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Regular vertex operator subalgebras and compressions of intertwining operators

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ABSTRACT

Let V be a vertex operator subalgebra of U. Assume that U, V, and its commutant V^c in U are CFT-type, self-dual, and regular VOAs. Assume also that the double commutant V^{cc} equals V. We prove that any intertwining operator of V is a compression of intertwining operators of U.

0. Introduction

In [21], Krauel-Miyamoto showed that if V is a vertex operator subalgebra of U, if U, V, and the commutant V^c are CFT-type, self-dual, and regular VOAs, and if $V^{cc} = V$, then any irreducible V-module appears in some irreducible U-module. For example, by [5,6,2], these assumptions are satisfied when we take $V \subset U$ to be $L_{k+1}(\mathfrak{g}) \subset L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})$, where k is a positive integer, $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra of type ADE, and $L_k(\mathfrak{g})$ is the corresponding (unitary) affine VOA. In this case, V^c is a discrete series principle W-algebra $\mathcal{W}_l(\mathfrak{g})$.

The above results have important applications to the unitarity problems in VOAs: For example, we can conclude that any irreducible $\mathcal{W}_l(\mathfrak{g})$-module is unitarizable since this is true for any unitary affine VOA. Moreover, using these results (together with the

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techniques developed in [23–25]), Tener showed in [26] that the modular tensor categories associated to all unitary affine VOAs and type AE discrete series W-algebras are unitary, and solved a longstanding problem in subfactor theory and algebraic quantum field theory: that the conformal nets associated to unitary affine VOAs and type ADE discrete series W-algebras are completely rational.

In this paper, we generalize the result of [21] to intertwining operators: We show that any intertwining operator of V is a compression of intertwining operators of U (Theorem 4.4). To be more precise, let W_I, W_J, W_K be (ordinary) U-modules, which can also be regarded as weak V-modules. Suppose that $\mathcal{Y}^U$ is a type $\binom{W_K}{W_I W_J}$ intertwining operator of U, and W_i, W_j, W_k are graded irreducible V-submodules of W_I, W_J, W_K respectively, then one can find $\lambda \in \mathbb{Q}$ such that z^λ times the restriction of $\mathcal{Y}^U$ to W_i, W_j, W_k is an intertwining operator $\mathcal{Y}$ of V. (Note that without the factor z^λ, the restriction itself may not satisfy the L_{-1}-derivative property.) We then say that $\mathcal{Y}$ is a **compression** of $\mathcal{Y}^U$. (See Definition 3.4 for more details.) Our main result of this article is that any intertwining operator of V can be written as a (finite) sum of those that are compressions of intertwining operators of U.

Our result can be applied to prove many important functional analytic properties for intertwining operators. One such property is the (polynomial) energy bounds condition [7,13,14], which says roughly that the smeared intertwining operators are bounded by L_0^n for some $n \geq 0$. Proving energy bounds condition for intertwining operators is a key step in relating the tensor structures of VOA modules and the corresponding conformal net modules; see [28,27,12,15]. On the other hand, one may deduce the energy bounds condition of the compressed intertwining operator $\mathcal{Y}$ from that of $\mathcal{Y}^U$. Since, by our main result, any intertwining operator of $L_k(\mathfrak{g})$ or $\mathcal{W}_l(\mathfrak{g})$ (when $\mathfrak{g}$ is of type ADE) is a compression of tensor products of intertwining operators of $L_1(\mathfrak{g})$, and since the latter were proved in [27] to be energy bounded, we can conclude that all intertwining operators of $L_k(\mathfrak{g})$ or $\mathcal{W}_l(\mathfrak{g})$ are energy bounded. This result will be used in [15] to show that if V is $L_k(\mathfrak{g})$ or $\mathcal{W}_l(\mathfrak{g})$, and if $\mathcal{A}$ is the corresponding conformal net, then the tensor and braid structures of the representation categories of V and $\mathcal{A}$ are compatible.

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1. Intertwining operators and tensor categories

Let $(V, Y, \mathbf{1}, \nu)$ be a vertex operator algebra (VOA) where $\mathbf{1}$ is the vacuum vector and ν is the conformal vector. For any $v \in V$, write $Y(v, z) = \sum_{n\in\mathbb{Z}} Y(v)_n z^{-n-1}$ where $Y(v)_n \in \mathrm{End}(V)$. Then $L_n := Y(\nu)_{n+1}$ satisfy the Virasoro relation

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}c,$$

where c is the central charge of V. We shall always assume that V is CFT-type, namely, V has L_0-grading $V = \bigoplus_{n\in\mathbb{N}} V(n)$ and $V(0) = \mathbb{C}\mathbf{1}$. We also assume that V is self-dual and regular (equivalently, self-dual, rational and C_2-cofinite [1]). Note that the self-dual condition is equivalent to the existence of a non-degenerate invariant bilinear form. As a consequence of CFT-type and being self-dual, V is simple. (See, for example, [7] proposition 6.4-(iv).) Moreover, any (ordinary) V-module is semisimple, and the category of V-modules is a rigid modular tensor category [20].

We write V-modules as $W_i, W_j, W_k, \dots$ whose vertex operators are denoted by $Y_i, Y_j, Y_k, \dots$ respectively. V itself as a V-module (the vacuum module) will also be written as W_0. We write $Y_i(v, z) = \sum_{n\in\mathbb{Z}} Y_i(v)_n z^{-n-1}$ where each $Y_i(v)_n \in \mathrm{End}(W_i)$. Again, any V-module has L_0-grading $W_i = \bigoplus_{n\in\mathbb{C}} W(n)$. Recall that a homogeneous vector of W_i is, by definition, an eigenvector of L_0. In the case that W_i is irreducible (i.e. simple), we furthermore have $W_i = \bigoplus_{n\in\mathbb{N}+\alpha} W(n)$ for some $\alpha \in \mathbb{C}$. (Indeed, $\alpha \in \mathbb{Q}$ by [4,9].) We let P_n denote the projection of W_i onto $W_i(n)$. We also let

$$W_i(\le n) = \bigoplus_{\mathrm{Re}(m)\le n} W(m),$$

and let $P_{\le n}$ be the projection of W_i onto $W_i(\le n)$.

Let $W_{\bar{i}}$ denote the contragredient module of W_i. Recall that as a vector space, $W_{\bar{i}} = \bigoplus_{n\in\mathbb{C}} W_i(n)^*$. (See [10] for more details.) The evaluation between $w' \in W_{\bar{i}}$ and $w \in W_i$ is written as $\langle w, w'\rangle$ or $\langle w', w\rangle$. (The same notation will be used if one of w, w' is in the algebraic completion.) Since V is self-dual, we identify the vacuum module $V = W_0$ and its contragredient module $W_{\bar{0}}$. $W_{\bar{\bar{i}}}$ is identified with W_i in an obvious way.

Recall that if W_i, W_j, W_k are V-modules, an intertwining operator $\mathcal{Y}$ of type $\binom{W_k}{W_i W_j}$ (or $\binom{k}{i\ j}$ for short) is a linear map

$$W_i \to \mathrm{End}(W_j, W_k)\{z\}$$
$$w_i \mapsto \mathcal{Y}(w^{(i)}, z) = \sum_{n\in\mathbb{C}} \mathcal{Y}(w^{(i)})_n z^{-n-1}$$

where the sum above is the formal sum, each $\mathcal{Y}(w^{(i)})_n$ is in $\mathrm{End}(W_j, W_k)$, and the following conditions are satisfied:

(a) (Lower truncation) For any $w^{(j)} \in W_j$, $\mathcal{Y}(w^{(i)})_n w^{(j)} = 0$ when $\mathrm{Re}(n)$ is sufficiently large.

(b) (Jacobi identity) For any $u \in V, w^{(i)} \in W_i, m, n \in \mathbb{Z}, s \in \mathbb{C}$, we have

$$\sum_{l\in\mathbb{N}} \binom{m}{l} \mathcal{Y}\big(Y_i(u)_{n+l} w^{(i)}\big)_{m+s-l}$$
$$= \sum_{l\in\mathbb{N}} (-1)^l \binom{n}{l} Y_k(u)_{m+n-l} \mathcal{Y}(w^{(i)})_{s+l}$$

$$-\sum_{l\in\mathbb{N}}(-1)^{l+n}\binom{n}{l}\mathcal{Y}(w^{(i)})_{n+s-l}Y_j(u)_{m+l}. \tag{1.1}$$

(c) (L_{-1}-derivative) $\frac{d}{dz}\mathcal{Y}(w^{(i)},z)=\mathcal{Y}(L_{-1}w^{(i)},z)$ for any $w^{(i)}\in W_i$.

We say that W_i,W_j,W_k are respectively the **charge space**, the **source space**, and the **target space** of $\mathcal{Y}$. If W_i,W_j,W_k are all irreducible, we say that $\mathcal{Y}$ is an **irreducible intertwining operator**.

Recall that given $\mathcal{Y}\in\mathcal{V}\binom{k}{i\ j}$, one can define $\mathbb{B}\mathcal{Y}\in\mathcal{V}\binom{k}{j\ i}$ and $\complement\mathcal{Y}\in\mathcal{V}\binom{\bar{j}}{i\ \bar{k}}$, called the (positively) **braided** intertwining operator and the **contragredient** intertwining operator of $\mathcal{Y}$, by choosing any $w^{(i)}\in W_i, w^{(j)}\in W_j, w^{(\bar{k})}\in W_{\bar{k}}$ and setting

$$\mathbb{B}\mathcal{Y}(w^{(j)},z)w^{(i)}=e^{zL_{-1}}\mathcal{Y}(w^{(i)},e^{\mathbf{i}\pi}z)w^{(j)},$$

$$\langle\complement\mathcal{Y}(w^{(i)},z)w^{(\bar{k})},w^{(j)}\rangle=\langle w^{(\bar{k})},\mathcal{Y}(e^{zL_1}(e^{\mathbf{i}\pi}z^{-2})^{L_0}w^{(i)},z^{-1})w^{(j)}\rangle.$$

In particular, $Y_{\bar{i}}\in\mathcal{V}\binom{\bar{i}}{0\ \bar{i}}$ (the vertex operator for $W_{\bar{i}}$) is contragredient to Y_i. See [10] for details.

We refer the reader to [17] and the references therein for the definition and basic properties of the tensor category Rep(V) of V-modules. (See also [13] for a sketch of the Huang-Lepowsky tensor product theory.) Roughly speaking, Rep(V) is defined such that the fusion rules are exactly the dimensions of the spaces of intertwining operators, and the R- and F-matrices are described by the braid and the fusion relations of intertwining operators.

To be more precise, the tensor functor (fusion product) $\boxtimes$ is defined in such a way that there is a functorial isomorphism

$$\mathrm{Hom}_V(W_i\boxtimes W_j,W_k)\xrightarrow{\simeq}\mathcal{V}\binom{k}{i\ j},\qquad \alpha\mapsto\mathcal{Y}_\alpha \tag{1.2}$$

for any V-modules W_i,W_j,W_k. By saying this map is functorial, we mean that if $F\in\mathrm{Hom}_V(W_{i'},W_i)$, $G\in\mathrm{Hom}_V(W_{j'},W_j)$, and $H\in\mathrm{Hom}_V(W_k,W_{k'})$, then for any $w^{(i')}\in W_{i'}$ and $w^{(j')}\in W_{j'}$,

$$\mathcal{Y}_{H\alpha(F\otimes G)}(w^{(i')},z)w^{(j')}=H\mathcal{Y}_\alpha(Fw^{(i')},z)Gw^{(j')}.$$

In particular, we denote by $\mathcal{L}_{i,j}\in\mathcal{V}\binom{i\boxtimes j}{i\ j}=\mathcal{V}\binom{W_i\boxtimes W_j}{W_i\ W_j}$ the value of the identity element $\mathbf{1}_{i\boxtimes j}\in\mathrm{Hom}_V(W_i\boxtimes W_j,W_i\boxtimes W_j)$ under (1.2), i.e.,

$$\mathcal{L}_{i,j}=\mathcal{Y}_{\mathbf{1}_{i\boxtimes j}}.$$

Here we adopt the notation

$$W_{i\boxtimes j}=W_i\boxtimes W_j.$$

Then for any $\alpha \in \mathrm{Hom}_V(W_i \boxtimes W_j, W_k)$, by the functoriality of (1.2) and that $\alpha = \alpha \cdot \mathbf{1}_{W_i \boxtimes W_j}$, it is clear that

$$\mathcal{Y}_\alpha = \alpha \mathcal{L}_{i,j}.$$

(In the papers of Huang-Lepowsky, $\mathcal{L}_{i,j}(w^{(i)}, z)w^{(j)}$ is written as $w^{(i)} \boxtimes_{P(z)} w^{(j)}$, regarded as a fusion product of the vectors $w^{(i)}, w^{(j)}$.)

Note that $Y_i \in \mathcal{V}\binom{i}{0\ i}$. The left unitor $W_0 \boxtimes W_i \xrightarrow{\simeq} W_i$ is defined such that it is sent by (1.2) to the element Y_i. The right unitor

$$\kappa(i) : W_i \boxtimes W_0 \xrightarrow{\simeq} W_i$$

is defined such that

$$\mathcal{Y}_{\kappa(i)} = \mathbb{B} Y_i.$$

We call $\mathcal{Y}_{\kappa(i)} \in \mathcal{V}\binom{i}{i\ 0}$ the **creation operator** of W_i.

The braid isomorphism $\mathbb{B} = \mathbb{B}_{i,j} : W_i \boxtimes W_j \xrightarrow{\simeq} W_j \boxtimes W_i$ is defined such that

$$\mathcal{Y}_{\mathbb{B}_{i,j}} = \mathbb{B} \mathcal{L}_{i,j}.$$

We will not use braiding in this article. To describe the associativity isomorphisms, we first notice:

Proposition 1.1. *Choose any $z \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Then for any $n \in \mathbb{C}$,*

$$\operatorname*{Span}_{w^{(i)} \in W_i, w^{(j)} \in W_j} P_{\leq n} \cdot \mathcal{L}_{i,j}(w^{(i)}, z)w^{(j)} = (W_i \boxtimes W_j)(\leq n). \tag{1.3}$$

This proposition was proved in [13] section A.2.[1] The main idea of the proof is as follows. Set $W_k = W_i \boxtimes W_j$. Then it is equivalent to proving that for any $w^{(\overline{k})} \in W_{\overline{k}}$, if

$$\langle w^{(\overline{k})}, \mathcal{L}_{i,j}(w^{(i)}, z)w^{(j)} \rangle = 0$$

for any $w^{(i)} \in W_i$ and $w^{(j)} \in W_j$, then $w^{(\overline{k})} = 0$. To see this, let $\mathcal{W} \subset W_{\overline{k}}$ be the subspace of all $w^{(\overline{k})}$ satisfying the above identity. Using the Jacobi identity for intertwining operators, it is easy to see that $\mathcal{W}$ is V-invariant, i.e. it is a V-submodule of $W_{\overline{k}}$. Assume $\mathcal{W}$ is non-trivial. Choose an irreducible module W_l such that $\mathcal{W}$ has an irreducible submodule isomorphic to $W_{\overline{l}}$. Then there is a non-zero $T \in \mathrm{Hom}_V(W_k, W_l)$ whose transpose $T^{\mathrm{t}} \in \mathrm{Hom}_V(W_{\overline{l}}, W_{\overline{k}})$ maps $W_{\overline{l}}$ into $\mathcal{W}$. Using the definition of $\mathcal{W}$ and the fact that

[1] Proposition 1.1 is similar to but slightly stronger than [18] Lemma 14.9. That lemma says that (1.3) holds with $P_{\leq s}$ replaced by P_s and $(W_i \boxtimes W_j)(\leq s)$ by $(W_i \boxtimes W_j)(s)$. Huang's result is enough for applications in our paper.

$T^{\mathrm{t}} w^{(\overline{l})} \in \mathcal{W}$ for each $w^{(\overline{l})} \in W_{\overline{l}}$, it is easy to see that $T\mathcal{L}_{i,j} = 0$. Recall $W_k = W_i \boxtimes W_j$. So we can write $T\mathcal{L}_{i,j} = \mathcal{Y}_T$. Therefore $T = 0$, which gives a contradiction.

Corollary 1.2. *Let W_i, W_j, W_s be V-modules, and assume that W_s is isomorphic to an irreducible submodule of $W_i \boxtimes W_j$. If $\Xi^s_{i,j}$ is a basis of $\mathrm{Hom}_V(W_i \boxtimes W_j, W_s)$, $\alpha \in \Xi^s_{i,j}$, then there exist homogeneous vectors $w^{(i)}_1, \dots, w^{(i)}_m \in W_i$, $w^{(j)}_1, \dots, w^{(j)}_m \in W_j$, $w^{(\overline{s})} \in W_{\overline{s}}$, and constants $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$, such that for any $\beta \in \Xi^s_{i,j}$, the expression*

$$\sum_{l=1}^{m} z^{\lambda_l} \langle \mathcal{Y}_\beta(w^{(i)}_l, z) w^{(j)}_l, w^{(\overline{s})} \rangle \tag{1.4}$$

is a constant (where z is a complex variable), and it is non-zero if and only if $\beta = \alpha$.

Proof. Choose $n \in \mathbb{C}$ such that $W_s(n)$ is non-trivial. Choose for each $\alpha \in \Xi^s_{i,j}$ a morphism $\breve{\alpha} \in \mathrm{Hom}_V(W_s, W_i \boxtimes W_j)$ such that $\alpha\breve{\beta} = \delta_{\alpha,\beta} \mathbf{1}_s$ for any $\alpha, \beta \in \Xi^s_{i,j}$. Now we fix $\alpha \in \Xi^s_{i,j}$, and choose a non-zero vector $w^{(s)} \in W_s(n)$. By Proposition 1.1, there exist homogeneous vectors $w^{(i)}_1, \dots, w^{(i)}_m \in W_i$ and $w^{(j)}_1, \dots, w^{(j)}_m \in W_j$ such that

$$\sum_{l=1}^{m} P_n \mathcal{L}_{i,j}(w^{(i)}_l, 1) w^{(j)}_l = \breve{\alpha} w^{(s)}.$$

Choose $w^{(\overline{s})} \in W_{\overline{s}}(n)$ such that $\langle w^{(s)}, w^{(\overline{s})} \rangle \neq 0$. (Recall that $W_{\overline{s}}(n)$ is the dual vector space of $W_s(n)$.) Then it is easy to check that the constant

$$\sum_{l=1}^{m} \langle \mathcal{Y}_\beta(w^{(i)}_l, 1) w^{(j)}_l, w^{(\overline{s})} \rangle \tag{1.5}$$

is non-zero (in which case equals $\langle w^{(s)}, w^{(\overline{s})} \rangle$) if and only if $\beta = \alpha$. Now let

$$\lambda_l = \mathrm{wt}(w^{(s)}_l) - \mathrm{wt}(w^{(i)}_l) - \mathrm{wt}(w^{(j)}_l),$$

where the three terms on the right hand side are the weights (i.e. the L_0-eigenvalues) of the corresponding homogeneous vectors. Then (1.5) equals (1.4). This proves the claim of this corollary. □

For any irreducible equivalence class of V-modules we choose a representing element and let them form a (finite) set $\mathcal{E}$. We assume $W_0 \in \mathcal{E}$. Choose V-modules W_i, W_j, W_k, W_l. If $W_s \in \mathcal{E}$, we write $s \in \mathcal{E}$ for short. For each $r \in \mathcal{E}$, we choose bases $\Xi^r_{j,k}$ of $\mathrm{Hom}_V(W_j \boxtimes W_k, W_r)$ and $\Xi^l_{i,r}$ of $\mathrm{Hom}_V(W_i \boxtimes W_r, W_l)$. Choose $z, \zeta \in \mathbb{C}^\times$ with $0 < |\zeta| < |z|$. Then for any $w^{(i)} \in W_i, w^{(j)} \in W_j$, and any $\alpha \in \Xi^l_{i,r}, \beta \in \Xi^r_{j,k}$ the product

$$\mathcal{Y}_\alpha(w^{(i)}, z) \mathcal{Y}_\beta(w^{(j)}, \zeta) \tag{1.6}$$

converges absolutely [19], in the sense that for any $w^{(k)} \in W_k, w^{(\bar{l})} \in W_{\bar{l}}$,

$$\sum_{n\in\mathbb{C}} \left|\langle \mathcal{Y}_\alpha(w^{(i)}, z) P_n \mathcal{Y}_\beta(w^{(j)}, \zeta) w^{(k)}, w^{(\bar{l})}\rangle\right| < +\infty.$$

Moreover, consider the expression

$$\langle \mathcal{Y}_\alpha(w^{(i)}, z) \mathcal{Y}_\beta(w^{(j)}, \zeta) w^{(k)}, w^{(\bar{l})}\rangle \tag{1.7}$$

as an element of $(W_i \otimes W_j \otimes W_k \otimes W_{\bar{l}})^*$. (Note that this element depends also on the arguments $\arg z$ and $\arg \zeta$.) By linearity, we have a linear map

$$\Psi_{z,\zeta} : \bigoplus_{r\in\mathcal{E}} \mathrm{Hom}_V(W_i \boxtimes W_r, W_l) \otimes \mathrm{Hom}_V(W_j \boxtimes W_k, W_r) \to (W_i \otimes W_j \otimes W_k \otimes W_{\bar{l}})^* \tag{1.8}$$

sending $\alpha \otimes \beta$ to the linear functional defined by (1.7). This map is well-known to be injective. Indeed, choose any $\mathfrak{X}$ in the domain of $\Psi_{z,\zeta}$. If $\Psi_{z,\zeta}(\mathfrak{X})$ equals 0 for one pair (z, ζ), then, by the existence of differential equations as in [19], $\Psi_{z,\zeta}(\mathfrak{X})$ equals 0 for all z, ζ satisfying $0 < |\zeta| < |z|$. (See [13] the paragraphs after theorem 2.4 for a detailed explanation.) Then, using Proposition 1.1, it is not hard to show $\mathfrak{X} = 0$. (See for instance [13] proposition 2.3 whose proof is in section A.2.)

Similarly, when $0 < |z - \zeta| < |\zeta|$, one can define an injective linear map

$$\Phi_{z,\zeta} : \bigoplus_{s\in\mathcal{E}} \mathrm{Hom}_V(W_s \boxtimes W_k, W_l) \otimes \mathrm{Hom}_V(W_i \boxtimes W_j, W_s) \to (W_i \otimes W_j \otimes W_k \otimes W_{\bar{l}})^* \tag{1.9}$$

sending each $\gamma \otimes \delta$ to the linear functional determined by the following iterate of intertwining operators:

$$\langle \mathcal{Y}_\gamma(\mathcal{Y}_\delta(w^{(i)}, z - \zeta) w^{(j)}, \zeta) w^{(k)}, w^{(\bar{l})}\rangle. \tag{1.10}$$

Again, this map depends on the choice of arguments: $\arg(z-\zeta)$ and $\arg(\zeta)$, and the above expression converges absolutely in an appropriate sense. By a deep result of [18,19], $\Phi_{z,\zeta}$ and $\Psi_{z,\zeta}$ have the same image.

We now assume

$$0 < |z - \zeta| < |\zeta| < |z|, \qquad \arg(z - \zeta) = \arg \zeta = \arg z. \tag{1.11}$$

In particular, ζ, z are on the same ray starting from the origin. If we also choose bases $\Xi^l_{s,k}$ and $\Xi^s_{i,j}$ of $\mathrm{Hom}_V(W_s \boxtimes W_k, W_l)$ and $\mathrm{Hom}_V(W_i \boxtimes W_j, W_s)$ respectively, then we have a matrix $\{F^{\alpha\beta}_{\gamma\delta}\}$ (the fusion matrix) representing the invertible map $\Psi_{z,\zeta}\Phi^{-1}_{z,\zeta}$. Equivalently, we have a unique number $F_{\gamma\delta}^{\alpha\beta}$ for each $\alpha, \beta, \gamma, \delta$ such that for each $s \in \mathcal{E}$ and each $\gamma \in \Xi_{s,k}^{l}, \delta \in \Xi_{i,j}^{s}$, the fusion relation

$$\mathcal{Y}_{\gamma}(\mathcal{Y}_{\delta}(w^{(i)}, z-\zeta)w^{(j)}, \zeta) = \sum_{r\in\mathcal{E}} \sum_{\alpha\in\Xi_{i,r}^{l}, \beta\in\Xi_{j,k}^{r}} F_{\gamma\delta}^{\alpha\beta} \cdot \mathcal{Y}_{\alpha}(w^{(i)}, z)\mathcal{Y}_{\beta}(w^{(j)}, \zeta) \quad (1.12)$$

holds for each $w^{(i)} \in W_i, w^{(j)} \in W_j$. This fusion matrix is independent of the particular choice of z, ζ satisfying the above mentioned conditions. The associativity isomorphisms of $\mathrm{Rep}(V)$ are defined in such a way that after making $\mathrm{Rep}(V)$ strict, we have

$$\gamma(\delta \otimes \mathbf{1}_k) = \sum_{r\in\mathcal{E}} \sum_{\alpha\in\Xi_{i,r}^{l}, \beta\in\Xi_{j,k}^{r}} F_{\gamma\delta}^{\alpha\beta} \cdot \alpha(\mathbf{1}_i \otimes \beta), \quad (1.13)$$

namely, F is also an F-matrix of $\mathrm{Rep}(V)$.

2. Fusion of annihilation and vertex operators

Let W_i, W_j be V-modules. For each $W_s \in \mathcal{E}$, there is a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathrm{Hom}_V(W_i \boxtimes W_j, W_s) \otimes \mathrm{Hom}_V(W_s, W_i \boxtimes W_j)$ such that if $\alpha \in \mathrm{Hom}_V(W_i \boxtimes W_j, W_s)$ and $T \in \mathrm{Hom}_V(W_s, W_i \boxtimes W_j)$, then

$$\alpha T = \langle \alpha, T\rangle \mathbf{1}_s. \quad (2.1)$$

This bilinear form gives an isomorphism

$$\mathrm{Hom}_V(W_s, W_i \boxtimes W_j) \xrightarrow{\simeq} \mathrm{Hom}_V(W_i \boxtimes W_j, W_s)^*. \quad (2.2)$$

We shall always identify $\mathrm{Hom}_V(W_s, W_i \boxtimes W_j)$ and $\mathrm{Hom}_V(W_i \boxtimes W_j, W_s)^*$ using the above isomorphism.

Recall from the last section that $\Xi_{i,j}^{s}$ is a basis of $\mathrm{Hom}_V(W_i \boxtimes W_j, W_s)$. Then we can choose a dual basis $\{\breve{\alpha} : \alpha \in \Xi_{i,j}^{s}\}$. Namely, for each $\alpha \in \Xi_{i,j}^{s}$, we have $\breve{\alpha} \in \mathrm{Hom}_V(W_s, W_i \boxtimes W_j)$, and if $\beta \in \Xi_{i,j}^{s}$, then $\langle \alpha, \breve{\beta}\rangle = \delta_{\alpha,\beta}$. So we also have

$$\alpha\breve{\beta} = \delta_{\alpha,\beta}\mathbf{1}_s. \quad (2.3)$$

This implies that

$$\mathbf{1}_{i\boxtimes j} = \sum_{s\in\mathcal{E}} \sum_{\alpha\in\Xi_{i,j}^{s}} \breve{\alpha}\alpha, \quad (2.4)$$

since, by (2.3), the left multiplications of both sides of (2.4) by any $\beta \in \Xi_{i,j}^{s}$ equal β.

In [16], Huang-Kong used the rigidity of $\mathrm{Rep}(V)$ to define a natural isomorphism $\mathrm{Hom}_V(W_{\bar{i}} \boxtimes W_{\bar{j}}, W_{\bar{s}}) \xrightarrow{\simeq} \mathrm{Hom}_V(W_i \boxtimes W_j, W_s)^*$. Since $\mathcal{V}\binom{\bar{s}}{\bar{i}\ \bar{j}}$ is isomorphic to $\mathcal{V}\binom{j}{\bar{i}\ s}$ by sending $\mathcal{Y}$ to $\mathfrak{C}\mathcal{Y}$, we also have an isomorphism

$$\text{ɥ} : \mathrm{Hom}_V(W_i \boxtimes W_j, W_s)^* \xrightarrow{\simeq} \mathrm{Hom}_V(W_{\bar{i}} \boxtimes W_s, W_j). \tag{2.5}$$

In the following, we review the construction of this isomorphism.

In [20], Huang showed that $\mathrm{Rep}(V)$ is rigid, and the (categorical) dual object of any V-module W_i could be chosen to be the contragredient module $W_{\bar{i}}$. Moreover, if we define

$$\mathrm{ev}_{\bar{i},i} \in \mathrm{Hom}_V(W_{\bar{i}} \boxtimes W_i, V)$$

such that

$$\mathcal{Y}_{\mathrm{ev}_{\bar{i},i}} = \mathfrak{C}\mathcal{Y}_{\kappa(\bar{i})},$$

(Recall that $\mathcal{Y}_{\kappa(\bar{i})}$ is the creation operator of $W_{\bar{i}}$, which is of type $\binom{\bar{i}}{\bar{i}\ 0}$. $\mathcal{Y}_{\mathrm{ev}_{\bar{i},i}}$, which is of type $\binom{0}{\bar{i}\ i}$, is called the **annihilation operator** of W_i.) then there is a (unique) morphism

$$\mathrm{coev}_{i,\bar{i}} \in \mathrm{Hom}_V(V, W_i \boxtimes W_{\bar{i}})$$

satisfying the conjugate equations

$$(\mathbf{1}_i \otimes \mathrm{ev}_{\bar{i},i})(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_i) = \mathbf{1}_i,$$
$$(\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{i}} \otimes \mathrm{coev}_{i,\bar{i}}) = \mathbf{1}_{\bar{i}}.$$

This is also true for $W_{\bar{i}}$. Thus we have $\mathrm{ev}_{i,\bar{i}}$ and $\mathrm{coev}_{\bar{i},i}$ defined by $\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{\bar{i}},\bar{i}}$ and $\mathrm{coev}_{\bar{i},i} = \mathrm{coev}_{\bar{i},\bar{\bar{i}}}$.

Recall the identification (2.2). We define

$$\text{ɥ} : \mathrm{Hom}_V(W_s, W_i \boxtimes W_j) \xrightarrow{\simeq} \mathrm{Hom}_V(W_{\bar{i}} \boxtimes W_s, W_j),$$
$$T \mapsto \text{ɥ}(T) = (\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_j)(\mathbf{1}_{\bar{i}} \otimes T). \tag{2.6}$$

That ɥ is an isomorphism follows from the conjugate equations. Using the definition of ɥ and equation (2.4), it is easy to see

$$\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_j = \sum_{s\in\mathcal{E}} \sum_{\alpha\in\Xi^s_{i,j}} \text{ɥ}(\breve{\alpha})(\mathbf{1}_{\bar{i}} \otimes \alpha). \tag{2.7}$$

Thus, by (1.12) and (1.13), we have the following fusion relation which will play an important role in later sections: Let $z, \zeta \in \mathbb{C}$ satisfy (1.11). Then for any $w^{(i)} \in W_i, w^{(\bar{i})} \in W_{\bar{i}}$,

$$Y_j(\mathcal{Y}_{\mathrm{ev}_{\bar{i},i}}(w^{(\bar{i})}, z-\zeta)w^{(i)}, \zeta) = \sum_{s\in\mathcal{E}} \sum_{\alpha\in\Xi^s_{i,j}} \mathcal{Y}_{\text{ɥ}(\breve{\alpha})}(w^{(\bar{i})}, z)\mathcal{Y}_{\alpha}(w^{(i)}, \zeta). \tag{2.8}$$

Note that for each $s \in \mathcal{E}$, $\{\eta(\breve{\alpha}) : \alpha \in \Xi^s_{i,j}\}$ is a basis of $\mathrm{Hom}_V(W_{\bar{i}} \boxtimes W_s, W_j)$. Roughly speaking, this fusion relation says that any intertwining operator arises from fusing the annihilation operators and the vertex operators. This is parallel to the fact that any V-module character occurs in the sum resulting from the modular transformation $\tau \mapsto -1/\tau$ of the vacuum module character.

3. Compressions of intertwining operators

Assume that V is a vertex operator subalgebra (sub-VOA for short) of another CFT-type VOA U with vertex operator Y^U and conformal vector ω. This means that V is a subspace of U, V and U share the same vacuum vector $\mathbf{1}$, and that $Y^U(v_1, z)v_2 = Y(v_1, z)v_2$ when $v_1, v_2 \in V$. Let $L^U_n = Y^U(\omega)_{n+1}$. We shall always assume the additional condition that

$$L^U_0 \nu = 2\nu, \qquad L^U_1 \nu = 0. \tag{3.1}$$

Then by [11] or [22] theorem 3.11.12, $(V^c, Y', \mathbf{1}, \nu')$ is a sub-VOA of U, where V^c is the set of all $u \in U$ such that $Y(v)_n u = 0$ for all $v \in V$ and $n \in \mathbb{N}$, Y' is the restriction of Y^U to V^c, and $\nu' = \omega - \nu$. We set $L'_n = Y'(\nu')_{n+1}$.

Assume that V^c is self-dual, CFT-type, and regular. Then $V \otimes V^c$ is also CFT-type and self-dual (and also regular). Thus it is simple. Therefore, the homomorphism of $V \otimes V^c$-modules

$$V \otimes V^c \to U, \qquad v \otimes v' \mapsto Y(v)_{-1} Y(v')_{-1} \mathbf{1} \tag{3.2}$$

(cf. [22] proposition 3.12.7) must be injective. Thus, we can regard $V \otimes V^c$ as a *conformal* sub-VOA of U sharing the same conformal vector $\omega = \nu + \nu' = \nu \otimes \mathbf{1} + \mathbf{1} \otimes \nu'$. Note that by the identification $v \otimes v' = Y(v)_{-1} Y(v')_{-1} \mathbf{1}$, we have $\mathbf{1} = \mathbf{1} \otimes \mathbf{1}, v = v \otimes \mathbf{1}, v' = \mathbf{1} \otimes v'$.

Recall that by [10] chapter 4, any irreducible $V \otimes V^c$-module is the tensor product of a V-module and a V^c-module. Moreover, by [3] theorem 2.10, any irreducible intertwining operator of $V \otimes V^c$ can be written as a sum of tensor products of irreducible intertwining operators of V and of V^c. Therefore, any U-module, considered as a $V \otimes V^c$-module, is a direct sum of those of the form $W_i \otimes W_{i'}$, where W_i is an irreducible V-module and $W_{i'}$ is an irreducible V^c-module. Theorem 2.10 also implies that any intertwining operator of U can be decomposed as a sum of $\mathcal{Y}_\alpha \otimes \mathcal{Y}_{\alpha'}$, where $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{\alpha'}$ are irreducible intertwining operators of V and V^c respectively.

In the following, $W_I, W_J, W_K, \ldots$ will denote U-modules, and $W_{i'}, W_{j'}, W_{k'}, \ldots$ will denote V^c-modules. $\mathcal{V}^U\binom{K}{I\ J}$ and $\mathcal{V}'\binom{k'}{i'\ j'}$ will denote the corresponding vector spaces of intertwining operators of U and V^c respectively. Note that W_I can not be regarded as a V-module (unless when $\omega = \nu$) but only as a weak V-module (see [8] for the definition.)

Definition 3.1. Let W_i be an irreducible V-module and W_I be a U-module. Let $\varphi : W_i \to W_I$ and $\psi : W_I \to W_i$ be homomorphisms of weak V-modules, i.e., they intertwine the actions of V. We say that φ is **grading-preserving** if φ maps each L_0-eigenspace of W_i into an L_0^U-eigenspace of W_I. We say that ψ is **grading-preserving** if the preimage under ψ of any L_0^U-eigenspace of W_I is contained in an L_0-eigenspace of W_i.

Remark 3.2. We have seen that there is an identification of $V \otimes V^c$-modules:

$$W_I \simeq \bigoplus_{s\in\mathcal{E}} W_s \otimes W_{\sigma(s)} \tag{3.3}$$

where for each $s \in \mathcal{E}$, $W_{\sigma(s)}$ is a (non-necessarily irreducible) V^c-module. We can also regard (3.3) as a decomposition of W_I into irreducible weak V-modules, where for each $s \in \mathcal{E}$, $W_{\sigma(s)}$ is the multiplicity space of W_s. (Note that different elements in $\mathcal{E}$ give rise to non-equivalent irreducible modules.) Choose any $s \in \mathcal{E}$. Choose $\varphi : W_s \to W_I$ and $\psi : W_I \to W_s$ to be homomorphisms of weak V-modules. Then it is not hard to see that we can find $w^{(\sigma(s))} \in W_{\sigma(s)}$ and $\varpi \in W^*_{\sigma(s)}$ (note that $W^*_{\sigma(s)}$ is the dual vector space of $W_{\sigma(s)}$) such that

$$\varphi = \mathbf{1}_s \otimes w^{(\sigma(s))}, \qquad \psi = \mathbf{1}_s \otimes \varpi, \tag{3.4}$$

where $w^{(\sigma(s))}$ is considered as the linear map $\mathbb{C} \to W_{\sigma(s)}$ sending 1 to $w^{(\sigma(s))}$.[2] Moreover, outside $W_s \otimes W_{\sigma(s)}$, $\mathbf{1}_s \otimes \varpi$ is defined to be the zero functional. Thus, it is clear that *φ (resp. ψ) is grading-preserving if any only if $w^{(\sigma(s))}$ (resp. ϖ) equals an (L_0'-) homogeneous vector of $W_{\sigma(s)}$ (resp. $W_{\overline{\sigma(s)}}$).*

Definition 3.3. Let W_i be an irreducible V-module and W_I be a U-module. We say that W_i is a **compression** of W_I if W_I (considered as a weak V-module) has an irreducible weak V-submodule isomorphic to W_i. Equivalently, the $V \otimes V^c$-module W_I has a (non-trivial) irreducible submodule isomorphic to $W_i \otimes W_{i'}$ for some irreducible V^c-module $W_{i'}$.

Definition 3.4. Let $\mathcal{Y} \in \mathcal{V}\binom{k}{i\ j}$ be an irreducible intertwining operator of V.

(a) Let $\mathcal{Y}^U \in \mathcal{V}^U\binom{K}{I\ J}$ be an intertwining operator of U. We say that $\mathcal{Y}$ is a **compression** of $\mathcal{Y}^U$, if there exit $\lambda \in \mathbb{Q}$ and grading-preserving homomorphisms of weak V-modules $\varphi : W_i \to W_I$, $\phi : W_j \to W_J$, and $\psi : W_K \to W_k$, such that for any $w^{(i)} \in W_i$ and $z \in \mathbb{C}^\times$

$$\mathcal{Y}(w^{(i)}, z) = z^\lambda \cdot \psi\mathcal{Y}^U(\varphi w^{(i)}, z)\phi.$$

[2] To see that φ can be written in this way, choose any $t \in \mathcal{E}$ and $\omega \in W^*_{\sigma(t)}$, and consider the homomorphism of irreducible V-modules $T_\omega : W_s \to W_t$ defined by $T_\omega = (\mathbf{1}_t \otimes \omega) \circ \varphi$. Then $T_\omega = 0$ whenever $s \neq t$ (since $W_s \not\simeq W_t$). So the image of φ is in $W_s \otimes W_{\sigma(s)}$. Now assume $t = s$. Then T_ω is a scalar. Choose a basis $\{e_1, e_2, \dots\}$ of W_s, and write $\varphi(e_1) = \sum_n e_n \otimes w_n$ where each w_n is in $W_{\sigma(s)}$. Then, for each ω, $T_\omega(e_1) = \sum_n \omega(w_n)e_n$ is a scalar multiple of e_1, which shows that $w_n = 0$ when $n > 1$. Thus $\varphi = \mathbf{1}_s \otimes w_1$. That ψ has the desired form can be proved similarly.

(b) If W_I, W_J are U-modules, we say that $\mathcal{Y}$ is a **compression of type $\binom{\bullet}{I\ J}$ intertwining operators of** U, if $\mathcal{Y}$ is a (finite) sum of compressions of intertwining operators of U whose charge spaces are W_I and source spaces are W_J.

(c) If $\mathcal{Y}$ is a (finite) sum of compressions of intertwining operators of U, we simply say that $\mathcal{Y}$ is a **compression of intertwining operators of** U

Proposition 3.5. *Let W_I, W_J, W_K be U-modules with $V \otimes V^c$-irreducible decompositions*

$$W_I \simeq \bigoplus W_i \otimes W_{i'}, \qquad W_J \simeq \bigoplus W_j \otimes W_{j'}, \qquad W_K \simeq \bigoplus W_k \otimes W_{k'}.$$

Then, according to these decompositions, any $\mathcal{Y}^U \in \mathcal{V}^U\binom{K}{I\ J}$ can be written as a sum of elements of the form $\mathcal{Y} \otimes \mathcal{Y}'$, where $\mathcal{Y} \in \mathcal{V}\binom{k}{i\ j}$ is the compression of a type $\binom{K}{I\ J}$ intertwining operator of U, and $\mathcal{Y}'$ is an irreducible intertwining operator of V^c.

Proof. We fix irreducible $V \otimes V^c$-submodules $W_i \otimes W_{i'}, W_j \otimes W_{j'}, W_k \otimes W_{k'}$ of W_I, W_J, W_k respectively. Let $\Theta^{k'}_{i',j'}$ be a basis of $\mathrm{Hom}_{V^c}(W_{i'} \boxtimes W_{j'}, W_{k'})$. We have a (functorial) isomorphism

$$\mathrm{Hom}_{V^c}(W_{i'} \boxtimes W_{j'}, W_{k'}) \xrightarrow{\simeq} \mathcal{V}'\binom{k'}{i'\ j'}, \qquad \alpha' \mapsto \mathcal{Y}'_{\alpha'} \tag{3.5}$$

similar to (1.2). Consider $\mathcal{Y}^U$ as an intertwining operator of $V \otimes V^c$, and restrict it to $W_i \otimes W_{i'}, W_j \otimes W_{j'}, W_k \otimes W_{k'}$. Then, by [3] theorem 2.10, this restriction is a sum of tensor products of V- and V^c-intertwining operators. Assume without loss of generality that this restriction is non-zero. Then $W_k, W_{k'}$ must be irreducible submodules of $W_i \boxtimes W_j, W_{i'} \boxtimes W_{j'}$ respectively. Now, for each $\alpha' \in \Theta^{k'}_{i',j'}$, we can find $\alpha \in \mathrm{Hom}_V(W_i \boxtimes W_j, W_k)$ (not necessarily in $\Xi^k_{i,j}$) such that the restriction of $\mathcal{Y}^U$ equals

$$\sum_{\alpha' \in \Theta^{k'}_{i',j'}} \mathcal{Y}_\alpha \otimes \mathcal{Y}'_{\alpha'}.$$

We shall show that each $\mathcal{Y}_\alpha$ is a sum of compressions of type $\binom{K}{I\ J}$ intertwining operators of U.

Choose $\alpha' \in \Theta^{k'}_{i',j'}$ and apply Corollary 1.2 to V^c. Then there exist homogeneous vectors $w^{(i')}_1, \dots, w^{(i')}_m \in W_{i'}$, $w^{(j')}_1, \dots, w^{(j')}_m \in W_{j'}$, $w^{(\overline{k'})} \in W_{\overline{k'}}$, and constants $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$, such that for any $\beta' \in \Theta^{k'}_{i',j'}$, the expression

$$\sum_{l=1}^{m} z^{\lambda_l} \langle \mathcal{Y}'_{\beta'}(w^{(i')}_l, z) w^{(j')}_l, w^{(\overline{k'})} \rangle \tag{3.6}$$

is a constant (over the complex variable z), and this constant is non-zero if and only if $\beta' = \alpha'$. By scaling the vector $w^{(\overline{k'})}$, we may assume that when $\beta' = \alpha'$, the above constant is 1. Now, for each $l = 1, 2, \dots, m$, define grading-preserving homomorphisms of weak V-modules $\varphi_l : W_i \to W_I, \phi_l : W_j \to W_J, \psi : W_K \to W_k$ by

$$\varphi_l = \mathbf{1}_i \otimes w_l^{(i')}, \qquad \phi_l = \mathbf{1}_j \otimes w_l^{(j')}, \qquad \psi = \mathbf{1}_k \otimes w^{(\overline{k'})}.$$

Then we have

$$\mathcal{Y}_\alpha(w^{(i)}, z) = \sum_{l=1}^{m} z^{\lambda_l} \cdot \psi \mathcal{Y}^U(\varphi_l w^{(i)}, z)\phi_l.$$

This finishes the proof. $\square$

4. Proof of the main result

In this section, we assume that $V^{cc} = V$. Let W_I, W_J be V-modules, and we fix irreducible decompositions of $V \otimes V^c$-modules:

$$U \simeq (V \otimes V^c) \oplus \Big(\bigoplus W_a \otimes W_{a'}\Big), \tag{4.1}$$

$$W_I \simeq (V \otimes V^c) \oplus \Big(\bigoplus W_i \otimes W_{i'}\Big), \tag{4.2}$$

$$W_J \simeq (V \otimes V^c) \oplus \Big(\bigoplus W_j \otimes W_{j'}\Big). \tag{4.3}$$

We first recall the following obvious fact.

Proposition 4.1. *If $W_a \otimes W_{a'}$ is an irreducible $V \otimes V^c$-submodule of $U \ominus (V \otimes V^c)$, then W_a is not isomorphic to V and $W_{a'}$ is not isomorphic to V^c.*

Proof. If $W_{a'}$ is isomorphic to V^c, then $W_{a'}$ contains a non-zero homogeneous vector w_2 equivalent to the vacuum vector of V^c. Choose any non-zero $w_1 \in W_a$. Then for any $v' \in V^c$ and $n \in \mathbb{N}$, $Y'(v')_n w_2 = 0$. Therefore $Y^U(v')_n(w_1 \otimes w_2) = w_1 \otimes Y'(v')_n w_2 = 0$. Thus $w_1 \otimes w_2 \in V^{cc} = V = V \otimes \mathbf{1}$, which is impossible since $w_1 \otimes w_2$ is not in $V \otimes V^c$. So $W_{a'}$ is not isomorphic to V^c. Since $V^{ccc} = V^c$, for a similar reason, W_a is also not isomorphic to V. $\square$

For each irreducible V^c-module, we choose a representing element, and let them form a finite set $\mathcal{E}'$. Assume $W_{0'} := V^c$ is in $\mathcal{E}'$. If $W_{i'}, W_{j'}, W_{k'}$ are V^c-modules, we choose a basis $\Theta_{i,j}^k$ of $\mathrm{Hom}_{V^c}(W_{i'} \boxtimes W_{j'}, W_{k'})$. The linear isomorphism

$$\mathfrak{q} : \mathrm{Hom}_{V^c}(W_{i'} \boxtimes W_{j'}, W_{k'})^* \xrightarrow{\simeq} \mathrm{Hom}_{V^c}(W_{\overline{i'}} \boxtimes W_{k'}, W_{j'}) \tag{4.4}$$

and the morphism

$$\mathrm{ev}_{\overline{i'}, i'} \in \mathrm{Hom}_{V^c}(W_{\overline{i'}} \boxtimes W_{i'}, V^c)$$

are defined as in section 2. Then $\mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}$ is the annihilation operator of $W_{i'}$.

According to the decomposition for W_I, $W_{\overline{I}}$ also has the corresponding decomposition:

$$W_{\overline{I}} \simeq (V \otimes V^c) \oplus \Big(\bigoplus W_{\overline{i}} \otimes W_{\overline{i'}}\Big).$$

Let

$$\mathrm{ev}_{\overline{I},I} \in \mathrm{Hom}_U(W_{\overline{I}} \boxtimes W_I, U)$$

which corresponds to the annihilation operator $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ of the U-module W_I. Suppose that in the above decompositions, $W_i \boxtimes W_{i'}$ is an irreducible submodule of $W_I \ominus (V \otimes V^c)$. If we regard $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ as an intertwining operator of $V \otimes V^c$, then it is easy to see that the restriction of $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ to the charge subspace $W_{\overline{i}} \boxtimes W_{\overline{i'}}$ source subspace $W_i \boxtimes W_{i'}$ and the target subspace $V \otimes V^c$ is $\mathcal{Y}_{\mathrm{ev}_{\overline{i},i}} \otimes \mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}$, the tensor product of the annihilation operators of W_i and of $W_{i'}$.

Theorem 4.2. *Let V be a vertex operator subalgebra of U satisfying (3.1). Assume that U, V, and V^c are CFT type, self-dual, and regular VOAs. Assume also that $V^{cc} = V$. Let W_I, W_J be U-modules. Let W_i, W_j be irreducible V-modules that are compressions of W_I and W_J respectively. Then any irreducible intertwining operator of V with charge space W_i and source space W_j is a compression of type $\binom{\bullet}{I\ J}$ intertwining operators of U.*

Proof. Let $W_i \otimes W_{i'}$ and $W_j \otimes W_{j'}$ be irreducible $V \otimes V^c$-submodules of W_I, W_J respectively. Assume that $k \in \mathcal{E}$ and not all type $\binom{k}{i\ j}$ intertwining operators of V are compressions of type $\binom{\bullet}{I\ J}$ intertwining operators of U. Let $\mathscr{V}\binom{k}{i\ j}$ be a subspace of $\mathcal{V}\binom{k}{i\ j}$ with codimension 1 containing all elements of $\mathcal{V}\binom{k}{i\ j}$ that are compressions of type $\binom{\bullet}{I\ J}$ intertwining operators of U. Choose a nonzero element $\mathfrak{A} \in \mathrm{Hom}_V(W_i \boxtimes W_j, W_k)$ such that $\mathcal{Y}_{\mathfrak{A}} \notin \mathscr{V}\binom{k}{i\ j}$. We assume that the basis $\Xi^k_{i,j}$ of $\mathrm{Hom}_V(W_i \boxtimes W_j, W_k)$ is chosen such that $\mathfrak{A} \in \Xi^k_{i,j}$, and that $\mathcal{Y}_\alpha \in \mathscr{V}\binom{k}{i\ j}$ for any $\alpha \in \Xi^k_{i,j}$ not equal to $\mathfrak{A}$.

Choose $z, \zeta \in \mathbb{C}$ satisfying (1.11). Recall that Y^U_I is the U-vertex operator of W_I and $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ is the U-annihilation operator of W_I. In the following, we shall calculate the fusion relation for the iterate of $V \otimes V^c$-intertwining operators Y^U_J and $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ (with restricted charge, source, and target spaces) in two ways. These two methods will give incompatible results, which therefore lead to a contradiction. Let $\pi_{j\otimes j'}$ be the projection of the algebraic completion of W_J onto the one of $W_j \otimes W_{j'}$.

Step 1. Note that for each $s \in \mathcal{E}, s' \in \mathcal{E}'$, the set $\Xi^s_{i,j} \times \Theta^{s'}_{i',j'}$ (more precisely, $\{\mathcal{Y}_\alpha \otimes \mathcal{Y}'_{\alpha'} : \alpha \in \Xi^s_{i,j}, \alpha' \in \Theta^{s'}_{i',j'}\}$) is a basis of the vector space of type $\binom{W_s\otimes W_{s'}}{W_i\otimes W_{i'}\ \ W_j\otimes W_{j'}}$ intertwining operators of $V \otimes V^c$. (See [3] theorem 2.10; it is also an easy consequence of Corollary 1.2.) Thus, for any $s \in \mathcal{E}, s' \in \mathcal{E}'$ and $\alpha, \beta \in \mathrm{Hom}_V(W_i \boxtimes W_j, W_s), \alpha', \beta' \in \mathrm{Hom}_{V^c}(W_{i'} \boxtimes W_{j'}, W_{s'})$, there is a *unique* constant $\lambda_{\alpha,\beta,\alpha',\beta'} \in \mathbb{C}$ such that for any

$$w_1 \in W_{\overline{i}}, w_2 \in W_{\overline{i'}}, w_3 \in W_i, w_4 \in W_{i'}, w_5 \in W_j, w_6 \in W_{j'}, \tag{4.5}$$

the following fusion relation of $V\otimes V^c$-intertwining operators holds:

$$\begin{aligned}&\pi_{j\otimes j'}\cdot Y^U_J\big(\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}(w_1\otimes w_2,z-\zeta)(w_3\otimes w_4),\zeta\big)(w_5\otimes w_6)\\ =&\sum_{\substack{s\in\mathcal{E}\\ s'\in\mathcal{E}'}}\sum_{\substack{\alpha,\beta\in\Xi^s_{i,j}\\ \alpha',\beta'\in\Theta^{s'}_{i',j'}}}\lambda_{\alpha,\beta,\alpha',\beta'}\cdot\mathcal{Y}_{\natural(\breve{\beta})}(w_1,z)\mathcal{Y}_{\alpha}(w_3,\zeta)w_5\otimes\mathcal{Y}'_{\natural(\breve{\beta'})}(w_2,z)\mathcal{Y}'_{\alpha'}(w_4,\zeta)w_6.\end{aligned}\tag{4.6}$$

(Recall (1.2) and (3.5) for the notations $\mathcal{Y},\mathcal{Y}'$.) On the other hand, by (2.8), the iterate of the U-intertwining operators Y^U_J and $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ equals a sum of products of type $\binom{J}{\overline{I}\ \bullet}$ intertwining operators and type $\binom{\bullet}{I\ J}$ intertwining operators of U. Therefore, by Proposition 3.5 and the uniqueness of fusion coefficients, we have

$$\lambda_{\mathfrak{A},\beta,\alpha',\beta'}=0\tag{4.7}$$

for any $s'\in\mathcal{E}'$, $\beta\in\Xi^k_{i,j}$, and $\alpha',\beta'\in\Theta^{s'}_{i',j'}$. In particular, the right hand side of (4.6) has no terms containing $\mathcal{Y}_{\natural(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5$.

Step 2. We calculate the iterate of $\pi_{j\otimes j'}\cdot Y^U_J$ and $\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}$ using a different method, and show that some terms containing $\mathcal{Y}_{\natural(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5$ will appear. By the paragraph before the theorem, we know that for any $w_1,w_2,\dots,w_6$ as in (4.5),

$$\begin{aligned}&\mathcal{Y}^U_{\mathrm{ev}_{\overline{I},I}}(w_1\otimes w_2,z-\zeta)(w_3\otimes w_4)\\ =&\mathcal{Y}_{\mathrm{ev}_{\overline{i},i}}(w_1,z-\zeta)w_3\otimes\mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}(w_2,z-\zeta)w_4\\ &+\sum_{W_a\otimes W_{a'}}\sum_{\gamma,\gamma'}\mathcal{Y}_{\gamma}(w_1,z-\zeta)w_3\otimes\mathcal{Y}'_{\gamma'}(w_2,z-\zeta)w_4\end{aligned}\tag{4.8}$$

where the first sum is over all irreducible $V\otimes V^c$-submodules of $U\ominus(V\otimes V^c)$ as in the decomposition (4.1), $\gamma\in\mathrm{Hom}_V(W_{\overline{i}}\boxtimes W_i,W_a)$, and $\gamma'\in\mathrm{Hom}_{V^c}(W_{\overline{i'}}\boxtimes W_{i'},W_{a'})$. We shall now calculate the iterate of $\pi_{j\otimes j'}\cdot Y^U_J$ with each term on the right hand side of (4.8).

The first term is in the algebraic completion of $V\otimes V^c$. Moreover, the restriction of Y^U_J (regarded as a $V\otimes V^c$-intertwining operator) to $V\otimes V^c,W_j\otimes W_{j'},W_j\otimes W_{j'}$ equals $Y_j\otimes Y'_{j'}$, where $Y_j,Y'_{j'}$ are respectively the vertex operators of the V-module W_j and the V^c-module $W_{j'}$. Therefore, by (2.8),

$$\begin{aligned}&\pi_{j\otimes j'}\cdot Y^U_J\big(\mathcal{Y}_{\mathrm{ev}_{\overline{i},i}}(w_1,z-\zeta)w_3\otimes\mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}(w_2,z-\zeta)w_4,\zeta\big)(w_5\otimes w_6)\\ =&Y_j\big(\mathcal{Y}_{\mathrm{ev}_{\overline{i},i}}(w_1,z-\zeta)w_3,\zeta\big)w_5\otimes Y'_{j'}\big(\mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}(w_2,z-\zeta)w_4,\zeta\big)w_6\\ =&\sum_{s\in\mathcal{E}}\sum_{\alpha\in\Xi^s_{i,j}}\mathcal{Y}_{\natural(\breve{\alpha})}(w_1,z)\mathcal{Y}_{\alpha}(w_3,\zeta)w_5\otimes Y'_{j'}\big(\mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}(w_2,z-\zeta)w_4,\zeta\big)w_6.\end{aligned}\tag{4.9}$$

In the above expression, (the sum of) all the terms containing $\mathcal{Y}_{\eta(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5$ is

$$\mathcal{Y}_{\eta(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5 \otimes Y'_{j'}\big(\mathcal{Y}'_{\mathrm{ev}_{\overline{i'},i'}}(w_2,z-\zeta)w_4,\zeta\big)w_6. \tag{4.10}$$

On the other hand, suppose that when restricted to the charge subspace $W_a\otimes W_{a'}$ (where $W_a\otimes W_{a'}$ is an irreducible submodule of $U\ominus(V\otimes V^c)$) and source and target subspace $W_j\otimes W_{j'}$, the $V\otimes V^c$-intertwining operator Y^U_J could be written as $\sum_{\delta,\delta'}\mathcal{Y}_\delta\otimes\mathcal{Y}'_{\delta'}$, where each $\mathcal{Y}_\delta$ is of type $\binom{j}{a\ j}$ and $\mathcal{Y}'_{\delta'}$ has type $\binom{j'}{a'\ j'}$. Then the iterate of $\pi_{j\otimes j'}\cdot Y^U_J$ with the second term of (4.8) is

$$\begin{aligned}&\sum_{W_a\otimes W_{a'}}\sum_{\gamma,\gamma'}\pi_{j\otimes j'}\cdot Y^U_J\big(\mathcal{Y}_\gamma(w_1,z-\zeta)w_3\otimes\mathcal{Y}_{\gamma'}(w_2,z-\zeta)w_4,\zeta\big)(w_5\otimes w_6)\\ =&\sum_{W_a\otimes W_{a'}}\sum_{\substack{\gamma,\gamma'\\ \delta,\delta'}}\mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w_1,z-\zeta)w_3,\zeta\big)w_5\otimes\mathcal{Y}'_{\delta'}\big(\mathcal{Y}'_{\gamma'}(w_2,z-\zeta)w_4,\zeta\big)w_6.\end{aligned} \tag{4.11}$$

If we write each $\mathcal{Y}_\delta\big(\mathcal{Y}_\gamma(w_1,z-\zeta)w_3,\zeta\big)w_5$ as a sum of products of V-intertwining operators under the bases $\Xi^s_{i,j}$ and $\{\eta(\breve{\alpha}):\alpha\in\Xi^s_{i,j}\}$ (over all $s\in\mathcal{E}$) similar to part of (4.6), then the sum of all the terms containing $\mathcal{Y}_{\eta(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5$ should be

$$\mathcal{Y}_{\eta(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5\otimes\sum_{W_a\otimes W_{a'}}\sum_{\substack{\gamma,\gamma'\\ \delta,\delta'}}\kappa_{\gamma,\delta}\cdot\mathcal{Y}'_{\delta'}\big(\mathcal{Y}'_{\gamma'}(w_2,z-\zeta)w_4,\zeta\big)w_6 \tag{4.12}$$

where each $\kappa_{\gamma,\delta}$ is a constant. By Proposition 4.1, every $W_{a'}$ (which is irreducible) is not isomorphic to V^c. Therefore, as the linear map $\Phi_{z,\zeta}$ (see (1.9)) is injective, the sum of (4.10) and (4.12) is not zero for some $w_1,\dots,w_6$ satisfying (4.5). This shows that (4.6) (which is the sum of (4.9) and (4.11)) has non-zero terms containing $\mathcal{Y}_{\eta(\breve{\mathfrak{A}})}(w_1,z)\mathcal{Y}_{\mathfrak{A}}(w_3,\zeta)w_5$. In other words, $\lambda_{\mathfrak{A},\mathfrak{A},\alpha',\beta'}\neq 0$ for some $s'\in\mathcal{E}'$ and $\alpha',\beta'\in\Theta^{s'}_{i',j'}$. This gives a contradiction. □

The following result was proved in [21]:

Theorem 4.3. *Let V be a vertex operator subalgebra of U satisfying (3.1). Assume that U, V, and V^c are CFT type, self-dual, and regular VOAs. Assume also that $V^{cc}=V$. Then any irreducible V-module is the compression of a U-module.*

The above two theorems imply immediately the following:

Theorem 4.4. *Under the assumption of Theorem 4.3, any irreducible intertwining operator of V is a compression of intertwining operators of U.*

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