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Polynomial energy bounds for type F_4 WZW-models

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We prove that for any type F_4 unitary affine VOA $V^l_{\mathfrak{f}_4}$, sufficiently many intertwining operators satisfy polynomial energy bounds. This finishes the Wassermann type analysis of intertwining operators for all WZW-models.

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0. Introduction

In his breakthrough work [24], Wassermann computed the fusion rules for type A WZW conformal nets, thus establishing the first important relation between the tensor structures of VOA modules and conformal net ones. One of the key steps in [24] is to prove that sufficiently many intertwining operators have bounded smeared primary (i.e. lowest weight) fields. The boundedness condition does not hold in general, and should be replaced by the *(polynomial) energy bounds condition* [5], which says roughly that the smeared intertwining operator is bounded by some nth power of the energy operator L_0. By generalizing the ideas of Wassermann, Toledano-Laredo established in [22] the energy bounds condition for sufficiently many intertwining operators of type D unitary affine VOAs. Similar results for type BCG were proved recently by the author in [12]. The main strategies are the same (which we call the compression method): One first show the energy bounds for level 1 intertwining operators. The higher level cases are treated inductively by considering the diagonal embedding $V^{l+1}_{\mathfrak{g}} \subset V^1_{\mathfrak{g}} \otimes V^l_{\mathfrak{g}}$ and showing that sufficiently many $V^{l+1}_{\mathfrak{g}}$-intertwining operators are compressions (i.e. restrictions) of $V^1_{\mathfrak{g}} \otimes V^l_{\mathfrak{g}}$-intertwining operators. Since the latter are energy-bounded by induction, the former are so.

The most difficult and technical part of this method is to verify that compressing the intertwining operators of the larger VOA (say $V^1_{\mathfrak{g}} \otimes V^l_{\mathfrak{g}}$) produce enough intertwining operators of the smaller VOA (say $V_{\mathfrak{g}}^{l+1}$). In [12, 22, 24] this is mostly done by Lie algebraic methods, which require case by case studies. Generalizing such analysis to type E (especially E_8) WZW-models will be still possible but significantly more difficult. Fortunately, when $\mathfrak{g}$ is of type ADE, due to the facts (cf. [2–4]) that $V_{\mathfrak{g}}^{l+1}$ and its commutant in $V_{\mathfrak{g}}^1 \otimes V_{\mathfrak{g}}^l$ are both regular, and that $V_{\mathfrak{g}}^{l+1}$ equals its double commutant, it is always possible to produce enough intertwining operators via compressions (cf. [14, 17]). This proves the energy bounds condition for *all* WZW-intertwining operators of type ADE (which, even in the case of type AD, is stronger than the results in [22, 24]). Moreover, this method is vertex algebraic and does not require case by case studies. Unfortunately, it is not known if the main results in [4] can be applied to other Lie types or not. Thus, so far, for the remaining type F_4, one can only prove the energy bounds using Lie theoretic methods. This is the goal of this paper.

Let $\mathfrak{f}_4$ be the type F_4 complex simple Lie algebra. Let $l \in \mathbb{Z}_+$. The main result of this paper is that sufficiently many intertwining operators of the unitary affine VOA $V_{\mathfrak{f}_4}^l$ is energy-bounded. To be more precise, let λ_4 be the highest weight associated to the smallest non-trivial irreducible $\mathfrak{f}_4$-module $L_{\mathfrak{f}_4}(\lambda_4)$ (which is of dimension 26 and admissible at level 1), and $L_{\mathfrak{f}_4}(\lambda_4, l)$ is the irreducible representation of the affine VOA $V_{\mathfrak{f}_4}^l$ with highest weight λ_4. We show that any intertwining operator with charge space $L_{\mathfrak{f}_4}(\lambda_4, l)$ is energy-bounded (Theorem 2.1), and that any irreducible $V_{\mathfrak{f}_4}^l$-module is a submodule of a tensor (fusion) product of $L_{\mathfrak{f}_4}(\lambda_4, l)$ (Theorem 5.1). (The second result says that considering only intertwining operators with charge space $L_{\mathfrak{f}_4}(\lambda_4, l)$ is "sufficient".) Theorem 5.1 is an easy consequence of $V_{\mathfrak{f}_4}^l$-fusion rules well known in the literature; Theorem 2.1 is the non-trivial part of our main result. In the following, we briefly explain the strategies of the proof by comparing $\mathfrak{f}_4$ with $\mathfrak{g}_2$ studied in [12].

One might guess that $\mathfrak{f}_4$ and $\mathfrak{g}_2$ can be treated in a similar way due to the conformal embedding $V_{\mathfrak{f}_4}^1 \otimes V_{\mathfrak{g}_2}^1 \subset V_{\mathfrak{e}_8}^1$. However, it turns out that $\mathfrak{f}_4$ is very different from $\mathfrak{g}_2$ and from all the classical Lie types in the following two aspects.

(1) For any complex simple Lie algebra $\mathfrak{g}$ not of type F_4 or E_8, the weight multiplicities of the smallest non-trivial irreducibles are bounded by 1. By contrast, $L_{\mathfrak{f}_4}(\lambda_4)$ has weight 0 with multiplicity 2. Consequently, the tensor product rules of $L_{\mathfrak{f}_4}(\lambda_4)$ exceed 1, which makes the analysis of $\mathfrak{f}_4$ more subtle.
(2) It seems very difficult (if not impossible) to reduce the higher level cases to the level 1 case using the Lie algebraic methods as in [12, 22, 24]. Thus, one also needs to treat level 2 separately. However, the method for $\mathfrak{g}_2$ level 1 (see [12] especially Lemma 5.5) can be applied only to $\mathfrak{f}_4$ level 1 but not level 2.

We resolve the second issue by exploiting the conformal embeddings

$$V_{\mathfrak{sl}_2}^1 \otimes V_{\mathfrak{sp}_6}^1 \subset V_{\mathfrak{f}_4}^1,$$

$$\mathrm{Vir}^{c_9} \otimes V_{\mathfrak{sl}_2}^2 \otimes V_{\mathfrak{sp}_6}^2 \subset V_{\mathfrak{f}_4}^2,$$

where Vir^{c_9} is the (regular) unitary Virasoro VOA with central charge $c_9 < 1$. Since the intertwining operators of the smaller VOAs are energy-bounded, so are those of the larger ones. See Sec. 3 for more details. As for the first issue, we show that the 2-dimensional weight 0 subspace $L_{\mathfrak{f}_4}(\lambda_4)[0]$ is spanned by two particular vectors (called $F_{\rho_3}v_{\rho_3}, F_{\rho_4}v_{\rho_4}$) killed by the homomorphisms in $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \lambda_4, \lambda_4)$ and in $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \lambda_3, \lambda_3)$, respectively. (Here λ_3 is the highest weight for the irreducible 273-dimensional representation.) The key result is Lemma 4.2, for which we give two different proofs. The importance of this result is that any homomorphism of the form $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \lambda, \lambda)$ (where λ is a dominant integral weight of $\mathfrak{f}_4$) reduces to a linear combination of those in $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \lambda_4, \lambda_4)$ and $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \lambda_3, \lambda_3)$. (For instance, see the proof of Proposition 6.1.)

Section 6 contains the most technical part of this paper, which reduces the four types of intertwining operators in Proposition 6.3 to those of levels 1 and 2. The main idea is the same as case (II) of [12, Sec. 5.2]. In particular, Proposition 6.2 and Lemma 6.1, on which the proof of Proposition 6.3 relies, also appears either explicitly or implicitly in [12]. Indeed, Proposition 6.2 is a (slight) generalization of [12, Lemma 2.15], whereas Lemma 6.1 generalizes the arguments in the proof of [12, Lemma 5.6].

The main result of this paper, together with those in [12, 14, 22, 24], is crucial for showing the equivalence of the VOA tensor categories and the conformal net tensor categories associated to WZW-models and their regular cosets. This is the main topic of [13] and will not be discussed in this paper.

1. Energy-Bounded Intertwining Operators

We refer the readers to [5, 7, 11] for the basics of unitary VOAs, unitary modules, and unitary intertwining operators. Let V be a unitary VOA. Thus, V is equipped with an inner product $\langle\cdot|\cdot\rangle$ (antilinear on the second variable) and an antilinear automorphism Θ (preserving the vacuum vector Ω and the conformal vector ν) such that for every $v, v_1, v_2 \in V$,

$$\langle v_1 | Y(v,\overline{z})v_2\rangle = \langle Y(e^{zL_1}(-z^{-2})^{L_0}\Theta v, z^{-1})v_1 | v_2\rangle.$$

Moreover, Θ is anti-unitary, and $\Theta^2 = \mathbf{1}_V$. A V-module (W_i, Y_i) is called unitary if W_i is equipped with an inner product $\langle\cdot|\cdot\rangle$ under which the above relation holds for any $v \in V, v_1, v_2 \in W_i$. In particular, V is a unitary V-module. The eigenvalues of L_0 on W_i are non-negative.

For each $v \in V$, we write $Y(v,z) = \sum_{n\in\mathbb{Z}} Y(v)_n z^{-n-1}$. Recall that if W_i, W_j, W_k are unitary V-modules, a type $\binom{W_k}{W_i W_j} = \binom{k}{i\ j}$ (unitary) intertwining operator $\mathcal{Y}$ is a linear map

$$\begin{aligned} W_i &\to \mathrm{End}(W_j, W_k)\{z\},\\ w_i &\mapsto \mathcal{Y}(w^{(i)}, z) = \sum_{n\in\mathbb{R}} \mathcal{Y}(w^{(i)})_n z^{-n-1}, \end{aligned}$$

where the sum above is the formal sum, each $\mathcal{Y}(w^{(i)})_n$ is in $\mathrm{End}(W_j, W_k)$, and the following conditions are satisfied:

(a) (Lower truncation) For any $w^{(j)} \in W_j$, $\mathcal{Y}(w^{(i)})_n w^{(j)} = 0$ when n is sufficiently large.
(b) (Jacobi identity) For any $u \in V, w^{(i)} \in W_i, m, n \in \mathbb{Z}, s \in \mathbb{R}$, we have

$$\sum_{l\in\mathbb{N}} \binom{m}{l} \mathcal{Y}(Y_i(u)_{n+l}w^{(i)})_{m+s-l},$$
$$= \sum_{l\in\mathbb{N}} (-1)^l \binom{n}{l} Y_k(u)_{m+n-l}\mathcal{Y}(w^{(i)})_{s+l}$$
$$- \sum_{l\in\mathbb{N}} (-1)^{l+n} \binom{n}{l} \mathcal{Y}(w^{(i)})_{n+s-l} Y_j(u)_{m+l}. \qquad (1)$$

(c) (L_{-1}-derivative) $\frac{d}{dz}\mathcal{Y}(w^{(i)}, z) = \mathcal{Y}(L_{-1}w^{(i)}, z)$.

We say that W_i, W_j, W_k are , respectively, the **charge space**, the **source space**, and the **target space** of $\mathcal{Y}$. If W_i, W_j, W_k are all irreducible, we say that $\mathcal{Y}$ is an **irreducible intertwining operator**.

Let $w^{(i)} \in W_i$ be a homogeneous vector (for simplicity), i.e. $w^{(i)}$ is an eigenvector of L_0. We say that $\mathcal{Y}(w^{(i)}, z)$ is **energy-bounded** (or satisfies polynomial energy bounds) if there exist $M, t, r \geq 0$ such that for any $w^{(j)} \in W_j, n \in \mathbb{R}$,

$$\|\mathcal{Y}(w^{(i)})_n w^{(j)}\| \leq M(1 + |n|)^t \|(1 + L_0)^r w^{(j)}\|,$$

where the norms $\|\cdot\|$ are defined by the inner products of W_j, W_k. We say that $\mathcal{Y}$ is energy-bounded if $\mathcal{Y}(w^{(i)}, z)$ is so for any homogeneous $w^{(i)} \in W_i$. We say that V is strongly energy-bounded if Y_i is energy-bounded for any unitary V-module W_i.

Suppose that $\mathfrak{g}$ is a finite dimensional (unitary) complex simple Lie algebra. Let $(\cdot|\cdot)$ be the (unique) invariant inner product under which the longest roots of $\mathfrak{g}$ have length $\sqrt{2}$. For each $l \in \mathbb{Z}_+ = \{1, 2, 3, \ldots\}$, the **affine VOA** $V^l_{\mathfrak{g}}$ is the unique unitary VOA generated by the weight 1 subspace $V(1)$ such that $V(1)$ (with the naturally defined Lie algebra structure) is equivalent to $\mathfrak{g}$, and that the inner product $\langle\cdot|\cdot\rangle$ on $V(1)$ equals l times $(\cdot|\cdot)$. $V^l_{\mathfrak{g}}$ is strongly energy-bounded. (See [5, Example 8.7]) Thus, by [11, Corollary 3.7(a)], we have

Proposition 1.1. *Let $\mathcal{Y}$ be an irreducible $V^l_{\mathfrak{g}}$-module with charge space W_i. If there exists a homogeneous nonzero $w^{(i)} \in W_i$ such that $\mathcal{Y}(w^{(i)}, z)$ is energy-bounded, then $\mathcal{Y}$ is energy-bounded.*

In the remaining part of this paper, unless otherwise stated, we shall let $\mathfrak{g}$ denote the type F_4 complex simple Lie algebra $\mathfrak{f}_4$.

2. The 26-Dimensional Representation $L_{\mathfrak{g}}(\lambda_4)$

We follow the notations in [12]. Recall that we set $\mathfrak{g} = \mathfrak{f}_4$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}^*$ is the space of weights of $\mathfrak{g}$. Let θ be the longest root of $\mathfrak{f}_4$. Recall that the invariant inner product $(\cdot|\cdot)$ on $\mathfrak{f}_4$ is chosen such that $(\theta|\theta) = 2$. One then have an isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ induced by this inner product. For any root α and weight λ, we set

$$n_{\lambda,\alpha} = \frac{2(\lambda|\alpha)}{(\alpha|\alpha)}. \tag{2}$$

$\mathfrak{g}$ has an involution $*$ such that $X^* = -X$ when X is an element in the compact real form. Then, with the $*$-structure, $\mathfrak{g}$ is a unitary Lie algebra. Thus, given a unitary representation W of $\mathfrak{g}$ where W is equipped with an inner product $\langle\cdot|\cdot\rangle$, we have for any $X \in \mathfrak{g}, u, v \in W$ that

$$\langle Xu|v\rangle = \langle u|X^*v\rangle.$$

In particular, the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}$ is unitary. Hence, for any $X, Y, Z \in \mathfrak{g}$, we have

$$([X, Y]|Z) = (Y|[X^*, Z]).$$

Let $l = \mathbb{Z}_+$. Let $P_+(\mathfrak{g})$ be the set of dominant integral weights of $\mathfrak{g}$. Let $P_+(\mathfrak{g}, l)$ be the set of all $\lambda \in P_+(\mathfrak{g})$ admissible at level l (i.e. $(\lambda|\theta) \leq l$). Let $V^l_{\mathfrak{g}}$ be the level l unitary affine VOA associated to $\mathfrak{g}$. Then, the irreducible unitary representations of $V^l_{\mathfrak{g}}$ are precisely those equivalent to some highest weight (equivalently, lowest energy) representation $L_{\mathfrak{g}}(\lambda, l)$, where $\lambda \in P_+(\mathfrak{g}, l)$. (See [9]) $L_{\mathfrak{g}}(\lambda)$ denotes the finite dimensional (unitary) irreducible representation of $\mathfrak{g}$ with highest weight λ. We also identify $L_{\mathfrak{g}}(\lambda)$ with the lowest energy subspace of $L_{\mathfrak{g}}(\lambda, l)$. If $\mu \in \mathfrak{h}$ is a weight, then $L_{\mathfrak{g}}(\lambda)[\mu]$ the μ-weight space of $\mathfrak{g}$, i.e. $L_{\mathfrak{g}}(\lambda)[\mu]$ consists of $v \in L_{\mathfrak{g}}(\lambda)$ such that $hv = (h|\mu)v$ for any $h \in \mathfrak{h}$.

$\mathfrak{f}_4$ has the following simple roots (described by the "orthogonal basis")

$$\rho_1 = [0, 1, -1, 0], \quad \rho_2 = [0, 0, 1, -1], \quad \rho_3 = [0, 0, 0, 1], \quad \rho_4 = \frac{1}{2}[1, -1, -1, -1].$$

The corresponding fundamental weights are

$$\lambda_1 = [1, 1, 0, 0], \quad \lambda_2 = [2, 1, 1, 0], \quad \lambda_3 = \frac{1}{2}[3, 1, 1, 1], \quad \lambda_4 = [1, 0, 0, 0].$$

We also have $\theta = \lambda_1 = [1, 1, 0, 0]$. We also adopt the following notation

$$(n_1, n_2, n_3, n_4) = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_4 + n_4\lambda_4.$$

$\mathfrak{f}_4$ has 24 positive roots, whose lengths are either 1 or $\sqrt{2}$. Those with length 1 fall into the following two groups. The roots in **Group A** are orthogonal to the highest root $\theta = \lambda_1$:

$$(0, -1, 2, -1) = [0, 0, 0, 1] \quad (= \rho_3)$$
$$(-1, 1, 0, -1) = [0, 0, 1, 0]$$
$$(0, 0, -1, 2) = \frac{1}{2}[1, -1, -1, -1] \quad (= \rho_4)$$
$$(0, -1, 1, 1) = \frac{1}{2}[1, -1, -1, 1]$$
$$(-1, 1, -1, 1) = \frac{1}{2}[1, -1, 1, -1]$$
$$(-1, 0, 1, 0) = \frac{1}{2}[1, -1, 1, 1]$$

Group B consists of those not orthogonal to θ. There are also six elements in group B:

$$[0, 1, 0, 0], \quad [1, 0, 0, 0], \quad \frac{1}{2}[1, 1, \pm 1, \pm 1].$$

Any positive root with length $\sqrt{2}$ is of the form $[a, b, c, d]$ where only two of a, b, c, d are nonzero; the first nonzero number is 1; the second one is 1 or -1. If α is a root, we choose a raising operator E_α. Then $F_\alpha := E_\alpha^*$ is a lowering operator. We normalize E_α such that $E_\alpha, F_\alpha, H_\alpha = [E_\alpha, F_\alpha]$ satisfy $[H_\alpha, E_\alpha] = 2E_\alpha$. Thus $[H_\alpha, F_\alpha] = -2E_\alpha$. Indeed, if β is also a root, then $[H_\alpha, E_\beta] = n_{\beta,\alpha} E_\beta, [H_\alpha, F_\beta] = -n_{\beta,\alpha} F_\beta$. More generally, due to the fact that for any $\lambda \in \mathfrak{h}^* \simeq \mathfrak{h}$, we have

$$(H_\alpha|\lambda) = n_{\lambda,\alpha},$$

if v is a λ-weight vector, then

$$H_\alpha v = n_{\lambda,\alpha} v.$$

We also have

$$(E_\alpha|E_\alpha) = (F_\alpha|F_\alpha) = \frac{2}{(\alpha|\alpha)}.$$

One can assume furthermore that

$$E_{-\alpha} = F_\alpha, \quad F_{-\alpha} = E_\alpha, \quad H_{-\alpha} = -H_\alpha.$$

See [12, Sec. 1.2] for more details.

$L_\mathfrak{g}(\lambda_4)$ is 26-dimensional and has 25 weights: the zero weight, the 12 roots in groups A and B, and there negatives. If a weight μ of $L_\mathfrak{g}(\lambda_4)$ is not zero, then $\dim L_\mathfrak{g}(\lambda_4)[\mu] = 1$. On the other hand, $\dim L_\mathfrak{g}(\lambda_4)[0] = 2$. These facts can be checked by LieART (see Sec. Appendix A).

The main result of this paper is:

Theorem 2.1. *Any irreducible intertwining operator of* $V^l_{\mathfrak{f}_4}$ *with charge space* $L_{\mathfrak{f}_4}(\lambda_4, l)$ *is energy-bounded.*

3. The Case $l = 1, 2$

In this section, we prove Theorem 2.1 when $l = 1, 2$. The proof for level 1 is similar but slightly simpler than level 2. So we will mainly focus on level 2. $\mathfrak{g} = \mathfrak{f}_4$ has a unitary Lie subalgebras $\mathfrak{a}_1$ and $\mathfrak{c}_3$ where $\mathfrak{a}_1$ is generated by the raising and the lowering operators of λ_1 and $\mathfrak{c}_3$ is generated by those of ρ_2, ρ_3, ρ_4. Then $\mathfrak{a}_1$ and $\mathfrak{c}_3$ are simple Lie algebras of type A_1, C_3, respectively, i.e. $\mathfrak{a}_1 = \mathfrak{sl}_2, \mathfrak{c}_3 = \mathfrak{sp}_6$. By the fact that $\rho_2 \pm \lambda_1, \rho_3 \pm \lambda_1, \rho_2 \pm \lambda_1$ are not roots of $\mathfrak{f}_4$ since there lengths exceed $\sqrt{2}$, we have $[\mathfrak{a}_1, \mathfrak{c}_3] = 0$, which shows that there is an embedding of unitary Lie algebras $\mathfrak{a}_1 \oplus \mathfrak{c}_3 \subset \mathfrak{f}_4$. Moreover, the long roots of $\mathfrak{a}_1, \mathfrak{c}_3$ both have lengths $\sqrt{2}$ under the normalized invariant inner product of $\mathfrak{g}$. Thus, the Dynkin indexes of $\mathfrak{a}_1 \subset \mathfrak{f}_4$ and $\mathfrak{c}_3 \subset \mathfrak{f}_4$ are both 1, and one thus have unitary vertex operator subalgebras

$$V^l_{\mathfrak{a}_1} \otimes V^l_{\mathfrak{c}_3} \subset V^l_{\mathfrak{f}_4}.$$

(See, for example, [12, Sec. 2.1]) By comparing the central charges, one sees that when $l = 1$, the above subalgebra is a conformal subalgebra, i.e. both sides have the same central charge and hence the same conformal vector. When $l = 2$, the difference between the two central charges is $c_9 = 1 - \frac{6}{9\cdot 10}$. Thus, we have conformal subalgebras

$$V^1_{\mathfrak{a}_1} \otimes V^1_{\mathfrak{c}_3} \subset V^1_{\mathfrak{f}_4},$$
$$\mathrm{Vir}^{c_9} \otimes V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3} \subset V^2_{\mathfrak{f}_4},$$

where Vir^{c_9} is the unitary Virasoro VOA with central charge c_9. By [6], unitary affine VOAs and unitary Virasoro VOAs with central charge $c < 1$ are regular. So $\mathrm{Vir}^{c_9} \otimes V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$ is regular. Thus, any unitary $V^2_{\mathfrak{f}_4}$-module, as a unitary $\mathrm{Vir}^{c_9} \otimes V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$-module, has a finite orthogonal irreducible decomposition, and each irreducible component has the form $W_1 \otimes W_2 \otimes W_3$, where W_1, W_2, W_3 are irreducible representations of $\mathrm{Vir}^{c_9}, V^2_{\mathfrak{a}_1}, V^2_{\mathfrak{c}_3}$, respectively. (Cf. [21, Proposition 2.20]).

Lemma 3.1. *$L_{\mathfrak{f}_4}(\lambda_4, 2)$ has an irreducible unitary $\mathrm{Vir}^{c_9} \otimes V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$-submodule*

$$W_1 \otimes L_{\mathfrak{a}_1}(\square, 2) \otimes L_{\mathfrak{c}_3}(\vartheta_1, 2), \tag{3}$$

where W_1 is an irreducible unitary Vir^{c_9}-module, $\square$ is the highest weight of the (2-dimensional) vector representation of $\mathfrak{sl}_2$, and ϑ_1 is the highest weight of the (6-dimensional) vector representation of $\mathfrak{sp}_6$.

The case of level 1 is similar and is left to the reader.

Proof. The lowest energy subspace of $L_{\mathfrak{f}_4}(\lambda_4, 2)$ is $L_{\mathfrak{f}_4}(\lambda_4)$. Since the restrictions of λ_4 to the Cartan subalgebras of $\mathfrak{a}_1$ and $\mathfrak{c}_3$ equal $\square$ and ϑ_1, respectively, the $\mathfrak{f}_4$-module $L_{\mathfrak{f}_4}(\lambda_4)$ has an irreducible $(\mathfrak{a}_1 \oplus \mathfrak{c}_3)$-submodule $L_{\mathfrak{a}_1}(\square) \otimes L_{\mathfrak{c}_3}(\vartheta_1)$. Thus, the weak $V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$-submodule of $L_{\mathfrak{f}_4}(\lambda_4, 2)$ generated by $L_{\mathfrak{a}_1}(\square) \otimes L_{\mathfrak{c}_3}(\vartheta_1)$ is equivalent to $L_{\mathfrak{a}_1}(\square, 2) \otimes L_{\mathfrak{c}_3}(\vartheta_1, 2)$. Recall that $L_{\mathfrak{f}_4}(\lambda_4, 2)$ is a finite sum of irreducible $\mathrm{Vir}^{c_9} \otimes V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$-submodules of the form $W_1 \otimes W_2 \otimes W_3$ which, as a weak $V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$-module, is a direct sum of $W_2 \otimes W_3$. Therefore, there must be an irreducible submodule $W_1 \otimes W_2 \otimes W_3$ with $W_2 \otimes W_3 \simeq L_{\mathfrak{a}_1}(\square, 2) \otimes L_{\mathfrak{c}_3}(\vartheta_1, 2)$. □

Proposition 3.1. *Theorem 2.1 holds when l equals 1 or 2.*

Proof. We only discuss the case of $l = 2$ since the other case can be treated in a similar way. Let $\mathcal{Y}$ be any intertwining operator of $V^2_{\mathfrak{f}_4}$ with charge space $L_{\mathfrak{f}_4}(\lambda_4, 2)$. By Proposition 1.1, it suffices to prove that $\mathcal{Y}(w, z)$ is energy-bounded for some nonzero homogeneous vector $w \in L_{\mathfrak{f}_4}(\lambda_4, 2)$. If we regard $\mathcal{Y}$ as an intertwining operator of $\mathrm{Vir}^{c_9} \otimes V^2_{\mathfrak{a}_1} \otimes V^2_{\mathfrak{c}_3}$, then, by the above lemma, one can restrict the charge space of $\mathcal{Y}$ of a charge subspace of the form (3). Let $\mathcal{Y}_0$ be the restriction of $\mathcal{Y}$ to this charge subspace. Then it suffices to prove that $\mathcal{Y}_0$ is energy-bounded.

By [1, Theorem 2.10], $\mathcal{Y}_0$ is a finite sum of intertwining operators of the form $\mathcal{Y}_1 \otimes \mathcal{Y}_2 \otimes \mathcal{Y}_3$ where $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ are irreducible intertwining operators of $\mathrm{Vir}^{c_9}, V^2_{\mathfrak{a}_1}, V^2_{\mathfrak{c}_3}$ with charge spaces $W_1, L_{\mathfrak{a}_1}(\square, 2), L_{\mathfrak{c}_3}(\vartheta_1, 2)$, respectively. By [12, Theorem 4.2], $\mathcal{Y}_3$ is energy-bounded. By [24], $\mathcal{Y}_2$ is energy-bounded. By [18, Proposition IV.1.3], any intertwining operator of a unitary $c < 1$ Virasoro VOA is energy-bounded.[a] So $\mathcal{Y}_1$ is energy-bounded. By the arguments in [5, Sec. 6], or [11, Proposition 3.5], tensor products of energy-bounded intertwining operators are energy-bounded. So $\mathcal{Y}_0$ is energy-bounded. □

Remark 3.1. The method in this section can be used to prove a similar result for the level 1 affine type G_2 VOA $V^1_{\mathfrak{g}_2}$. Let $\alpha_1 = [\sqrt{2/3}, 0], \alpha_2 = [-\sqrt{3/2}, \sqrt{1/2}]$ be the simple roots of $\mathfrak{g}_2$. Then $\alpha_3 = [0, \sqrt{2}]$ is a root of $\mathfrak{g}$ with squared length 2. We have embedding $\mathfrak{a}_1 \oplus \mathfrak{a}_1 \subset \mathfrak{g}_2$ with the first $\mathfrak{a}_1$ generated by the raising and the lowering operators of α_1, and the second one generated by those of α_3. This embedding induces a conformal extension $V^3_{\mathfrak{a}_1} \otimes V^1_{\mathfrak{a}_1} \subset V^1_{\mathfrak{g}_2}$. Let $\varsigma = [\sqrt{1/6}, \sqrt{1/2}]$ which is the highest weight of the 7-dimensional irreducible $\mathfrak{g}_2$-module. Then the $V^1_{\mathfrak{g}_2}$-module $L_{\mathfrak{g}_2}(\varsigma, 1)$ has an irreducible $V^3_{\mathfrak{a}_1} \otimes V^1_{\mathfrak{a}_1}$-submodule $L_{\mathfrak{a}_1}(\square, 3) \otimes L_{\mathfrak{a}_1}(\square, 1)$. Since the intertwining operators of $V^3_{\mathfrak{a}_1} \otimes V^1_{\mathfrak{a}_1}$ with such charge space are energy-bounded, so are those of $V^1_{\mathfrak{g}_2}$ with charge space $L_{\mathfrak{g}_2}(\varsigma, 1)$.

4. The Tensor (Fusion) Product Rule N^{ν}_{μ}

For each $\lambda \in P_+(\mathfrak{g}, l)$, we let Δ_λ be the conformal weight of $L_{\mathfrak{g}}(\lambda, l)$, i.e. Δ_λ is the smallest eigenvalue of L_0 on $L_{\mathfrak{g}}(\lambda, l)$. Note that Δ_λ depends on l. Indeed, we have $\Delta_\lambda = \frac{C_\lambda}{2(l+h^\vee)}$ where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, and C_λ is the Casimir number of $L_{\mathfrak{g}}(\lambda)$.

We denote by $\mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}$ the vector space of type $\binom{\nu}{\lambda\ \mu} = \binom{L_{\mathfrak{g}}(\nu,l)}{L_{\mathfrak{g}}(\lambda,l)\ L_{\mathfrak{g}}(\mu,l)}$ intertwining operators of $V^l_{\mathfrak{g}}$, assuming that λ, μ, ν are admissible at level l. We let $N^{\nu}_{\lambda\mu}$ be

[a] This also follows from the fact that any intertwining operator of a type A discrete series W-algebra is energy-bounded. See the introduction of [14].

the dimension of this vector space. Since in this paper, we will be mainly interested in the case that $\lambda = \lambda_4$, we let

$$N_\mu^\nu := N_{\lambda_4\mu}^\nu = \dim \mathcal{V}_{\mathfrak{g}}^l \binom{\nu}{\lambda_4\ \mu}. \tag{4}$$

Set

$$\Delta_{\lambda\mu}^\nu = \Delta_\lambda + \Delta_\mu - \Delta_\nu.$$

We also write $\mathrm{Hom}_{\mathfrak{g}}(L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu))$ as $\mathrm{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$ for short. Then for any $\mathcal{Y} \in \mathcal{V}_{\mathfrak{g}}^l\binom{\nu}{\lambda\ \mu}$ and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda), v^{(\mu)} \in L_{\mathfrak{g}}(\mu)$, we have $\mathcal{Y}(u^{(\lambda)})_{\Delta_{\lambda\mu}^\nu - 1} \cdot u^{(\mu)} \in L_{\mathfrak{g}}(\nu)$. We define the linear map

$$\Psi : \mathcal{V}_{\mathfrak{g}}^l \binom{\nu}{\lambda\ \mu} \to \mathrm{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \tag{5}$$

by sending each element $\mathcal{Y}$ to $\Psi\mathcal{Y}$ satisfying

$$\Psi\mathcal{Y}(u^{(\lambda)} \otimes u^{(\mu)}) = \mathcal{Y}(u^{(\lambda)})_{\Delta_{\lambda\mu}^\nu - 1} \cdot u^{(\mu)}.$$

It is well known that Ψ is injective. Moreover, if $(\lambda|\theta) = 1$ (for example, if $\lambda = \lambda_4$) then Ψ is also surjective. (See [10, (5.9)]) In this case, the fusion rule agrees with the truncated tensor product rule:

$$N_\lambda^\nu = \dim \mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \mu, \nu).$$

In particular, N_μ^ν is independent of the level l at which μ, ν are admissible.

Thus, to study the dimension of $\mathcal{V}_{\mathfrak{g}}^l\binom{\nu}{\lambda_4\ \mu}$, it suffices to understand the tensor product rules of $L_{\mathfrak{g}}(\lambda_4)$. Let $v_\mu \in L_{\mathfrak{g}}(\mu)[\mu], v_\nu \in L_{\mathfrak{g}}(\nu)[\nu]$ be (nonzero) highest weight vectors of $L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu)$, respectively. Define a linear map

$$\Gamma : \mathrm{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu) \to L_{\mathfrak{g}}(\lambda)[\nu - \mu]^*, \tag{6}$$

such that for any $T \in \mathrm{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$, ΓT, as a linear functional on $L_{\mathfrak{g}}(\lambda)[\nu - \mu]$, is defined by

$$(\Gamma T)(u^{(\lambda)}) = \langle T(u^{(\lambda)} \otimes v_\mu) | v_\nu \rangle \tag{7}$$

for any $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]$. Then, Γ is injective.

The image of Γ can be described as follows. Let $K_{\mathfrak{g}}^\mu(\lambda)[\nu - \mu]$ be the subspace of $L_{\mathfrak{g}}(\lambda)[\nu - \mu]$ spanned by vectors of the form $F_\alpha^{n_{\mu,\alpha}+1} u^{(\lambda)}$, where α is a simple root of $\mathfrak{g}$, and $u^{(\lambda)} \in L_{\mathfrak{g}}(\lambda)$ has weight $\nu - \mu + (n_{\mu,\alpha} + 1)\alpha$. Denote by $K_{\mathfrak{g}}^\mu(\lambda)[\nu - \mu]^\perp$ the set of elements of $L_{\mathfrak{g}}(\lambda)[\nu - \mu]^*$ vanishing on $K_{\mathfrak{g}}^\mu(\lambda)[\nu - \mu]$. Then by [12, Proposition 1.17], we have

$$\boxed{\mathrm{Im}(\Gamma) = K_{\mathfrak{g}}^\mu(\lambda)[\nu - \mu]^\perp.} \tag{8}$$

Thus,

$$N_{\lambda\mu}^\nu = \dim L_{\mathfrak{g}}(\lambda)[\nu - \mu] - \dim K_{\mathfrak{g}}^\mu(\lambda)[\nu - \mu]. \tag{9}$$

For example, fix nonzero vectors

$$v_{\rho_3} \in L_{\mathfrak{g}}(\lambda_4)[\rho_3], \quad v_{\rho_4} \in L_{\mathfrak{g}}(\lambda_4)[\rho_4]. \tag{10}$$

Since we know that a simple root α must be one of $\rho_1, \rho_2, \rho_3, \rho_4$, it is easy to see that

$$K_{\mathfrak{g}}^{\lambda_3}(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_4} v_{\rho_4}, \quad K_{\mathfrak{g}}^{\lambda_4}(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_3} v_{\rho_3}. \tag{11}$$

As we shall see immediately, these two vectors are nonzero and (indeed) linearly independent. Thus, we have $N_{\lambda_3}^{\lambda_3} = N_{\lambda_4}^{\lambda_4} = 1$. More generally, we have:

Lemma 4.1. *Let $\mu = n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 + n_4\lambda_4$ be a dominant integral weight.*

(a) *If $n_3 = n_4 = 0$, then $N_\mu^\mu = 0$.*
(b) *If $n_3 > 0$ and $n_4 = 0$, then $N_\mu^\mu = 1$ and $K_{\mathfrak{g}}^\mu(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_4} v_{\rho_4}$.*
(c) *If $n_3 = 0$ and $n_4 > 0$, then $N_\mu^\mu = 1$ and $K_{\mathfrak{g}}^\mu(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_3} v_{\rho_3}$.*
(d) *If $n_3, n_4 > 0$, then $N_\mu^\mu = 2$.*

We shall prove this lemma together with the following one:

Lemma 4.2. *$F_{\rho_3} v_{\rho_3}$ and $F_{\rho_4} v_{\rho_4}$ form a basis of $L_{\mathfrak{g}}(\lambda_4)[0]$.*

Proof. We know that any vector in $K_{\mathfrak{g}}^\mu(\lambda_4)[0]$ must be of the form $F_\alpha^{n_{\mu,\alpha}+1} u$ where $u \in L_{\mathfrak{g}}(\lambda_4)$ has weight $(n_{\mu,\alpha}+1)\alpha$ and α is one of $\rho_1, \ldots, \rho_4$. Since $(n_{\mu,\alpha}+1)\alpha$ is one of the 25 weights of $L_{\mathfrak{g}}(\lambda_4)$, the only possible case is that $n_{\mu,\alpha} = 0$ (equivalently, $(\mu|\alpha) = 0$) and $\alpha \in \{\rho_3, \rho_4\}$. (Note that ρ_1, ρ_2 are not weights of $L_{\mathfrak{g}}(\lambda_4)$.)

It is easy to calculate that

$$(\mu|\rho_3) = \frac{n_3}{2}, \quad (\mu|\rho_4) = \frac{n_4}{2}.$$

In case (a), μ is orthogonal to both ρ_3, ρ_4. It follows that $K_{\mathfrak{g}}^\mu(\lambda_4)[0]$ is spanned by $F_{\rho_3} v_{\rho_3}, F_{\rho_4} v_{\rho_4}$. In particular, $K_{\mathfrak{g}}^0(\lambda_4)[0]$ is spanned by the two vectors. Since $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes 0, 0)$ is clearly trivial, by (9), we must have $K_{\mathfrak{g}}^0(\lambda_4)[0] = L_{\mathfrak{g}}(\lambda)[0]$ whose dimension is 2. Thus, $F_{\rho_3} v_{\rho_3}, F_{\rho_4} v_{\rho_4}$ span $L_{\mathfrak{g}}(\lambda_4)[0]$. This proves Lemma 4.2. By this lemma, $K_{\mathfrak{g}}^\mu(\lambda_4)[0]$, which has dimension 2, equals $L_{\mathfrak{g}}(\lambda_4)[0]$. So $N_\mu^\mu = 0$.

Assume case (b). Then μ is orthogonal to ρ_4 but not ρ_3. It follows that $K_{\mathfrak{g}}^\mu(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_4} v_{\rho_4}$. Similarly, in case (c), μ is orthogonal to ρ_3 but not ρ_4. So $K_{\mathfrak{g}}^\mu(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_3} v_{\rho_3}$. Finally, in case (d), neither ρ_3 nor ρ_4 is orthogonal to μ. So $K_{\mathfrak{g}}^\mu(\lambda_4)[0]$ must be trivial. □

Similar to Lemma 4.2, we have:

Lemma 4.3. *Let $\rho_5 = \frac{1}{2}[-1, 1, 1, -1]$ whose negative is a root in group A. Choose a nonzero $v_{\rho_5} \in L_{\mathfrak{g}}(\lambda_4)[\rho_5]$. Then $F_{\rho_3} v_{\rho_3}$ and $F_{\rho_5} v_{\rho_5}$ form a basis of $L_{\mathfrak{g}}(\lambda_4)[0]$.*

Proof. For each root α, let ϖ_α be the element in the Weyl group defined by the reflection along the hyperplane orthogonal to α. In other words, for each $\lambda \in \mathfrak{h}$,

$\varpi_\alpha(\lambda) = \lambda - n_{\lambda,\alpha} \cdot \alpha$. Now we set $\alpha = [0, 1, 1, 0]$ and $\beta = [1, 0, 0, 0]$. Set $\varpi = \varpi_\alpha \varpi_\beta$. Then $\varpi(\rho_3) = \rho_3$ and $\varpi(\rho_4) = \rho_5$. Thus, $F_{\rho_3} v_{\rho_3}$ and $F_{\rho_5} v_{\rho_5}$ are linearly independent since $F_{\rho_3} v_{\rho_3}$ and $F_{\rho_4} v_{\rho_4}$ are so. Indeed, one can show that two vectors are not parallel by calculating the angle between them. Due to the equivalence induced by reflections, the angles between $F_{\rho_3} v_{\rho_3}, F_{\rho_5} v_{\rho_5}$ and between $F_{\rho_3} v_{\rho_3}, F_{\rho_4} v_{\rho_4}$ can be calculated using the same algorithm and share similar properties. (See Sec. Appendix B for an instance of calculating the angle.) □

When $\nu - \mu$ is a nonzero weight of $L_{\mathfrak{g}}(\lambda_4)$, the weight space $L_{\mathfrak{g}}(\lambda_4)[\nu - \mu]$ has dimension 1. In this case, the number N_μ^ν can be calculated by the following method. (Cf. [12, Corollary 1.18]).

Proposition 4.1. *Let $\mu, \nu \in P_+(\mathfrak{g})$. Assume that $\nu - \mu$ is a nonzero weight of $L_{\mathfrak{g}}(\lambda_4)$. Then $N_\mu^\nu = 1$ if and only if for any $\alpha \in \{\rho_1, \rho_2, \rho_3, \rho_4\}$,*

$$\dim L_{\mathfrak{g}}(\lambda_4)[\nu - \mu + (n_{\mu,\alpha} + 1)\alpha] = 0.$$

Otherwise, $N_\mu^\nu = 0$.

5. The Fundamental Types

We say that an intertwining operator in $\mathcal{V}_{\mathfrak{g}}^l \binom{\nu}{\lambda_4 \ \mu}$ is of type $\binom{\nu}{\mu}$ level l.

We have shown that the level 1 intertwining operators with charge space $L_{\mathfrak{g}}(\lambda_4, 1)$ are energy-bounded. Thus, by [12, Proposition 2.14] and that $(\lambda_4|\theta) = 1$, to prove the energy bounds condition for any type $\binom{\nu}{\mu}$ level l intertwining operator, it suffices to consider the case that $(\mu|\theta) = (\nu|\theta) = l$. In this case, $\nu - \mu$, which is orthogonal to θ, is either 0 or one of the six roots in group A. Also, since $N_{\lambda_4\lambda_4}^0 = 1$ (by, for example, Proposition 4.1), $L_{\mathfrak{g}}(\lambda_4, l)$ is self-dual. Thus, any type $\binom{\nu}{\mu}$ level l intertwining operator is the adjoint of a type $\binom{\mu}{\nu}$ level l intertwining operator. (See [11, Sec. 1.3] for the definition of adjoint intertwining operators.) Thus, by [11, Corollary 3.7(d)] type $\binom{\nu}{\mu}$ level l intertwining operators are energy-bounded if and only if type $\binom{\mu}{\nu}$ level l intertwining operators are so.

Definition 5.1. Assume $k, l \in \mathbb{Z}_+$ and $k \leq l$. Let $\mu_0, \nu_0 \in P_+(\mathfrak{g}, k)$ and $\mu, \nu \in P_+(\mathfrak{g}, l)$. We say that type $\binom{\nu}{\mu}$ level l **reduces to** type $\binom{\nu_0}{\mu_0}$ level k, if the following conditions are satisfied:

(a) $N_\mu^\nu \leq N_{\mu_0}^{\nu_0}$.
(b) There exists $\rho \in P_+(\mathfrak{g}, l - k)$ such that $\mu = \mu_0 + \rho$ and $\nu = \nu_0 + \rho$.

The following result generalizes [12, Lemma 2.15(a)], and applies to all Lie types but not just $\mathfrak{f}_4$.

Proposition 5.1. *Suppose that type $\binom{\nu}{\mu}$ level l reduces to type $\binom{\nu_0}{\mu_0}$ level k. Then $N_\mu^\nu = N_{\mu_0}^{\nu_0}$. Moreover, all type $\binom{\nu}{\mu}$ level l intertwining operators are energy-bounded if all type $\binom{\nu_0}{\mu_0}$ level k intertwining operators are so.*

Proof. The main idea is the same as in the proof of [12, Lemma 2.15(a)]. Define a linear map $\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda_4\ \mu_0} \to \mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda_4\ \mu}$, $\mathcal{Y} \mapsto \widetilde{\mathcal{Y}}$ as follows. Set $k' = l - k$. Set irreducible unitary $V^k_{\mathfrak{g}} \otimes V^{k'}_{\mathfrak{g}}$-modules

$$W_1 = L_{\mathfrak{g}}(\lambda_4, k) \otimes L_{\mathfrak{g}}(0, k'),$$

$$W_2 = L_{\mathfrak{g}}(\mu_0, k) \otimes L_{\mathfrak{g}}(\rho, k'),$$

$$W_3 = L_{\mathfrak{g}}(\nu_0, k) \otimes L_{\mathfrak{g}}(\rho, k').$$

Note that the vertex operator Y_ρ associated to the $V^{k'}_{\mathfrak{g}}$-module $L_{\mathfrak{g}}(\rho, k')$ is of type $\binom{\rho}{0\ \rho}$. For each $\mathcal{Y}$ of type $\binom{\nu_0}{\mu_0}$ level k,

$$\mathfrak{Y} := \mathcal{Y} \otimes Y_\rho$$

is an intertwining operator of $V^k_{\mathfrak{g}} \otimes V^{k'}_{\mathfrak{g}}$ of type $\binom{W_3}{W_1 W_2}$. Since $V^{k'}_{\mathfrak{g}}$ is strongly energy-bounded, the vertex operator Y_ρ is energy-bounded. So $\mathfrak{Y}$ is energy-bounded if $\mathcal{Y}$ is so.

Let $v_{\lambda_4}, v_{\mu_0}, v_{\nu_0}, \Omega, v_\rho$ be the highest weight (and lowest energy) vectors of $L_{\mathfrak{g}}(\lambda_4, k), L_{\mathfrak{g}}(\mu_0, k), L_{\mathfrak{g}}(\nu_0, k), L_{\mathfrak{g}}(0, k'), L_{\mathfrak{g}}(\rho, k')$. (Note that Ω can be chosen to be the vacuum vector of the VOA $V^{k'}_{\mathfrak{g}} = L_{\mathfrak{g}}(0, k')$.) Set

$$w_1 = v_{\lambda_4} \otimes \Omega, \quad w_2 = v_{\mu_0} \otimes v_\rho, \quad w_3 = v_{\nu_0} \otimes v_\rho$$

which are homogeneous vectors in W_1, W_2, W_3, respectively. Consider the diagonal embedding $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$ which induces $V^l_{\mathfrak{g}} \subset V^k_{\mathfrak{g}} \otimes V^{k'}_{\mathfrak{g}}$. Then, w_1, w_2, w_3 are highest weight vectors of the level l affine Lie algebra $\widehat{\mathfrak{g}}_l$ with weights λ_4, μ, ν, respectively. Let $\widetilde{\mathcal{Y}}$ be the "restriction" of $\mathfrak{Y}$ to the irreducible $V^l_{\mathfrak{g}}$-submodules generated by w_1, w_2, w_3, respectively. To be more precise, note that the unitary $\mathfrak{g}$-modules $\mathfrak{g}w_1, \mathfrak{g}w_2, \mathfrak{g}w_3$ are equivalent to $L_{\mathfrak{g}}(\lambda_4), L_{\mathfrak{g}}(\mu), L_{\mathfrak{g}}(\nu)$, respectively. Let e_3 be the orthogonal projection of W_3 onto $\mathfrak{g}w_3$. Let $s = \Delta^{\nu_0}_{\lambda_4 \mu_0} - 1$, and define $T_{\mathcal{Y}} \in \mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \mu, \nu)$ by

$$\begin{aligned} T_{\mathcal{Y}} : L_{\mathfrak{g}}(\lambda_4) \otimes L_{\mathfrak{g}}(\mu) &\to L_{\mathfrak{g}}(\nu) \\ u^{(\lambda_4)} \otimes u^{(\mu)} &\mapsto e_3 \mathfrak{Y}(u^{(\lambda_4)})_s u^{(\mu)}. \end{aligned}$$

Then, by [12, Theorem 2.12], there is a unique $\widetilde{\mathcal{Y}}$ of type $\binom{\nu}{\mu}$ level l such that $\Psi\widetilde{\mathcal{Y}} = T_{\mathcal{Y}}$, (Recall that the injective map Ψ is defined in (5).) and $\widetilde{\mathcal{Y}}$ is energy-bounded if $\mathcal{Y}$ is so. (Indeed, by the proof of that theorem, $\widetilde{\mathcal{Y}}$ is chosen to be $z^a \cdot p_3 \mathfrak{Y}(p_1, z) p_2$ for some $a \in \mathbb{R}$, where p_1, p_2, p_3 are the orthogonal projections of W_3 onto $\widehat{\mathfrak{g}}w_1, \widehat{\mathfrak{g}}w_2, \widehat{\mathfrak{g}}w_3$, respectively.)

From the definition of $\mathfrak{Y}$ and $T_{\mathcal{Y}}$, it is not hard to see that $\Gamma\Psi\mathcal{Y}$ equals $\Gamma T_{\mathcal{Y}}$. Therefore, as $\Psi\widetilde{\mathcal{Y}} = T_{\mathcal{Y}}$, we have $\Gamma\Psi\mathcal{Y} = \Gamma\Psi\widetilde{\mathcal{Y}}$. Thus, by the injectivity of Ψ and Γ, we have $\mathcal{Y} = 0$ if and only if $\widetilde{\mathcal{Y}} = 0$. Therefore, the linear map $\mathcal{Y} \mapsto \widetilde{\mathcal{Y}}$ is injective. By condition (a), this map is bijective. $\square$

Proposition 5.2. *Assume that $(\nu|\theta) = (\mu|\theta) = l$. Then any intertwining operator of type $\binom{\nu}{\mu}$ level l reduces to one of the following **fundamental types**:*

(1) *Type $\binom{2\lambda_3}{\lambda_2+\lambda_4}$ level 4 and its adjoint.*
(2) *Type $\binom{\lambda_2}{\lambda_1+\lambda_4}$ level 3 and its adjoint.*
(3) *Type $\binom{2\lambda_4}{\lambda_3}$ level 2 and its adjoint.*
(4) *Type $\binom{\lambda_3+\lambda_4}{\lambda_2}$ level 3 and its adjoint.*
(5) *Type $\binom{\lambda_2+\lambda_4}{\lambda_1+\lambda_3}$ level 4 and its adjoint.*
(6) *Type $\binom{\lambda_3}{\lambda_1}$ level 2 and its adjoint.*
(7) *Type $\binom{\lambda_3+\lambda_4}{\lambda_3+\lambda_4}$ level 3.*
(8) *Type $\binom{\lambda_3}{\lambda_3}$ level 2.*
(9) *Type $\binom{\lambda_4}{\lambda_4}$ level 1.*

Proof. Assume $N_\mu^\nu > 0$. By Lemma 4.1, if $\nu = \mu$, then any type reduces to one of (7)– (9). Now assume $\nu \neq \mu$. Then $N_\mu^\nu \leq 1$, and either $\nu - \mu$ or its negative is one of the six roots in group A. (See Sec. 2.) By Proposition 4.1 or by LieART computations (see Sec. Appendix A), it is not hard to check that each of the first six cases has fusion/tensor product rule 1. Moreover, these six cases correspond to the six positive roots in group A. Assume $\nu - \mu$ is positive. So $\nu - \mu$ is a positive root in group A. Then $\binom{\nu}{\mu}$ level l reduces the one of (1)–(6) corresponding to $\nu - \mu$. If $\nu - \mu$ is negative, then $\binom{\mu}{\nu}$, which is adjoint to $\binom{\nu}{\mu}$, reduces to the one of (1)–(6). □

Note that only case (7) has fusion rule 2. All the other fundamental types have fusion rule 1 by (for instance) Proposition 4.1.

We close this section with an application of Proposition 5.1. Given $\lambda, \mu \in P_+(\mathfrak{f}_4, l)$, one can define the fusion product

$$L_{\mathfrak{f}_4}(\lambda, l) \boxtimes L_{\mathfrak{f}_4}(\mu, l) = \bigoplus_{\nu \in P_+(\mathfrak{f}_4, l)} L_{\mathfrak{f}_4}(\lambda, l) \otimes \mathcal{V}_{\mathfrak{f}_4}^l \begin{pmatrix} \nu \\ \lambda \ \mu \end{pmatrix}^*,$$

which we write as $\lambda \boxtimes \mu$ for short. (Note that $\lambda \boxtimes \mu$ depends on the level l.) Thus, the multiplicity of ν in $\lambda \boxtimes \mu$ is the fusion rule $N_{\lambda\mu}^\nu$. Fusion products of more than two irreducible modules can be defined inductively. The associativity of $\boxtimes$ (i.e. $(\lambda \boxtimes \mu) \boxtimes \nu \simeq \lambda \boxtimes (\mu \boxtimes \nu)$) follows from, say, [23] or [15, 16] or [20].

Theorem 5.1. *For any $l = 1, 2, 3, \ldots$, the irreducible $V_{\mathfrak{f}_4}^l$-module $L_{\mathfrak{f}_4}(\lambda_4, l)$ $\boxtimes$-generates the category of (semisimple) $V_{\mathfrak{f}_4}^l$-modules. Namely, any irreducible $V_{\mathfrak{f}_4}^l$-module is equivalent to a submodule of a (finite) fusion product of $L_{\mathfrak{f}_4}(\lambda_4, l)$.*

Proof. Let Q be a subset of $P_+(\mathfrak{f}_4, l)$ consisting of all ν such that $L_{\mathfrak{f}_4}(\nu, l)$ appears as an irreducible submodule of a fusion product of $L_{\mathfrak{f}_4}(\lambda_4, l)$. Equivalently, Q is the smallest subset containing λ_4 and satisfying that if $\mu \in Q$ and $N_\mu^\nu \equiv N_{\lambda_4\ \mu}^\nu > 0$ then $\nu \in Q$. Note that $0 \in Q$ since λ_4 is self-dual.

We first claim that if $\lambda, \mu \in Q$ and $\lambda + \mu$ is admissible at level l then $\lambda + \mu \in Q$. By the associativity of $\boxtimes$, it suffices to show that $N_{\lambda\ \mu}^{\lambda+\mu} > 0$. Suppose $N_{\lambda\ \mu}^{\lambda+\mu} = 0$. Set $k = (\theta|\lambda), \nu = \lambda + \mu, \mu_0 = 0, \nu_0 = \lambda$. Then, type $\binom{\nu}{\lambda\ \mu}$ level l reduces to type $\binom{\lambda}{\lambda\ 0}$ level k in the same sense of Definition 5.1, namely, we have (a) $N_{\lambda\mu}^{\nu} \leq N_{\lambda\mu_0}^{\nu_0}$, and (b) there exists $\rho \in P_+(\mathfrak{f}_4, l-k)$ such that $\mu = \mu_0 + \rho$ and $\nu = \nu_0 + \rho$. (Set $\rho = \mu$.) Thus, by the proof of Proposition 5.1, we have $N_{\lambda\mu}^{\nu} = N_{\lambda\mu_0}^{\nu_0}$, i.e., $N_{\lambda\ \mu}^{\lambda+\mu} = N_{\lambda\ 0}^{\lambda}$, which equals 1. This gives a contradiction.

We now use the above result to prove $Q = P_+(\mathfrak{f}_4, l)$. This is clearly true when $l = 1$ (since both sets equal $\{0, \lambda_4\}$). Assume $l \geq 2$. Then, by the above result, $2\lambda_4 \in Q$. By for instance Proposition 4.1, we have $N_{2\lambda_4}^{\lambda_3} \equiv N_{\lambda_4\ 2\lambda_4}^{\lambda_3} = 1$ (cf. Proposition 5.2(3)). So $\lambda_3 \in Q$. Since $N_{\lambda_3}^{\lambda_1} = 1$ (cf. Propostion 5.2(6)), $\lambda_1 \in Q$. Thus $Q = P_+(\mathfrak{f}_4, l) = \{0, \lambda_4, 2\lambda_4, \lambda_1, \lambda_3\}$ when $l = 2$. Assume $l \geq 3$. Then $\lambda_1 + \lambda_4 \in Q$ by the above paragraph. Thus, as $N_{\lambda_1+\lambda_4}^{\lambda_2} = 1$ (cf. Proposition 5.2(2)), we conclude $\lambda_2 \in Q$. By the above paragraph, any $n_1\lambda_1 + \cdots + n_4\lambda_4$, if admissible at level l, is in Q. This proves $Q = P_+(\mathfrak{f}_4, l)$ in general. □

6. Proof for the Fundamental Types

By the results in the last section, to prove Theorem 2.1, it suffices to prove the energy bounds condition for any intertwining operator of fundamental type. Since the case $l \leq 2$ has already been proved, it remains to prove the cases (1) (2) (4) (5) (7) in Proposition 5.2. We first discuss the easier case.

Proof for case (7)

Proposition 6.1. *The intertwining operators of type* $\binom{\lambda_3+\lambda_4}{\lambda_3+\lambda_4}$ *level* 3 *are energy-bounded.*

Proof. Choose a nonzero intertwining operator $\mathcal{Y}_1$ of type $\binom{\lambda_3}{\lambda_3}$ level 2. Then $\mathcal{Y}_1$ is energy-bounded by Proposition 3.1. As in the proof of Proposition 5.1, one can construct $\widetilde{\mathcal{Y}}_1$ of type $\binom{\lambda_3+\lambda_4}{\lambda_3+\lambda_4}$ level 3 by compressing $\mathcal{Y}_1 \otimes Y_{\lambda_4}$, where Y_{λ_4} is the vertex operator of $L_{\mathfrak{g}}(\lambda_4, 1)$. Moreover, $\widetilde{\mathcal{Y}}_1$ is energy-bounded, and by the last paragraph of that proof, $\Gamma\Psi\widetilde{\mathcal{Y}}_1 = \Gamma\Psi\mathcal{Y}_1$. Now, $\Psi\mathcal{Y}_1$ is a nonzero element in $\mathrm{Hom}_{\mathfrak{g}}(\lambda_4 \otimes \lambda_3, \lambda_3)$. Thus, by (8), $\Gamma\Psi\widetilde{\mathcal{Y}}_1$ is a nonzero linear functional on $L_{\mathfrak{g}}(\lambda_4)[0]$ orthogonal to $K_{\mathfrak{g}}^{\mu_3}(\lambda_3)[0]$. Therefore, by Lemma 4.1, $\Gamma\Psi\widetilde{\mathcal{Y}}_1$ kills $F_{\rho_4}v_{\rho_4}$.

Similarly, we choose a nonzero $\mathcal{Y}_2$ of type $\binom{\lambda_4}{\lambda_4}$ level 1. Then one can construct $\widetilde{\mathcal{Y}}_2$ of type $\binom{\lambda_3+\lambda_4}{\lambda_3+\lambda_4}$ level 3 by compressing $\mathcal{Y}_2 \otimes Y_{\lambda_3}$ where Y_{λ_3} is the vertex operator of $L_{\mathfrak{g}}(\lambda_3, 2)$. Then $\widetilde{\mathcal{Y}}_2$ is energy-bounded, and, by (8) and Lemma 4.1, $\Gamma\Psi\widetilde{\mathcal{Y}}_2$ kills $F_{\rho_3}v_{\rho_3}$. Therefore, by Lemma 4.2, $\widetilde{\mathcal{Y}}_1$ and $\widetilde{\mathcal{Y}}_2$ are linearly independent. So the 2-dimensional vector space $\mathcal{V}_{\mathfrak{g}}^3\binom{\lambda_3+\lambda_4}{\lambda_4\ \lambda_3+\lambda_4}$ is spanned by the energy-bounded intertwining operators $\widetilde{\mathcal{Y}}_1$ and $\widetilde{\mathcal{Y}}_2$. This completes the proof. □

Proof for cases (1) (2) (4) (5)

Our proof of these four cases relies on the following proposition which holds for any complex (unitary) finite dimensional simple Lie algebra. The statement and the proof are similar to (but slightly more general than) [12, Lemma 2.15b]. It is also a generalization of Proposition 5.1.

Proposition 6.2. *Choose* $\lambda,\mu,\nu\in P_+(\mathfrak{g},l)$ *satisfying* $\dim\mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}>0$. *Assume that there exist* $\rho,\mu_0,\nu_0\in P_+(\mathfrak{g},l)$ *satisfying the following conditions.* (*We set* $k=\max\{(\lambda|\theta),(\mu_0|\theta),(\nu_0|\theta)\}$.)

(a) $\dim\mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}\leq\dim\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}$.
(b) $\mu=\mu_0+\rho$, *and* $\dim L_{\mathfrak{g}}(\nu_0)[\nu-\rho]=\dim\mathrm{Hom}_{\mathfrak{g}}(\nu_0\otimes\rho,\nu)=1$.
(c) $(\rho|\theta)+k\leq l$.
(d) *For any nonzero* $\mathcal{Y}\in\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}$,

$$(\Psi\mathcal{Y})(L_{\mathfrak{g}}(\lambda)[\nu-\mu]\otimes L_{\mathfrak{g}}(\mu_0)[\mu_0])\neq 0. \tag{12}$$

(*Note that the above expression is a subspace of* $L_{\mathfrak{g}}(\nu_0)[\nu-\rho]$.) *Then* $\dim\mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}=\dim\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}$, *and all the intertwining operators in* $\mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}$ *are energy-bounded if those in* $\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}$ *are so.*

Proof. As in the proof of Proposition 5.1, define a linear map $\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}\to\mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}$, $\mathcal{Y}\mapsto\widetilde{\mathcal{Y}}$ as follows. Set $k'=l-k$. Then, by (b), $\rho\in P_+(\mathfrak{g},k')$. Set irreducible unitary $V^k_{\mathfrak{g}}\otimes V^{k'}_{\mathfrak{g}}$-modules

$$W_1=L_{\mathfrak{g}}(\lambda,k)\otimes L_{\mathfrak{g}}(0,k'),$$
$$W_2=L_{\mathfrak{g}}(\mu_0,k)\otimes L_{\mathfrak{g}}(\rho,k'),$$
$$W_3=L_{\mathfrak{g}}(\nu_0,k)\otimes L_{\mathfrak{g}}(\rho,k').$$

For each $\mathcal{Y}\in\mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}$,

$$\mathfrak{Y}:=\mathcal{Y}\otimes Y_\rho$$

is an intertwining operator of $V^k_{\mathfrak{g}}\otimes V^{k'}_{\mathfrak{g}}$ of type $\binom{W_3}{W_1W_2}$.

We now define the homogeneous vectors w_1,w_2,w_3 as in the proof of Proposition 5.1. Let $v_\lambda,v_{\mu_0},v_\rho$ be the highest weight (and lowest energy) vectors of $L_{\mathfrak{g}}(\lambda,k),L_{\mathfrak{g}}(\mu_0,k),L_{\mathfrak{g}}(\rho,k')$, respectively. Ω is the vacuum vector and highest weight vector of $L_{\mathfrak{g}}(0,k')$. Set

$$w_1=v_\lambda\otimes\Omega,\quad w_2=v_{\mu_0}\otimes v_\rho.$$

The lowest (L_0-) energy subspace of W_3 is $L_{\mathfrak{g}}(\nu_0)\otimes L_{\mathfrak{g}}(\rho)$. By (b), the $\mathfrak{g}$-module $L_{\mathfrak{g}}(\nu_0)\otimes L_{\mathfrak{g}}(\rho)$ has a unique irreducible submodule (equivalent to) $L_{\mathfrak{g}}(\nu)$. We let w_3 be a (nonzero) highest weight vector of $L_{\mathfrak{g}}(\nu)$. Consider again the diagonal embedding $V^l_{\mathfrak{g}}\subset V^k_{\mathfrak{g}}\otimes V^{k'}_{\mathfrak{g}}$. Then w_1,w_2,w_3 are highest weight vectors of the affine

Lie algebra $\widehat{\mathfrak{g}}_l$ with weights λ, μ, ν, respectively. Let e_3 be the orthogonal projection of W_3 onto $\mathfrak{g}w_3$. Let $s = \Delta^{\nu_0}_{\lambda\mu_0} - 1$, and define $T_{\mathcal{Y}} \in \mathrm{Hom}_{\mathfrak{g}}(\lambda \otimes \mu, \nu)$ by

$$\begin{aligned} T_{\mathcal{Y}} : L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}}(\mu) &\to L_{\mathfrak{g}}(\nu) \\ u^{(\lambda)} \otimes u^{(\mu)} &\mapsto e_3 \mathfrak{Y}(u^{(\lambda)})_s u^{(\mu)}. \end{aligned}$$

Again, by [12, Theorem 2.12], there is a unique $\widetilde{\mathcal{Y}} \in \mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu}$ such that $\Psi\widetilde{\mathcal{Y}} = T_{\mathcal{Y}}$, and that $\widetilde{\mathcal{Y}}$ is energy-bounded if $\mathcal{Y}$ is so.

It remains to check that $\mathcal{Y} \mapsto \widetilde{\mathcal{Y}}$ is injective. Suppose $\widetilde{\mathcal{Y}} = 0$. Then $T_{\mathcal{Y}} = 0$. Identify $L_{\mathfrak{g}}(\lambda, k)$ with $L_{\mathfrak{g}}(\lambda, k) \otimes \Omega$ and hence $L_{\mathfrak{g}}(\lambda)$ with $L_{\mathfrak{g}}(\lambda) \otimes \Omega$. For each $u \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]$,

$$T_{\mathcal{Y}}(u \otimes w_2) = e_3(\mathcal{Y} \otimes Y_\rho)(u \otimes \Omega)_s w_2 = e_3(\mathcal{Y}(u)_s v_{\mu_0} \otimes v_\rho),$$

which is 0. Since $\mathcal{Y}(u)_s v_{\mu_0} \otimes v_\rho \in L_{\mathfrak{g}}(\nu_0) \otimes v_\rho$, e_3 restricts to and can be regarded as the projection of $L_{\mathfrak{g}}(\nu_0) \otimes L_{\mathfrak{g}}(\rho)$ onto $L_{\mathfrak{g}}(\nu) \simeq \mathfrak{g}w_3$. Thus, e_3 is a nonzero element in $\mathrm{Hom}_{\mathfrak{g}}(\nu_0 \otimes \rho, \nu)$. Since $\mathcal{Y}(u)_s v_{\mu_0} \in L_{\mathfrak{g}}(\nu_0)[\nu - \rho]$, one has $e_3(\mathcal{Y}(u)_s v_{\mu_0} \otimes v_\rho) = (\Gamma e_3)(\mathcal{Y}(u)_s v_{\mu_0})$, which equals 0. Since Γ is injective, $\Gamma e_3 \neq 0$. So $\mathcal{Y}(u)_s v_{\mu_0} = 0$ since $L_{\mathfrak{g}}(\nu_0)[\nu - \rho]$ is one- dimensional. To summarize, we have proved that $\mathcal{Y}(u)_s v_{\mu_0} = 0$ for any $u \in L_{\mathfrak{g}}(\lambda)[\nu - \mu]$. Therefore, by condition (d), we must have $\mathcal{Y} = 0$. □

Lemma 6.1. *In Proposition* 6.2, *we set* $\alpha = \nu_0 - \nu + \rho$ *and* $\eta = \nu - \mu$. *Then condition* (*d*) *holds if the following are satisfied*:

(i) $\dim \mathrm{Hom}_{\mathfrak{g}}(\lambda \otimes \mu_0, \nu_0) = 1$.
(ii) α *is a positive root of* $\mathfrak{g}$.
(iii) *There exists* $u \in L_{\mathfrak{g}}(\lambda)[\eta]$ *such that* $E_\alpha u \notin K^{\mu_0}_{\mathfrak{g}}(\lambda)[\nu_0 - \mu_0]$.

Moreover, assume (*i*) *and* (*ii*). *Then* (*iii*) *holds if the following is satisfied*:

(iii′) $\dim L_{\mathfrak{g}}(\lambda)[\nu_0 - \mu_0] = 1$ *and* $(\eta|\alpha) < 0$.

We will show that cases (1) and (4) satisfy (i)–(iii), and that cases (2) and (5) satisfy (i)–(iii').

Proof. Choose any nonzero $\mathcal{Y} \in \mathcal{V}^k_{\mathfrak{g}}\binom{\nu_0}{\lambda\ \mu_0}$. Then $T := \Psi\mathcal{Y}$ is nonzero. Choose any nonzero $u \in L_{\mathfrak{g}}(\lambda)[\eta]$ satisfying condition (iii). As usual, we let v_{μ_0} and v_{ν_0} be (nonzero) highest weight vectors of $L_{\mathfrak{g}}(\mu_0), L_{\mathfrak{g}}(\nu_0)$, respectively. Since α is a positive root, we have a lowering operator F_α. Then

$$\langle T(u \otimes v_{\mu_0}) | F_\alpha v_{\nu_0}\rangle = \langle T(E_\alpha u \otimes v_{\mu_0}) | v_{\nu_0}\rangle + \langle T(u \otimes E_\alpha v_{\mu_0}) | v_{\nu_0}\rangle = \langle T(E_\alpha u \otimes v_{\mu_0}) | v_{\nu_0}\rangle.$$

Since $E_\alpha u$ has weight $\eta + \alpha = \nu_0 - (\mu - \rho) = \nu_0 - \mu_0$, we have

$$\langle T(u \otimes v_{\mu_0}) | F_\alpha v_{\nu_0}\rangle = (\Gamma T)(E_\alpha u).$$

By (i) and (8), $K^{\mu_0}_{\mathfrak{g}}(\lambda)[\nu_0 - \mu_0]$ has codimension 1, which must be the kernel of the nonzero linear functional ΓT. So $(\Gamma T)(E_\alpha u)$ is not zero since $E_\alpha u$ is not in this kernel. This proves (12).

Assume that (i)–(iii') are true. Since we have assumed $\dim \mathcal{V}^l_{\mathfrak{g}}\binom{\nu}{\lambda\ \mu} > 0$ in Proposition 6.2, $L_{\mathfrak{g}}(\lambda)[\nu-\mu] = L_{\mathfrak{g}}(\lambda)[\eta]$ is non-trivial. Choose any nonzero $u \in L_{\mathfrak{g}}(\lambda)[\eta]$. Since $(\eta|\alpha) < 0$ and hence $n_{\eta,\alpha} < 0$, we have $E_\alpha u \neq 0$. Indeed, this follows either by a standard $\mathfrak{sl}_2$-argument or by the following calculation:

$$\begin{aligned}\langle E_\alpha u|E_\alpha u\rangle = \langle F_\alpha E_\alpha u|u\rangle = \langle F_\alpha u|F_\alpha u\rangle - \langle H_\alpha u|u\rangle \\ \geq -\langle H_\alpha u|u\rangle = -n_{\eta,\alpha}(u|u) > 0.\end{aligned}$$

Since $\dim L_{\mathfrak{g}}(\lambda)[\nu_0-\mu_0] = \dim \mathrm{Hom}_{\mathfrak{g}}(\lambda\otimes\mu_0, \nu_0) = 1$, the subspace $K^{\mu_0}_{\mathfrak{g}}(\lambda)[\nu_0-\mu_0]$ is trivial by (9). This proves (iii). □

We return to the Lie algebra $\mathfrak{g} = \mathfrak{f}_4$. We now prove the energy bounds condition for the cases (1), (2), (4) and (5). To be more specific, we prove:

Proposition 6.3. *The intertwining operators of the following types are energy-bounded*:

(1) *Type* $\binom{\lambda_2+\lambda_4}{2\lambda_3}$ *level* 4.
(2) *Type* $\binom{\lambda_2}{\lambda_1+\lambda_4}$ *level* 3.
(4) *Type* $\binom{\lambda_2}{\lambda_3+\lambda_4}$ *level* 3.
(5) *Type* $\binom{\lambda_2+\lambda_4}{\lambda_1+\lambda_3}$ *level* 4.

Then their adjoint types are also energy-bounded. (See the beginning of Sec. 5.)

Proof. For all these cases, we let $\lambda = \lambda_4$. Recall the notations in Sec. 2. Choose $\mu, \nu, \rho, \mu_0, \nu_0$, and calculate $\nu - \rho$,

$$k = \max\{(\lambda|\theta), (\mu_0|\theta), (\nu_0|\theta)\}, \quad \alpha = \nu_0 - \nu + \rho, \quad \eta = \nu - \mu.$$

Case (1), $k = 2$.

$$\begin{gathered}\mu = 2\lambda_3, \quad \nu = \lambda_2 + \lambda_4, \quad \rho = \lambda_3, \quad \mu_0 = \lambda_3, \quad \nu_0 = \lambda_3, \\ \nu - \rho = (0, 1, -1, 1), \quad \alpha = [0, 0, 0, 1], \quad \eta = [0, 0, 0, -1].\end{gathered}$$

Case (2), $k = 2$.

$$\begin{gathered}\mu = \lambda_1 + \lambda_4, \quad \nu = \lambda_2, \quad \rho = \lambda_4, \quad \mu_0 = \lambda_1, \quad \nu_0 = \lambda_3, \\ \nu - \rho = (0, 1, 0, -1), \quad \alpha = \frac{1}{2}[1, -1, -1, 1], \quad \eta = [0, 0, 1, 0].\end{gathered}$$

Case (4), $k = 1$.

$$\begin{gathered}\mu = \lambda_3 + \lambda_4, \quad \nu = \lambda_2, \quad \rho = \lambda_3, \quad \mu_0 = \lambda_4, \quad \nu_0 = \lambda_4, \\ \nu - \rho = \frac{1}{2}[1, 1, 1, -1], \quad \alpha = \frac{1}{2}[1, -1, -1, 1], \quad \eta = \frac{1}{2}[-1, 1, 1, -1].\end{gathered}$$

Case (5), $k = 2$.

$$\mu = \lambda_1 + \lambda_3, \quad \nu = \lambda_2 + \lambda_4, \quad \rho = \lambda_3, \quad \mu_0 = \lambda_1, \quad \nu_0 = \lambda_3,$$

$$\nu - \rho = (0, 1, -1, 1), \quad \alpha = [0, 0, 0, 1], \quad \eta = \frac{1}{2}[1, -1, 1, -1].$$

We know that $N_\mu^\nu = 1$ since all the cases in Proposition 5.2, except case (7), have fusion (tensor product) rule 1. Thus, $\dim \mathcal{V}_\mathfrak{g}^l\binom{\nu}{\lambda\ \mu} = 1$. We now check that conditions (a)–(c) of Proposition 6.2 and (i)–(iii) (or (iii')) of Lemma 5 are satisfied. By the fusion rules in Proposition 5.2, in each case, $N_{\mu_0}^{\nu_0} = \dim \mathcal{V}_\mathfrak{g}^k\binom{\nu_0}{\lambda\ \mu_0} = 1$. Thus (a) and (i) are satisfied. (c) and (ii) are obvious. In case (4), $\nu - \rho$ is in group B, which is a weight $L_\mathfrak{g}(\nu_0) = L_\mathfrak{g}(\lambda_4)$ with multiplicity 1. So $\dim L_\mathfrak{g}(\nu_0)[\nu - \rho] = 1$. One can show that $\dim \mathrm{Hom}_\mathfrak{g}(\nu_0 \otimes \rho, \nu) = 1$ using Proposition 4.1. This proves (b). For the other cases, condition (b) can be checked using LieART. (See Sec. Appendix A for details.)

It remains to check condition (iii) or (iii') of Lemma 6.1. We first discuss cases (2) and (5). Then $\nu_0 - \mu_0 = (-1, 0, 1, 0)$ is a nonzero weight of $L_\mathfrak{g}(\lambda) = L_\mathfrak{g}(\lambda_4)$. So $\dim L_\mathfrak{g}(\lambda)[\nu_0 - \mu_0] = 1$. It is easy to check that $(\eta|\alpha) < 0$. This proves (iii'). Now, we assume case (1). Recall the definition of v_{ρ_3} and v_{ρ_4} in (10). Then $K_\mathfrak{g}^{\mu_0}(\lambda)[\nu_0 - \mu_0] = K_\mathfrak{g}^{\lambda_3}(\lambda_4)[0]$, which, by Lemma 4.1, is spanned by $F_{\rho_4} v_{\rho_4}$. We have $\eta = -\alpha = -\rho_3$. Choose any nonzero $u \in L_\mathfrak{g}(\lambda)[\eta] = L_\mathfrak{g}(\lambda_4)[-\rho_3]$. Then $E_\alpha u = E_{\rho_3} u$, which has weight 0. By a standard $\mathfrak{sl}_2$-argument (where $\mathfrak{sl}_2$ is generated by E_{ρ_3} and F_{ρ_3}), $E_{\rho_3} u$ and $F_{\rho_3} v_{\rho_3}$ are proportional since the (irreducible) $\mathfrak{sl}_2$-subrepresentations generated by u and by v_{ρ_3} agree. Thus, by lemma 4.2, $E_\alpha u$ and $F_{\rho_4} v_{\rho_4}$ are linearly independent. So $E_\alpha u$ is not in $K_\mathfrak{g}^{\mu_0}(\lambda)[\nu_0 - \mu_0]$. This proves (iii). Similarly, in case (4), by Lemma 4.1 we have $K_\mathfrak{g}^{\mu_0}(\lambda)[\nu_0 - \mu_0] = K_\mathfrak{g}^{\lambda_4}(\lambda_4)[0] = \mathbb{C} \cdot F_{\rho_3} v_{\rho_3}$. Choose $\rho_5 = \frac{1}{2}[-1, 1, 1, -1]$ and nonzero $v_{\rho_5} \in L_\mathfrak{g}(\lambda_4)[\rho_5]$. Then $L_\mathfrak{g}(\lambda)[\eta] = L_\mathfrak{g}(\lambda_4)[\rho_5]$. We have $\alpha = -\rho_5$. So $E_\alpha v_{\rho_5} = E_{-\rho_5} v_{\rho_5} = F_{\rho_5} v_{\rho_5}$, which, by Lemma 4.3, is not in $\mathbb{C} \cdot F_{\rho_3} v_{\rho_3}$. This again proves (iii). □

Thus, we have proved that the intertwining operators of fundamental types are energy-bounded. This proves Theorem 2.1.

Appendix A. Computations by LieART

One can calculate the weight multiplicities and the tensor product rules using LiE [19][b] or the Mathematica package LieART [8]. The LieART codes used in this paper are provided below.

The following LieART (v.2.0.0) code shows the root system of $\mathfrak{g} = \mathfrak{f}_4$.

RootSystem[F4]//OrthogonalBasis

[b]The LiE onlie service can be found on the personal website of M.A.A. van Leuwen. See http://wwwmathlabo.univ-poitiers.fr/ maavl/LiE/form.html.

The weights and their multiplicities of $L_{\mathfrak{g}}(\lambda_4) = L_{\mathfrak{g}}((0,0,0,1))$ can be calculated by the code:

WeightSystem[Irrep[F4][0,0,0,1]]//OrthogonalBasis

The following codes are used in the proof of Proposition 6.3 to check the first half of condition (b), namely, $\dim L_{\mathfrak{g}}(\nu_0)[\nu - \rho] = 1$. The outputs of these codes are all 1.

- Case (1) input:

WeightMultiplicity[Weight[F4][0,1,−1,1], Irrep[F4][0,0,1,0]]

- Case (2) input:

WeightMultiplicity[Weight[F4][0,1,0,−1], Irrep[F4][0,0,1,0]]

- Case (4) input:

WeightMultiplicity[Weight[F4][0,1,−1,0], Irrep[F4][0,0,0,1]]

- Case (5) input:

WeightMultiplicity[Weight[F4][0,1,−1,1], Irrep[F4][0,0,1,0]]

The code for case (1) computes the multiplicity of the weight $(0,1,-1,1)$ in $L_{\mathfrak{g}}((0,0,1,0)) = L_{\mathfrak{g}}(\lambda_3)$. The other codes are understood in a similar way.

To check the second half of condition (b), namely, $\mathrm{Hom}_{\mathfrak{g}}(\nu_0 \otimes \rho, \nu) = 1$, we calculate $\nu_0 \otimes \rho$, which shows that ν (boxed in the outputs) appears precisely once in the tensor product. We write (n_1, n_2, n_3, n_4) as $(n_1 n_2 n_3 n_4)$ when none of the four integers exceeds 9.

- Case (1) input:

DecomposeProduct[Irrep[F4][0,0,1,0], Irrep[F4][0,0,1,0]]//StandardForm

Output:

$$\begin{aligned}(0010) \otimes (0010) = {} & (0000) + (0001) + (1000) + 2(0010) + 2(0002) + 2(1001) \\ & + (2000) + (0100) + (0003) + 2(0011) + (1010) + (1002) \\ & + \boxed{(0101)} + (0020)\end{aligned}$$

- Case (2) input:

DecomposeProduct[Irrep[F4][0,0,1,0], Irrep[F4][0,0,0,1]]//StandardForm

Output:

$$(0010) \otimes (0001) = (0001) + (1000) + (0010) + (0002) + (1001) + \boxed{(0100)} + (0011)$$

- Case (4): Same as case (2).
- Case (5): Same as case (1).

Appendix B. Another Proof of Lemma 4.2

Due to the importance of Lemma 4.2, we give in this section an alternate proof of this lemma. Let $v \in L_{\mathfrak{g}}(\lambda_4)[\lambda_4]$ be a highest weight vector with length 1. Set

$$\alpha = [1, 0, 0, -1], \quad \beta = \frac{1}{2}[1, 1, 1, 1]$$

which are roots of $\mathfrak{g}$. Recall $\rho_3 = [0, 0, 0, 1], \rho_4 = \frac{1}{2}[1, -1, -1, -1]$. Then $\alpha + \rho_3 = \beta + \rho_4 = [1, 0, 0, 0] = \lambda_4$. By scaling v_{ρ_3}, v_{ρ_4}, we may assume that $v_{\rho_3} = F_\alpha v$ and $v_{\rho_4} = F_\beta v$. We shall show that $F_{\rho_3} F_\alpha v$ and $F_{\rho_4} F_\beta v$ are not parallel by calculating the angle between them.

We first calculate the square length

$$\langle F_{\rho_3} F_\alpha v | F_{\rho_3} F_\alpha v \rangle = \langle F_\alpha v | E_{\rho_3} F_{\rho_3} F_\alpha v \rangle.$$

Since $[E_{\rho_3}, F_{\rho_3}] = H_{\rho_3}$, $E_{\rho_3} v = 0$, and since $\rho_3 - \alpha = [-1, 0, 0, 2]$ is not a root, the above expression equals

$$\begin{aligned} \langle F_\alpha v | H_{\rho_3} F_\alpha v \rangle &= n_{\lambda_4 - \alpha, \rho_3} \langle F_\alpha v | F_\alpha v \rangle = n_{\rho_3, \rho_3} \langle F_\alpha v | F_\alpha v \rangle \\ &= 2 \langle F_\alpha v | F_\alpha v \rangle = 2 \langle v | H_\alpha v \rangle = 2 n_{\lambda_4, \alpha} \|v\|^2 = 2. \end{aligned}$$

A similar calculation shows

$$\langle F_{\rho_4} F_\beta v | F_{\rho_4} F_\beta v \rangle = 2 n_{\lambda_4, \beta} \|v\|^2 = 2.$$

To show that the two vectors are not parallel, we need to show that the absolute value of $\langle F_{\rho_3} F_\alpha v | F_{\rho_4} F_\beta v \rangle$ is not equal to 2.

Set $\gamma = \beta - \rho_3 = \alpha - \rho_4 = \frac{1}{2}[1, 1, 1, -1]$ which is a positive root. We would like to express $[E_{\rho_3}, F_\beta]$ in terms of F_γ. Note that $(F_\gamma | F_\gamma) = 2/(\gamma|\gamma) = 2$. Since $[F_{\rho_3}, F_\beta] = 0$ as $\rho_3 + \beta = \frac{1}{2}[1, 1, 1, 3]$ is not a root, we have

$$\begin{aligned} ([E_{\rho_3}, F_\beta] | [E_{\rho_3}, F_\beta]) &= (F_\beta | [F_{\rho_3}, [E_{\rho_3}, F_\beta]]) = (F_\beta | [[F_{\rho_3}, E_{\rho_3}], F_\beta]) \\ &= -(F_\beta | [H_{\rho_3}, F_\beta]) = n_{\beta, \rho_3} (F_\beta | F_\beta) = 1 \cdot 2 = 2. \end{aligned}$$

Thus, $[E_{\rho_3}, F_\beta] = k_1 F_\gamma$ for some constant k_1 satisfying $|k_1| = 1$. Similarly, since $\rho_4 + \alpha = \frac{1}{2}[3, -1, -1, -3]$ is not a root, and since $(F_\alpha | F_\alpha) = 2/(\alpha|\alpha) = 1$, we have

$$([E_{\rho_4}, F_\alpha] | [E_{\rho_4}, F_\alpha]) = -(F_\alpha | [H_{\rho_4}, F_\alpha]) = n_{\alpha, \rho_4} (F_\alpha | F_\alpha) = 2 \cdot 1 = 2.$$

Thus $[E_{\rho_4}, F_\alpha] = k_2 F_\gamma$ where $|k_2| = 1$. Now, using the fact that $\rho_3 - \rho_4 = \frac{1}{2}[-1, 1, 1, 3]$ is not a root, we find

$$\begin{aligned} \langle F_{\rho_3} F_\alpha v | F_{\rho_4} F_\beta v \rangle &= \langle F_\alpha v | E_{\rho_3} F_{\rho_4} F_\beta v \rangle = \langle F_\alpha v | F_{\rho_4} [E_{\rho_3}, F_\beta] v \rangle = k_1 \langle F_\alpha v | F_{\rho_4} F_\gamma v \rangle \\ &= k_1 \langle E_{\rho_4} F_\alpha v | F_\gamma v \rangle = k_1 k_2 \langle F_\gamma v | F_\gamma v \rangle = k_1 k_2 \langle v | H_\gamma v \rangle \\ &= k_1 k_2 n_{\lambda_4, \gamma} = k_1 k_2 \end{aligned}$$

whose absolute value is 1. This finishes the proof.

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