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Q-Systems and Extensions of Completely Unitary Vertex Operator Algebras

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Complete unitarity is a natural condition on a CFT-type regular vertex operator algebra (VOA), which ensures that its modular tensor category (MTC) is unitary. In this paper we show that any CFT-type unitary (conformal) extension U of a completely unitary VOA V is completely unitary. Our method is to relate U with a Q-system A_U in the C^*-tensor category $\mathrm{Rep}^{\mathrm{u}}(V)$ of unitary V-modules. We also update the main result of [30] to the unitary cases by showing that the tensor category $\mathrm{Rep}^{\mathrm{u}}(U)$ of unitary U-modules is equivalent to the tensor category $\mathrm{Rep}^{\mathrm{u}}(A_U)$ of unitary A_U-modules as unitary MTCs. As an application, we obtain infinitely many new (regular and) completely unitary VOAs including all CFT-type $c < 1$ unitary VOAs. We also show that the latter are in one-to-one correspondence with the (irreducible) conformal nets of the same central charge c, the classification of which is given by [29].

Introduction

This is the 1st part in a series of papers to study the relations between unitary vertex operator algebra (VOA) extensions and conformal net extensions. We will always focus on rational conformal field theories, so our VOAs are assumed to be CFT type, self-dual, and regular, so that the categories of VOA modules are modular tensor categories (MTCs).

Although both unitary VOAs and conformal nets are mathematical formulations of unitary chiral CFTs, they are defined and studied in rather different ways, with the

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former being more algebraic and geometric, and the latter mainly functional analytic. A systematic study to relate these two approaches was initiated by Carpi–Kawahigashi–Longo–Weiner [7] and followed by [9, 10, 21, 43–45], etc. In these works the methods of relating the two approaches are transcendental and have a lot of analytic subtleties. Due to these subtleties, certain models (such as unitary Virasoro VOAs, and unitary affine VOAs especially of type A) are easier to analyze than the others. On the other hand, when studying the extensions and conformal inclusions of chiral CFTs, the main tools in the two approaches are quite similar: both are (commutative) associative algebras in a tensor category $\mathcal{C}$ (called $\mathcal{C}$-algebras); see [8, 23, 30] for VOA extensions, and [5, 29, 31] for conformal net extensions; see also [16, 17] for the general notion of algebra objects inside a tensor category. In this and the following papers, we will see that $\mathcal{C}$-algebras are also powerful tools for relating unitary VOA extensions and conformal net extensions in the above-mentioned systematic and transcendental settings.

There is, however, one important difference between the $\mathcal{C}$-algebras used in the two approaches: for conformal net extensions the $\mathcal{C}$-algebras are *unitary*. Unitarity is an essential property for conformal nets and operator algebras but not quite necessary for VOAs. However, it is impossible to relate VOAs and conformal nets without adding unitary structures on VOAs (and their representation categories). This point is already clear in [21], where we have seen that to relate the tensor categories of VOAs and conformal nets, one has to first make the VOA tensor categories unitary.

In this paper our main goal is to relate the C^*-tensor categories of unitary VOA extensions with those of unitary $\mathcal{C}$-algebras (also called C^*-Frobenius algebras or (under slightly stronger condition) Q-systems [33]). As applications, we prove many important unitary properties of VOA extensions, the most important of which are the complete unitarity of VOAs as defined below.

Complete unitarity of unitary VOA extensions

A CFT-type regular VOA V is called **completely unitary** if the following conditions are satisfied.

- V is unitary [11], which means roughly that V is equipped with an inner product and an antiunitary antiautomorphism Θ, which relates the vertex operators of V to their adjoints.
- Any V-module admits a unitary structure. Since V-modules are semisimple, it suffices to assume the unitarizability of irreducible V-modules.

- For any irreducible unitary V-modules W_i, W_j, W_k, the non-degenerate *invariant sesquilinear form* Λ introduced in [19] and defined on the dual vector space of type $\binom{k}{i\,j}$ intertwining operators of V is positive.

The importance of complete unitarity lies in the following theorem.

Theorem 0.1 ([19] theorem 7.9). If V is a CFT-type, regular, and completely unitary VOA, then the unitary V-modules form a unitary MTC.

However, compared to unitarity, complete unitarity is much harder to prove since not only vertex operators but also intertwining operators need to be taken care of. In this paper, our main result as follows provides a powerful tool for proving the complete unitarity.

Theorem 0.2. Suppose that V is a CFT-type, regular, and completely unitary VOA, and U is a CFT-type unitary VOA extension of V. Then U is also completely unitary.

Roughly speaking, if we know that a unitary VOA U is an extension of a completely unitary VOA V, then U is also completely unitary. As applications, since the complete unitarity has been established for unitary affine VOAs and $c < 1$ unitary Virasoro VOAs (minimal models) as well as their tensor products (proposition 3.31), we know that all their unitary extensions are completely unitary. In particular, these extensions have unitary MTCs. We also show that these tensor categories are equivalent to the unitary MTCs associated to the corresponding Q-system. To be more precise, we prove the following:

Theorem 0.3. Let V be a CFT-type, regular, and completely unitary VOA, and let U be a CFT-type unitary extension of V whose Q-system is A_U. If $\mathrm{Rep}^{\mathrm{u}}(U)$ is the category of unitary U-modules, and $\mathrm{Rep}^{\mathrm{u}}(A_U)$ is the category of unitary A_U-modules, then $\mathrm{Rep}^{\mathrm{u}}(U)$ is naturally equivalent to $\mathrm{Rep}^{\mathrm{u}}(A_U)$ as unitary MTCs.

A non-unitary version of the above theorem has already been proved in [8]: $\mathrm{Rep}^{\mathrm{u}}(U)$ is known to be equivalent to $\mathrm{Rep}^{\mathrm{u}}(A_U)$ as MTCs. The above theorem says that the unitary structures of the two MTCs are also equivalent in a natural way. We remark that the unitary tensor structures compatible with the $*$-structure of $\mathrm{Rep}^{\mathrm{u}}(U)$ are unique by [6, 41]. Thus [6, 41] provide a different method of proving the above theorem.

We would like to point out that our results on the relation between unitary VOA extensions and Q-systems also provide a new method of proving the unitarity of VOAs. For instance, the irreducible $c < 1$ conformal nets are classified as (finite index) extensions of Virasoro nets by Kawahigashi–Longo in [29] (table 3), and their VOA counterparts are given by Dong–Lin in [12]. However, Dong–Lin were not able to prove the unitarity of two exceptional cases: the types (A_{10}, E_6) and (A_{28}, E_8) (see remark 4.16 of [12]). But since these two types are realized as commutative Q-systems in [29], we can now show that the corresponding VOAs are actually unitary. This proves that the $c < 1$ CFT-type unitary VOAs are in one-to-one correspondence with the irreducible conformal nets with the same central charge c.

We have left several important questions unanswered in this paper. We see that Q-systems can relate unitary VOA extensions and conformal net extensions. But [7] also provides a uniform way of relating unitary VOAs and conformal nets using smeared vertex operators. Are these two relations compatible? Moreover, do the VOA extensions and the corresponding conformal net extensions have the same tensor categories? Answers to these questions are out of scope in this paper, so we leave them to future works.

Outline of the paper

In chapter 1 we review the construction and basis properties of VOA tensor categories due to Huang–Lepowsky. We also review various methods of constructing new intertwining operators from old ones and translate them into tensor categorical language. The translation of adjoint and conjugate intertwining operators is the most important result of this chapter. Unitary VOAs, unitary representations, and the unitary structure on VOA tensor categories are also reviewed.

In chapter 2 we relate unitary VOA extensions and Q-systems as well as their (unitary) representations. The 1st two sections serve as background materials. In section 2.1 we review the relation between VOA extensions and commutative $\mathcal{C}$-algebras as in [23]. Their results are adapted to our unitary setting. In section 2.2 we review various notions concerning dualizable objects in C^*-tensor categories. Most importantly, we review the construction of standard evaluations and coevaluations in C^*-tensor categories necessary for defining quantum traces and quantum dimensions. Standard reference for this topic is [32]. We also explain why the naturally defined evaluations and coevaluations in the tensor categories associated to completely unitary VOAs are standard. In section 2.3 we define a notion of unitary $\mathcal{C}$-algebras, which is a direct translation of unitary VOA extensions in categorical language. This notion is related to C^*-Frobenius algebras and Q-systems in section 2.4. The equivalence of $c < 1$ unitary VOAs and conformal nets is also proved in that section. A VOA U is called strongly unitary if it satisfies the 1st two of the three conditions defining complete unitarity. Therefore, strong unitarity means the unitarity of U and the unitarizability of all U-modules. In section 2.5, we give two proofs that any unitary extension U of a completely unitary VOA V is strongly unitary. The 1st proof uses induced representations, and the 2nd one uses a result of standard representations of Q-systems in [5].

In chapter 3 we use the C^*-tensor categories of the bimodules of Q-systems to prove the complete unitarity of unitary VOA extensions. We review the construction and basic properties of these C^*-tensor categories in the 1st four sections. Although these results are known to experts (cf. [37] chapter 6 or [39] section 4.1), we provide detailed and self-contained proofs of all the relevant facts, which we hope are helpful to the readers who are not familiar with tensor categories. We present the theory in such a way that it can be directly compared with the (Hermitian) tensor categories of unitary VOA modules. So in some sense our approach is closer in spirit to [8, 30]. In section 3.5 we prove the main results of this paper: theorems 0.2 and 0.3 (which are theorem 3.30 of that section). Finally, applications are given in section 3.6.

1 Intertwining Operators and Tensor Categories of Unitary VOAs

1.1 Braiding, fusion, and contragredient intertwining operators

Let V be a self-dual VOA with vacuum vector Ω and conformal vector ν. For any $v \in V$, its vertex operator is written as $Y(v,z) = \sum_{n\in\mathbb{Z}} Y(v)_n z^{-n-1}$. Then $\{L_n = Y(\nu)_{n+1} : n \in \mathbb{Z}\}$ are the Virasoro operators. Throughout this paper, we assume that the grading of V satisfies $V = \mathbb{C}\Omega \oplus \left(\bigoplus_{n\in\mathbb{Z}_{>0}} V(n)\right)$ where $V(n)$ is the eigenspace of L_0 with eigenvalue n, that is, V is of CFT type. We assume also that V is regular, which is equivalent to that V is rational and C_2-cofinite. (See [13] for the definition of these terminologies as well as the equivalence theorem.) Such condition guarantees that the intertwining operators of V satisfy the braiding and fusion relations [25, 26] and the modular invariance [27, 48] and that the category $\mathrm{Rep}(V)$ of (automatically semisimple) V-modules is indeed an MTC [28].

We refer the reader to [4, 14, 46] for the general theory of tensor categories, and [28] for the construction of the tensor category $\mathrm{Rep}(V)$ of V-modules. A brief review of this construction can also be found in [18] section 2.4 or [21] section 4.1. Here we outline some of the key properties of $\mathrm{Rep}(V)$, which will be used in the future.

Representations of V are written as W_i, W_j, W_k, etc. If a V-module W_i is given, its contragredient module (cf. [15] section 5.2) is written as $W_{\bar{i}}$. $W_{\bar{\bar{i}}}$, the contragredient module of $W_{\bar{i}}$, can naturally be identified with W_i. So we write $\bar{\bar{i}} = i$. Note that the symbol i is now reserved for representations. So we write the imaginary unit $\sqrt{-1}$ as $\mathbf{i}$. Let W_0 be the vacuum V-module, which is also the identity object in $\mathrm{Rep}(V)$. The product of V-modules is constructed in such a way that for any W_i, W_j, W_k there is a canonical isomorphism of linear spaces

$$\mathcal{Y}:\mathrm{Hom}(W_i\boxtimes W_j, W_k)\xrightarrow{\simeq}\mathcal{V}\binom{k}{i\,j},\qquad \alpha\mapsto\mathcal{Y}_\alpha \tag{1.1}$$

where $\mathcal{V}\binom{k}{i\,j}=\mathcal{V}\binom{W_k}{W_iW_j}$ is the (finite-dimensional) vector space of intertwining operators of V. For any $w^{(i)}\in W_i$, we write $\mathcal{Y}_\alpha(w^{(i)},z)=\sum_{s\in\mathbb{C}}\mathcal{Y}_\alpha(w^{(i)})_s z^{-s-1}$ where z is a complex variable defined in $\mathbb{C}^\times:=\mathbb{C}\setminus\{0\}$ and $\mathcal{Y}_\alpha(w^{(i)})_s:W_j\to W_k$ is the s-th mode of the intertwining operator. We say that W_i, W_j, W_k are respectively the charge space, the source space, and the target space of the intertwining operator $\mathcal{Y}_\alpha$. Tensor products of morphisms are defined such that the following condition is satisfied: if $F\in\mathrm{Hom}(W_{i'},W_i), G\in\mathrm{Hom}(W_{j'},W_j), K\in\mathrm{Hom}(W_k,W_{k'})$, then for any $w^{(i')}\in W_{i'}$,

$$\mathcal{Y}_{K\alpha(F\otimes G)}(w^{(i')},z)=K\mathcal{Y}_\alpha(Fw^{(i')},z)G. \tag{1.2}$$

One way to realize the above properties is as follows: notice first of all that V has finitely many equivalence classes of irreducible unitary V-modules. Fix, for each equivalence class, a representing element, and let them form a finite set $\mathcal{E}$. We assume that the vacuum unitary module $V=W_0$ is in $\mathcal{E}$. If $W_t\in\mathcal{E}$, we will use the notation $t\in\mathcal{E}$ to simplify formulas. We then define $W_i\boxtimes W_j$ to be $\bigoplus_{t\in\mathcal{E}}\mathcal{V}\binom{t}{i\,j}^*\otimes W_t$ [24] (here $\mathcal{V}\binom{t}{i\,j}^*$ is the dual vector space of $\mathcal{V}\binom{t}{i\,j}$). Then for each $t\in\mathcal{E}$ there is a natural identification between $\mathrm{Hom}(W_i\boxtimes W_j,W_t)$ and $\mathcal{V}\binom{t}{i\,j}$, which can be extended to the general case $\mathrm{Hom}(W_i\boxtimes W_j,W_k)\simeq\mathcal{V}\binom{k}{i\,j}$ via the canonical isomorphisms

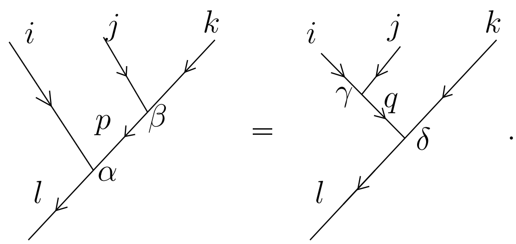
$$\mathrm{Hom}(W_i\boxtimes W_j,W_k)\simeq\bigoplus_{t\in\mathcal{E}}\mathrm{Hom}(W_i\boxtimes W_j,W_t)\otimes\mathrm{Hom}(W_t,W_k), \tag{1.3}$$

$$\mathcal{V}\binom{k}{i\,j}\simeq\bigoplus_{t\in\mathcal{E}}\mathcal{V}\binom{t}{i\,j}\otimes\mathrm{Hom}(W_t,W_k). \tag{1.4}$$

The tensor structure of $\mathrm{Rep}(V)$ is defined in such a way that it is related to the fusion relations of the intertwining operators of V as follows. Choose non-zero z,ζ with the same arguments (notation: $\arg z=\arg\zeta$) satisfying $0<|z-\zeta|<|\zeta|<|z|$. In particular, we assume that z,ζ are on a common ray stemming from the origin. We also choose $\arg(z-\zeta)=\arg\zeta=\arg z$. Suppose that we have W_i,W_j,W_k,W_l,W_p,W_q in $\mathrm{Rep}(V)$, and intertwining operators $\mathcal{Y}_\alpha\in\mathcal{V}\binom{l}{i\,p},\mathcal{Y}_\beta\in\mathcal{V}\binom{p}{j\,k},\mathcal{Y}_\gamma\in\mathcal{V}\binom{q}{i\,j},\mathcal{Y}_\delta\in\mathcal{V}\binom{l}{q\,k}$, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, the following fusion relation holds when acting on W_k:

$$\mathcal{Y}_\alpha(w^{(i)}, z)\mathcal{Y}_\beta(w^{(j)}, \zeta) = \mathcal{Y}_\delta\left(\mathcal{Y}_\gamma(w^{(i)}, z-\zeta)w^{(j)}, \zeta\right). \tag{1.5}$$

Then, under the identification of $W_i \boxtimes (W_j \boxtimes W_k)$ and $(W_i \boxtimes W_j) \boxtimes W_k$ (which we denote by $W_i \boxtimes W_j \boxtimes W_k$) via the associativity isomorphism, we have the identity $\alpha(\mathbf{1}_i \otimes \beta) = \delta(\gamma \otimes \mathbf{1}_k)$, which can be expressed graphically as



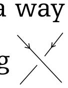
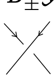
(1.6)

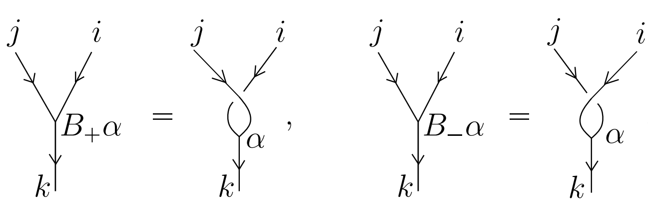
Here we take the convention that morphisms go from top to bottom.

Convention 1.1. When we consider fusion relations in the form (1.5), we always assume $0 < |z-\zeta| < |\zeta| < |z|$ and $\arg(z-\zeta) = \arg\zeta = \arg z$.

The braided and the contragredient intertwining operators are two major ways of constructing new intertwining operators from old ones [15]. As we shall see, they can all be translated into operations on morphisms. We first discuss braiding. Given $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, we can define **braided intertwining operators** $B_+\mathcal{Y}_\alpha, B_-\mathcal{Y}_\alpha$ of type $\mathcal{V}\binom{k}{j\,i}$ in the following way: choose any $w^{(i)} \in W_i, w^{(j)} \in W_j$. Then

$$(B_\pm\mathcal{Y}_\alpha)(w^{(j)}, z)w^{(i)} = e^{zL_{-1}}\mathcal{Y}_\alpha(w^{(i)}, e^{\pm \mathbf{i}\pi}z)w^{(j)}. \tag{1.7}$$

Then the braid isomorphism $ß = ß_{i,j} : W_i \boxtimes W_j \to W_j \boxtimes W_i$ is constructed in such a way that $B_\pm\mathcal{Y}_\alpha = \mathcal{Y}_{\alpha\circ ß^{\pm 1}}$. Write $B_+\mathcal{Y}_\alpha = \mathcal{Y}_{B_+\alpha}$ and $B_-\mathcal{Y}_\alpha = \mathcal{Y}_{B_-\alpha}$. Then $B_\pm\alpha = \alpha \circ ß^{\pm 1}$. Using  and  to denote ß and $ß^{-1}$, respectively, this formula can be pictured as



(1.8)

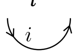
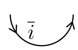
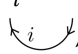
Let $Y_i = Y_i(v, z)$ be the vertex operator associated to the module W_i. Then Y_i is also a type $\binom{i}{0\ i}$ intertwining operator. It's easy to verify that $B_+Y_i = B_-Y_i$ as type $\mathcal{V}\binom{i}{i\ 0}$ intertwining operators, which we denote by $\mathcal{Y}_{\kappa(i)}$ and call the **creation operator** of W_i. Then the canonical isomorphism of the left multiplication by identity $W_0 \boxtimes W_i \xrightarrow{\simeq} W_i$ is defined to be the one corresponding to Y_i. Similarly the right multiplication by identity $W_i \boxtimes W_0 \xrightarrow{\simeq} W_i$ is chosen to be $\kappa(i)$.

One can also construct **contragredient intertwining operators** $C_+\mathcal{Y}_\alpha \equiv \mathcal{Y}_{C_+\alpha}, C_-\mathcal{Y}_\alpha \equiv \mathcal{Y}_{C_-\alpha}$ of $\mathcal{Y}_\alpha$, which are of type $\binom{\bar{j}}{i\ \bar{k}}$, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j, w^{(\bar{k})} \in W_{\bar{k}}$,

$$\langle \mathcal{Y}_{C_\pm\alpha}(w^{(i)}, z)w^{(\bar{k})}, w^{(j)}\rangle = \langle w^{(\bar{k})}, \mathcal{Y}_\alpha(e^{zL_1}(e^{\mp \mathrm{i}\pi}z^{-2})^{L_0}w^{(i)}, z^{-1})w^{(j)}\rangle. \tag{1.9}$$

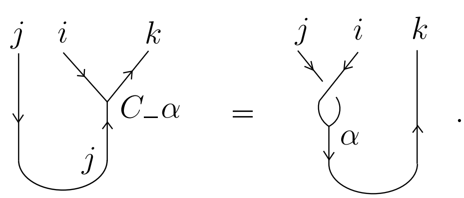
Here, and also throughout this paper, we follow the convention $\arg z^r = r \arg z$ ($r \in \mathbb{R}$) unless otherwise stated. To express contragredient intertwining operators graphically, we first introduce, for any V-module W_i (together with its contragredient module $W_{\bar{i}}$), two important intertwining operators $\mathcal{Y}_{\mathrm{ev}_{i,\bar{i}}} \in \mathcal{V}\binom{0}{i\ \bar{i}}$ and $\mathcal{Y}_{\mathrm{ev}_{\bar{i},i}} \in \mathcal{V}\binom{0}{\bar{i}\ i}$, called the **annihilation operators** of $W_{\bar{i}}$ and W_i, respectively. Recall that V is self-dual. Fix an isomorphism $V = W_0 \simeq W_{\bar{0}}$ and identify W_0 and $W_{\bar{0}}$ through this isomorphism. We now define

$$\mathcal{Y}_{\mathrm{ev}_{i,\bar{i}}} = C_-\mathcal{Y}_{\kappa(i)} = C_-B_\pm Y_i. \tag{1.10}$$

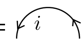
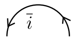
The type of $\mathcal{Y}_{\mathrm{ev}_{i,\bar{i}}}$ shows that $\mathrm{ev}_{i,\bar{i}} \in \mathrm{Hom}(W_i \boxtimes W_{\bar{i}}, V)$, which plays the role of the evaluation map of $W_{\bar{i}}$. $\mathrm{ev}_{\bar{i},i} \in \mathrm{Hom}(W_{\bar{i}} \boxtimes W_i, V)$ can be defined in a similar way. We write $\mathrm{ev}_{i,\bar{i}} =$ and $\mathrm{ev}_{\bar{i},i} =$  $=$  $=$ , following the convention that a vertical line with label i but upward-pointing arrow means (the identity morphism of) $W_{\bar{i}}$. We can now give a categorical description of C_α with the help of the following fusion relation (cf. [19] remark 5.4)

$$\mathcal{Y}_{\mathrm{ev}_{j,\bar{j}}}(w^{(j)}, z)\mathcal{Y}_{C_-\alpha}(w^{(i)}, \zeta) = \mathcal{Y}_{\mathrm{ev}_{k,\bar{k}}}\left(\mathcal{Y}_{B_-\alpha}(w^{(j)}, z-\zeta)w^{(i)}, \zeta\right), \tag{1.11}$$

which can be translated to



(1.12)

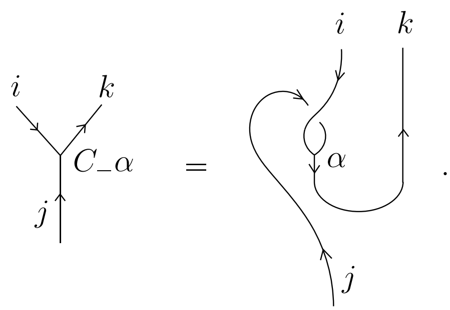
By [28] section 3, there exist coevaluation maps $\mathrm{coev}_{i,\bar{i}} =$  $\in \mathrm{Hom}(V, W_i \boxtimes W_{\bar{i}})$ and $\mathrm{coev}_{\bar{i},i} =$  $=$  $\in \mathrm{Hom}(V, W_{\bar{i}} \boxtimes W_i)$ satisfying

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathbf{1}_i)(\mathbf{1}_i \otimes \mathrm{coev}_{\bar{i},i}) = \mathbf{1}_i = (\mathbf{1}_i \otimes \mathrm{ev}_{\bar{i},i})(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_i), \tag{1.13}$$

$$(\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{i}} \otimes \mathrm{coev}_{i,\bar{i}}) = \mathbf{1}_{\bar{i}} = (\mathbf{1}_{\bar{i}} \otimes \mathrm{ev}_{i,\bar{i}})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}}). \tag{1.14}$$

Thus, let $W_{\bar{j}}$ tensor both sides of equation (1.12) from the left, and then apply $\mathrm{coev}_{\bar{j},j} \otimes \mathbf{1}_i \otimes \mathbf{1}_k$ to the tops, we obtain the following:

Proposition 1.2. For any V-modules W_i, W_j, W_k, and any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$,



(1.15)

Finally, we remark that the ribbon structure on $\mathrm{Rep}(V)$ is defined by the twist $\vartheta = \vartheta_i := e^{2i\pi L_0} \in \mathrm{End}(W_i)$ for any V-module W_i.

1.2 Unitary VOAs and unitary representations

In this section, we only assume that V is of CFT type and discuss the unitary conditions on V and its tensor category. We do not assume, at the beginning, that V is self-dual. In particular, we do not identify W_0 with $W_{\bar{0}}$. As we shall see, the unitary structure on V is closely related to certain V-module isomorphism $\epsilon : W_0 \to W_{\bar{0}}$.

Suppose V is equipped with a normalized inner product $\langle\cdot|\cdot\rangle$ ("normalized" means $\langle\Omega|\Omega\rangle = 1$). We say that $(V, \langle\cdot|\cdot\rangle)$ (simply V in the future) is **unitary** [7, 11], if there exists an anti-unitary anti-automorphism Θ of V (called **PCT operator**) satisfying that

$$\langle Y(v,z)v_1|v_2\rangle = \langle v_1|Y(e^{\bar{z}L_1}(-\overline{z^{-2}})^{L_0}\Theta v, \overline{z^{-1}})v_2\rangle. \tag{1.16}$$

We abbreviate the above relation to

$$Y(v,z) = Y(e^{\bar{z}L_1}(-\overline{z^{-2}})^{L_0}\Theta v, \overline{z^{-1}})^\dagger, \tag{1.17}$$

with † understood as formal adjoint.

By our definition, $\Theta : V \to V$ is an antilinear (i.e., conjugate linear) bijective map satisfying

$$\langle u|v\rangle = \langle \Theta v|\Theta u\rangle, \tag{1.18}$$

$$\Theta Y(u,z)v = Y(\Theta u,\overline{z})\Theta v \tag{1.19}$$

for all $u, v \in V$. Such Θ is unique by [7] proposition 5.1. Note that $\Theta v = \Theta Y(\Omega, z)v = Y(\Theta\Omega,\overline{z})\Theta v$, which implies $\Theta\Omega = \Omega$. Also $\Theta\nu = \nu$ by [7] corollary 4.11. Indeed, by that corollary, under the assumption of anti-automorphism, anti-unitarity is equivalent to $\Theta\nu = \nu$, and also equivalent to that Θ preserves the grading of V. Note also that Θ is uniquely determined by V and its inner product $\langle\cdot|\cdot\rangle$, and that $\Theta^2 = \mathbf{1}_V$ ([7] proposition 5.1).

Later we will discuss the unitarity of VOA extensions using tensor-categorical methods. For that purpose the map Θ is difficult to deal with due to its anti-linearity. So let us give an equivalent description of unitarity using linear maps. Let V^* be the dual vector space of V. The inner product $\langle\cdot|\cdot\rangle$ on V induces naturally an antilinear injective map $\complement : V \to V^*, v \mapsto \langle\cdot|v\rangle$. We set $\overline{V} = \complement V$, and define an inner product on $\overline{V}$, also denoted by $\langle\cdot|\cdot\rangle$, under which $\complement$ becomes anti-unitary. Equivalently, we require $\langle u|v\rangle = \langle \complement v|\complement u\rangle$ for all $u, v \in V$. The adjoint (which is also the inverse) of $\complement : V \to \overline{V}$ is denoted by $\complement : \overline{V} \to V$. The reason we use the same symbol for the two conjugation maps is to regard $\complement$ as an involution. We also adopt the notation $\overline{v} = \complement v$ and $\overline{\overline{v}} = v$. We will use both $\overline{v}$ and $\complement$ for conjugation very frequently, but the latter is used more often when no specific vectors are mentioned.

Proposition 1.3. $(V, \langle\cdot|\cdot\rangle)$ is unitary if and only if there exists a (unique) unitary map $\epsilon : V \to \overline{V}$, called the **reflection operator** of V, such that the following two relations hold for all $v \in V$:

$$Y(v,z) = Y(e^{\overline{z}L_1}(-\overline{z^{-2}})^{L_0}\overline{\epsilon v}, \overline{z^{-1}})^\dagger, \tag{1.20}$$

$$\epsilon Y(\epsilon^*\overline{v}, z)\epsilon^* = \complement Y(v,\overline{z})\complement. \tag{1.21}$$

When V is unitary, we have

$$\complement\epsilon = \epsilon^*\complement. \tag{1.22}$$

Note that the 1st and the 3rd equations are acting on V, while the 2nd one on $\overline{V}$.

Proof. ϵ and Θ are related by $\epsilon = \complement\Theta$. Then $\overline{\epsilon v} = \Theta v, \epsilon^*\overline{v} = \Theta^{-1}v$. It is easy to see that (1.20) and (1.21) are equivalent to (1.17) and (1.19), respectively. The uniqueness of ϵ follows from that of Θ. When V is unitary, $\complement\epsilon = \Theta = \Theta^{-1} = \epsilon^*\complement$. ∎

In the following we always assume V to be unitary. Let W_i be a V-module, and assume that the vector space W_i is equipped with an inner product $\langle\cdot|\cdot\rangle$. We say that $(W_i, \langle\cdot|\cdot\rangle)$ (or just W_i) is a **unitary** V-module, if for any $v \in V$ we have $Y_i(v,z) = Y_i(e^{\overline{z}L_1}(-\overline{z^{-2}})^{L_0}\Theta v, \overline{z^{-1}})^\dagger$ when acting on W_i. Equivalently,

$$Y_i(v,z) = Y_i(e^{\overline{z}L_1}(-\overline{z^{-2}})^{L_0}\overline{\epsilon v}, \overline{z^{-1}})^\dagger. \tag{1.23}$$

A V-module is called **unitarizable** if it can be equipped with an inner product under which it becomes a unitary V-module. An intertwining operator $\mathcal{Y}_\alpha$ of V is called **unitary** if it is among unitary V-modules.

Note that just as the conjugations between V and $\overline{V}$, given a unitary V-module W_i, we have a natural conjugation $\complement : W_i \to W_{\overline{i}}$ whose inverse is also written as $\complement : W_{\overline{i}} \to W_i$. (Indeed, we first have an injective antilinear map $\complement : W_i \to W_i^*$. That $\complement W_i = W_{\overline{i}}$, that is, that vectors in $\complement W_i$ are precisely those with finite conformal weights, follows from the fact that $\langle L_0 \cdot |\cdot\rangle = \langle\cdot|L_0\cdot\rangle$, which is a consequence of (1.23) and the fact that $\overline{\epsilon\nu} = \Theta\nu = \nu$.) Fix an inner product on $W_{\overline{i}}$ under which $\complement$ becomes anti-unitary. Then $W_{\overline{i}}$ is also a unitary V-module. For any $w^{(i)} \in W_i$, write $\overline{w^{(i)}} = \complement w^{(i)}, \overline{\overline{w^{(i)}}} = \complement\overline{w^{(i)}} = w^{(i)}$. Recall by our notation that Y_i and $Y_{\overline{i}}$ are the vertex operators of W_i and $W_{\overline{i}}$, respectively. The relation between these two operators is pretty simple: $Y_{\overline{i}}(\Theta v, z) = \complement Y_i(v,\overline{z})\complement$, acting on $W_{\overline{i}}$. (See [18] formula (1.19).) Equivalently,

$$Y_{\overline{i}}(\overline{\epsilon v}, z) = \complement Y_i(v,\overline{z})\complement. \tag{1.24}$$

This equation (applied to $i = 0$), together with (1.21) and (1.22), shows for any $v \in V$ that $\epsilon Y_0(v,z)\epsilon^* = Y_{\overline{0}}(v,z)$, where $Y_0 = Y$ is the vertex operator of the vacuum module V, and $Y_{\overline{0}}$ is that of its contragredient module. We conclude the following:

Proposition 1.4. If V is a unitary CFT-type VOA, then V is self-dual. The vacuum module $V = W_0$ is unitarily equivalent to its contragredient module $\overline{V} = W_{\overline{0}}$ via the reflection operator ϵ.

Convention 1.5. Unless otherwise stated, if V is unitary and CFT type, the isomorphism $W_0 \xrightarrow{\simeq} W_{\overline{0}}$ is always chosen to be the reflection operator ϵ, and the identification of $V = W_0$ and $\overline{V} = W_{\overline{0}}$ is always assumed to be through ϵ.

1.3 Adjoint and conjugate intertwining operators

We have seen in section 1.1 two ways of producing new intertwining operators from old ones: the braided and the contragredient intertwining operators. In the unitary case there are two extra methods: the conjugate and the adjoint intertwining operators. In this section, our main goal is to derive tensor-categorical descriptions of these two constructions of intertwining operators.

First we review the definition of these two constructions; see [20] section 1.3 for more details and basic properties. Let V be unitary and CFT type. Choose unitary V-modules W_i, W_j, W_k. For any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, its **conjugate intertwining operator** $\complement\mathcal{Y}_\alpha \equiv \mathcal{Y}_{\complement\alpha} \equiv \mathcal{Y}_{\overline{\alpha}}$ is of type $\mathcal{V}\binom{\overline{k}}{\overline{i}\,\overline{j}}$ defined by

$$\mathcal{Y}_{\overline{\alpha}}(\overline{w^{(i)}}, z) = \complement\mathcal{Y}_\alpha(w^{(i)}, \overline{z})\complement \tag{1.25}$$

for any $w^{(i)} \in W_i$. Despite its simple form, one cannot directly translate this definition into tensor categorical language, again due to the anti-linearity of $\complement$. Therefore we need to first consider the **adjoint intertwining operator** $\mathcal{Y}_\alpha^\dagger \equiv \mathcal{Y}_{\alpha^\dagger} \in \mathcal{V}\binom{j}{\overline{i}\,k}$, defined by

$$\alpha^\dagger = \overline{C_+\alpha} = C_-\overline{\alpha}, \tag{1.26}$$

which will be closely related to the $*$-structures of the C^*-tensor categories. Then for any $w^{(i)} \in W_i$,

$$\mathcal{Y}_{\alpha^\dagger}(\overline{w^{(i)}}, z) = \mathcal{Y}_\alpha(e^{\overline{z}L_1}(e^{-\mathrm{i}\pi}\overline{z^{-2}})^{L_0}\overline{w^{(i)}}, \overline{z^{-1}})^\dagger. \tag{1.27}$$

Note that $\dagger$ is an involution: $\alpha^{\dagger\dagger} = \alpha$. Moreover, by unitarity, up to equivalence of the charge spaces $\epsilon : V \xrightarrow{\simeq} \overline{V}$, Y_i is equal to its adjoint intertwining operator, and obviously also equal to the conjugate intertwining operator of $Y_{\overline{i}}$. Another important relation is

$$\mathrm{ev}_{\overline{i},i} = \kappa(i)^\dagger \tag{1.28}$$

by [18] formula (1.44). Recall from section 1.1 that, up to the isomorphism $W_0 \simeq W_{\overline{0}}$, $\mathrm{ev}_{\overline{i},i}$ is defined to be $C_-\kappa(\overline{i})$. Due to convention 1.5, the more precise definition is $\mathrm{ev}_{\overline{i},i} = \epsilon^{-1} \circ C_-\kappa(\overline{i})$. We will use (1.28) more often than this definition in our paper.

We shall now relate $\alpha^\dagger$ with the unitarity structure of the tensor category of unitary V-modules. Let V be a CFT-type VOA. Assume that V is **strongly unitary** [42], which means that V is unitary, and any V-module is unitarizable. Then $\mathrm{Rep}^{\mathrm{u}}(V)$, the category of unitary V-modules, is a C^*-category, whose $*$ structure is defined as follows: if W_i, W_j are unitary and $T \in \mathrm{Hom}(W_i, W_j)$, then $T^* \in \mathrm{Hom}(W_j, W_i)$ is simply the adjoint of T, defined with respect to the inner products of W_i and W_j.

Assume also that V is regular. To make $\mathrm{Rep}^{\mathrm{u}}(V)$ a C^*-tensor category, we have to choose, for any unitary V-modules W_i, W_j, a suitable unitary structure on $W_i \boxtimes W_j$. Note that it is already known that $W_i \boxtimes W_j$ is unitarizable by the strong unitary of V. But here the unitary structure has to be chosen such that the structural isomorphisms become unitary. So, for instance, if W_k is also unitary, the associativity isomorphism $(W_i \boxtimes W_j) \boxtimes W_k \xrightarrow{\simeq} W_i \boxtimes (W_j \boxtimes W_k)$ and the braid isomorphism $ß : W_i \boxtimes W_j \xrightarrow{\simeq} W_j \boxtimes W_i$ have to be unitary. Then we can identify $(W_i \boxtimes W_j) \boxtimes W_k$ and $W_i \boxtimes (W_j \boxtimes W_k)$ as the same unitary V-module called $W_i \boxtimes W_j \boxtimes W_k$.

To fulfill these purposes, recall that $W_i \boxtimes W_j$ is defined to be $\bigoplus_{t\in\mathcal{E}} \mathcal{V}{t \choose i\,j}^* \otimes W_t$. Since every W_t already has a unitary structure, it suffices to assume that the direct sum is orthogonal, and define a suitable inner product Λ (called invariant inner product) on each $\mathcal{V}{t \choose i\,j}^*$.

Let us first assume that we can always find Λ, which make $\mathrm{Rep}^{\mathrm{u}}(V)$ a braided C^*-tensor category. Then we can define an inner product on $\mathcal{V}{t \choose i\,j}$, also denoted by Λ, under which the anti-linear map $\mathcal{V}{t \choose i\,j} \to \mathcal{V}{t \choose i\,j}^*$, $\mathcal{Y}_\alpha \mapsto \Lambda(\cdot|\mathcal{Y}_\alpha)$ becomes anti-unitary. Here we assume $\Lambda(\cdot|\cdot)$ to be linear on the 1st variable and anti-linear on the 2nd one. Let $\langle \mathcal{Y}_\alpha : \alpha \in \Xi^t_{i,j}\rangle$ be a basis of $\mathcal{V}{t \choose i\,j}$, and let $\langle \check{\mathcal{Y}}^\alpha : \alpha \in \Xi^t_{i,j}\rangle$ be its dual basis in $\mathcal{V}{t \choose i\,j}^*$. In other words we assume for any $\alpha, \beta \in \Xi^t_{i,j}$ that $\langle \mathcal{Y}_\alpha, \check{\mathcal{Y}}^\beta\rangle = \delta_{\alpha,\beta}$. The following proposition relates Λ with the categorical inner product.

Proposition 1.6. For any $\mathcal{Y}_\alpha, \mathcal{Y}_\beta \in \mathcal{V}{t \choose i\,j}$ we have

$$\alpha\beta^* = \Lambda(\mathcal{Y}_\alpha|\mathcal{Y}_\beta)\mathbf{1}_t. \tag{1.29}$$

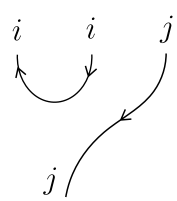
Proof. Assume that $\Xi^t_{i,j}$ is orthonormal under Λ. Then so is $\langle \check{\mathcal{Y}}^\alpha : \alpha \in \Xi^t_{i,j}\rangle$. For any $\alpha \in \Xi^t_{i,j}$, let P_α be the projection mapping $W_i \boxtimes W_j = \bigoplus^\perp_{t\in\mathcal{E}} \mathcal{V}{t \choose i\,j}^* \otimes W_t$ onto $\check{\mathcal{Y}}^\alpha \otimes W_t$, and let U_α be the unitary map $\check{\mathcal{Y}}^\alpha \otimes W_t \to W_t$, $\check{\mathcal{Y}}^\alpha \otimes w^{(t)} \mapsto w^{(t)}$. Then we clearly have $\alpha = U_\alpha P_\alpha$. So $\alpha\alpha^* = \mathbf{1}_t$. If $\beta \in \Xi^t_{i,j}$ and $\beta \neq \alpha$, then clearly $P_\beta P_\alpha = 0$, and hence $\alpha\beta^* = 0$. Thus (1.28) holds for basis vectors, and hence holds in general. ■

We warn the reader the difference of the two notations $\alpha^\dagger$ and α^*. If $\mathcal{Y}_\alpha \in \mathcal{V}{t \choose i\,j}$ then $\alpha^\dagger \in \mathrm{Hom}(W_{\bar{i}} \boxtimes W_t, W_j)$, while α^*, defined using the $*$-structure of $\mathrm{Rep}^{\mathrm{u}}(V)$, is in $\mathrm{Hom}(W_t, W_i \boxtimes W_j)$. Thus, $\mathcal{Y}_{\alpha^\dagger}$ is the adjoint intertwining operator while $\mathcal{Y}_{\alpha^*}$ makes no sense. This is different from the notations used in [18, 19], where $\alpha^\dagger$ is not defined but $\mathcal{Y}_{\alpha^*}$ (also written as $\mathcal{Y}_\alpha^\dagger$) denotes the adjoint intertwining operator. Despite such difference, $\alpha^\dagger$ and α^*, the VOA adjoint and the categorical adjoint, should be related in a natural way. And it is time to review the construction of invariant Λ introduced in [19].

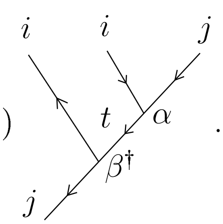
Definition 1.7. Let W_i, W_j be unitary V-modules. Then the **invariant sesquilinear form** Λ on $\mathcal{V}\binom{t}{i\,j}^*$ for any $t \in \mathcal{E}$ is defined such that the fusion relation holds for any $w_1^{(i)}, w_2^{(i)} \in W_i$:

$$Y_j\left(\mathcal{Y}_{\mathrm{ev}_{\bar{i},i}}(\overline{w_2^{(i)}}, z-\zeta)w_1^{(i)}, \zeta\right) = \sum_{t\in\mathcal{E}}\sum_{\alpha,\beta\in\Xi_{i,j}^t} \Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta)\cdot \mathcal{Y}_{\beta^\dagger}(\overline{w_2^{(i)}}, z)\mathcal{Y}_\alpha(w_1^{(i)}, \zeta). \tag{1.30}$$

Equation (1.30) can equivalently be presented as



$$= \sum_{t\in\mathcal{E}}\sum_{\alpha,\beta\in\Xi_{i,j}^t} \Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta)$$



. (1.31)

It is not too hard to show that Λ is Hermitian (i.e., $\Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta) = \overline{\Lambda(\check{\mathcal{Y}}^\beta|\check{\mathcal{Y}}^\alpha)}$). Indeed, one can prove this by applying [19] formula (5.34) to the adjoint of (1.30). (The intertwining operator $\mathcal{Y}_{\tilde{\sigma}}$ in that formula could be determined from the proofs of [19] corollary 5.7 and theorem 5.5.) Moreover, from the rigidity of $\mathrm{Rep}(V)$, one can deduce that Λ is non-degenerate; see [22] theorem 3.4 (The relation between the bilinear form in [22] theorem 3.4 and the sesquilinear form Λ is explained in [19] section 8.3.), or step 3 of the proof of [19] theorem 6.7. However, to make Λ an inner product, one has to prove that Λ is positive. In [18, 20] we have proved the positivity of Λ for many examples of VOAs. As mentioned in the introduction, one of our main goal of this paper is to prove that all unitary extensions of these examples have positive Λ. We first introduce the following definition.

Definition 1.8. Let V be a regular and CFT-type VOA. We say that V is **completely unitary**, if V is strongly unitary (i.e., V is unitary and any V-module is unitarizable), and if for any unitary V-modules W_i, W_j and any $t \in \mathcal{E}$, the invariant sesquilinear form Λ defined on $\mathcal{V}\binom{t}{i\,j}^*$ is positive. In this case we call Λ the **invariant inner product** of V.

As we have said, since Λ is non-degenerate, when V is completely unitary Λ becomes an inner product on $\mathcal{V}\binom{t}{i\,j}^*$ (and hence also on $\mathcal{V}\binom{t}{i\,j}$), which can be extended to an inner product on $W_i \boxtimes W_j$, also denoted by Λ. Then $W_i \boxtimes W_j$ becomes a unitary (but not just unitarizable) V-module. In other words it is an object not just in $\mathrm{Rep}(V)$ but also in $\mathrm{Rep}^{\mathrm{u}}(V)$. Moreover, as shown in [19], Λ is the right inner product, which makes all structural isomorphisms unitary. More precisely:

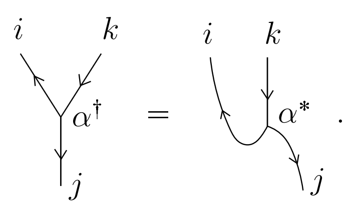
Theorem 1.9 (cf. [19] theorem 7.9). If V is regular, CFT type, and completely unitary, then $\mathrm{Rep}^{\mathrm{u}}(V)$ is a unitary MTC.

In the remaining part of this paper, we assume, unless otherwise stated, that V is regular, CFT type, and completely unitary. The following relation is worth noting; see [19] section 7.3.

Proposition 1.10. If W_i is unitary then $\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}^*_{i,\bar{i}}$.

We are now ready to state the main results of this section.

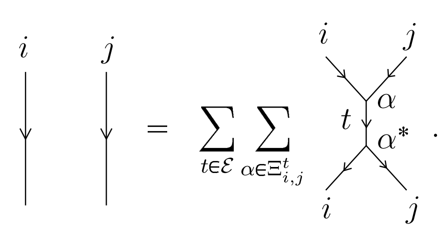
Theorem 1.11. For any unitary V-modules W_i, W_j, W_k and any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\,j}$, we have $\alpha^\dagger = (\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_j)(\mathbf{1}_{\bar{i}} \otimes \alpha^*)$. In other words,



(1.32)

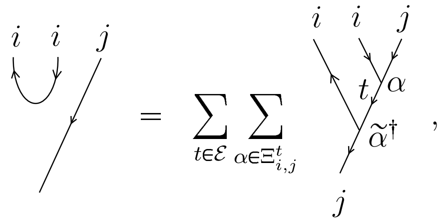
Proof. It suffices to assume W_k to be irreducible. So let us prove (1.32) for all $k = t \in \mathcal{E}$ and any basis vector $\alpha \in \Xi^t_{i,j}$. Here we assume $\Xi^t_{i,j}$ to be orthonormal under Λ. Choose $\widetilde{\alpha} \in \mathrm{Hom}(W_i \boxtimes W_j, W_t)$ such that $\widetilde{\alpha}^\dagger$ equals $(\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_j)(\mathbf{1}_{\bar{i}} \otimes \alpha^*)$. We want to show $\widetilde{\alpha} = \alpha$.

By proposition 1.6 we have $\alpha\beta^* = \delta_{\alpha,\beta}\mathbf{1}_t$ for any $\alpha, \beta \in \Xi^t_{i,j}$. This implies



(1.33)

Tensor $\mathbf{l}_{\bar{i}}$ from the left and apply $\mathrm{ev}_{\bar{i},i} \otimes \mathbf{l}_j$ to the bottom, we obtain



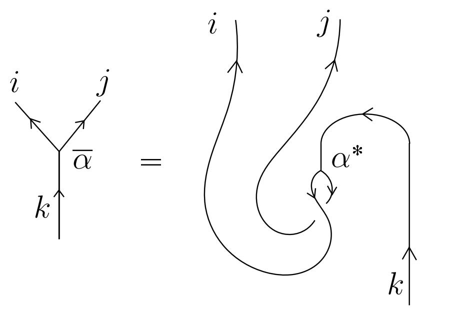
(1.34)

which, together with equation (1.31), implies

$$\widetilde{\alpha}^\dagger = \sum_{\beta\in\Xi^t_{i,j}} \Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta)\beta^\dagger = \sum_{\beta\in\Xi^t_{i,j}} \delta_{\alpha,\beta}\beta^\dagger = \alpha^\dagger.$$

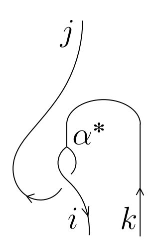
Thus $\widetilde{\alpha} = \alpha$. ■

Corollary 1.12. For any unitary V-modules W_i, W_j, W_k and any $\mathcal{Y}_\alpha \in \mathcal{V}\binom{k}{i\ j}$,



(1.35)

Proof. We have $\overline{\alpha} = \overline{C_+ C\_\alpha} = (C\_\alpha)^\dagger$. By propositions 1.2, 1.10, and the unitarity of ß, $(C\_\alpha)^*$ equals



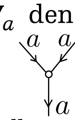
(1.36)

By theorem 1.11, one obtains $(C\_\alpha)^\dagger$ from $(C\_\alpha)^*$ by bending the leg i to the top. Thus (1.36) becomes the right hand side of (1.35). ■

2 Unitary VOA Extensions, C^*-Frobenius Algebras, and Their Representations

2.1 Preunitary VOA extensions and commutative $\mathcal{C}$-algebras

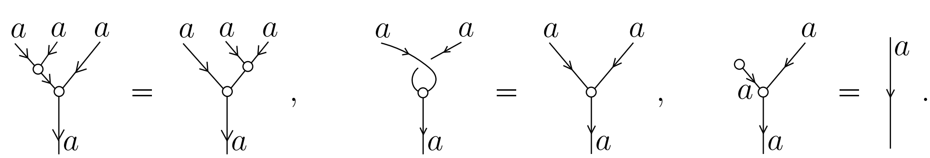
In this chapter, we fix a regular, CFT-type, and completely unitary VOA V. $W_{\overline{0}} = \overline{V}$ is identified with $W_0 = V$ via the reflection operator ϵ. Thus $\epsilon = 1$. Let U be a (VOA) extension of V, whose vertex operator is denoted by $\mathcal{Y}_\mu$. By definition, U and V share the same conformal vector ν and vacuum vector Ω, and V is a vertex operator subalgebra of U. As in the previous chapter, the symbol Y is reserved for the vertex operator of V. Then Y is the restriction of the action $\mathcal{Y}_\mu : U \curvearrowright U$ to $V \curvearrowright V$. More generally, let Y_a be the restriction of $\mathcal{Y}_\mu$ to $V \curvearrowright U$. Then (U, Y_a) becomes a representation of V. We write this V-module as (W_a, Y_a), or W_a for short. (So by our notation, U equals W_a as a vector space.) Since all V-modules are unitarizable, we fix a unitary structure (i.e., an inner product) $\langle \cdot | \cdot \rangle$ on the V-module W_a whose restriction to V is the one $\langle \cdot | \cdot \rangle$ of the unitary VOA V. Such $(U, \langle \cdot | \cdot \rangle)$, or U for short, is called a **preunitary (VOA) extension** of V. It is clear that any extension of V is preunitarizable. Finally, we notice that $\mathcal{Y}_\mu$ is a type $\binom{a}{a\,a} = \binom{W_a}{W_a W_a}$ unitary intertwining operator of V.

The above discussion can be summarized as follows: the preunitary VOA extension U is a unitary V module (W_a, Y_a); its vertex operator $\mathcal{Y}_\mu$ is in $\mathcal{V}\binom{a}{a\,a}$. Thus $\in \mathrm{Hom}(W_a \boxtimes W_a, W_a)$ by our notation in the last chapter. Let $\iota : W_0 \to W_a$ denote the embedding of V into U. Then clearly $\iota \in \mathrm{Hom}(W_0, W_a)$. We write $\iota =$  .
Set $A_U = (W_a, \mu, \iota)$. Then A_U is a **commutative associative algebra in $\mathrm{Rep}^{\mathrm{u}}(V)$** (or **commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra** for short) [30, 40], which means

- (Associativity) $\mu(\mu \otimes \mathbf{1}_a) = \mu(\mathbf{1}_a \otimes \mu)$.
- (Commutativity) $\mu \circ ß = \mu$.
- (Unit) $\mu(\iota \otimes \mathbf{1}_a) = \mathbf{1}_a$.

Note that the associators and the unitors of $\mathrm{Rep}^{\mathrm{u}}(V)$ have been suppressed to simplify discussions. We will also do so in the remaining part of this article.

Recall that ß is the braid isomorphism $ß_{a,a} : W_a \boxtimes W_a \to W_a \boxtimes W_a$. These three conditions can respectively be pictured as



Indeed, associativity and commutativity are equivalent to the Jacobi identity for $\mathcal{Y}_\mu$. The unit property follows from the fact that $\mathcal{Y}_\mu$ restricts to Y_a. See [23] for more details. Note that we also have

$$\mu \circ ß^n = \mu, \qquad \mu(\mathbf{1}_a \otimes \iota) = \mathbf{1}_a \tag{2.1}$$

for any $n \in \mathbb{Z}$. The 1st equation follows from induction and that $\mu = \mu ßß^{-1} = \mu ß^{-1}$. The 2nd equation holds because $\mu(\mathbf{1}_a \otimes \iota) = \mu ß_{a,a}(\mathbf{1}_a \otimes \iota) = \mu(\iota \otimes \mathbf{1}_a)ß_{a,0} = \mathbf{1}_a ß_{a,0} = \mathbf{1}_a$.

Recall that the twist is defined by $e^{2\mathrm{i}\pi L_0}$. Since L_0 has only integral eigenvalues on W_a, the unitary V-module W_a has trivial twist: $\vartheta_a = \mathbf{1}_a$. We also notice that A is **normalized** (i.e., $\iota^*\iota = \mathbf{1}_0$) since the inner product on U restricts to that of V. We thus conclude: *if U is a preunitary extension of V then A_U is a normalized commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra with trivial twist. Conversely, any such commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra arises from a preunitary CFT-type extension.* Moreover, if U is of CFT type, then A_U is **haploid**, which means that $\dim \mathrm{Hom}(W_0, W_a) = 1$. Indeed, if W_i is a unitary V-submodule of W_a equivalent to W_0, then the lowest conformal weight of W_i is 0, which implies that $\Omega \in W_i$ and hence that $W_i = W_0$. Therefore $\dim \mathrm{Hom}(W_0, W_a) = \dim \mathrm{Hom}(W_0, W_0) = 1$ by the simpleness of V. Conversely, if A_U is haploid, then U is of CFT type provided that any irreducible V-module not equivalent to W_0 has no homogeneous vectors with conformal weight 0. This converse statement will not be used in this paper (except in corollary 2.22). We thus content ourselves with the following result.

Proposition 2.1 (cf. [23] theorem 3.2). If U is a preunitary CFT-type VOA extension of V, then A_U is a normalized haploid commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra with trivial twist.

A detailed discussion of unitary VOA extensions will be given in the following sections. For now we first give the definition:

Definition 2.2. Let U be a CFT-type VOA extension of V, and assume that the vector space U is equipped with a normalized inner product $\langle \cdot | \cdot \rangle$. We say that the extension $V \subset U$ is unitary (equivalently, that U is a **unitary (VOA) extension** of V), if $\langle \cdot | \cdot \rangle$ restricts to the normalized inner product of V, and $(U, \langle \cdot | \cdot \rangle)$ is a unitary VOA.

A unitary VOA extension is clearly preunitary. Another useful fact is the following:

Proposition 2.3. If U is a CFT-type unitary VOA extension of V, then the PCT operator Θ_U of U restricts to the one Θ_V of V. In particular, V is a Θ_U-invariant subspace of U.

Thus we can let Θ denote unambiguously both the PCT operators of U and of V.

Proof. By relation (1.17) and the fact that $\Theta_U^2 = \mathbf{1}_U, \Theta_V^2 = \mathbf{1}_V$, for any $v \in V \subset U$ we have

$$Y(\Theta_U v, z) = Y(e^{\overline{z}L_1}(-\overline{z^{-2}})^{L_0}v, \overline{z^{-1}})^\dagger = Y(\Theta_V v, z)$$

when evaluating between vectors in V. It should be clear to the reader how the condition that the normalized inner product of U restricts to that of V is used in the above equations. Thus $\Theta_U|_V = \Theta_V$. ■

2.2 Duals and standard evaluations in C^*-tensor categories

Dualizable objects

Let $\mathcal{C}$ be a C^*-tensor category (cf. [47]) whose identity object W_0 is simple. We assume tacitly that $\mathcal{C}$ is closed under finite orthogonal direct sums and orthogonal subobjects, which means that for a finite collection $\{W_s : s \in \mathfrak{S}\}$ of objects in $\mathcal{C}$ there exists an object W_i and partial isometries $\{u_s \in \mathrm{Hom}(W_i, W_s) : s \in \mathfrak{S}\}$ satisfying $u_t u_s^* = \delta_{s,t}\mathbf{1}_s$ ($\forall s, t \in \mathfrak{S}$) and $\sum_{s\in\mathfrak{S}} u_s^* u_s = \mathbf{1}_i$, and that for any object W_i and a projection $p \in \mathrm{End}(W_i)$ there exists an object W_j and a partial isometry $u \in \mathrm{Hom}(W_i, W_j)$ such that $uu^* = \mathbf{1}_j, u^*u = p$ (A morphism $u \in \mathrm{Hom}(W_i, W_j)$ is called a partial isometry if u^*u and uu^* are projections. A morphism $e \in \mathrm{End}(W_i)$ is called a projection if $e^2 = e = e^*$.).

Assume that an object W_i in $\mathcal{C}$ has a right dual $W_{\bar{i}}$, which means that there exist evaluation $\mathrm{ev}_i \in \mathrm{Hom}(W_{\bar{i}} \boxtimes W_i, W_0)$ and coevaluation $\mathrm{coev}_i \in \mathrm{Hom}(W_0, W_i \boxtimes W_{\bar{i}})$ satisfying $(\mathbf{1}_i \otimes \mathrm{ev}_i)(\mathrm{coev}_i \otimes \mathbf{1}_i) = \mathbf{1}_i$ and $(\mathrm{ev}_i \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{i}} \otimes \mathrm{coev}_i) = \mathbf{1}_{\bar{i}}$. Set $\mathrm{ev}_{\bar{i},i} = \mathrm{ev}_i, \mathrm{coev}_{i,\bar{i}} = \mathrm{coev}_i$, and set also $\mathrm{ev}_{i,\bar{i}} = (\mathrm{coev}_{i,\bar{i}})^*, \mathrm{coev}_{\bar{i},i} = (\mathrm{ev}_{\bar{i},i})^*$. Then equations (1.13) and (1.14) are satisfied, which shows that $W_{\bar{i}}$ is also a left dual of W_i. In this case we say that W_i is **dualizable**. Note that $\mathrm{ev}_{i,\bar{i}}$ determines the remaining three ev and coev. In general, we say that $\mathrm{ev}_{i,\bar{i}} \in \mathrm{Hom}(W_i \boxtimes W_{\bar{i}}, W_0), \mathrm{ev}_{\bar{i},i} \in \mathrm{Hom}(W_{\bar{i}} \boxtimes W_i, W_0)$ are **evaluations** (or simply **ev**) of W_i and $W_{\bar{i}}$ if equations (1.13) and (1.14) are satisfied when setting $\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}_{i,\bar{i}}^*, \mathrm{coev}_{\bar{i},i} = \mathrm{ev}_{\bar{i},i}^*$. In the case that W_i is self-dual, we say that $\mathrm{ev}_{i,i}$ is an **evaluation** (**ev**) of W_i, if, by setting $\mathrm{coev}_{i,i} = \mathrm{ev}_{i,i}^*$, we have $(\mathrm{ev}_{i,i} \otimes \mathbf{1}_i)(\mathbf{1}_i \otimes \mathrm{coev}_{i,i}) = \mathbf{1}_i$. Taking adjoint, we also have $(\mathbf{1}_i \otimes \mathrm{ev}_{i,i})(\mathrm{coev}_{i,i} \otimes \mathbf{1}_i) = \mathbf{1}_i$.

Assume that W_i, W_j are dualizable with duals $W_{\bar{i}}, W_{\bar{j}}$, respectively. Choose evaluations $\mathrm{ev}_{i,\bar{i}}, \mathrm{ev}_{\bar{i},i}, \mathrm{ev}_{j,\bar{j}}, \mathrm{ev}_{\bar{j},j}$. Then $W_{i\boxtimes j} := W_i \boxtimes W_j$ is also dualizable with a dual $W_{\bar{j}\boxtimes\bar{i}} := W_{\bar{j}} \boxtimes W_{\bar{i}}$ and evaluations

$$\mathrm{ev}_{i\boxtimes j,\bar{j}\boxtimes\bar{i}} = \mathrm{ev}_{i,\bar{i}}(\mathbf{1}_i \otimes \mathrm{ev}_{j,\bar{j}} \otimes \mathbf{1}_{\bar{i}}), \qquad \mathrm{ev}_{\bar{j}\boxtimes\bar{i},i\boxtimes j} = \mathrm{ev}_{\bar{j},j}(\mathbf{1}_{\bar{j}} \otimes \mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_j). \tag{2.2}$$

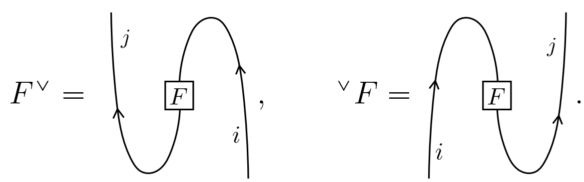
Convention 2.4. Unless otherwise stated, if the ev for $W_i, W_{\bar{i}}$ and $W_j, W_{\bar{j}}$ are chosen, then we always define the ev for $W_{i\boxtimes j}, W_{\bar{j}\boxtimes\bar{i}}$ using equations (2.2).

Using ev and coev for W_i, W_j and their duals $W_{\bar{i}}, W_{\bar{j}}$, one can define for any $F \in \mathrm{Hom}(W_i, W_j)$ a pair of transposes $F^\vee, {}^\vee F$ by

$$F^\vee = (\mathrm{ev}_{\bar{j},j} \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{j}} \otimes F \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{j}} \otimes \mathrm{coev}_{i,\bar{i}}),$$

$${}^\vee F = (\mathbf{1}_{\bar{i}} \otimes \mathrm{ev}_{j,\bar{j}})(\mathbf{1}_{\bar{i}} \otimes F \otimes \mathbf{1}_{\bar{j}})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{j}}).$$

Pictorially,



(2.3)

One easily checks that ${}^\vee(F^\vee) = F = ({}^\vee F)^\vee$, $(FG)^\vee = G^\vee F^\vee$, ${}^\vee(FG) = ({}^\vee G)({}^\vee F)$, $(F^\vee)^* = {}^\vee(F^*)$.

Standard evaluations

The evaluations defined above are not unique even up to unitaries. For any $\mathrm{ev}_{i,\bar{i}}, \mathrm{ev}_{\bar{i},i}$ of $W_i, W_{\bar{i}}$, and any invertible $K \in \mathrm{End}(W_i)$, $\widetilde{\mathrm{ev}}_{i,\bar{i}} := \mathrm{ev}_{i,\bar{i}}(K\otimes\mathbf{1}_{\bar{i}})$ and $\widetilde{\mathrm{ev}}_{\bar{i},i} := \mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}}\otimes(K^*)^{-1})$ are also evaluations. Thus one can normalize evaluations to satisfy certain nice conditions. In C^*-tensor categories, the ev that attract most interest are the so called standard evaluations. It is known that for dualizable objects, standard ev always exist and are unique up to unitaries, and that the two transposes defined by standard ev are equal. We refer the reader to [3, 32, 47] for these results. In the following, we review an explicit method of constructing standard evaluations following [47]. Since the evaluations for VOA tensor categories defined in the previous chapter can be realized by this construction, these evaluations are standard (see proposition 2.5).

Define scalars $\mathrm{Tr}_L(F)$ and $\mathrm{Tr}_R(F)$ for each $F \in \mathrm{End}(W_i)$ such that $\mathrm{ev}_{i,\bar{i}}(F \otimes \mathbf{1}_{\bar{i}})\mathrm{coev}_{i,\bar{i}} = \mathrm{Tr}_L(F)\mathbf{1}_0$ and $\mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}} \otimes F)\mathrm{coev}_{\bar{i},i} = \mathrm{Tr}_R(F)\mathbf{1}_0$. Then Tr_L is a positive linear functional on $\mathrm{End}(W_i)$. Moreover, Tr_L is faithful: if $\mathrm{Tr}_L(F^*F) = 0$, then $\mathrm{ev}_{i,\bar{i}} \circ (F \otimes \mathbf{1}_{\bar{i}})$ is zero (since its absolute value is zero). So $F = 0$. Similar things can be said about Tr_R. We say that $\mathrm{ev}_{i,\bar{i}}$ is a **standard evaluation** if $\mathrm{Tr}_L(F) = \mathrm{Tr}_R(F)$ for all $F \in \mathrm{End}(W_i)$. Since $\mathrm{Tr}_L(F^\vee) = \mathrm{Tr}_R(F) = \mathrm{Tr}_L(F^\vee)$ by easy graphical calculus, it is easy to see that $\mathrm{ev}_{i,\bar{i}}$ is standard if and only if $\mathrm{ev}_{\bar{i},i}$ is so. Standard ev are unique up to unitaries: If

$W_{i'} \simeq W_{\bar{i}}, T \in \mathrm{Hom}(W_{i'}, W_{\bar{i}})$ is unitary, and $\widetilde{\mathrm{ev}}_{i,i'}, \widetilde{\mathrm{ev}}_{i',i}$ are also standard, then we may find a unitary $K \in \mathrm{End}(W_i)$ satisfying $\widetilde{\mathrm{ev}}_{i,i'} = \mathrm{ev}_{i,\bar{i}}(K \otimes T), \widetilde{\mathrm{ev}}_{i',i} = \mathrm{ev}_{\bar{i},i}(T \otimes K)$. (See [47] lemma 3.9-(iii).)

If W_i is simple, a standard evaluation is easy to construct by multiplying $\mathrm{ev}_{i,\bar{i}}$ by some nonzero constant λ (and hence multiplying $\mathrm{ev}_{\bar{i},i}$ by $\overline{\lambda^{-1}}$) such that $\mathrm{Tr}_L(\mathbf{1}_i) = \mathrm{Tr}_R(\mathbf{1}_i)$. In general, if W_i is dualizable and hence semisimple, we have orthogonal irreducible decomposition $W_i \simeq \bigoplus^{\perp}_{s\in\mathfrak{S}} W_s$ where each irreducible subobject W_s is dualizable (with a dual $W_{\bar{s}}$). Choose partial isometries $\{u_s \in \mathrm{Hom}(W_i, W_s) : s \in \mathfrak{S}\}$ and $\{v_{\bar{s}} \in \mathrm{Hom}(W_{\bar{i}}, W_{\bar{s}}) : s \in \mathfrak{S}\}$ satisfying $u_t u_s^* = \delta_{s,t}\mathbf{1}_s, v_{\bar{t}} v_{\bar{s}}^* = \delta_{s,t}\mathbf{1}_{\bar{s}}$ and $\sum_s u_s^* u_s = \mathbf{1}_i, \sum_s v_{\bar{s}}^* v_{\bar{s}} = \mathbf{1}_{\bar{i}}$. Then we define

$$\mathrm{ev}_{i,\bar{i}} = \sum_{s\in\mathfrak{S}} \mathrm{ev}_{s,\bar{s}}(u_s \otimes v_{\bar{s}}), \qquad \mathrm{ev}_{\bar{i},i} = \sum_{s\in\mathfrak{S}} \mathrm{ev}_{\bar{s},s}(v_{\bar{s}} \otimes u_s) \tag{2.4}$$

(where $\mathrm{ev}_{s,\bar{s}}$ and $\mathrm{ev}_{\bar{s},s}$ are standard for all s), define coev using adjoint. Then $\mathrm{ev}_{i,\bar{i}}$ and $\mathrm{ev}_{\bar{i},i}$ are standard. (See [47] lemma 3.9 for details.) (In [47] the categories are assumed to be rigid. Thus any orthogonal subobject of W_i, that is, any object W_s, which is associated with a partial isometry $u : W_i \to W_s$ satisfying $uu^* = \mathbf{1}_s$, is dualizable. This fact is also true without assuming $\mathcal{C}$ to be rigid. (Cf. [1] lemma 4.20.) Here is one way to see this. Notice that we may assume Tr_L is tracial by multiplying $\mathrm{ev}_{i,\bar{i}}$ by $K \otimes \mathbf{1}_{\bar{i}}$ where K is a positive invertible element of $\mathrm{End}(W_i)$. (Cf. the proof of [1] Thm. 4.12.) Thus, for any $F, G \in \mathrm{End}(W_i)$, we have $\mathrm{Tr}_L(GF) = \mathrm{Tr}_L(F^{\vee\vee} \cdot G)$ in general and $\mathrm{Tr}_L(FG) = \mathrm{Tr}_L(GF)$ by tracialness, which shows $F^{\vee\vee} = F$ and hence $F^\vee = {}^\vee F$. Thus, $(F^\vee)^* = (F^*)^\vee$. So F is a projection iff $F^\vee$ is so. Let $p = u^*u$. Then $p^\vee \in \mathrm{End}(W_{\bar{i}})$ is a projection. Thus, there exist an object $W_{\bar{s}}$ and $v \in \mathrm{Hom}(W_{\bar{i}}, W_{\bar{s}})$ satisfying $vv^* = \mathbf{1}_{\bar{s}}, v^*v = p^\vee$. Then $W_{\bar{s}}$ is dual to W_s since one can choose evaluations $\mathrm{ev}_{s,\bar{s}} := \mathrm{ev}_{i,\bar{i}}(u^* \otimes v^*), \mathrm{ev}_{\bar{s},s} := \mathrm{ev}_{\bar{i},i}(v^* \otimes u^*)$.)

Proposition 2.5. If V is a regular, CFT-type, and completely unitary VOA, then the ev and coev defined in chapter 1 (same as those in [18, 19]) for any object W_i in $\mathrm{Rep}^{\mathrm{u}}(V)$ and its contragredient module $W_{\bar{i}}$ are standard.

Proof. First of all, assume W_i is irreducible. By [19] proposition 7.7 and the paragraph before that, we have $\mathrm{Tr}_L(\mathbf{1}_i) = d_i = d_{\bar{i}} = \mathrm{Tr}_R(\mathbf{1}_i)$. So $\mathrm{ev}_{i,\bar{i}}$ is standard. Now, assume W_i is semisimple with finite orthogonal irreducible decomposition $W_i = \bigoplus^{\perp}_s W_s$. For each irreducible summand W_s, define u_s (resp. $v_{\bar{s}}$) to be the projection of W_i onto W_s (resp. $W_{\bar{i}}$ onto $W_{\bar{s}}$). (Note that $W_{\bar{i}}$ and $W_{\bar{s}}$ are respectively the contragredient modules of W_i, W_s.) Using (1.10), it is easy to see that $\mathcal{Y}_{\mathrm{ev}_{i,\bar{i}}}(w, z) = \sum_s \mathcal{Y}_{\mathrm{ev}_{s,\bar{s}}}(u_s w, z) v_{\bar{s}}$ for any $w \in W_i$. So the 1st equation of (2.4) is satisfied. Similarly, the 2nd one of (2.4) holds. ∎

Convention 2.6. Unless otherwise stated, for any object W_i in $\mathrm{Rep}^{\mathrm{u}}(V)$, $W_{\bar{i}}$ is always understood as the contragredient module of W_i, and the standard ev and coev for $W_i, W_{\bar{i}}$ are always defined as in chapter 1.

It is worth noting that standardness is preserved by tensor products: If standard ev are chosen for W_i, W_j and their dual $W_{\bar{i}}, W_{\bar{j}}$, then the ev of $W_{i\boxtimes j}, W_{\bar{j}\boxtimes\bar{i}}$ defined by (2.2) are also standard.

Standard ev are also characterized by minimizing quantum dimensions. Define constants $d_i, d_{\bar{i}}$ satisfying $\mathrm{ev}_{i,\bar{i}}\mathrm{coev}_{i,\bar{i}} = d_i \mathbf{1}_0, \mathrm{ev}_{\bar{i},i}\mathrm{coev}_{\bar{i},i} = d_{\bar{i}}\mathbf{1}_0$. Then standard ev are precisely those minimizing $d_i d_{\bar{i}}$ and satisfying $d_i = d_{\bar{i}}$ (cf. [32]). We will always assume $d_i, d_{\bar{i}}$ to be those defined by standard evaluations, and call them the **quantum dimensions** of $W_i, W_{\bar{i}}$.

2.3 Unitarity of $\mathcal{C}$-algebras and VOA extensions

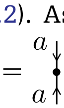
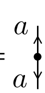
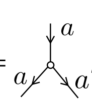
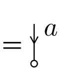
Let W_a be an object in $\mathcal{C}$, choose $\mu \in \mathrm{Hom}(W_a \boxtimes W_a, W_a), \iota \in \mathrm{Hom}(W_0, W_a)$, and assume that $A = (W_a, \mu, \iota)$ is an **associative algebra in** $\mathcal{C}$ (also called $\mathcal{C}$-**algebra** (In [8, 23, 30], commutativity is required in the definition of $\mathcal{C}$-algebras when $\mathcal{C}$ is braided. This is not assumed in our paper.)), which means

- (Associativity) $\mu(\mu \otimes \mathbf{1}_a) = \mu(\mathbf{1}_a \otimes \mu)$.
- (Unit) $\mu(\iota \otimes \mathbf{1}_a) = \mathbf{1}_a = \mu(\mathbf{1}_a \otimes \iota)$.

Since W_0 is simple, we can choose $D_A > 0$ (called the **quantum dimension** of A) satisfying $\iota^*\mu\mu^*\iota = D_A\mathbf{1}_0$. We say that A is

- **haploid** if $\dim \mathrm{Hom}(W_0, W_a) = 1$;
- **normalized** if $\iota^*\iota = \mathbf{1}_0$;
- **special** if $\mu\mu^* \in \mathbb{C}\mathbf{1}_a$; in this case we set scalar $d_A > 0$ such that $\mu\mu^* = d_A\mathbf{1}_a$;
- **standard** if A is special, W_a is dualizable (with quantum dimension d_a), and $D_A = d_a$.

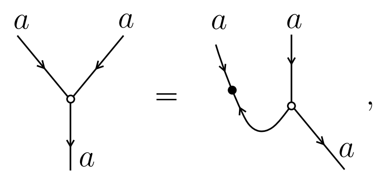
Note that any $\mathcal{C}$-algebra A is clearly equivalent to a normalized one.

Assume that W_a has a dual $W_{\bar{a}}$, together with (not necessarily standard) $\mathrm{ev}_{a,\bar{a}}, \mathrm{ev}_{\bar{a},a}$. Define ev for $W_a \boxtimes W_a$ and $W_{\bar{a}} \boxtimes W_{\bar{a}}$ using (2.2). Assume also that W_a is self-dual, that is, $W_a \simeq W_{\bar{a}}$. Choose a unitary morphism $\varepsilon =$  $\in \mathrm{Hom}(W_a, W_{\bar{a}})$, and write its adjoint as $\varepsilon^* =$ . Write also $\mu^* =$ , $\iota^* =$ .

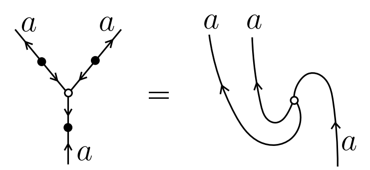
Definition 2.7. A unitary $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$ is called a **reflection operator** of A (with respect to the chosen ev of $W_a, W_{\overline{a}}$), if the following two equations are satisfied:

$$\mu = (\mathrm{ev}_{\overline{a},a} \otimes \mathbf{1}_a)(\varepsilon \otimes \mu^*), \qquad \varepsilon\mu(\varepsilon^* \otimes \varepsilon^*) = (\mu^*)^\vee. \tag{2.5}$$

Pictorially,



(2.6)



(2.7)

Proposition 2.8. The reflection operator ε is uniquely determined by the dual $W_{\overline{a}}$ and the ev of W_a and $W_{\overline{a}}$.

Proof. Apply ι^* to the bottom of (2.6), and then apply the unit property, we have $\mathrm{ev}_{\overline{a},a}(\varepsilon \otimes \mathbf{1}_a) = \iota^*\mu$, which, by rigidity, implies

$$\varepsilon = (\iota^*\mu \otimes \mathbf{1}_{\overline{a}})(\mathbf{1}_a \otimes \mathrm{coev}_{a,\overline{a}}). \tag{2.8}$$

■

Definition 2.9. Let $\mathcal{C}$ be a C^*-tensor category with simple W_0. A $\mathcal{C}$-algebra $A = (W_a, \mu, \iota)$ in $\mathcal{C}$ is called **unitary**, if W_a is dualizable, and for a choice of $W_{\overline{a}}$ dual to W_a and ev of $W_a, W_{\overline{a}}$, there exists a reflection operator $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$. If, moreover, the ev of $W_a, W_{\overline{a}}$ are standard, we say that A is **s-unitary** (The letter "s" stands for several closely related notions: standard evaluations, spherical tensor categories, symmetric Frobenius algebras [16].)[5].

The definition of s-unitary $\mathcal{C}$-algebras is independent of the choice of duals and standard ev, as shown below:

Proposition 2.10. If A is s-unitary, then for any $W_{a'}$ dual to W_a, and any standard $\widetilde{\mathrm{ev}}$ of $W_a, W_{a'}$, there exists a reflection operator $\widetilde{\varepsilon} : W_a \to W_{a'}$

Proof. Since A is s-unitary, we can choose $W_{\overline{a}}$ dual to W_a and standard ev of $W_a, W_{\overline{a}}$ such that there exists a reflection operator $\varepsilon : W_a \to W_{\overline{a}}$. Let us define $\widetilde{\varepsilon} = (\iota^*\mu \otimes \mathbf{1}_{\overline{a}})(\mathbf{1}_a \otimes \widetilde{\mathrm{coev}}_{a,a'})$ and show that $\widetilde{\varepsilon}$ is a reflection operator. We first show that $\widetilde{\varepsilon}$ is unitary. Choose a unitary $T \in \mathrm{Hom}(W_{a'}, W_{\overline{a}})$. Then by the up to unitary uniqueness of standard ev, there exists a unitary $K \in \mathrm{Hom}(W_a, W_a)$ such that $\widetilde{\mathrm{ev}}_{a,a'} = \mathrm{ev}_{a,\overline{a}}(K \otimes T), \widetilde{\mathrm{ev}}_{a',a} = \mathrm{ev}_{\overline{a},a}(T \otimes K)$. By (2.3), $\mathrm{ev}_{a,\overline{a}}(K \otimes \mathbf{1}_{\overline{a}}) = \mathrm{ev}_{a,\overline{a}}(\mathbf{1}_a \otimes K^\vee)$. Therefore $\widetilde{\mathrm{ev}}_{a,a'} = \mathrm{ev}_{a,\overline{a}}(\mathbf{1}_a \otimes (K^\vee)T)$. Thus $\widetilde{\varepsilon} = (\iota^*\mu \otimes \mathbf{1}_{\overline{a}})(\mathbf{1}_a \otimes T^*(K^\vee)^*)(\mathbf{1}_a \otimes \mathrm{coev}_{a,\overline{a}}) = (\mathbf{1}_0 \otimes T^*(K^\vee)^*)(\iota^*\mu \otimes \mathbf{1}_{\overline{a}})(\mathbf{1}_a \otimes \mathrm{coev}_{a,\overline{a}})$, which, together with (2.8), implies $\widetilde{\varepsilon} = T^*(K^\vee)^*\varepsilon$. Thus $\widetilde{\varepsilon}$ is unitary since $\varepsilon, K^\vee, T$ are unitary.

Now, from the definition of $\widetilde{\varepsilon}$, we see that $\widetilde{\mathrm{ev}}_{a',a}(\widetilde{\varepsilon} \otimes \mathbf{1}_a) = \iota^*\mu$. Therefore $\widetilde{\mathrm{ev}}_{a',a}(\widetilde{\varepsilon} \otimes \mathbf{1}_a) = \mathrm{ev}_{\overline{a},a}(\varepsilon \otimes \mathbf{1}_a)$. Using this fact, one can now easily check that (2.6) and (2.7) hold for $\widetilde{\varepsilon}$ and the standard $\widetilde{\mathrm{ev}}$ of $W_a, W_{a'}$. ■

We now relate s-unitary $\mathcal{C}$-algebras with unitary VOA extensions. First we need a lemma.

Lemma 2.11. Let U be a preunitary CFT-type extension of V, $A_U = (W_a, \mu, \iota)$ the associated $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra, and $W_{\overline{a}}$ the contragredient module of W_a. Then U is a unitary VOA if and only if there exists a unitary $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$ satisfying for all $w^{(a)} \in W_a$ that

$$\mathcal{Y}_\mu(w^{(a)}, z) = \mathcal{Y}_{\mu^\dagger}(\varepsilon w^{(a)}, z), \tag{2.9}$$

$$\varepsilon \mathcal{Y}_\mu(\varepsilon^* \overline{w^{(a)}}, z)\varepsilon^* = \mathcal{Y}_{\overline{\mu}}(\overline{w^{(a)}}, z). \tag{2.10}$$

Proof. Suppose that such ε exists, then by proposition 1.3 and the definition of adjoint and conjugate intertwining operators, U is unitary. Conversely, if U is unitary, then by proposition 1.3 there exists a unitary map $\varepsilon : W_a \to W_{\overline{a}}$ such that (2.9) and (2.10) are true. Moreover, by proposition 1.4, ε is a homomorphism of U-modules. So it is also a homomorphism of V-modules. This proves $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$. ■

The following is the main result of this section.

Theorem 2.12. Let V be a regular, CFT-type, and completely unitary VOA. Let U be a CFT-type preunitary extension of V, and let $A_U = (W_a, \mu, \iota)$ be the haploid commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra associated to U. Then U is unitary if and only if A_U is s-unitary.

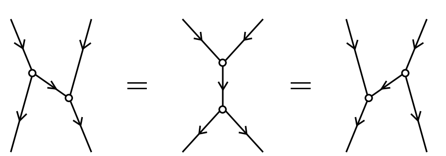
Proof. Following convention 2.6, we let $W_{\overline{a}}$ be the contragredient V-module of W_a, and choose standard ev for $W_a, W_{\overline{a}}$ as in chapter 1. By lemma 2.11 and relation (1.2), the unitarity of U is equivalent to the existence of a unitary $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$ satisfying

$$\mu = \mu^\dagger(\varepsilon \otimes \mathbf{1}_a), \qquad \varepsilon\mu(\varepsilon^* \otimes \varepsilon^*) = \overline{\mu}. \tag{2.11}$$

Now, by theorem 1.11, $\mu^\dagger = (\mathrm{ev}_{\overline{a},a} \otimes \mathbf{1}_a)(\mathbf{1}_{\overline{a}} \otimes \mu^*)$. Thus $\mu^\dagger(\varepsilon \otimes \mathbf{1}_a) = (\mathrm{ev}_{\overline{a},a} \otimes \mathbf{1}_a)(\varepsilon \otimes \mu^*)$. Therefore the 1st equation of (2.11) is equivalent to the 1st one of (2.5). By corollary 1.12, $\overline{\mu} = (\text{ß}_{a,a}\mu^*)^\vee = ((\mu\text{ß}_{a,a}^{-1})^*)^\vee$, which equals $(\mu^*)^\vee$ by the commutativity of A_U. Therefore the 2nd equation of (2.11) is also equivalent to that of (2.5). We conclude that ε satisfies (2.11) if and only if ε is a reflection operator. Thus the unitarity of U is equivalent to the existence of a reflection operator under the standard ev, which is precisely the s-unitarity of A_U. ■

2.4 Unitary $\mathcal{C}$-algebras and C^*-Frobenius algebras

Let $A = (W_a, \mu, \iota)$ be a $\mathcal{C}$-algebra. A is called a C^***-Frobenius algebra** in $\mathcal{C}$ if $(\mathbf{1}_a \otimes \mu)(\mu^* \otimes \mathbf{1}_a) = \mu^*\mu$. By taking adjoint we have the equivalent condition $\mu^*\mu = (\mu \otimes \mathbf{1}_a)(\mathbf{1}_a \otimes \mu^*)$. Assume in this section that all line segments in the pictures are labeled by a. Then these two equations read



(2.12)

A special (i.e., $\mu\mu^* \in \mathbb{C}\mathbf{1}_a$) C^*-Frobenius algebra is called a **Q-system** (We warn the reader that in the literature there is no agreement on whether standardness is required in the definition of Q-systems. For example, the Q-systems in [5] are in fact standard Q-systems in our paper.)[5]. We remark that the $\mathcal{C}$-algebra A is a Q-system if and only if it is special. In other words, the Frobenius relations (2.12) are consequences of the unit property, the associativity, and the specialness of A; see [32] or [5] lemma 3.7.

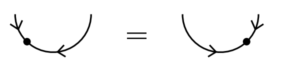
The main goal of this section is to relate (s-)unitarity to Frobenius property. More precisely, we shall show

$$
\begin{array}{rcl}
\text{Unitary}\mathcal{C}\text{-algebras} & = & C^*\text{-Frobenius algebras} \\
& \cup & \\
\text{Special unitary}\mathcal{C}\text{-algebras} & = & \text{Q-systems} \\
& \cup & \\
\text{Special s-unitary}\mathcal{C}\text{-algebras} & = & \text{standard Q-systems.}
\end{array} \tag{2.13}
$$

Moreover, under the assumption of haploid condition, all these notions are equivalent. In the process of the proof we shall also see that (2.7) is a consequence of (2.6). Thus the definition of reflection operator can be simplified to assume only (2.6).

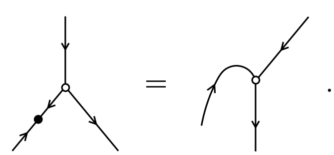
To begin with, let us fix a dual $W_{\overline{a}}$ of W_a together with evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ of $W_a, W_{\overline{a}}$. Choose a unitary $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$.

Proposition 2.13. If ε satisfies (2.6), then $\mathrm{ev}_{\overline{a},a}(\varepsilon \otimes \mathbf{1}_a) = \mathrm{ev}_{a,\overline{a}}(\mathbf{1}_a \otimes \varepsilon)$; equivalently,

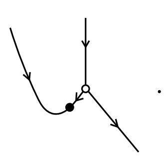


(2.14)

Proof. Take the adjoint of (2.6) and apply $(\varepsilon \otimes \mathbf{1}_a)$ to the bottom, we have



Bending the left legs to the top proves that μ equals



(2.15)

We thus see that the right hand side of (2.6) equals (2.15). Finally, we apply ι^* to their bottoms and use the unit property. This proves equation (2.14). ■

Corollary 2.14. If ε and the evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ of $W_a, W_{\overline{a}}$ satisfy (2.6), then there exists an evaluation $\widetilde{\mathrm{ev}}_{a,a}$ for the self-dual object W_a such that $\widetilde{\varepsilon} := \mathbf{1}_a \in \mathrm{End}(W_a)$ and

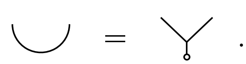
$\widetilde{\mathrm{ev}}_{a,a}$ also satisfy (2.6). Moreover, if $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ are standard, then one can also choose $\widetilde{\mathrm{ev}}_{a,a}$ to be standard.

Proof. We remind the reader that the definition of the evaluations of a self-dual object is given at the beginning of section 2.2. Assuming that $\varepsilon : W_a \to W_{\overline{a}}$ satisfies (2.6), we simply define $\widetilde{\mathrm{ev}}_{a,a} \in \mathrm{Hom}(W_a \boxtimes W_a, W_0)$ to be the left and also the right hand side of (2.14), that is,

$$\mathrm{ev}_{a,a} := \mathrm{ev}_{\overline{a},a}(\varepsilon \otimes \mathbf{1}_a) = \mathrm{ev}_{a,\overline{a}}(\mathbf{1}_a \otimes \varepsilon). \tag{2.16}$$

Then one easily checks that $(\widetilde{\mathrm{ev}}_{a,a} \otimes \mathbf{1}_a)(\mathbf{1}_a \otimes \widetilde{\mathrm{coev}}_{a,a}) = \mathbf{1}_a$ where $\widetilde{\mathrm{coev}}_{a,a} := (\widetilde{\mathrm{ev}}_{a,a})^*$, and that $\mathbf{1}_a$ and $\widetilde{\mathrm{ev}}_{a,a}$ also satisfy (2.6). If the ev for $W_a, W_{\overline{a}}$ are standard, then $\widetilde{\mathrm{ev}}_{a,a}$ is also standard by the unitarity of ε. ■

We shall write $\widetilde{\mathrm{ev}}_{a,a}$ as $\mathrm{ev}_{a,a}$ instead. Then the above corollary says that when (2.6) holds, we may well assume that $\overline{a} = a$, $\mathrm{ev}_{a,\overline{a}} = \mathrm{ev}_{\overline{a},a}$ (which is written as $\mathrm{ev}_{a,a}$), and $\varepsilon = \mathbf{1}_a$. Pictorially, we may remove the arrows and the • on the strings to simplify calculations. Moreover, by (2.6) and the unit property, one has $\mathrm{ev}_{a,a} = \iota^*\mu$:



(2.17)

We now prove the main results of this section. Recall that $\mathcal{C}$ is a C^*-tensor category with simple W_0 and $A = (W_a, \mu, \iota)$ is a $\mathcal{C}$-algebra.

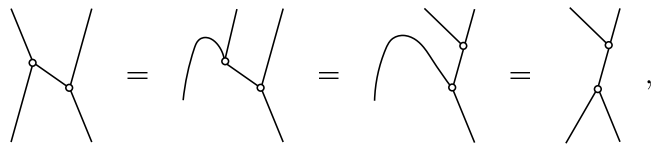
Theorem 2.15. *A is a unitary $\mathcal{C}$-algebra if and only if A is a C^*-Frobenius algebra in $\mathcal{C}$.*

Theorem 2.16. *If there exists $W_{\overline{a}}$ dual to W_a, evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ of $W_a, W_{\overline{a}}$, and a unitary $\varepsilon \in \mathrm{Hom}(W_a, W_{\overline{a}})$ satisfying (2.6), then ε also satisfies (2.7). Consequently, ε is a reflection operator, and A is unitary.*

We prove the two theorems simultaneously.

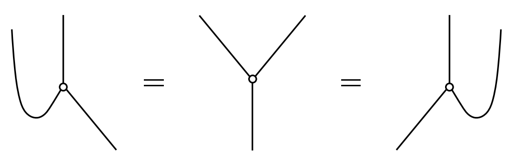
Proof. Step 1. Suppose there exists a dual object $W_{\overline{a}}$, evaluations of $W_a, W_{\overline{a}}$, and a unitary morphism $\varepsilon : W_a \to W_{\overline{a}}$ satisfying (2.6). We assume that $\overline{a} = a$, $\mathrm{ev}_{a,a} := \mathrm{ev}_{a,\overline{a}} =$

$\mathrm{ev}_{\overline{a},a}$ is an evaluation of W_a, and $\varepsilon = \mathbf{1}_a$. Then



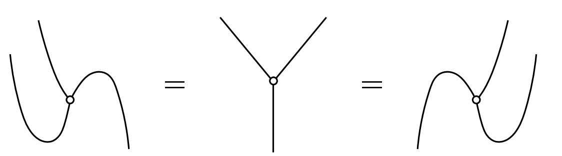
where we have used successively equation (2.6), associativity, and again equation (2.6) in the above equations. This proves the 1st and hence also the 2nd equation of (2.12). We conclude that A is a C^*-Frobenius algebra.

Step 2. Assume that A is a C^*-Frobenius algebra in $\mathcal{C}$. We shall show that W_a is self-dual, and construct a reflection operator. Define $\mathrm{ev}_{a,a} \in \mathrm{Hom}(W_a \boxtimes W_a, W_0)$ to be $\mathrm{ev}_{a,a} = \iota^*\mu$ (see (2.17)). One then easily verifies $(\mathrm{ev}_{a,a} \otimes \mathbf{1}_a)(\mathbf{1}_a \otimes (\mathrm{ev}_{a,a})^*) = \mathbf{1}_a$ by applying respectively $\mathbf{1}_a \otimes \iota$ and $\iota^* \otimes \mathbf{1}_a$ to the top and the bottom of the 2nd equation of (2.12), and then applying the unit property. This shows that W_a is self-dual and $\mathrm{ev}_{a,a} = \iota^*\mu$ is an evaluation of W_a. Therefore we can also omit arrows. Apply $\iota^* \otimes \mathbf{1}_a$ and $\mathbf{1}_a \otimes \iota^*$ to the bottoms of the 2nd and the 1st equation of (2.12), respectively, and then use the unit property and equation (2.17), we obtain



(2.18)

which proves that $\mathbf{1}_a$ and $\mathrm{ev}_{a,a}$ satisfy equation (2.6). Since the 1st and the 3rd items of (2.18) are equal, we take the adjoint of them and bend their left or right legs to the top to obtain



(2.19)

In other words, μ is invariant under clockwise and anticlockwise "1-click rotations." Thus μ equals the clockwise 1-click rotation of the left hand side of (2.18), which proves (2.7) for $\mathbf{1}_a$ and $\mathrm{ev}_{a,a}$. By (2.16), ε and the original evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ also satisfy equation (2.7). Therefore ε is a reflection operator of A with respect to $W_{\overline{a}}$ and the given evaluations of $W_a, W_{\overline{a}}$. This finishes the proof of the two theorems. ■

Corollary 2.17. A is a special unitary $\mathcal{C}$-algebra if and only if A is a Q-system.

Proof. Q-systems are by definition special C^*-Frobenius algebras. ■

We now relate s-unitarity and standardness. We have seen that if A is unitary then $\mathrm{ev}_{a,a} = \iota^*\mu$ is an evaluation of W_a. Therefore, setting $\mathrm{coev}_{a,a} = \mathrm{ev}_{a,a}^*$, we have $\mathrm{ev}_{a,a}\mathrm{coev}_{a,a} = D_A \mathbf{1}_0$ by the definition of D_A. By the minimizing property of standard evaluations, we have $D_A \geq d_a$, with equality holds if and only if $\mathrm{ev}_{a,a}$ is standard. Note that $\mathbf{1}_a$ is a reflection operator with respect to $\mathrm{ev}_{a,a}$. Therefore we have the following:

Proposition 2.18. Let A be unitary. Then $\mathrm{ev}_{a,a} := \iota^*\mu$ is an evaluation of the self-dual object W_a, and $D_A \geq d_a$. Moreover, we have $D_A = d_a$ if and only if A is s-unitary.

Corollary 2.19. A is a special s-unitary $\mathcal{C}$-algebra if and only if A is a standard Q-system.

Proof. By theorem 2.15 and the definition of Q-systems, A is a standard Q-system if and only if A is a special unitary $\mathcal{C}$-algebra satisfying $D_A = d_a$. Thus the corollary follows immediately from the above proposition. ■

Thus we've finished proving the relations (2.13) given at the beginning of this section.

Proposition 2.20. Assuming haploid condition, the six notions in (2.13) are equivalent.

Proof. Let A be a haploid C^*-Frobenius algebra in $\mathcal{C}$. Then by [5] lemma 3.3, A is special. The standardness follows from [32] section 6 (see also [36] remark 5.6-3, or [38] theorem 2.9). Therefore A is a standard Q-system. ■

We can now restate the main result of the last section (theorem 2.12) in the following way:

Theorem 2.21. Let V be a regular, CFT-type, and completely unitary VOA. Let U be a CFT-type preunitary extension of V, and let $A_U = (W_a, \mu, \iota)$ be the haploid commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebra associated to U. Then U is a unitary extension of V if and only if A_U is a C^*-Frobenius algebra. If this is true then A_U is also a standard Q-system.

Let us give an application of this theorem.

Corollary 2.22. The $c < 1$ CFT-type unitary VOAs are in one-to-one correspondence with the irreducible conformal nets with the same central charge c. Their classifications are given by [29] table 3.

Proof. As shown in [29] proposition 3.5, $c < 1$ irreducible conformal nets are precisely irreducible finite-index extensions of the Virasoro net $\mathcal{A}_c$ with central charge c. Thus they are in 1–1 correspondence with the haploid commutative Q-systems in $\mathrm{Rep}^{\mathrm{ss}}(\mathcal{A}_c)$, where $\mathrm{Rep}^{\mathrm{ss}}(\mathcal{A}_c)$ is the unitary MTC of the semisimple representations of $\mathcal{A}_c$. By [21] theorem 5.1, $\mathrm{Rep}^{\mathrm{ss}}(\mathcal{A}_c)$ is unitarily equivalent to $\mathrm{Rep}^{\mathrm{u}}(V)$, where V is the unitary Virasoro VOA with central charge c. By [23] theorem 3.2, haploid commutative $\mathrm{Rep}^{\mathrm{u}}(V)$-algebras with trivial twist are in 1–1 correspondence with CFT-type extensions of V. Thus, by theorem 2.21 and the equivalence of unitary MTCs, CFT-type unitary extensions of $V \Leftrightarrow$ haploid commutative Q-systems in $\mathrm{Rep}^{\mathrm{ss}}(\mathcal{A}_c)$. (The trivial twist condition is redundant; see theorem 3.25.) But also CFT-type unitary extensions of V $\Leftrightarrow$ unitary VOAs with central charge c by [11] theorem 5.1. This proves the desired result. ■

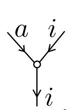
2.5 Strong unitarity of unitary VOA extensions

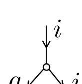
Starting from this section, $A = (W_a, \mu, \iota)$ is assumed to be a unitary $\mathcal{C}$-algebra, or equivalently, a C^*-Frobenius algebra in $\mathcal{C}$. We say that (W_i, μ_L) (resp. (W_i, μ_R)) (Later we will write μ_L and μ_R as μ_L^i and μ_R^i to emphasize the dependence of μ_L, μ_R on the W_i.) is a **left A-module** (resp. **right A-module**), if W_i is an object in $\mathcal{C}$, and $\mu_L \in \mathrm{Hom}(W_a \boxtimes W_i, W_i)$ (resp. $\mu_R \in \mathrm{Hom}(W_i \boxtimes W_a, W_i)$) satisfies the *unit property*

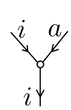
$$\mu_L(\iota \otimes \mathbf{1}_a) \qquad \text{resp.} \qquad \mu_R(\mathbf{1}_a \otimes \iota) = \mathbf{1}_a \tag{2.20}$$

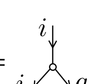
and the *associativity*:

$$\mu_L(\mathbf{1}_a \otimes \mu_L) = \mu_L(\mu \otimes \mathbf{1}_i) \qquad \text{resp.} \qquad \mu_R(\mu_R \otimes \mathbf{1}_a) = \mu_R(\mathbf{1}_i \otimes \mu). \tag{2.21}$$

We write $\mu_L =$ 

, $\mu_L^* =$ 

, $\mu_R =$ 

, $\mu_R^* =$ 

. If (W_i, μ_L) is a left A-module and (W_i, μ_R) is a right A-module, we say that (W_i, μ_L, μ_R) is an **A-bimodule** if the following bimodule associativity holds:

$$\mu_R(\mu_L \otimes \mathbf{1}_a) = \mu_L(\mathbf{1}_a \otimes \mu_R). \tag{2.22}$$

We leave it to the reader to draw the pictures of associativity and unit property. We abbreviate (W_i, μ_L), (W_i, μ_R), or (W_i, μ_L, μ_R) to W_i when no confusion arises.

Set $\mathrm{ev}_{a,a} = \iota^*\mu$ as in the last section. A left (resp. right) A-module (W_i, μ_L) (resp. (W_i, μ_R)) is called **unitary**, if

$$\mu_L = (\mathrm{ev}_{a,a} \otimes \mathbf{1}_i)(\mathbf{1}_a \otimes \mu_L^*) \qquad \text{resp.} \qquad \mu_R = (\mathbf{1}_i \otimes \mathrm{ev}_{a,a})(\mu_R^* \otimes \mathbf{1}_a). \tag{2.23}$$

An A-bimodule (W_a, μ_L, μ_R) is called **unitary** if (W_a, μ_L) is a unitary left A-module and (W_a, μ_R) is a unitary right A-module. Unitarity can be stated for any evaluations and reflection operators:

Proposition 2.23. Let $W_{\overline{a}}$ be dual to W_a, $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ evaluations of $W_{a,\overline{a}}, W_{\overline{a},a}$, and $\varepsilon : W_a \to W_{\overline{a}}$ a reflection operator. Then a left (resp. right) A-module (W_a, μ_L) (resp. (W_a, μ_R)) is unitary if and only if

$$\mu_L = (\mathrm{ev}_{\overline{a},a} \otimes \mathbf{1}_i)(\varepsilon \otimes \mu_L^*) \qquad \text{resp.} \qquad \mu_R = (\mathbf{1}_i \otimes \mathrm{ev}_{a,\overline{a}})(\mu_R^* \otimes \varepsilon). \tag{2.24}$$

Graphically,

(2.25)

Proof. This is obvious since we have equations (2.16). ∎

If W_i and W_j are left (resp. right) A-modules, then a morphism $F \in \mathrm{Hom}(W_i, W_j)$ of $\mathcal{C}$ is called a **left (resp. right) A-module morphism**, if

$$\mu_L(\mathbf{1}_a \otimes F) = F\mu_L \qquad \text{resp.} \qquad \mu_R(F \otimes \mathbf{1}_a) = F\mu_R, \tag{2.26}$$

pictorially,

(2.27)

We let $\mathrm{Hom}_{A,-}(W_i, W_j)$ (resp. $\mathrm{Hom}_{-,A}(W_i, W_j)$) be the vector space of left (resp. right) A-module morphisms. If W_i, W_j are A-bimodules, then we set $\mathrm{Hom}_A(W_i, W_j) = \mathrm{Hom}_{A,-}(W_i, W_j) \cap \mathrm{Hom}_{-,A}(W_i, W_j)$ to be the vector space of **A-bimodule morphisms**. In the case $W_i = W_j$, we write these spaces of morphisms as $\mathrm{End}_{A,-}(W_i), \mathrm{End}_{-,A}(W_i)$, $\mathrm{End}_A(W_i)$, respectively. W_i is called a **simple** or **irreducible** left A-module (resp. right A-module, A-bimodule) if $\mathrm{End}_{A,-}(W_i)$ (resp. $\mathrm{End}_{-,A}(W_i)$, $\mathrm{End}_A(W_i)$) is spanned by $\mathbf{1}_i$. The following proposition is worth noting:

Proposition 2.24 (cf. [39] section 6.1). The category of unitary left A-modules (resp. right A-modules, A-bimodules) is a C^*-category whose $*$-structure inherits from that of $\mathcal{C}$. In particular, this category is closed under finite orthogonal direct sums and subobjects.

Proof. If W_i, W_j are unitary left A-modules and $F \in \mathrm{Hom}_{A,-}(W_i, W_j)$, one can easily check that $F^* \in \mathrm{Hom}_{A,-}(W_j, W_i)$ using figures (2.25) and (2.27). Hence, the C^*-ness of the category of left A-modules follows from that of $\mathcal{C}$. Existence of finite orthogonal direct sums follow from that of $\mathcal{C}$. If $p \in \mathrm{End}_{A,-}(W_i)$ is a projection of the unitary left A-module (W_i, μ_L^i), we choose an object W_k in $\mathcal{C}$ and a partial isometry $u \in \mathrm{Hom}(W_i, W_k)$ such that $uu^* = \mathbf{1}_k, u^*u = p$. Then $(W_k, u\mu_L^i u^*)$ is easily verified to be a unitary left A-module. Note that the fact that p intertwines the left action of A is used to verify the associativity.

The cases of right modules and bimodules are proved in a similar way. ■

A left A-module (resp. right A-module, A-bimodule) W_i is called **$\mathcal{C}$-dualizable** if W_i is a dualizable object in $\mathcal{C}$. W_i is called **unitarizable** if there exists a unitary left A-module (resp. right A-module, A-bimodule) W_j and an invertible $F \in \mathrm{Hom}_{A,-}(W_i, W_j)$ (resp. $F \in \mathrm{Hom}_{-,A}(W_i, W_j)$, $F \in \mathrm{Hom}_A(W_i, W_j)$).

Corollary 2.25. The category of unitary $\mathcal{C}$-dualizable left A-modules (resp. right A-modules, A-bimodules) is a semisimple C^*-category whose $*$-structure inherits from that of $\mathcal{C}$.

Proof. Suppose W_i is a $\mathcal{C}$-dualizable left A-module. Then $\mathrm{End}_A(W_i)$ is a C^*-subalgebra of the finite-dimensional C^*-algebra $E(W_i)$. Thus $\mathrm{End}_A(W_i)$ is a direct sum of matrix algebras, which implies that W_i is a finite orthogonal direct sum of irreducible left A-modules. The other types of modules are treated in a similar way. ■

We are now going to prove the 1st main result of this section, that any $\mathcal{C}$-dualizable module is unitarizable. First we need a lemma.

Lemma 2.26. Let W_i, W_k be $\mathcal{C}$-dualizable left A-modules (resp. right A-modules, A-bimodules). If W_k is unitary, and there exists a surjective $F \in \mathrm{Hom}_{A,-}(W_k, W_i)$ (resp. $F \in \mathrm{Hom}_{-,A}(W_k, W_i)$, $F \in \mathrm{Hom}_A(W_k, W_i)$), then W_i is unitarizable. In particular, W_i is semisimple as a left A-module (resp. right A-module, A-bimodule).

We remark that this lemma is obvious when W_i is already known to be semisimple as a left, right, or bi A-module, which is enough for our application to representations of VOA extensions. (Indeed, the extension U of V considered in this paper is always regular, hence its modules are semisimple.) Those who are only interested in the application to VOAs can skip the following proof.

Proof. We only prove this for left modules, since the other cases can be proved similarly. Write the two modules as $(W_i, \mu_L^i), (W_k, \mu_L^k)$. Note that F^*F is a positive element in the finite-dimensional C^*-algebra $\mathrm{End}(W_k)$. So $\lim_{n\to\infty}(F^*F)^{1/n}$ converges under the C^*-norm to a projection $P \in \mathrm{End}(W_k)$, which is the range projection of F^*. Set $G = P\mu_L^k$ and $H = P\mu_L^k(\mathbf{1}_a \otimes P)$, which are morphisms in $\mathrm{Hom}(W_a \boxtimes W_k, W_k)$. Then using (2.27) and the fact that $F = FP$, we obtain $FG = FH$, since both equal $F\mu_L^k$. Therefore $(F^*F)^n G = (F^*F)^n H$ for any integer $n > 0$, and hence $(F^*F)^{1/n}G = (F^*F)^{1/n}H$ by polynomial interpolation. Thus $G = PG = PH = H$. We conclude

$$P\mu_L^k = P\mu_L^k(\mathbf{1}_a \otimes P). \tag{2.28}$$

This equation, together with (2.25), shows $(\mathbf{1}_a \otimes P)(\mu_L^k)^* = (\mathbf{1}_a \otimes P)(\mu_L^k)^*P$, whose adjoint is

$$\mu_L^k(\mathbf{1}_a \otimes P) = P\mu_L^k(\mathbf{1}_a \otimes P). \tag{2.29}$$

We can therefore combine (2.28) and (2.29) to get $P\mu_L^k = \mu_L^k(\mathbf{1}_a \otimes P)$. In other words, P is a projection in $\mathrm{End}_{A,-}(W_k)$. Thus, by proposition 2.24, one can find a unitary left A-module W_j and a partial isometry $K \in \mathrm{Hom}_{A,-}(W_j, W_k)$ satisfying $K^*K = \mathbf{1}_j$ and $KK^* = P$. Therefore the left A-module W_i is equivalent to W_j since $FK \in \mathrm{Hom}_{A,-}(W_j, W_i)$ is invertible. ■

Theorem 2.27. $\mathcal{C}$-dualizable left A-modules, right A-modules, and A-bimodules are unitarizable.

In particular, when $\mathcal{C}$ is rigid, any left A-module, right A-module, or A-bimodule is unitarizable.

Proof. For any $\mathcal{C}$-dualizable object W_i in $\mathcal{C}$ the induced left A-module $(W_a \boxtimes W_i, \mu \otimes \mathbf{1}_i)$, abbreviated to $W_a \boxtimes W_i$, is clearly $\mathcal{C}$-dualizable. By the unitarity of A, one easily checks that $W_a \boxtimes W_i$ is a unitary left A-module. Now assume that (W_i, μ_L) is a left A-module. Then $\mu_L \in \mathrm{Hom}_{A,-}(W_a \boxtimes W_i, W_i)$. Moreover, μ_L is surjective since $(\iota \otimes \mathbf{1}_i)\mu_L = \mathbf{1}_i$ is surjective. Therefore W_i is unitarizable by lemma 2.26. The case of right modules is proved in a similar way. In the case that (W_i, μ_L, μ_R) is a $\mathcal{C}$-dualizable A-bimodule, we notice that $(W_a \boxtimes W_i \boxtimes W_a, \mu \otimes \mathbf{1}_i \otimes \mathbf{1}_a, \mathbf{1}_a \otimes \mathbf{1}_i \otimes \mu)$ is a unitary A-bimodule, and $\mu_R(\mu_L \otimes \mathbf{1}_a) = \mu_L(\mathbf{1}_a \otimes \mu_R) \in \mathrm{Hom}_A(W_a \boxtimes W_i \boxtimes W_a, W_i)$ whose surjectivity follows again from the unit property. Thus, again, W_i is a unitarizable A-bimodule. ∎

In the case that A is special, there is another proof of unitarizability due to [5], which does not require dualizability. To begin with, we say that a left A-module (W_i, μ_L) is **standard** if $\mu_L \mu_L^* \in \mathbb{C}\mathbf{1}_i$. We now follow the argument of [5] lemma 3.22. For any left A-module (W_i, μ_L), $\Delta := (\mu_L \mu_L^*)^{1/2}$ is invertible. Therefore (W_i, μ_L) is equivalent to $(W_i, \widetilde{\mu}_L)$ where $\widetilde{\mu}_L = \Delta^{-1}\mu_L(\mathbf{1}_a \otimes \Delta)$. Using associativity and the fact that $\mu\mu^* = d_A \mathbf{1}_a$, one can check that $(W_i, \widetilde{\mu}_L)$ is standard and $\widetilde{\mu}_L \widetilde{\mu}_L^* = d_A \mathbf{1}_i$. In particular, if (W_i, μ_L) is standard then Δ is a constant and hence $\widetilde{\mu}_L = \mu_L$. Thus we must have $\mu_L \mu_L^* = d_A \mathbf{1}_i$. This proves that any left A-module is equivalent to a standard left A-module, that any standard left A-module must satisfy $\mu_L \mu_L^* = d_A \mathbf{1}_a$. Moreover, by [5] formula (3.4.5), any standard left A-module is unitary. Conversely, if (W_i, μ_L) is unitary, one can check that $\mu_L \mu_L^* \in \mathrm{End}_{A,-}(W_i)$ and hence $\Delta \in \mathrm{End}_{A,-}(W_i)$. This proves that $\widetilde{\mu}_L = \mu_L$ and hence that (W_i, μ_L) is standard. Right A-modules can be proved in a similar way. When (W_i, μ_L, μ_R) is an A-bimodule, we define $\mu_{LR} = \mu_R(\mu_L \otimes \mathbf{1}_a) = \mu_L(\mathbf{1}_a \otimes \mu_R)$, and say that the A-bimodule W_i is **standard** if $\mu_{LR}\mu_{LR}^* \in \mathbb{C}\mathbf{1}_i$ (cf. [5] section 3.6). With the help of $\Delta := (\mu_{LR}\mu_{LR}^*)^{1/2}$ one can prove similar results as of left A-modules, with the only exception being that $\mu_{LR}\mu_{LR}^* = d_A^2 \mathbf{1}_i$. We summarize the discussion in the following theorems:

Theorem 2.28. If A is a Q-system in $\mathcal{C}$, then all left A-modules, right A-modules, and A-bimodules are unitarizable.

Theorem 2.29. Let A be a Q-system in $\mathcal{C}$, and (W_i, μ_L) (resp. $(W_i, \mu_R), (W_i, \mu_L, \mu_R)$) a left A-module (resp. right A-module, A-bimodule). Then the following statements are equivalent.

- W_i is unitary.
- W_i is standard.
- $\mu_L \mu_L^* = d_A \mathbf{1}_i$ (left A-module case), or $\mu_R \mu_R^* = d_A \mathbf{1}_i$ (right A-module case), or $\mu_{LR} \mu_{LR}^* = d_A^2 \mathbf{1}_i$ where $\mu_{LR} = \mu_R(\mu_L \otimes \mathbf{1}_a) = \mu_L(\mathbf{1}_a \otimes \mu_R)$ (A-bimodule case).

If $\mathcal{C}$ is braided with braid operator ß, and if A is commutative, a left A-module (W_i, μ_L) is called **single-valued** if $\mu_L = \mu_L ß^2$ (more precisely, $\mu_L = \mu_L ß_{i,a} ß_{a,i}$). If (W_i, μ_L) is a single-valued left A-module, then (W_i, μ_R) is a right A-module where $\mu_R = \mu_L ß_{i,a} = \mu_L ß_{a,i}^{-1}$. Moreover, by the associativity of (W_i, μ_L), (W_i, μ_L, μ_R) is an A-bimodule. We summarize that when $\mathcal{C}$ is braided and A is commutative, any single-valued left A-module is an A-bimodule. Moreover, if W_i is a unitary single-valued left A-module, the it is also a unitary A-bimodule. The category of (unitary) single-valued left A-module is naturally a full subcategory of the (C^*-)categories of (unitary) left A-modules, (unitary) right A-modules, and (unitary) A-bimodules.

With the results obtained so far, we prove that any CFT-type unitary extension U of V is strongly unitary. Let $A_U = (W_a, \mu, \iota)$ be the Q-system associated to U. Suppose that W_i is a U-module with vertex operator $\mathcal{Y}_{\mu_L}$. Then W_i is also a V-module, thus we may fix an inner product $\langle \cdot | \cdot \rangle$ on the vector space W_i under which W_i becomes a unitary V-module. We call W_i, together with $\langle \cdot | \cdot \rangle$, a **preunitary U-module**. Moreover, $\mathcal{Y}_{\mu_L}$ can be regarded as a unitary intertwining operator of V of type $\binom{i}{a\ i}$. Thus $\mu_L \in \mathrm{Hom}(W_a \boxtimes W_i, W_i)$. One can check that (W_i, μ_L) is a single-valued left A_U-module. Indeed, unit property is obvious, associativity follows from the Jacobi identity for $\mathcal{Y}_{\mu_L}$, and single-valued property follows from the fact that $\mathcal{Y}_{\mu_L}(\cdot, z)$ has only integer powers of z. Conversely, any preunitary U-module arises from a single-valued A_U-module. Thus the category of preunitary U-modules is naturally equivalent to the category of single-valued left A_U-modules. See [23] for more details.

To discuss the unitarizability of U-modules, the following is needed:

Theorem 2.30. Assume that V is a CFT-type, regular, and completely unitary VOA, U is a CFT-type unitary extension of V, and W_i is a preunitary U-module. Then W_i is a unitary U-module if and only if W_i is a unitary left A_U-module. If this is true then W_i is also a unitary A_U-bimodule.

Proof. Let $W_{\overline{a}}$ be the contragredient module of W_a, $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$ the evaluations of $W_a, W_{\overline{a}}$ defined in chapter 1, and $\varepsilon : U = W_a \to \overline{U} = W_{\overline{a}}$ the reflection operator with respect to the chosen dual and evaluations. By equation (1.23) and the definition of adjoint intertwining operators, W_i is unitary if and only if $\mathcal{Y}_{\mu_L}(w^{(a)}, z) = \mathcal{Y}_{\mu_L^\dagger}(\varepsilon w^{(a)}, z)$ for any $w^{(a)} \in W_a = U$. From equation (1.2) we know that W_i is unitary if and only if $\mu_L = \mu_L^\dagger(\varepsilon \otimes \mathbf{1}_i)$. With the help of theorem 1.11, this equation is equivalent to $\mu_L = (\mathrm{ev}_{\overline{a},a} \otimes \mathbf{1}_i)(\varepsilon \otimes \mu_L^*)$, which by proposition 2.23 means precisely the unitarity of the left A_U-module W_i. If we already have that W_i is a unitary left A_U-module, then, since W_i is singlevalued, it is also a unitary A-bimodule. ■

We now prove the strong unitarity of U.

Theorem 2.31. If V is a CFT-type, regular, and completely unitary VOA, and U is a CFT-type unitary extension of V, then U is strongly unitary, that is, any U-module is unitarizable.

Proof. Since U-modules are clearly preunitarizable, we choose a preunitary U-module W_i. Then by either theorem 2.27 or theorem 2.28, W_i is unitarizable as a left A_U-module. Thus, by equation (1.2) and theorem 2.30, W_i is unitarizable as a U-module. ■

3 C^*-Tensor Categories Associated to Q-Systems and Unitary VOA Extensions

3.1 Unitary tensor products of unitary bimodules of Q-systems

In this chapter, $\mathcal{C}$ is a C^*-tensor category with simple W_0 as before, and $A = (W_a, \mu, \iota)$ is a Q-system (i.e., special C^*-Frobenius algebra) in $\mathcal{C}$. Set evaluation $\mathrm{ev}_{a,a} = \iota^*\mu$ as usual. We suppress the label a in diagram calculus. Let $\mathrm{BIM}^{\mathrm{u}}(A)$ be the C^*-category of unitary A-bimodules whose morphisms are A-bimodule morphisms. In this and the next sections, we review the construction of a C^*-tensor structure on $\mathrm{BIM}^{\mathrm{u}}(A)$. See [8, 30, 37] for reference. Note that our setting is slightly more general than that of [37], since we do not assume $\mathcal{C}$ is rigid or A is standard. Nevertheless, many ideas in [37] still work in our setting. To make our article self-contained, we include detailed proofs for all the relevant results.

Choose unitary A-bimodules $(W_i, \mu_L^i, \mu_R^i), (W_j, \mu_L^j, \mu_R^j)$. Then $W_i \boxtimes W_j$ is a unitary A-bimodule with left action $\mu_L^i \otimes \mathbf{1}_j$ and right action $\mathbf{1}_i \otimes \mu_R^j$. Define $\Psi_{i,j} \in \mathrm{Hom}_A(W_i \boxtimes W_a \boxtimes W_j, W_i \boxtimes W_j)$ and $\chi_{i,j} \in \mathrm{End}_A(W_i \boxtimes W_j)$ to be

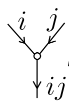
$$\Psi_{i,j} = \mu_R^i \otimes \mathbf{1}_j - \mathbf{1}_i \otimes \mu_L^j, \tag{3.1}$$

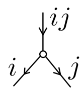
$$\chi_{i,j} = (\mathbf{1}_i \otimes \mu_L^j)((\mu_R^i)^* \otimes \mathbf{1}_j) = (\mu_R^i \otimes \mathbf{1}_j)(\mathbf{1}_i \otimes (\mu_L^j)^*). \tag{3.2}$$

Definition 3.1. Let W_i, W_j be unitary A-bimodules. We say that $(W_{ij}, \mu_{i,j})$ (abbreviated to W_{ij} when no confusion arises) is a **tensor product** of W_i, W_j over A (cf. [8]), if

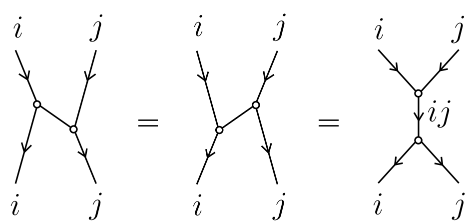
- $W_{ij} = (W_{ij}, \mu_L^{ij}, \mu_R^{ij})$ is an A-bimodule, $\mu_{i,j} \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_{ij})$ (One should not confuse $i \boxtimes j$ and ij. By our notation, $W_{i\boxtimes j} = W_i \boxtimes W_j$ is different from W_{ij}.), and $\mu_{i,j}\Psi_{i,j} = 0$.
- (Universal property) If (W_k, μ_L^k, μ_R^k) is a unitary A-bimodule, $\alpha \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_k)$, and $\alpha\Psi_{i,j} = 0$ (Such α is called a categorical intertwining operator in [8].), then there exists a unique $\widetilde{\alpha} \in \mathrm{Hom}_A(W_{ij}, W_k)$ satisfying $\alpha = \widetilde{\alpha}\mu_{i,j}$. In this case, we say that $\widetilde{\alpha}$ is **induced by** α via the tensor product W_{ij}.

The tensor product $(W_{ij}, \mu_{i,j})$ is called **unitary** if W_{ij} is a unitary A-bimodule and $\chi_{i,j} = \mu_{i,j}^*\mu_{i,j}$.

We write $\mu_{i,j} =$ 

, $\mu_{i,j}^* =$ 

. Then the equation $\chi_{i,j} = \mu_{i,j}^*\mu_{i,j}$ reads



, (3.3)

which is a special case of the Frobenius relations for unitary tensor products to be proved later (theorem 3.14). (Note that the 1st equality of (3.3) follows from the unitarity of W_i and W_j.)

The existence of a tensor product is clear if one assumes moreover that $\mathcal{C}$ is abelian: Let W_{ij} be a cokernel of $\Psi_{i,j}$, and define the bimodule structure on W_{ij} using that of $W_i \boxtimes W_j$. Then W_{ij} becomes a tensor product over A of W_i and W_j; see [31] for more details. However, to make the tensor product unitary one has to be more careful when choosing the cokernel. In the following, we proceed in a slightly different way motivated by [5] section 3.7, and we do not require the abelianess of $\mathcal{C}$. To begin with, using $\mu\mu^* = d_A\mathbf{1}_a$, one verifies easily that $\chi_{i,j}^2 = d_A\chi_{i,j}$. Therefore,

Lemma 3.2 ([5] lemma 3.36). $d_A^{-1}\chi_{i,j} \in \mathrm{End}_A(W_i \boxtimes W_j)$ is a projection.

By proposition 2.24, there exists a unitary A-bimodule W_{ij} and a partial isometry $u_{i,j} \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_{ij})$ satisfying $u_{i,j}u_{i,j}^* = \mathbf{1}_{ij}$ and $u_{i,j}^*u_{i,j} = d_A^{-1}\chi_{i,j}$. Setting $\mu_{i,j} = \sqrt{d_A}u_{i,j}$, one obtains $\mu_{i,j}^*\mu_{i,j} = \chi_{i,j}$ (equations (3.3)) and $\mu_{i,j}\mu_{i,j}^* = d_A\mathbf{1}_{ij}$. We now show that $(W_{ij}, \mu_{i,j})$ is a unitary tensor product.

Proposition 3.3. Let W_{ij} be a unitary A-bimodule, and $\mu_{i,j} \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_{ij})$. Then $(W_{ij}, \mu_{i,j})$ is a unitary tensor product of W_i, W_j over A if and only if $\mu_{i,j}^*\mu_{i,j} = \chi_{i,j}$ and $\mu_{i,j}\mu_{i,j}^* = d_A\mathbf{1}_{ij}$.

Proof. "If": Since $\mu_{i,j}^*\mu_{i,j} = \chi_{i,j}$ and the unitarity of the A-bimodule W_{ij} are assumed, it suffices to show that W_{ij} is a tensor product. Since $\chi_{i,j}\Psi_{i,j}$ clearly equals 0, we compute

$$\mu_{i,j}\Psi_{i,j} = d_A^{-1}\mu_{i,j}\mu_{i,j}^*\mu_{i,j}\Psi_{i,j} = d_A^{-1}\mu_{i,j}\chi_{i,j}\Psi_{i,j} = 0. \tag{3.4}$$

If W_k is a unitary A-bimodule, $\alpha \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_k)$, and $\alpha\Psi_{i,j} = 0$, then one can set $\widetilde{\alpha} = d_A^{-1}\alpha\mu_{i,j}^*$ and compute

$$\widetilde{\alpha}\mu_{i,j} = d_A^{-1}\alpha\mu_{i,j}^*\mu_{i,j} = d_A^{-1}\alpha\chi_{i,j} = d_A^{-1}\alpha(\mu_R^i(\mu_R^i)^* \otimes \mathbf{1}_j) = \alpha,$$

where we have used $\alpha\Psi_{i,j} = 0$ and $\mu_R^i(\mu_R^i)^* = d_A\mathbf{1}_i$ (theorem 2.29) respectively to prove the 3rd and the 4th equalities. If there is another $\widehat{\alpha}$ satisfying also $\alpha = \widehat{\alpha}\mu_{i,j}$, then $\widehat{\alpha} = d_A^{-1}\widehat{\alpha}\mu_{i,j}\mu_{i,j}^* = d_A^{-1}\alpha\mu_{i,j}^* = \widetilde{\alpha}$. Thus the universal property is checked.

"Only if": This will be proved after the next theorem. ∎

Theorem 3.4. Let W_i, W_j be unitary A-bimodules. Then unitary tensor products over A of W_i, W_j exist and are unique up to unitaries. More precisely, uniqueness means that if $(W_{ij}, \mu_{i,j})$ and $(W_{i\bullet j}, \eta_{i,j})$ are unitary tensor products of W_i, W_j over A, then there exists a (unique) unitary $u \in \mathrm{Hom}_A(W_{ij}, W_{i\bullet j})$ such that $\eta_{i,j} = u\mu_{i,j}$.

Proof. Existence has already been proved. We now prove the uniqueness. Since $\eta_{i,j} \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_{i\bullet j})$ is annihilated by $\Psi_{i,j}$, by the universal property for $(W_{ij}, \mu_{i,j})$ there exists a unique $u \in \mathrm{Hom}_A(W_{ij}, W_{i\bullet j})$ satisfying $\eta_{i,j} = u\mu_{i,j}$. In other words u is the A-bimodule morphism induced by $\eta_{i,j}$. It remains to prove that u is unitary.

We first show that u is invertible. By the universal property for $(W_{i\bullet j}, \eta_{i,j})$, there exists $v \in \mathrm{Hom}_A(W_{i\bullet j}, W_{ij})$ such that $\mu_{i,j} = v\eta_{i,j}$. Thus $\eta_{i,j} = uv\eta_{i,j}$. Therefore, uv is induced by $\eta_{i,j}$ via the tensor product $W_{i\bullet j}$. But $\mathbf{1}_{i\bullet j}$ is clearly also induced by $\eta_{i,j}$ via $W_{i\bullet j}$. Therefore $uv = \mathbf{1}_{i\bullet j}$. Similarly $vu = \mathbf{1}_{ij}$. This proves that u is invertible.

We now calculate

$$\mu_{i,j}^* u^* u \mu_{i,j} = \eta_{i,j}^* \eta_{i,j} = \chi_{i,j} = \mu_{i,j}^* \mu_{i,j}.$$

By the universal property, $\mu_{i,j}^* u^* u$ and $\mu_{i,j}^*$ are equal since they are induced by the same morphism via W_{ij}. So $u^* u \mu_{i,j} = \mu_{i,j}$. By the universal property again, we have $u^* u = \mathbf{1}_{ij}$. Therefore u is unitary. ∎

Proof of the "only if" part of Proposition 3.3. By the "if" part of proposition 3.3 and the paragraph before that, there exists a unitary tensor product $(W_{ij}, \mu_{i,j})$ satisfying $\mu_{i,j}\mu_{i,j}^* = d_A \mathbf{1}_{ij}$. By uniqueness up to unitaries, this equation holds for any unitary tensor product. ∎

In the remaining part of this section, we generalize the notion of unitary tensor product to more than two unitary A-bimodules. For simplicity we only discuss the case of three bimodules. The more general cases can be treated in a similar fashion and are thus left to the reader.

Choose unitary A-bimodules W_i, W_j, W_k with left actions $\mu_L^i, \mu_L^j, \mu_L^k$ and right actions $\mu_R^i, \mu_R^j, \mu_R^k$, respectively. Then $(W_i \boxtimes W_j \boxtimes W_k, \mu_L^i \otimes \mathbf{1}_j \otimes \mathbf{1}_k, \mathbf{1}_i \otimes \mathbf{1}_j \otimes \mu_R^k)$ is a unitary A-bimodule.

Lemma 3.5. $\chi_{i,j} \otimes \mathbf{1}_k$ and $\mathbf{1}_i \otimes \chi_{j,k}$ commute. Define $\chi_{i,j,k} \in \mathrm{End}_A(W_i \boxtimes W_j \boxtimes W_k)$ to be their product. Then $d_A^{-2}\chi_{i,j,k}$ is a projection.

Proof. The commutativity of these two morphisms is verified using the commutativity of the left and right actions of W_j. Thus $d_A^{-1}\chi_{i,j} \otimes \mathbf{1}_k$ and $d_A^{-1}\mathbf{1}_i \otimes \chi_{j,k}$ are commuting projections, whose product is therefore also a projection. ∎

Definition 3.6. $(W_{ijk}, \mu_{i,j,k})$ (or W_{ijk} for short) is called a **unitary tensor product** of W_i, W_j, W_k over A, if

- $W_{ijk} = (W_{ijk}, \mu_L^{ijk}, \mu_R^{ijk})$ is a unitary A-bimodule, $\mu_{i,j,k} \in \mathrm{Hom}_A(W_i \boxtimes W_j \boxtimes W_k, W_{ijk})$, and $\mu_{i,j,k}(\Psi_{i,j} \otimes \mathbf{1}_k) = \mu_{i,j,k}(\mathbf{1}_i \otimes \Psi_{j,k}) = 0$.
- (Universal property) If (W_l, μ_L^l, μ_R^l) is a unitary A-bimodule, $\alpha \in \mathrm{Hom}_A(W_i \boxtimes W_j \boxtimes W_k, W_l)$, and $\alpha(\Psi_{i,j} \otimes \mathbf{1}_k) = \alpha(\mathbf{1}_i \otimes \Psi_{j,k}) = 0$, then there exists a unique $\widetilde{\alpha} \in$

$\mathrm{Hom}_A(W_{ijk}, W_l)$ satisfying $\alpha = \widetilde{\alpha}\mu_{i,j,k}$. In this case, we say that $\widetilde{\alpha}$ is **induced by** α via the tensor product W_{ijk}.

- (Unitarity) $\chi_{i,j,k} = \mu_{i,j,k}\mu_{i,j,k}^*$.

Proposition 3.7. Let W_{ijk} be a unitary A-bimodules, and $\mu_{i,j,k} \in \mathrm{Hom}_A(W_i \boxtimes W_j \boxtimes W_k, W_{ijk})$. Then $(W_{ijk}, \mu_{i,j,k})$ is a unitary tensor product of W_i, W_j, W_k over A if and only if $\mu_{i,j,k}^*\mu_{i,j,k} = \chi_{i,j,k}$ and $\mu_{i,j,k}\mu_{i,j,k}^* = d_A^2\mathbf{1}_{ijk}$.

Theorem 3.8. Unitary tensor products of W_i, W_j, W_k exist and are unique up to unitaries.

We omit the proofs of these two results since they can be proved in a similar way as proposition 3.3 and theorem 3.4.

3.2 C^*-tensor categories associated to Q-systems

We are now ready to define the unitary tensor structure on the C^*-category $\mathrm{BIM}^{\mathrm{u}}(A)$ of unitary A-bimodules. The tensor bifunctor $\boxtimes_A$ is defined as follows. For any unitary A-bimodules W_i, W_j, we choose a unitary tensor product $(W_{ij}, \mu_{i,j})$. Then $W_i \boxtimes_A W_j$ is just the unitary A-bimodule W_{ij}. To define tensor product of morphisms, we choose another pair of unitary A-bimodules $W_{i'}, W_{j'}$, and choose any $F \in \mathrm{Hom}_A(W_i, W_{i'})$ and $G \in \mathrm{Hom}_A(W_j, W_{j'})$. Of course, there is also a chosen unitary tensor product $(W_{i'j'}, \mu_{i',j'})$ of $W_{i'}, W_{j'}$ over A. Since $F \otimes G : W_i \boxtimes W_j \to W_{i'} \boxtimes W_{j'}$ is clearly an A-bimodule morphism, we have $\mu_{i',j'}(F \otimes G) \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_{i'j'})$, and one can easily show that $\mu_{i',j'}(F \otimes G)\Psi_{i,j} = 0$. Therefore, by universal property, there exists a unique morphism in $\mathrm{Hom}_A(W_{ij}, W_{i'j'})$, denoted by $F \otimes_A G$, such that

$$\mu_{i',j'}(F \otimes G) = (F \otimes_A G)\mu_{i,j}. \tag{3.5}$$

This defines the tensor product of F and G in $\mathrm{BIM}^{\mathrm{u}}(A)$. We now show that $\boxtimes_A$ is a $*$-bifunctor. Notice that $F^* \otimes_A G^*$ is defined by $\mu_{i,j}(F^* \otimes G^*) = (F^* \otimes_A G^*)\mu_{i',j'}$. Therefore, using $(F \otimes G)^* = F^* \otimes G^*$ we compute

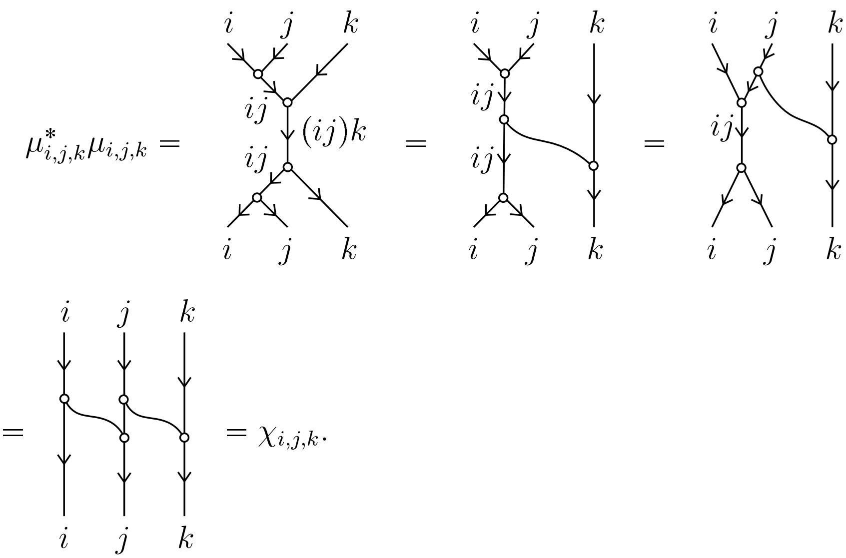
$$\begin{aligned}(F \otimes_A G)^* &= d_A^{-1}\mu_{ij}\mu_{ij}^*(F \otimes_A G)^* = d_A^{-1}\mu_{ij}(F \otimes G)^*\mu_{i',j'}^* = d_A^{-1}\mu_{ij}(F^* \otimes G^*)\mu_{i',j'}^* \\ &= d_A^{-1}(F^* \otimes_A G^*)\mu_{i',j'}\mu_{i',j'}^* = F^* \otimes_A G^*.\end{aligned}$$

To construct associativity isomorphisms we need the following:

Proposition 3.9. Let W_i, W_j, W_k be unitary A-bimodules. Then $(W_{(ij)k}, \mu_{ij,k}(\mu_{i,j}\otimes \mathbf{1}_k))$ and $(W_{i(jk)}, \mu_{i,jk}(\mathbf{1}_i \otimes \mu_{j,k}))$ are unitary tensor products of W_i, W_j, W_k over A.

Note that here $W_{(ij)k}$ is understood as the unitary tensor product of W_{ij} and W_k over A, and $W_{i(jk)}$ is understood similarly.

Proof. The two cases can be treated in a similar way. So we only prove the 1st one. Set $\mu_{i,j,k} = \mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k)$. By proposition 3.7, it suffices to prove $\mu^*_{i,j,k}\mu_{i,j,k} = \chi_{i,j,k}$ and $\mu_{i,j,k}\mu^*_{i,j,k} = d_A^2\mathbf{1}_{ijk}$. The 2nd equation follows directly from that $\mu_{i,j}\mu^*_{i,j} = d_A\mathbf{1}_{ij}$ and $\mu_{ij,k}\mu^*_{ij,k} = d_A\mathbf{1}_{(ij)k}$. To prove the 1st one, we compute (recalling that we have suppressed the label a)



■

Corollary 3.10. For any unitary A-bimodules W_i, W_j, W_k there exists a (unique) unitary $\mathfrak{A}_{i,j,k} \in \mathrm{Hom}_A(W_{(ij)k}, W_{i(jk)})$ satisfying

$$\mu_{i,jk}(\mathbf{1}_i \otimes \mu_{j,k}) = \mathfrak{A}_{i,j,k} \cdot \mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k). \tag{3.6}$$

We define the unitary **associativity isomorphism** $W_{(ij)k} \to W_{i(jk)}$ to be $\mathfrak{A}_{i,j,k}$.

Proof. This follows immediately from the above proposition and theorem 3.8. ■

Proposition 3.11 (Pentagon axiom). Let W_i, W_j, W_k, W_l be unitary A-bimodules. Then

$$(\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mathfrak{A}_{i,jk,l}(\mathfrak{A}_{i,j,k} \otimes_A \mathbf{1}_l) = \mathfrak{A}_{i,j,kl}\mathfrak{A}_{ij,k,l}. \tag{3.7}$$

Proof. One can define unitary tensor products of W_i, W_j, W_k, W_l over A in a similar way as those of three unitary A-bimodules. Moreover, using the argument of proposition 3.9 one shows that $(W_{((ij)k)l}, \mu_{(ij)k,l}(\mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k) \otimes \mathbf{1}_l))$ is a unitary tensor product of W_i, W_j, W_k, W_l over A. We now compute

$$\begin{aligned}
&(\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mathfrak{A}_{i,jk,l}(\mathfrak{A}_{i,j,k} \otimes_A \mathbf{1}_l) \cdot \mu_{(ij)k,l}(\mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k) \otimes \mathbf{1}_l)\\
\overset{(3.5)}{=\!=}\;& (\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mathfrak{A}_{i,jk,l} \cdot \mu_{i(jk),l}(\mathfrak{A}_{i,j,k} \otimes \mathbf{1}_l)(\mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k) \otimes \mathbf{1}_l)\\
=\;& (\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mathfrak{A}_{i,jk,l} \cdot \mu_{i(jk),l}((\mathfrak{A}_{i,j,k} \cdot \mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k)) \otimes \mathbf{1}_l)\\
\overset{(3.6)}{=\!=}\;& (\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mathfrak{A}_{i,jk,l} \cdot \mu_{i(jk),l}(\mu_{i,jk}(\mathbf{1}_i \otimes \mu_{j,k}) \otimes \mathbf{1}_l)\\
=\;& (\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mathfrak{A}_{i,jk,l} \cdot \mu_{i(jk),l}(\mu_{i,jk} \otimes \mathbf{1}_l)(\mathbf{1}_i \otimes \mu_{j,k} \otimes \mathbf{1}_l)\\
\overset{(3.6)}{=\!=}\;& (\mathbf{1}_i \otimes_A \mathfrak{A}_{j,k,l})\mu_{i,(jk)l}(\mathbf{1}_i \otimes \mu_{jk,l})(\mathbf{1}_i \otimes \mu_{j,k} \otimes \mathbf{1}_l)\\
\overset{(3.5)}{=\!=}\;& \mu_{i,j(kl)}(\mathbf{1}_i \otimes \mathfrak{A}_{j,k,l})(\mathbf{1}_i \otimes \mu_{jk,l})(\mathbf{1}_i \otimes \mu_{j,k} \otimes \mathbf{1}_l)\\
=\;& \mu_{i,j(kl)}(\mathbf{1}_i \otimes (\mathfrak{A}_{j,k,l} \cdot \mu_{jk,l}(\mu_{j,k} \otimes \mathbf{1}_l)))\\
\overset{(3.6)}{=\!=}\;& \mu_{i,j(kl)}(\mathbf{1}_i \otimes \mu_{j,kl}(\mathbf{1}_j \otimes \mu_{k,l})),
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{A}_{i,j,kl}\mathfrak{A}_{ij,k,l} \cdot \mu_{(ij)k,l}(\mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k) \otimes \mathbf{1}_l)\\
=\;& \mathfrak{A}_{i,j,kl}\mathfrak{A}_{ij,k,l} \cdot \mu_{(ij)k,l}(\mu_{ij,k} \otimes \mathbf{1}_l)(\mu_{i,j} \otimes \mathbf{1}_k \otimes \mathbf{1}_l)\\
\overset{(3.6)}{=\!=}\;& \mathfrak{A}_{i,j,kl} \cdot \mu_{ij,kl}(\mathbf{1}_{ij} \otimes \mu_{k,l})(\mu_{i,j} \otimes \mathbf{1}_k \otimes \mathbf{1}_l)\\
=\;& \mathfrak{A}_{i,j,kl} \cdot \mu_{ij,kl}(\mu_{i,j} \otimes \mathbf{1}_{kl})(\mathbf{1}_i \otimes \mathbf{1}_j \otimes \mu_{k,l})\\
\overset{(3.6)}{=\!=}\;& \mu_{i,j(kl)}(\mathbf{1}_i \otimes \mu_{j,kl})(\mathbf{1}_i \otimes \mathbf{1}_j \otimes \mu_{k,l}) = \mu_{i,j(kl)}(\mathbf{1}_i \otimes \mu_{j,kl}(\mathbf{1}_j \otimes \mu_{k,l})).
\end{aligned}$$

Thus equation (3.7) holds when both sides are multiplied by $\mu_{(ij)k,l}(\mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k) \otimes \mathbf{1}_l)$. Hence equation (3.7) is true by the universal property for the unitary tensor products of W_i, W_j, W_k, W_l. ■

We choose the vacuum bimodule (W_a, μ, μ) to be the identity object of $\mathrm{BIM}^{\mathrm{u}}(A)$. Then by proposition 3.3, for any unitary A-bimodule (W_i, μ_L^i, μ_R^i), we have that (W_i, μ_L^i) is a unitary tensor product of $W_a \boxtimes W_i$ and (W_i, μ_R^i) is a unitary tensor product of $W_i \boxtimes W_a$. By uniqueness up to unitaries, there exist unique unitary $\mathfrak{l}_i \in \mathrm{Hom}_A(W_{ai}, W_i)$ and $\mathfrak{r}_i \in \mathrm{Hom}_A(W_{ia}, W_i)$ satisfying

$$\mu_L^i = \mathfrak{l}_i \mu_{a,i}, \qquad \mu_R^i = \mathfrak{r}_i \mu_{i,a}. \tag{3.8}$$

Proposition 3.12 (Triangle axiom). For any unitary A-bimodules W_i, W_j we have

$$(\mathbf{1}_i \otimes_A \mathfrak{l}_j)\mathfrak{A}_{i,a,j} = \mathfrak{r}_i \otimes_A \mathbf{1}_j. \tag{3.9}$$

Proof. Similar to (and simpler than) the proof of pentagon axiom, we show that

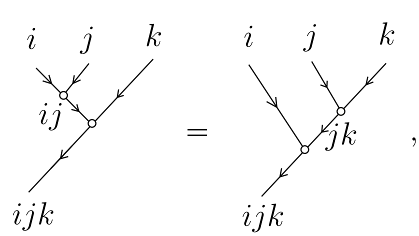
$$(\mathbf{1}_i \otimes_A \mathfrak{l}_j)\mathfrak{A}_{i,a,j} \cdot \mu_{ia,j}(\mu_{i,a} \otimes \mathbf{1}_j) = \mu_{i,j}(\mathbf{1}_i \otimes \mu_j^L) = \mu_{i,j}(\mu_i^R \otimes \mathbf{1}_j) = (\mathfrak{r}_i \otimes_A \mathbf{1}_j) \cdot \mu_{ia,j}(\mu_{i,a} \otimes \mathbf{1}_j),$$

which proves triangle axiom by universal property. ■

We conclude the following:

Theorem 3.13. With the $*$-bifunctor $\boxtimes_A$, the associativity isomorphisms, the unit object, and the left and right multiplications by unit defined above, $\mathrm{BIM}^{\mathrm{u}}(A)$ is a C^*-tensor category.

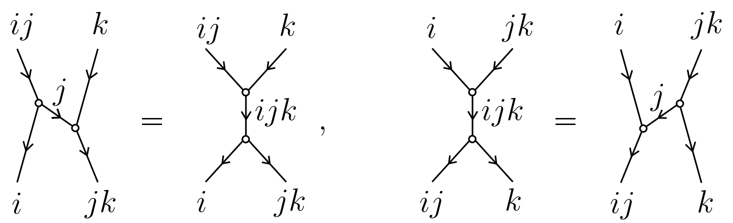
In the following, we identify different ways of unitary tensor products via associativity isomorphisms, and identify W_{ai} with W_i and W_{ia} with W_i via $\mathfrak{l}_i$ and $\mathfrak{r}_i$ respectively. Then $\mathrm{BIM}^{\mathrm{u}}(A)$ can be treated as if it is a strict C^*-tensor category. We have $(ij)k = i(jk)$, both denoted by ijk, and also $ai = i = ia$. Thus the $\mathfrak{A}_{i,j,k}, \mathfrak{l}_i, \mathfrak{r}_i$ are all identity morphisms. Therefore $\mu_L^i = \mu_{a,i}, \mu_R^i = \mu_{i,a}$, and in particular $\mu = \mu_{a,a}$. Moreover, equation (3.6) now reads



(3.10)

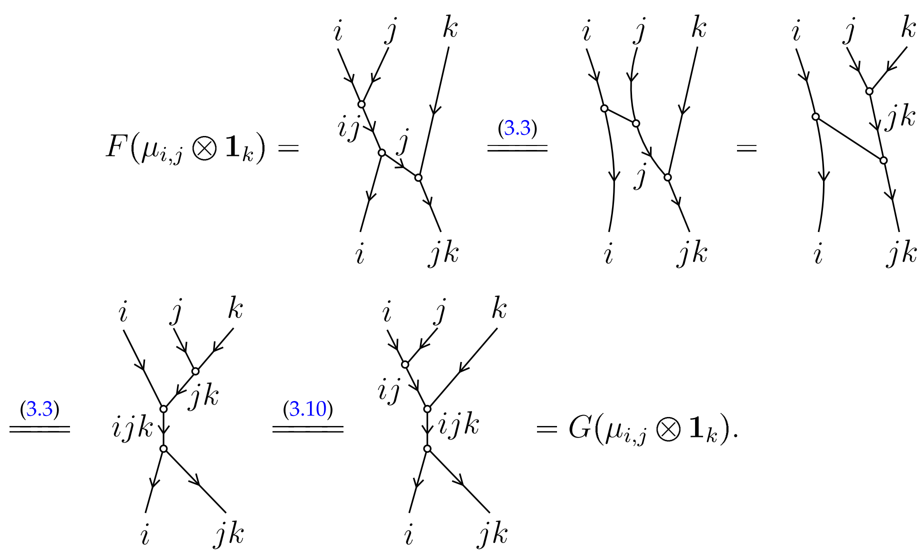
which means that the left action $i \curvearrowright j$ and the right action $j \curvearrowleft k$ commute. These two actions indeed commute *adjointly*, as indicated below.

Theorem 3.14 (Frobenius relations).



(3.11)

Proof. Let F and G be the 1st and the 2nd item of (3.11). One computes



Therefore $F = d_A^{-1} F(\mu_{i,j} \otimes \mathbf{1}_k)(\mu_{i,j} \otimes \mathbf{1}_k)^* = d_A^{-1} G(\mu_{i,j} \otimes \mathbf{1}_k)(\mu_{i,j} \otimes \mathbf{1}_k)^* = G$, which proves the 1st equation. The 2nd one is the adjoint of the 1st one. ■

The above Frobenius relations are the decisive property that makes a tensor product theory unitary. They are indeed closely related to the locality axiom of the categorical extensions of conformal nets [21] where the *adjoint commutativity* of left and right actions plays a central role. In subsequent works we will relate the C^*-tensor categories of conformal net extensions and unitary VOA extensions using Frobenius relations.

We close this section by showing that the C^*-tensor structure of $\mathrm{BIM}^{\mathrm{u}}(A)$ is independent of the choice of unitary tensor products. Suppose that we have two systems of unitary tensor products: for any objects W_i, W_j in $\mathrm{BIM}^{\mathrm{u}}(A)$ we have unitary tensor products $(W_{i\times j}, \mu_{i,j}), (W_{i\bullet j}, \eta_{i,j})$ of W_i, W_j over A, which define (strict) C^*-tensor categories $(\mathrm{BIM}^{\mathrm{u}}(A), \boxtimes_A), (\mathrm{BIM}^{\mathrm{u}}(A), \boxdot_A)$. Tensor products of morphisms are written as $\otimes_A, \odot_A$, respectively. By uniqueness up to unitaries, there exists a unique unitary $\Phi_{i,j} \in \mathrm{Hom}_A(W_{i\times j}, W_{i\bullet j})$ such that

$$\eta_{i,j} = \Phi_{i,j}\mu_{i,j}. \tag{3.12}$$

Proposition 3.15. Φ is functorial: for any unitary A-modules $W_i, W_{i'}, W_j, W_{j'}$ and any $F \in \mathrm{Hom}_A(W_i, W_{i'}), G \in \mathrm{Hom}_A(W_j, W_{j'})$,

$$\Phi_{i',j'}(F \otimes_A G) = (F \odot_A G)\Phi_{i,j}. \tag{3.13}$$

Proof. We compute

$$\Phi_{i',j'}(F \otimes_A G)\mu_{i,j} = \Phi_{i',j'} \cdot \mu_{i',j'}(F \otimes G) = \eta_{i',j'}(F \otimes G) = (F \odot_A G)\eta_{i,j} = (F \odot_A G)\Phi_{i,j} \cdot \mu_{i,j}.$$

Thus the desired equation is proved by universal property. ■

Theorem 3.16. Φ induces an equivalence of C^*-tensor categories $(\mathrm{BIM}^{\mathrm{u}}(A), \boxtimes_A) \simeq (\mathrm{BIM}^{\mathrm{u}}(A), \boxdot_A)$. More precisely, for any unitary A-bimodules W_i, W_j, W_k,

- The following diagram commutes.

$$\begin{tikzcd} W_{i\times j\times k} \arrow[r, "\mathbf{1}_i\otimes_A\Phi_{j,k}"] \arrow[d, "\Phi_{i,j}\otimes_A\mathbf{1}_k"'] & W_{i\times(j\bullet k)} \arrow[d, "\Phi_{i,j\bullet k}"'] \\ W_{(i\bullet j)\times k} \arrow[r, "\Phi_{i\bullet j,k}"] & W_{i\bullet j\bullet k} \end{tikzcd} . \tag{3.14}$$

- The following two morphisms equal $\mathbf{1}_i$.

$$W_i = W_{a\times i} \xrightarrow{\Phi_{a,i}} W_{a\bullet i} = W_i, \tag{3.15}$$

$$W_i = W_{i\times a} \xrightarrow{\Phi_{i,a}} W_{i\bullet a} = W_i. \tag{3.16}$$

Proof. To prove the 1st condition, we calculate

$$\Phi_{i,j\bullet k}(\mathbf{1}_i\otimes_A\Phi_{j,k})\cdot\mu_{i,j\times k}(\mathbf{1}_i\otimes\mu_{j,k})=\Phi_{i,j\bullet k}\cdot\mu_{i,j\bullet k}(\mathbf{1}_i\otimes\Phi_{j,k})(\mathbf{1}_i\otimes\mu_{j,k})$$
$$=\Phi_{i,j\bullet k}\cdot\mu_{i,j\bullet k}(\mathbf{1}_i\otimes\Phi_{j,k}\cdot\mu_{j,k})\xlongequal{(3.12)}\eta_{i,j\bullet k}(\mathbf{1}_i\otimes\eta_{j,k})\xlongequal{(3.10)}\eta_{i\bullet j,k}(\eta_{i,j}\otimes\mathbf{1}_k),$$

and also

$$\Phi_{i\bullet j,k}(\Phi_{i,j}\otimes_A\mathbf{1}_k)\cdot\mu_{i,j\times k}(\mathbf{1}_i\otimes\mu_{j,k})\xlongequal{(3.10)}\Phi_{i\bullet j,k}(\Phi_{i,j}\otimes_A\mathbf{1}_k)\cdot\mu_{i\times j,k}(\mu_{i,j}\otimes\mathbf{1}_k)$$
$$=\Phi_{i\bullet j,k}\cdot\mu_{i\bullet j,k}(\Phi_{i,j}\otimes\mathbf{1}_k)(\mu_{i,j}\otimes\mathbf{1}_k)=\Phi_{i\bullet j,k}\cdot\mu_{i\bullet j,k}(\Phi_{i,j}\mu_{i,j}\otimes\mathbf{1}_k)\xlongequal{(3.12)}\eta_{i\bullet j,k}(\eta_{i,j}\otimes\mathbf{1}_k).$$

This proves (3.14) since $(W_{i\times j\times k},\mu_{i,j\times k}(\mathbf{1}_i\otimes\mu_{j,k}))$ is a unitary tensor product of W_i,W_j,W_k over A by proposition 3.9.

Let μ_L^i,μ_R^i be the left and right actions of W_i. Then under the identifications $i=a\times i=i\times a=a\bullet i=i\bullet a$, we know by equations (3.8) that $\mu_{a,i},\eta_{a,i}$ both equal μ_L^i, and $\mu_{i,a},\eta_{i,a}$ both equal μ_R^i. Thus, by (3.12), $\Phi_{a,i}=\mathbf{1}_i=\Phi_{i,a}$. ■

3.3 Dualizable unitary bimodules

Let (W_i,μ_L^i,μ_R^i) be a unitary A-bimodule as usual. Recall that W_i is called $\mathcal{C}$-dualizable if it is dualizable as an object in $\mathcal{C}$. The notion of $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizability is understood in a similar way. In [37] section 6.2, it was shown that if W_i is $\mathcal{C}$-dualizable, then it is $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizable. The converse is also true by proposition 6.13 of [37] (Note that although A is assumed in [37] to be standard, the results there also apply to the non-standard case since any C^*-Frobenius algebra is isomorphic to a standard Q-system by [38] theorem 2.9.). In this section, we give a slightly different proof of this result; see theorem 3.18.

We first assume that W_i is $\mathcal{C}$-dualizable. Our proof of the $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizability is motivated by [30] lemma 1.16 and [8] proposition 2.77. Notice that $W_{a\boxtimes i\boxtimes a}=W_a\boxtimes W_i\boxtimes W_a$ is naturally a unitary A-bimodule with left action $\mu\otimes\mathbf{1}_i\otimes\mathbf{1}_a$ and right action $\mathbf{1}_a\otimes\mathbf{1}_j\otimes\mu$. Moreover, $\mu_{LR}^i:=\mu_R^i(\mu_L^i\otimes\mathbf{1}_a)=\mu_L^i(\mathbf{1}_a\otimes\mu_R^i):W_a\boxtimes W_i\boxtimes W_a\to W_i$ is an A-bimodule morphism, and $d_A^{-1}\mu_{LR}^i$ is a partial isometry with range $\mathbf{1}_i$. Therefore W_i is a sub A-bimodule of $W_a\boxtimes W_i\boxtimes W_a$, and hence it suffices to show that $W_a\boxtimes W_i\boxtimes W_a$ is $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizable.

Let $W_{\bar{i}}$ be a dual object of W_i, and choose $\mathrm{ev}_{i,\bar{i}},\mathrm{ev}_{\bar{i},i}$ of $W_i,W_{\bar{i}}$. Then a natural candidate of dual bimodule of $W_a\boxtimes W_i\boxtimes W_a$ is $W_a\boxtimes W_{\bar{i}}\boxtimes W_a$. Let us first understand their unitary tensor product over A. For this purpose, we choose a general unitary A-

bimodule (W_j, μ_L^j, μ_R^j), and check easily using proposition 3.3 that $(W_a \boxtimes W_i \boxtimes W_a \boxtimes W_j \boxtimes W_a, \mathbf{1}_a \otimes \mathbf{1}_i \otimes \mu \otimes \mathbf{1}_i \otimes \mathbf{1}_a)$ is a unitary tensor product of $W_a \boxtimes W_i \boxtimes W_a$ and $W_a \boxtimes W_j \boxtimes W_a$ over A. We thus define the unitary tensor product of $W_a \boxtimes W_i \boxtimes W_a$ and $W_a \boxtimes W_{\bar{i}} \boxtimes W_a$ over A in this way. Briefly, $(a \boxtimes i \boxtimes a)(a \boxtimes \bar{i} \boxtimes a) = a \boxtimes i \boxtimes a \boxtimes \bar{i} \boxtimes a$, and similarly, $(a \boxtimes \bar{i} \boxtimes a)(a \boxtimes i \boxtimes a) = a \boxtimes \bar{i} \boxtimes a \boxtimes i \boxtimes a$.

We now define $\mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a} \in \mathrm{Hom}_A(W_{a\boxtimes i\boxtimes a\boxtimes \bar{i}\boxtimes a}, W_a)$ and $\mathrm{ev}^A_{a\boxtimes \bar{i}\boxtimes a, a\boxtimes i\boxtimes a} \in \mathrm{Hom}_A(W_{a\boxtimes \bar{i}\boxtimes a\boxtimes i\boxtimes a}, W_a)$ by

$$\mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a} = \mu(\mathbf{1}_a \otimes (\mathrm{ev}_{i,\bar{i}}(\mathbf{1}_i \otimes \iota^* \otimes \mathbf{1}_{\bar{i}})) \otimes \mathbf{1}_a),$$

$$\mathrm{ev}^A_{a\boxtimes \bar{i}\boxtimes a, a\boxtimes i\boxtimes a} = \mu(\mathbf{1}_a \otimes (\mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}} \otimes \iota^* \otimes \mathbf{1}_i)) \otimes \mathbf{1}_a).$$

Since we also have $(a \boxtimes \bar{i} \boxtimes a)(a \boxtimes i \boxtimes a)(a \boxtimes \bar{i} \boxtimes a) = a \boxtimes \bar{i} \boxtimes a \boxtimes i \boxtimes a \boxtimes \bar{i} \boxtimes a$, we check using (3.5) and the associativity of A that

$$\mathbf{1}_{a\boxtimes \bar{i}\boxtimes a} \otimes_A \mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a} = \mathbf{1}_a \otimes \mathbf{1}_{\bar{i}} \otimes \mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a},$$

$$\mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a} \otimes_A \mathbf{1}_{a\boxtimes i\boxtimes a} = \mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a} \otimes \mathbf{1}_i \otimes \mathbf{1}_a,$$

and that $\mathrm{ev}^A_{a\boxtimes \bar{i}\boxtimes a, a\boxtimes i\boxtimes a}$ satisfies similar relations. Using these equations it is straightforward to check that $\mathrm{ev}^A_{a\boxtimes i\boxtimes a, a\boxtimes \bar{i}\boxtimes a}$ and $\mathrm{ev}^A_{a\boxtimes \bar{i}\boxtimes a, a\boxtimes i\boxtimes a}$ are evaluations in $\mathrm{BIM}^{\mathrm{u}}(A)$ of $W_a \boxtimes W_i \boxtimes W_a$ and $W_a \boxtimes W_{\bar{i}} \boxtimes W_a$, which proves that $W_a \boxtimes W_i \boxtimes W_a$ and hence W_i are $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizable.

To prove the inverse direction we need the following lemma.

Lemma 3.17. Let $W_{\bar{i}}$ be a unitary A-module, not yet known to be dual to W_i. Suppose that we have morphisms $\mathrm{ev}^A_{i,\bar{i}} \in \mathrm{Hom}_A(W_{i\bar{i}}, W_a)$ and $\mathrm{ev}^A_{\bar{i},i} \in \mathrm{Hom}_A(W_{\bar{i}i}, W_a)$. Set

$$\mathrm{ev}_{i,\bar{i}} = \iota^* \mathrm{ev}^A_{i,\bar{i}} \cdot \mu_{i,\bar{i}}, \qquad \mathrm{ev}_{\bar{i},i} = \iota^* \mathrm{ev}^A_{\bar{i},i} \cdot \mu_{\bar{i},i}, \tag{3.17}$$

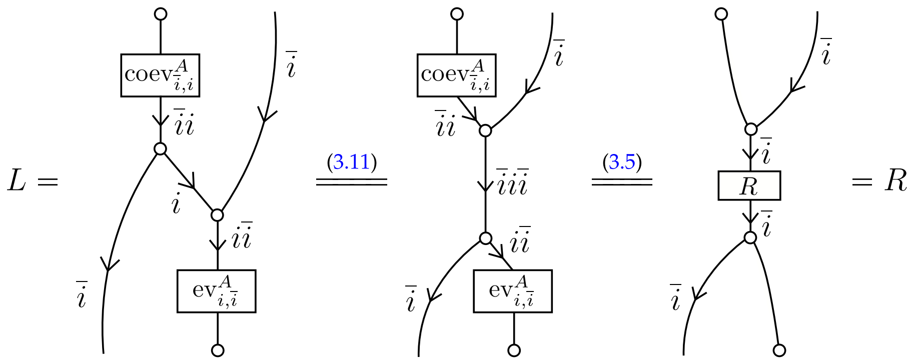
and also set $\mathrm{coev}^A_{i,\bar{i}} = (\mathrm{ev}^A_{i,\bar{i}})^*, \mathrm{coev}^A_{\bar{i},i} = (\mathrm{ev}^A_{\bar{i},i})^*, \mathrm{coev}_{i,\bar{i}} = (\mathrm{ev}_{i,\bar{i}})^*, \mathrm{coev}_{\bar{i},i} = (\mathrm{ev}_{\bar{i},i})^*$. Then

$$(\mathbf{1}_{\bar{i}} \otimes \mathrm{ev}_{i,\bar{i}})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}}) = (\mathbf{1}_{\bar{i}} \otimes_A \mathrm{ev}^A_{i,\bar{i}})(\mathrm{coev}^A_{\bar{i},i} \otimes_A \mathbf{1}_{\bar{i}}), \tag{3.18}$$

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathbf{1}_i)(\mathbf{1}_i \otimes \mathrm{coev}_{\bar{i},i}) = (\mathrm{ev}^A_{i,\bar{i}} \otimes_A \mathbf{1}_i)(\mathbf{1}_i \otimes_A \mathrm{coev}^A_{\bar{i},i}). \tag{3.19}$$

We say that $\mathrm{ev}_{i,\bar{i}}$ and $\mathrm{ev}^A_{i,\bar{i}}$, $\mathrm{ev}_{\bar{i},i}$ and $\mathrm{ev}^A_{\bar{i},i}$ are **correlated** if they satisfy (3.17).

Proof. The two equations can be proved in a similar way, so we only prove the 1st one. Let L, R be respectively the left and right hand sides of (3.18). Then



■

Now if W_i is $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizable, then we can find a unitary A-bimodule $W_{\bar{i}}$ dual to W_i, and evaluations $\mathrm{ev}^A_{i,\bar{i}}, \mathrm{ev}^A_{\bar{i},i}$ of $W_i, W_{\bar{i}}$ in $\mathrm{BIM}^{\mathrm{u}}(A)$. Define $\mathrm{ev}_{i,\bar{i}}, \mathrm{ev}_{\bar{i},i}$ by equations (3.17). Then equations (3.18) and (3.19) imply that $\mathrm{ev}_{i,\bar{i}}, \mathrm{ev}_{\bar{i},i}$ are evaluations of $W_i, W_{\bar{i}}$ in $\mathcal{C}$. Thus W_i is $\mathcal{C}$-dualizable. This finishes the proof of the following theorem.

Theorem 3.18. If W_i is a unitary A-bimodule, then W_i is $\mathcal{C}$-dualizable if and only if W_i is $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizable. Moreover, one can choose correlated evaluations $\mathrm{ev}, \mathrm{ev}^A$ in $\mathcal{C}$ and $\mathrm{BIM}^{\mathrm{u}}(A)$.

By the above theorem, we will no longer distinguish between $\mathcal{C}$- and $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizability. Using the same argument as lemma 3.17 one also proves that under correlated evaluations, the $\mathcal{C}$-transposes and $\mathrm{BIM}^{\mathrm{u}}(A)$-transposes of a unitary A-bimodule morphism F are equal. Therefore the symbols ${}^{\vee}F$ and $F^{\vee}$ are defined unambiguously. Compare [37] lemma 6.10.

Proposition 3.19. Let W_i, W_j be dualizable unitary A-bimodules. Choose dual objects $W_{\bar{i}}, W_{\bar{j}}$, and evaluations $\mathrm{ev}, \mathrm{ev}^A$ (with suitable subscripts) in $\mathcal{C}$ and $\mathrm{BIM}^{\mathrm{u}}(A)$, respectively. Assume that $\mathrm{ev}, \mathrm{ev}^A$ are correlated. Then for any $F \in \mathrm{Hom}_A(W_i, W_j)$, its transposes in $\mathcal{C}$ are the same as those in $\mathrm{BIM}^{\mathrm{u}}(A)$. More precisely, we have

$$\begin{aligned}&(\mathbf{1}_{\bar{i}} \otimes \mathrm{ev}_{j,\bar{j}})(\mathbf{1}_{\bar{i}} \otimes F \otimes \mathbf{1}_{\bar{j}})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{j}})\\ =&(\mathbf{1}_{\bar{i}} \otimes_A \mathrm{ev}^A_{j,\bar{j}})(\mathbf{1}_{\bar{i}} \otimes_A F \otimes_A \mathbf{1}_{\bar{j}})(\mathrm{coev}^A_{\bar{i},i} \otimes_A \mathbf{1}_{\bar{j}}),\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&(\mathrm{ev}_{\bar{j},j}\otimes\mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{j}}\otimes F\otimes\mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{j}}\otimes\mathrm{coev}_{i,\bar{i}})\\
=&(\mathrm{ev}^A_{\bar{j},j}\otimes_A\mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{j}}\otimes_A F\otimes_A\mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{j}}\otimes_A\mathrm{coev}^A_{i,\bar{i}}).
\end{aligned}
\tag{3.21}
$$

Recall that for any $F\in\mathrm{End}(W_i)$, one can define scalars $\mathrm{Tr}_L(F),\mathrm{Tr}_R(F)$ such that $\mathrm{ev}_{i,\bar{i}}(F\otimes\mathbf{1}_{\bar{i}})\mathrm{coev}_{i,\bar{i}}=\mathrm{Tr}_L(F)\mathbf{1}_0$ and $\mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}}\otimes F)\mathrm{coev}_{\bar{i},i}=\mathrm{Tr}_R(F)\mathbf{1}_0$. If A is **simple** in the sense that $\mathrm{End}_A(W_a)=\mathbb{C}\mathbf{1}_a$, and $F\in\mathrm{End}_A(W_i)$, one can similarly define scalars $\mathrm{Tr}^A_L(F),\mathrm{Tr}^A_R(F)$ such that $\mathrm{ev}^A_{i,\bar{i}}(F\otimes_A\mathbf{1}_{\bar{i}})\mathrm{coev}^A_{i,\bar{i}}=\mathrm{Tr}^A_L(F)\mathbf{1}_a$ and $\mathrm{ev}^A_{\bar{i},i}(\mathbf{1}_{\bar{i}}\otimes_A F)\mathrm{coev}^A_{\bar{i},i}=\mathrm{Tr}_R(F)\mathbf{1}_a$. In the case that ev^A and ev are correlated, these two traces satisfy very simple relations:

Proposition 3.20. If A is a simple Q-system, $W_i,W_{\bar{i}}$ are mutually dual unitary A-bimodules, and the ev and ev^A for W_i are correlated, then for any $F\in\mathrm{End}_A(W_i)$, we have

$$
\mathrm{Tr}_L(F)=D_A\mathrm{Tr}^A_L(F),\qquad \mathrm{Tr}_R(F)=D_A\mathrm{Tr}^A_R(F).
\tag{3.22}
$$

As a consequence, ev^A are standard if the correlated ev are so.

Proof. We only prove the relation for left traces.

$$
\begin{aligned}
&\mathrm{Tr}_L(F)\mathbf{1}_0=\mathrm{ev}_{i,\bar{i}}(F\otimes\mathbf{1}_{\bar{i}})\mathrm{coev}_{i,\bar{i}}\xlongequal{(3.17)}\iota^*\mathrm{ev}^A_{i,\bar{i}}\cdot\mu_{i,\bar{i}}(F\otimes\mathbf{1}_{\bar{i}})(\mu_{i,\bar{i}})^*\mathrm{coev}^A_{i,\bar{i}}\iota\\
=&\iota^*\mathrm{ev}^A_{i,\bar{i}}(F\otimes_A\mathbf{1}_{\bar{i}})\cdot\mu_{i,\bar{i}}(\mu_{i,\bar{i}})^*\mathrm{coev}^A_{i,\bar{i}}\iota=d_A\iota^*\mathrm{ev}^A_{i,\bar{i}}(F\otimes_A\mathbf{1}_{\bar{i}})\cdot\mathrm{coev}^A_{i,\bar{i}}\iota\\
=&d_A\mathrm{Tr}^A_L(F)\iota^*\iota=D_A\mathrm{Tr}^A_L(F)\mathbf{1}_0.
\end{aligned}
$$

■

Note that a simple C^*-Frobenius algebra is always a simple Q-system, since $\mu\mu^*$ is in $\mathrm{End}_A(W_a)$, which must be a scalar and hence proves the specialness. Examples of simple Q-systems include haploid C^*-Frobenius algebras, since in general we have $\dim\mathrm{Hom}(W_0,W_a)=\dim\mathrm{End}_{A,-}(W_a)\geq\dim\mathrm{End}_A(W_a)$ (cf. [38] remark 2.7-(1)). Recall that haploid C^*-Frobenius algebras are also standard ($D_A=d_a$) by proposition 2.20. As a consequence, the $\mathcal{C}$-algebra A_U associated to a unitary VOA extension U is haploid and hence a simple standard Q-system.

Construction of dual bimodules and correlated ev^A

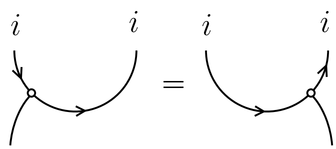
In the remaining part of this section we assume that A is standard. Then for a dualizable unitary A-bimodule (W_i,μ^i_L,μ^i_R) one can explicitly construct the dual bimodule and ev^A

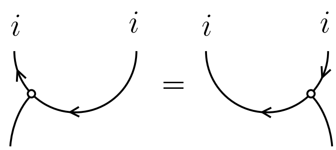
following [30] figures 9–11 or [39] section 4.1. This construction will be used in the next section to understand the ribbon structure of the unitary representation category of A.

Since A is now standard, $\mathrm{ev}_{a,a} := \iota^*\mu$ is a standard evaluation of W_a. Choose an object $W_{\bar{i}}$ in $\mathcal{C}$ dual to W_i, and choose standard ev for $W_i, W_{\bar{i}}$. Recall convention 2.4. Motivated by corollary 1.12, we define

$$\mu_L^{\bar{i}} = ((\mu_R^i)^*)^\vee, \qquad \mu_R^{\bar{i}} = ((\mu_L^i)^*)^\vee. \tag{3.23}$$

Then using graphical calculus it is not hard to verify that $(W_{\bar{i}}, \mu_L^{\bar{i}}, \mu_R^{\bar{i}})$ is a unitary A-bimodule. (Note that the standardness is used to verify the unitarity.) Moreover, using the above definition, and noting that $(\cdot)^\vee = {}^\vee(\cdot)$, one checks that

$\mathfrak{e}_{i,\bar{i}} :=$  $\in \mathrm{Hom}_A(W_i \boxtimes W_{\bar{i}}, W_a)$, (3.24)

$\mathfrak{e}_{\bar{i},i} :=$  $\in \mathrm{Hom}_A(W_{\bar{i}} \boxtimes W_i, W_a)$, (3.25)

and that $\mathfrak{e}_{i,\bar{i}}\Psi_{i,\bar{i}} = 0 = \mathfrak{e}_{\bar{i},i}\Psi_{\bar{i},i}$. Therefore there exist $\mathrm{ev}^A_{i,\bar{i}} \in \mathrm{Hom}_A(W_{i,\bar{i}}, W_a), \mathrm{ev}^A_{\bar{i},i} \in \mathrm{Hom}_A(W_{\bar{i},i}, W_a)$ satisfying

$$\mathfrak{e}_{i,\bar{i}} = \mathrm{ev}^A_{i,\bar{i}}\mu_{i,\bar{i}}, \qquad \mathfrak{e}_{\bar{i},i} = \mathrm{ev}^A_{\bar{i},i}\mu_{\bar{i},i}. \tag{3.26}$$

By unit property, $\mathrm{ev}^A_{i,\bar{i}}$ and $\mathrm{ev}_{i,\bar{i}}$, $\mathrm{ev}^A_{\bar{i},i}$ and $\mathrm{ev}_{\bar{i},i}$ are correlated. Therefore, by lemma 3.17, $\mathrm{ev}^A_{i,\bar{i}}$ and $\mathrm{ev}^A_{\bar{i},i}$ are evaluations of $W_i, W_{\bar{i}}$ in $\mathrm{BIM}^{\mathrm{u}}(A)$. If, moreover, A is simple, then ev^A are standard, and $\mathrm{Tr}_L(F) = \mathrm{Tr}_R(F) = d_a\mathrm{Tr}^A_L(F) = d_a\mathrm{Tr}^A_R(F)$ for any $F \in \mathrm{End}_A(W_i)$ by proposition 3.20. By the uniqueness up to unitaries of standard evaluations, the values of traces are independent of the choice of standard evaluations. Therefore we have (cf. [30] theorem 1.18 and [37] proposition 6.9.):

Theorem 3.21. If A is a simple and standard Q-system, W_i is a dualizable unitary A-bimodule, and $\mathrm{Tr} := \mathrm{Tr}_L = \mathrm{Tr}_R$ and $\mathrm{Tr}^A := \mathrm{Tr}^A_L = \mathrm{Tr}^A_R$ are defined using (not necessarily correlated) standard ev and standard ev^A, respectively. Then $\mathrm{Tr}(F) = d_a\mathrm{Tr}^A(F)$, where d_a is the ($\mathcal{C}$-)quantum dimension of W_a. In particular, the $\mathcal{C}$-quantum dimension of W_i equals d_a multiplied by the $\mathrm{BIM}^{\mathrm{u}}(A)$-quantum dimension of W_i.

3.4 Braiding and ribbon structures

In this section, $\mathcal{C}$ is a braided C^*-tensor category with (unitary) braiding ß and simple W_0, and A is a commutative Q-system in $\mathcal{C}$. Let $\mathrm{Rep}^{\mathrm{u}}(A)$ be the C^*-category of single-valued unitary left A-modules. As discussed in section 2.5, single-valued unitary left A-modules admits a canonical unitary bimodule structure, $\mathrm{Rep}^{\mathrm{u}}(A)$ is a full C^*-subcategory of $\mathrm{BIM}^{\mathrm{u}}(A)$, and $\mathrm{Hom}_{A,-}, \mathrm{Hom}_{-,A}, \mathrm{Hom}_A$ are the same for $\mathrm{Rep}^{\mathrm{u}}(A)$. If W_i, W_j are in $\mathrm{Rep}^{\mathrm{u}}(A)$, (W_k, μ_L^k, μ_R^k) is in $\mathrm{BIM}^{\mathrm{u}}(A)$, and $\alpha \in \mathrm{Hom}_A(W_i \boxtimes W_j, W_k)$ satisfies $\alpha\Psi_{i,j} = 0$, then one can show easily using graphical calculus that $\mu_L^k ß_{k,a}(\alpha \otimes \mathbf{1}_a) = \mu_R^k(\alpha \otimes \mathbf{1}_a)$. Now we choose a unitary tensor product $(W_{ij}, \mu_{i,j})$ of W_i, W_j over A, where W_{ij} is a unitary A-bimodule with left and right actions μ_L^{ij}, μ_R^{ij}. Set $W_k = W_{ij}, \alpha = \mu_{i,j}$. Then we have $\mu_L^{ij} ß_{ij,a} = \mu_R^{ij}$ since $(\alpha \otimes \mathbf{1}_a)(\alpha \otimes \mathbf{1}_a)^* = (\mu_{i,j} \otimes \mathbf{1}_a)(\mu_{i,j} \otimes \mathbf{1}_a)^* = d_A \mathbf{1}_{ij} \otimes \mathbf{1}_a$. Similar argument shows $\mu_L^{ij} ß_{a,ij}^{-1} = \mu_R^{ij}$. Therefore W_{ij} is single-valued with left and right actions related by ß. We conclude that $\mathrm{Rep}^{\mathrm{u}}(A)$ is closed under unitary tensor products. In other words, $\mathrm{Rep}^{\mathrm{u}}(A)$ is a full C^*-tensor subcategory of $\mathrm{BIM}^{\mathrm{u}}(A)$.

Proposition 3.22 (cf. [30] theorem 1.15). If A is standard, and W_i is an object in $\mathrm{Rep}^{\mathrm{u}}(A)$, then W_i is $\mathrm{Rep}^{\mathrm{u}}(A)$-dualizable if and only if W_i is $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizable (equivalently, $\mathcal{C}$-dualizable).

Proof. We have seen in theorem 3.18 that $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizability and $\mathcal{C}$-dualizability are the same. $\mathrm{Rep}^{\mathrm{u}}(A)$-dualizability clearly implies $\mathrm{BIM}^{\mathrm{u}}(A)$-dualizability. Now assume that W_i is $\mathcal{C}$-dualizable. In section 3.3 we have constructed a unitary A-bimodule $W_{\bar{i}}$ dual to W_i. It is easy to check that the left and right actions of $W_{\bar{i}}$ defined by (3.23) are related by the braiding ß of $\mathcal{C}$. In particular, $W_{\bar{i}}$ is an object in $\mathrm{Rep}^{\mathrm{u}}(A)$. Thus W_i is $\mathrm{Rep}^{\mathrm{u}}(A)$-dualizable. ∎

Braiding

We now define braiding for $\mathrm{Rep}^{\mathrm{u}}(A)$. Let W_i, W_j be objects in $\mathrm{Rep}^{\mathrm{u}}(A)$, and let $(W_{ij}, \mu_{i,j})$ and $(W_{ji}, \mu_{j,i})$ be respectively the unitary tensor products over A of W_i, W_j and W_j, W_i used to define the tensor structure of $\mathrm{BIM}^{\mathrm{u}}(A)$. Since the braiding of $\mathcal{C}$ is unitary, using proposition 3.3 one easily shows that $(W_{ij}, \mu_{j,i} ß_{i,j})$ is also a unitary tensor product of W_j, W_i over A. Hence there exists a unique unitary $ß_{i,j}^A \in \mathrm{End}_A(W_{ij}, W_{ji})$ such that

$$\mu_{j,i} ß_{i,j} = ß_{i,j}^A \mu_{i,j}. \tag{3.27}$$

Theorem 3.23. $(\mathrm{Rep}^{\mathrm{u}}(A), \boxtimes_A, ß^A)$ is a braided C^*-tensor category.

Proof. The hexagon axioms

$$(ß^A_{i,k} \otimes_A \mathbf{1}_j)(\mathbf{1}_i \otimes_A ß^A_{j,k}) = ß^A_{ij,k},$$
$$((ß^A_{k,i})^{-1} \otimes_A \mathbf{1}_j)(\mathbf{1}_i \otimes_A (ß^A_{k,j})^{-1}) = (ß^A_{k,ij})^{-1}$$

(for all W_i, W_j, W_k in $\mathrm{Rep}^{\mathrm{u}}(A)$) can be proved in a similar way as pentagon axiom (proposition 3.11): one shows that both sides are equal when multiplied from the right by $\mu_{i,jk}(\mathbf{1}_i \otimes \mu_{j,k}) = \mu_{ij,k}(\mu_{i,j} \otimes \mathbf{1}_k)$. ■

Now assume that we have two systems of unitary tensor products $(W_{i\times j}, \mu_{i,j})$, $(W_{i\bullet j}, \eta_{i,j})$, which define two braided C^*-tensor categories $(\mathrm{Rep}^{\mathrm{u}}(A), \boxtimes_A, ß^A)$ and $(\mathrm{Rep}^{\mathrm{u}}(A), \boxdot_A, \sigma^A)$. By theorem 3.16, the functorial unitary Φ defined by (3.12) induces an equivalence of the C^*-tensor categories. Indeed, it also preserves the braidings:

Theorem 3.24. The functorial unitary Φ defined by (3.12) induces an equivalence of the braided C^*-tensor categories $(\mathrm{Rep}^{\mathrm{u}}(A), \boxtimes_A, ß^A) \simeq (\mathrm{Rep}^{\mathrm{u}}(A), \boxdot_A, \sigma^A)$, which means that Φ satisfies the two conditions of theorem 3.16, together with the condition that for any objects W_i, W_j in $\mathrm{Rep}^{\mathrm{u}}(A)$,

$$\Phi_{j,i} ß^A_{i,j} = \sigma^A_{i,j} \Phi_{i,j}. \tag{3.28}$$

Proof. One verifies that $\Phi_{j,i} ß^A_{i,j} \mu_{i,j} = \sigma^A_{i,j} \eta_{i,j} = \sigma^A_{i,j} \Phi_{i,j} \mu_{i,j}$. ■

Ribbon structures

Let us now assume that $\mathcal{C}$ is rigid, which means that any object of $\mathcal{C}$ is dualizable. By [35] proposition 2.4, there is a canonical twist operator $\vartheta = \vartheta_i \in \mathrm{End}(W_i)$ for any object W_i in $\mathcal{C}$: Choose $W_{\bar{i}}$ dual to W_i, standard $\mathrm{ev}_{i,\bar{i}}, \mathrm{ev}_{\bar{i},i}$ and corresponding coev for $W_i, W_{\bar{i}}$. Then by standardness of ev one can show

$$\begin{aligned}\vartheta_i &:= (\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_i)(\mathbf{1}_{\bar{i}} \otimes ß_{i,i})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_i)\\ &= (\mathbf{1}_i \otimes \mathrm{ev}_{i,\bar{i}})(ß_{i,i} \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_i \otimes \mathrm{coev}_{i,\bar{i}}).\end{aligned} \tag{3.29}$$

(Note that by uniqueness up to unitaries of standard ev, ϑ_i is independent of the choice of standard evaluations.) By this relation, ϑ is unitary. Moreover, ϑ defines a ribbon structure on $\mathcal{C}$ (i.e., ϑ commutes with all morphisms, $\vartheta_{i\boxtimes j} = (\vartheta_i \otimes \vartheta_j) ß_{j,i} ß_{i,j}$, and $\vartheta_i^\vee = \vartheta_{\bar{i}}$).

Then $(\mathcal{C}, \boxtimes, ß, \vartheta)$ is a rigid C^*-ribbon category. Using the definition of ϑ_i, one easily shows

$$\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i} ß_{i,\bar{i}} (\vartheta_i \otimes \mathbf{1}_{\bar{i}}), \tag{3.30}$$

which completely determines the morphism ϑ_i. In the case of $\mathrm{Rep}^{\mathrm{u}}(V)$, we have shown in [19] section 7.3 (especially equation (7.30), which relies on [18] formula (1.41)) that $e^{2\mathbf{i}\pi L_0}$ satisfies the above equation. Thus the twist $\vartheta = e^{2\mathbf{i}\pi L_0}$ defined in the end of section 1.1 is the canonical twist of the braided C^*-fusion category (unitary braided fusion category) $\mathrm{Rep}^{\mathrm{u}}(V)$.

Suppose now that A is haploid. By the commutativity of A, haploidness is equivalent to simpleness since $\mathrm{End}_{A,-}(W_a) = \mathrm{End}_A(W_a)$. A is also standard by proposition 2.20. Therefore, by theorem 3.22, $\mathrm{Rep}^{\mathrm{u}}(A)$ is rigid. Thus $\mathrm{Rep}^{\mathrm{u}}(A)$ also admits a canonical twist ϑ^A under which $\mathrm{Rep}^{\mathrm{u}}(A)$ becomes a C^*-ribbon category. The twist satisfies

$$\begin{aligned}\vartheta_i :=& (\mathrm{ev}^A_{\bar{i},i} \otimes_A \mathbf{1}_i)(\mathbf{1}_{\bar{i}} \otimes_A ß^A_{i,i})(\mathrm{coev}^A_{\bar{i},i} \otimes_A \mathbf{1}_i)\\ =&(\mathbf{1}_i \otimes_A \mathrm{ev}^A_{i,\bar{i}})(ß^A_{i,i} \otimes_A \mathbf{1}_{\bar{i}})(\mathbf{1}_i \otimes_A \mathrm{coev}^A_{i,\bar{i}}),\end{aligned} \tag{3.31}$$

where $W_{\bar{i}}$ is an object of $\mathrm{Rep}^{\mathrm{u}}(A)$ dual to W_i, and $\mathrm{ev}^A_{i,\bar{i}}, \mathrm{ev}^A_{\bar{i},i}$ are standard evaluations for $W_i, W_{\bar{i}}$. We now show that the ribbon structures of $\mathcal{C}$ and $\mathrm{Rep}^{\mathrm{u}}(A)$ are compatible.

Theorem 3.25. Suppose that $\mathcal{C}$ is a rigid braided C^*-tensor category, A is a haploid commutative Q-system in $\mathcal{C}$, and ϑ and ϑ^A are the canonical unitary twists of $\mathcal{C}$ and $\mathrm{Rep}^{\mathrm{u}}(A)$, respectively. Then $\vartheta_i = \vartheta^A_i$ for any object W_i in $\mathrm{Rep}^{\mathrm{u}}(A)$.

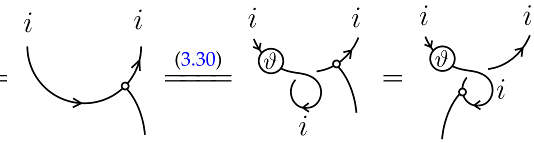
As an immediate consequence, W_a has trivial ($\mathcal{C}$-)twist since $\vartheta^A_a = \mathbf{1}_a$.

Proof. Choose any W_i in $\mathrm{Rep}^{\mathrm{u}}(A)$. Using the definition of twist one checks easily that $\vartheta_i \in \mathrm{End}_A(W_i)$. Let $W_{\bar{i}}$ be a dual object in $\mathcal{C}$, equip $W_{\bar{i}}$ with a unitary A-bimodule structure by (3.23), and use equations (3.24), (3.25), and (3.26) to define standard $\mathrm{Rep}^{\mathrm{u}}(A)$-evaluations ev^A correlated to standard $\mathcal{C}$-evaluations ev for $W_i, W_{\bar{i}}$. We now prove $\vartheta_i = \vartheta^A_i$ by showing

$$\mathrm{ev}^A_{i,\bar{i}} = \mathrm{ev}^A_{\bar{i},i} ß^A_{i,\bar{i}} (\vartheta_i \otimes_A \mathbf{1}_{\bar{i}}). \tag{3.32}$$

We compute

$$\mathrm{ev}^{A}_{i,\bar{i}}\cdot\mu_{i,\bar{i}} \overset{(3.26)}{=\!=} \mathfrak{e}_{i,\bar{i}} \overset{(3.24)}{=\!=}$$



$$\overset{(3.25),\,(3.26)}{=\!=\!=} \mathrm{ev}^{A}_{\bar{i},i}\cdot\mu_{\bar{i},i}\text{ß}_{i,\bar{i}}(\vartheta_i\otimes\mathbf{1}_{\bar{i}}) \overset{(3.27)}{=\!=} \mathrm{ev}^{A}_{\bar{i},i}\text{ß}^{A}_{i,\bar{i}}\cdot\mu_{i,\bar{i}}(\vartheta_i\otimes\mathbf{1}_{\bar{i}}) \overset{(3.5)}{=\!=} \mathrm{ev}^{A}_{\bar{i},i}\text{ß}^{A}_{i,\bar{i}}(\vartheta_i\otimes_A\mathbf{1}_{\bar{i}})\cdot\mu_{i,\bar{i}}.$$

■

An MTC is called *unitary* if it is a (rigid) braided C^*-tensor category, and if its twist is the canonical unitary twist associated to the rigid braided C^*-tensor structure.

Corollary 3.26. Let $(\mathcal{C},\boxtimes,\text{ß},\vartheta)$ be a unitary MTC. If A is a haploid commutative Q-system in $\mathcal{C}$, then $(\mathrm{Rep}^{\mathrm{u}}(A),\boxtimes_A,\text{ß}^A,\vartheta)$ is also a unitary MTC.

Proof. We have shown that A has trivial $\mathcal{C}$-twist. Thus by [30] theorem 4.5, $\mathrm{Rep}^{\mathrm{u}}(A)$ is an MTC (Our $\mathrm{Rep}^{\mathrm{u}}(A)$ is written as $\mathrm{Rep}^{0}(A)$ in [30].). By the above theorem, when restricted to $\mathrm{Rep}^{\mathrm{u}}(A)$, ϑ is the canonical unitary twist ϑ^A of $\mathrm{Rep}^{\mathrm{u}}(A)$. Thus $\mathrm{Rep}^{\mathrm{u}}(A)$ is a unitary MTC. ■

3.5 Complete unitarity of unitary VOA extensions

Recall that V is a CFT-type, regular, and completely unitary VOA. Let U be a CFT-type unitary extension of V, and let $A_U = (W_a,\mu,\iota)$ be the corresponding standard commutative Q-system. (Note that the trivial twist condition for A is now redundant by theorem 3.25.) Recall by theorem 2.30 that any unitary U-module is naturally a single-valued unitary left A_U-module, and any single-valued unitary left A_U-module can be regarded as a unitary U-module. Thus $\mathrm{Rep}^{\mathrm{u}}(U)$ is naturally equivalent to $\mathrm{Rep}^{\mathrm{u}}(A_U)$ as C^*-categories. Note that U is also regular (equivalently, rational and C_2-cofinite [1]) by the proof of [34] theorem 4.13 (Although [34] theorem 4.13 only discusses orbifold-type extensions, the argument there is quite general and clearly applies to the general case. Note that the nonzeroness of quantum dimensions required in that theorem is obvious in the unitary case.). Thus, just as V, the tensor category of U-modules is (rigid and) modular. In the following, we shall show that U is completely unitary, which implies that $\mathrm{Rep}^{\mathrm{u}}(U)$ is a unitary MTC. Moreover, we shall show that the unitary MTCs $\mathrm{Rep}^{\mathrm{u}}(U)$ and $\mathrm{Rep}^{\mathrm{u}}(A_U)$ are naturally equivalent.

Let W_i, W_j, W_k be unitary U-modules, which can be regarded respectively as unitary A_U-bimodules with left actions $\mu_L^i, \mu_L^j, \mu_L^k$ and right actions $\mu_R^i, \mu_R^j, \mu_R^k$ related by ß. Recall (1.1). Then $\mathcal{Y}_{\mu_L^i}, \mathcal{Y}_{\mu_L^j}, \mathcal{Y}_{\mu_L^k}$ are the vertex operators of U on W_i, W_j, W_k respectively. We let $\mathcal{V}_U\binom{k}{i\,j}$ be the vector space of type $\binom{k}{i\,j}$ intertwining operators of U. Again, $\mathcal{V}\binom{k}{i\,j}$ denotes the vector space of type $\binom{k}{i\,j}$ intertwining operators of V. Since any intertwining operator of U is also an intertwining operator of V, $\mathcal{V}_U\binom{k}{i\,j}$ is a subspace of $\mathcal{V}\binom{k}{i\,j}$. We give a categorical interpretation of $\mathcal{V}_U\binom{k}{i\,j}$. Note that $W_i \boxtimes W_j$ is naturally a unitary A_U-bimodule, with left and actions defined by the left action of W_i and the right action of W_j.

Lemma 3.27. Any $\alpha \in \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_k)$ satisfies $\alpha\Psi_{i,j} = 0$.

Proof. This is easy to prove using graphical calculus and the fact that the left actions of W_i, W_j, W_k are related by ß to the right ones. ■

Proposition 3.28 ([8] section 3.4). The map $\mathcal{Y} : \mathrm{Hom}(W_i \boxtimes W_j, W_k) \xrightarrow{\simeq} \mathcal{V}\binom{k}{i\,j}$, $\alpha \mapsto \mathcal{Y}_\alpha$ (see section 1.1) restricts to an isomorphism $\mathcal{Y} : \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_k) \xrightarrow{\simeq} \mathcal{V}_U\binom{k}{i\,j}$.

Proof. We sketch the proof here; details can be found in the reference provided. Choose any $\alpha \in \mathrm{Hom}(W_i \boxtimes W_j, W_k)$. Then $\mathcal{Y}_\alpha$ being an intertwining operator of U means precisely that $\mathcal{Y}_\alpha$ satisfies the Jacobi identity with the vertex operator of U. By contour integrals, the Jacobi identity is well known to be equivalent to the fusion relations

$$\mathcal{Y}_{\mu_L^k}(u,z)\mathcal{Y}_\alpha(w^{(i)},\zeta) = \mathcal{Y}_\alpha(\mathcal{Y}_{\mu_L^i}(u,z-\zeta)w^{(i)},\zeta), \tag{3.33}$$

$$\mathcal{Y}_\alpha(w^{(i)},\zeta)\mathcal{Y}_{\mu_L^k}(u,z) = \mathcal{Y}_\alpha(\mathcal{Y}_{\mu_L^i}(u,z-\zeta)w^{(i)},\zeta) \tag{3.34}$$

for any $u \in U = W_a, w^{(i)} \in W_i$, where $0 < |z-\zeta| < |\zeta| < |z|$ and $\arg(z-\zeta) = \arg\zeta = \arg z$ in the 1st equation, and $0 < |\zeta - z| < |z| < |\zeta|$ and $\arg z = \arg\zeta = \pi + \arg(z-\zeta)$ in the 2nd one. (See for instance [18] proposition 2.13.) By [18] proposition 2.9, (3.34) is equivalent to the fusion relation

$$\mathcal{Y}_\alpha(w^{(i)},\zeta)\mathcal{Y}_{\mu_L^k}(u,z) = \mathcal{Y}_\alpha(\mathcal{Y}_{\mu_R^i}(w^{(i)},\zeta-z)u,z). \tag{3.35}$$

The categorical interpretations of (3.33) and (3.35) are respectively $\alpha \in \mathrm{Hom}_{A_U,-}(W_i \boxtimes W_j, W_k)$ and $\alpha\Psi_{i,j} = 0$, which are clearly equivalent to that $\alpha \in \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_k)$. ■

Recall the definition of VOA modules in section 1.1. Recall by theorem 2.31 that any irreducible U-module admits a unitary structure. We choose a representative W_t for each equivalence class $[W_t]$ of irreducible unitary U-modules (equivalently, irreducible unitary single-valued left A_U-modules), and let all these W_t form a set $\mathcal{E}_U$. That $W_t \in \mathcal{E}_U$ is abbreviated to $t \in \mathcal{E}_U$. We also assume that the vacuum U-module W_a is in $\mathcal{E}_U$. Then the tensor product of U-modules W_i, W_j is

$$W_{ij} \equiv W_i \boxtimes_U W_j = \bigoplus_{t \in \mathcal{E}_U} \mathcal{V}_U \binom{t}{i\,j}^* \otimes W_t. \tag{3.36}$$

We choose an inner product Λ_U for any $\mathcal{V}_U\binom{t}{i\,j}^*$, and assume that the above direct sum is orthogonal. The vertex operator for W_{ij} is $\bigoplus_t \mathbf{1} \otimes \mathcal{Y}_{\mu_L^t}$, where $\mu_L^t \in \mathrm{Hom}_{A_U}(W_a \boxtimes W_t, W_t)$ is the left action of the A_U-bimodule W_t.

Define a U-intertwining operator $\mathcal{Y}_{\mu_{i,j}}$ of type $\binom{ij}{i\,j} = \binom{W_{ij}}{W_i W_j}$, such that for any $w^{(i)} \in W_i, w^{(j)} \in W_j, t \in \mathcal{E}_U, \mathcal{Y}_\alpha \in \mathcal{V}_U\binom{t}{i\,j}$, and $w^{(\bar{t})} \in W_{\bar{t}}$ (the contragredient unitary U-module of W_t),

$$\langle \mathcal{Y}_{\mu_{i,j}}(w^{(i)}, z) w^{(j)}, \mathcal{Y}_\alpha \otimes w^{(\bar{t})} \rangle = \langle \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)}, w^{(\bar{t})} \rangle. \tag{3.37}$$

To write the above definition more explicitly, we choose a basis $\Upsilon_{i,j}^t$ of the vector space $\mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_t)$. Then $\{\mathcal{Y}_\alpha : \alpha \in \Upsilon_{i,j}^k\}$ is a basis of $\mathcal{V}_U\binom{t}{i\,j}$ whose dual basis is denoted by $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Upsilon_{i,j}^k\}$. Then

$$\mathcal{Y}_{\mu_{i,j}}(w^{(i)}, z) w^{(j)} = \sum_{t \in \mathcal{E}_U} \sum_{\alpha \in \Upsilon_{i,j}^t} \check{\mathcal{Y}}^\alpha \otimes \mathcal{Y}_\alpha(w^{(i)}, z) w^{(j)}. \tag{3.38}$$

Note that $\mu_{i,j} \in \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_{ij})$. Then the above relation can also be written as

$$\mu_{i,j} = \sum_{t \in \mathcal{E}_U} \sum_{\alpha \in \Upsilon_{i,j}^t} \check{\mathcal{Y}}^\alpha \otimes \alpha. \tag{3.39}$$

By lemma 3.27 we have $\mu_{i,j}\Psi_{i,j} = 0$. We claim that $\mathcal{Y}_{\mu_{i,j}}$ satisfies the universal property that for any unitary U-module W_k and any $\mathcal{Y}_\alpha$ in $\mathcal{V}_U\binom{k}{i\,j}$ (equivalently, $\alpha \in \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_k)$) there exists a unique U-module homomorphism $\widetilde{\alpha} : W_{ij} \to W_k$ (equivalently, $\widetilde{\alpha} \in \mathrm{Hom}_{A_U}(W_{ij}, W_k)$) such that $\mathcal{Y}_\alpha = \widetilde{\alpha}\mathcal{Y}_{\mu_{i,j}}$ (equivalently, $\alpha = \widetilde{\alpha}\mu_{i,j}$ by (1.2)). Indeed, since the vector space $\mathrm{Hom}_U(W_{ij}, W_k)$ of U-module morphisms from W_{ij} to W_k is naturally identified with $\mathrm{Hom}_{A_U}(W_{ij}, W_k)$, just as (1.1), we have a natural isomorphism of vector spaces

$$\widetilde{\mathcal{Y}} : \mathrm{Hom}_U(W_{ij}, W_k) = \mathrm{Hom}_{A_U}(W_{ij}, W_k) \to \mathcal{V}_U\binom{k}{i\,j}, \qquad \widetilde{\alpha} \mapsto \widetilde{\mathcal{Y}}_{\widetilde{\alpha}}. \tag{3.40}$$

It is easy check for any $\widetilde{\alpha} \in \mathrm{Hom}_{A_U}(W_{ij}, W_k)$ that

$$\widetilde{\mathcal{Y}}_{\widetilde{\alpha}} = \widetilde{\alpha}\mathcal{Y}_{\mu_{i,j}}, \tag{3.41}$$

which by (1.2) also equals $\mathcal{Y}_{\widetilde{\alpha}\mu_{i,j}}$. Now, by proposition 3.28, for any $\alpha \in \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_k)$, we can find an $\widetilde{\alpha}$ satisfying

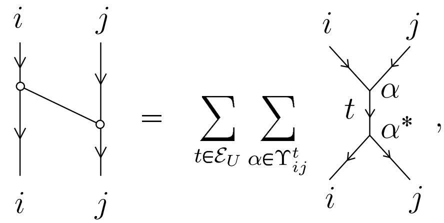
$$\mathcal{Y}_\alpha = \widetilde{\mathcal{Y}}_{\widetilde{\alpha}}. \tag{3.42}$$

Thus we have $\mathcal{Y}_\alpha = \widetilde{\mathcal{Y}}_{\widetilde{\alpha}} = \widetilde{\alpha}\mathcal{Y}_{\mu_{i,j}} = \mathcal{Y}_{\widetilde{\alpha}\mu_{i,j}}$, which shows $\alpha = \widetilde{\alpha}\mu_{i,j}$. Recalling definition 3.1, we conclude that $(W_{ij}, \mu_{i,j})$ is a tensor product of the A_U-bimodules W_i, W_j over A_U. Indeed, under suitable choice of Λ_U, the tensor products become unitary:

Theorem 3.29. There exists for each $t \in \mathcal{E}_U$ a unique inner product Λ_U on the vector space $\mathcal{V}_U\binom{t}{i\,j}^*$ such that $(W_{ij}, \mu_{i,j})$ becomes a unitary tensor product of the A_U-bimodules W_i, W_j over A_U. Moreover, Λ_U is the invariant sesquilinear form of U (cf. section 1.3).

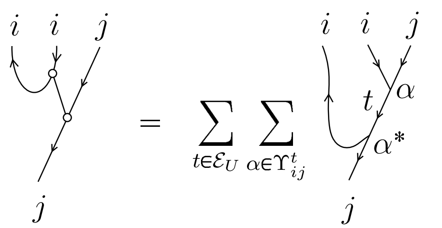
Proof. Let $(W_{i\bullet j}, \eta_{i,j})$ be a unitary tensor product of W_i, W_j over A_U. By the 1st half of the proof of theorem 3.4, tensor products of unitary bimodules of a Q-system are unique up to multiplications by invertible morphisms. Thus there exists an invertible $K \in \mathrm{Hom}_{A_U}(W_{ij}, W_{i\bullet j})$ such that $\eta_{i,j} = K\mu_{i,j}$. Therefore, the decomposition of $W_{i\bullet j}$ into irreducible single-valued left A_U-modules is the same as that of W_{ij}, which takes the form (3.36). Now, using linear algebra, one can easily find an inner product Λ_U on any $\mathcal{V}_U\binom{t}{i\,j}^*$, such that K becomes unitary. Then the tensor product $(W_{i\bullet j}, \eta_{i,j})$ defined by such Λ_U is clearly unitary. This proves the existence of Λ_U. The uniqueness of Λ_U follows from the uniqueness up to unitaries of the unitary tensor products of A_U-bimodules (theorem 3.4).

Now assume that $(W_{ij}, \mu_{i,j})$ is a unitary tensor product. We show that Λ_U is the invariant sesquilinear form. Assume that for each $t \in \mathcal{E}_U$, $\Upsilon^t_{i,j}$ is chosen in such a way that $\{\check{\mathcal{Y}}^{\alpha} : \alpha \in \Upsilon_{i,j}^{k}\}$ is an orthonormal basis of $\mathcal{V}_U\binom{t}{i\ j}^*$ under Λ_U. Then by $\chi_{i,j} = \mu_{i,j}^*\mu_{i,j}$ and equation (3.39), we have



(3.43)

and hence



(3.44)

which by theorem 1.11 implies for any $w_1^{(i)}, w_2^{(i)} \in W_i$ the fusion relation

$$\mathcal{Y}_{\mu_L^j}\left(\mathcal{Y}_{(\mu_R^i)^\dagger}(\overline{w_2^{(i)}}, z-\zeta)w_1^{(i)}, \zeta\right) = \sum_{t\in\mathcal{E}_U}\sum_{\alpha\in\Upsilon_{ij}^t} \mathcal{Y}_{\alpha^\dagger}(\overline{w_2^{(i)}}, z)\mathcal{Y}_\alpha(w_1^{(i)}, \zeta). \tag{3.45}$$

Recall that $\mathcal{Y}_{\mu_L^j}$ is the vertex operator of U on W_j. Since $\mu_R^i = \mu_L^i ß_{a,i}$, we have $\mathcal{Y}_{\mu_R^i} = B_+\mathcal{Y}_{\mu_L^i}$ by (1.8), which shows that $\mathcal{Y}_{\mu_R^i} \in \mathcal{V}_U\binom{i}{i\ a}$ is the creation operator of the U-module W_i. (See section (1.1) for the definition of creation and annihilation operators). Thus by (1.28), $\mathcal{Y}_{(\mu_R^i)^\dagger} \in \mathcal{V}_U\binom{a}{\bar{i}\ i}$ is the annihilation operator of the U-module W_j. Therefore, by definition 1.7, we see that (3.45) is the fusion relation that defines invariant sesquilinear forms for U. This shows that $\{\check{\mathcal{Y}}^{\alpha} : \alpha \in \Upsilon_{i,j}^{k}\}$ is also an orthonormal basis of $\mathcal{V}_U\binom{t}{i\ j}^*$ under the invariant sesquilinear form, which proves that the latter is positive definite and equals Λ_U. ∎

Theorem 3.30. Let V be a CFT-type, regular, and completely unitary VOA, and let U be a CFT-type unitary VOA extension of V. Then:

- U is also (regular and) completely unitary.
- Under the natural identification of $\mathrm{Rep}^{\mathrm{u}}(U)$ and $\mathrm{Rep}^{\mathrm{u}}(A_U)$ as C^*-categories, the monoidal, braiding, and ribbon structures of $\mathrm{Rep}^{\mathrm{u}}(U)$ agree with those of

$(\mathrm{Rep}^{\mathrm{u}}(A_U), \boxtimes_{A_U}, ß^{A_U}, \vartheta^{A_U})$ defined by the system of unitary tensor products $(W_{ij}, \mu_{i,j})$ (for any W_i, W_j in $\mathrm{Rep}^{\mathrm{u}}(A_U)$) as constructed in (3.36) and (3.37) under the invariant inner product Λ_U.

By "natural identification," we mean that each unitary U-module W_i is identified with the corresponding single-valued unitary left A_U-module W_i; a homomorphism $F : W_i \to W_j$ of unitary U-modules is identified with F, considered as a homomorphism of A_U-bimodules.

Proof. As mentioned at the beginning of this section, the regularity of U is proved in [34]. By theorems 2.31 and 3.29, U is completely unitary. Hence $\mathrm{Rep}^{\mathrm{u}}(U)$ is a unitary MTC. That $\mathrm{Rep}^{\mathrm{u}}(U)$ and $\mathrm{Rep}^{\mathrm{u}}(A_U)$ share the same tensor and braiding structures is proved in [8]. In order for this paper to be self-contained, we sketch the proof as follows.

For any objects W_i, W_j, W_k, let $\mathcal{A}_{i,j,k} : W_{(ij)k} = (W_i \boxtimes_U W_j) \boxtimes_U W_k \to W_{i(jk)} = W_i \boxtimes_U (W_j \boxtimes_U W_k)$ be the associativity isomorphism of $\mathrm{Rep}^{\mathrm{u}}(U)$. It is shown in [21] proposition 4.3 that under the identification of $W_{(ij)k}$ and $W_{i(jk)}$ via $\mathcal{A}_{i,j,k}$, one has the fusion relation

$$\mathcal{Y}_{\mu_{i,jk}}(w^{(i)}, z)\mathcal{Y}_{\mu_{j,k}}(w^{(j)}, \zeta) = \mathcal{Y}_{\mu_{ij,k}}(\mathcal{Y}_{\mu_{i,j}}(w^{(i)}, z-\zeta)w^{(j)}, \zeta) \tag{3.46}$$

for any $w^{(i)} \in W_i, w^{(j)} \in W_j$. This means that relation (3.10) holds under the identification via $\mathcal{A}_{i,j,k}$. But we know that due to equation (3.6), the same relations also hold under the identification via $\mathfrak{A}_{i,j,k}$, the associativity isomorphism of $\mathrm{Rep}^{\mathrm{u}}(A_U)$. Thus $\mathcal{A}_{i,j,k} = \mathfrak{A}_{i,j,k}$.

Next, we know that in $\mathrm{Rep}^{\mathrm{u}}(U)$, the identification $W_{ai} \simeq W_i$ is via the vertex operator $\mathcal{Y}_{\mu_L^i}$ of U on W_i, and the identification $W_{ia} \simeq W_i$ is via the creation operator of the U-module W_i, which is $\mathcal{Y}_{\mu_R^i}$, as argued at the end of the proof of theorem 3.29. Define $\mathfrak{l}_i \in \mathrm{Hom}_{A_U}(W_{ai}, W_i)$ and $\mathfrak{r}_i \in \mathrm{Hom}_{A_U}(W_{ia}, W_i)$ using equations (3.8). Then, from section 3.2, we know that $\mathfrak{l}_i$ and $\mathfrak{r}_i$ define respectively the equivalences $W_{ai} \simeq W_i$ and $W_{ia} \simeq W_i$ in $\mathrm{Rep}^{\mathrm{u}}(A_U)$ (as a full C^*-tensor subcategory of $\mathrm{BIM}^{\mathrm{u}}(A_U)$). On the other hand, by (3.42) we have $\mathcal{Y}_{\mu_L^i} = \widetilde{\mathcal{Y}}_{\mathfrak{l}_i}$. Therefore $\mathfrak{l}_i : W_{ai} = W_a \boxtimes_U W_i \to W_i$ is the U-module homomorphism corresponding to the vertex operator of the U-module W_i. Thus, by the definition of the monoidal structures of VOA tensor categories (see section 1.1), $\mathfrak{l}_i$ also defines the equivalence $W_{ai} \simeq W_i$ in $\mathrm{Rep}^{\mathrm{u}}(U)$. Similarly, $\mathfrak{r}_i$ is the U-module morphism corresponding to the creation operator of W_i. Hence it defines the equivalence $W_{ia} \simeq W_i$ in $\mathrm{Rep}^{\mathrm{u}}(U)$. We have now proved that the (C^*-)monoidal structure of $\mathrm{Rep}^{\mathrm{u}}(U)$ agrees with that of $\mathrm{Rep}^{\mathrm{u}}(A_U)$.

Let $ß^U_{i,j} \in \mathrm{Hom}_U(W_{ij}, W_{ji})$ be the braiding of $W_i \boxtimes_U W_j$ in Rep^U. We want to show that $ß^U$ equals the braiding $ß^{A_U}$ of $\mathrm{Rep}^{\mathrm{u}}(A_U)$. By (3.27) it suffices to check that for any object W_k in $\mathrm{Rep}^{\mathrm{u}}(A_U)$ and any $\widetilde{\alpha} \in \mathrm{Hom}_{A_U}(W_{ij}, W_k)$,

$$\widetilde{\alpha}\mu_{j,i}ß_{i,j} = \widetilde{\alpha}ß^U_{i,j}\mu_{i,j}, \tag{3.47}$$

where ß is the braiding of $\mathrm{Rep}^{\mathrm{u}}(V)$. Set $\alpha = \widetilde{\alpha}\mu_{j,i} \in \mathrm{Hom}_{A_U}(W_i \boxtimes W_j, W_k)$. Then by (1.8), $\mathcal{Y}_{\alpha ß_{i,j}} = B_+\mathcal{Y}_\alpha$, and similarly $\widetilde{\mathcal{Y}}_{\widetilde{\alpha}ß^U_{i,j}} = B_+\widetilde{\mathcal{Y}}_{\widetilde{\alpha}}$. Note that the braiding B_+ defined for V-intertwining operators and for U-intertwining operators are the same since U and V have the same Virasoro operators. We now compute

$$\mathcal{Y}_{\widetilde{\alpha}ß^U_{i,j}\mu_{i,j}} \xlongequal{(1.2)} \widetilde{\alpha}ß^U_{i,j}\mathcal{Y}_{\mu_{i,j}} \xlongequal{(3.41)} \widetilde{\mathcal{Y}}_{\widetilde{\alpha}ß^U_{i,j}} = B_+\widetilde{\mathcal{Y}}_{\widetilde{\alpha}} \xlongequal{(3.42)} B_+\mathcal{Y}_\alpha = \mathcal{Y}_{\alpha ß_{i,j}} = \mathcal{Y}_{\widetilde{\alpha}\mu_{j,i}ß_{i,j}}.$$

Finally, for both categories, the twists are defined by the rigid braided C^*-tensor structures. Therefore the ribbon structures agree. ■

3.6 Applications

To use theorem 3.30 in its full power, we first prove the complete unitarity for another type of extensions (which do not preserve conformal vectors).

Proposition 3.31. Let V and $\widetilde{V}$ be CFT-type and regular VOAs. Then $V \otimes \widetilde{V}$ is (regular and) completely unitary if and only if both V and $\widetilde{V}$ are completely unitary. If this is true then $\mathrm{Rep}^{\mathrm{u}}(V \otimes \widetilde{V})$ is the tensor product of $\mathrm{Rep}^{\mathrm{u}}(V)$ and $\mathrm{Rep}^{\mathrm{u}}(\widetilde{V})$.

Proof. Clearly $V \otimes \widetilde{V}$ is CFT type. Note that $V \otimes \widetilde{V}$ is also regular by [13] proposition 3.3. Assume first of all that V and $\widetilde{V}$ are completely unitary. By [11] proposition 2.9, $V \otimes \widetilde{V}$ is unitary. By [15] theorem 4.7.4, any irreducible $V \otimes \widetilde{V}$-module is the tensor product of an irreducible V-module and an irreducible $\widetilde{V}$-module, which by the strong unitarity of V and $\widetilde{V}$ are unitarizable. Therefore the $V \otimes \widetilde{V}$-module is also unitarizable, and hence $V \otimes \widetilde{V}$ is strongly unitary.

We now show that $V \otimes \widetilde{V}$ is completely unitary. Choose unitary V-modules W_i, W_j and unitary $\widetilde{V}$-modules $W_{\widetilde{i}}, W_{\widetilde{j}}$. Choose W_t in $\mathcal{E}$. Let also $\widetilde{\mathcal{E}}$ be a complete set of representatives of irreducible $\widetilde{V}$-modules, and choose any $W_{\widetilde{t}}$ in $\mathcal{E}$. Choose bases $\Xi^t_{i,j}$ of $\mathcal{V}\binom{t}{i\,j}$ and $\widetilde{\Xi}^{\widetilde{t}}_{\widetilde{i},\widetilde{j}}$ of $\widetilde{\mathcal{V}}\binom{\widetilde{t}}{\widetilde{i}\,\widetilde{j}}$ (the vector space of type $\binom{\widetilde{t}}{\widetilde{i}\,\widetilde{j}}$ intertwining operators of $\widetilde{V}$) so that their dual bases $\{\check{\mathcal{Y}}^\alpha : \alpha \in \Xi^t_{i,j}\}$ and $\{\check{\mathcal{Y}}^{\widetilde{\alpha}} : \widetilde{\alpha} \in \widetilde{\Xi}^{\widetilde{t}}_{\widetilde{i},\widetilde{j}}\}$ are orthonormal under the invariant inner products in $\mathcal{V}{t\choose i\,j}^*$ and $\widetilde{\mathcal{V}}{\widetilde{t}\choose \widetilde{i}\,\widetilde{j}}^*$, respectively. Then equation (1.30) holds for V with $\Lambda(\check{\mathcal{Y}}^\alpha|\check{\mathcal{Y}}^\beta)=\delta_{\alpha,\beta}$, and a similar relation holds for $\widetilde{V}$. Let $Y_{j\otimes\widetilde{j}}$ be the vertex operator of the $V\otimes\widetilde{V}$-module $W_j\otimes W_{\widetilde{j}}$, and let $\mathcal{Y}_{\mathrm{ev}_{\overline{i\otimes\widetilde{i}},i\otimes\widetilde{i}}}$ be the annihilation operator of $W_i\otimes W_{\widetilde{i}}$. Then $Y_{j\otimes\widetilde{j}}=Y_j\otimes Y_{\widetilde{j}}$ and $\mathcal{Y}_{\mathrm{ev}_{\overline{i\otimes\widetilde{i}},i\otimes\widetilde{i}}}=\mathcal{Y}_{\mathrm{ev}_{\overline{i},i}}\otimes\mathcal{Y}_{\mathrm{ev}_{\overline{\widetilde{i}},\widetilde{i}}}$. Set $\mathcal{Y}_{\alpha\otimes\widetilde{\alpha}}=\mathcal{Y}_\alpha\otimes\mathcal{Y}_{\widetilde{\alpha}}$, which is a type ${t\otimes\widetilde{t}\choose i\otimes\widetilde{i}\;j\otimes\widetilde{j}}$ intertwining operator of $V\otimes\widetilde{V}$ if $\alpha\in\Xi^t_{i,j}$, $\widetilde{\alpha}\in\widetilde{\Xi}^{\widetilde{t}}_{\widetilde{i},\widetilde{j}}$. Then for any $w_1^{(i)},w_2^{(i)}\in W_i$, $w_3^{(\widetilde{i})},w_4^{(\widetilde{i})}\in W_{\widetilde{i}}$ we have the fusion relation

$$\begin{aligned}&Y_{j\otimes\widetilde{j}}\left(\mathcal{Y}_{\mathrm{ev}_{\overline{i\otimes\widetilde{i}},i\otimes\widetilde{i}}}\left(\overline{w_2^{(i)}}\otimes\overline{w_4^{(\widetilde{i})}},z-\zeta\right)\left(w_1^{(i)}\otimes w_3^{(\widetilde{i})}\right),\zeta\right)\\&=\sum_{t\in\mathcal{E},\widetilde{t}\in\widetilde{\mathcal{E}}}\;\sum_{\alpha\in\Xi^t_{i,j},\widetilde{\alpha}\in\widetilde{\Xi}^{\widetilde{t}}_{\widetilde{i},\widetilde{j}}}\mathcal{Y}_{(\alpha\otimes\widetilde{\alpha})^\dagger}(\overline{w_2^{(i)}}\otimes\overline{w_4^{(\widetilde{i})}},z)\cdot\mathcal{Y}_{\alpha\otimes\widetilde{\alpha}}(w_1^{(i)}\otimes w_3^{(\widetilde{i})},\zeta).\end{aligned}\tag{3.48}$$

From this relations, we see that the invariant sesquilinear form Λ on $\mathcal{V}{t\otimes\widetilde{t}\choose i\otimes\widetilde{i}\;j\otimes\widetilde{j}}^*$ is positive. Moreover, by the non-degeneracy of this Λ, $\Xi^t_{i,j}\times\widetilde{\Xi}^{\widetilde{t}}_{\widetilde{i},\widetilde{j}}$ is a basis of $\mathcal{V}{t\otimes\widetilde{t}\choose i\otimes\widetilde{i}\;j\otimes\widetilde{j}}$ whose dual basis is therefore orthonormal in $\mathcal{V}{t\otimes\widetilde{t}\choose i\otimes\widetilde{i}\;j\otimes\widetilde{j}}^*$. Thus $\mathcal{V}{t\otimes\widetilde{t}\choose i\otimes\widetilde{i}\;j\otimes\widetilde{j}}=\mathcal{V}{t\choose i\,j}\otimes\widetilde{\mathcal{V}}{\widetilde{t}\choose\widetilde{i}\,\widetilde{j}}$, and the Λ on $\mathcal{V}{t\otimes\widetilde{t}\choose i\otimes\widetilde{i}\;j\otimes\widetilde{j}}^*$ equals $\Lambda\otimes\Lambda$ on $\mathcal{V}{t\choose i\,j}^*\otimes\widetilde{\mathcal{V}}{\widetilde{t}\choose\widetilde{i}\,\widetilde{j}}^*$. That $\mathrm{Rep}^{\mathrm{u}}(V\otimes\widetilde{V})=\mathrm{Rep}^{\mathrm{u}}(V)\otimes\mathrm{Rep}^{\mathrm{u}}(\widetilde{V})$ now follows easily. (It also follows from [2] theorem 2.10.)

We now prove the "only if" part. Assume that $V\otimes\widetilde{V}$ is completely unitary. We want to prove that V (or similarly $\widetilde{V}$) is completely unitary. Let $\widetilde{\Omega}$ be the vacuum vector of $\widetilde{V}$. Then V can be regarded as a (non-conformal) vertex subalgebra of $V\otimes\widetilde{V}$. The inner product on $V\otimes\widetilde{V}$ restricts to one on V, which makes V unitary. Now let W_i be an irreducible V-module. Write $\widetilde{V}=W_{\widetilde{0}}$ as the unitary vacuum $\widetilde{V}$-module. Then the $V\otimes\widetilde{V}$-module $W_i\otimes W_{\widetilde{0}}$ admits a unitary structure. The restriction of the inner product of $W_i\otimes W_{\widetilde{0}}$ to $W_i\otimes\widetilde{\Omega}$ produces a unitary structure on W_i (cf. [43] proposition 2.20).

Now choose unitary V-modules W_i,W_j, and choose W_t in $\mathcal{E}$ again. Notice the natural isomorphism $\mathcal{V}{t\choose i\,j}\xrightarrow{\simeq}\mathcal{V}{t\otimes\widetilde{0}\choose i\otimes\widetilde{0}\;j\otimes\widetilde{0}}$ sending $\mathcal{Y}_\alpha\in\mathcal{V}{t\choose i\,j}$ to $\mathcal{Y}_\alpha\otimes Y_{\widetilde{0}}$. Here $Y_{\widetilde{0}}$ is the vertex operator of $\widetilde{V}$ (on the vacuum module $W_{\widetilde{0}}$). Such map is clearly injective. It is also surjective, since any intertwining operator in $\mathcal{V}{t\otimes\widetilde{0}\choose i\otimes\widetilde{0}\;j\otimes\widetilde{0}}$ can be restricted to the subspaces $W_i\otimes\widetilde{\Omega},W_j\otimes\widetilde{\Omega},W_t\otimes\widetilde{\Omega}$ to produce the desired preimage. Note that the Λ on $\mathcal{V}{t\otimes\widetilde{0}\choose i\otimes\widetilde{0}\;j\otimes\widetilde{0}}^*$ is positive definite by the complete unitarity of $V\otimes\widetilde{V}$. Choose a basis in $\mathcal{V}{t\otimes\widetilde{0}\choose i\otimes\widetilde{0}\;j\otimes\widetilde{0}}$ whose dual basis is orthonormal under Λ. Using a suitable fusion relation, it is straightforward to check that the corresponding basis in $\mathcal{V}{t\choose i\,j}$ also has orthonormal dual basis in $\mathcal{V}{t\choose i\,j}^*$ under Λ. In particular, Λ is positive on $\mathcal{V}{t\choose i\,j}^*$. This proves the complete unitarity of V. ■

The above proposition implies a strategy of proving the completely unitarity of a non-conformal unitary extension U of V. Let V^c be the commutant of V in V^c (the coset subalgebra), which is unitary by [7]. Then U is a unitary (conformal) extension of $V \otimes V^c$ by [44] proposition 2.21. Now it suffices to show the regularity and complete unitarity of V and V^c, the proof of which might require a similar trick applied to V and V^c.

Corollary 3.32. Let V be a (finite) tensor product of $c < 1$ unitary Virasoro VOAs, affine unitary VOAs, and even lattice VOAs. Let U be a CFT-type unitary extension of V. Then U is regular and completely unitary. Consequently, the category of unitary U-modules is a unitary MTC.

Proof. The affine unitary VOAs of these types are regular by [13] and completely unitary by [19] theorem 8.4, [20] theorem 6.1, and [45] theorem 5.5. The $c < 1$ unitary Virasoro VOAs (resp. even lattice VOAs) are regular also by [13] and completely unitary by [19] theorem 8.1 (resp. [21] theorem 5.8). Therefore, by theorem 3.30 and proposition 3.31, CFT-type unitary extensions of their tensor products are also regular and completely unitary. ■

The above corollary is by no means in the most general form. For example, we know that W-algebras in discrete series of type A and E are completely unitary by [45] theorem 5.5. So one can definitely add these examples to the list in that corollary.

Corollary 3.33. Let U be a CFT-type unitary VOA with central charge $c < 1$. Then U is completely unitary. Consequently, the category of unitary U-modules is a unitary MTC.

Proof. By [11] theorem 5.1, U is a unitary extension of the unitary Virasoro VOA $L(c, 0)$. ■

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