus define an inner product $\langle \cdot | \cdot \rangle$ on U^* :

$$\langle C_U u | C_U v \rangle = \langle v|u \rangle \quad (\forall u, v \in U).$$

Then C_u becomes an anti-unitary map. It is easy to check that the action of \mathfrak{g} on U^* can be written as

$$(1.33) \quad X \cdot v' = -C_U X^* C_U^{-1} \cdot v' \quad (\forall v' \in U^*).$$

Thus U^* is also a unitary \mathfrak{g} -module.

Now, for any $\lambda \in P_+(\mathfrak{g})$, the dual representation of $L_{\mathfrak{g}}(\lambda)$ is also a finite-dimensional \mathfrak{g} -module, which is irreducible. Thus it is a unitary highest-weight \mathfrak{g} -module.

Convention 1.14. For any $\lambda \in P_+(\mathfrak{g})$, we let $\bar{\lambda}$ be the highest weight of the dual representation of $L_{\mathfrak{g}}(\lambda)$ and let $C_{\lambda} : L_{\mathfrak{g}}(\lambda) \rightarrow L_{\mathfrak{g}}(\bar{\lambda})$ be the canonical anti-unitary map.

Tensor products of representations. Recall that if U_1, U_2 are finite-dimensional unitary \mathfrak{g} -modules, then $U_1 \otimes U_2$ becomes a unitary \mathfrak{g} -module if the action of \mathfrak{g} is defined by

$$X(u_1 \otimes u_2) = Xu_1 \otimes u_2 + u_1 \otimes Xu_2 \quad (X \in \mathfrak{g}, u_1 \in U_1, u_2 \in U_2).$$

Now given two unitary highest-weight modules $L_{\mathfrak{g}}(\lambda), L_{\mathfrak{g}}(\mu)$, we are interested in the irreducible decomposition of $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$. So we define, for any $\nu \in P_+(\mathfrak{g})$,

$$\text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = \text{Hom}_{\mathfrak{g}}(L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu)).$$

The number $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$, being the multiplicity of $L_{\mathfrak{g}}(\nu)$ in $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$, is called a **tensor product rule** of \mathfrak{g} .

Tensor product rules share some similar properties with the fusion rules of VOAs. For example, for any $T \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$, we define a linear map $\text{Ad} T : L_{\mathfrak{g}}(\bar{\lambda}) \otimes L_{\mathfrak{g}}(\nu) \rightarrow L_{\mathfrak{g}}(\mu)$, such that for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda), u^{(\mu)} \in L_{\mathfrak{g}}(\mu), u^{(\nu)} \in L_{\mathfrak{g}}(\nu)$,

$$(1.34) \quad \langle \text{Ad} T(C_{\lambda} u^{(\lambda)} \otimes u^{(\nu)}) | u^{(\mu)} \rangle = \langle u^{(\nu)} | T(u^{(\lambda)} \otimes u^{(\mu)}) \rangle.$$

It is easy to check that $\text{Ad} T$ intertwines the action of \mathfrak{g} , so we have an antilinear map

$$(1.35) \quad \text{Ad} : \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \rightarrow \text{Hom}_{\mathfrak{g}}(\bar{\lambda} \otimes \nu, \mu),$$

which implies that

$$(1.36) \quad \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = \dim \text{Hom}_{\mathfrak{g}}(\bar{\lambda} \otimes \nu, \mu).$$

In particular, we have

$$\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, 0) = \dim \text{Hom}_{\mathfrak{g}}(\bar{\lambda} \otimes 0, \mu) = \dim \text{Hom}_{\mathfrak{g}}(L_{\mathfrak{g}}(\bar{\lambda}), L_{\mathfrak{g}}(\mu)) = \delta_{\bar{\lambda}, \mu}.$$

We thus conclude:

Proposition 1.15. $\bar{\lambda} = \mu$ if and only if $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$ contains an irreducible submodule equivalent to the trivial module \mathbb{C} . In that case the multiplicity of \mathbb{C} in $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$ is 1.

Computing tensor product rules. It will be very helpful to compute some tensor product rules of a simple Lie algebra. For this purpose, we fix highest-weight vectors $v_{\mu} \in L_{\mathfrak{g}}(\mu), v_{\nu} \in L_{\mathfrak{g}}(\nu)$. Let $L_{\mathfrak{g}}(\lambda)[\nu - \mu]$ be the weight subspace of $L_{\mathfrak{g}}(\lambda)$ with weight $\nu - \mu$; i.e., $L_{\mathfrak{g}}(\lambda)[\nu - \mu]$ is the subspace of vectors u satisfying that $Hu = \langle \nu - \mu, H \rangle u$ for any $H \in \mathfrak{h}$. Define a linear map

$$(1.37) \quad \Gamma : \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \rightarrow L_{\mathfrak{g}}(\lambda)[\nu - \mu]^*,$$

such that for any $T \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$, ΓT is defined by

$$(1.38) \quad (\Gamma T)(u^{(\lambda)}) = \langle T(u^{(\lambda)} \otimes v_{\mu}) | v_{\nu} \rangle \quad (\forall u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]).$$

We first give an upper bound for a tensor product rule. Let \mathfrak{g}_+ and \mathfrak{g}_- be the Lie subalgebras of raising operators and lowering operators, respectively.

Proposition 1.16. *The map Γ is injective. Consequently,*

$$(1.39) \quad \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \leq \dim L_{\mathfrak{g}}(\lambda)[\nu - \mu].$$

Proof. Assume that (1.38) equals 0 for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]$. Then (1.38) obviously equals 0 for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$. Since v_{μ} is a cyclic vector of $L_{\mathfrak{g}}(\mu)$ under the action of \mathfrak{g}_- , and since v_{ν} is annihilated by \mathfrak{g}_+ , one can easily show that $\langle T(u^{(\lambda)} \otimes u^{(\mu)})|v_{\nu}\rangle = 0$ for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda), u^{(\mu)} \in L_{\mathfrak{g}}(\mu)$. Using the fact that $v_{\nu} \in L_{\mathfrak{g}}(\nu)$ is also \mathfrak{g}_- -cyclic, one proves that $T = 0$. \square

We now describe the range of Γ . Recall that for any root α , its dual root is defined by $\check{\alpha} = 2\alpha/\|\alpha\|^2$. For any $\rho \in \mathfrak{h}^*$, let $n_{\rho,\alpha} = (\rho|\check{\alpha})$.

Proposition 1.17. *Let $K_{\mathfrak{g}}^{\mu}(\lambda)$ be the subspace of $L_{\mathfrak{g}}(\lambda)$ spanned by vectors of the form $F_{\alpha}^{n_{\mu,\alpha}+1}u^{(\lambda)}$, where $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, α is a simple root of \mathfrak{g} , and $F_{\alpha} \in \mathfrak{g}_{-\alpha}$. Let $K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu]$ be the weight subspace of $K_{\mathfrak{g}}^{\mu}(\lambda)$ with weight $\nu - \mu$. Then an element $\varphi \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]^*$ is in the range of Γ if and only if $\varphi \perp K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu]$. As a consequence,*

$$(1.40) \quad \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = \dim L_{\mathfrak{g}}(\lambda)[\nu - \mu] - \dim K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu].$$

Proof. The “only if” part is easy to prove. Indeed, assume that $\varphi = \Gamma T$. Then, since $F_{\alpha}^{n_{\mu,\alpha}+1}v_{\mu} = 0$ and $F_{\alpha}^*v_{\nu} = 0$, for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$ such that $F_{\alpha}^{n_{\mu,\alpha}+1}u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]$, we have

$$\varphi(F_{\alpha}^{n_{\mu,\alpha}+1}u^{(\lambda)}) = \langle T(F_{\alpha}^{n_{\mu,\alpha}+1}u^{(\lambda)} \otimes v_{\mu})|v_{\nu}\rangle = -\langle T(u^{(\lambda)} \otimes F_{\alpha}^{n_{\mu,\alpha}+1}v_{\mu})|v_{\nu}\rangle = 0.$$

Thus $\varphi \perp K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu]$.

The proof of the “if” part is postponed to section 1.A. \square

Corollary 1.18. *Let $\lambda, \mu, \nu \in P_+(\mathfrak{g})$. Assume that $\dim L_{\mathfrak{g}}(\lambda)[\nu - \mu] = 1$. Then $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 1$ if and only if for any simple root α ,*

$$(1.41) \quad \dim L_{\mathfrak{g}}(\lambda)[\nu - \mu + (n_{\mu,\alpha} + 1)\alpha] = 0.$$

Otherwise, $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 0$.

Proof. Assume firstly that $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 0$. Then by Proposition 1.17, $\dim K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu] = 1$. Choose a simple root α and a weight vector $u^{(\lambda)}$ of $L_{\mathfrak{g}}(\lambda)$ such that $F_{\alpha}^{n_{\mu,\alpha}+1}u^{(\lambda)}$ is a non-zero weight $\nu - \mu$ vector of $L_{\mathfrak{g}}(\lambda)$. Then $u^{(\lambda)}$ is a non-zero vector of $L_{\mathfrak{g}}(\lambda)$ of weight $\nu - \mu + (n_{\mu,\alpha} + 1)\alpha$. Therefore

$$\dim L_{\mathfrak{g}}(\lambda)[\nu - \mu + (n_{\mu,\alpha} + 1)\alpha] > 0.$$

Conversely, assume that there exists a simple root α such that

$$\dim L_{\mathfrak{g}}(\lambda)[\nu - \mu + (n_{\mu,\alpha} + 1)\alpha] > 0.$$

Consider the unitary representation of \mathfrak{sl}_2 on

$$U = \bigoplus_{n \in \mathbb{Z}} L_{\mathfrak{g}}(\lambda)[\nu - \mu + n\alpha],$$

where the \mathfrak{sl}_2 is generated by the elements $E_{\alpha}, F_{\alpha}, H_{\alpha}$ defined by (1.29). Note that both $L_{\mathfrak{g}}(\lambda)[\nu - \mu]$ and $L_{\mathfrak{g}}(\lambda)[\nu - \mu + (n_{\mu,\alpha} + 1)\alpha]$ are non-trivial eigensubspaces of H_{α} and their eigenvalues differ by an even number. Therefore, from representation theory of \mathfrak{sl}_2 , one can easily show the existence of $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\nu - \mu + (n_{\mu,\alpha} + 1)\alpha]$ such that $F_{\alpha}^{n_{\mu,\alpha}+1}u^{(\lambda)}$ is non-zero. Hence $\dim K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu] > 0$. By Proposition 1.17, $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 0$. \square

1.3. Unitary affine VOAs.

The affine Lie algebra $\widehat{\mathfrak{g}}$ and its positive energy representations. Given a finite-dimensional unitary complex simple Lie algebra \mathfrak{g} , we let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$, where t is a formal variable and K is a formal vector. Set $X(n) = X \otimes t^n$ for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The Lie algebra structure on $\widehat{\mathfrak{g}}$ is given by

$$(1.42) \quad [X(m), Y(n)] = [X, Y](m+n) + m(X|Y^*)\delta_{m,-n}K \quad (X, Y \in \mathfrak{g}, m, n \in \mathbb{Z}),$$

$$(1.43) \quad [K, X(n)] = 0 \quad (X \in \mathfrak{g}, n \in \mathbb{Z}).$$

(Recall that by the anti-unitarity of $*$, $(X|Y^*) = (Y|X^*)$.) $\widehat{\mathfrak{g}}$ is called the **affine Lie algebra** of \mathfrak{g} . We define an involution $*$: $\widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$ satisfying

$$(1.44) \quad X(n)^* = X^*(-n) \quad (X \in \mathfrak{g}, n \in \mathbb{Z}),$$

$$(1.45) \quad K^* = K.$$

Again we have

$$[\mathfrak{X}, \mathfrak{Y}]^* = [\mathfrak{Y}^*, \mathfrak{X}^*] \quad (\mathfrak{X}, \mathfrak{Y} \in \widehat{\mathfrak{g}}).$$

Unitary representations (unitary modules) of $\widehat{\mathfrak{g}}$ are defined in the same way as those of \mathfrak{g} . A unitary representation W of $\widehat{\mathfrak{g}}$ is called a **positive energy representation** (PER) if the vector space W has an orthogonal grading $W = \bigoplus_{n \in \mathbb{N}}^\perp W(n)$, and for any $n \in \mathbb{N}$,

$$\dim W(n) < \infty,$$

$$X(m)W(n) \subset W(n-m) \quad (X \in \mathfrak{g}, m \in \mathbb{Z}),$$

$$KW(n) \subset W(n).$$

We now discuss briefly irreducible PERs of $\widehat{\mathfrak{g}}$. Details can be found in [Was10]. Let W be an irreducible PER of $\widehat{\mathfrak{g}}$. Choose $n_0 \in \mathbb{N}$ such that $\dim W(n_0) > 0$ and $\dim W(n) = 0$ for any $n < n_0$. $W(n_0)$ is called the **lowest-energy subspace** of W . Identify \mathfrak{g} with $\widehat{\mathfrak{g}}(0)$ by setting

$$X = X(0) \quad (X \in \mathfrak{g}).$$

Then \mathfrak{g} is a unitary Lie subalgebra of $\widehat{\mathfrak{g}}$, and $W(n_0)$ is a finite-dimensional unitary representation of \mathfrak{g} , called the **lowest-energy \mathfrak{g} -module**. It is easy to see that *the lowest-energy \mathfrak{g} -module of an irreducible PER of $\widehat{\mathfrak{g}}$ is irreducible*.

The action of K on $W(n_0)$ commutes with the irreducible action $\mathfrak{g} \curvearrowright W(n_0)$. Therefore, by Schur’s lemma,

$$K|_{W(n_0)} = l \cdot \text{id}_{W(n_0)}$$

for some $l \in \mathbb{R}$. That l is real follows from the self-adjointness of K . We say that l is the **level** of the irreducible PER W . (In general, a general PER of $\widehat{\mathfrak{g}}$ is said to have **level** l if the action of K on the pre-Hilbert space is the identity operator multiplied by l .) Since $\widehat{\mathfrak{g}} \cdot W(n_0) = W$, K must be the constant l on the entire vector space W .

It is not hard to show that $\mathfrak{g} \curvearrowright W(n_0)$ and l completely characterizes the PER W ; i.e., if two irreducible PERs reduce to equivalent lowest-energy \mathfrak{g} -modules and if their levels are equal, then they are (unitary) equivalent. We also know that finite-dimensional irreducible representations of \mathfrak{g} must be highest-weight representations and are determined, up to equivalence, by their highest weights. So the equivalence

class of W is determined by the pair (λ, l) , where $\lambda \in \mathfrak{h}^*$ is the highest weight of $\mathfrak{g} \curvearrowright W(n_0)$.

There is also a notion of highest weightness for PERs of $\widehat{\mathfrak{g}}$. Let \mathfrak{g}_+ (resp., \mathfrak{g}_-) be the Lie subalgebra of raising operators (resp., lowering operators) in \mathfrak{g} . Let $\widehat{\mathfrak{g}}_{>0}$ (resp., $\widehat{\mathfrak{g}}_{<0}$) be the Lie subalgebra of $\widehat{\mathfrak{g}}$ spanned by $X(n)$, where $X \in \mathfrak{g}$ and $n \in \mathbb{Z}_{>0}$ (resp., $n \in \mathbb{Z}_{<0}$). Let W be a PER of $\widehat{\mathfrak{g}}$. A non-zero vector $v \in W$ is called a **highest-weight vector** of W if the following conditions are satisfied:

- (a) There exists a grading of W such that $v \in W(n)$ for some $n \in \mathbb{N}$.
- (b) There exists $\lambda \in \mathfrak{h}^*$ such that $Hv = \langle \lambda, H \rangle v$ for any $H \in \mathfrak{h}$.
- (c) There exists $l \in \mathbb{R}$ such that $Kv = lv$.
- (d) $\mathfrak{g}_+v = 0, \widehat{\mathfrak{g}}_{>0}v = 0$.

In this case, λ is called the **highest weight** of v , and l is called the **level** of v . If v is a cyclic vector, i.e., $W = U(\widehat{\mathfrak{g}}) \cdot v$ where $U(\widehat{\mathfrak{g}})$ is the universal enveloping algebra of $\widehat{\mathfrak{g}}$, we say that W is a **highest-weight PER** of $\widehat{\mathfrak{g}}$.

If W is an irreducible PER, then the highest-weight vector v of its lowest-energy \mathfrak{g} -module is a highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright W$. So W is a highest-weight PER of $\widehat{\mathfrak{g}}$. Conversely, if W is a highest-weight PER and v is a highest-weight vector with highest weight λ and level l , then it is easy to see that W is irreducible. So *irreducible PERs and highest-weight PERs are the same for $\widehat{\mathfrak{g}}$* . Moreover, v is a highest-weight vector of the lowest-energy \mathfrak{g} -module of W . So W has a unique up to scalar multiplication highest-weight vector, a unique highest weight, and a unique level.

The remaining question is to know the possible λ and l for irreducible PERs. Clearly $\lambda \in P_+(\mathfrak{g})$. But extra conditions are required. Let $\theta \in \Phi$ be the highest root of \mathfrak{g} , i.e., the highest weight of the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}$. Since θ is a longest root, we have $\|\theta\|^2 = 2$. Choose $E_{-\theta} \in \mathfrak{g}_{-\theta}, F_{-\theta} \in \mathfrak{g}_\theta, H_{-\theta} \in \mathfrak{h}$ as in (1.29). Then we have $\|E_{-\theta}\|^2 = \|F_{-\theta}\|^2 = 1$ and $H_{-\theta} = h_{-\theta}$. Using these facts, one can easily check that $E_{-\theta}(1), F_{-\theta}(-1), H_{-\theta}(0) + K$ satisfy the unitary \mathfrak{sl}_2 relation. Since the highest-weight vector v of W is also a highest-weight vector of this \mathfrak{sl}_2 with highest weight $-(\lambda|\theta) + l$ (which is also easy to check), we must have

$$-(\lambda|\theta) + l \in \mathbb{Z}_{\geq 0}.$$

Since $\lambda, \theta \in P_+(\mathfrak{g})$, we must have $(\lambda|\theta) \in \mathbb{Z}_{\geq 0}$. Therefore, l is a non-negative integer no less than $(\lambda|\theta)$. We conclude that if λ, l are the highest weight and the level of an irreducible PER of $\widehat{\mathfrak{g}}$, then condition (1.46) holds. The converse is also true: if condition (1.46) is satisfied, then there exists an irreducible PER with highest weight λ and level l . See [Kac90] or [Was10, chapter III]. We conclude the following.

Theorem 1.19. *Let W be a PER of $\widehat{\mathfrak{g}}$. Then W is irreducible if and only if it is a highest-weight PER. In this case, W has a unique up to scalar multiplication highest-weight vector v_λ with highest weight λ and level l . λ and l satisfy*

$$(1.46) \quad \lambda \in P_+(\mathfrak{g}), \quad l \in \mathbb{Z}_{\geq 0}, \quad (\lambda|\theta) \leq l,$$

and they completely determine the PER W up to unitary equivalence. The lowest-energy \mathfrak{g} -module $W(n_0)$ of W is irreducible. We have $K = l \cdot \text{id}_W, v_\lambda \in W(n_0)$, and v_λ is also a highest-weight vector of $\mathfrak{g} \curvearrowright W(n_0)$ with highest weight λ .

Conversely, if λ and l satisfy (1.46), then there exists an irreducible PER of $\widehat{\mathfrak{g}}$ with highest weight λ and level l .

Due to this theorem, for any $l \in \mathbb{Z}_{\geq 0}$, we say that a dominant integral weight λ of \mathfrak{g} is **admissible at level l** if $(\lambda|\theta) \leq l$. For such λ , we choose $L_{\mathfrak{g}}(\lambda, l)$ to be a standard PER of $\widehat{\mathfrak{g}}$ with highest weight λ and level l .

Unitary affine VOAs. The PER $L_{\mathfrak{g}}(0, l)$ of $\widehat{\mathfrak{g}}$ is called the level l **vacuum representation** of $\widehat{\mathfrak{g}}$. We now extend such representation of $\widehat{\mathfrak{g}}$ to a unitary VOA structure.

Theorem 1.20. *Let \mathfrak{g} be a finite-dimensional complex simple unitary Lie algebra. Fix $l \in \mathbb{Z}_{\geq 0}$, and fix a highest-weight vector $\Omega \in L_{\mathfrak{g}}(0, l)$ satisfying $\|\Omega\| = 1$. Then there exists a unique unitary VOA structure of CFT type on the pre-Hilbert space $V_{\mathfrak{g}}^l = L_{\mathfrak{g}}(0, l)$, such that Ω is the vacuum vector, and*

$$(1.47) \quad Y(X(-1)\Omega, x) = \sum_{n \in \mathbb{Z}} X(n)x^{-n-1} \quad (X \in \mathfrak{g}).$$

$V_{\mathfrak{g}}^l$ is called the **level l unitary affine VOA of \mathfrak{g}** . The conformal vector ν of $V_{\mathfrak{g}}^l$ is given by the formula

$$(1.48) \quad \nu = \frac{1}{2(l + h^\vee)} \sum_i X_i^*(-1)X_i(-1)\Omega,$$

where h^\vee is the dual Coxeter number of \mathfrak{g} and $\{X_i\}$ is a set of orthonormal basis of \mathfrak{g} under the normalized invariant inner product $(\cdot|\cdot)$ (see Convention 1.13). The PCT operator Θ is determined by

$$(1.49) \quad \Theta X(-1)\Omega = -X^*(-1)\Omega \quad (X \in \mathfrak{g}).$$

Proof. The existence of a VOA structure on $V_{\mathfrak{g}}^l$ such that relation (1.47) holds, Ω is the vacuum vector, and the ν defined by (1.49) is the conformal vector is given in [FZ92, Theorem 2.4.1]. In this case, the set

$$(1.50) \quad E = \{X(-1)\Omega : X \in \mathfrak{g}\}$$

generates $V_{\mathfrak{g}}^l$. So any antilinear automorphism on the VOA $V_{\mathfrak{g}}^l$ is determined by its values on E . That $V_{\mathfrak{g}}^l$ is unitary with PCT operator Θ determined by (1.49) is proved in [DL14, section 4.2]. One can also prove the unitarity of $V_{\mathfrak{g}}^l$ using [CKLW18, Proposition 5.17], which is a general criterion on the unitarity of a VOA. To use this proposition, one needs to show that $L_n^\dagger = L_{-n}$ and that there exists a generating set F of quasi-primary vectors in $V_{\mathfrak{g}}^l$ such that

$$(1.51) \quad Y(v, x)^\dagger = x^{-2\Delta_v} Y(v, x^{-1}) \quad (\forall v \in F).$$

Indeed, using (1.48) and (1.10), one easily obtains Sugawara’s formulae:

$$(1.52) \quad L_0 = \frac{1}{l + h^\vee} \left(\frac{1}{2} \sum_i X_i^*(0)X_i(0) + \sum_{n>0} \sum_i X_i^*(-n)X_i(n) \right),$$

$$(1.53) \quad L_m = \frac{1}{2(l + h^\vee)} \sum_{n \in \mathbb{Z}} \sum_i X_i^*(m - n)X_i(n) \quad (m \neq 0).$$

Direct computation now gives $L_m^\dagger = L_{-m}$. Using (1.42) and (1.44), it is not hard to check that the set E is quasi-primary and that its subset $F = \{X(-1)\Omega : X \in \mathfrak{g}, X = X^*\}$, which is also generating, satisfies condition (1.51).

Since E is generating, (1.47) uniquely determines the vertex-algebraic structure of $V_{\mathfrak{g}}^l$. Uniqueness of Θ and ν follows from Proposition 1.1. □

Theorem 1.21. Fix $l \in \mathbb{Z}_{\geq 0}$.

(1) For any $\lambda \in P_+(\mathfrak{g})$ satisfying $(\lambda|\theta) \leq l$, there exists a unique unitary representation of $V_{\mathfrak{g}}^l$ on the pre-Hilbert space $L_{\mathfrak{g}}(\lambda, l)$, the vertex operator Y_{λ} of which satisfies

$$(1.54) \quad Y_{\lambda}(X(-1)\Omega, x)w^{(\lambda)} = \sum_{n \in \mathbb{Z}} X(n)w^{(\lambda)} \cdot x^{-n-1} \quad (X \in \mathfrak{g}, w^{(\lambda)} \in \widehat{L}_{\mathfrak{g}}(\lambda, l)).$$

The $V_{\mathfrak{g}}^l$ -module $L_{\mathfrak{g}}(\lambda, l)$ is irreducible. $L_{\mathfrak{g}}(\lambda_1, l)$ is unitarily equivalent to $L_{\mathfrak{g}}(\lambda_2, l)$ if and only if $\lambda_1 = \lambda_2$.

(2) Any $V_{\mathfrak{g}}^l$ -module is a (finite) direct sum of irreducible representations of the form $L_{\mathfrak{g}}(\lambda, l)$, where $\lambda \in P_+(\mathfrak{g})$ is admissible at level l . In particular, any $V_{\mathfrak{g}}^l$ -module is unitarizable.

Proof. (1) (Cf. [DL14, Proposition 4.9].) Existence of a $V_{\mathfrak{g}}^l$ -module $L_{\mathfrak{g}}(\lambda, l)$ satisfying (1.54) is proved in [FZ92, section 3.1]. Since $L_{\mathfrak{g}}(\lambda)$ is unitary as a $\widehat{\mathfrak{g}}$ module, by (1.44), (1.47), and (1.49), the vertex operator Y_{λ} satisfies relation (1.5) when $v \in E = \{X(-1)\Omega : X \in \mathfrak{g}\}$. Since any $v \in E$ is quasi-primary, relation (1.4) also holds for Y_{λ} and any $v \in E$. Therefore, by [Gui19, Proposition 1.10], $L_{\mathfrak{g}}(\lambda, l)$ is a unitary $V_{\mathfrak{g}}^l$ -module. The rest of part (1) is obvious.

(2) This was also proved in [FZ92, section 3.1]. □

Remark 1.22. Let $L_{\mathfrak{g}}(\lambda, l)$ be a unitary irreducible $V_{\mathfrak{g}}^l$ -module, where $\lambda \in P_+(\mathfrak{g})$ is admissible at level l . Recall that \mathfrak{g} is regarded as a Lie subalgebra of $\widehat{\mathfrak{g}}$ by identifying $X \in \mathfrak{g}$ with $X(0)$, and the action of \mathfrak{g} on the lowest-energy subspace of $L_{\mathfrak{g}}(\lambda, l)$ is equivalent to $L_{\mathfrak{g}}(\lambda)$, the unitary highest-weight representation of \mathfrak{g} with highest weight λ . So we identify the vector space $L_{\mathfrak{g}}(\lambda)$ with the lowest-energy subspace of $L_{\mathfrak{g}}(\lambda, l)$.

Choose $\Delta_{\lambda} \in \mathbb{R}$ such that $L_0|_{L_{\mathfrak{g}}(\lambda)} = \Delta_{\lambda} \cdot \text{id}_{L_{\mathfrak{g}}(\lambda)}$; i.e., Δ_{λ} is the lowest conformal weight of $L_{\mathfrak{g}}(\lambda, l)$. By Sugawara’s construction, $\Delta_{\lambda} > 0$. We say that Δ_{λ} is the **conformal weight** of $L_{\mathfrak{g}}(\lambda, l)$. Then the conformal weight Δ_w of any non-zero homogeneous vector $w \in L_{\mathfrak{g}}(\lambda, l)$ must be in $\Delta_{\lambda} + \mathbb{Z}_{\geq 0}$.

Remark 1.23. It is easy to see that the contragredient $V_{\mathfrak{g}}^l$ -module of $L_{\mathfrak{g}}(\lambda, l)$ is $L_{\mathfrak{g}}(\bar{\lambda}, l)$. As an immediate consequence, we have

$$(1.55) \quad (\lambda|\theta) = (\bar{\lambda}|\theta) \quad (\forall \lambda \in P_+(\mathfrak{g})).$$

Indeed, we can choose $l = (\lambda|\theta)$. Then since $L_{\mathfrak{g}}(\bar{\lambda}, l)$ is also an irreducible V -module, we must have $(\bar{\lambda}|\theta) \leq l = (\lambda|\theta)$. Replacing λ with $\bar{\lambda}$, we get $(\lambda|\theta) = (\bar{\lambda}|\theta) \leq (\bar{\lambda}|\theta)$.

Theorem 1.24. Let $\lambda \in P_+(\mathfrak{g})$ be admissible at level l . Then $L_{\mathfrak{g}}(\lambda, l)$ is energy-bounded. Moreover, for any $X \in \mathfrak{g}$, $Y_{\lambda}(X(-1)\Omega, x)$ satisfies first order energy bounds.

Proof. The argument in [BSM90, section 2] shows that $Y_{\lambda}(X(-1)\Omega, x)$ satisfies first order energy bounds. Indeed, one can prove the $\frac{1}{2}$ -th order energy bounds condition for $Y_{\lambda}(X(-1)\Omega, x)$ using Sugawara’s formula (1.52), as shown in [TL04, Proposition II.1.2.1]. Now the energy boundedness of Y_{λ} follows from Proposition 1.10. □

1.A. Proof of Proposition 1.17. In this appendix section, we prove the “if” part of Proposition 1.17 using a method similar to [TK88, Proposition 2.3].

We first recall some basic facts about Verma modules. Let $\mathbf{A}\mathfrak{g}$ be the free associative algebra generated by the *vector space* \mathfrak{g} , i.e., $\mathbf{A}\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{g}^{\otimes n}$. Let $I_{\mathfrak{g}}$ be the ideal of $\mathbf{A}\mathfrak{g}$ generated by $XY - YX - [X, Y]$ ($X, Y \in \mathfrak{g}$). The universal enveloping algebra of \mathfrak{g} is $U\mathfrak{g} = \mathbf{A}\mathfrak{g}/I_{\mathfrak{g}}$. Now for any $\eta \in \mathfrak{h}^*$, we choose an arbitrary non-zero vector v_{η} . Then $U\mathfrak{g} \otimes v_{\eta}$ is a (left) $U\mathfrak{g}$ -module. Identify $1 \otimes v_{\eta}$ with v_{η} . Let J_{η} be the $U\mathfrak{g}$ -submodule of $U\mathfrak{g} \otimes v_{\eta}$ generated by the vectors $Hv_{\eta} - \langle \eta, H \rangle v_{\eta}$ ($H \in \mathfrak{h}$) and Xv_{η} ($X \in \mathfrak{g}_{+}$). The quotient module $M_{\eta} = U\mathfrak{g} \otimes v_{\eta} / J_{\eta}$ is a Verma module of \mathfrak{g} with highest weight η . M_{η} , as a $U\mathfrak{g}_{-}$ -module, is canonically isomorphic to $U\mathfrak{g}_{-} \otimes v_{\eta}$, since, by the Poincaré-Birkhoff-Witt theorem, $U\mathfrak{g} = U\mathfrak{g}_{-} \otimes U\mathfrak{h} \otimes U\mathfrak{g}_{+}$. Lastly, let $\overline{M_{\eta}}$ be the complex conjugate of M_{η} , and let $C_{\eta} : M_{\eta} \rightarrow \overline{M_{\eta}}, u^{(\eta)} \mapsto C_{\eta}u^{(\eta)} = \overline{u^{(\eta)}}$ be the antilinear isomorphism. We let \mathfrak{g} act on $\overline{M_{\eta}}$ by setting $X\overline{u^{(\eta)}} = -\overline{X^*u^{(\eta)}}$ ($u^{(\eta)} \in M_{\eta}$). Then $\overline{v_{\eta}}$ is a lowest-weight vector of $\overline{M_{\eta}}$.

Now, for any $\varphi \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]^*$, we extend it to a \mathfrak{g} -invariant tri-linear form on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes \overline{M_{\nu}}$. First, we extend φ to an element in $L_{\mathfrak{g}}(\lambda)^*$ by setting $\varphi(u^{(\lambda)}) = 0$ when $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\rho]$ and $\rho \neq \nu - \mu$. We now regard φ as a linear map $L_{\mathfrak{g}}(\lambda) \otimes v_{\mu} \otimes \overline{v_{\nu}} \rightarrow \mathbb{C}$. Next, identify v_{μ} with $1 \otimes v_{\mu} \in \mathbf{A}\mathfrak{g}_{-} \otimes v_{\mu}$, and extend φ to a linear map $L_{\mathfrak{g}}(\lambda) \otimes (\mathbf{A}\mathfrak{g}_{-} \otimes v_{\mu}) \otimes \overline{v_{\nu}} \rightarrow \mathbb{C}$, satisfying

$$(1.56) \quad \varphi(u^{(\lambda)} \otimes X_1 \cdots X_n v_{\mu} \otimes \overline{v_{\nu}}) = (-1)^n \varphi(X_n \cdots X_1 u^{(\lambda)} \otimes v_{\mu} \otimes \overline{v_{\nu}})$$

for any $n \in \mathbb{Z}_{\geq 0}, X_1, \dots, X_n \in \mathfrak{g}_{-}$. One easily checks that φ vanishes on $L_{\mathfrak{g}}(\lambda) \otimes (I_{\mathfrak{g}_{-}} \otimes v_{\mu}) \otimes \overline{v_{\nu}}$, where $I_{\mathfrak{g}_{-}}$ is the ideal of \mathfrak{g}_{-} generated by $XY - YX - [X, Y]$ ($X, Y \in \mathfrak{g}_{-}$). So φ factors through a linear map, also denoted by φ , mapping $L_{\mathfrak{g}}(\lambda) \otimes (U\mathfrak{g}_{-} \otimes v_{\mu}) \otimes \overline{v_{\nu}} \rightarrow \mathbb{C}$. Identify $U\mathfrak{g}_{-} \otimes v_{\mu}$ with M_{μ} . Thus φ is a linear functional on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes \overline{v_{\nu}}$.

Extend φ to a linear functional on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes (\overline{\mathbf{A}\mathfrak{g} \otimes v_{\nu}})$ by setting, inductively,

$$(1.57) \quad \begin{aligned} \varphi(u^{(\lambda)} \otimes u^{(\mu)} \otimes \overline{X_1 X_2 \cdots X_n v_{\nu}}) &= \varphi(X_1^* u^{(\lambda)} \otimes u^{(\mu)} \otimes \overline{X_2 \cdots X_n v_{\nu}}) \\ &+ \varphi(u^{(\lambda)} \otimes X_1^* u^{(\mu)} \otimes \overline{X_2 \cdots X_n v_{\nu}}), \end{aligned}$$

where $X_1, \dots, X_n \in \mathfrak{g}$. Clearly φ vanishes on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes \overline{I_{\mathfrak{g}} \otimes v_{\nu}}$. So φ factors through a linear functional on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes \overline{U\mathfrak{g} \otimes v_{\nu}}$, also denoted by φ . Using (1.56) and the fact that φ vanishes on $L_{\mathfrak{g}}(\lambda)[\rho] \otimes v_{\mu} \otimes \overline{v_{\nu}}$ when $\rho + \mu \neq \nu$, it is not hard to see that φ also vanishes on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes \overline{J_{\nu} \otimes v_{\nu}}$. So it factors through a linear functional on $L_{\mathfrak{g}}(\lambda) \otimes M_{\mu} \otimes \overline{M_{\nu}}$, denoted again by φ . By (1.57), φ is clearly \mathfrak{g} -invariant.

Let N_{μ} be the \mathfrak{g}_{-} -submodule of M_{μ} generated by $F_{\alpha}^{n_{\mu, \alpha} + 1} v_{\mu}$ (α is a simple root, $F_{\alpha} \in \mathfrak{g}_{-\alpha}$). Then it is well-known that each $F_{\alpha}^{n_{\mu, \alpha} + 1} v_{\mu}$ is a highest-weight vector, implying that N_{μ} is \mathfrak{g} -invariant. We also know that $L_{\mathfrak{g}}(\mu) = M_{\mu} / N_{\mu}$. Hence, if we assume that $\varphi \perp K_{\mathfrak{g}}^{\mu}(\lambda)[\nu - \mu]$, then by (1.56), φ vanishes on $L_{\mathfrak{g}}(\lambda) \otimes N_{\mu} \otimes \overline{v_{\nu}}$ and hence on $L_{\mathfrak{g}}(\lambda) \otimes N_{\mu} \otimes \overline{M_{\nu}}$ by the \mathfrak{g} -invariance of φ . So φ can be regarded as a \mathfrak{g} -invariant linear functional on $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) \otimes \overline{M_{\nu}}$. Let

$$O_{\nu} = \{v^{(\nu)} \in M_{\nu} : \varphi \perp L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) \otimes \overline{v^{(\nu)}}\}.$$

Clearly O_{ν} is a \mathfrak{g} -submodule of M_{ν} . For any $\rho \in \mathfrak{h}^*$, $O_{\nu}[\rho] = M_{\nu}[\rho]$ when ρ is not a weight of $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$. So M_{ν} / O_{ν} is a finite-dimensional \mathfrak{g} -module. Assume, without loss of generality, that $\varphi \neq 0$. Then M_{ν} / O_{ν} is non-trivial. So $M_{\nu} / O_{\nu} \simeq L_{\mathfrak{g}}(\nu)$. Thus φ is a \mathfrak{g} -invariant linear functional on $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) \otimes L_{\mathfrak{g}}(\overline{\nu})$,

which is equivalent to saying that $\varphi \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$. It is obvious that $\Gamma\varphi$ is our original φ . Thus the “if” part of Proposition 1.17 is proved.

2. COMPRESSION PRINCIPLE

2.1. Unitary affine vertex subalgebras. It is well-known that for a VOA V , its weight-1 subspace $V(1)$ has a natural Lie algebra structure. In this chapter we let V be a unitary VOA of CFT type with PCT operator Θ . We shall see that $V(1)$ is a unitary Lie algebra.

Lemma 2.1. *Vectors in $V(1)$ are quasi-primary.*

Proof. Choose any $v \in V(1)$. Then $L_1v \in V(0) = \mathbb{C}\Omega$. But $\langle L_1v | \Omega \rangle = \langle v | L_{-1}\Omega \rangle = 0$. So $L_1v = 0$. □

Proposition 2.2. *The vector space $V(1)$, equipped with the bracket relation*

$$[u, v] = Y(u, 0)v \quad (u, v \in V(1))$$

*and the *-structure*

$$(2.1) \quad v^* = -\Theta v \quad (v \in V(1)),$$

is a finite-dimensional unitary complex Lie algebra. The inner product of V restricts to an invariant inner product of $V(1)$.

Proof. Equation (1.11) shows that

$$(2.2) \quad [Y(u, 0), Y(v, 0)] = Y(Y(u, 0)v, 0).$$

Therefore, for any $u, v, w \in V(1)$,

$$\begin{aligned} [u, [v, w]] - [v, [u, w]] &= [u, Y(v, 0)w] + [v, Y(u, 0)w] = Y(u, 0)Y(v, 0)w \\ &\quad - Y(v, 0)Y(u, 0)w \\ &= Y(Y(u, 0)v, 0)w = [Y(u, 0)v, w] = [[u, v], w]. \end{aligned}$$

Thus the Jacobi identity is proved.

Now (2.1) and equation (1.5) imply that

$$(2.3) \quad Y(u, x)^\dagger = x^{-2}Y(u^*, x^{-1})$$

or equivalently

$$(2.4) \quad Y(u, n)^\dagger = Y(u^*, -n) \quad (n \in \mathbb{Z}).$$

Hence

$$\langle [u, v] | w \rangle = \langle Y(u, 0)v | w \rangle = \langle v | Y(u^*, 0)w \rangle = \langle v | [u^*, w] \rangle.$$

Therefore the inner product on $V(1)$ inherited from V is invariant. So the Lie algebra $V(1)$ equipped with the involution $*$ defined by (2.1) is unitary. □

Now we assume that \mathfrak{g} is a unitary simple Lie subalgebra of $V(1)$. By definition, \mathfrak{g} is preserved by the involution $*$ of $V(1)$. The invariant inner product $\langle \cdot | \cdot \rangle$ on $V(1)$ is also an invariant one of \mathfrak{g} . Recall that our normalized invariant inner product $(\cdot | \cdot)$ of \mathfrak{g} is chosen so that the length of longest roots of \mathfrak{g} is $\sqrt{2}$. Since \mathfrak{g} is simple, there exists $l > 0$ such that $\langle X_1 | X_2 \rangle = l(X_1 | X_2)$ for any $X_1, X_2 \in \mathfrak{g}$. We call l **the level of \mathfrak{g} in V** .

Lemma 2.3. For any $X_1, X_2 \in \mathfrak{g} \subset V(1)$,

$$(2.5) \quad Y(X_1, 1)X_2 = l(X_1|X_2^*)\Omega.$$

Proof. Clearly $Y(X_1, 1)X_2 \in V(0)$. So there exists $c \in \mathbb{C}$ such that $Y(X_1, 1)X_2 = c\Omega$. We now compute, using relation (2.4), that

$$\begin{aligned} c &= \langle c\Omega|\Omega \rangle = \langle Y(X_1, 1)X_2|\Omega \rangle = \langle X_2|Y(X_1^*, -1)\Omega \rangle = \langle X_2|X_1^* \rangle = l(X_2|X_1^*) \\ &= l(X_1|X_2^*). \end{aligned}$$

□

In the following theorem we show that the unitary simple Lie subalgebra $\mathfrak{g} \subset V(1)$ generates a unitary affine vertex subalgebra of V . This theorem can be regarded as the unitary analogue of [DM06, Theorem 3.1].

Theorem 2.4. We have $l \in \mathbb{Z}_{\geq 0}$. The map $\pi : \widehat{\mathfrak{g}} \rightarrow \text{End}(V)$ defined by

$$\begin{aligned} X(n) &\mapsto Y(X, n) \quad (X \in \mathfrak{g}), \\ K &\mapsto l \cdot \text{id}_V \end{aligned}$$

is a PER of $\widehat{\mathfrak{g}}$. Its unitary submodule $\widehat{\mathfrak{g}} \curvearrowright \widehat{\mathfrak{g}}\Omega$ is unitarily equivalent to the vacuum representation $V_{\mathfrak{g}}^l \equiv L_{\mathfrak{g}}(0, l)$ of $\widehat{\mathfrak{g}}$, and Ω is a highest-weight vector of $\widehat{\mathfrak{g}}\Omega$. Let $\varphi : V_{\mathfrak{g}}^l \rightarrow \widehat{\mathfrak{g}}\Omega$ be the unitary $\widehat{\mathfrak{g}}$ -module isomorphism mapping $\Omega \in V_{\mathfrak{g}}^l$ to $\Omega \in \widehat{\mathfrak{g}}\Omega \subset V$. Then φ embeds $V_{\mathfrak{g}}^l$ into V as a unitary vertex subalgebra.

Proof. For any $X_1, X_2 \in \mathfrak{g}, k = 2, 3, 4, \dots, Y(X_1, k)X_2 \in V(1 - k) = 0$. Therefore, by equations (1.11) and (2.5),

$$\begin{aligned} [\pi(X_1, m), \pi(X_2, n)] &= [Y(X_1, m), Y(X_2, n)] = \sum_{k \geq 0} \binom{m}{k} Y(Y(X_1, k)X_2, m + n - k) \\ &= Y(Y(X_1, 0)X_2, m + n) + mY(Y(X_1, 1)X_2, m + n - 1) \\ &= Y([X_1, X_2], m + n) + ml(X_1|X_2^*)Y(\Omega, m + n - 1) \\ &= Y([X_1, X_2], m + n) + ml(X_1|X_2^*)\delta_{m, -n} \text{id}_V \\ &= \pi([X_1, X_2](m + n) + m(X_1|X_2^*)\delta_{m, -n}K) \\ &= \pi([X_1(m), X_2(n)]). \end{aligned}$$

Clearly $\pi(K)$ commutes with any $Y(X, n)$. By (2.4), π is a unitary representation of $\widehat{\mathfrak{g}}$. So π is a level l PER of $\widehat{\mathfrak{g}}$. In particular, $l \in \mathbb{Z}_{\geq 0}$. By state-field correspondence, $Y(X, n)\Omega = 0$ when $n \geq 0$. So Ω is a highest-weight vector with highest weight 0. By Theorem 1.19, there exists an isometry $\varphi : V_{\mathfrak{g}}^l \rightarrow V$, such that $\varphi(\Omega) = \Omega$, and $\varphi X(n) = \pi(X(n))\varphi$ for any $X \in \mathfrak{g}, n \in \mathbb{Z}$. It is clear that φ preserves the gradings of the two vector spaces.

Write $U = V_{\mathfrak{g}}^l$ for simplicity. For clarity, we use different symbols Y_U and Y_V to denote vertex operators of U and V . We also let Θ_U and Θ_V be the PCT operators of U and V , respectively, and let Ω_U, Ω_V be the vacuum vectors of U and V , respectively. To show that φ embeds U into V as a vertex subalgebra, we still need to check that

$$(2.6) \quad \varphi Y_U(u, x) = Y_V(\varphi u, x)\varphi,$$

$$(2.7) \quad \varphi \Theta_U \cdot u = \Theta_V \varphi \cdot u$$

for any $u \in U$.

First, note that

$$\varphi Y_U(X, n) = \varphi X(n) = \pi(X(n))\varphi = Y_V(X, n)\varphi$$

and also that

$$\begin{aligned} \varphi\Theta_U \cdot X(-1)\Omega_U &= -\varphi X^*(-1)\Omega_U = -\pi(X^*(-1))\varphi\Omega_U \\ &= -Y_V(X^*, -1)\Omega_V = Y_V(\Theta_V X, -1)\Theta_V\Omega_V \\ &= \Theta_V Y_V(X, -1)\Omega_V = \Theta_V \pi(X(-1))\varphi\Omega_U = \Theta_V \varphi \cdot X(-1)\Omega_U. \end{aligned}$$

So (2.6) and (2.7) hold for any $u \in U(1)$. Since $U(1)$ generates U , using Jacobi identity (1.10), one can easily show that (2.6) and (2.7) hold for any $u \in U$. \square

In the special case when $\mathfrak{g} = V(1)$ and $V(1)$ generates V , we have $V = \widehat{\mathfrak{g}}\Omega$. So φ is a unitary equivalence between $V_{\mathfrak{g}}^l$ and V . The above theorem can be regarded as a uniqueness theorem for unitary affine VOA: If V is a CFT type unitary VOA generated by the set $V(1)$ of weight-1 vectors, and if the Lie algebra $\mathfrak{g} = V(1)$ is simple, then the unitary VOA V is equivalent to $V_{\mathfrak{g}}^l$ for some $l \in \mathbb{Z}_{\geq 0}$. More generally, if $V(1)$ generates V and has decomposition (1.28), it is not hard to show the following equivalence of unitary VOAs:

$$(2.8) \quad V \simeq V_{\mathfrak{z}}^1 \otimes V_{\mathfrak{g}_1}^{l_1} \otimes \cdots \otimes V_{\mathfrak{g}_n}^{l_n},$$

where $l_1, \dots, l_n \in \mathbb{Z}_{\geq 0}$, and $V_{\mathfrak{z}}^1$ is the unitary (level 1) Heisenberg VOA associated to the unitary abelian Lie algebra \mathfrak{z} whose unitary structure inherits from that of $V(1)$. (A finite-dimensional complex vector space \mathfrak{h} becomes a unitary abelian Lie algebra when \mathfrak{h} is equipped with an inner product and an anti-unitary map $\Theta : \mathfrak{h} \rightarrow \mathfrak{h}$ satisfying $\Theta^2 = \text{id}_{\mathfrak{h}}$.)

Examples. Let \mathfrak{p} be a finite-dimensional unitary simple Lie algebra, and assume that \mathfrak{g} is a unitary simple Lie subalgebra of \mathfrak{p} . As our first example, we study the unitary affine vertex subalgebra arising from $\mathfrak{g} \subset V_{\mathfrak{p}}^l$. Let $\langle \cdot | \cdot \rangle_{\mathfrak{g}}$ and $\langle \cdot | \cdot \rangle_{\mathfrak{p}}$ be the normalized invariant inner product of \mathfrak{g} and \mathfrak{p} , respectively. Then $\langle \cdot | \cdot \rangle_{\mathfrak{p}}$ also restricts to an invariant inner product of \mathfrak{g} . So there exists $k > 0$ such that

$$(2.9) \quad \langle X_1 | X_2 \rangle_{\mathfrak{p}} = k \langle X_1 | X_2 \rangle_{\mathfrak{g}} \quad (\forall X_1, X_2 \in \mathfrak{g}).$$

k is called the **Dynkin index** of the embedding $\mathfrak{g} \subset \mathfrak{p}$ and is denoted by $[\mathfrak{p} : \mathfrak{g}]$.

Proposition 2.5. *For any $l \in \mathbb{Z}_{\geq 0}$, the level of \mathfrak{g} in $V_{\mathfrak{p}}^l$ is kl , where $k = [\mathfrak{p} : \mathfrak{g}]$. Hence we have a unitary VOA embedding $V_{\mathfrak{g}}^{kl} \subset V_{\mathfrak{p}}^l$.*

Proof. Since the level of \mathfrak{p} in $V_{\mathfrak{p}}^l$ is l , the invariant inner product $\langle \cdot | \cdot \rangle$ on \mathfrak{p} inherited from the inner product of $V_{\mathfrak{p}}^l$ is $l \langle \cdot | \cdot \rangle_{\mathfrak{p}}$. When further restricted to \mathfrak{g} , it becomes $kl \langle \cdot | \cdot \rangle_{\mathfrak{g}}$. Therefore the level of \mathfrak{g} in $V_{\mathfrak{p}}^l$ is kl . \square

Corollary 2.6. *The only possible values of Dynkin index $[\mathfrak{p} : \mathfrak{g}]$ are $1, 2, 3, \dots$*

Now let us turn to the second example. Choose $l_1, \dots, l_n \in \mathbb{Z}_{\geq 0}$, and take $V = V_{\mathfrak{g}}^{l_1} \otimes \cdots \otimes V_{\mathfrak{g}}^{l_n}$. Then $V(1) = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$. Regard \mathfrak{g} as a unitary Lie subalgebra of $V(1)$ by embedding \mathfrak{g} diagonally into $V(1)$:

$$(2.10) \quad \begin{aligned} \mathfrak{g} &\subset \mathfrak{g} \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}, \\ X &\mapsto (X, X, \dots, X). \end{aligned}$$

The following result is easy to check.

Proposition 2.7. *The level of \mathfrak{g} in $V_{\mathfrak{g}}^{l_1} \otimes \cdots \otimes V_{\mathfrak{g}}^{l_n}$ corresponding to the diagonal embedding (2.10) is $l_1 + \cdots + l_n$. So we have $V_{\mathfrak{g}}^{l_1+\cdots+l_n} \subset V_{\mathfrak{g}}^{l_1} \otimes \cdots \otimes V_{\mathfrak{g}}^{l_n}$.*

Unitary representations. We now go back to general theory and regard $V_{\mathfrak{g}}^l$ as a unitary vertex subalgebra inside V . The following proposition is obvious.

Proposition 2.8. *Let W_i be a unitary representation of V . The map $\pi_i : \widehat{\mathfrak{g}} \rightarrow \text{End}(W_i)$ defined by*

$$\begin{aligned} X(n) &\mapsto Y_i(X, n) & (X \in \mathfrak{g}), \\ K &\mapsto l \cdot \text{id}_{W_i} \end{aligned}$$

is a PER of $\widehat{\mathfrak{g}}$. We say that $V \curvearrowright W_i$ **restricts to the PER** $\widehat{\mathfrak{g}} \curvearrowright W_i$.

Clearly, the $V_{\mathfrak{g}}^l$ -module $L_{\mathfrak{g}}(\lambda, l)$ described in Theorem 1.21 restricts to the PER $\widehat{\mathfrak{g}} \curvearrowright L_{\mathfrak{g}}(\lambda, l)$.

Theorem 2.9. *Let W_i be a unitary V -module, let $\lambda \in P_+(\mathfrak{g})$, and let $w_{\lambda} \in W_i$ be a non-zero homogeneous vector. If w_{λ} is a λ -highest-weight vector of the restricted PER $\widehat{\mathfrak{g}} \curvearrowright W_i$, then $\widehat{\mathfrak{g}}w_{\lambda}$ is $V_{\mathfrak{g}}^l$ -invariant, and the action $V_{\mathfrak{g}}^l \curvearrowright \widehat{\mathfrak{g}}w_{\lambda}$ is a unitary $V_{\mathfrak{g}}^l$ -module equivalent to $L_{\mathfrak{g}}(\lambda, l)$.*

Proof. Clearly $V_{\mathfrak{g}}^l$ leaves $\widehat{\mathfrak{g}}w_{\lambda}$ invariant. By Remark 1.5, $V_{\mathfrak{g}}^l \curvearrowright W_i$ is a unitary $V_{\mathfrak{g}}^l$ -module. Therefore $V_{\mathfrak{g}}^l \curvearrowright \widehat{\mathfrak{g}}w_{\lambda}$ is also a unitary $V_{\mathfrak{g}}^l$ -module.

Choose a highest-weight vector v_{λ} of $L_{\mathfrak{g}}(\lambda, l)$ satisfying $\|v_{\lambda}\| = \|w_{\lambda}\|$. Since $V_{\mathfrak{g}}^l \curvearrowright \widehat{\mathfrak{g}}w_{\lambda}$ restricts to a highest-weight PER of $\widehat{\mathfrak{g}}$ with highest-weight λ and level l , there exists an (unitary) equivalence $\varphi : L_{\mathfrak{g}}(\lambda, l) \rightarrow \widehat{\mathfrak{g}}w_{\lambda}$ of PERs of $\widehat{\mathfrak{g}}$ satisfying $\varphi v_{\lambda} = w_{\lambda}$. Clearly φ intertwines the actions of $V(1)$ on $L_{\mathfrak{g}}(\lambda, l)$ and on $\widehat{\mathfrak{g}}w_{\lambda}$. Since $V(1)$ generates $V_{\mathfrak{g}}^l$, φ also intertwines the actions of $V_{\mathfrak{g}}^l$. Therefore, φ is an equivalence between the unitary $V_{\mathfrak{g}}^l$ -modules $L_{\mathfrak{g}}(\lambda, l)$ and $\widehat{\mathfrak{g}}w_{\lambda}$. \square

The following lemma will be used later.

Lemma 2.10. *In Theorem 2.9, regard $L_{\mathfrak{g}}(\lambda, l)$ as a V -submodule of W_i by identifying $L_{\mathfrak{g}}(\lambda, l)$ with $\widehat{\mathfrak{g}}w_{\lambda}$. Let $\{L_n\}$ and $\{L_n^{\mathfrak{g}}\}$ be the Virasoro operators of V and $V_{\mathfrak{g}}^l$, respectively. Then there exists $a \in \mathbb{R}$ such that*

$$(2.11) \quad L_0 w^{(\lambda)} = (L_0^{\mathfrak{g}} + a)w^{(\lambda)} \quad (\forall w^{(\lambda)} \in \widehat{\mathfrak{g}}w_{\lambda}).$$

Proof. Regard $V_{\mathfrak{g}}^l$ as inside V as usual. It is obvious that $L_0 u = L_0^{\mathfrak{g}} u$ for any $u \in V_{\mathfrak{g}}^l$. Therefore, by the translation property, for any $w^{(\lambda)} \in \widehat{\mathfrak{g}}w_{\lambda}$, $u \in V_{\mathfrak{g}}^l$,

$$\begin{aligned} [L_0, Y_i(u, x)]w^{(\lambda)} &= (Y_i(L_0 u, x) + x \frac{d}{dx} Y_i(u, x))w^{(\lambda)} \\ &= (Y_i(L_0^{\mathfrak{g}} u, x) + x \frac{d}{dx} Y_i(u, x))w^{(\lambda)} \\ &= [L_0^{\mathfrak{g}}, Y_i(u, x)]w^{(\lambda)}. \end{aligned}$$

So $T := (L_0 - L_0^{\mathfrak{g}})|_{\widehat{\mathfrak{g}}w_{\lambda}}$ commutes with the action of $V_{\mathfrak{g}}^l$ on $\widehat{\mathfrak{g}}w_{\lambda}$. In particular, T commutes with $L_0|_{\widehat{\mathfrak{g}}w_{\lambda}}$. So T leaves the lowest-energy subspace $\mathfrak{g}w_{\lambda}$ of $\widehat{\mathfrak{g}}w_{\lambda}$ invariant. Since $\mathfrak{g}w_{\lambda}$ is finite-dimensional, we can choose an eigenvalue $a \in \mathbb{C}$ of $T|_{\mathfrak{g}w_{\lambda}}$. So $\text{Ker}(T - a)$ is a non-trivial subspace of $\widehat{\mathfrak{g}}w_{\lambda}$ which is clearly $V_{\mathfrak{g}}^l$ -invariant. Since $\widehat{\mathfrak{g}}w_{\lambda}$ is an irreducible $V_{\mathfrak{g}}^l$ -module, $\text{Ker}(T - a)$ must be $\widehat{\mathfrak{g}}w_{\lambda}$. Thus we've proved (2.11). Since the eigenvalues of L_0 and $L_0^{\mathfrak{g}}$ are non-negative numbers, a must be real. \square

2.2. Intertwining operators of unitary affine VOAs. By Theorem 1.21, to study intertwining operators of $V_{\mathfrak{g}}^l$, it is sufficient to study the irreducible ones. So we let $L_{\mathfrak{g}}(\lambda, l), L_{\mathfrak{g}}(\mu, l), L_{\mathfrak{g}}(\nu, l)$ be three irreducible representations of $V_{\mathfrak{g}}^l$, where $\lambda, \mu, \nu \in P_+(\mathfrak{g})$ are admissible at level l . $\mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$ denotes the vector space of type $\binom{\nu}{\lambda \mu} \equiv \binom{L_{\mathfrak{g}}(\lambda, l)}{L_{\mathfrak{g}}(\mu, l) L_{\mathfrak{g}}(\nu, l)}$ intertwining operators of $V_{\mathfrak{g}}^l$. If $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$, we call

$$\Delta_{\alpha} := \Delta_{\lambda} + \Delta_{\mu} - \Delta_{\nu}$$

the **conformal weight** of \mathcal{Y}_{α} .

Consider the lowest-energy subspaces $L_{\mathfrak{g}}(\lambda), L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu)$ of $L_{\mathfrak{g}}(\lambda, l), L_{\mathfrak{g}}(\mu, l), L_{\mathfrak{g}}(\nu, l)$, respectively. By (1.14), for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda), u^{(\mu)} \in L_{\mathfrak{g}}(\mu)$, the conformal weight of $\mathcal{Y}_{\alpha}(u^{(\lambda)}, \Delta_{\alpha} - 1)u^{(\mu)}$ is Δ_{ν} . So $\mathcal{Y}_{\alpha}(u^{(\lambda)}, \Delta_{\alpha} - 1)u^{(\mu)} \in L_{\mathfrak{g}}(\nu)$. We thus define

$$\Psi : \mathcal{V}_{\mathfrak{g}}^l \left(\binom{\nu}{\lambda \mu} \right) \rightarrow \text{Hom}(L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu))$$

by sending each $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$ to the element $\Psi\mathcal{Y}_{\alpha}$ satisfying

$$\begin{aligned} \Psi\mathcal{Y}_{\alpha} : L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) &\rightarrow L_{\mathfrak{g}}(\nu), \\ \Psi\mathcal{Y}_{\alpha}(u^{(\lambda)} \otimes u^{(\mu)}) &= \mathcal{Y}_{\alpha}(u^{(\lambda)}, \Delta_{\alpha} - 1)u^{(\mu)}. \end{aligned}$$

By Jacobi identity (1.12), for any $X \in \mathfrak{g}$ we have

$$X \cdot \mathcal{Y}_{\alpha}(u^{(\lambda)}, \Delta_{\alpha} - 1)u^{(\mu)} - \mathcal{Y}_{\alpha}(u^{(\lambda)}, \Delta_{\alpha} - 1)Xu^{(\mu)} = \mathcal{Y}_{\alpha}(Xu^{(\lambda)}, \Delta_{\alpha} - 1)u^{(\mu)}.$$

So $\Psi\mathcal{Y}_{\alpha}$ intertwines the actions of \mathfrak{g} on the tensor product \mathfrak{g} -module $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$ and on $L_{\mathfrak{g}}(\nu)$. Thus the image of Ψ is inside $\text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$. We thus regard Ψ as a map from $\mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$ to $\text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$.

The following well-known proposition (cf. [TK88, Proposition 2.1], [FZ92, Theorem 3.2.3]) allows us to study the intertwining operators of affine VOAs using Lie-algebraic methods. To make this paper self-contained, we present a proof here following the argument in [TK88].

Proposition 2.11. *The map $\Psi : \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu) \rightarrow \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$ is injective. As a consequence, the fusion rule $N_{\lambda\mu}^{\nu} \equiv \dim \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$ of $V_{\mathfrak{g}}^l$ is a finite number and satisfies*

$$N_{\lambda\mu}^{\nu} \leq \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu).$$

Proof. Suppose $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$ and $\Psi\mathcal{Y}_{\alpha} = 0$. We show that for any $s \in \mathbb{R}$,

$$(2.12) \quad \langle \mathcal{Y}_{\alpha}(w^{(\lambda)}, s)w^{(\mu)} | w^{(\nu)} \rangle = 0$$

for any $w^{(\lambda)} \in L_{\mathfrak{g}}(\lambda, l), w^{(\mu)} \in L_{\mathfrak{g}}(\mu, l), w^{(\nu)} \in L_{\mathfrak{g}}(\nu, l)$. By Jacobi identity (1.10), it suffices to prove this when $w^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$. Let

$$k(w^{(\mu)}, w^{(\nu)}) = \Delta_{w^{(\mu)}} - \Delta_{\mu} + \Delta_{w^{(\nu)}} - \Delta_{\nu} \in \mathbb{Z}_{\geq 0},$$

where $\Delta_{w^{(\mu)}}, \Delta_{w^{(\nu)}}$ are the conformal weights of $w^{(\mu)}, w^{(\nu)}$, respectively. We prove (2.12) by induction on $k(w^{(\mu)}, w^{(\nu)})$.

If $k(w^{(\mu)}, w^{(\nu)}) = 0$, then $w^{(\mu)}, w^{(\nu)}$ are also in the lowest-energy subspaces. By (1.14), $\mathcal{Y}_{\alpha}(w^{(\lambda)}, s)w^{(\mu)}$ is a homogeneous vector with conformal weight $\Delta_{\lambda} + \Delta_{\mu} - s - 1 = \Delta_{\alpha} + \Delta_{\nu} - s - 1$. So clearly (2.12) holds when $s \neq \Delta_{\alpha} - 1$. Since

$\Psi\mathcal{Y}_\alpha = 0$, (2.12) is also true when $s = \Delta_\alpha - 1$. So we've proved (2.12) for any s when $k(w^{(\mu)}, w^{(\nu)}) = 0$.

Now choose an arbitrary $N \in \mathbb{Z}_{>0}$ and assume that for any s , (2.12) holds whenever $k(w^{(\mu)}, w^{(\nu)}) = 0, 1, 2, \dots, N - 1$. We show that this is also true when $k(w^{(\mu)}, w^{(\nu)}) = N$. Since $N > 0$, either $w^{(\mu)}$ or $w^{(\nu)}$ must have non-lowest conformal weight (energy). We first assume that $w^{(\mu)} \notin L_{\mathfrak{g}}(\mu)$. Then $w^{(\mu)}$ can be written as a linear combination of vectors of the form $X(-n)w_0^{(\mu)}$, where $X \in \mathfrak{g}, n \in \mathbb{Z}_{>0}$, and $w_0^{(\mu)} \in L_{\mathfrak{g}}(\mu, l)$ is homogeneous. So $\Delta_{w_0^{(\mu)}} = \Delta_{w^{(\mu)}} - n$. By Jacobi identity (1.11),

$$\begin{aligned} & \langle \mathcal{Y}_\alpha(w^{(\lambda)}, s)X(-n)w_0^{(\mu)} | w^{(\nu)} \rangle \\ &= \langle \mathcal{Y}_\alpha(w^{(\lambda)}, s)w_0^{(\mu)} | X^*(n)w^{(\nu)} \rangle - \sum_{l \in \mathbb{Z}_{\geq 0}} \binom{-n}{l} \langle \mathcal{Y}_\alpha(X(l)w^{(\lambda)}, -n + s - l)w_0^{(\mu)} | w^{(\nu)} \rangle. \end{aligned}$$

Since $w^{(\lambda)}$ has lowest energy, the right hand side of the above equation becomes

$$\langle \mathcal{Y}_\alpha(w^{(\lambda)}, s)w_0^{(\mu)} | X^*(n)w^{(\nu)} \rangle - \langle \mathcal{Y}_\alpha(Xw^{(\lambda)}, -n + s)w_0^{(\mu)} | w^{(\nu)} \rangle.$$

Since $k(w_0^{(\mu)}, X^*(n)w^{(\nu)}) < N$ and $k(w_0^{(\mu)}, w^{(\nu)}) < N$, by induction, the above expression is 0. So we've proved (2.12) for $k(w^{(\mu)}, w^{(\nu)}) = N$ when $w^{(\mu)} \notin L_{\mathfrak{g}}(\mu)$. The case $w^{(\nu)} \notin L_{\mathfrak{g}}(\nu)$ is treated in a similar way. \square

Thus the vector space $\mathcal{V}_{\mathfrak{g}}^l(\lambda^\nu_\mu)$ can be identified with its image under the map Ψ . For most examples considered in this paper, Ψ is in fact surjective. But this is not true in general. We refer the reader to [FZ92, Theorem 3.2.3] for a general description of the image of Ψ .

Theorem 2.12 (Compression principle). *Let V be a unitary VOA of CFT type, and let \mathfrak{g} be a unitary simple Lie subalgebra of $V(1)$. Let l be the level of \mathfrak{g} in V . Let W_i, W_j, W_k be unitary representations of V . Assume that w_λ (resp., w_μ, w_ν) is a (non-zero) homogeneous highest weight vector of $\widehat{\mathfrak{g}} \curvearrowright W_i$ (resp., $\widehat{\mathfrak{g}} \curvearrowright W_j, \widehat{\mathfrak{g}} \curvearrowright W_k$) with highest-weight λ (resp., μ, ν), and choose isometric \mathfrak{g} -module homomorphisms $\varphi_\lambda : L_{\mathfrak{g}}(\lambda) \rightarrow W_i$ (resp., $\varphi_\mu : L_{\mathfrak{g}}(\mu) \rightarrow W_j, \varphi_\nu : L_{\mathfrak{g}}(\nu) \rightarrow W_k$) with image $\mathfrak{g}w_\lambda$ (resp., $\mathfrak{g}w_\mu, \mathfrak{g}w_\nu$). Let $\varphi_\nu^\dagger : W_k \rightarrow L_{\mathfrak{g}}(\nu)$ be the formal adjoint of φ_ν ; i.e., φ_ν^\dagger satisfies*

$$\langle \varphi_\nu u^{(\nu)} | w^{(k)} \rangle = \langle u^{(\nu)} | \varphi_\nu^\dagger w^{(k)} \rangle \quad (\forall u^{(\nu)} \in L_{\mathfrak{g}}(\nu), w^{(k)} \in W_k).$$

If there exist $s \in \mathbb{R}$ and a type $\binom{k}{i \ j}$ intertwining operator \mathfrak{Y} of V , such that the map

$$(2.13) \quad \begin{aligned} & \varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda, s)\varphi_\mu : L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) \rightarrow L_{\mathfrak{g}}(\nu), \\ & u^{(\lambda)} \otimes u^{(\mu)} \mapsto \varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda u^{(\lambda)}, s)\varphi_\mu u^{(\mu)} \end{aligned}$$

is non-zero, then $\varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda, s)\varphi_\mu \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$, and there exists a type $\binom{\nu}{\lambda \ \mu}$ intertwining operator \mathcal{Y} of $V_{\mathfrak{g}}^l$ such that

$$(2.14) \quad \Psi\mathcal{Y} = \varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda, s)\varphi_\mu.$$

If, moreover, $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$ and $\mathfrak{Y}(\varphi_\lambda u^{(\lambda)}, x)$ satisfies r -th order energy bounds for some $r \geq 0$, then $\mathcal{Y}(u^{(\lambda)}, x)$ also satisfies r -th order energy bounds.

Proof. By Theorem 2.9, we can extend φ_λ (resp., φ_μ, φ_ν) to an isometric $V_{\mathfrak{g}}^l$ -module homomorphism $\phi_\lambda : L_{\mathfrak{g}}(\lambda, l) \rightarrow W_i$ (resp., $\phi_\mu : L_{\mathfrak{g}}(\mu, l) \rightarrow W_j, \phi_\nu : L_{\mathfrak{g}}(\nu, l) \rightarrow W_k$) with image $\widehat{\mathfrak{g}}w_\lambda$ (resp., $\widehat{\mathfrak{g}}w_\mu, \widehat{\mathfrak{g}}w_\nu$). Define

$$\begin{aligned} \mathcal{Y}_0 : L_{\mathfrak{g}}(\lambda, l) \otimes L_{\mathfrak{g}}(\mu, l) &\rightarrow L_{\mathfrak{g}}(\nu, l)\{x\}, \\ v^{(\lambda)} \otimes v^{(\mu)} \mapsto \mathcal{Y}_0(v^{(\lambda)}, x)v^{(\mu)} &:= \phi_\nu^\dagger \mathfrak{Y}(\phi_\lambda v^{(\lambda)}, x)\phi_\mu v^{(\mu)}. \end{aligned}$$

Then \mathcal{Y}_0 clearly satisfies the Jacobi identity (1.9) with the vertex operator of $V_{\mathfrak{g}}^l$. Note that $\widehat{\mathfrak{g}}w_\lambda, \widehat{\mathfrak{g}}w_\mu, \widehat{\mathfrak{g}}w_\nu$ are graded subspaces of W_i, W_j, W_k , respectively. Let $\{L_n\}$ and $\{L_n^{\mathfrak{g}}\}$ be the Virasoro operators of V and $V_{\mathfrak{g}}^l$, respectively. Then these subspaces are L_0 invariant. By the translation property (1.13) for \mathfrak{Y} , for any $v^{(\lambda)} \in L_{\mathfrak{g}}(\lambda, l)$,

$$\phi_\nu^\dagger L_0 \phi_\nu \cdot \mathcal{Y}_0(v^{(\lambda)}, x) - \mathcal{Y}_0(v^{(\lambda)}, x) \cdot \phi_\mu^\dagger L_0 \phi_\mu = \mathcal{Y}_0(\phi_\lambda^\dagger L_0 \phi_\lambda v^{(\lambda)}, x) + x \frac{d}{dx} \mathcal{Y}_0(v^{(\lambda)}, x).$$

By Lemma 2.10, the action of $\phi_\lambda^\dagger L_0 \phi_\lambda - L_0^{\mathfrak{g}}$ (resp., $\phi_\mu^\dagger L_0 \phi_\mu - L_0^{\mathfrak{g}}, \phi_\nu^\dagger L_0 \phi_\nu - L_0^{\mathfrak{g}}$) on $L_{\mathfrak{g}}(\lambda, l)$ (resp., $L_{\mathfrak{g}}(\mu, l), L_{\mathfrak{g}}(\nu, l)$) is a real constant. So we can easily find $t \in \mathbb{R}$ such that the map

$$\begin{aligned} \mathcal{Y} : L_{\mathfrak{g}}(\lambda, l) \otimes L_{\mathfrak{g}}(\mu, l) &\rightarrow L_{\mathfrak{g}}(\nu, l)\{x\}, \\ v^{(\lambda)} \otimes v^{(\mu)} \mapsto \mathcal{Y}(v^{(\lambda)}, x)v^{(\mu)} &:= x^t \mathcal{Y}_0(v^{(\lambda)}, x)v^{(\mu)} \end{aligned}$$

satisfies the translation property

$$[L_0^{\mathfrak{g}}, \mathcal{Y}(v^{(\lambda)}, x)] = \mathcal{Y}(L_0^{\mathfrak{g}}v^{(\lambda)}, x) + x \frac{d}{dx} \mathcal{Y}(v^{(\lambda)}, x).$$

So \mathcal{Y} is a type $\binom{\nu}{\lambda \mu}$ intertwining operator of $V_{\mathfrak{g}}^l$. One can easily check that $\Psi\mathcal{Y} = \mathfrak{Y}$ and that for any homogeneous $v^{(\lambda)} \in L_{\mathfrak{g}}(\lambda, l)$, $\mathcal{Y}(v^{(\lambda)}, x)$ satisfies r -th order energy bounds if $\mathfrak{Y}(\phi_\lambda v^{(\lambda)}, x)$ does. \square

In this article, the compression principle is more often used in the following simpler form.

Corollary 2.13. *Let $V, \mathfrak{g}, l, W_i, W_j, W_k, w_\lambda, w_\mu, w_\nu$ be as in Theorem 2.12. Identify $\widehat{\mathfrak{g}}w_\lambda, \widehat{\mathfrak{g}}w_\mu, \widehat{\mathfrak{g}}w_\nu$ with $L_{\mathfrak{g}}(\lambda, l), L_{\mathfrak{g}}(\mu, l), L_{\mathfrak{g}}(\nu, l)$, respectively. If*

$$\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \leq 1$$

and there exists a type $\binom{k}{i j}$ intertwining operator \mathfrak{Y} of V such that

$$\langle \mathfrak{Y}(\mathfrak{g}w_\lambda, x)\mathfrak{g}w_\mu | \mathfrak{g}w_\nu \rangle \neq 0,$$

then

$$N_{\lambda\mu}^\nu = \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 1,$$

where $N_{\lambda\mu}^\nu := \dim \mathcal{V}_{\mathfrak{g}}^l \binom{\nu}{\lambda \mu}$ is a fusion rule of $V_{\mathfrak{g}}^l$. Moreover, let $u^{(\lambda)} \in \mathfrak{g}w_\lambda$, and suppose that $\mathfrak{Y}(u^{(\lambda)}, x)$ satisfies r -th order energy bounds for some $r \geq 0$. Then for any type $\binom{\nu}{\lambda \mu}$ intertwining operator \mathcal{Y} of $V_{\mathfrak{g}}^l$, $\mathcal{Y}(u^{(\lambda)}, x)$ satisfies r -th order energy bounds.

As an application of the compression principle, we now show that when the level l is large, the fusion rules of $V_{\mathfrak{g}}^l$ equal the corresponding tensor product rules of \mathfrak{g} , and the energy bounds conditions for general intertwining operators can be deduced from those for creation operators. To begin with, we choose $\lambda, \mu, \nu \in P_+(\mathfrak{g})$ admissible at level l . As usual, we let θ be the highest root of \mathfrak{g} .

Proposition 2.14. *Let $a = (\lambda|\theta)$. Assume that $(\mu|\theta) \leq l - a$ or $(\nu|\theta) \leq l - a$. Then*

$$N_{\lambda\mu}^\nu = \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu);$$

i.e., the map $\Psi : \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu) \rightarrow \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$ is bijective. Let $\mathcal{Y}_{\kappa(\lambda)} \in \mathcal{V}_{\mathfrak{g}}^a(\lambda \ 0)$ be the creation operator of $L_{\mathfrak{g}}(\lambda, a)$. Choose $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$. Assume, moreover, that there exists $r \geq 0$ such that $\mathcal{Y}_{\kappa(\lambda)}(u^{(\lambda)}, x)$ satisfies r -th order energy bounds. Then for any $\mathcal{Y}_\alpha \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu \mu)$, $\mathcal{Y}_\alpha(u^{(\lambda)}, x)$ also satisfies r -th order energy bounds.

Proof. Let $b = l - a$. Suppose $(\mu|\theta) \leq b$. Choose non-zero $T \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$. Since $L_{\mathfrak{g}}(\nu)$ is irreducible, $TT^* = c \cdot \text{id}_{L_{\mathfrak{g}}(\nu)}$ for some $c > 0$. Assume, without loss of generality, that $c = 1$. So $T^* : L_{\mathfrak{g}}(\nu) \rightarrow L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$ is an isometry. Let $V = V_{\mathfrak{g}}^a \otimes V_{\mathfrak{g}}^b$, and choose unitary V -modules

$$W_i = L_{\mathfrak{g}}(\lambda, a) \otimes L_{\mathfrak{g}}(0, b), \quad W_j = L_{\mathfrak{g}}(0, a) \otimes L_{\mathfrak{g}}(\mu, b), \quad W_k = L_{\mathfrak{g}}(\lambda, a) \otimes L_{\mathfrak{g}}(\mu, b).$$

Let Y_μ be the vertex operator of $V_{\mathfrak{g}}^b$ on $L_{\mathfrak{g}}(\mu, b)$. So $Y_\mu \in \mathcal{V}_{\mathfrak{g}}^b(0 \ \mu)$. Now it is clear that

$$\mathfrak{Y} := \mathcal{Y}_{\kappa(\lambda)} \otimes Y_\mu$$

is a type $\binom{k}{i \ j}$ intertwining operator of V . Let v_λ, v_μ, v_ν be the highest-weight vectors of $L_{\mathfrak{g}}(\lambda), L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu)$ of unit length, respectively. Choose an isometric \mathfrak{g} -module homomorphism $\varphi_\lambda : L_{\mathfrak{g}}(\lambda) \rightarrow W_i$ (resp., $\varphi_\mu : L_{\mathfrak{g}}(\mu) \rightarrow W_j$) with image $L_{\mathfrak{g}}(\lambda) \otimes \Omega$ (resp., $\Omega \otimes L_{\mathfrak{g}}(\mu)$) satisfying $\varphi_\lambda u^{(\lambda)} = u^{(\lambda)} \otimes \Omega$ (resp., $\varphi_\mu u^{(\mu)} = \Omega \otimes u^{(\mu)}$) for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$ (resp., $u^{(\mu)} \in L_{\mathfrak{g}}(\mu)$). Note that $L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$ is the lowest-energy subspace of W_k . We extend $T^* : L_{\mathfrak{g}}(\nu) \rightarrow L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu)$ to $\varphi_\nu : L_{\mathfrak{g}}(\nu) \rightarrow W_k$. So φ_ν is also an isometric \mathfrak{g} -module homomorphism, and the image of φ_ν has the lowest conformal weight (energy). It follows that $\varphi_\nu v_\nu$ is a highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright W_k$ with highest weight ν . Now consider the \mathfrak{g} -module homomorphism

$$\varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda, -1)\varphi_\mu : L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) \rightarrow L_{\mathfrak{g}}(\nu).$$

For any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda), u^{(\mu)} \in L_{\mathfrak{g}}(\mu)$, we compute

$$\begin{aligned} \varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda u^{(\lambda)}, -1)\varphi_\mu u^{(\mu)} &= \varphi_\nu^\dagger \mathfrak{Y}(u^{(\lambda)} \otimes \Omega, -1)(\Omega \otimes u^{(\mu)}) \\ &= \varphi_\nu^\dagger (\mathcal{Y}_{\kappa(\lambda)}(u^{(\lambda)}, -1) \otimes \text{id})(\Omega \otimes u^{(\mu)}) \\ &= \varphi_\nu^\dagger (u^{(\lambda)} \otimes u^{(\mu)}) = T(u^{(\lambda)} \otimes u^{(\mu)}). \end{aligned}$$

Hence $\varphi_\nu^\dagger \mathfrak{Y}(\varphi_\lambda, -1)\varphi_\mu = T$. By Theorem 2.12, there exists $\mathcal{Y}_\alpha \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \ \nu \ \mu)$ satisfying $\Psi \mathcal{Y}_\alpha = T$, and if for a vector $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, $\mathcal{Y}_{\kappa(\lambda)}(u^{(\lambda)}, x) \otimes \text{id} = \mathfrak{Y}(u^{(\lambda)} \otimes \Omega, x)$ satisfies r -th order energy bounds, then the same property holds for $\mathcal{Y}_\alpha(u^{(\lambda)}, x)$.

Now suppose that $(\nu|\theta) \leq l - a$. Since $(\bar{\lambda}|\theta) = (\lambda|\theta) = a$, we have $N_{\bar{\lambda}\nu}^\mu = \dim \text{Hom}_{\mathfrak{g}}(\bar{\lambda} \otimes \nu, \mu)$. Therefore

$$N_{\lambda\mu}^\nu = N_{\bar{\lambda}\nu}^\mu = \dim \text{Hom}_{\mathfrak{g}}(\bar{\lambda} \otimes \nu, \mu) = \dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu).$$

Since the conjugate intertwining operator $\mathcal{Y}_{\kappa(\lambda)}^\overline{(\cdot)}(C_\lambda u^{(\lambda)}, x)$ of $\mathcal{Y}_{\kappa(\lambda)}(u^{(\lambda)}, x)$ clearly satisfies r -th order energy bounds, by what we've proved above, $\mathcal{Y}_{\alpha^*}(C_\lambda u^{(\lambda)}, x)$ satisfies r -th order energy bounds for any $\mathcal{Y}_\alpha \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \ \nu \ \mu)$. By Proposition 1.8, $\mathcal{Y}_\alpha(u^{(\lambda)}, x)$ also satisfies r -th order energy bounds. □

The situation becomes more complicated when l is not so large, say, when $(\mu|\theta) \leq l \leq (\lambda|\theta) + (\mu|\theta)$. In this case, the following technical lemma will be helpful. The reader might temporarily skip this lemma and return to it later.

Lemma 2.15. *Let $l \in \mathbb{Z}_{\geq 0}$. Choose $\lambda, \mu, \nu, \rho, \mu_1, \nu_1 \in P_+(\mathfrak{g})$ admissible at level l , and choose highest-weight vectors $v_{\mu_1} \in L_{\mathfrak{g}}(\mu_1)$ and $v_{\nu_1} \in L_{\mathfrak{g}}(\nu_1)$. Set $a = \max\{(\lambda|\theta), (\mu_1|\theta), (\nu_1|\theta)\}$. Assume that the weight spaces of $L_{\mathfrak{g}}(\lambda)$ have dimensions at most 1, that*

$$(2.15) \quad a \leq l, \quad (\rho|\theta) \leq l - a,$$

that $\dim \mathcal{V}_{\mathfrak{g}}^a(\lambda \begin{smallmatrix} \nu_1 \\ \mu_1 \end{smallmatrix}) = 1$, and that there exists $r \geq 0$, such that for any $\mathcal{Y}_{\sigma} \in \mathcal{V}_{\mathfrak{g}}^a(\lambda \begin{smallmatrix} \nu_1 \\ \mu_1 \end{smallmatrix})$ and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, $\mathcal{Y}_{\sigma}(u^{(\lambda)}, x)$ satisfies r -th order energy bounds. If one of the three conditions stated below holds, then $N_{\lambda\mu}^{\nu} := \dim \mathcal{V}_{\mathfrak{g}}^l(\lambda \begin{smallmatrix} \nu \\ \mu \end{smallmatrix}) = 1$, and for any $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{g}}^a(\lambda \begin{smallmatrix} \nu \\ \mu \end{smallmatrix})$ and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, $\mathcal{Y}_{\alpha}(u^{(\lambda)}, x)$ satisfies r -th order energy bounds. The three conditions are:

- (a) $\mu = \mu_1 + \rho$ and $\nu = \nu_1 + \rho$.
- (b) $\mu = \mu_1 + \rho$, $L_{\mathfrak{g}}(\nu)$ is equivalent to a submodule of $L_{\mathfrak{g}}(\nu_1) \otimes L_{\mathfrak{g}}(\rho)$, the weight spaces of $L_{\mathfrak{g}}(\nu_1)$ have dimensions at most 1, and

$$(2.16) \quad T(L_{\mathfrak{g}}(\lambda)[\nu - \mu] \otimes v_{\mu_1}) \neq 0$$

for any non-zero $T \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu_1, \nu_1)$.

- (c) $\nu = \nu_1 + \rho$, $L_{\mathfrak{g}}(\mu)$ is equivalent to a submodule of $L_{\mathfrak{g}}(\mu_1) \otimes L_{\mathfrak{g}}(\rho)$, the weight spaces of $L_{\mathfrak{g}}(\mu_1)$ have dimensions at most 1, and for any non-zero $T \in \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu_1, \nu_1)$, there exist $u_{\nu-\mu}^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]$, $u^{(\mu_1)} \in L_{\mathfrak{g}}(\mu_1)$ such that

$$(2.17) \quad \langle T(u_{\nu-\mu}^{(\lambda)} \otimes u^{(\mu_1)}) | v_{\nu_1} \rangle \neq 0.$$

Proof. Let $b = l - a$. Then ρ is admissible at level b . Let $V = V_{\mathfrak{g}}^a \otimes V_{\mathfrak{g}}^b$, and choose unitary V -modules

$$W_i = L_{\mathfrak{g}}(\lambda, a) \otimes L_{\mathfrak{g}}(0, b), \quad W_j = L_{\mathfrak{g}}(\mu_1, a) \otimes L_{\mathfrak{g}}(\rho, b), \quad W_k = L_{\mathfrak{g}}(\nu_1, a) \otimes L_{\mathfrak{g}}(\rho, b).$$

Choose any non-zero $\mathcal{Y}_{\sigma} \in \mathcal{V}_{\mathfrak{g}}^a(\lambda \begin{smallmatrix} \nu_1 \\ \mu_1 \end{smallmatrix})$. Then

$$\mathfrak{Y} = \mathcal{Y}_{\sigma} \otimes Y_{\rho}$$

is a type $\begin{pmatrix} k \\ i \ j \end{pmatrix}$ intertwining operator of V . Consider the diagonal Lie subalgebra $\mathfrak{g} \subset V(1) = \mathfrak{g} \oplus \mathfrak{g}$. Then \mathfrak{g} has level $l = a + b$ in V .

Let v_{λ} and v_{ρ} be the highest-weight vectors of $L_{\mathfrak{g}}(\lambda)$ and $L_{\mathfrak{g}}(\rho)$, respectively. Let $w_{\lambda} = v_{\lambda} \otimes \Omega$. Then w_{λ} is a level l highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright W$ with highest weight λ . Since $\mathfrak{g}w_{\lambda} = L_{\mathfrak{g}}(\lambda) \otimes \Omega$, $\mathfrak{Y}(w^{(i)}, x)$ satisfies r -th order energy bounds for any $w^{(i)} \in \mathfrak{g}w_{\lambda}$. In the following, we shall construct, for cases (a) and (b), highest-weight vectors w_{μ}, w_{ν} of the level l $\widehat{\mathfrak{g}}$ -modules W_j, W_k with highest weights μ, ν , respectively, and show that

$$(2.18) \quad \langle \mathfrak{Y}(\mathfrak{g}w_{\lambda}, x) \mathfrak{g}w_{\mu} | \mathfrak{g}w_{\nu} \rangle \neq 0.$$

Then Corollary 2.13 will imply the statement claimed in this proposition. Case (c) will follow from case (b).

(a) Let $w_\mu = v_{\mu_1} \otimes v_\rho$ and $w_\nu = v_{\nu_1} \otimes v_\rho$. Then these two vectors are highest-weight vectors with weights μ and ν , respectively. Note that by Propositions 1.16 and 2.11,

$$\begin{aligned} 1 &= \dim \mathcal{V}_\mathfrak{g}^a \begin{pmatrix} \nu_1 \\ \lambda \mu_1 \end{pmatrix} \leq \dim \text{Hom}_\mathfrak{g}(\lambda \otimes \mu_1, \nu_1) \leq \dim L_\mathfrak{g}(\lambda)[\nu_1 - \mu_1] \\ &= \dim L_\mathfrak{g}(\lambda)[\nu - \mu] \leq 1. \end{aligned}$$

So all the intermediate terms in the above inequality equal 1. We choose a non-zero vector $u_{\nu-\mu}^{(\lambda)} \in L_\mathfrak{g}(\lambda)[\nu - \mu]$. Then $u_{\nu-\mu}^{(\lambda)} \otimes \Omega \in \mathfrak{g}w_\lambda$. Let Δ_σ be the conformal weight of \mathcal{Y}_σ , and set $s = \Delta_\sigma - 1$. We compute

$$\begin{aligned} &\langle \mathfrak{Y}(u_{\nu-\mu}^{(\lambda)} \otimes \Omega, s)w_\mu | w_\nu \rangle \\ &= \langle \mathfrak{Y}(u_{\nu-\mu}^{(\lambda)} \otimes \Omega, s)(v_{\mu_1} \otimes v_\rho) | v_{\nu_1} \otimes v_\rho \rangle = \|v_\rho\|^2 \langle \mathcal{Y}_\sigma(u_{\nu-\mu}^{(\lambda)}, s)v_{\mu_1} | v_{\nu_1} \rangle, \end{aligned}$$

which must be non-zero since the map $\Gamma \circ \Psi : \mathcal{V}_\mathfrak{g}^a \begin{pmatrix} \nu_1 \\ \lambda \mu_1 \end{pmatrix} \rightarrow L_\mathfrak{g}(\lambda)[\nu_1 - \mu_1]^*$ is injective. Thus (2.18) follows.

(b) Condition (2.16) implies that $L_\mathfrak{g}(\lambda)[\nu - \mu]$ has positive dimension, which must be 1. We again let $u_{\nu-\mu}^{(\lambda)}$ be a non-zero vector of $L_\mathfrak{g}(\lambda)[\nu - \mu]$. So $u_{\nu-\mu}^{(\lambda)} \otimes \Omega$ is inside $\mathfrak{g}w_\lambda$. Let $w_\mu = v_{\mu_1} \otimes v_\rho$ be a μ -highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright W_j$. We now define w_ν . Since the non-trivial weight spaces of $L_\mathfrak{g}(\nu_1)$ have dimension 1, $\dim \text{Hom}_\mathfrak{g}(\nu_1 \otimes \rho, \nu) = 1$. Thus, up to scalar multiplication, there is a unique ν -highest-weight vector of $\mathfrak{g} \curvearrowright L_\mathfrak{g}(\nu_1) \otimes L_\mathfrak{g}(\rho)$. We let w_ν be such a vector. Identify $L_\mathfrak{g}(\nu_1) \otimes L_\mathfrak{g}(\rho)$ as the lowest-energy (conformal weight) subspace of W_k . Then w_ν is a ν -highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright W_k$.

Since $\mathcal{Y}_\sigma \in \mathcal{V}_\mathfrak{g}^a \begin{pmatrix} \nu_1 \\ \lambda \mu_1 \end{pmatrix}$ is non-zero, $T = \Psi(\mathcal{Y}_\sigma)$ is a non-zero element in $\text{Hom}_\mathfrak{g}(\lambda \otimes \mu_1, \nu_1)$. Let again $s = \Delta_\sigma - 1$. Then the vector

$$(2.19) \quad u_{\nu-\rho}^{(\nu_1)} := \mathcal{Y}_\sigma(u_{\nu-\mu}^{(\lambda)}, s)v_{\mu_1}$$

is non-zero by condition (2.16). Clearly $u_{\nu-\rho}^{(\nu_1)}$ is a weight vector of $L_\mathfrak{g}(\nu_1)$ with weight $\nu - \mu + \mu_1 = \nu - \rho$. So $u_{\nu-\rho}^{(\nu_1)} \in L_\mathfrak{g}(\nu_1)[\nu - \rho]$. Since the map

$$\Gamma : \text{Hom}_\mathfrak{g}(\nu_1 \otimes \rho, \nu) \rightarrow L_\mathfrak{g}(\nu_1)[\nu - \rho]^*$$

is injective, we must have $\langle u_{\nu-\rho}^{(\nu_1)} \otimes v_\rho | w_\nu \rangle \neq 0$. We now compute

$$\begin{aligned} &\langle \mathfrak{Y}(u_{\nu-\mu}^{(\lambda)} \otimes \Omega, s)w_\mu | w_\nu \rangle = \langle \mathfrak{Y}(u_{\nu-\mu}^{(\lambda)} \otimes \Omega, s)(v_{\mu_1} \otimes v_\rho) | w_\nu \rangle \\ &= \langle \mathcal{Y}_\sigma(u_{\nu-\mu}^{(\lambda)}, s)v_{\mu_1} \otimes v_\rho | w_\nu \rangle = \langle u_{\nu-\rho}^{(\nu_1)} \otimes v_\rho | w_\nu \rangle \neq 0. \end{aligned}$$

Thus (2.18) is proved.

(c) Choose an arbitrary non-zero $T \in \text{Hom}_\mathfrak{g}(\lambda \otimes \mu_1, \nu_1)$. Then $\text{Ad } T \in \text{Hom}_\mathfrak{g}(\overline{\lambda} \otimes \nu_1, \mu_1)$, and the condition that (2.17) holds for some $u_{\nu-\mu}^{(\lambda)} \in L_\mathfrak{g}(\lambda)$, $u^{(\mu_1)} \in L_\mathfrak{g}(\mu_1)$ is equivalent to

$$(2.20) \quad \text{Ad } T(L_\mathfrak{g}(\overline{\lambda})[\mu - \nu] \otimes v_{\nu_1}) \neq 0.$$

Here we use the fact that $C_\lambda \cdot L_\mathfrak{g}(\lambda)[\eta] = L_\mathfrak{g}(\overline{\lambda})[-\eta]$ for any $\eta \in \mathfrak{h}^*$, which is easy to check using the definition of dual representations, and we let in particular η be $\nu - \mu$. Now, $\dim \mathcal{V}_\mathfrak{g}^a \begin{pmatrix} \mu_1 \\ \lambda \nu_1 \end{pmatrix} = \dim \mathcal{V}_\mathfrak{g}^a \begin{pmatrix} \nu_1 \\ \lambda \mu_1 \end{pmatrix} = 1$. By Proposition 1.8, $\mathcal{Y}_{\sigma^*}(C_\lambda u^{(\lambda)}, x)$ satisfies r -th order energy bounds. Thus case (b) implies that $N_{\lambda\nu}^\mu = 1$, and for any $\mathcal{Y}_\alpha \in \mathcal{V}_\mathfrak{g}^a \begin{pmatrix} \nu \\ \lambda \mu \end{pmatrix}$ and $u^{(\lambda)} \in L_\mathfrak{g}(\lambda)$, $\mathcal{Y}_{\alpha^*}(C_\lambda u^{(\lambda)}, x)$ satisfies r -th order energy bounds.

Therefore, $N_{\lambda\mu}^\nu = N_{\lambda\nu}^\mu = 1$ and, by Proposition 1.8 again, $\mathcal{Y}_\alpha(u^{(\lambda)}, x)$ satisfies r -th order energy bounds. □

3. TYPE B

3.1. $\mathfrak{so}_{2n+1} \subset \mathfrak{so}_{2n+2}$. Let $n = 2, 3, 4, \dots$. \mathfrak{so}_{2n+2} be the unitary Lie algebra of skew-symmetric linear operators on \mathbb{C}^{2n+2} . Likewise, elements in \mathfrak{so}_{2n+1} are skew-symmetric linear operators on \mathbb{C}^{2n+1} . Let $e_1, e_2, \dots, e_{2n+2}$ be the standard orthonormal basis of \mathbb{C}^{2n+2} . Regard \mathbb{C}^{2n+1} as a subspace of \mathbb{C}^{2n+2} spanned by e_1, \dots, e_{2n+1} . For any $X \in \mathfrak{so}_{2n+1}$, we extend it to a linear operator on \mathbb{C}^{2n+2} by setting $Xe_{2n+2} = 0$. In this way, we embed \mathfrak{so}_{2n+1} as a unitary Lie subalgebra of \mathfrak{so}_{2n+2} .

For any $i, j = 1, 2, \dots, 2n + 2$, we choose $E_{i,j} \in \text{End}(\mathbb{C}^{2n+2})$ to satisfy $E_{i,j}e_k = \delta_{j,k}e_i$ ($\forall k = 1, 2, \dots, 2n + 2$). Then for any $X = \sum c_{ij}E_{i,j} \in \text{End}(\mathbb{C}^{2n+2})$, $X \in \mathfrak{so}_{2n+2}$ if and only if $c_{ij} = -c_{ji}$ for all $i, j = 1, \dots, 2n + 2$. If $X \in \mathfrak{so}_{2n+2}$, then $X \in \mathfrak{so}_{2n+1}$ if and only if $c_{k,2n+2} = c_{2n+2,k} = 0$ for all $k = 1, \dots, 2n + 2$.

Write $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{p} = \mathfrak{so}_{2n+2}$ for simplicity. In this section, we collect some basic facts about these two Lie algebras. First we choose Cartan subalgebras. See, for instance, [FH91] for more details. For any $i = 1, \dots, n + 1$, we let

$$S_i = -\mathbf{i}E_{2i+1,2i+2} + \mathbf{i}E_{2i+2,2i+1}.$$

Here \mathbf{i} is the imaginary number $\sqrt{-1}$, not to be confused with the index i . Let $\mathfrak{t} = \text{Span}_{\mathbb{C}}\{S_1, \dots, S_{n+1}\}$ and $\mathfrak{h} = \text{Span}_{\mathbb{C}}\{S_1, \dots, S_n\}$. Then \mathfrak{t} is a Cartan subalgebra of \mathfrak{p} , and \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} , which is a unitary Lie subalgebra of \mathfrak{t} at the same time. Let $\{\theta_1, \dots, \theta_{n+1}\}$ be the dual basis of $\{S_1, \dots, S_{n+1}\}$; i.e., for any $i, j = 1, \dots, n + 1$, $\theta_i \in \mathfrak{t}^*$ satisfies $\langle \theta_i, S_j \rangle = \delta_{ij}$. Let $\vartheta_1, \dots, \vartheta_n$ be respectively the restrictions of $\theta_1, \dots, \theta_n$ on \mathfrak{h} . Then $\{\vartheta_1, \dots, \vartheta_n\}$ is a basis of \mathfrak{h}^* , which is the dual basis of $\{S_1, \dots, S_n\} \subset \mathfrak{h}$. Clearly the restriction of θ_{n+1} on \mathfrak{h} is 0.

We also introduce the relation \succsim between real numbers to simplify our discussion. Given $a, b \in \mathbb{R}$, we say $b \succsim a$ if $b - a \in \{0, 1, 2, \dots\}$.

Facts about $\mathfrak{p} = \mathfrak{so}_{2n+2}$.

Simple roots: $\theta_1 - \theta_2, \theta_2 - \theta_3, \dots, \theta_{n-1} - \theta_n, \theta_n \pm \theta_{n+1}$.

Roots: $\pm\theta_i \pm \theta_j$ ($i \neq j$).

\mathfrak{p}_+ : matrices whose non-zero entries are above the diagonal 2×2 blocks S_1, \dots, S_{n+1} .

Fundamental weights:

$$\begin{aligned} \theta_1 + \dots + \theta_i & \quad (i = 1, 2, \dots, n - 1), \\ \varsigma_+ & = \frac{1}{2}(\theta_1 + \dots + \theta_n + \theta_{n+1}), \\ \varsigma_- & = \frac{1}{2}(\theta_1 + \dots + \theta_n - \theta_{n+1}). \end{aligned}$$

Dominant integral weights: $\lambda = f_1\theta_1 + \dots + f_{n+1}\theta_{n+1}$, where

$$(3.1) \quad f_1 \succsim \dots \succsim f_n \succsim |f_{n+1}| \succsim 0 \quad (\text{single-valued})$$

$$(3.2) \quad \text{or } f_1 \succsim \dots \succsim f_n \succsim |f_{n+1}| \succsim \frac{1}{2} \quad (\text{double-valued}).$$

Highest root: $\theta_1 + \theta_2$.

The normalized invariant inner product satisfies $(S_i|S_j) = (\theta_i|\theta_j) = \delta_{i,j}$.

Weights of $L_{\mathfrak{p}}(\theta_1)$: $\pm\theta_1, \dots, \pm\theta_n, \pm\theta_{n+1}$.

Weights of $L_{\mathfrak{p}}(\zeta_{\pm})$:

$$\frac{(-1)^{\sigma_1}\theta_1 + \dots + (-1)^{\sigma_{n+1}}\theta_{n+1}}{2} \quad (\sigma_1, \dots, \sigma_{n+1} \in \mathbb{Z}),$$

$$\sigma_1 + \dots + \sigma_{n+1} \in \begin{cases} 2\mathbb{Z} & \text{for } \zeta_+, \\ 2\mathbb{Z} + 1 & \text{for } \zeta_-. \end{cases}$$

The weight spaces of $L_{\mathfrak{p}}(\theta_1)$, $L_{\mathfrak{p}}(\zeta_+)$, and $L_{\mathfrak{p}}(\zeta_-)$ have dimensions at most 1.

Facts about $\mathfrak{g} = \mathfrak{so}_{2n+1}$.

Simple roots: $\vartheta_1 - \vartheta_2, \vartheta_2 - \vartheta_3, \dots, \vartheta_{n-1} - \vartheta_n, \vartheta_n$.

Roots: $\pm\vartheta_i \pm \vartheta_j$ ($i \neq j$), $\pm\vartheta_i$.

$\mathfrak{g}_+ = \mathfrak{p}_+ \cap \mathfrak{g}$.

Fundamental weights:

$$\vartheta_1 + \dots + \vartheta_i \quad (i = 1, 2, \dots, n - 1),$$

$$\varsigma = \frac{1}{2}(\vartheta_1 + \dots + \vartheta_n).$$

Dominant integral weights: $\lambda = f_1\vartheta_1 + \dots + f_n\vartheta_n$, where

$$(3.3) \quad f_1 \gtrsim \dots \gtrsim f_n \gtrsim 0 \quad (\text{single-valued})$$

$$(3.4) \quad \text{or } f_1 \gtrsim \dots \gtrsim f_n \gtrsim \frac{1}{2} \quad (\text{double-valued}).$$

Highest root: $\vartheta_1 + \vartheta_2$.

The normalized invariant inner product satisfies $(S_i|S_j) = (\vartheta_i|\vartheta_j) = \delta_{i,j}$.

Weights of $L_{\mathfrak{g}}(\vartheta_1)$: $\pm\vartheta_1, \dots, \pm\vartheta_n, 0$.

Weights of $L_{\mathfrak{g}}(\varsigma)$:

$$(3.5) \quad \frac{(-1)^{\sigma_1}\vartheta_1 + \dots + (-1)^{\sigma_n}\vartheta_n}{2} \quad (\sigma_1, \dots, \sigma_n \in \mathbb{Z}).$$

The weight spaces of $L_{\mathfrak{g}}(\vartheta_1)$ and $L_{\mathfrak{g}}(\varsigma)$ have dimensions at most 1.

It is clear that the Dynkin index of $\mathfrak{g} \subset \mathfrak{p}$ is 1.

3.2. Main result. The dominant integral weights of \mathfrak{p} admissible at level 1 are $\theta_1, \zeta_+, \zeta_-$. Choose $l = 1, 2, \dots$. The following results were essentially proved in [TL04]. The case $l = 1$ should be proved using Frenkel-Kac construction and lattice VOAs; see Corollary A.7. The higher level cases can be reduced to the level 1 case using the compression principle; see [TL04, chapter VI, sections 2 and 3].

Theorem 3.1. *Let $\mathfrak{p} = \mathfrak{so}_{2n}$ ($n \geq 3$), and choose $\lambda, \mu, \nu \in P_+(\mathfrak{p})$ admissible at level l .*

(a) *If $\lambda = \theta_1$ or ζ_{\pm} , then the maps $\Psi : \mathcal{V}_{\mathfrak{p}}^l(\lambda \begin{smallmatrix} \nu \\ \mu \end{smallmatrix}) \rightarrow \text{Hom}_{\mathfrak{p}}(\lambda \otimes \mu, \nu)$ and $\Gamma : \text{Hom}_{\mathfrak{p}}(\lambda \otimes \mu, \nu) \rightarrow L_{\mathfrak{p}}(\lambda)[\nu - \mu]^*$ are bijective.*

(b) *The $V_{\mathfrak{p}}^l$ -modules $L_{\mathfrak{p}}(\zeta_+, l)$ and $L_{\mathfrak{p}}(\zeta_-, l)$ are generating (see Definition 1.7).*

(c) *If $\lambda = \zeta_{\pm}$, then any $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{p}}^l(\lambda \begin{smallmatrix} \nu \\ \mu \end{smallmatrix})$ is energy-bounded.*

(d) *If $\lambda = \theta_1$, then for any $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{p}}^l(\lambda \begin{smallmatrix} \nu \\ \mu \end{smallmatrix})$ and $u^{(\lambda)} \in L_{\mathfrak{p}}(\lambda)$, $\mathcal{Y}_{\alpha}(u^{(\lambda)}, x)$ satisfies 0-th order energy bounds. In particular, \mathcal{Y}_{α} is energy-bounded.*

Now we assume that λ, μ, ν are dominant integral weights of \mathfrak{g} admissible at level l and that $\lambda = \vartheta_1$ or ς . Then the weight spaces of $L_{\mathfrak{g}}(\lambda)$ have dimensions at most 1. We shall calculate the fusion rule $N_{\lambda\mu}^{\nu} = \dim \mathcal{V}_{\mathfrak{g}}^l(\lambda \begin{smallmatrix} \nu \\ \mu \end{smallmatrix})$ and establish

the energy bounds condition for type $(\begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix})$ intertwining operators of $V_{\mathfrak{g}}^l$. Since $N_{\lambda\mu}^{\nu} \leq \dim L_{\mathfrak{g}}(\lambda)[\nu - \mu] \leq 1$, in order to know when $N_{\lambda\mu}^{\nu}$ equals 1, we can assume that $\nu - \mu$ is a weight of $L_{\mathfrak{g}}(\lambda)$.

Case $\lambda = \varsigma$. First we assume that μ is single-valued. Then, since $\nu - \mu$ is a weight of $L_{\mathfrak{g}}(\lambda)$, which can be written as (3.5), μ must be double-valued. Write

$$\begin{aligned} \nu - \mu &= \frac{1}{2}((-1)^{\sigma_1}\vartheta_1 + \cdots + (-1)^{\sigma_n}\vartheta_n), \\ \mu &= f_1\vartheta_1 + \cdots + f_n\vartheta_n, \\ \nu &= g_1\vartheta_1 + \cdots + g_n\vartheta_n. \end{aligned}$$

Then f_1, \dots, f_n satisfy (3.3), g_1, \dots, g_n satisfy (3.4), and $g_i = f_i + (-1)^{\sigma_i}/2$ ($i = 1, \dots, n$). Let

$$\begin{aligned} \tilde{\lambda} &= \begin{cases} \varsigma_+ & (\text{if } \sigma_1 + \cdots + \sigma_n \text{ is even}), \\ \varsigma_- & (\text{if } \sigma_1 + \cdots + \sigma_n \text{ is odd}), \end{cases} \\ \tilde{\mu} &= f_1\theta_1 + \cdots + f_n\theta_n, \\ \tilde{\nu} &= g_1\theta_1 + \cdots + g_n\theta_n + \frac{\theta_{n+1}}{2}. \end{aligned}$$

Then by (3.1) and (3.2), $\tilde{\mu}$ and $\tilde{\nu}$ are dominant integral weights of \mathfrak{p} , which are clearly admissible at level l , and

$$\tilde{\nu} - \tilde{\mu} = \frac{1}{2}((-1)^{\sigma_1}\theta_1 + \cdots + (-1)^{\sigma_n}\theta_n + \theta_{n+1})$$

is a weight of $L_{\mathfrak{p}}(\tilde{\lambda})$. So by Theorem 3.1(a), $\dim \mathcal{V}_{\mathfrak{p}}^l(\tilde{\lambda} \begin{smallmatrix} \tilde{\nu} \\ \tilde{\mu} \end{smallmatrix}) = 1$.

Choose a non-zero $\mathfrak{Y} \in \mathcal{V}_{\mathfrak{p}}^l(\tilde{\lambda} \begin{smallmatrix} \tilde{\nu} \\ \tilde{\mu} \end{smallmatrix})$. Then \mathfrak{Y} is energy-bounded by Theorem 3.1(c). Let w_{λ}, w_{μ} , and w_{ν} be respectively highest-weight vectors of $\hat{\mathfrak{p}} \curvearrowright L_{\mathfrak{p}}(\tilde{\lambda}, l), L_{\mathfrak{p}}(\tilde{\mu}, l), L_{\mathfrak{p}}(\tilde{\nu}, l)$. Then they are also highest-weight vectors of the action of $\hat{\mathfrak{g}}$ on these vector spaces with weights λ, μ, ν , respectively. If we can show that

$$(3.6) \quad \langle \mathfrak{Y}(\mathfrak{g}w_{\lambda}, x)\mathfrak{g}w_{\mu} | \mathfrak{g}w_{\nu} \rangle \neq 0,$$

then by Corollary 2.13, $\dim \mathcal{V}_{\mathfrak{g}}^l(\begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix}) = 1$, and for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\begin{smallmatrix} \nu \\ \lambda \mu \end{smallmatrix})$ and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, $\mathcal{Y}(u^{(\lambda)}, x)$ is energy-bounded. Therefore, by Proposition 1.11 and Theorem 1.24, \mathcal{Y} is energy-bounded.

To prove (3.6), we first choose a non-zero $u^{(\tilde{\lambda})} \in L_{\mathfrak{p}}(\tilde{\lambda})[\tilde{\nu} - \tilde{\mu}]$. Since the map $\Gamma\Psi : \mathcal{V}_{\mathfrak{p}}^l(\tilde{\lambda} \begin{smallmatrix} \tilde{\nu} \\ \tilde{\mu} \end{smallmatrix}) \rightarrow L_{\mathfrak{p}}(\tilde{\lambda})[\tilde{\nu} - \tilde{\mu}]^*$ is injective, we can choose a suitable $s \in \mathbb{R}$ such that $\langle \mathfrak{Y}(u^{(\tilde{\lambda})}, s)w_{\mu} | w_{\nu} \rangle \neq 0$. Since we know all the possible weights of $\mathfrak{g}w_{\lambda} \simeq L_{\mathfrak{g}}(\varsigma)$, and since each weight space has dimension no greater than 1, we can easily compute that $\mathfrak{g}w_{\lambda}$ has dimension 2^n . But $L_{\mathfrak{p}}(\tilde{\lambda})$ also has dimension 2^n , which can be computed in a similar way. Therefore $\mathfrak{g}w_{\lambda} = L_{\mathfrak{p}}(\tilde{\lambda})$, and hence $u^{(\tilde{\lambda})} \in \mathfrak{g}w_{\lambda}$. Thus (3.6) is proved.

The case that μ is double-valued (and hence ν is single-valued) can be treated in a similar way. Finally, using the knowledge of fusion rules, one can easily check that $L_{\mathfrak{g}}(\varsigma, l)$ is a generating $V_{\mathfrak{g}}^l$ -module.

Case $\lambda = \vartheta_1$. As we have seen, our analysis of the case $\lambda = \varsigma$ relies on the fact that the action $\mathfrak{p} \curvearrowright L_{\mathfrak{p}}(\varsigma_{\pm})$, when restricted to \mathfrak{g} , is also irreducible and equivalent to $L_{\mathfrak{g}}(\varsigma)$. This is achieved by a calculation of dimensions. However, we don't have a similar result when $\lambda = \vartheta_1$. Let v_{θ_1} be a highest-weight vector of $\mathfrak{p} \curvearrowright L_{\mathfrak{p}}(\theta_1)$. Then v_{θ_1} is a weight- ϑ_1 highest-weight vector of $\mathfrak{g} \curvearrowright L_{\mathfrak{p}}(\theta_1)$, and hence $\mathfrak{g}v_{\theta_1}$ is equivalent to $L_{\mathfrak{g}}(\vartheta_1)$. We have $\dim \mathfrak{g}v_{\theta_1} = \dim L_{\mathfrak{g}}(\vartheta_1) = 2n + 1$, and $\dim L_{\mathfrak{p}}(\theta_1) = 2n + 2$. So $\mathfrak{g}v_{\theta_1}$ is a proper subspace of $L_{\mathfrak{p}}(\theta_1)$. We still have the following.

Lemma 3.2. *Identify $\mathfrak{g}v_{\theta_1}$ with $L_{\mathfrak{g}}(\vartheta_1)$.*

(a) *For any $i = 1, 2, \dots, n$, we have $L_{\mathfrak{g}}(\vartheta_1)[\vartheta_i] = L_{\mathfrak{p}}(\theta_1)[\theta_i]$ and $L_{\mathfrak{g}}(\vartheta_1)[-\vartheta_i] = L_{\mathfrak{p}}(\theta_1)[-\theta_i]$.*

(b) *For any non-zero $u_0 \in L_{\mathfrak{g}}(\vartheta_1)[0]$, there exist non-zero vectors $u_+ \in L_{\mathfrak{p}}(\theta_1)[\theta_{n+1}]$ and $u_- \in L_{\mathfrak{p}}(\theta_1)[-\theta_{n+1}]$ such that $u_0 = u_+ + u_-$.*

Proof. It is clear that for any $i = 1, \dots, n$, $L_{\mathfrak{p}}(\theta_1)[\theta_i]$ is the weight- ϑ_i subspace of $\mathfrak{g} \curvearrowright L_{\mathfrak{p}}(\theta_1)$, $L_{\mathfrak{p}}(\theta_1)[-\theta_i]$ is the weight- $(-\vartheta_i)$ subspace of $\mathfrak{g} \curvearrowright L_{\mathfrak{p}}(\theta_1)$, and $L_{\mathfrak{p}}(\theta_1)[\theta_{n+1}] \oplus L_{\mathfrak{p}}(\theta_1)[-\theta_{n+1}]$ is the weight 0 subspace of $\mathfrak{g} \curvearrowright L_{\mathfrak{p}}(\theta_1)$. Thus

$$(3.7) \quad L_{\mathfrak{g}}(\vartheta_1)[\vartheta_i] \subset L_{\mathfrak{p}}(\theta_1)[\theta_i],$$

$$(3.8) \quad L_{\mathfrak{g}}(\vartheta_1)[-\vartheta_i] \subset L_{\mathfrak{p}}(\theta_1)[-\theta_i],$$

$$(3.9) \quad L_{\mathfrak{g}}(\vartheta_1)[0] \subset L_{\mathfrak{p}}(\theta_1)[\theta_{n+1}] \oplus L_{\mathfrak{p}}(\theta_1)[-\theta_{n+1}].$$

Since both sides of (3.7) and (3.8) have dimension 1, we actually have $L_{\mathfrak{g}}(\vartheta_1)[\vartheta_i] = L_{\mathfrak{p}}(\theta_1)[\theta_i]$ and $L_{\mathfrak{g}}(\vartheta_1)[-\vartheta_i] = L_{\mathfrak{p}}(\theta_1)[-\theta_i]$.

Now identify $L_{\mathfrak{p}}(\theta_1)$ with the standard representation $\mathfrak{so}_{2n+2} \curvearrowright \mathbb{C}^{2n+2}$, and let e_1, \dots, e_{2n+2} be the standard orthonormal basis. Then it is clear that $L_{\mathfrak{g}}(\vartheta_1)[0] = \mathbb{C} \cdot e_{2n+1}$. Clearly e_{2n+1} is not a weight vector of $\mathfrak{p} \curvearrowright L_{\mathfrak{p}}(\theta)$. So

$$L_{\mathfrak{g}}(\vartheta_1)[0] \not\subset L_{\mathfrak{p}}(\theta_1)[\theta_{n+1}] \cup L_{\mathfrak{p}}(\theta_1)[-\theta_{n+1}].$$

This proves (b). □

We are now ready to compute fusion rules and prove the energy bounds condition. Write $\mu = \sum_{j=1}^n f_j \vartheta_j$ and $\nu = \sum_{j=1}^n g_j \vartheta_j$ as usual. Let $\tilde{\lambda} = \theta_1$. In the following, we shall either show that $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 0$ (which indicates $\dim \mathcal{V}_{\mathfrak{g}}^l(\lambda \overset{\nu}{\mu}) = 0$) or choose $\tilde{\mu}, \tilde{\nu} \in P_+(\mathfrak{p})$ admissible at level l , such that the restrictions of these weights to \mathfrak{h}^* are μ and ν and $\tilde{\nu} - \tilde{\mu}$ is a weight of $L_{\mathfrak{p}}(\tilde{\lambda})$. Then we choose highest-weight vectors $w_{\tilde{\lambda}}, w_{\tilde{\mu}}, w_{\tilde{\nu}}$ of $\hat{\mathfrak{p}} \curvearrowright L_{\mathfrak{p}}(\tilde{\lambda}, l), L_{\mathfrak{p}}(\tilde{\mu}, l), L_{\mathfrak{p}}(\tilde{\nu}, l)$, respectively, which are also highest-weight vectors of $\hat{\mathfrak{g}} \curvearrowright L_{\mathfrak{p}}(\tilde{\lambda}, l), L_{\mathfrak{p}}(\tilde{\mu}, l), L_{\mathfrak{p}}(\tilde{\nu}, l)$ with highest weights $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$, respectively. Choose a non-zero weight- $(\nu - \mu)$ vector u of $\mathfrak{g} \curvearrowright \mathfrak{g}w_{\tilde{\lambda}}$. Choose a non-zero $\mathfrak{Y} \in \mathcal{V}_{\mathfrak{p}}^l(\tilde{\lambda} \overset{\tilde{\nu}}{\tilde{\mu}})$. We shall show that $\langle \mathfrak{Y}(u, x)w_{\tilde{\mu}} | w_{\tilde{\nu}} \rangle \neq 0$. This will imply, by Corollary 2.13 and Theorem 3.1(d), that $\dim \mathcal{V}_{\mathfrak{g}}^l(\lambda \overset{\nu}{\mu}) = 1$ and that for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \overset{\nu}{\mu})$ and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, $\mathcal{Y}(u^{(\lambda)}, x)$ satisfies 0-th order energy bounds.

Subcase 1: $\nu - \mu$ equals ϑ_i or $-\vartheta_i$ ($i = 1, \dots, n$), and μ is single-valued. Then, since $\nu - \mu$ is a weight of $L_{\mathfrak{g}}(\vartheta_1)$, ν must also be single-valued. Set

$$\tilde{\mu} = \sum_j^n f_j \theta_j, \quad \tilde{\nu} = \sum_j^n g_j \theta_j.$$

Then $\tilde{\nu} - \tilde{\mu}$ equals θ_i or $-\theta_i$, which is a weight of $L_{\mathfrak{p}}(\theta_1)$. If $u \in (\mathfrak{g}w_{\tilde{\lambda}})[\nu - \mu]$ is non-zero, then by Lemma 3.2(a), $u \in L_{\mathfrak{p}}(\tilde{\lambda})[\tilde{\nu} - \tilde{\mu}]$. Therefore, as $\Gamma\Psi$ is injective, $\langle \mathfrak{Y}(u, x)w_{\tilde{\mu}} | w_{\tilde{\nu}} \rangle \neq 0$.

Subcase 2: $\nu - \mu$ equals ϑ_i or $-\vartheta_i$ ($i = 1, \dots, n$), and μ is double-valued. This subcase can be treated in a similar way as we treated Subcase 1, except that we let

$$\tilde{\mu} = \sum_j^n f_j \theta_j + \frac{1}{2} \theta_{n+1}, \quad \tilde{\nu} = \sum_j^n g_j \theta_j + \frac{1}{2} \theta_{n+1}.$$

Subcase 3: $\mu = \nu$ are single-valued, and $f_n = g_n \geq 1$. Set

$$\tilde{\mu} = \sum_j^n f_j \theta_j, \quad \tilde{\nu} = \sum_j^n g_j \theta_j + \theta_{n+1} (= \tilde{\mu} + \theta_{n+1}).$$

Choose a non-zero $u \in (\mathfrak{g}w_\lambda)[\nu - \mu]$. Then by Lemma 3.2(b), there exist non-zero $u_+ \in L_{\mathfrak{p}}(\theta_1)[\theta_{n+1}]$ and $u_- \in L_{\mathfrak{p}}(\theta_1)[-\theta_{n+1}]$ satisfying $u_0 = u_+ + u_-$. By injectivity of $\Gamma\Psi$, $\langle \mathfrak{Y}(u_+, s)w_\mu | w_\nu \rangle \neq 0$ for some $s \in \mathbb{R}$. The vector $\mathfrak{Y}(u_-, s)w_\mu$ is a weight- $(\tilde{\mu} - \theta_{n+1})$ vector of $\mathfrak{p} \curvearrowright L_{\mathfrak{p}}(\tilde{\nu}, l)$. So we must have $\langle \mathfrak{Y}(u_-, s)w_\mu | w_\nu \rangle = 0$. Therefore $\langle \mathfrak{Y}(u, s)w_\mu | w_\nu \rangle \neq 0$.

Subcase 4: $\mu = \nu$ are double-valued. We can prove this by choosing

$$\tilde{\mu} = \sum_j^n f_j \theta_j - \frac{1}{2} \theta_{n+1}, \quad \tilde{\nu} = \sum_j^n g_j \theta_j + \frac{1}{2} \theta_{n+1}$$

and using a similar argument as in Subcase 3.

Subcase 5: $\mu = \nu$ are single-valued, and $f_n = g_n = 0$. We know that $\alpha = \vartheta_n$ is a simple root of \mathfrak{g} . Then $n_{\mu, \alpha} := 2(\mu | \vartheta_n) / (\vartheta_n | \vartheta_n) = 0$, and hence

$$\dim L_{\mathfrak{g}}(\lambda)[\nu - \mu + (n_{\mu, \alpha} + 1)\alpha] = \dim L_{\mathfrak{g}}(\vartheta_1)[\vartheta_n] = 1.$$

By Corollary 1.18, $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 0$. Therefore $\dim \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu) = 0$.

Now that we know the fusion rules, we can easily show that $L_{\mathfrak{g}}(\varsigma, l)$ is generating. Thus we've proved the main result of this chapter.

Theorem 3.3. *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ ($n \geq 2$), and choose $\lambda, \mu, \nu \in P_+(\mathfrak{g})$ admissible at level l .*

- (a) *If $\lambda = \vartheta_1$ or ς , then $\Psi : \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu) \rightarrow \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$ is bijective.*
- (b) *If $\lambda = \varsigma$, then $\Gamma : \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \rightarrow L_{\mathfrak{g}}(\lambda)[\nu - \mu]^*$ is bijective. If $\lambda = \vartheta_1$, then Γ is bijective if and only if $(\mu | \vartheta_n) > 0$; otherwise $\dim \text{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) = 0$.*
- (c) *$L_{\mathfrak{g}}(\varsigma, l)$ is a generating $V_{\mathfrak{g}}^l$ -module.*
- (d) *If $\lambda = \varsigma$, then any $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu)$ is energy-bounded.*
- (e) *If $\lambda = \vartheta_1$, then for any $\mathcal{Y}_{\alpha} \in \mathcal{V}_{\mathfrak{g}}^l(\lambda \nu)$ and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$, $\mathcal{Y}_{\alpha}(u^{(\lambda)}, x)$ satisfies 0-th order energy bounds. In particular, \mathcal{Y}_{α} is energy-bounded.*

4. TYPE C

4.1. $\mathfrak{sp}_{2n} \subset \mathfrak{so}_{4n}$. Choose $n = 2, 3, 4, \dots$. Let I_n be the $n \times n$ identity matrix, and let $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The complex symplectic Lie algebra \mathfrak{sp}_{2n} is the unitary Lie subalgebra of $\mathfrak{gl}_{2n} = \text{End}(\mathbb{C}^{2n})$ whose elements $X \in \mathfrak{gl}_{2n}$ satisfy $J_{2n}X + X^t J_{2n} = 0$. Write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $n \times n$ matrices. Then $X \in \mathfrak{sp}_{2n}$ if and only if $D = -A^t$ and B and C are symmetric.

We first choose a Cartan subalgebra of $\mathfrak{g} := \mathfrak{sp}_{2n}$. As in the last section, we let e_1, \dots, e_{2n} be the standard orthonormal basis of \mathbb{C}^{2n} . For any $i, j = 1, \dots, 2n$,

we define $E_{i,j} \in \text{End}(\mathbb{C}^{2n})$ to satisfy $E_{i,j}e_k = \delta_{j,k}e_i$ ($\forall k = 1, \dots, 2n$). For any $i = 1, \dots, n$, we let

$$(4.1) \quad T_i = E_{i,i} - E_{n+i,n+i}.$$

A Cartan subalgebra can be chosen to be $\mathfrak{h} = \text{Span}_{\mathbb{C}}\{T_1, \dots, T_n\}$. Choose a basis $\{\vartheta_1, \dots, \vartheta_n\}$ of \mathfrak{h}^* to be the dual basis of $\{T_1, \dots, T_n\}$.

As in the last section, given two real numbers a, b , we say $b \gtrsim a$ if $b - a = 0, 1, 2, \dots$

Facts about $\mathfrak{g} = \mathfrak{sp}_{2n}$.

Simple roots: $\vartheta_1 - \vartheta_2, \vartheta_2 - \vartheta_3, \dots, \vartheta_{n-1} - \vartheta_n, 2\vartheta_n$.

Roots: $\pm\vartheta_i \pm \vartheta_j$ ($i \neq j$), $\pm 2\vartheta_i$.

Positive roots: $\vartheta_i \pm \vartheta_j$ ($i < j$), $2\vartheta_i$.

If α is a positive root, then the root space

$$\mathfrak{g}_{\alpha} = \begin{cases} \mathbb{C}(E_{i,n+j} + E_{j,n+i}) & (\text{if } \alpha = \vartheta_i + \vartheta_j, 1 \leq i, j \leq n), \\ \mathbb{C}(E_{i,j} - E_{n+j,n+i}) & (\text{if } \alpha = \vartheta_i - \vartheta_j, 1 \leq i < j \leq n). \end{cases}$$

Fundamental weights: $\vartheta_1 + \dots + \vartheta_i$ ($i = 1, 2, \dots, n$).

Dominant integral weights:

$$(4.2) \quad \lambda = f_1\vartheta_1 + \dots + f_n\vartheta_n \quad (f_1 \gtrsim \dots \gtrsim f_n \gtrsim 0).$$

Highest root: $2\vartheta_1$.

The normalized invariant inner product satisfies $(T_i|T_j) = 2\delta_{i,j}$, $(\vartheta_i|\vartheta_j) = \frac{1}{2}\delta_{i,j}$.

Weights of $L_{\mathfrak{g}}(\vartheta_1)$: $\pm\vartheta_1, \dots, \pm\vartheta_n$.

The weight spaces of $L_{\mathfrak{g}}(\vartheta_1)$ have dimensions at most 1.

Embedding $\mathfrak{sp}_{2n} \subset \mathfrak{so}_{4n}$. Note that $L_{\mathfrak{g}}(\vartheta_1)$ is equivalent to the standard representation $\mathfrak{g} \curvearrowright \mathbb{C}^{2n}$. We now consider the representation $L_{\mathfrak{g}}(\vartheta_1) \oplus L_{\mathfrak{g}}(\overline{\vartheta_1})$. Equivalently, we consider the embedding

$$(4.3) \quad \mathfrak{sp}_{2n} \hookrightarrow \text{End}(\mathbb{C}^{4n}), \quad X \mapsto \begin{pmatrix} X & 0 \\ 0 & -X^t \end{pmatrix}.$$

We now regard \mathfrak{g} as a unitary Lie subalgebra of $\text{End}(\mathbb{C}^{4n})$ in this way. Then a $4n \times 4n$ matrix X is an element of \mathfrak{g} if and only if

$$X = \begin{pmatrix} A & B & 0 & 0 \\ C & -A^t & 0 & 0 \\ 0 & 0 & -A^t & -C \\ 0 & 0 & -B & A \end{pmatrix},$$

where A, B, C, D are $n \times n$ matrices and B, C are symmetric. Then (4.1) should be replaced by

$$(4.4) \quad T_i = E_{i,i} - E_{n+i,n+i} - E_{2n+i,2n+i} + E_{3n+i,3n+i} \quad (1 \leq i \leq n).$$

For a positive root α of \mathfrak{g} , the root space

$$\mathfrak{g}_{\alpha} = \begin{cases} \mathbb{C}(E_{i,n+j} + E_{j,n+i} - E_{3n+i,2n+j} - E_{3n+j,2n+i}) & (\text{if } \alpha = \vartheta_i + \vartheta_j, 1 \leq i, j \leq n), \\ \mathbb{C}(E_{i,j} - E_{n+j,n+i} - E_{2n+j,2n+i} + E_{3n+i,3n+j}) & (\text{if } \alpha = \vartheta_i - \vartheta_j, 1 \leq i < j \leq n). \end{cases}$$

Set $K_{4n} = \begin{pmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{pmatrix}$, where I_{2n} is the identity $2n \times 2n$ matrix. Let \mathfrak{p} be the unitary Lie subalgebra of $\text{End}(\mathbb{C}^{4n})$ whose elements $Y \in \text{End}(\mathbb{C}^{4n})$ satisfy $K_{4n}Y + Y^t K_{4n} = 0$. Then \mathfrak{p} is $*$ -isomorphic to \mathfrak{so}_{4n} . Indeed, if we choose $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{2n} & I_{2n} \\ \mathbf{i}I_{2n} & -\mathbf{i}I_{2n} \end{pmatrix}$, then $U^t U = K_{4n}$. The Lie algebra isomorphism can be defined as

$$\mathfrak{p} \rightarrow \mathfrak{so}_{4n}, \quad Y \mapsto UYU^{-1}.$$

Since U is unitary, i.e., $U^* = U^{-1}$, this isomorphism preserves the $*$ -structures of \mathfrak{p} and \mathfrak{so}_{4n} , both defined by the adjoint of operators.

A $4n \times 4n$ matrix Y is an element in \mathfrak{p} if and only if $Y = \begin{pmatrix} P & Q \\ R & -P^t \end{pmatrix}$, where P, Q, R are $2n \times 2n$ matrices and Q, R are skew-symmetric. Let e_1, \dots, e_{4n} be the standard orthonormal basis of \mathbb{C}^{4n} as usual, and define the $4n \times 4n$ matrix $E_{i,j}$ ($i \leq j, j \leq 4n$) as usual. For any $1 \leq i \leq 2n$ we set

$$S_i = E_{i,i} - E_{2n+i,2n+i}.$$

Then $\mathfrak{t} = \text{Span}_{\mathbb{C}}\{S_1, \dots, S_{2n}\}$ is a Cartan subalgebra of \mathfrak{p} and $\mathfrak{h} \subset \mathfrak{t}$. We let $\theta_1, \dots, \theta_{2n}$ be the dual basis of S_1, \dots, S_{2n} . Then for any $1 \leq i \leq n$, the restrictions of $\theta_i, \theta_{n+i} \in \mathfrak{t}^*$ to \mathfrak{h} are $\vartheta_i, -\vartheta_i$, respectively.

The facts about \mathfrak{p} described in section 3.1 also hold for $\theta_1, \dots, \theta_{2n}$. However, if we choose simple roots of \mathfrak{p} as in section 3.1, then \mathfrak{g}_+ will no longer be a subset of \mathfrak{p}_+ . So let us choose another set of simple roots instead and consider relevant facts below. This new set of simple roots is chosen with respect to the order $1, 2, \dots, n, 2n, 2n - 1, 2n - 2, \dots, n + 2, n + 1$ rather than the standard $1, 2, \dots, 2n$ considered in section 3.1.

Simple roots: $\theta_1 - \theta_2, \theta_2 - \theta_3, \dots, \theta_{n-1} - \theta_n, \theta_n - \theta_{2n}, \theta_{2n} - \theta_{2n-1}, \theta_{2n-1} - \theta_{2n-2}, \dots, \theta_{n+2} - \theta_{n+1}, \theta_{n+2} + \theta_{n+1}$.

Roots: $\pm\theta_i \pm \theta_j$ ($i \neq j$).

Positive roots: $\theta_i + \theta_j$ for all $1 \leq i \neq j \leq 2n$; $\theta_i - \theta_j, -\theta_{n+i} + \theta_{n+j}$ for all $1 \leq i < j \leq n$; $\theta_i - \theta_{n+j}$ for all $1 \leq i, j \leq n$.

If α is a positive root of \mathfrak{p} , then the root space

$$(4.5) \quad \mathfrak{g}_\alpha = \begin{cases} \mathbb{C}(E_{i,2n+j} - E_{j,2n+i}) & (\text{if } \alpha = \theta_i + \theta_j, 1 \leq i \neq j \leq 2n), \\ \mathbb{C}(E_{i,j} - E_{2n+j,2n+i}) & (\text{if } \alpha = \theta_i - \theta_j, 1 \leq i < j \leq n), \\ \mathbb{C}(E_{n+j,n+i} - E_{3n+i,3n+j}) & (\text{if } \alpha = -\theta_{n+i} + \theta_{n+j}, 1 \leq i < j \leq n), \\ \mathbb{C}(E_{i,n+j} - E_{3n+j,2n+i}) & (\text{if } \alpha = \theta_i - \theta_{n+j}, 1 \leq i, j \leq n). \end{cases}$$

Non-zero dominant integral weights admissible at level 1:

$$\theta_1, \quad \varsigma_\pm = \frac{1}{2} \left(\pm \theta_{n+1} + \sum_{i \neq n+1} \theta_i \right).$$

Highest root: $\theta_1 + \theta_2$.

The normalized invariant inner product satisfies $(S_i|S_j) = (\theta_i|\theta_j) = \delta_{i,j}$.

Weights of $L_{\mathfrak{p}}(\theta_1)$: $\pm\theta_1, \dots, \pm\theta_{2n}$.

Weights of $L_{\mathfrak{p}}(\varsigma_{\pm})$:

$$\frac{(-1)^{\sigma_1}\theta_1 + \dots + (-1)^{\sigma_{2n}}\theta_{2n}}{2} \quad (\sigma_1, \dots, \sigma_{2n} \in \mathbb{Z}),$$

$$\sigma_1 + \dots + \sigma_{2n} \in \begin{cases} 2\mathbb{Z} & \text{for } \varsigma_+, \\ 2\mathbb{Z} + 1 & \text{for } \varsigma_-. \end{cases}$$

The weight spaces of $L_{\mathfrak{p}}(\theta_1), L_{\mathfrak{p}}(\varsigma_+)$, and $L_{\mathfrak{p}}(\varsigma_-)$ have dimensions at most 1.

Using this information, one can check that $\mathfrak{g} \subset \mathfrak{p}$ has Dynkin index 1 and that

$$(4.6) \quad \mathfrak{g}_+ \subset \mathfrak{p}_+ \cap \left(\bigoplus_{1 \leq i \neq j \leq 2n}^{\perp} \mathfrak{p}_{\theta_i + \theta_j} \right)^{\perp}$$

$$= \left(\bigoplus_{1 \leq i < j \leq 2n}^{\perp} \mathfrak{p}_{\theta_i - \theta_j} \right) \oplus^{\perp} \left(\bigoplus_{1 \leq i < j \leq 2n}^{\perp} \mathfrak{p}_{-\theta_{n+i} + \theta_{n+j}} \right) \oplus^{\perp} \left(\bigoplus_{1 \leq i, j \leq 2n}^{\perp} \mathfrak{p}_{\theta_i - \theta_{n+j}} \right).$$

4.2. Main result. In this section, we show, for any $\mu, \nu \in P_+(\mathfrak{g})$ admissible at level l , that the fusion rule $N_{\vartheta_1 \mu}^{\nu}$ equals 1 when $\nu - \mu$ is a weight of $L_{\mathfrak{g}}(\vartheta_1)$ and equals 0 otherwise. This will imply that $L_{\mathfrak{g}}(\vartheta_1, l)$ is a generating $V_{\mathfrak{g}}^l$ -module. We also prove the 0-th order energy bounds condition for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1, \mu)$. We first do this for level 1.

Level 1. Dominant integral weights of \mathfrak{g} admissible at level 1 are $0, \vartheta_1, \vartheta_1 + \vartheta_2, \dots, \vartheta_1 + \vartheta_2 + \dots + \vartheta_n$. Now assume μ, ν are among these weights. Since $\Gamma\Psi : \mathcal{V}_{\mathfrak{g}}^1(\vartheta_1, \mu) \rightarrow L_{\mathfrak{g}}(\vartheta_1)[\nu - \mu]^*$ is injective, we can assume that $\nu - \mu$ is a weight of $L_{\mathfrak{g}}(\vartheta_1)$. So there exists $k = 1, \dots, n$ such that $\nu - \mu = \pm\vartheta_k$.

Let us first assume that $\nu - \mu = \vartheta_k$. Then we must have

$$\mu = \vartheta_1 + \dots + \vartheta_{k-1}, \quad \nu = \vartheta_1 + \dots + \vartheta_k$$

(set $\mu = 0$ when $k = 1$). Set

$$\begin{aligned} \varsigma_{\mu} &= \varsigma_+, & \varsigma_{\nu} &= \varsigma_- & (\text{if } k \text{ is odd}), \\ \varsigma_{\mu} &= \varsigma_-, & \varsigma_{\nu} &= \varsigma_+ & (\text{if } k \text{ is even}). \end{aligned}$$

We now choose highest-weight vectors of \mathfrak{g} in $L_{\mathfrak{p}}(\theta_1, 1), L_{\mathfrak{p}}(\varsigma_{\mu}, 1), L_{\mathfrak{p}}(\varsigma_{\nu}, 1)$, respectively. Identify $\mathfrak{p} \curvearrowright L_{\mathfrak{p}}(\theta_1)$ with the standard representation $\mathfrak{p} \curvearrowright \mathbb{C}^{4n}$, and choose $w_{\vartheta_1} = e_{3n+1}$. One can easily check that w_{ϑ_1} is a ϑ_1 -highest-weight vector of the level 1 representation $\widehat{\mathfrak{g}} \curvearrowright L_{\mathfrak{p}}(\theta_1, 1)$.

Let w_{μ} (resp., w_{ν}) be a non-zero-weight vector $L_{\mathfrak{p}}(\varsigma_{\mu})$ (resp., $L_{\mathfrak{p}}(\varsigma_{\nu})$) with weight

$$\left(\sum_{i=1}^n \theta_i - \sum_{i=n+1}^{n+k-1} \theta_i + \sum_{i=n+k}^{2n} \theta_i \right) / 2,$$

$$\text{(resp., } \left(\sum_{i=1}^n \theta_i - \sum_{i=n+1}^{n+k} \theta_i + \sum_{i=n+k+1}^{2n} \theta_i \right) / 2).$$

Now, using (4.6), we can check that w_{μ} and w_{ν} are highest-weight vectors of $\widehat{\mathfrak{g}} \curvearrowright L_{\mathfrak{p}}(\varsigma_{\mu}, 1), L_{\mathfrak{p}}(\varsigma_{\nu}, 1)$ with highest weights μ and ν , respectively. Suppose that we can prove Lemma 4.1. Then, since $e_{3n+k} \in L_{\mathfrak{p}}(\theta_1)[-\theta_{n+k}]$, by the injectivity of Ψ , for any non-zero $\mathfrak{Y} \in \mathcal{V}_{\mathfrak{p}}^1(\theta_1, \varsigma_{\mu})$ (which exists due to Theorem 3.1(a)), we have $\langle \mathfrak{Y}(e_{3n+k}, x)w_{\mu} | w_{\nu} \rangle \neq 0$. Since $e_{3n+k} \in 0 \oplus \mathbb{C}^{2n} = \mathfrak{g}e_{3n+1} = \mathfrak{g}w_{\vartheta_1}$, we actually have $\langle \mathfrak{Y}(\mathfrak{g}w_{\vartheta_1}, x)w_{\mu} | w_{\nu} \rangle \neq 0$. Therefore, by Corollary 2.13 and Theorem 3.1(d),

$\dim \mathcal{V}_{\mathfrak{g}}^1(\nu_\mu) = 1$, and for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^1(\nu_\mu)$ and $u^{(\theta_1)} \in L_{\mathfrak{g}}(\vartheta_1)$, $\mathcal{Y}(u^{(\theta_1)}, x)$ satisfies 0-th order energy bounds.

Lemma 4.1. *For any non-zero $T \in \text{Hom}_{\mathfrak{p}}(\theta_1 \otimes \zeta_\mu, \zeta_\nu)$, inequality $\langle T(e_{3n+k} \otimes w_\mu) | w_\nu \rangle \neq 0$ holds.*

Proof. We write $w_{k-1} = w_\mu, w_k = w_\nu$ and prove

$$(4.7) \quad \langle T(e_{3n+k} \otimes w_{k-1}) | w_k \rangle \neq 0$$

by induction on k . For any positive root α of \mathfrak{p} , we choose $E_\alpha, F_\alpha, H_\alpha$ as in (1.29). Now for any $i = 1, \dots, n - 1$ we choose $\alpha_i = \theta_{n+i} + \theta_{n+i+1}$. Then, by (4.5),

$$(4.8) \quad E_{\alpha_i} \in \mathbb{C}(E_{n+i, 3n+i+1} - E_{n+i+1, 3n+i}),$$

$$(4.9) \quad F_{\alpha_i} \in \mathbb{C}(E_{3n+i+1, n+i} - E_{3n+i, n+i+1}).$$

Given a general k , we first show that inequality

$$(4.10) \quad \langle T(e_{3n+k-2} \otimes w_{k-3}) | w_{k-2} \rangle \neq 0$$

implies (4.7). Assume (4.10) is true. Then

$$(4.11) \quad \begin{aligned} \langle T(e_{3n+k} \otimes F_{\alpha_{k-2}} w_{k-3}) | F_{\alpha_{k-1}} w_{k-2} \rangle &= \langle T(E_{\alpha_{k-1}} e_{3n+k} \otimes F_{\alpha_{k-2}} w_{k-3}) | w_{k-2} \rangle \\ &+ \langle T(e_{3n+k} \otimes E_{\alpha_{k-1}} F_{\alpha_{k-2}} w_{k-3}) | w_{k-2} \rangle. \end{aligned}$$

However, the weight of $E_{\alpha_{k-1}} F_{\alpha_{k-2}} w_{k-3}$ is $\dots + \frac{3}{2}\theta_{n+k} + \dots$, which is not a weight of $L_{\mathfrak{p}}(\zeta_{\pm})$. Therefore, the second summand in equation (4.11) vanishes. Thus (4.11) equals

$$(4.12) \quad \begin{aligned} \langle T(E_{\alpha_{k-1}} e_{3n+k} \otimes F_{\alpha_{k-2}} w_{k-3}) | w_{k-2} \rangle &= - \langle T(F_{\alpha_{k-2}} E_{\alpha_{k-1}} e_{3n+k} \otimes w_{k-3}) | w_{k-2} \rangle \\ &+ \langle T(E_{\alpha_{k-1}} e_{3n+k} \otimes w_{k-3}) | E_{\alpha_{k-2}} w_{k-2} \rangle. \end{aligned}$$

Again, the weight of $E_{\alpha_{k-2}} w_{k-2}$ is $\dots + \frac{3}{2}\theta_{n+k-1} + \dots$. So the second summand of (4.12) equals 0. By (4.8) and (4.9), there exists $c \in \mathbb{C} \setminus \{0\}$ such that $F_{\alpha_{k-2}} E_{\alpha_{k-1}} e_{3n+k} = c \cdot e_{3n+k-2}$. Therefore (4.11) equals the left hand side of (4.10) multiplied by $-c$. So (4.11) is non-zero. Since $F_{\alpha_{k-2}} w_{k-3}$ and w_{k-1} have the same weight, they must be proportional. Similarly, $F_{\alpha_{k-1}} w_{k-2}$ and w_k are proportional. Thus inequality (4.7) is proved.

Now, from what we have proved above, in order to finish the proof of this lemma, it suffices to prove (4.7) when $k = 1, 2$. If $k = 1$, then w_{k-1} and w_k are highest-weight vectors of $L_{\mathfrak{p}}(\zeta_\mu) = L_{\mathfrak{p}}(\zeta_+)$ and $L_{\mathfrak{p}}(\zeta_\nu) = L_{\mathfrak{p}}(\zeta_-)$, respectively. Then (4.7) follows from the injectivity of $\Gamma : \text{Hom}_{\mathfrak{p}}(\theta_1 \otimes \zeta_+, \zeta_-) \rightarrow L_{\mathfrak{p}}(\theta_1)[- \theta_{n+1}]^*$. If $k = 2$, then w_1 is the highest-weight vector, but w_2 is not. Choose a highest-weight vector w_0 of $L_{\mathfrak{p}}(\zeta_\nu) = L_{\mathfrak{p}}(\zeta_+)$, Then, since $E_{\alpha_1} w_1 = 0$,

$$(4.13) \quad \langle T(e_{3n+2} \otimes w_1) | F_{\alpha_1} w_0 \rangle = \langle T(E_{\alpha_1} e_{3n+2} \otimes w_1) | w_0 \rangle.$$

By (4.8), the right hand side of (4.13) equals $\langle T(e_{n+1} \otimes w_1) | w_0 \rangle$ multiplied by a non-zero complex number. Since the difference of the weights of w_0 and w_1 is $\zeta_+ - \zeta_- = \theta_{n+1}$, which is the same as the weight of e_{n+1} , from the injectivity of $\Gamma : \text{Hom}_{\mathfrak{p}}(\theta_1 \otimes \zeta_-, \zeta_+) \rightarrow L_{\mathfrak{p}}(\theta_1)[\theta_{n+1}]^*$, we find that $\langle T(e_{n+1} \otimes w_1) | w_0 \rangle$ is non-zero. So (4.13) is also non-zero. Since $F_{\alpha_1} w_0$ and w_2 have the same weight, they are proportional. Thus (4.7) is proved when $k = 2$. □

The case $\nu - \mu = -\vartheta_k$ can be treated in a similar way. Alternatively, we first show $\dim \text{Hom}_{\mathfrak{g}}(\vartheta_1 \otimes \vartheta_1, 0) = 1$ using Corollary 1.18, which implies $\overline{\vartheta_1} = \vartheta_1$ by Proposition 1.15. Then we use the result of case $\nu - \mu = -\vartheta_k$ to prove this case by applying Proposition 1.8.

Level l . We assume $\nu - \mu = \vartheta_k$ ($k = 1, \dots, n$). The case $\nu - \mu = -\vartheta_k$ can be treated either in a similar way or by applying Proposition 1.8. Write $\mu = \sum_{i=1}^n f_i \vartheta_i, \nu = \sum_{i=1}^n g_i \vartheta_i$. Then $g_k = f_k + 1$, and $f_i = g_i$ when $i \neq k$. Let $a_n = f_n, b_n = g_n$, and let $a_i = f_i - f_{i+1}, b_i = g_i - g_{i+1}$ for any $i = 1, \dots, n - 1$. Consider the fundamental weights $\lambda_1 = \vartheta_1, \lambda_2 = \vartheta_1 + \vartheta_2, \dots, \lambda_n = \vartheta_1 + \vartheta_2 + \dots + \vartheta_n$. Then $\mu = \sum_{i=1}^n a_i \lambda_i, \nu = \sum_{i=1}^n b_i \lambda_i$.

If $2 \leq k \leq n - 1$, then $b_{k-1} = a_{k-1} - 1, b_k = a_k + 1$, and $b_i = a_i$ for any $i \neq k - 1, k$. Let

$$\rho = (a_{k-1} - 1)\lambda_{k-1} + \sum_{i \neq k-1} a_i \lambda_i, \quad \mu_1 = \lambda_{k-1}, \quad \nu_1 = \lambda_k.$$

Then $\rho, \mu_1, \nu_1 \in P_+(\mathfrak{g})$, μ_1, ν_1 are admissible at level 1, ρ is admissible at level $l - 1$, and $\mu = \rho + \mu_1, \nu = \rho + \nu_1$. Thus, by the result of level 1 and Lemma 2.15(a), $N_{\vartheta_1 \mu}^\nu = 1$, and for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1, \mu)$ and $u^{(\vartheta_1)} \in L_{\mathfrak{g}}(\vartheta_1)$, $\mathcal{Y}(u^{(\vartheta_1)}, x)$ satisfies 0-th order energy bounds.

Now assume that $k = 1$. Then $b_1 = a_1 + 1$, and $b_i = a_i$ for any $i \neq k - 1$. By setting $\rho = \mu, \mu_1 = 0, \nu_1 = \lambda_1 = \vartheta_1$, we again prove the desired result using Lemma 2.15(a). We summarize the result of this section as follows.

Theorem 4.2. *Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ ($n \geq 2$), and choose $\mu, \nu \in P_+(\mathfrak{g})$ admissible at level l .*

- (a) *The maps $\Psi : \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1, \mu) \rightarrow \text{Hom}_{\mathfrak{g}}(\vartheta_1 \otimes \mu, \nu)$ and $\Gamma : \text{Hom}_{\mathfrak{g}}(\vartheta_1 \otimes \mu, \nu) \rightarrow L_{\mathfrak{g}}(\vartheta_1)[\nu - \mu]^*$ are bijective.*
- (b) *$L_{\mathfrak{g}}(\vartheta_1)$ is a generating $V_{\mathfrak{g}}^l$ -module.*
- (c) *For any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1, \mu)$ and $u^{(\vartheta_1)} \in L_{\mathfrak{g}}(\vartheta_1)$, $\mathcal{Y}(u^{(\vartheta_1)}, x)$ satisfies 0-th order energy bounds. In particular, \mathcal{Y} is energy-bounded.*

Remark 4.3. Our arguments in this section also work pretty well for affine \mathfrak{sl}_n VOAs. Indeed, one has a similar embedding $\mathfrak{sl}_n \subset \mathfrak{so}_{2n}$ with Dynkin index 1 defined by (4.3). Then one can show, similarly, that any irreducible intertwining operator of $V_{\mathfrak{sl}_n}^1$ whose charge space restricts to the standard representation $\mathbb{C}^n = L_{\mathfrak{sl}_n}(\vartheta_1)$ of \mathfrak{sl}_n is a compression of an irreducible intertwining operator of $V_{\mathfrak{so}_{2n}}^1$ whose charge space is $L_{\mathfrak{so}_{2n}}(\theta_1, 1)$ and whose source and target spaces are $L_{\mathfrak{so}_{2n}}(\zeta_{\pm}, 1)$. The higher level case can similarly be reduced to the level 1 case using Lemma 2.15(a).

We remark that the method used in [Was98] for type A also works for our type C . This method also uses essentially the index 1 embedding $\mathfrak{sl}_n \subset \mathfrak{so}_{2n}$, but it uses free fermions instead of the intertwining operators of affine $V_{\mathfrak{so}_{2n}}^1$. In fact, if we let F_{2n} be the $2n$ -dimensional free Fermion unitary vertex superalgebra whose even part is a unitary VOA, then its weight 1 subspace $F_{2n}(1)$, as a unitary Lie algebra, is equivalent to \mathfrak{so}_{2n} , and the level of \mathfrak{so}_{2n} in (the even part of) F_{2n} is 1. Therefore, the level of \mathfrak{sl}_n in F_{2n} is also 1. One can then realize the intertwining operators of $V_{\mathfrak{sl}_n}^1$ whose charge spaces are $L_{\mathfrak{sl}_n}(\vartheta_1, 1)$ as compressions of the vertex operator of F_{2n} , thus proving the results for $V_{\mathfrak{sl}_n}^1$. This method can easily be adapted to $\mathfrak{sp}_{2n} \subset \mathfrak{so}_{4n} \subset F_{4n}$ to prove the results for $V_{\mathfrak{sp}_{2n}}^1$. One thus sees that types A and C are similar in many aspects. However, the lattice VOA method works only for type A , and (clearly) not for type C .

For the reader’s convenience, we list the result for type A as follows.

Theorem 4.4 (Cf. [Was98]). *Let $\mathfrak{g} = \mathfrak{sl}_n$ ($n \geq 2$), and choose $\mu, \nu \in P_+(\mathfrak{g})$ admissible at level l . Let ϑ_1 be the integral dominant weight of \mathfrak{g} so that $L_{\mathfrak{g}}(\vartheta_1)$ is the standard representation \mathbb{C}^n of \mathfrak{sl}_n . Then (a), (b), (c) of Theorem 4.2 hold verbatim.*

5. TYPE G_2

5.1. $\mathfrak{g}_2 \subset \mathfrak{e}_8$.

Facts about \mathfrak{g}_2 . The root system of \mathfrak{g}_2 is shown in Figure 1. Among these roots, we choose α_1, α_2 to be a set of simple roots. Then ϑ_1 and ϑ_2 are the fundamental weights, and ϑ_2 is the highest root of \mathfrak{g}_2 . Let $(\cdot | \cdot)$ be the normalized invariant inner product of \mathfrak{g}_2 . Then clearly $(\vartheta_1 | \vartheta_2) = 1$, and $(\vartheta_2 | \vartheta_2) = 2$.

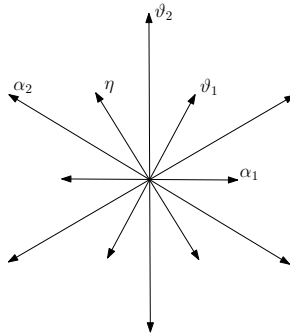


FIGURE 1. G_2 root system

It is well-known that $L_{\mathfrak{g}_2}(\vartheta_1)$ has dimension 7, its weights are $\pm\vartheta_1, \pm\eta, \pm\alpha_1, 0$, and for any weight the weight space has dimension 1. (Here $\eta = \alpha_2 - \alpha_1$. See Figure 1.) We now determine, for any integral dominant weights μ, ν of \mathfrak{g}_2 , the tensor product rule $\dim \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \mu, \nu)$. If this number is non-zero, then $\nu - \mu$ must be one of the 7 weights of $L_{\mathfrak{g}_2}(\vartheta_1)$. Write $\nu = n_1\vartheta_1 + n_2\vartheta_2$ with $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. Then, using Corollary 1.18, one can show that $\dim \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \mu, \nu) = 1$ if $\nu - \mu \in \{\pm\vartheta_1, \pm\eta, \pm\alpha_1\}$ or if $\mu = \nu$ and $n_1 > 0$; otherwise $\dim \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \mu, \nu) = 0$. This result can be summarized by the tensor graph in Figure 2.

Graph 2 is read as follows. Dots represent dominant integral weights of \mathfrak{g}_2 . If two dots represent $\mu, \nu \in P_+(\mathfrak{g}_2)$, then a line segment connecting these two points means $\dim \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \mu, \nu) = \dim \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \nu, \mu) = 1$. The equality of these two dimensions stems from the fact that $\vartheta_1 = \overline{\vartheta_1}$, which is a consequence of $\dim \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \vartheta_1, 0) = 1$ and Proposition 1.15. When two dots coincide, a line segment becomes a loop.

Embedding $\mathfrak{g}_2 \oplus \mathfrak{f}_4 \subset \mathfrak{e}_8$. In [Dyn52], E.B. Dynkin gave a systematic treatment of maximal Lie subalgebras of a simple Lie algebra. We will use the example $\mathfrak{g}_2 \oplus \mathfrak{f}_4 \subset \mathfrak{e}_8$ to prove the energy bounds conditions of intertwining operators of \mathfrak{g}_2 . Let us first review the construction of the subalgebra $\mathfrak{g}_2 \oplus \mathfrak{f}_4$ given in [Dyn52, Table 35].

The simple roots ρ_1, \dots, ρ_8 of \mathfrak{e}_8 and β_1, \dots, β_4 of \mathfrak{f}_4 are chosen as in Figure 3. We choose raising operators $P_1 \in (\mathfrak{e}_8)_{\rho_1}, \dots, P_8 \in (\mathfrak{e}_8)_{\rho_8}$ of unit length. Now, for

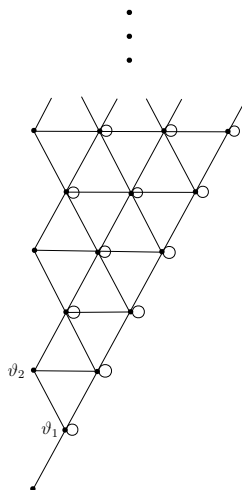


FIGURE 2. $L_{\mathfrak{g}}(\vartheta_1)$ tensor graph

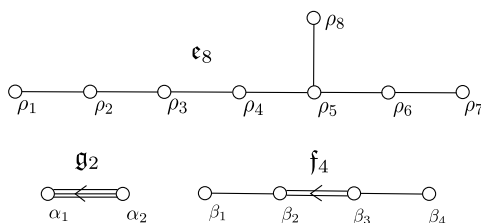


FIGURE 3

any $i_1, i_2, \dots, i_n \in \{1, \dots, 8\}$, we write $P_{i_1 i_2 \dots i_n} = [P_{i_1}, [P_{i_2}, \dots, [P_{i_{n-1}}, P_{i_n}]]]$. Set

$$(5.1) \quad A_1 = P_{234568543} - P_{234567854} + P_{234567856}, \quad A_2 = P_1,$$

$$(5.2) \quad B_1 = P_3 + P_7, \quad B_2 = P_4 + P_6, \quad B_3 = P_5, \quad B_4 = P_8.$$

Theorem 5.1 ([Dyn52, Theorem 13.1 and Table 35]). *The unitary Lie subalgebra of \mathfrak{e}_8 generated by $A_1, A_2, B_1, \dots, B_4$ is equivalent to $\mathfrak{g}_2 \oplus \mathfrak{f}_4$. It is a maximal unitary Lie subalgebra of \mathfrak{e}_8 . Moreover, $A_1 \in (\mathfrak{g}_2)_{\alpha_1}, A_2 \in (\mathfrak{g}_2)_{\alpha_2}, B_1 \in (\mathfrak{f}_4)_{\beta_1}, \dots, B_4 \in (\mathfrak{f}_4)_{\beta_4}$, and these elements are non-zero.*

Note that a unitary Lie subalgebra generated by some elements is the Lie subalgebra generated by these elements together with their adjoints.

Irreducible decomposition of $\mathfrak{g}_2 \curvearrowright \mathfrak{e}_8$. Since \mathfrak{g}_2 is a unitary Lie subalgebra of \mathfrak{e}_8 , the adjoint representation $\mathfrak{g}_2 \curvearrowright \mathfrak{e}_8$ (which is the restriction of the adjoint representation $\mathfrak{e}_8 \curvearrowright \mathfrak{e}_8$ to \mathfrak{g}_2) is a unitary representation of \mathfrak{g}_2 . We now study the irreducible decomposition of this representation.

Lemma 5.2. *The embedding $\mathfrak{g}_2 \subset \mathfrak{e}_8$ has Dynkin index 1.*

Proof. Let $(\cdot|\cdot)$ be the normalized invariant inner product of \mathfrak{e}_8 . Recall that $(P_1|P_1) = 1$. Let $H_1 = [P_1, P_1^*]$. As $(\rho_1|\rho_1) = 2$, by (1.29), P_1, P_1^*, H_1 satisfy the unitary \mathfrak{sl}_2 -relation. Since $A_1 = P_1$ and $H_1 = [A_1, A_1^*]$, by (1.29) and (1.32), there exists

$c \in \mathbb{C}$ of unit length such that $cA_1 = \sqrt{2}A_1/(\|A_1\|\|\alpha_2\|) = \sqrt{2}A_1/\|\alpha_2\|$. Comparing the norms of both sides, we get $\|\alpha_2\| = \sqrt{2}$; i.e., the longest roots of \mathfrak{g}_2 have norm $\sqrt{2}$ under the normalized invariant inner product of \mathfrak{e}_8 . Hence $(\cdot|\cdot)$ restricts to the normalized invariant inner product of \mathfrak{g}_2 . □

Lemma 5.3. *Let \mathfrak{p} be a unitary simple Lie algebra, and let \mathfrak{g} be a unitary simple Lie subalgebra of \mathfrak{p} . Assume that $\mathfrak{g} \subset \mathfrak{p}$ has Dynkin index k . Consider the adjoint action of \mathfrak{g} on the orthogonal complement $\mathfrak{g}^\perp = \mathfrak{p} \ominus \mathfrak{g}$. If $v_\lambda \in \mathfrak{g}^\perp$ is a highest-weight vector of $\mathfrak{g} \curvearrowright \mathfrak{g}^\perp$ with highest weight λ , then λ is admissible at level k .*

Proof. The PER $\widehat{\mathfrak{p}} \curvearrowright L_{\mathfrak{p}}(0, 1)$ restricts to a level k PER $\widehat{\mathfrak{g}} \curvearrowright L_{\mathfrak{p}}(0, 1)$. Let Ω be the vacuum vector of $L_{\mathfrak{p}}(0, 1)$. If we can show that $v_\lambda(-1)\Omega$ is a weight λ highest-weight vector of this PER, then $\widehat{\mathfrak{g}} \curvearrowright \widehat{\mathfrak{g}}v_\lambda(-1)\Omega$ is unitarily equivalent to $\widehat{\mathfrak{g}} \curvearrowright L_{\mathfrak{g}}(\lambda, k)$, and hence λ is admissible at level k .

Choose any $X \in \mathfrak{g}$ and $n = 1, 2, \dots$. If $n > 1$, then the conformal weight of $X(n)v_\lambda(-1)\Omega$ is negative, which immediately shows that $X(n)v_\lambda(-1)\Omega = 0$. In the case that $n = 1$, we compute

$$X(1)v_\lambda(-1)\Omega = [X(1), v_\lambda(-1)]\Omega = [X, v_\lambda](0)\Omega + (X|v_\lambda^*)\Omega = (v_\lambda|X^*)\Omega.$$

Since $v_\lambda \in \mathfrak{g}^\perp$, the right hand side of the above equation vanishes. Therefore $X(1)v_\lambda(-1)\Omega = 0$.

Now let us assume $X \in \mathfrak{g}_+$. Then $[X, v_\lambda] = \text{Ad}(X)v_\lambda = 0$. Therefore

$$X(0)v_\lambda(-1)\Omega = [X(0), v_\lambda(-1)]\Omega = [X, v_\lambda](-1)\Omega = 0.$$

Finally, if $H \in \mathfrak{h}$ where \mathfrak{h} is the Cartan subalgebra of \mathfrak{g} , then $[H, v_\lambda] = \langle \lambda, H \rangle v_\lambda$. Thus

$$H(0)v_\lambda(-1)\Omega = [H, v_\lambda](-1)\Omega = \langle \lambda, H \rangle v_\lambda(-1)\Omega.$$

Our proof is finished. □

Now we are ready to study the irreducible decomposition of $\mathfrak{g}_2 \curvearrowright \mathfrak{e}_8$. In the following, we let $(\cdot|\cdot)$ be the normalized invariant inner product of \mathfrak{e}_8 , which restricts to the normalized invariant inner product of \mathfrak{g}_2 . Consider the orthogonal decomposition of the unitary \mathfrak{g}_2 -module:

$$(5.3) \quad \mathfrak{e} = \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp.$$

The adjoint action $\mathfrak{g}_2 \curvearrowright \mathfrak{g}_2$ is unitarily equivalent to $\mathfrak{g}_2 \curvearrowright L_{\mathfrak{g}_2}(\vartheta_2)$. Since $[\mathfrak{g}_2, \mathfrak{f}_4] = 0$, the action $\mathfrak{g}_2 \curvearrowright \mathfrak{f}_4$ is trivial; i.e., it is an orthogonal direct sum of $\dim \mathfrak{f}_4 = 52$ pieces of irreducible trivial \mathfrak{g}_2 -modules.

It remains to study $\mathfrak{g}_2 \curvearrowright (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$. We first show that this representation has no trivial irreducible submodules. Let $\mathfrak{k} = \{X \in \mathfrak{e}_8 : [\mathfrak{g}_2, X] = 0\}$. Then

$$\begin{aligned} [\mathfrak{g}_2, [\mathfrak{k}, \mathfrak{k}]] &= [[\mathfrak{g}_2, \mathfrak{k}], \mathfrak{k}] + [\mathfrak{k}, [\mathfrak{g}_2, \mathfrak{k}]] = 0, \\ [\mathfrak{g}_2, \mathfrak{k}^*] &= [\mathfrak{k}, \mathfrak{g}_2^*]^* = [\mathfrak{k}, \mathfrak{g}_2]^* = 0, \end{aligned}$$

which shows that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $\mathfrak{k}^* \subset \mathfrak{k}$; i.e., \mathfrak{k} is a unitary Lie subalgebra of \mathfrak{e}_8 . Moreover, since \mathfrak{g}_2 is simple,

$$(\mathfrak{g}_2|\mathfrak{k}) = (([\mathfrak{g}_2, \mathfrak{g}_2]|\mathfrak{k})) = (\mathfrak{g}_2|[\mathfrak{g}_2^*, \mathfrak{k}]) = (\mathfrak{g}_2|[\mathfrak{g}_2, \mathfrak{k}]) = 0.$$

Thus \mathfrak{k} is orthogonal to \mathfrak{g}_2 . Hence \mathfrak{e}_8 contains a unitary Lie subalgebra $\mathfrak{g}_2 \oplus \mathfrak{k}$. Obviously $\mathfrak{f}_4 \subset \mathfrak{k}$. Since $\mathfrak{g}_2 \oplus \mathfrak{f}_4$ is a maximal unitary Lie subalgebra of \mathfrak{e}_8 , \mathfrak{f}_4 must equal \mathfrak{k} . Therefore, for any $v \in (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$, if \mathfrak{g}_2 acts trivially on v (which is equivalent

to saying that $[\mathfrak{g}_2, v] = 0$), then $v \in \mathfrak{k} = \mathfrak{f}_4$, which forces v to be 0. We conclude that $\mathfrak{g}_2 \curvearrowright (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$ has no irreducible submodule equivalent to $L_{\mathfrak{g}_2}(0) = \mathbb{C}$.

Now, by Lemmas 5.2 and 5.3, any irreducible submodule of $\mathfrak{g}_2 \curvearrowright (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$ must be a non-trivial irreducible \mathfrak{g}_2 -module whose highest weight is admissible at level 1. The only choice of such weight is ϑ_1 . Therefore $(\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$ is equivalent to a direct sum of $L_{\mathfrak{g}_2}(\vartheta_1)$. Using the dimensions of $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_8$, and $L_{\mathfrak{g}_2}(\vartheta_1)$, one checks that the multiplicity of $L_{\mathfrak{g}_2}(\vartheta_1)$ in $(\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$ is $(248 - 14 - 52)/7 = 26$. We conclude the following.

Theorem 5.4. *The adjoint representation $\mathfrak{g}_2 \curvearrowright \mathfrak{e}_8$ has the irreducible decomposition*

$$(5.4) \quad \mathfrak{e}_8 \simeq 52\mathbb{C} \oplus 26L_{\mathfrak{g}_2}(\vartheta_1) \oplus L_{\mathfrak{g}_2}(\vartheta_2).$$

Moreover, we have the following identifications:

$$(5.5) \quad 52\mathbb{C} = \mathfrak{f}_4, \quad 26L_{\mathfrak{g}_2}(\vartheta_1) = (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp, \quad L_{\mathfrak{g}_2}(\vartheta_2) = \mathfrak{g}_2.$$

5.2. Main result. In this section we set $\mathfrak{g} = \mathfrak{g}_2$ and $\mathfrak{p} = \mathfrak{e}_8$.

We first prove the energy bounds condition for the creation operator of $L_{\mathfrak{g}}(\vartheta_1, 1)$. Choose an arbitrary $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^1(\vartheta_1, 0)$. We show that for any $u^{(\vartheta_1)} \in L_{\mathfrak{g}}(\vartheta_1) \subset L_{\mathfrak{g}}(\vartheta_1, 1)$, $\mathcal{Y}(u^{(\vartheta_1)}, x)$ satisfies first order energy bounds. Let $Y_{\mathfrak{p}}$ be the vertex operator of $V_{\mathfrak{p}}^1$ representing the vacuum module. Then $Y_{\mathfrak{p}} \in \mathcal{V}_{\mathfrak{p}}^1(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix})$. Let Ω be the vacuum vector of $V_{\mathfrak{p}}^1$. Then Ω is a 0-highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright V_{\mathfrak{p}}^1$. Now, by Theorem 5.4, we can choose a ϑ_1 -highest-weight vector v_{ϑ_1} of $\mathfrak{g} \curvearrowright (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$. Using the argument in the proof of Lemma 5.3, one can show that $v_{\vartheta_1}(-1)\Omega$ is a ϑ_1 -highest-weight vector of $\widehat{\mathfrak{g}} \curvearrowright V_{\mathfrak{p}}^1$. Now the creation property $\lim_{x \rightarrow 0} Y_{\mathfrak{p}}(v_{\vartheta_1}(-1)\Omega, x)\Omega = v_{\vartheta_1}(-1)\Omega$ gives $\langle Y_{\mathfrak{p}}(v_{\vartheta_1}(-1)\Omega, x)\Omega | v_{\vartheta_1}(-1)\Omega \rangle \neq 0$. By Theorem 1.24, for any $u \in \mathfrak{g}v_{\vartheta_1}(-1)\Omega \subset V_{\mathfrak{p}}^1(1)$, $Y_{\mathfrak{p}}(u, x)$ satisfies first order energy bounds. Thus we can prove the desired results using Corollary 2.13.

We now fix an arbitrary $l = 1, 2, \dots$. For any $\mu, \nu \in P_+(\mathfrak{g})$ admissible at level l , we shall show that $\dim \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1, \begin{smallmatrix} \nu \\ \mu \end{smallmatrix}) = \dim \text{Hom}_{\mathfrak{g}}(\vartheta_1 \otimes \mu, \nu)$ and that for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1, \begin{smallmatrix} \nu \\ \mu \end{smallmatrix})$ and $u^{(\vartheta_1)} \in L_{\mathfrak{g}}(\vartheta_1)$, $\mathcal{Y}(u^{(\vartheta_1)}, x)$ satisfies first order energy bounds. Note that when $(\mu|\vartheta_2) < l$ or $(\nu|\vartheta_2) < l$, this result follows from Proposition 2.14 and the energy bounds condition for level 1 creation operators. Therefore we may well assume that $(\mu|\vartheta_2) = (\nu|\vartheta_2) = l$. This means that on the tensor graph, Figure 2, μ and ν have the same height l (if we assume that the height of ϑ_1 is 1).

Now we assume that $\dim \text{Hom}_{\mathfrak{g}}(\vartheta_1 \otimes \mu, \nu) = 1$. Then on the tensor graph, μ and ν are connected by either a horizontal line segment or a loop. (In the later case $\mu = \nu$.) Using the tensor graph, it is not hard to find $a \in \mathbb{Z}_{>0}$ and $\rho, \mu_1, \nu_1 \in P_+(\mathfrak{g})$, such that $a \leq l$, ρ is admissible at level $l - a$, μ_1, ν_1 are admissible at level a , $\mu = \rho + \mu_1$, $\nu = \rho + \nu_1$, and one of the following three cases holds: (a) $a = 1, \mu_1 = \nu_1 = \vartheta_1$; (b) $a = 2, \mu_1 = 2\vartheta_1, \nu_1 = \vartheta_2$; (c) $a = 2, \mu_1 = \vartheta_2, \nu_1 = 2\vartheta_1$. Therefore, to prove the result for intertwining operators of general types, it suffices, by Lemma 2.15(a), to prove this result in the following three special cases:

- (I) $l = 1, \mu = \nu = \vartheta_1$.
- (II) $l = 2, \mu = 2\vartheta_1, \nu = \vartheta_2$.
- (III) $l = 2, \mu = \vartheta_2, \nu = 2\vartheta_1$.

Since $\vartheta_1 = \overline{\vartheta_1}$, case (III) will follow from case (II) and Proposition 1.8. Thus we only need to prove cases (I) and (II).

Case (I). To prove this case we need the following lemma.

Lemma 5.5. *There exist ϑ_1 -highest-weight vectors $v_{\vartheta_1}^1, v_{\vartheta_1}^2, v_{\vartheta_1}^3 \in (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$ satisfying $([\mathfrak{g}_2 v_{\vartheta_1}^1, \mathfrak{g}_2 v_{\vartheta_1}^2] | \mathfrak{g}_2 v_{\vartheta_1}^3) \neq 0$.*

Proof. Let $M = (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$. Recall that the raising operators P_1, \dots, P_8 of \mathfrak{e}_8 have unit length. Choose $X = P_2, Y = P_4 - P_6 \in \mathfrak{e}_8$. Then $[X, Y] = P_{24} - P_{26}$. By comparing them with (5.1) and (5.2), one can easily check that $X, Y, [X, Y] \in M$. This proves $[M, M] \cap M \neq 0$, and hence $([M, M] | M) \neq 0$. By Theorem 5.4, the representation $\mathfrak{g}_2 \curvearrowright M$ has the orthogonal irreducible decomposition $M = \bigoplus_{i=1}^{26} M_i$, where each M_i is unitarily equivalent to $L_{\mathfrak{g}_2}(\vartheta_1)$. So there exist $i, j, k = 1, \dots, 26$ such that $([M_i, M_j] | M_k)$. Choose highest-weight vectors $v_{\vartheta_1}^1, v_{\vartheta_1}^2, v_{\vartheta_1}^3$ of M_i, M_j, M_k , respectively. Then $([\mathfrak{g}_2 v_{\vartheta_1}^1, \mathfrak{g}_2 v_{\vartheta_1}^2] | \mathfrak{g}_2 v_{\vartheta_1}^3) \neq 0$. \square

Again let Y_p be the vertex operator for the vacuum module of V_p^l . Choose $v_{\vartheta_1}^1, v_{\vartheta_1}^2, v_{\vartheta_1}^3 \in (\mathfrak{g}_2 \oplus \mathfrak{f}_4)^\perp$ as in Lemma 5.5. From the proof of Lemma 5.3, $v_{\vartheta_1}^1(-1)\Omega, v_{\vartheta_1}^2(-1)\Omega, v_{\vartheta_1}^3(-1)\Omega$ are level 1 ϑ_1 -highest-weight vectors of $\widehat{\mathfrak{g}} \curvearrowright V_p^1$. Since $\mathfrak{g}v_{\vartheta_1}^1(-1)\Omega \subset V_p^1(1)$, by Theorem 1.24, $Y_p(u, x)$ satisfies first order energy bounds for any $u \in \mathfrak{g}v_{\vartheta_1}^1(-1)\Omega$. By Lemma 5.5, we can choose $u_1 \in \mathfrak{g}v_{\vartheta_1}^1, u_2 \in \mathfrak{g}v_{\vartheta_1}^2, u_3 \in \mathfrak{g}v_{\vartheta_1}^3$ satisfying $([u_1, u_2] | u_3) \neq 0$. Hence

$$\begin{aligned} \langle Y(u_1(-1)\Omega, 0)u_2(-1)\Omega | u_3(-1)\Omega \rangle &= \langle u_1(0)u_2(-1)\Omega | u_3(-1)\Omega \rangle \\ &= \langle [u_1(0), u_2(-1)\Omega] | u_3(-1)\Omega \rangle = \langle [u_1, u_2](-1)\Omega | u_3(-1)\Omega \rangle \\ &= \langle u_3(-1)^* [u_1, u_2](-1)\Omega | \Omega \rangle \\ &= \langle u_3^*(1) [u_1, u_2](-1)\Omega | \Omega \rangle = \langle [u_3^*, [u_1, u_2]](0)\Omega | \Omega \rangle + (u_3^* | [u_1, u_2]^*) \langle \text{id} \Omega | \Omega \rangle \\ &= ([u_1, u_2] | u_3) \neq 0. \end{aligned}$$

We can therefore use Corollary 2.13 to prove the desired result.

Case (II). To prove this case we also need a lemma.

Lemma 5.6. *Choose a non-zero $T \in \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \vartheta_1, \vartheta_1)$. Let u_1 be a non-zero element in $L_{\mathfrak{g}_2}(\vartheta_1)[- \alpha_1]$, and let $u_2 \in L_{\mathfrak{g}_2}(\vartheta_1)[\vartheta_1]$ be a highest-weight vector of $L_{\mathfrak{g}_2}(\vartheta_1)$. Then $T(u_1 \otimes u_2) \neq 0$.*

Proof. Our notation is as in Theorem 5.1, where A_1 is a non-zero element in $(\mathfrak{g}_2)_{\alpha_1}$. Then $A_1 u_2 = 0$. By analyzing the \mathfrak{sl}_2 -module $L_{\mathfrak{g}_2}(\vartheta_1)[\alpha_1] \oplus L_{\mathfrak{g}_2}(\vartheta_1)[0] \oplus L_{\mathfrak{g}_2}(\vartheta_1)[- \alpha_1]$, it is easy to see that $A_1 u_1 \neq 0$. Therefore

$$(5.6) \quad \langle T(u_1 \otimes u_2) | A_1^* u_2 \rangle = - \langle T(A_1 u_1 \otimes u_2) | u_2 \rangle.$$

Now $A_1 u_1$ is a non-zero element in $L_{\mathfrak{g}_2}(\vartheta_1)[0]$. Hence, by the injectivity of $\Gamma : \text{Hom}_{\mathfrak{g}_2}(\vartheta_1 \otimes \vartheta_1, \vartheta_1) \rightarrow L_{\mathfrak{g}_2}(\vartheta_1)[0]$, the right hand side of equation (5.6) is non-zero. Therefore $T(u_1 \otimes u_2) \neq 0$. \square

Now for case (II), we let $a = 1, \rho = \vartheta_1, \mu_1 = \nu_1 = \vartheta_1$. Then $\mu = \rho + \mu_1, L_{\mathfrak{g}}(\nu) \subset L_{\mathfrak{g}}(\rho) \otimes L_{\mathfrak{g}}(\nu_1)$. By Lemma 5.6, condition (2.16) is satisfied. By case (I), $\dim \mathcal{V}_{\mathfrak{g}}^a(\vartheta_1, \mu_1) = \dim \mathcal{V}_{\mathfrak{g}}^1(\vartheta_1, \vartheta_1) = 1$, and for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^a(\vartheta_1, \mu_1)$ and $u \in L_{\mathfrak{g}}(\vartheta_1)$, $\mathcal{Y}(u, x)$ satisfies first order energy bounds. Therefore case (II) can be proved using Lemma 2.15(b).

Now, since we know the fusion rules of $L_{\mathfrak{g}}(\vartheta_1)$ with another irreducible $V_{\mathfrak{g}}^l$ -module, we can deduce that $L_{\mathfrak{g}}(\vartheta_1, l)$ is generating. The main result of this chapter is thus proved.

Theorem 5.7. *Let $\mathfrak{g} = \mathfrak{g}_2$, and choose $\mu, \nu \in P_+(\mathfrak{g})$ admissible at level l .*

(a) *The map $\Psi : \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1^\nu, \mu) \rightarrow \text{Hom}_{\mathfrak{g}}(\vartheta_1 \otimes \mu, \nu)$ is bijective. The dimension of these two isomorphic spaces can be read from Figure 2.*

(b) *$L_{\mathfrak{g}}(\vartheta_1)$ is a generating $V_{\mathfrak{g}}^l$ -module.*

(c) *For any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l(\vartheta_1^\nu, \mu)$ and $u^{(\vartheta_1)} \in L_{\mathfrak{g}}(\vartheta_1)$, $\mathcal{Y}(u^{(\vartheta_1)}, x)$ satisfies first order energy bounds. In particular, \mathcal{Y} is energy-bounded.*

6. CONCLUDING REMARKS

Assume that V is a unitary rational VOA whose representation category exists as a modular tensor category in the sense of [Hua08]. We've defined, in [Gui19, Gui17], a sesquilinear form Λ on each dual vector space of intertwining operators of V , which is automatically non-degenerate. To obtain a unitary tensor product theory, one needs to prove the positivity of Λ . In those two papers, we gave two criteria on the positivity of Λ , both concerning the energy bounds condition for intertwining operators and vertex operators. See conditions A and B in [Gui17, section 5.3].

Now let V be a unitary affine VOA $V_{\mathfrak{g}}^l$. If there exists a generating set \mathcal{F} of irreducible $V_{\mathfrak{g}}^l$ -modules such that any (irreducible) intertwining operator of $V_{\mathfrak{g}}^l$ whose charge space lies inside \mathcal{F} is energy-bounded, then condition A is satisfied, and hence, by [Gui17, Theorem 7.8], Λ is positive, and the modular tensor category is unitary. Now by the results of [Was98] and [TL04] and by the main results we've obtained in this paper (see Theorems 3.1, 3.3, 4.2, 4.4, 5.7), $V_{\mathfrak{g}}^l$ satisfies such requirement when \mathfrak{g} is of type A, B, C, D , or G_2 . Therefore we have

Theorem 6.1. *Let \mathfrak{g} be a (unitary) complex simple Lie algebra of type A, B, C, D , or G_2 . Choose $l = 0, 1, 2, \dots$. Then on any dual vector space of intertwining operators of $V_{\mathfrak{g}}^l$, the sesquilinear form Λ defined in [Gui17, section 6.2] is positive-definite. Moreover Λ induces a unitary structure on the modular tensor category of $V_{\mathfrak{g}}^l$.*

The remaining unsolved cases are types E and F_4 . Note that the type E level 1 cases are known due to Corollary A.7. The methods we have used for G_2 in this article, in particular, the embedding $\mathfrak{g}_2 + \mathfrak{f}_4 \subset \mathfrak{e}_8$, might also be helpful in the F_4 level 1 case. However, the analysis might be more complicated due to the increase of the dimension. For types E_6, E_7, F_4 , it is also possible, though much more complicated, to reduce the higher level cases to the level 1 cases using the compression principle. This method, however, does not work for E_8 , as the representation theory of type E_8 level 1 is trivial. The smallest non-trivial situation for E_8 is when the level equals 2, in which case there will be fusion rules greater than 1. This phenomenon tremendously increases the difficulty of analysis, so one needs to look for other methods to work with this case.

APPENDIX A. HEISENBERG AND LATTICE VOAs

In this appendix we discuss the energy bounds conditions for the intertwining operators of Heisenberg VOAs and even lattice VOAs.

Unitary Heisenberg VOAs. Let \mathfrak{h} be a unitary abelian Lie algebra of dimension n . In particular, an inner product $(\cdot | \cdot)$ and an anti-unitary involution $*$ is chosen. We let $\mathfrak{h}_{\mathbb{R}} = \{X \in \mathfrak{h} : X^* = -X\}$. The affine Lie algebra $\hat{\mathfrak{h}}$ is defined in much the same way as we define $\hat{\mathfrak{g}}$ when \mathfrak{g} is simple. The irreducible PER theory of

$\widehat{\mathfrak{h}}$ is also similar to $\widehat{\mathfrak{g}}$, except that for any $\lambda \in (i\mathfrak{h}_{\mathbb{R}})^*$ and $l > 0$, there exists a unique (up to equivalence) unitary $\widehat{\mathfrak{h}}$ -module $L_{\widehat{\mathfrak{h}}}(\lambda, l)$, which is also isomorphic to $L_{\widehat{\mathfrak{h}}}(\lambda, 1)$. So we can always assume that the level $l = 1$. Similarly to unitary affine VOAs, there exists a unitary VOA structure $V_{\widehat{\mathfrak{h}}}^1$ on $L_{\widehat{\mathfrak{h}}}(0, 1)$ of CFT type, and equation (1.48) gives the conformal vector of $V_{\widehat{\mathfrak{h}}}^1$ by setting $h^\vee = 0$ (and of course $l = 1$). $V_{\widehat{\mathfrak{h}}}^1$ is called the **unitary Heisenberg VOA** associated to $\widehat{\mathfrak{h}}$. Any unitary irreducible $V_{\widehat{\mathfrak{h}}}^1$ -module can be restricted to the unitary $\widehat{\mathfrak{h}}$ -module $L_{\widehat{\mathfrak{h}}}(\lambda, 1)$ for some $\lambda \in (i\mathfrak{h}_{\mathbb{R}})^*$. The conformal weight of $L_{\widehat{\mathfrak{h}}}(\lambda, 1)$ is $(\lambda|\lambda)/2$. Conversely, for any $\lambda \in (i\mathfrak{h}_{\mathbb{R}})^*$, the irreducible PER $L_{\widehat{\mathfrak{h}}}(\lambda, 1)$ of $\widehat{\mathfrak{h}}$ can be extended to a unitary $V_{\widehat{\mathfrak{h}}}^1$ -module. See [LL04] for more details. Unfortunately, there are uncountably many irreducible $V_{\widehat{\mathfrak{h}}}^1$ -modules, so $V_{\widehat{\mathfrak{h}}}^1$ is not a rational VOA.

The inner product on \mathfrak{h} identifies the dual space \mathfrak{h}^* of \mathfrak{h} as the conjugate of \mathfrak{h} , which in turn identifies the \mathbb{R} -vector space $i\mathfrak{h}_{\mathbb{R}}$ with its (real) dual space $(i\mathfrak{h}_{\mathbb{R}})^*$. We assume this identification in the following. One of the most important problems in the theory of Heisenberg VOAs and lattice VOAs was to determine the intertwining operators of $V_{\mathfrak{h}}^1$, as well as their braid and fusion relations. This was mainly achieved in [DL93]. See also [TZ12] for a brief exposition of this theory. The result can be summarized as below. For any $\alpha, \beta, \gamma \in i\mathfrak{h}_{\mathbb{R}}$, if $\gamma = \alpha + \beta$, then the fusion rule $N_{\alpha\beta}^\gamma = \dim \mathcal{V}_{\mathfrak{h}}^1(\begin{smallmatrix} \gamma \\ \alpha \ \beta \end{smallmatrix}) = 1$. Otherwise $N_{\alpha\beta}^\gamma = 0$. Intertwining operators of type $(\begin{smallmatrix} \alpha+\beta \\ \alpha \ \beta \end{smallmatrix})$ can be described as follows. First, we fix for any $\lambda \in i\mathfrak{h}_{\mathbb{R}}$ a unital highest-weight vector v_λ in $L_{\mathfrak{h}}(\lambda, 1)$. We choose v_0 to be the vacuum vector $\Omega \in V_{\mathfrak{h}}^1 = L_{\mathfrak{h}}(0, 1)$. Then for any $\mu \in i\mathfrak{h}_{\mathbb{R}}$, one can easily define a unitary map $c_\lambda : L_{\mathfrak{h}}(\mu, 1) \rightarrow L_{\mathfrak{h}}(\lambda + \mu, 1)$ such that

$$(A.1) \quad c_\lambda \cdot X_1(-n_1) \cdots X_k(-n_k)v_\mu = X_1(-n_1) \cdots X_k(-n_k)v_{\lambda+\mu}$$

for any $X_1, \dots, X_k \in \mathfrak{h}, n_1, \dots, n_k \in \mathbb{Z}_{>0}$. It follows that $[X(n), c_\lambda] = \delta_{n,0}(X|\lambda)c_\lambda$ for any $X \in \mathfrak{h}$ and $n \in \mathbb{Z}$. We also have $v_\lambda = c_\lambda\Omega$. Let $x^{\lambda(0)}$ be an operator-valued formal series of x , so that $x^{\lambda(0)} = x^{(\lambda|\mu)} \cdot \text{id}$ when acting on $L_{\mathfrak{h}}(\mu, 1)$. We also define operator-valued formal power series

$$(A.2) \quad E^\pm(\lambda, x) = \exp(\mp \sum_{n \in \mathbb{Z}_{>0}} \frac{\lambda(\pm n)}{n} x^{\mp n})$$

acting on each $L_{\mathfrak{h}}(\mu, 1)$. Then a standard (non-zero) type $(\begin{smallmatrix} \alpha+\beta \\ \alpha \ \beta \end{smallmatrix})$ intertwining operator $\mathcal{Y}_{\alpha,\beta}^{\alpha+\beta}$ can be defined to be

$$(A.3) \quad \mathcal{Y}_{\alpha,\beta}^{\alpha+\beta}(c_\alpha u, x) = c_\alpha E^-(\alpha, x)Y(u, x)E^+(\alpha, x)x^{\alpha(0)},$$

where $u \in L_{\mathfrak{h}}(0, 1) = V_{\mathfrak{h}}^1$, and $Y(u, x)$ is the vertex operator of $V_{\mathfrak{h}}^1$. To simplify our notation, we let $\mathcal{Y}_\alpha(c_\alpha u, x)$ act on any $L_{\mathfrak{h}}(\mu, 1)$ (where $\mu \in i\mathfrak{h}_{\mathbb{R}}$) to be $\mathcal{Y}_{\alpha,\mu}^{\alpha+\mu}(c_\alpha u, x)$. Now we can let x be a complex variable z in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Then $\mathcal{Y}_\alpha(c_\alpha u, z)$ is a multivalued function of z depending on its argument $\arg z$. The following fusion and braid relations are due to [DL93, Theorem 5.1].

Theorem A.1. *Choose $\alpha, \beta, \gamma \in i\mathfrak{h}_{\mathbb{R}}, u, v \in V_{\mathfrak{h}}^1, z_1, z_2 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Then the following fusion and braid relations hold when acting on $L_{\mathfrak{h}}(\gamma, 1)$.*

(a) *If $0 < |z_1 - z_2| < |z_2| < |z_1|$ and $\arg z_1 = \arg z_2 = \arg(z_1 - z_2)$, then*

$$(A.4) \quad \mathcal{Y}_\alpha(c_\alpha u, z_1)\mathcal{Y}_\beta(c_\beta v, z_2) = \mathcal{Y}_{\alpha+\beta}(\mathcal{Y}_\alpha(c_\alpha u, z_1 - z_2)c_\beta v, z_2).$$

(b) If $|z_1| = |z_2| = 1$ and $\arg z_2 < \arg z_1 < \arg z_2 + 2\pi$, then

$$(A.5) \quad \mathcal{Y}_\alpha(c_\alpha u, z_1)\mathcal{Y}_\beta(c_\beta v, z_2) = e^{i\pi(\alpha|\beta)}\mathcal{Y}_\beta(c_\beta v, z_2)\mathcal{Y}_\alpha(c_\alpha u, z_1),$$

where the left hand side is defined by first replacing z_1 with rz_1 where $r > 1$, then taking the product of the two intertwining operators, and finally letting $r \rightarrow 1$. The right hand side is defined similarly, except that we replace z_2 with rz_2 .

We now discuss adjoint intertwining operators. First, let Θ be the PCT operator of $V_{\mathfrak{h}}^1$. For any $\alpha \in i\mathfrak{h}_{\mathbb{R}}$ we define an anti-unitary map $- : L_{\mathfrak{h}}(\alpha, 1) \rightarrow L_{\mathfrak{h}}(-\alpha, 1)$ satisfying $\overline{c_\alpha u} = c_{-\alpha}\Theta u$ for any $u \in V_{\mathfrak{h}}^1$. So more explicitly, for any $X_1, \dots, X_k \in \mathfrak{h}, n_1, \dots, n_k \in \mathbb{Z}$ we have

$$(A.6) \quad \overline{X_1(n_1) \cdots X_k(n_k)v_\alpha} = (-1)^k X_1^*(n_1) \cdots X_k^*(n_k)v_{-\alpha}.$$

It is easy to check that $-$ induces a unitary $V_{\mathfrak{h}}^1$ -module equivalence between $L_{\mathfrak{h}}(-\alpha, 1)$ and the contragredient module $L_{\mathfrak{h}}(\overline{\alpha}, 1)$ of $L_{\mathfrak{h}}(\alpha, 1)$. Now one can easily compute that the adjoint intertwining operator $(\mathcal{Y}_{\alpha,\beta}^{\alpha+\beta})^\dagger$ of $\mathcal{Y}_{\alpha,\beta}^{\alpha+\beta}$ is⁶

$$(A.7) \quad (\mathcal{Y}_{\alpha,\beta}^{\alpha+\beta})^\dagger = e^{i\pi \cdot \frac{(\alpha|\alpha)}{2}} \mathcal{Y}_{-\alpha,\alpha+\beta}^\beta.$$

Theorem 1.24 clearly holds with \mathfrak{g} replaced by \mathfrak{h} ; i.e., for any $\lambda \in i\mathfrak{h}_{\mathbb{R}}, L_{\mathfrak{h}}(\lambda, 1)$ is energy-bounded, and for any $X \in \mathfrak{h}, Y_\lambda(X(-1)\Omega, x)$ satisfies first order energy bounds. The energy bounds condition of intertwining operators was established in [TL04].

Theorem A.2. Choose any $\alpha, \beta \in i\mathfrak{h}_{\mathbb{R}}$.

- (a) If $(\alpha|\alpha) \leq 1$, then $\mathcal{Y}_\alpha(c_\alpha \Omega, x)$ satisfies 0-th order energy bounds.
- (b) If $(\alpha|\alpha) \leq 2$, then $\mathcal{Y}_\alpha(c_\alpha \Omega, x)$ satisfies first order energy bounds.
- (c) For general α and any $u \in V_{\mathfrak{h}}^1, \mathcal{Y}_\alpha(c_\alpha u, x)$ is energy-bounded.

The proof of this theorem can be sketched as follows. First, if $(\alpha|\alpha) = 1$, the operator $Y_\alpha(x) := E^-(\alpha, x)E^+(\alpha, x)$ and its adjoint satisfy a free-fermion-like anticommuting relation: Write $Y_\alpha(x) = \sum_{n \in \mathbb{Z}} Y_\alpha(n)x^{-n}$; then

$$Y_\alpha(n)Y_\alpha(m)^\dagger + Y_\alpha(m-1)^\dagger Y_\alpha(n-1) = \delta_{n,m}.$$

From this we can deduce the 0-th order energy bounds. See [TL04, chapter VI, Proposition 1.2.1] for more details.⁷ (Note that it doesn't matter which $L_{\mathfrak{h}}(\mu, 1)$ the operator $Y_\alpha(x)$ is acting on, since $Y_\alpha(x)$ commutes with the unitary operator c_β for any $\beta \in i\mathfrak{h}_{\mathbb{R}}$.) If $(\alpha|\alpha) = 2$, then a similar argument shows that $Y_\alpha(x)$ and its adjoint satisfy an affine Lie algebra-like relation. So the first order energy bounds can be established using a similar argument as in Theorem 1.24. Now suppose that $(\alpha|\alpha) < 1$ or < 2 . Identify \mathfrak{h} with $\mathfrak{h} \oplus 0$ in $\mathfrak{h}^1 = \mathfrak{h} \oplus \mathfrak{h}$. Now choose $\beta \in 0 \oplus i\mathfrak{h}_{\mathbb{R}}$ so that $(\beta|\beta) + (\alpha|\alpha) = 1$ or 2 , and let $\gamma = \alpha + \beta \in i\mathfrak{h}_{\mathbb{R}}^1$. Then $(\gamma|\gamma) = 1$ or 2 , and $Y_\gamma(x) = Y_\alpha(x) \otimes Y_\beta(x)$. Then the 0-th order or first order energy bounds condition

⁶An easy way to see this relation is to verify it for the vector $v_\alpha = c_\alpha \Omega$. This is sufficient since $(\mathcal{Y}_{\alpha,\beta}^{\alpha+\beta})^\dagger$ must be a type $\begin{pmatrix} \beta \\ -\alpha & \alpha+\beta \end{pmatrix}$ intertwining operator, which is therefore proportional to $\mathcal{Y}_{-\alpha,\alpha+\beta}^\beta$.

⁷Another way to see this is to note that the lattice vertex superalgebra for the integral lattice \mathbb{Z} is the 1-dimensional free-fermion vertex superalgebra. Likewise, when $(\alpha|\alpha) = 2$, one has the first order energy bounds condition, since the lattice VOA for $\sqrt{2}\mathbb{Z}$ is equivalent to the affine VOA $V_{\mathfrak{sl}_2}^1$.

of $Y_\alpha(x)$ follows from that of $Y_\gamma(x)$. Thus (a) and (b) are proved.⁸ Note that conversely, if we know the energy bounds condition of $Y_\alpha(x)$ and $Y_\beta(x)$, then we can also easily know the energy bounds condition of $Y_\gamma(x)$. Thus by induction and (a), one can easily prove (c) when $u = \Omega$, and hence for any u due to Proposition 1.11. Hence this theorem is proved.

Even lattice VOAs. In some sense, the relation between lattice VOAs and Heisenberg VOAs is similar to that between simple Lie algebras and their Cartan subalgebras. Let us choose a non-degenerate even lattice Λ in $i\mathfrak{h}_\mathbb{R}$. Here “non-degenerate” means that the rank of Λ equals the dimension of $i\mathfrak{h}_\mathbb{R}$, and “even” means that $(\alpha|\alpha) \in 2\mathbb{Z}$ for any $\alpha \in \Lambda$. In particular $(\Lambda|\Lambda) \subset \mathbb{Z}$. The reason we require evenness is to ensure that the conformal weights of $V_\mathfrak{h}^1$ -modules are integers.

We now extend the action of $V_\mathfrak{h}^1$ on $V_\Lambda = \bigoplus_{\alpha \in \Lambda} L_\mathfrak{h}(\alpha, 1)$ to a VOA structure. The vacuum vector and the conformal vector of V_Λ equals those of $L_\mathfrak{h}(0, 1) = V_\mathfrak{h}^1$. For any $\alpha \in \Lambda$ and $u \in V_\mathfrak{h}^1$, we define the vertex operator $Y^\Lambda(c_\alpha u, x)$ of V_Λ to be

$$(A.8) \quad Y^\Lambda(c_\alpha u, x)w^{(\mu)} = \epsilon(\alpha, \mu)\mathcal{Y}_\alpha(c_\alpha u, x)w^{(\mu)}$$

for any $w^{(\mu)} \in L_\mathfrak{h}(\mu, 1)$ where $\mu \in \Lambda$. Here $\epsilon(\alpha, \mu) \in S^1 \subset \mathbb{C}$ will be determined shortly. Y^Λ satisfies the translation property since each \mathcal{Y}_α does. So in order for V_Λ to be a VOA, the sufficient and necessary condition is that Y^Λ satisfies the creation property and the Jacobi identity. The Creation property is equivalent to

$$(A.9) \quad \epsilon(\alpha, 0) = 1 \quad (\forall \alpha \in \Lambda).$$

The Jacobi identity on Y^Λ is equivalent to the following associativity and commutativity properties:

$$(A.10) \quad Y^\Lambda(c_\alpha u, z_1)Y^\Lambda(c_\beta v, z_2) = Y^\Lambda(Y^\Lambda(c_\alpha u, z_1 - z_2)c_\beta v, z_2),$$

$$(A.11) \quad Y^\Lambda(c_\alpha u, z_1)Y^\Lambda(c_\beta v, z_2) = Y^\Lambda(c_\beta v, z_2)Y^\Lambda(c_\alpha u, z_1),$$

which are understood in a similar way as in Theorem A.1. By the fusion relation (A.4), equation (A.10) is equivalent to

$$(A.12) \quad \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) \quad (\forall \alpha, \beta, \gamma \in \Lambda);$$

i.e., ϵ is an S^1 -valued cocycle on Λ . If we assume (A.12), then, by the braid relation (A.5), the commutativity (A.11) is equivalent to

$$(A.13) \quad \epsilon(\alpha, \beta) = (-1)^{(\alpha|\beta)}\epsilon(\beta, \alpha).$$

The existence of ϵ satisfying (A.9), (A.12), and (A.13) is guaranteed by the following well-known proposition (cf. [FLM88, Proposition 5.2.3] or [Kac98, Lemma 5.5]).

Proposition A.3. *If $\Lambda \simeq \mathbb{Z}^n$, (A, \cdot) is an abelian group, and $\omega : \Lambda \times \Lambda \rightarrow A$ satisfies $\omega(\alpha + \beta, \gamma) = \omega(\alpha, \gamma)\omega(\beta, \gamma)$ and $\omega(\alpha, \beta) = \omega(\beta, \alpha)^{-1}$ for any α, β, γ , then there is an A -valued cocycle ϵ on Λ such that $\epsilon(\alpha, \beta) = \omega(\alpha, \beta)\epsilon(\beta, \alpha)$; i.e., the commutator of ϵ is ω . Moreover, if $b : \Lambda \times \Lambda \rightarrow A$ is a coboundary, i.e., $b(\alpha, \beta) = f(\alpha)f(\beta)f(\alpha + \beta)^{-1}$ for some function $f : \Lambda \rightarrow A$, then $\epsilon' = eb$ is also a cocycle with commutant ω .*

⁸This tensor product trick is also due to [TL04, Proposition VI.1.2.1].

Hence, once such ϵ is chosen, we have a VOA V_Λ with vertex operator Y^Λ . Define an anti-unitary map $\Theta^\Lambda : V_\Lambda \rightarrow V_\Lambda$ such that for any $\alpha \in \Lambda, u \in V_\mathfrak{h}^1$,

$$(A.14) \quad \Theta^\Lambda c_\alpha u = (-1)^{\frac{(\alpha|\alpha)}{2}} \epsilon(-\alpha, \alpha)^{-1} \overline{c_\alpha u} = (-1)^{\frac{(\alpha|\alpha)}{2}} \epsilon(-\alpha, \alpha)^{-1} c_{-\alpha} \Theta u.$$

Then using (A.9), (A.12), and (A.13), one can check that Θ^Λ is an anti-automorphism. Using relation (A.7), one arrives at the following (see also [DL14, Theorem 4.12]).

Theorem A.4. *(V_Λ, Y^Λ) is a unitary VOA of CFT type with PCT operator Θ^Λ .*

Note that if we choose f, b, ϵ' as in Proposition A.3 and assume that $f(0) = 1$, then the unitary VOA V_Λ defined by ϵ is clearly equivalent to the one defined by ϵ' , with the unitary equivalence map defined by f .

Let Λ° be the dual lattice of Λ , i.e., the lattice in $i\mathfrak{h}_\mathbb{R}$ generated by the dual basis of a basis of Λ . So $\Lambda \subset \Lambda^\circ$, and we have quotient map $[\cdot] : \Lambda^\circ \rightarrow \Lambda^\circ/\Lambda, \lambda \mapsto [\lambda]$. Choose a cocycle ϵ on Λ° whose commutator ω satisfies $\omega(\alpha, \beta) = (-1)^{(\alpha|\beta)}$ for any $\alpha, \beta \in \Lambda$. As above, we let $W_{\Lambda^\circ} = \bigoplus_{\lambda \in \Lambda^\circ} L_\mathfrak{h}(\lambda, 1)$ and let V_Λ^1 act on W_{Λ° with vertex operator Y^Λ defined by (A.8) for any $\mu \in \Lambda^\circ$. Then, using a similar argument as above, one can show that V_{Λ° is a unitary V_Λ -module. For any $\lambda \in \Lambda^\circ, W_{[\lambda]} = \bigoplus_{\alpha \in \Lambda} L_\mathfrak{h}(\lambda + \alpha, 1)$ is clearly an irreducible unitary V_Λ -submodule of W_{Λ° , and any irreducible unitary V_Λ -module is of this form. (See also [DL14, Theorem 4.12].)

We now describe intertwining operators of V_Λ . If $\lambda_0, \mu_0, \nu_0 \in \Lambda^\circ$, then any type $\begin{pmatrix} [\nu_0] \\ [\lambda_0] & [\mu_0] \end{pmatrix}$ intertwining operator of V_Λ is clearly also an intertwining operator of $V_\mathfrak{h}^1$. From this we can easily see that there are not type $\begin{pmatrix} [\nu_0] \\ [\lambda_0] & [\mu_0] \end{pmatrix}$ intertwining operators when $\nu_0 - \lambda_0 - \mu_0 \notin \Lambda_0$. Now we look at type $\begin{pmatrix} [\lambda_0 + \mu_0] \\ [\lambda_0] & [\mu_0] \end{pmatrix}$ intertwining operators. If $\mathfrak{Y} \in \mathcal{V} \begin{pmatrix} [\lambda_0 + \mu_0] \\ [\lambda_0] & [\mu_0] \end{pmatrix}$, then, since it is an intertwining operator of $V_\mathfrak{h}^1$, there exists $\kappa : (\lambda_0 + \Lambda) \times (\mu_0 + \Lambda) \rightarrow \mathbb{C}$ satisfying that for any $\lambda \in \lambda_0 + \Lambda, \mu \in \mu_0 + \Lambda, w^{(\mu)} \in L_\mathfrak{h}(\mu, 1), v \in V_\mathfrak{h}^1$,

$$(A.15) \quad \mathfrak{Y}(c_\lambda v, x)w^{(\mu)} = \kappa(\lambda, \mu)\mathfrak{Y}_\lambda(c_\alpha v, x)w^{(\mu)}.$$

\mathfrak{Y} should satisfy the Jacobi identity, which says that for any $\alpha \in \Lambda, u \in V_\mathfrak{h}^1$,

$$(A.16) \quad Y^\Lambda(c_\alpha u, z_1)\mathfrak{Y}(c_\lambda v, z_2) = \mathfrak{Y}(Y^\Lambda(c_\alpha u, z_1 - z_2)c_\lambda v, z_2),$$

$$(A.17) \quad Y^\Lambda(c_\alpha u, z_1)\mathfrak{Y}(c_\lambda v, z_2) = \mathfrak{Y}(c_\lambda v, z_2)Y^\Lambda(c_\alpha u, z_1).$$

By the fusion and braid relations (A.4), (A.5), these two relations are equivalent to the condition that for any $\alpha \in \Lambda, \lambda \in \lambda_0 + \Lambda, \mu \in \mu_0 + \Lambda$,

$$(A.18) \quad \epsilon(\alpha, \lambda + \mu)\kappa(\lambda, \mu) = \epsilon(\alpha, \lambda)\kappa(\alpha + \lambda, \mu) = (-1)^{(\alpha|\lambda)}\kappa(\lambda, \alpha + \mu)\epsilon(\alpha, \mu).$$

From this it is easy to see that the value of κ is uniquely determined by $\kappa(\lambda_0, \mu_0)$. Hence the dimension of $\mathcal{V} \begin{pmatrix} [\lambda_0 + \mu_0] \\ [\lambda_0] & [\mu_0] \end{pmatrix}$ is at most 1. Conversely, if we define, for any $\lambda \in \lambda_0 + \Lambda, \mu \in \mu_0 + \Lambda$,

$$(A.19) \quad \kappa(\lambda, \mu) = \epsilon(\lambda, \mu)\omega(\mu - \mu_0, \lambda)e^{i\pi(\mu - \mu_0|\lambda)}$$

(cf. [TL04, Chapter V, Equation (5.3.1)]), then κ satisfies condition (A.18). Hence \mathfrak{Y} satisfies the Jacobi identity. Since \mathfrak{Y} clearly satisfies the translation property as it is an intertwining operator of $V_\mathfrak{h}^1$, \mathfrak{Y} is a non-zero type $\begin{pmatrix} [\lambda_0 + \mu_0] \\ [\lambda_0] & [\mu_0] \end{pmatrix}$ intertwining

operator of V_Λ . We arrive at the following theorem proved in [DL93, Proposition 12.2].

Theorem A.5. *Choose $\lambda_0, \mu_0, \nu_0 \in \Lambda^\circ$. If $\nu_0 - \lambda_0 - \mu_0 \in \Lambda$, then the fusion rule $N_{[\lambda_0][\mu_0]}^{[\nu_0]} = 1$, and a non-zero type $\binom{[\nu_0]}{[\lambda_0] \quad [\mu_0]}$ intertwining operator of V_Λ can be defined such that for any $\lambda \in \lambda_0 + \Lambda, \mu \in \mu_0 + \Lambda, w^{(\mu)} \in L_{\mathfrak{h}}(\mu, 1), v \in V_{\mathfrak{h}}^1$,*

$$(A.20) \quad \mathfrak{Y}(c_\lambda v, x)w^{(\mu)} = \epsilon(\lambda, \mu)\omega(\mu - \mu_0, \lambda)e^{i\pi(\mu - \mu_0|\lambda)}\mathcal{Y}_\lambda(c_\lambda v, x)w^{(\mu)}.$$

If $\nu_0 - \lambda_0 - \mu_0 \notin \Lambda$, then $N_{[\lambda_0][\mu_0]}^{[\nu_0]} = 0$.

This theorem, together with Theorem A.2, immediately implies the following.

Theorem A.6. *Let $\lambda_0 \in \Lambda^\circ$, and let \mathfrak{Y} be an intertwining operator of V_Λ with charge space $W_{[\lambda_0]}$. Then \mathfrak{Y} is energy-bounded. Moreover, for any $\lambda \in \lambda_0 + \Lambda$, $\mathfrak{Y}(c_\lambda \Omega, x)$ satisfies 0-th order energy bounds when $(\lambda|\lambda) \leq 1$ and satisfies first order energy bounds when $(\lambda|\lambda) \leq 2$.*

Corollary A.7 (Cf. [TL04]). *Let \mathfrak{g} be a complex unitary simple Lie algebra of type A, D , or E . The inner product on \mathfrak{g} as well as on its Cartan subalgebra \mathfrak{h} is chosen such that the length of the roots of \mathfrak{g} are $\sqrt{2}$. Let Λ be the root lattice of \mathfrak{g} , and choose dominant integral weights λ, μ, ν of \mathfrak{g} admissible at level 1. If $\nu - \lambda - \mu \in \Lambda$, then $\dim \mathcal{V}_{\mathfrak{g}}^1(\lambda \quad \nu \quad \mu) = 1$, and any type $\binom{\nu}{\lambda \quad \mu}$ intertwining operator of $V_{\mathfrak{g}}^1$ is energy-bounded. Moreover, let d be the distance between 0 and $\lambda + \Lambda$. Choose any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^1(\lambda \quad \nu \quad \mu), u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$. If $d^2 \leq 1$, then $\mathcal{Y}(u^{(\lambda)}, x)$ satisfies 0-th order energy bounds. If $d^2 \leq 2$, then $\mathcal{Y}(u^{(\lambda)}, x)$ satisfies first order energy bounds.*

If, however, $\nu - \lambda - \mu \notin \Lambda$, then $\dim \mathcal{V}_{\mathfrak{g}}^1(\lambda \quad \nu \quad \mu) = 0$.

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