On Examples of Supersingular and Rationally Chain Connected Threefolds

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Abstract

In this work, we show that for a certain class of threefolds in positive characteristics, rational-chain-connectivity is equivalent to supersingularity. The same result is known for K3 surfaces with elliptic fibrations. And there are examples of threefolds that are both supersingular and rationally chain connected.

1 Introduction

The goal of this article is to generalize the following result to a certain class of threefolds in positive characteristics.

Theorem 1.1. *([9], Problem 12, p.11 and p.12) Let k be an algebraically closed field in positive characteristic. If a K3 surface X over k admits an elliptic fibration, then X is supersingular if and only if any two points on X can be connected by a chain of finitely many rational curves.*

To generalize Theorem 1.1 to threefolds, it is natural to consider the class of threefolds with K3-fibrations. There are examples of Calabi-Yau threefolds with smooth proper K3-fibrations over $\mathbb{P}^1_k$ constructed by Stefan Schröer in [23], which are both supersingular and unirational (Theorem 3.1 and Theorem 3.3). In this article, we will show that supersingularity is equivalent to rational-chain-connectivity for a broader class of threefolds that are generalizations of Schröer's examples.

1.1 Preliminaries

To state our results, we first fix some conventions. Let k be an algebraically closed field of characteristic $p > 0$. By a Calabi-Yau n-fold X, we mean a smooth, projective k-scheme of dimension n such that $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$ and that $\omega_X \simeq \mathcal{O}_X$. When n equals two, a Calabi-Yau 2-fold is a K3 surface. And when n is one, a Calabi-Yau 1-fold is just an elliptic curve.

Definition 1.2. Let X be a smooth projective variety over a perfect field k in characteristic $p > 0$. Let $W = W(k)$ be its Witt vectors. Denote by K the fraction field of W. We say that X is *supersingular* if the slope $[0, 1)$ part of $\mathrm{H}^i_{\mathrm{rig}}(X/K)$ is vanishing for all $i > 0$. And we say that X is H^i*-supersingular* is the slope $[0, 1)$ part of $\mathrm{H}^i_{\mathrm{rig}}(X/K)$ is vanishing.

Remark 1.3. Note that for a smooth and proper variety X over a perfect field k with $\operatorname{char} k > 0$, we have the isomorphism between rigid cohomology and crystalline cohomology

$$\mathrm{H}^i_{\mathrm{rig}}(X/K) \simeq \mathrm{H}^i_{\mathrm{cris}}(X/W) \otimes_W K,$$

as F-isocrystals, for every i (see [4], 1.1, p.687). Moreover, the slopes of $\mathrm{H}^i_{\mathrm{rig}}(X/K)$ and $\mathrm{H}^i_{\mathrm{cris}}(X/W)$ are defined by applying the Dieudonné-Manin classification Theorem (see [3] p.94, or [1], 1.1.2, or [8]) to the F-isocrystals $\mathrm{H}^i_{\mathrm{rig}}(X/K)$ and $\mathrm{H}^i_{\mathrm{cris}}(X/W) \otimes_W K$ respectively. Thus, both of these two cohomology theories give the same slopes. Since we consider only the slopes of cohomologies, in the rest of this article, we will use rigid cohomology and crystalline cohomology equivalently.

Remark 1.4. Let n be the dimension of X. Some authors use the definition that X is supersingular if it is H^n-supersingular as in our definition (see [27], Definition 13, p.8). And, for a Calabi-Yau n-fold X, H^n-supersingularity is equivalent to saying that the formal group $\Phi^n_X(\mathbb{G}_m)$ of X has infinite height, see Corollary II.4.3 of [3] and p.98 of [11].

One of the reasons for the study of supersingularity is its connection with unirationality for K3 surfaces conjectured by Artin, Shioda, etc.

Conjecture 1.5. *(Artin, Rudakov, Shafarevich, Shioda) Let k be an algebraically closed field in characteristic $p > 0$. A K3 surface over k is supersingular if and only if it is unirational.*

This conjecture holds for Kummer surfaces when $p \geq 3$ and for any K3 surfaces when $p = 2$. Note that any Kummer surface has an elliptic fibration. Motivated by Schröer's examples, we make the following definition for K3-fibrations.

Definition 1.6. Let X be a smooth and projective threefold over a field k. We say that $\pi : X \to \mathbb{P}^1_k$ is a *K3-fibration* if π is smooth, proper and for every $s \in \mathbb{P}^1_k$ the geometric fiber $X_{\overline{\kappa(s)}}$ is a K3 surface.

On the other hand, in Theorem 1.1, the condition equivalent to supersingularity is the rational-chain-connectivity of the K3 surface. In general, let X be a variety over a field k. Recall that a *tree of rational curves C* on X is a k-morphism $u_C : C \to X$ where C is a proper, geometrically connected curve of arithmetic genus zero at worst nodal, defined over a field extension k_C of k, and every irreducible component of $C_{\overline{k_C}}$ is rational. We say that two distinct points x_1 and x_2 on X, not necessarily closed, are connected by the tree of rational curves C if they are in the image of $C_{\overline{k_C}} \to X_{\overline{k}} \to X$. Moreover, we say that C is a *rational curve* if $C_{\overline{k_C}}$ is irreducible and rational.

Definition 1.7. Let X be a variety over a field k. We say that X is *rationally chain connected (resp. rationally connected)* if every pair of distinct general points x_1, x_2 on X, including non-closed points, is connected by a tree of rational curves (resp. by a rational curve).

Note that Definition 1.7 agrees with Definition (3.2) in [15], p.199, since we include non-closed points in the definition. When k is an uncountable algebraically closed field, rational-chain-connectivity (resp. rational-connectivity) is equivalent to that every pair of two distinct very general closed points can be connected by a connected tree of rational curves (resp. by a rational curve), see [15], Proposition (3.6), p.201.

Now, we can state our main theorem.

Theorem 1.8. *Fix an algebraically closed field k with $\operatorname{char} k \geq 3$. Let X be a smooth, projective threefold over k with a K3-fibration. Denote by $k(t)$ the function field of $\mathbb{P}^1_k$. Suppose that $X_{k(t)}$ is a Kummer surface. Then, the following are equivalent:*

(i) the slopes of $\mathrm{H}^2_{rig}(X/K)$ are all 1,

(ii) X is supersingular,

(iii) X is rationally chain connected.

In particular, if X is supersingular, then X is uniruled.

1.2 Outline of the article

In Section 2, we show that rationally chain connected K3 surfaces are unirational if we assume Conjecture 1.5 (Corollary 2.6). In Section 3, we go over the constructions of Calabi-Yau threefolds in Schröer's article [23] and show that they are all unirational (Theorem 3.1 and Theorem 3.3). Section 4 is the technical part about crystalline cohomology, which is used to show that every K3 fiber will be supersingular if we assume the total space is supersingular (Proposition 4.2). Finally, in Section 5, we give the proof of Theorem 1.8.

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2 Rationally chain connected K3 surfaces

The definition of rational-chain-connectivity directly leads to following result.

Theorem 2.1. *([15], Theorem (3.13.1), p.206) Let X be a rationally chain connected variety over an algebraically closed field k. Then, $A_0(X) \simeq \mathbb{Z}$.*

There are several notions of supersingularity for K3 surfaces in literatures which are all equivalent.

Definition 2.2. Let X be a K3 surface over an algebraically closed field. We say that X is *Shioda-supersingular* if its Picard rank is 22. And X is called *Artin-supersingular* if the height of its Artin-Mazur formal Brauer group is infinity, that is, $h(\widehat{Br}(X)) = \infty$.

The following result characterizes Shioda-supersingularity and Artin-supersingularity in terms of the slopes of crystalline cohomology.

Theorem 2.3. *([17], Theorem 2.3) Let X be a K3 surface over an algebraically closed field k in odd characteristic. Denote by $W = W(k)$ the ring of Witt vectors of k. Then, the following are equivalent:*

(i) X is Shioda-supersingular.

(ii) X is Artin-supersingular.

(iii) For all i, the F-crystal $\mathrm{H}^i_{cris}(X/W)$ is of slope $i/2$.

(iv) The slope $[0, 1)$ part of $\mathrm{H}^2_{cris}(X/W)$ is vanishing.

Therefore, we just say that a K3 surface X is *supersingular* if it satisfies one of the conditions in Theorem 2.3.

Fix a perfect field k with characteristic $p > 0$. We cite the following result about vanishing of slope $[0, 1)$ part of rigid cohomology.

Theorem 2.4. *([10], Theorem 1.1, p.188) Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$. Let $K(X)$ be the function field of X. If the Chow group of 0-cycles $A_0(X \times_k \overline{K(X)})$ is equal to $\mathbb{Z}$, then the slope $[0, 1)$ part of $\mathrm{H}^i_{rig}(X/K)$ is vanishing for $i > 0$.*

As a direct consequence, we have the following theorem.

Theorem 2.5. *Let X be a rationally chain connected K3 surface over an algebraically closed field of characteristic $p \geq 3$. Then, X is supersingular.*

Proof. Since X is rationally chain connected, $X \times_k \overline{K(X)}$ is rationally chain connected. Thus, the result follows from Theorem 2.4 and Theorem 2.1 since the slopes of rigid cohomology and crystalline cohomology agree on X. □

We note that, when $p = 2$, a K3 surface is supersingular if and only if it is unirational (combine the results in [22], [21], [2] and [18]). It is known that unirational surfaces are supersingular ([25], Corollary 2, p.235). However, Theorem 2.5 shows that a much weaker condition is sufficient to obtain the supersingularity for K3 surfaces, i.e., rational-chain-connectivity. This leads us to consider the equivalence of supersingularity and rational-chain-connectivity for higher dimensional varieties, for example, threefolds. As we will see in Section 3, this is true for the Calabi-Yau threefolds constructed in [23] since they are unirational.

As a corollary of Theorem 2.5, we have the following result.

Corollary 2.6. *Let X be a K3 surface over an algebraically closed field k of characteristic $p \geq 3$. Assume that Conjecture 1.5 is true. Then, the following are equivalent:*

- *X is supersingular;*
- *X is unirational;*
- *X is rationally connected;*
- *X is rationally chain connected.*

In particular, these conditions are all equivalent if X is a Kummer surface.

Proof. Suppose that X is rationally chain connected. Then, by Theorem 2.5, X is supersingular. Thus, X is unirational if Conjecture 1.5 holds for X, so X is rationally connected. Hence X is rationally chain connected. Moreover, Conjecture 1.5 holds when X is a Kummer surface and char $k \geq 3$ ([26], Theorem 1.1, p.154), so the conditions are all equivalent for Kummer surfaces. □

3 Schröer's examples

In this section, we review the constructions of Schröer's examples and show that they are unirational. Note that by construction these examples exist only in characteristics two and three.

Fix an algebraically closed field k of characteristic $p > 0$. Let A be a *superspecial* Abelian surface, that is, A is a two-dimensional Abelian variety isomorphic to the product of two supersingular elliptic curves ([23], p.1583); here supersingular elliptic curves are defined as in [24], p.145. Denote X' by the variety $A \times_k \mathbb{P}^1_k$. As explained in [23], fix an integer $n \geq 1$ and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_k}(-n) \xrightarrow{r,s} \mathcal{O}_{\mathbb{P}^1_k}^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1_k}(n) \to 0 \tag{$\dagger$}$$

given by two homogeneous quadratic polynomials r, $s \in H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n))$ without common zeros. Let $H \subset X'$ be the relative radical subgroup scheme of height one whose p-Lie algebra is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(-n) \subset \mathcal{O}_{\mathbb{P}^1_k}^{\oplus 2}$ ([23], p.1583). Then, the quotient $X = X'/H$ has $\omega_X = \mathcal{O}_X$ if $p = 3$ and $n = 1$, or $p = 2$ and $n = 2$ ([23], Corollary 3.2, p.1584). And the quotient X is an Abelian scheme over $\mathbb{P}^1_k$, whose fibers X_t, $t \in \mathbb{P}^1_k$, are supersingular Abelian surfaces.

3.1 Characteristic three case

Now, take $p = 3$ and $n = 1$ in ($\dagger$) as in [23], Section 4, p.1584. Construct $Z \to \mathbb{P}^1_k$ as the quotient of X by the involution $[-1]$. Denote by $f : X \to Z$ the quotient $\mathbb{P}^1_k$-morphism. For every $t \in \mathbb{P}^1_k$, the fiber Z_t is a singular Kummer surface with 16 singular points that are rational double points of type A_1. Let $D \subset Z$ be the schematic closure of the union of singular points. Then, the blowing up $Y \to Z$

of Z with center D gives a smooth Calabi-Yau threefold, and the geometric fibers $Y_{\bar{t}}$, $t \in \mathbb{P}^1_k$, are supersingular Kummer K3 surfaces ([23], Proposition 4.1 and Proposition 4.2, p.1585).

Theorem 3.1. *Keep the notations in [23], Section 4, p.1584. The Calabi-Yau threefold Y constructed above is unirational.*

Proof. Since $X' \to X$ is a homomorphism of group schemes, we have the commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r] & Z \end{tikzcd}$$

where Z' is the quotient of X' by $[-1]$, and $Z' \to Z$ is surjective. Note that Z' is a constant family of singular Kummer surfaces. Blow up the singular locus of Z' and Z to get Y' and Y. There is a domaint rational map from Y' to Y. The smooth threefold Y' is a constant family of Kummer K3 surfaces. By [26], Theorem 1.1, p.154, the Kummer surface associated to a supersingular Abelian surface is unirational when $\operatorname{char} k > 2$. Thus, there is a rational dominant map from $\mathbb{P}^2_k \times_k \mathbb{P}^1_k$ to Y'. Since the variety $\mathbb{P}^2_k \times_k \mathbb{P}^1_k$ is rational, Y' is unirational. Therefore, Y is unirational. □

Remark 3.2. By [23], Proposition 8.1, p.1590, the Artin-Mazur formal group of the Calabi-Yau threefold constructed in [23], Section 4 and Section 7, is isomorphic to the formal additive group $\hat{\mathbb{G}}_a$, whose height is infinity. Therefore, Theorem 3.1 gives examples of H^3-supersingular Calabi-Yau threefolds (see Remark 1.4) that are unirational when $\operatorname{char} k = 3$. Moreover, by Theorem 2.4, Y is supersingular as defined in Definition 1.2.

3.2 Characteristic two case

Let k be an algebraically closed field of characteristic 2. Let E be the supersingular elliptic curve given by the Weierstrass equation $x^3 = y^2 + y$. Take a primitive third root of unity $\zeta \in k$. The automorphism $\varphi : E \to E$ via $(x, y) \mapsto (\zeta x, y)$ gives an action of $G = \mathbb{Z}/3\mathbb{Z}$ on E. The action has three fixed points, $(0, 0)$, $(0, 1)$ and ∞. Consider the supersingular Abelian surface $A = E \times E$, endowed with the action of G via $\phi = (\varphi, \varphi)$. The morphsim $\phi : A \to A$ has 9 fixed points, which correspondes to the singularities on A/G. Note that A/G is a proper normal surface over k.

Let X' be $A \times_k \mathbb{P}^1_k$. Take $p = 2$ and $n = 2$ in (†). The quotient $X = X'/H$ has $\omega_X \simeq \mathcal{O}_X$. The fiberwise action of G on $X' \to \mathbb{P}^1_k$ descends to a fiberwise action on $X \to \mathbb{P}^1_k$. Set $Z = X/G$ and let $Y \to Z$ be the minimal resolution of singularities. Then, Y is a Calabi-Yau threefold in characteristic two, and every geometric fiber $Y_{\bar{t}}$ of $Y \to \mathbb{P}^1_k$ has Picard number $\rho(Y_{\bar{t}}) = 22$.

Theorem 3.3. *Keep the notations as above ([23], Section 7, p.1589). The Calabi-Yau threefold Y constructed above is unirational.*

Proof. Let x be the origin of A. Since A is smooth, $\widehat{\mathcal{O}}_{A,x}$ is isomorphic to $k[[u,v]]$ as k-algebras, where u, v is a regular system of parameters of $\widehat{\mathcal{O}}_{A,x}$. Denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_{A,x}$. As remarked in Proposition 5.1, [23], p.1585, the action of ϕ on $\mathfrak{m}/\mathfrak{m}^2$ is given by the multiplication by ζ. So ϕ maps u to ζu and v to ζv in $\mathfrak{m}/\mathfrak{m}^2$. Thus, $\mathcal{O}_{A,x}/G$ is not Gorenstein. Then, the corresponding singular point of x on A/G is not a rational double point since rational double points are Gorenstein. By [12], Theorem 2.11, p.10, A/G is a rational surface. Then, the same argument as in Theorem 3.1 shows that the Calabi-Yau threefold Y is unirational. $\square$

4 K3 fibrations

Let $f : X \to Y$ be a smooth proper morphism of smooth schemes over a perfect field of characteristic $p > 0$. Let K be the field of fractions of $W(k)$. In [20] Theorem (3.1), p.800, Ogus constructed F-isocrystals denoted by $R^i f_* \mathcal{O}_{X/K}$ on Y that satisfies the following characterizations ([20], Remark (3.7.1), p.803):

(1). the formation of $R^i f_* \mathcal{O}_{X/K}$ is compatible with any base change $Y' \to Y$,

(2). the fiber of $R^i f_* \mathcal{O}_{X/K}$ over a closed point y of Y is

$$\mathrm{H}^i_{\mathrm{cris}}(X_y, \mathcal{O}_{X_y}/W(\kappa(y))) \otimes \mathbb{Q}.$$

However, from the original construction in [20], it is not obvious that there exists a Leray spectral sequence

$$E_2^{i,j} = \mathrm{H}^i_{\mathrm{rig}}(Y, R^j f_* \mathcal{O}_{X/K}) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{rig}}(X). \tag{$\dagger$}$$

When Y is a smooth affine curve, $k = \bar{k}$, and $f : X \to Y$ admits a semi-stable compactification $\overline{X} \to \overline{Y}$ of proper, smooth k-schemes, [19] Remark 2.8, p.14, proves the existence of such a spectral sequence by using A. Shiho's theory of log convergent cohomology.

In the rest of this section, let k be an algebraically closed field of characteristic $p > 0$, K be the field of fractions of $W(k)$. Let $\pi : X \to \mathbb{P}^1_k$ be a *smooth*, *projective* morphism over k. Denote by $\mathcal{E}^i$ the convergent F-isocrystals $R^i \pi_* \mathcal{O}_{X/K}$ on $\mathbb{P}^1_k$ as discussed above. Then we have that

a) for any perfect-field-valued closed point $t : \mathrm{Spec}(F) \to \mathbb{P}^1_k$, $t^* \mathcal{E}^i = \mathrm{H}^i_{\mathrm{rig}}(X_t)$ (rigid cohomology over the field $W(F)[1/p]$), cf. [20], Remark (3.7.1),

b) there exists a Leray spectral sequence, which is ($\dagger$) above,

$$E_2^{i,j} = \mathrm{H}^i_{\mathrm{rig}}(\mathbb{P}^1_k, \mathcal{E}^j) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{rig}}(X) \tag{$*$}$$

Note that our hypotheses, smooth and projective, are much stronger than semi-stable as needed in [19] Remark 2.8.

Lemma 4.1. *In the situation above, the convergent F-isocrystals $\mathcal{E}^i$ are trivial.*

Proof. Since $\mathbb{P}^1_k$ admits a lift $\mathbb{P}^1_K$, a convergent isocrystal $\mathcal{E}$ on $\mathbb{P}^1_k$ is the same as a (necessarily integrable) connection $(E^{\mathrm{an}}, \nabla)$ on the rigid analytic space $\mathbb{P}^{1,\mathrm{an}}_K$. With some translation between formal and rigid geometry, this assertion is precisely Proposition 1.15 of [20].

Since $\mathbb{P}^{1,\mathrm{an}}_K$ is smooth and proper, the rigid analytic GAGA theorem (cf. [16]) implies that E^{an} comes from an algebraic vector bundle E. If we can show that E already has an algebraic connection, then choosing a connection on E amounts to choosing an $End(E^{\mathrm{an}})$-valued 1-form on $\mathbb{P}^{1,\mathrm{an}}$; the connection is thereby necessarily algebraic by applying GAGA again.

Hence, one needs to show that a vector bundle E on $\mathbb{P}^1_K$ admits an algebraic connection, given that E^{an} admits a connection. This follows from the fact that in both rigid and algebraic geometry, only trivial vector bundles on $\mathbb{P}^1$ allow connections.

Finally, the algebraic connection (E, ∇) can be defined on a field that is finitely generated over $\mathbb{Q}$ since K is of characteristic zero, so we can embed this field into $\mathbb{C}$. Recall that every connection on $\mathbb{P}^1_{\mathbb{C}}$ is flat. Then, since $\mathbb{P}^1_{\mathbb{C}}$ is simply connected, the flat connection (E, ∇) must be trivial by Riemann-Hilbert correspondence ([7] Theorem 2.17, p.12, and [14] Remark 4.9.2, p.63). □

Proposition 4.2. *Let X be a smooth and projective threefold over an algebraically closed field k in characteristic $p > 0$. Assume that X admits a K3-fibration and the slopes of $\mathrm{H}^2_{rig}(X)$ are all 1, then all the geometric fibers of π are supersingular K3 surfaces.*

Proof. By the K3 hypothesis, $\mathcal{E}^1 = \mathcal{E}^3 = 0$. Thus in the spectral sequence $(*)$, $E_2^{2,1} = E_2^{0,3} = E_2^{0,1} = E_2^{1,1} = 0$. By Lemma 4.1, $\mathcal{E}^2$ is a constant crystal. Recall that the rigid cohomology of $\mathbb{A}^1$ with respect to the constant isocrystal is trivial ([13] Remark 23.1.3, p.353). By Mayer-Vietoris sequence for rigid cohomology ([28], Theorem 7.1.2, p.963), it follows that

$$E_2^{1,2} = \mathrm{H}^1_{\mathrm{rig}}(\mathbb{P}^1_k, \mathcal{E}^2) = 0.$$

By Property a) and the constancy of $\mathcal{E}^2$, all the k-valued fibers S of π have the same rigid cohomology. Thus

$$E_2^{0,2} = \mathrm{H}^0_{\mathrm{rig}}(\mathbb{P}^1_k, \mathcal{E}^2) = \mathrm{H}^2_{\mathrm{rig}}(S).$$

It follows from the above vanishing that the spectral sequence $(*)$ degenerates at E_2. Thus $\mathrm{H}^2_{\mathrm{rig}}(X)$ maps surjectively onto $\mathrm{H}^2_{\mathrm{rig}}(S)$ for any fiber S of π. Since all the slopes of $\mathrm{H}^2_{\mathrm{rig}}(X)$ are 1, it follows that any k-valued fiber S of π is a supersingular K3 surface. The assertion for any geometric fiber then follows from Grothendieck's specialization theorem, spelled out in our needed form in by Theorem 2.1 of [5]. □

5 Proof of the main theorem

We first show that if a family of varieties over $\mathbb{P}^1$ admits a section then the rational-chain-connectivity of the generic fiber implies that the total space is also rationally chian connected.

Proposition 5.1. *Fix an algebraically closed field k. Let $\pi : X \to \mathbb{P}^1_k$ be a flat, proper morphism from an irreducible variety X. Denote by K the function field of $\mathbb{P}^1_k$. Assume that $X_{\overline{K}}$ is irreducible, rationally chain connected, and there is a section of π. Then, X is rationally chain connected.*

Proof. We can assume that k is an uncountable algebraically closed field, see [15] (3.2.5), p.199. Since π is flat, it is equidimensional. So, from [15] (3.5.1), p.201, there are countably many closed subvarieties $S_i \subseteq \mathbb{P}^1_k$ such that X_s is rationally chain connected if and only if $s \in \cup S_i$. By our hypothesis, the generic point of $\mathbb{P}^1_k$ is in S_i. However, the only closed subvariety of $\mathbb{P}^1_k$ containing the generic point is the whole $\mathbb{P}^1_k$, hence X_s is rationally chain connected for each $s \in \mathbb{P}^1_k$. Since X_s is proper, any two closed points can be connected by a chain of rational curves, see [6], Remark 4.22.(2), p.100. Then any two closed points of X can be connected by the section of π and chains of rational curves on the fibers. Therefore, X is rationally chain connected by [15] (3.6.1), p.201. □

Now we give the proof of Theorem 1.8.

Proof. (iii) $\Rightarrow$ (ii). If X is rationally chain connected, then the slopes of $\mathrm{H}^2_{\mathrm{cris}}(X)$ are all 1 by Theorem 2.1 and Theorem 2.4.

(ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii). Suppose that the slopes of $\mathrm{H}^2_{\mathrm{cris}}(X)$ are all 1. Let $\pi : X \to \mathbb{P}^1_k$ be the K3-fibration. By Proposition 4.2, all the geometric fibers of π are supersingular K3 surfaces. In particular, $X_{\overline{k(t)}}$ is a supersingular Kummer surface. By [26] Theorem 1.1 on p.154, $X_{\overline{k(t)}}$ is unirational, so rationally connected. Since the Kummer surface $X_{k(t)}$ is the minimal resolution of a quotient of an Abelian surface over $k(t)$ under the involution, $X_{k(t)}$ has a $k(t)$-point, which gives a section of π. Therefore, by Proposition 5.1, the total space X is rationally chain connected.

Finally, by [15], (3.3.4), p.200, X is uniruled. □

Remark 5.2. Theorem 1.8 also applies to the following situation. Let k be an algebraically closed field with $\operatorname{char} k \geq 3$. Let X be a smooth, projective threefold over k with a K3-fibration. Denote by $k(t)$ the function field of $\mathbb{P}^1_k$. Assume that $X \to \mathbb{P}^1_k$ has a section and Conjecture 1.5 holds. Then, X is rationally chain connected if and only if the slopes of $\mathrm{H}^2_{\mathrm{cris}}(X)$ are all 1 if and only if it is supersingular.

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