NONCOMMUTATIVE ENHANCEMENTS OF CONTRACTIONS

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Abstract. Given a contraction of a variety X to a base Y, we enhance the locus in Y over which the contraction is not an isomorphism with a certain sheaf of noncommutative rings $\mathcal{D}$, under mild assumptions which hold in the case of (1) crepant partial resolutions admitting a tilting bundle with trivial summand, and (2) all contractions with fibre dimension at most one. In all cases, this produces a global invariant. In the crepant setting, we then apply this to study derived autoequivalences of X. We work generally, dropping many of the usual restrictions, and so both extend and unify existing approaches. In full generality we construct a new endofunctor of the derived category of X by twisting over $\mathcal{D}$, and then, under appropriate restrictions on singularities, give conditions for when it is an autoequivalence. We show that these conditions hold automatically when the non-isomorphism locus in Y has codimension 3 or more, which covers determinantal flops, and we also control the conditions when the non-isomorphism locus has codimension 2, which covers 3-fold divisor-to-curve contractions.

Contents

1. Introduction

It has become increasingly clear that various aspects of birational geometry should be enhanced into a mildly noncommutative setting. One example of this, namely associating noncommutative algebras to certain contractions, has recently yielded many new results, including: algorithmic ways to relate minimal models based on cluster theory [W], new invariants for flips and flops [DW1] linked to Gopakumar–Vafa invariants [T2], the braiding of flop functors [DW3], noncommutative versions of curve counting [T3], a full conjectural analytic classification of 3-fold flops [DW1, HT], and, in addition, the first new examples of 3-fold flops since 1983 [BW, K1].

However, a major technical and philosophical drawback of many of these constructions is that they are local in nature, or apply only to contractions of curves. In this paper, we work in a much more general setting. We take a contraction $f\colon X \to Y$ satisfying mild characteristic-free assumptions, and enhance the non-isomorphism locus Z in Y with a certain sheaf of noncommutative rings. Amongst other things, we use this sheaf to give a new class of derived autoequivalences.

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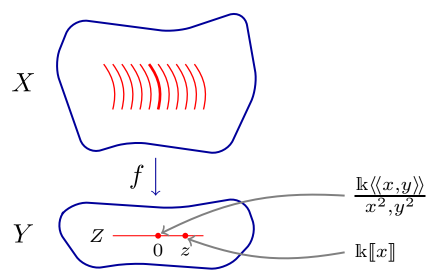
In this paper we accomplish the following.

- In an axiomatic framework sketched in §1.1, we construct a noncommutative ringed space $(Z, \mathcal{D})$ where Z is the non-isomorphism locus in Y. In 3-folds this applies to flopping, flipping, and divisor-to-curve contractions, but also works in arbitrary dimension, and for contractions with higher-dimensional fibres.
- In a crepant setting given in §1.2, using the sheaf of noncommutative rings $\mathcal{D}$ above, we construct an endofunctor Twist_X of $\mathrm{D}(\mathrm{Qcoh}\, X)$ associated to the contraction f, and establish criteria for it to be an autoequivalence.

The main benefit of this approach is its generality: we show in 2.5(1) that the axiomatic framework applies when f is a crepant resolution of a Gorenstein d-fold Y, with mild restrictions which hold in many known settings, including those of Haiman [H1], Bezrukavnikov–Kaledin [BK], Procesi bundles [L], Toda–Uehara [TU], Springer resolutions of determinantal varieties [BLV], and all known 3-fold crepant resolutions including derived McKay correspondence [BKR]. We note in 2.7 that it covers all 3-fold projective crepant resolutions, if the Craw–Ishii conjecture holds. Further we show in 2.5(2) that our framework applies to all contractions with fibres of dimension at most one.

As the construction of $\mathcal{D}$ is the most subtle part of the paper, we first briefly outline the local-to-global problems that arise in one specific example. Consider the following crepant divisor-to-curve contraction, given explicitly in 2.25. Y is singular along a one-dimensional locus Z, and above every point in Z is an irreducible curve.



At every closed point $z \in Z$, [DW2] constructs a noncommutative deformation algebra $\mathrm{A}_{\mathrm{con},z}$, which induces a universal sheaf $\mathcal{E}_z$ on the formal fibre above z. In the above example, at every closed point of Z away from the origin, $\mathrm{A}_{\mathrm{con},z}$ is isomorphic to $\Bbbk[\![x]\!]$, and at the origin it has the form shown above. The question is whether there exists a global sheaf of algebras $\mathcal{D}$ on Z which specialises, complete locally, to the algebras $\mathrm{A}_{\mathrm{con},z}$. Similarly one can ask whether the universal sheaves $\mathcal{E}_z$, as z varies over Z, can be glued into a single coherent sheaf $\mathcal{E}$ on X.

The key insight in this paper is that this can be done, but not on the nose; we construct a global $\mathcal{D}$ that recovers the algebras $\mathrm{A}_{\mathrm{con},z}$ up morita equivalence, and in (1.B) a global $\mathcal{E}$ that recovers the $\mathcal{E}_z$ up to additive equivalence. In the above example this means that $\mathcal{D}$ completed at a point z away from the origin is the ring of 2×2 matrices over $\Bbbk[\![x]\!]$, instead of $\Bbbk[\![x]\!]$ itself. This is rather harmless, since to extract local invariants, we pass to the basic algebra [DW1, §3], and for construction of derived autoequivalences, passing through morita equivalences is a mild and necessary procedure.

1.1. **Construction of Invariant** $(Z, \mathcal{D})$ **and Global** $\mathcal{E}$**.** We sketch this briefly, before describing our main results and applications. The construction does not use deformation theory, or any restrictions on singularities or fibre dimension. Full details are given in §2.

By a contraction, we mean a projective birational map $f\colon X \to Y$ between d-dimensional normal varieties over an algebraically closed field $\Bbbk$, satisfying $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_Y$, where we assume that Y is quasi-projective. Write Z for the locus of points of Y above which f is not an isomorphism.

We then assume that there is a vector bundle $\mathcal{P}$ on X containing $\mathcal{O}_X$ as a summand, such that

(1) The natural map $f_*\mathcal{E}nd_X(\mathcal{P}) \to \mathcal{E}nd_Y(f_*\mathcal{P})$ is an isomorphism.
(2) The functor
$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P},-)\colon \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \mathcal{B}) \tag{1.A}$$
is an equivalence, where $\mathcal{B} := f_*\mathcal{E}nd_X(\mathcal{P})$ is considered as a sheaf of $\mathcal{O}_Y$-algebras.

Given this setup, we write $\mathcal{Q} := f_*\mathcal{P}$, so that $\mathcal{B} \cong \mathcal{E}nd_Y(\mathcal{Q})$. We then define a subsheaf $\mathcal{I}$ of $\mathcal{B}$ consisting of local sections that at each stalk factor through a finitely generated projective module, in 2.8. We show in 2.9 that $\mathcal{I}$ is naturally a subsheaf of two-sided ideals of $\mathcal{B}$.

Thus we may also view $\mathcal{I}$ as a subsheaf of $\mathcal{B}^{\mathrm{op}}$, and define $\mathcal{D} := \mathcal{B}^{\mathrm{op}}/\mathcal{I}$, which we call the *sheafy contraction algebra*. For most purposes, taking the opposite ring structure can be ignored. In 2.11 we give two alternative descriptions of $\mathcal{I}$, which are important since they allow us to control local sections of $\mathcal{D}$, to such an extent that we can prove the following.

Theorem 1.1 (=2.16, Global Contraction Theorem)**.** $\operatorname{Supp}_Y \mathcal{D} = Z$.

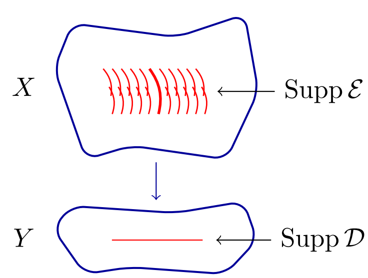
It follows that $\mathcal{D}$ is naturally a sheaf of algebras on the locus Z, and thus we can view the ringed space $(Z, \mathcal{D})$ as a noncommutative enhancement of Z.

Remark 1.2. When the locus Z can be realized as a moduli spaces $\mathcal{M}$ of stable sheaves, Toda [T4] constructs, under restrictions on $\dim Z$, another noncommutative enhancement of Z which is different to ours, since it is unusual for the stalks of $\mathcal{D}$ to be local rings; see also 2.20 and 2.26.

The noncommutative sheaf of rings $\mathcal{D}$ constructed is naturally a sheaf on the base Y of the contraction $f\colon X \to Y$. We next lift this to give an object on X, by observing that by construction $\mathcal{D}$ is a factor of $\mathcal{B}^{\mathrm{op}}$, so it also carries the natural structure of a $\mathcal{B}$-bimodule. In particular, we can view $\mathcal{D}$ as an object of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \mathcal{B}^{\mathrm{op}})$, and hence, across a dual version of (1.A), it gives an object
$$\mathcal{E} := f^{-1}\mathcal{D} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{B}^{\mathrm{op}}} \mathcal{P}^* \tag{1.B}$$
of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. In general $\mathcal{E}$ need not be a sheaf. In our most general setup, we prove the following.

Proposition 1.3 (=3.2)**.** $\mathbf{R}f_*\mathcal{E} = 0$. *In particular,* $\operatorname{Supp}_X \mathcal{E}$ *lies in the exceptional locus.*

The case of one-dimensional fibres is particularly pleasant, and a typical example is sketched below.



The following are our main results in this setting. In particular, this globalises the local noncommutative deformation theory as studied in [DW1, DW2, BB, K2].

Theorem 1.4. *Suppose that $f\colon X \to Y$ is a contraction where the fibres have dimension at most one. Then the following hold.*

(1) (=2.24) *The completion of* $\mathcal{D}$ *at a closed point* $z \in Z$ *is morita equivalent to the algebra that prorepresents noncommutative deformations of the reduced fibre over* z.
(2) (=3.5) $\mathcal{E}$ *is a sheaf.*
(3) (=3.7) $\operatorname{Supp}_X \mathcal{E}$ *equals the exceptional locus of* f.
(4) (=3.8) *The restriction of* $\mathcal{E}$ *to the formal fibre above a closed point* $z \in Z$ *recovers the universal sheaf* $\mathcal{E}_z$ *from noncommutative deformation theory, up to additive equivalence.*

The fact that in the one-dimensional fibre setting $\mathcal{E}$ is a sheaf is our main motivation for considering $\mathcal{B}^{\mathrm{op}}$ instead of $\mathcal{B}$.

1.2. **Applications to Autoequivalences** Twist_X**.** We next turn our attention to crepant contractions and autoequivalences. One benefit of the construction of $\mathcal{E}$ on X is that, using the natural exact sequence

$$0 \to \mathcal{I} \to \mathcal{B}^{\mathrm{op}} \to \mathcal{D} \to 0,$$

we can construct in §5, under our most general setup in §1.1, a functor Twist_X, which sits in a functorial triangle

$$f^{-1}\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{E}, a) \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}} \mathcal{E} \to a \to \mathsf{Twist}_X(a) \to .$$

For a precise description of the terms in these expressions, we refer the reader to §5, but remark here that when Z is a point, Twist_X reduces to a noncommutative twist over a contraction algebra, as studied in [DW1, DW3]. The existence of a global sheafy contraction algebra $\mathcal{D}$ allows us to avoid delicate local-to-global gluing arguments.

In general, Twist_X is not an equivalence, but we have the following criterion in terms of a Cohen–Macaulay property of $\mathcal{D}$ on Y, under a restriction to hypersurface singularities. There are two main reasons for this restriction on singularities (although it is not used everywhere), outlined in 2.20 below: note however that it holds automatically in the two main applications in this paper, namely to Springer resolutions of determinantal varieties of $n \times n$ matrices in 1.6, and to 3-fold divisor-to-curve contractions in 1.8.

Theorem 1.5 (=5.7)**.** *Under the general assumptions of 2.3, assume that* f *is crepant, and* $\widehat{\mathcal{O}}_{Y,z}$ *are hypersurfaces for all closed points* $z \in Z$. *Then the following are equivalent.*
(1) $\mathcal{D}$ *is a Cohen–Macaulay sheaf on* Y, *and* $\mathcal{E}$ *is a perfect complex on* X.
(2) $\mathcal{D}$ *is relatively spherical (in the sense of 5.6) for all closed points* $z \in Z$.

If these conditions hold, and they are automatic provided that $\dim Z \leq \dim Y - 3$, *then the functor* Twist_X *is an autoequivalence of* $\mathrm{D^b}(\operatorname{coh} X)$.

We remark that both parts of 1.5(1) can fail in general, and indeed if Z is not equidimensional, then $\mathcal{D}$ is not relatively spherical for some $z \in Z$.

The last statement in 1.5 ensures that our framework recovers previous autoequivalences associated to flopping contractions [T1, DW1, BB], since the assumptions of 1.5 are satisfied in the one-dimensional fibre flops setting described in [V1, Theorem C], but the main advantage of 1.5 is that it includes other interesting settings, which we briefly outline here.

Our first new application is to the varieties of singular $n \times n$ matrices, namely the determinantal varieties $\mathbb{k}[x_{ij}]/(\det x)$. For each such variety, the Springer resolution admits a suitable tilting bundle by work of Buchweitz, Leuschke, and Van den Bergh [BLV].

Corollary 1.6 (=5.8)**.** *Consider the Springer resolution* $X \to Y$ *of the variety of singular* $n \times n$ *matrices. Then* Twist_X *is an autoequivalence of* X.

We then establish results for one-dimensional fibre contractions which are not an isomorphism in codimension two, where the conditions in 1.5 are not automatic. Amongst others, this includes partial resolutions of Kleinian singularities, and 3-fold crepant divisor-to-curve examples. Leveraging our control of $\mathcal{E}$ in this setting, which by 1.4 is a sheaf with support coinciding with the exceptional locus, we reformulate the first part of 1.5(1), namely $\mathcal{D}$ being a Cohen–Macaulay sheaf on Y, in terms of a Cohen–Macaulay property for $\mathcal{E}$ on X.

Theorem 1.7 (=6.2)**.** *With the general setup in 1.5, suppose further that the fibres of f have dimension at most one. Then the following are equivalent.*

1. *$\mathcal{D}$ is a Cohen–Macaulay sheaf on Y.*
2. *For all $y \in Z$, and all $x \in f^{-1}(y)$, we have $\mathcal{E}_x \in \mathrm{CM}_{\dim Z_y+1}\,\mathcal{O}_{X,x}$.*

In particular, if these conditions hold, then above every $y \in Z$, the exceptional locus is equidimensional of dimension $\dim Z_y + 1$.

Combining with 1.5, the dimension criterion above gives an easy-to-check obstruction to $\mathcal{D}$ being relatively spherical, and we demonstrate this in various examples, such as 6.4. Furthermore, the conditions above enable us to prove that Twist_X is a equivalence in the motivating divisor-to-curve contraction example.

Theorem 1.8 (=6.5)**.** *With the setup in 1.7, suppose in addition that X is smooth and $\dim X = 3$, such that every reduced fibre above a closed point in Z contains precisely one irreducible curve. Then Twist_X is an autoequivalence of X.*

Remark 1.9. We expect that, when Twist_X is an autoequivalence, it can be expressed as a twist of a spherical functor as in [A2, A1, AL, K3]. However, we do not address this question in this paper, as it would not simplify our proofs.

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1.4. **Conventions.** Throughout we work over an algebraically closed field $\Bbbk$. Unqualified uses of the word 'module' refer to right modules, and $\operatorname{mod} A$ denotes the category of finitely generated right A-modules. We use the functional convention for composing homomorphisms, so $f \circ g$ means g then f. In particular, naturally this makes $M \in \operatorname{mod} A$ into an $\operatorname{End}_A(M)^{\mathrm{op}}$-module.

For $a \in \mathcal{A}$ an abelian category, we let $\operatorname{add} a$ denote all possible summands of finite direct sums of a. Given two objects $a, b \in \mathcal{A}$ where a is a summand of b, we then write $[a]$ for the two-sided ideal of $\operatorname{End}_{\mathcal{A}}(b)$ consisting of all morphisms that factor through $\operatorname{add} a$.

2. Global Thickenings

In this section we will noncommutatively enhance the non-isomorphism locus of a contraction $f\colon X \to Y$ which satisfies some mild conditions. We first recall what this means, and fix the setting.

Definition 2.1. *By a contraction, we will mean a projective birational morphism $f\colon X \to Y$ between normal varieties over $\Bbbk$, satisfying $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_Y$. It will be assumed that Y is quasi-projective.*

Notation 2.2. Given a contraction $f\colon X \to Y$, write Z for the locus of (not necessarily closed) points of Y above which f is not an isomorphism.

Given a contraction, we will furthermore assume the following condition.

Assumption 2.3. *For a contraction $f\colon X \to Y$, where $d = \dim X \geq 2$, assume that there is vector bundle $\mathcal{P} = \mathcal{O}_X \oplus \mathcal{P}_0$ on X, such that the following conditions hold.*

1. *The natural map $f_*\mathcal{E}nd_X(\mathcal{P}) \to \mathcal{E}nd_Y(f_*\mathcal{P})$ is an isomorphism.*
2. *The bundle $\mathcal{P}$ is tilting relative to Y, namely*
$$\mathbf{R}f_*\mathcal{E}nd_X(\mathcal{P}) = f_*\mathcal{E}nd_X(\mathcal{P}) =: \mathcal{B},$$
and furthermore there is an equivalence
$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P}, -)\colon\ \mathrm{D^b}(\operatorname{coh} X) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{coh}(Y, \mathcal{B})). \tag{2.A}$$

Here $\mathcal{B}$ is considered as a sheaf of $\mathcal{O}_Y$-algebras in the sense of [KS, §18.1]*, and the bounded derived category of modules over $\mathcal{B}$ is denoted by $\mathrm{D^b}(\mathrm{coh}(Y,\mathcal{B}))$.*

With mind to our applications, we will show in 2.5 below that the assumption is rather mild, and holds in general settings. This requires the following easy consequence of 2.3(2).

Lemma 2.4. *Suppose that 2.3(2) holds, V is an affine open subset of Y, and $U := f^{-1}(V)$. Then $\mathcal{P}|_U$ is a tilting bundle on U.*

Proof. Setting $\Lambda := \mathrm{End}_U(\mathcal{P}|_U)$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{D^b}(\mathrm{coh}\, X) \arrow[r, "\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P},-)", "\sim"'] \arrow[d, "|_U"] & \mathrm{D^b}(\mathrm{coh}(Y,\mathcal{B})) \arrow[d, "|_V"] \\
\mathrm{D^b}(\mathrm{coh}\, U) \arrow[r, "\mathbf{R}\mathrm{Hom}_U(\mathcal{P}|_U,-)"] & \mathrm{D^b}(\mathrm{mod}\,\Lambda)
\end{tikzcd}$$

In particular, $\mathbf{R}\mathrm{Hom}_U(\mathcal{P}|_U,\mathcal{P}|_U)\cong\mathcal{B}|_V\cong\Lambda$, which gives Ext vanishing. For generation, suppose that $a\in\mathrm{D}(\mathrm{Qcoh}\,U)$ with $\mathbf{R}\mathrm{Hom}_U(\mathcal{P}|_U,a)=0$. We must show that $a=0$. By adjunction $\mathbf{R}\mathrm{Hom}_X(\mathcal{P},\mathbf{R}i_*a)=0$ where $i\colon U\hookrightarrow X$ is the open inclusion. Since $\mathcal{P}$ compactly generates $\mathrm{D}(\mathrm{Qcoh}\,X)$, it follows that $\mathbf{R}i_*a=0$, and thus $a=0$ since $\mathbf{R}i_*$ is fully faithful. □

Proposition 2.5. *Assumption 2.3 is guaranteed to hold in the following two settings.*

(1) *Y is a Gorenstein d-fold, f is crepant, and X admits a relative tilting bundle containing $\mathcal{O}_X$ as a summand.*
(2) *X is a d-fold, and the fibres of f have dimension at most one.*

Proof. (1) Take $\mathcal{P}$ to be the relative tilting bundle on X, with $\mathcal{P}=\mathcal{O}_X\oplus\mathcal{P}_0$. Condition 2.3(2) is immediate. Condition 2.3(1) is local, therefore it suffices to consider affine $Y=\mathrm{Spec}\,R$, in which case the condition translates into showing that

$$\mathrm{End}_X(\mathcal{P})\to\mathrm{End}_R(f_*\mathcal{P}), \tag{2.B}$$

is an isomorphism. Note that $\mathrm{End}_X(\mathcal{P})$ is a maximal Cohen–Macaulay R-module since f is crepant [IW3, 4.8]. We next claim that $f_*\mathcal{P}$ is also maximal Cohen–Macaulay. The tilting property for $\mathcal{P}$ implies that $H^i(\mathcal{P}_0)=0=H^i(\mathcal{P}_0^*)$ for all $i>0$, and so

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(f_*\mathcal{P}_0,R) &\cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}f_*\mathcal{P}_0,\mathcal{O}_R)\\
&\cong \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P}_0,f^!\mathcal{O}_R)\\
&\cong \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{P}_0,\mathcal{O}_X)\\
&= \mathbf{R}f_*(\mathcal{P}_0^*)=f_*(\mathcal{P}_0^*),
\end{aligned}$$

where we use Grothendieck duality and crepancy. Thus $\mathbf{R}\mathrm{Hom}_R(f_*\mathcal{P}_0,R)$ is concentrated in degree zero, hence $f_*\mathcal{P}_0$ is maximal Cohen–Macaulay, and thence $f_*\mathcal{P}$ is also. In particular, (2.B) is a morphism between reflexive R-modules, so since it is an isomorphism in codimension two, it must be an isomorphism.
(2) Let $\mathcal{P}$ denote the bundle constructed by Van den Bergh in [V1, 3.3.2], which is relatively generated by global sections. Condition 2.3(2) is [V1, 3.3.1]. For condition 2.3(1), note first that f_* gives rise to the natural morphism

$$\mathcal{B}=f_*\mathcal{E}nd_X(\mathcal{P})\to\mathcal{E}nd_Y(f_*\mathcal{P}).$$

It suffices to check that this is an isomorphism on open affines V of Y, on which it is just the following, where we write $U:=f^{-1}(V)$.

$$f_*\colon\ \mathrm{End}_U(\mathcal{P}|_U)\to\mathrm{End}_V(f_*\mathcal{P}|_V) \tag{2.C}$$

By 2.4 $\mathcal{P}|_U$ is tilting, so $\operatorname{Ext}^1_U(\mathcal{P}|_U, \mathcal{P}|_U) = 0$. Since further Y is normal, f is projective birational, and $\mathcal{P}|_U$ is generated by global sections, it follows from [DW2, 4.3] that (2.C) is an isomorphism. $\square$

Remark 2.6. The relative tilting assumption from 2.5(1) reduces, in the case of affine $Y = \operatorname{Spec} R$, to the condition that $\operatorname{Ext}^i_X(\mathcal{P}, \mathcal{P}) = 0$ for $i > 0$, and $\mathcal{P}$ generates.

Remark 2.7. We remark that if $d = 3$ in 2.5(1), the assumption that X admits a tilting bundle is automatic if X is constructed as the moduli of an NCCR of an affine Y [V2, §6]. It is expected that the tilting bundle condition in 2.5(1) for $d = 3$ always holds, as a consequence of the Craw–Ishii conjecture and work of Iyama and the second author [IW2, 4.18(2)].

2.1. **Construction of $\mathcal{I}$.** In this subsection we work under the general assumptions of 2.3, and define an ideal subsheaf $\mathcal{I}$ of $\mathcal{B} := f_*\mathcal{E}nd_X(\mathcal{P}) \cong \mathcal{E}nd_Y(f_*\mathcal{P})$. To ease notation, we put $\mathcal{Q} := f_*\mathcal{P}$, so that $\mathcal{B} \cong \mathcal{E}nd_Y(\mathcal{Q})$. For $\mathcal{F}$ a sheaf, $s \in \mathcal{F}(V)$, and $v \in V$, we write $\langle s, V\rangle_v$ for s viewed in the stalk $\mathcal{F}_v$.

Definition 2.8. *With notation as above, for each open subset V of Y, define*

$$\mathcal{I}(V) := \{s \in \operatorname{End}_V(\mathcal{Q}|_V) \mid \mathcal{Q}_v \xrightarrow{\langle s,V\rangle_v} \mathcal{Q}_v \text{ factors through } \operatorname{add}\mathcal{O}_{Y,v} \text{ for all } v \in V\},$$

which is a two-sided ideal of $\mathcal{B}(V)$. That is, $\mathcal{I}$ consists of local sections of $\mathcal{B}$ that at each stalk factor through a finitely generated projective $\mathcal{O}_{Y,v}$-module.

Proposition 2.9. *$\mathcal{I}$ is a subsheaf of $\mathcal{B}$ consisting of two-sided ideals.*

Proof. We just need to prove that $\mathcal{I}$ is a subsheaf. For opens $U \subseteq V$ of Y, denote the restriction morphisms of $\mathcal{B}$ by ρ_{VU}. Note that if $s \in \mathcal{I}(V)$ then since

$$\langle s, V\rangle_u = \langle \rho_{VU}(s), U\rangle_u,$$

and this factors through a projective for all $u \in U$, it follows that ρ_{VU} takes $\mathcal{I}(V)$ to $\mathcal{I}(U)$. The presheaf axioms of $\mathcal{B}$ then imply that $\mathcal{I}$ is a subpresheaf of $\mathcal{B}$, so it suffices to check the two sheaf axioms.

Suppose then that $U = \bigcup U_i$ and $s \in \mathcal{I}(U)$ is such that $s|_{U_i} = 0$ for all i. Viewing this in $\mathcal{B}$, it follows that $s = 0$.

Lastly, suppose that $U = \bigcup U_i$ and there is a collection $s_i \in \mathcal{I}(U_i) \subseteq \mathcal{B}(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all i, j. Then since $\mathcal{B}$ is a sheaf certainly there exists $s \in \mathcal{B}(U)$ such that $s|_{U_i} = s_i$ for all i. We claim that $s \in \mathcal{I}(U)$, that is $\langle s, U\rangle_u$ factors through $\operatorname{add}\mathcal{O}_{Y,u}$ for all $u \in U$. Thus let $u \in U$, then certainly $u \in U_i$ for some i. But then

$$\langle s, U\rangle_u = \langle s|_{U_i}, U_i\rangle_u,$$

which factors through a projective since $s|_{U_i} = s_i \in \mathcal{I}(U_i)$. $\square$

We now show that on affine open subsets, $\mathcal{I}$ takes a particularly nice form. To simplify notation, for an affine open subset $V = \operatorname{Spec} R$ of Y, write Q for the R-module corresponding to $\mathcal{Q}|_{\operatorname{Spec} R}$, so that $\mathcal{B}(\operatorname{Spec} R) \cong \operatorname{End}_R(Q)$. Choose a surjection $h\colon F \twoheadrightarrow Q$ with F a free R-module, and write

$$\operatorname{Hom}_R(Q, F) \xrightarrow{\alpha := (h\circ)} \operatorname{Hom}_R(Q, Q).$$

It is standard that this localises to

$$\operatorname{Hom}_{R_\mathfrak{p}}(Q_\mathfrak{p}, F_\mathfrak{p}) \xrightarrow{\alpha_\mathfrak{p} = (h_\mathfrak{p}\circ)} \operatorname{Hom}_{R_\mathfrak{p}}(Q_\mathfrak{p}, Q_\mathfrak{p}).$$

By the defining property of projective modules, any morphism from an object in $P \in \operatorname{add} R$ to Q has to factor as

$$P \dashrightarrow F \xrightarrow{h} \!\!\!\!\to Q,$$

and any morphism from an object of $P \in \operatorname{add} R_{\mathfrak{p}}$ to $Q_{\mathfrak{p}}$ has to factor as

$$P \dashrightarrow F_{\mathfrak{p}} \xrightarrow{h_{\mathfrak{p}}} Q_{\mathfrak{p}}.$$

Hence

$$\{g\colon Q \to Q \mid g \text{ factors through } \operatorname{add} R\} = \operatorname{Im}(\alpha), \tag{2.D}$$

and

$$\begin{aligned}\mathcal{I}(\operatorname{Spec} R) &= \{g\colon Q \to Q \mid g_{\mathfrak{p}} \text{ factors through } \operatorname{add} R_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\} \\ &= \{g\colon Q \to Q \mid g_{\mathfrak{p}} \in \operatorname{Im}(\alpha_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}.\end{aligned} \tag{2.E}$$

The following lemma says that a morphism factors stalk-locally if and only if it factors affine-locally.

Lemma 2.10. *If* $\operatorname{Spec} R$ *is an affine open subset of* Y*, then*

$$\mathcal{I}(\operatorname{Spec} R) = \{g\colon Q \to Q \mid g \textit{ factors through } \operatorname{add} R\}.$$

Proof. By (2.D) and (2.E), it suffices to prove that

$$g \in \operatorname{Im} \alpha \iff g_{\mathfrak{p}} \in \operatorname{Im} \alpha_{\mathfrak{p}}$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. The direction ($\Rightarrow$) is clear. For the ($\Leftarrow$) direction, suppose that $g \in \operatorname{Im}(\alpha_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$, and consider $g + \operatorname{Im} \alpha \in \operatorname{End}_R(Q)/\operatorname{Im} \alpha$. Then stalk-locally this element is zero, hence it is zero (see e.g. [E, Lemma 2.8(a)]), and thus $g \in \operatorname{Im} \alpha$. □

It follows that we can define $\mathcal{I}$ in various equivalent ways, without referring to stalks.

Corollary 2.11. *Suppose that* V *is an open subset of* Y*, and* $s \in \mathcal{B}(V)$*. Then the following conditions are equivalent.*

1. $s \in \mathcal{I}(V)$.
2. *There exists an open affine cover* $V = \bigcup V_i$ *such that* $s|_{V_i}$ *factors through* $\operatorname{add} \mathcal{O}_{V_i}$ *for all* i*.*
3. *For every open affine cover* $V = \bigcup V_i$*,* $s|_{V_i}$ *factors through* $\operatorname{add} \mathcal{O}_{V_i}$ *for all* i*.*

Proof. Since $\mathcal{I}$ is a sheaf by 2.9, if $s \in \mathcal{I}(V)$ then $s|_{V_i} \in \mathcal{I}(V_i)$. Hence (1)⇒(3) follows from 2.10. (3)⇒(2) is obvious, and (2)⇒(1) is clear. □

2.2. Sheafy Contraction Algebras.

In this subsection we continue to work under the general assumptions of 2.3, but consider the dual bundle $\mathcal{V} := \mathcal{P}^*$, and $\mathcal{A} := f_*\mathcal{E}nd_X(\mathcal{V})$. Although not strictly necessary (up to taking opposite algebras, we show in the proof of 2.15 below that it gives the same objects), taking the dual is convenient since in the setting 2.5(2) of fibre dimension at most one, it will allow us to relate the above construction to our previous work, and deformation theory, in an easier way.

By 2.9, $\mathcal{I}$ is a sheaf of two-sided ideals of $\mathcal{B} := f_*\mathcal{E}nd_X(\mathcal{P})$. Since $\mathcal{A} \cong \mathcal{B}^{\mathrm{op}}$, we can also view $\mathcal{I}$ as a subsheaf of two-sided ideals of $\mathcal{A}$. Thus we can consider the presheaf quotient of $\mathcal{A}$ by $\mathcal{I}$, which is naturally a presheaf of algebras, and hence its sheafification $\mathcal{D} := \mathcal{A}/\mathcal{I}$ is a sheaf of algebras on Y.

Definition 2.12. *We call* $\mathcal{D} := \mathcal{A}/\mathcal{I}$ *the sheaf of contraction algebras on* Y*.*

Note that there is a natural $\mathcal{O}_Y$-algebra structure on $\mathcal{D}$ given by a morphism of sheaves of rings $\mathcal{O}_Y \to \mathcal{A}$, as in [KS, §18.1], however this morphism is not injective, as a consequence of 2.16 below. Locally, the sections of $\mathcal{D}$ have a particularly easy form, which we now describe. For an affine neighbourhood V in Y, consider the following Zariski local setup obtained by base change.

Setup 2.13. (Zariski local) Suppose that $f\colon U \to V = \operatorname{Spec} R$ is a projective birational morphism between normal varieties over $\mathbb{k}$, with $\mathbf{R}f_*\mathcal{O}_U = \mathcal{O}_V$.

We set $\Lambda_V := \operatorname{End}_U(\mathcal{V}|_U)$. The following is an extension of [DW2, §3.1], which only considered contractions with fibre dimension at most one.

Definition 2.14. *Write I_{con} for the two-sided ideal of Λ_V consisting of those morphisms $\varphi\colon \mathcal{V}|_U \to \mathcal{V}|_U$ which factor through an object $\mathcal{F} \in \operatorname{add}\mathcal{O}_U$, as follows.*

$$\begin{tikzcd} \mathcal{V}|_U \arrow[rr, "\varphi"] \arrow[dr, dotted] & & \mathcal{V}|_U \\ & \mathcal{F} \arrow[ur, dotted] & \end{tikzcd}$$

Then the contraction algebra *for Λ_V is defined to be $(\Lambda_V)_{\mathrm{con}} := \Lambda_V / I_{\mathrm{con}}$.*

The following proposition is important, and will be heavily used later.

Proposition 2.15. *If $V = \operatorname{Spec} R$ is an affine open subset of Y, then $\mathcal{D}(V) \cong (\Lambda_V)_{\mathrm{con}}$.*

Proof. For all affine open subsets $V = \operatorname{Spec} R$ of Y

$$0 \to \mathcal{I}(V) \to \mathcal{A}(V) \to \mathcal{D}(V) \to 0$$

is exact. It follows that, for $U := f^{-1}(V)$,

$$\begin{aligned} \mathcal{D}(V) = \mathcal{A}(V)/\mathcal{I}(V) &= \big(\mathcal{B}(V)/\mathcal{I}(V)\big)^{\mathrm{op}} \\ &\cong \big(\operatorname{End}_R(f_*\mathcal{P}|_U)/[R]\big)^{\mathrm{op}} && \text{(by (2.C), 2.10)} \\ &= \big(\operatorname{End}_U(\mathcal{P}|_U)/[\mathcal{O}_U]\big)^{\mathrm{op}} && \text{(by 2.3(1))} \end{aligned}$$

which is $\operatorname{End}_U(\mathcal{V}|_U)/[\mathcal{O}_U] = (\Lambda_V)_{\mathrm{con}}$. □

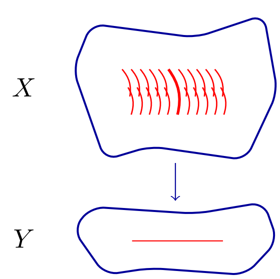
A benefit of our global construction of $\mathcal{D}$, which we will make use of later, is that the contraction theorem in [DW2] can be globalised. Although [DW2, 4.4–4.7] was stated in the one-dimensional fibre setting, the proof works word-for-word in the more general setup here.

Corollary 2.16 (Global Contraction Theorem)**.** $\operatorname{Supp}_Y \mathcal{D} = Z$.

Proof. By [DW2, 4.7], which does not require the one-dimensional fibre assumption, we see that $\operatorname{Supp}_V(\Lambda_V)_{\mathrm{con}} = Z \cap V$ for all affine opens V in Y. Since $\operatorname{Supp}_V(\Lambda_V)_{\mathrm{con}} = \operatorname{Supp}_V \mathcal{D}|_V$ by 2.15, the result follows. □

Remark 2.17. By construction $\mathcal{D}$ is a sheaf of algebras on Y. By 2.16 it is naturally a sheaf of algebras on Z, and thus we can view the ringed space $(Z, \mathcal{D})$ as a noncommutative enhancement of the locus Z. Note further that $\operatorname{coh}(Y, \mathcal{D}) = \operatorname{coh}(Z, \mathcal{D})$.

Example 2.18. Consider $R := \mathbb{C}[x,y,z,t]/(x^2t + y^3 - z^2)$ and $Y = \operatorname{Spec} R$. This is singular along $x = y = z = 0$, and has a crepant resolution sketched below, obtained by blowing up the ideal (x,y,z).



In this example X is derived equivalent to $\Lambda := \operatorname{End}_R(R \oplus M)$, where M is the cokernel of the following 4×4 matrix.

$$R^4 \xrightarrow{\begin{pmatrix} x & z & -y & 0 \\ z & xt & 0 & y \\ y^2 & 0 & xt & z \\ 0 & y^2 & -z & -x \end{pmatrix}} R^4 \tag{2.F}$$

We now calculate the stalks of the sheaf $\mathcal{D}$ at the closed points of Z. It is clear that, away from the origin, complete locally R is given by the equation $x^2 + y^3 - z^2$. This is the A_2 surface singularity crossed with the affine line, and thus the generic fibre is just the minimal resolution of A_2 crossed with the affine line. Since M has rank two, it follows that for all points of Z away from the origin, the completion of the stalk of $\mathcal{D}$ is given by the completion of the following quiver with relations.

$$\begin{tikzcd} \bullet \arrow[loop left, "s"] \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \arrow[loop right, "t"] \end{tikzcd} \qquad \begin{aligned} a \circ s &= t \circ a \\ b \circ t &= s \circ b \\ a \circ b &= 0 \\ b \circ a &= 0 \end{aligned}$$

On the other hand, the origin is a cD_4 point, and calculating the stalk of $\mathcal{D}$ there is a little trickier. Using the matrix in (2.F), it is easy to first present Λ as a quiver with relations, as in [AM, §4], then calculate the contraction algebra, as in [DW1, 1.3]. Doing this, one finds that the completion of $\mathcal{D}$ at the origin is the completion of

$$\frac{\mathbb{C}\langle a, b\rangle}{ab + ba, a^2}$$

at the ideal (a, b).

By 2.17 there is a noncommutative ringed space $(Z, \mathcal{D})$, and it is natural to compare this to the usual commutative ringed space $(Z, \mathcal{O}_Z)$, where $\mathcal{O}_Z$ is the structure sheaf that endows Z with its reduced subscheme structure. A key property of $(Z, \mathcal{O}_Z)$ is that for every point $z \in Z$, the stalk $\mathcal{O}_{Z,z}$ is local, that is, it admits only one simple module. This is not true in general for the stalk $\mathcal{D}_z$. However, in the crepant setting of 2.5(1), the following holds.

Proposition 2.19. *Suppose the crepant setting of 2.5(1), let $z \in Z$ be a closed point, and furthermore assume that $d = 3$ and X is smooth. If $\widehat{\mathcal{D}}_z$ is local, then $\widehat{\mathcal{O}}_{Y,z}$ is a hypersurface.*

Proof. Put $\mathfrak{R} := \widehat{\mathcal{O}}_{Y,z}$, and write $\widehat{\mathcal{D}}_z = \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus M)$. Since by assumption $\widehat{\mathcal{D}}_z$ is local, necessarily M must be indecomposable. Then by [IW2, 6.25(3)] mutation at the indecomposable summand M must be an involution, and so $\Omega^2 M \cong M$. But this implies that the complexity of M must be less than or equal to one, which easily (see for example the proof of [W, 6.14]) implies that $\mathfrak{R}$ is a hypersurface. $\square$

Remark 2.20. The converse to 2.19 is false, since if R is complete locally a hypersurface, $\widehat{\mathcal{D}}_z$ need not be local. Nevertheless, being the only situation with smooth X where a local $\widehat{\mathcal{D}}_z$ is possible, the situation where Y is complete locally a hypersurface has special status. It is also the only situation where our $\mathcal{D}$ can possibly coincide with Toda's noncommutative thickening [T4]. This partially motivates the hypersurface assumptions in §4 below.

2.3. **$\mathcal{D}$ and Deformations.** In the setting 2.5(2) when f has fibres of dimension at most one, in this subsection we briefly relate $\mathcal{D}$ to noncommutative deformation theory. For this, recall that an *n-pointed $\Bbbk$-algebra* Γ is an associative $\Bbbk$-algebra equipped with $\Bbbk$-algebra morphisms

$$\Bbbk^n \xrightarrow{i} \Gamma \xrightarrow{p} \Bbbk^n$$

which compose to the identity. Note that the morphisms p and i allow us to lift the canonical idempotents $\{e_1, \dots, e_n\}$ of $\Bbbk^n$ to Γ. We refer to Γ as *artinian* if it is finite-dimensional as a vector space over $\Bbbk$, and the ideal $\operatorname{Ker} p$ is nilpotent. Such algebras naturally form a category Art_n, and we write pArt_n for the category of pro-artinian algebras (see for instance [DW2, §2] for the precise construction).

Recall that a DG category is a graded category A whose morphism spaces have the structure of DG vector spaces. We write DG_n for the category which has as objects the DG categories with precisely n objects. Given a DG category $\mathsf{A} \in \mathsf{DG}_n$ and an algebra $\Gamma \in \mathsf{Art}_n$, we now recall an appropriate notion of deformations over Γ, according to the standard Maurer–Cartan formulation. Writing $\mathfrak{n} = \operatorname{Ker} p \subset \Gamma$, first consider the DG category $\mathsf{A} \underline{\otimes} \mathfrak{n} \in \mathsf{DG}_n$ with objects $\{1, \dots, n\}$, morphisms

$$\operatorname{Hom}_{\mathsf{A} \underline{\otimes} \mathfrak{n}}(i,j) := \operatorname{Hom}_{\mathsf{A}}(i,j) \otimes_{\Bbbk} (e_j \mathfrak{n} e_i),$$

and differential induced from A. (Note that the convention used here for notating compositions in Γ is opposite to that of [DW2].) Given a degree-1 morphism $\xi \in (\mathsf{A} \underline{\otimes} \mathfrak{n})^1$ consider the Maurer–Cartan equation

$$\operatorname{MC}(\xi) := d\xi + \tfrac{1}{2}[\xi, \xi] \in (\mathsf{A} \underline{\otimes} \mathfrak{n})^2 = 0$$

where d is the differential, and the bracket is induced from the commutator brackets on A and $\mathfrak{n}$.

Definition 2.21. *Given $\mathsf{A} \in \mathsf{DG}_n$, the associated* deformation functor *is given by*

$$\begin{aligned} \mathcal{D}ef^{\mathsf{A}} \colon \mathsf{Art}_n &\to \mathsf{Sets} \\ \Gamma &\mapsto \left\{ \xi \in (\mathsf{A} \underline{\otimes} \mathfrak{n})^1 \mid \operatorname{MC}(\xi) = 0 \right\} / \sim \end{aligned}$$

where the standard gauge equivalence relation $\sim$ is given in e.g. [DW2, 2.6].

Given objects $E_1, \dots, E_n \in \operatorname{coh}(X)$, and injective resolutions $E_i \to I^i_\bullet$, then

$$\mathsf{A} := \operatorname{End}^{\mathsf{DG}} \left(\bigoplus I^i_\bullet \right)$$

naturally lies in DG_n, and the associated deformation functor $\mathcal{D}ef^{\mathsf{A}}$ describes the simultaneous noncommutative deformations of the collection $\{E_i\}$.

Definition 2.22. *A functor $F \colon \mathsf{Art}_n \to \mathsf{Sets}$ is said to be* prorepresentable *by $\Gamma \in \mathsf{pArt}_n$ if the restriction of $\operatorname{Hom}_{\mathsf{pArt}_n}(\Gamma, -)$ to Art_n is naturally isomorphic to F.*

Given a contraction f with at most one-dimensional fibres as in 2.5(2), for every $z \in Z$, $C = f^{-1}(z)$ is a curve, and recall that we write $C^{\mathrm{red}} = \bigcup_{i=1}^n C_i$ where each $C_i \cong \mathbb{P}^1$. We put $E_i = \mathcal{O}_{C_i}(-1)$ and form A as above.

The following theorem was shown in [DW2, 1.1].

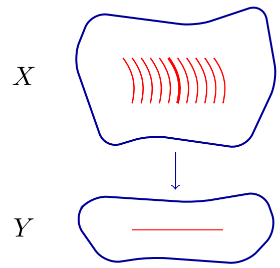
Theorem 2.23. *In the setting 2.5(2), for each closed point $z \in Z$ the functor of noncommutative deformations $\mathcal{D}ef^{\mathsf{A}}$ is represented by an algebra $\mathrm{A}_{\mathrm{con},z}$.*

The precise form of $\mathrm{A}_{\mathrm{con},z}$ is not important for now, but we do explain this in §4.2. Recall from §2.2 that for an affine open V of Y, and $U := f^{-1}(V)$, we write $\Lambda_V := \operatorname{End}_U(\mathcal{P}^*|_U)$. It was shown in [DW2, 1.1, 1.2] that the completion of $\mathcal{D}|_V = (\Lambda_V)_{\mathsf{con}}$ at a closed point $z \in Z \cap V$ is morita equivalent to $\mathrm{A}_{\mathrm{con},z}$. The following is then an immediate corollary of 2.15.

Corollary 2.24. *In the setting 2.5(2), let $z \in Z$ be a closed point. Then the completion of the stalk $\mathcal{D}_z$ is morita equivalent to $\mathrm{A}_{\mathrm{con},z}$.*

The morita equivalence above is illustrated in the following example.

Example 2.25. Consider $R := \mathbb{C}[x,y,z,t]/(x^3 - xyt - y^3 + z^2)$ and $Y = \operatorname{Spec} R$, which is singular along the t-axis. It has a crepant resolution sketched as follows.



In this example X is derived equivalent to $\Lambda = \operatorname{End}_R(R \oplus M)$, where M is the cokernel of the following 4×4 matrix.

$$R^4 \xrightarrow{\begin{pmatrix} x & z & -y & 0 \\ 0 & y^2 & -z & x \\ -z & x^2 - yt & 0 & y \\ -y^2 & 0 & x^2 - yt & z \end{pmatrix}} R^4$$

Away from the origin, the singular locus is just the A_1 surface singularity crossed with the affine line, so since M has rank two, complete locally away from the origin it must split into two isomorphic copies of the same rank-one CM module L, so that

$$\widehat{\Lambda} \cong \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus L^{\oplus 2}).$$

Hence, away from the origin, the completion of the stalk of $\mathcal{D}$ is the 2×2 matrices over $\mathbb{C}[t]$, since $\mathbb{C}[t]$ is the contraction algebra of $\operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus L)$.

At the origin, complete locally M is indecomposable, so that the stalk of $\mathcal{D}$ at the origin must be a local algebra. Using the same method as in 2.18, it is not difficult to show that the completion of the stalk is isomorphic to the completion of $\mathbb{C}\langle a,b \rangle/(a^2, b^2)$ at the ideal (a,b).

Remark 2.26. This remark explains why our thickening is different to the one constructed by Toda [T4]. In the above example, a generic exceptional fibre consists of a single smooth projective curve. Take its structure sheaf and consider its Hilbert polynomial to obtain a moduli space of simple sheaves. All generic fibres have this Hilbert polynomial (by flatness), and the moduli space has a point for each one of these, plus a point corresponding to the origin, to give a pure dimension one scheme. The completion of Toda's sheaf $\mathcal{D}'$ at each point abelianizes to give the completion of the stalk of the structure sheaf of the moduli space at that point. In particular, since the moduli space has pure dimension one, this stalk cannot be finite dimensional. But $\mathcal{D}$ abelianizes at the central point to give $\mathbb{C}[[a,b]]/(a^2, b^2)$, which is finite dimensional, and hence the sheaves $\mathcal{D}$ and $\mathcal{D}'$ are different.

3. A Universal Sheaf $\mathcal{E}$

In this section we use $\mathcal{D}$ from 2.12 to construct a complex of sheaves $\mathcal{E}$ on X, and present some of its basic properties. In the case of fibre dimension at most one, 2.5(2), we will show that $\mathcal{E}$ is a sheaf, with support equal to the exceptional locus.

3.1. General Construction of $\mathcal{E}$.

In this subsection we work under the general assumptions of 2.3. By definition, 2.12, $\mathcal{D}$ is a sheaf on Y. However, in the setting 2.5(2), consider $\mathcal{V} := \mathcal{P}^*$, which induces an equivalence

$$\mathrm{D^b}(\operatorname{coh} X) \xrightarrow[\sim]{\mathbf{R}f_* \mathbf{R}\mathcal{H}om_X(\mathcal{V}, -)} \mathrm{D^b}(\operatorname{coh}(Y, \mathcal{A})). \tag{3.A}$$

The sheaf of algebras $\mathcal{D} := \mathcal{A}/\mathcal{I}$ constructed in 2.12 is in particular an $\mathcal{A}$-module, and thus an object of $\mathrm{D^b}(\mathrm{coh}(Y,\mathcal{A}))$.

Notation 3.1. We write $\mathcal{E}$ for the object in $\mathrm{D^b}(\mathrm{coh}\,X)$ which corresponds to $\mathcal{D}$ across the equivalence (3.A).

Proposition 3.2. *Under the general assumptions of 2.3, for an affine open $V = \operatorname{Spec} R \subseteq Y$, set $U = f^{-1}(V)$ and write $f' \colon U \to V$ for the restricted map. Then the following hold.*

(1) $\mathbf{R}f'_*(\mathcal{E}|_U) = 0$.
(2) $\mathbf{R}f_*\mathcal{E} = 0$.

In particular, $\operatorname{Supp}_X \mathcal{E}$ *is contained in the exceptional locus.*

Proof. (1) There is a commutative diagram as follows.

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{coh}\,X) \arrow[r, "\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{V},-)", "\sim"'] \arrow[d, "|_U"] & \mathrm{D^b}(\mathrm{coh}(Y,\mathcal{A})) \arrow[d, "|_V"] \\ \mathrm{D^b}(\mathrm{coh}\,U) \arrow[r, "\mathbf{R}\mathrm{Hom}_U(\mathcal{V}|_U,-)", "\sim"'] & \mathrm{D^b}(\mathrm{mod}\,\Lambda_V) \end{tikzcd} \tag{3.B}$$

Since $\mathcal{E}$ corresponds to $\mathcal{D}$ on the top equivalence, it follows that $\mathcal{E}|_U$ corresponds to $\mathcal{D}(V)$ across the bottom equivalence. Hence by 2.15 $\mathcal{E}|_U$ corresponds to $(\Lambda_V)_{\mathrm{con}}$.

Let e denote the idempotent in Λ_V corresponding to R. Then the following diagram commutes.

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{coh}\,U) \arrow[r, "\mathbf{R}\mathrm{Hom}_U(\mathcal{V}|_U,-)", "\sim"'] \arrow[d, "\mathbf{R}f'_*"] & \mathrm{D^b}(\mathrm{mod}\,\Lambda_V) \arrow[d, "(-)e"] \\ \mathrm{D^b}(\mathrm{coh}\,\mathrm{Spec}\,R) \arrow[r, equal] & \mathrm{D^b}(\mathrm{mod}\,R) \end{tikzcd} \tag{3.C}$$

Since $(\Lambda_V)_{\mathrm{con}} \in \mathrm{mod}\,\Lambda_V$ satisfies $(\Lambda_V)_{\mathrm{con}}e = 0$, it follows that $\mathcal{E}|_U$ satisfies $\mathbf{R}f'_*(\mathcal{E}|_U) = 0$.
(2) Since $\mathbf{R}^i f_*\mathcal{E}|_V = \mathbf{R}^i f'_*(\mathcal{E}|_U)$ by flat base change (see e.g. [H2, III.8.2]), it follows from (1) that $\mathbf{R}f_*\mathcal{E} = 0$. $\square$

It may be the case that $\operatorname{Supp}_X \mathcal{E}$ always equals the exceptional locus, but this seems tricky to prove without more control over where the simple $(\Lambda_V)_{\mathrm{con}}$-modules go under the derived equivalence. Controlling the support of $\mathcal{E}$ is important, since later it will give an easy-to-check obstruction to $\mathcal{D}$ being relatively spherical.

3.2. The Sheaf $\mathcal{E}$ and Fibre Dimension One.

This subsection considers the setting of 2.5(2) when f has fibres of dimension at most one, and proves that $\mathcal{E}$ is a sheaf whose support $\operatorname{Supp}_X \mathcal{E}$ equals the exceptional locus.

In the setting of 2.5(2), recall from [B, §3] and [V1, §3.1] that perverse coherent sheaves on X may be defined as follows.

Definition 3.3. *The category ${}^{-1}\mathcal{P}er(X,Y)$, respectively ${}^{0}\mathcal{P}er(X,Y)$, consists of those objects $a \in \mathrm{D^b}(\mathrm{coh}\,X)$ such that*

(1) $H^i(a) = 0$ *if* $i \neq 0, -1$,
(2) $f_*H^{-1}(a) = 0$, $\mathbf{R}^1 f_*H^0(a) = 0$,
(3) $\mathrm{Hom}_X(H^0(a), c) = 0$, *respectively* $\mathrm{Hom}_X(c, H^{-1}(a)) = 0$, *for all* $c \in \mathcal{C}^0$

where $\mathcal{C} := \ker \mathbf{R}f_*$ and $\mathcal{C}^0$ denotes the full subcategory of $\mathcal{C}$ whose objects have cohomology only in degree 0.

It follows from [V1, 3.3.1, proof of 3.3.2] that the $\mathcal{P}$ in 2.5(2) is a local progenerator of ${}^{-1}\mathcal{P}er(X/Y)$, and $\mathcal{V} = \mathcal{P}^*$ likewise for ${}^{0}\mathcal{P}er(X/Y)$.

To show that $\mathcal{E}$ is a sheaf requires the following lemma, which is well known.

Lemma 3.4. *Suppose that $a \in \mathrm{D^b}(\mathrm{coh}\,X)$ such that $\mathbf{R}f_*a = 0$.*

(1) *$a \in {}^{-1}\mathcal{P}er(X,Y)$ if and only if a is concentrated in degree -1.*
(2) *$a \in {}^{0}\mathcal{P}er(X,Y)$ if and only if a is concentrated in degree 0.*

Proof. (1) ($\Rightarrow$) By definition $H^i(a) = 0$ unless $i = -1$ or 0, so we need only show $H^0(a) = 0$. As f has at most one-dimensional fibres, $\mathbf{R}f_* a = 0$ implies that $\mathbf{R}f_* H^0(a) = 0$ using [B, 3.1]. But $\mathrm{Hom}_X(H^0(a), c) = 0$ for all sheaves c such that $\mathbf{R}f_* c = 0$, in particular for $c = H^0(a)$, and the claim follows.
($\Leftarrow$) This follows immediately from the definition.
(2) The proof is similar. $\square$

Corollary 3.5. *In the setting of 2.5(2), the following statements hold.*

(1) *$\mathcal{E}|_U \cong (\Lambda_V)_{\mathrm{con}} \otimes_{\Lambda_V} \mathcal{V}|_U$.*
(2) *$\mathcal{E}$ is a sheaf in degree zero.*

Proof. (1) This follows from the proof of 3.2, in particular (3.B).
(2) This is shown using 3.2 and 3.4(2) applied to ${}^{0}\mathcal{P}er(U,V)$. $\square$

For a closed point $z \in Z$, consider an affine neighbourhood $V = \operatorname{Spec} R$ of z, and consider $U := f^{-1}(V)$ and $f|_U \colon U \to V$. The fibre above each closed point in Z, with reduced scheme structure, decomposes into curves C_i say, each isomorphic to $\mathbb{P}^1$.

Proposition 3.6. *In the setting of 2.5(2), if x is a closed point in the exceptional locus of $f|_U$, then $x \in \operatorname{Supp}_U \mathcal{E}|_U$.*

Proof. Certainly x sits on some C_i, so $x \in \operatorname{Supp} \mathcal{O}_{C_i}(-1)$. Write T for the Λ_V-module corresponding to $\mathcal{O}_{C_i}(-1)$ across the bottom equivalence in (3.B), then by [V1, 3.5.8] T is a simple Λ_V-module. But since $\mathbf{R}f_*\mathcal{O}_{C_i}(-1) = 0$, it follows from (3.C) that T is also a $(\Lambda_V)_{\mathrm{con}}$-module, and hence a simple $(\Lambda_V)_{\mathrm{con}}$-module. As such, there exists a surjection $(\Lambda_V)_{\mathrm{con}} \twoheadrightarrow T$, and so back across the bottom equivalence in (3.B) there is a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{E}|_U \to \mathcal{O}_{C_i}(-1) \to 0$$

in ${}^{0}\mathcal{P}er(U,V)$. Since the last two terms are sheaves, it follows from the long exact sequence in ordinary cohomology that so too is $\mathcal{K}$. Hence since passing to stalks is exact, the stalk of $\mathcal{E}$ at x must be non-zero, since it surjects onto the stalk of $\mathcal{O}_{C_i}(-1)$ at x, which is non-zero. It follows that $x \in \operatorname{Supp} \mathcal{E}|_U$. $\square$

Corollary 3.7. *In the setting of 2.5(2), $\operatorname{Supp}_X \mathcal{E}$ is equal to the exceptional locus.*

Proof. By 3.6, by varying $z \in Z$ we see that the set of closed points in the support of $\mathcal{E}$ contains the set of closed points in the exceptional locus. Since $\mathbf{R}f_*\mathcal{E} = 0$ by 3.2(2), $\operatorname{Supp}\mathcal{E}$ does not contain any further closed points. Since X is a variety, and so Zariski locally every prime ideal is the intersection of maximal ideals, we are done. $\square$

We next show that the global $\mathcal{E}$ recovers the universal sheaves from noncommutative deformation theory in the setting of 2.5(2). For a closed point $z \in Z$, write $\mathfrak{R}_z$ for the completion of the structure sheaf of Y at z, and consider the following flat base change diagram.

$$\begin{tikzcd} \mathfrak{X}_z \arrow[r, "j_z"] \arrow[d, "f_z"'] & X \arrow[d, "f"] \\ \operatorname{Spec}\mathfrak{R}_z \arrow[r, "i_z"] & Y \end{tikzcd}$$

The deformation theory in §2.3 gives rise to a universal sheaf $\mathcal{E}_z \in \operatorname{coh}\mathfrak{X}_z$ for every $z \in Z$.

Proposition 3.8. *In the setting of 2.5(2), for every $z \in Z$, $\operatorname{add} j_z^*\mathcal{E} = \operatorname{add}\mathcal{E}_z$.*

Proof. This is implicit in [DW2, 3.7], but we sketch the argument here. To ease notation, write $\Lambda = \Lambda_V$.

As in [DW2, 3.7] the algebra $\widehat{\Lambda}$ is morita equivalent to an algebra A, via a functor $\mathbb{F}$, and furthermore there are decompositions as $\widehat{\Lambda}$-modules

$$\mathbb{F}\mathrm{A}_{\mathrm{con}} = P_1 \oplus \ldots \oplus P_n \quad \text{and} \quad \widehat{\Lambda}_{\mathrm{con}} = P_1^{\oplus a_1} \oplus \ldots \oplus P_n^{\oplus a_n} \tag{3.D}$$

for some $a_i \geq 1$, where the P_i are the projective $\widehat{\Lambda}_{\mathrm{con}}$-modules. Now

$$\mathbf{R}\mathrm{Hom}_{\mathfrak{X}_z}(j_z^*\mathcal{V}, -)\colon \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,\mathfrak{X}_z) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\widehat{\Lambda})$$

is an equivalence, and it is easy to see using flat base change that $j_z^*\mathcal{E}$ corresponds to $\widehat{\Lambda}_{\mathrm{con}}$. As in [W, 4.14] and [DW1, §3.2], $\mathcal{E}_z$ corresponds across the equivalence to $\mathbb{F}\mathrm{A}_{\mathrm{con}}$. Since (3.D) implies that $\mathrm{add}\,\widehat{\Lambda}_{\mathrm{con}} = \mathrm{add}\,\mathbb{F}\mathrm{A}_{\mathrm{con}}$, it follows that $\mathrm{add}\,j_z^*\mathcal{E} = \mathrm{add}\,\mathcal{E}_z$. □

4. Spherical Properties via Cohen–Macaulay Modules

This section considers the crepant contractions of 2.5(1) for the Zariski local case and characterises, under some assumptions on singularities, when the noncommutative enhancement $\mathcal{D}$ is relatively spherical. These results are globalised in §5. All this involves the Cohen–Macaulay property, which we now review.

4.1. **Cohen–Macaulay Modules.** Recall that if $(R, \mathfrak{m})$ is a commutative local noetherian ring, with $M \in \mathrm{mod}\,R$, then the *depth* of M is defined to be

$$\mathrm{depth}_R\, M := \min\{i \mid \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\},$$

and for all $\mathfrak{p} \in \mathrm{Ass}(M)$ there is a chain of inequalities

$$\mathrm{depth}_R\, M \leq \dim R/\mathfrak{p} \leq \dim_R M.$$

Then M is called a *Cohen–Macaulay* module if either $M = 0$, or $M \neq 0$ and

$$\mathrm{depth}_R\, M = \dim_R M.$$

We write $\mathrm{CM}_{\mathcal{S}}\, R$ for the category of such modules. It is clear that if M is Cohen–Macaulay, necessarily M is equidimensional (i.e. $\dim R/\mathfrak{p} = \dim_R M$ for every minimal prime $\mathfrak{p} \in \mathrm{Supp}\, M$), and has no embedded primes (i.e. every associated prime is minimal).

If R is local Gorenstein, and $M \neq 0$, then by local duality (see e.g. [BH, 3.5.11])

$$M \in \mathrm{CM}_{\mathcal{S}}\, R \iff \mathbf{R}\mathrm{Hom}_R(M, R) \text{ is concentrated in degree } \dim R - \dim_R M.$$

When R is not necessarily local, but is still Gorenstein and equi-codimensional, we say that $M \in \mathrm{mod}\,R$ is CM if $M_{\mathfrak{m}} \in \mathrm{CM}_{\mathcal{S}}\, R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R, and again write $\mathrm{CM}_{\mathcal{S}}\, R$ for the category of CM R-modules. Since $\dim_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$ can vary between maximal ideals, $\mathbf{R}\mathrm{Hom}_R(M, R)$ may have cohomology in more than one degree.

We say that $M \in \mathrm{mod}\,R$ is *maximal Cohen–Macaulay* if $\mathrm{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = \dim R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \mathrm{Max}\, R$, and we write $\mathrm{MCM}\, R$ for the category of maximal Cohen–Macaulay R-modules.

4.2. **Setting.** The remainder of this section considers the following refinement of the crepant contraction setting of 2.5(1) to the affine case.

Setup 4.1. Suppose that $f\colon X \to Y$ is a contraction as in 2.1, where in addition $Y = \mathrm{Spec}\, R$ is affine and Gorenstein, $d = \dim X \geq 2$, f is crepant, and X admits a tilting bundle $\mathcal{P} = \mathcal{O}_X \oplus \mathcal{P}_0$.

Necessarily R is equi-codimensional by [E, 13.4], and a Gorenstein normal domain by assumption.

Notation 4.2. We set $M := f_*\mathcal{P}_0$, which is an R-module, and consider $\Lambda := \mathrm{End}_R(R \oplus M)$, which is isomorphic to $\mathrm{End}_X(\mathcal{P})$ by 2.5(1). Further, since f is crepant necessarily $\Lambda \in \mathrm{MCM}\, R$ by [IW3, 4.8], and thus $M \cong \mathrm{Hom}_R(R, M) \in \mathrm{MCM}\, R$, being a summand of Λ. We write $\Lambda_{\mathrm{con}} := \Lambda / I_{\mathrm{con}}$, where $I_{\mathrm{con}} := [R]$.

We will often reduce arguments about Λ_{con} to the completions of closed points, in which case the following notation will be useful. For a closed point $z = \mathfrak{m} \in \operatorname{Spec} R$, write $\mathfrak{R}$ for the completion of R at z, and consider the Krull–Schmidt decomposition

$$\widehat{M} = \mathfrak{R}^{\oplus a_0} \oplus M_1^{\oplus a_1} \oplus \ldots \oplus M_n^{\oplus a_n}.$$

Then we write $K = M_1 \oplus \ldots \oplus M_n$, and $\mathrm{A} := \operatorname{End}_{\mathfrak{R}}(\mathfrak{R} \oplus K)$. This depends on z, but we suppress this from the notation. We define

$$\mathrm{A}_{\mathrm{con},z} := \mathrm{A}/[\mathfrak{R}],$$

but throughout this section, to ease notation we will usually refer to this as simply $\mathrm{A}_{\mathrm{con}}$.

Definition 4.3. *We say that Λ_{con} is t-*relatively spherical *if*

$$\operatorname{Ext}^j_\Lambda(\Lambda_{\mathrm{con}}, T) \cong \begin{cases} \Bbbk & \text{if } j = 0, t \\ 0 & \text{else,} \end{cases}$$

for all simple Λ_{con}-modules T. There is no requirement that Λ_{con} is perfect.

There is an obvious variant of 4.3 for $\mathrm{A}_{\mathrm{con}}$. The following two subsections characterise when Λ_{con} and $\mathrm{A}_{\mathrm{con}}$ are relatively spherical, under the assumption that complete locally R has only hypersurface singularities. This additional assumption is motivated in part by 2.20, in part by the fact that in characteristic zero 3-dimensional Gorenstein terminal singularities have this property [R83, 0.6(I)], and in part since one of our main applications later in §6.2 will be to crepant divisor-to-curve contractions of 3-folds, in which case the hypersurface singularity condition holds automatically.

4.3. **Spherical via CM R-modules I.** The following three results are elementary.

Lemma 4.4. *In the setup 4.1, with $\mathfrak{p} \in \operatorname{Spec} R$, the following are equivalent.*

(1) $\mathfrak{p} \in Z$.
(2) $\mathfrak{p} \in \operatorname{Supp} \Lambda_{\mathrm{con}}$.
(3) $M_{\mathfrak{p}}$ *is a non-free $R_{\mathfrak{p}}$-module.*

Proof. (1)$\Leftrightarrow$(2) is the local version of 2.16.
(2)$\Leftrightarrow$(3) $(\Lambda_{\mathrm{con}})_{\mathfrak{p}} \cong \underline{\operatorname{End}}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, which is zero if and only if $M_{\mathfrak{p}}$ is free. $\square$

The following is also elementary, and is a simple consequence of the depth lemma.

Lemma 4.5. *In the setup 4.1, if $\dim_R \Lambda_{\mathrm{con}} \leq d-3$, then $\operatorname{Ext}^1_R(M, M) = 0$.*

Proof. By the assumption and 4.4, for $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} = 2$, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module, and hence certainly $\operatorname{Ext}^1_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$. This implies that for all $\mathfrak{q} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{q} = 3$, $\operatorname{Ext}^1_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}, M_{\mathfrak{q}})$ is a finite length $R_{\mathfrak{q}}$-module.

But since $\operatorname{End}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \in \operatorname{MCM} R_{\mathfrak{q}}$, by the depth lemma $\operatorname{Ext}^1_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}, M_{\mathfrak{q}}) = 0$. In turn, this implies that the Ext group for height four primes has finite length. Again, by the depth lemma, this must be zero. By induction the result follows. $\square$

Corollary 4.6. *In the setup 4.1, if $\dim_R \operatorname{Sing} R \leq d-3$, then $\operatorname{Ext}^1_R(M, M) = 0$.*

Proof. Since $M \in \operatorname{CM} R$, it is clear from 4.4 that $\dim_R \Lambda_{\mathrm{con}} \leq \dim_R \operatorname{Sing} R$. The result then follows immediately from 4.5. $\square$

The remainder of this subsection considers the case when Λ_{con} does not have maximal dimension, that is when $\dim_R \Lambda_{\mathrm{con}} \leq d-3$. The case $\dim_R \Lambda_{\mathrm{con}} = d-2$ is trickier, and will be dealt with in the next subsection.

Lemma 4.7. *In the setup 4.1, suppose further that Y is complete locally a hypersurface. If $\dim_R \Lambda_{\mathrm{con}} \leq d-3$, then at every closed point $z \in Z$,*

(1) $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con},z} = 3$.
(2) $\mathrm{A}_{\mathrm{con},z} \in \operatorname{CM}_{d-3} \mathfrak{R}$, *in particular* $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con},z} = d-3$.

Furthermore, the following statements hold.

(3) *The stalk of* $\mathcal{O}_Y$ *at every closed point of* Z *has dimension* $d-3$*.*
(4) $\operatorname{pd}_\Lambda \Lambda_{\mathrm{con}} = 3$*.*
(5) Λ_{con} *is* 3*-relatively spherical.*

Proof. (1)(2) By matrix factorisation there is an exact sequence

$$0 \to K \to F \to F \to K \to 0,$$

where F is a free $\mathfrak{R}$-module. By assumption and 4.5, we know that $\operatorname{Ext}^1_{\mathfrak{R}}(K,K) = 0$, hence applying $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, -)$ to the above sequence gives a projective resolution

$$0 \to P_K \to P_0^{\oplus b} \to P_0^{\oplus b} \to P_K \to \mathrm{A}_{\mathrm{con}} \to 0.$$

Thus $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} \le 3$, from which Auslander–Buchsbaum implies that $\operatorname{depth} \mathrm{A}_{\mathrm{con}} \ge d-3$. It thus follows that $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{d-3}\,\mathfrak{R}$, and so $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} = 3$.
(3) By (2) $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = d-3$, and by 4.4 $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = \dim_{\mathfrak{R}} Z$.
(4) This follows from (1), since projective dimension can be checked complete locally.
(5) This follows by checking complete locally, and using the projective resolution of $\mathrm{A}_{\mathrm{con}}$ given above. ∎

4.4. **Spherical via CM R-modules II.** Seeking a more general version of 4.7 when Λ_{con} has maximal dimension is subtle, for two reasons. First, $\operatorname{pd}_\Lambda \Lambda_{\mathrm{con}} = \infty$ can occur, and second the dimension of Z at closed points may vary, in which case asking for $\Lambda_{\mathrm{con}} \in \mathrm{CM}_{\mathcal{S}}\, R$ is more natural than specifying a particular $\mathrm{CM}_{d-t}\, R$.

Before extracting a global statement, we first work complete locally, and extend 4.7 as follows.

Theorem 4.8. *In the setup 4.1, suppose further that R is complete locally a hypersurface. Then* $\mathrm{A}_{\mathrm{con}}$ *is* t*-relatively spherical if and only if*

(1) $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} < \infty$.
(2) $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{d-t}\,\mathfrak{R}$.

In this case necessarily $t = d - \dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}}$*, which is either* 2 *or* 3*, and furthermore the assumptions* (1) *and* (2) *hold when* $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} \le d-3$*.*

Proof. ($\Leftarrow$) *Case* $t=2$: By Auslander–Buchsbaum, $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} = 2$. It then follows from [W, A.3] that $\Omega K \cong K$, and so applying $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K,-)$ to the exact sequence

$$0 \to \Omega K \to F \to K \to 0$$

gives the minimal projective resolution

$$0 \to P_K \to P_0^{\oplus a} \to P_K \to \mathrm{A}_{\mathrm{con}} \to 0.$$

Hence $\mathrm{A}_{\mathrm{con}}$ is 2-relatively spherical, and since by assumption $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{d-2}\,\mathfrak{R}$, we have $t = d - \dim \mathrm{A}_{\mathrm{con}}$.
Case $t \ge 3$: Since $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_t\,\mathfrak{R}$, necessarily $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} \le d-3$, so by 4.7 $\mathrm{A}_{\mathrm{con}}$ is 3-relatively spherical, and $t = d - \dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}}$.
($\Rightarrow$) *Case* $t=2$: Consider the beginning of the minimal projective resolution of $\mathrm{A}_{\mathrm{con}}$. Since $\mathrm{A}_{\mathrm{con}}$ is 2-relatively spherical, this has the form

$$\to P_K \to P_0^{\oplus a} \xrightarrow{\psi} P_K \to \mathrm{A}_{\mathrm{con}} \to 0.$$

Certainly $\operatorname{Ker}\psi = \operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, \Omega K)$, hence the projective cover of $\operatorname{Ker}\psi$ is obtained by applying $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K,-)$ to a minimal $\operatorname{add}(\mathfrak{R}\oplus K)$-approximation

$$f\colon U \to \Omega K.$$

But since the projective cover of $\operatorname{Ker}\psi$ is P_K, it follows that $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, U) \cong P_K := \operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, K)$, so by reflexive equivalence $U \cong K$. Further, since f is in particular an $\operatorname{add}\mathfrak{R}$-approximation, it is necessarily surjective. But $\mathfrak{R}$ is a hypersurface, so the rank of

K equals the rank of ΩK, hence $\operatorname{Ker} f = 0$ and thus f is an isomorphism. In turn, this implies that $\operatorname{Ker}\psi \cong P_K$, and hence $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} = 2$.

By Auslander–Buchsbaum, $\operatorname{depth}_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = d-2$. But as in 4.6, and since $\mathfrak{R}$ is normal

$$d-2 = \operatorname{depth}_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} \leq \dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} \leq \dim \operatorname{Sing} \mathfrak{R} \leq d-2.$$

Hence equality holds throughout, and $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{d-2}\,\mathfrak{R}$.
Case $t \geq 3$: Again, consider the beginning of the minimal projective resolution of $\mathrm{A}_{\mathrm{con}}$, which now has the form

$$\to P_0^{\oplus b} \xrightarrow{\phi} P_0^{\oplus c} \xrightarrow{\psi} P_K \to \mathrm{A}_{\mathrm{con}} \to 0.$$

The morphism $P_0^{\oplus b} \to \operatorname{Ker}\psi = \operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, \Omega K)$ is induced from a morphism of the form $f\colon F \to \Omega K$ (where F is free), which is a minimal $\operatorname{add}(\mathfrak{R}\oplus K)$-approximation. Again, necessarily this f is surjective, and its kernel is $\Omega\Omega K \cong K$, since $\mathfrak{R}$ is a hypersurface. But then, applying $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, -)$ to the exact sequence

$$0 \to K \to F \xrightarrow{f} \Omega K \to 0$$

and using the fact that f is an $\operatorname{add}(\mathfrak{R}\oplus K)$-approximation,

$$0 \to \operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, K) \to \operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, F) \twoheadrightarrow \operatorname{Hom}_{\mathfrak{R}}(\mathfrak{R}\oplus K, \Omega K) \to \operatorname{Ext}^1_{\mathfrak{R}}(K,K) \to 0$$

is exact. It follows that $\operatorname{Ext}^1_{\mathfrak{R}}(K,K) = 0$, and $\operatorname{Ker}\phi = P_K$. Consequently, $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} = 3$ and $\mathrm{A}_{\mathrm{con}}$ is 3-relatively spherical, thus $t = 3$.

By Auslander–Buchsbaum, $\operatorname{depth}_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = d-3$. We claim that $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{d-3}\,\mathfrak{R}$, so we just need to prove that $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} \neq d-2$. If there exists $\mathfrak{p} \in \operatorname{Supp} \mathrm{A}_{\mathrm{con}}$ with height two, then by 4.4 $K_{\mathfrak{p}}$ is a non-free $R_{\mathfrak{p}}$-module. But by the above

$$\operatorname{Hom}_{\underline{\mathrm{CM}}\, R_{\mathfrak{p}}}(K_{\mathfrak{p}}, K_{\mathfrak{p}}[1]) \cong \operatorname{Ext}^1_{\mathfrak{R}_{\mathfrak{p}}}(K_{\mathfrak{p}}, K_{\mathfrak{p}}) = 0,$$

and further since $\mathfrak{R}_{\mathfrak{p}}$ is a Gorenstein surface with only an isolated singular point, $\underline{\mathrm{CM}}\,\mathfrak{R}_{\mathfrak{p}}$ is 1-CY. This implies that $\underline{\operatorname{Hom}}_{\mathfrak{R}_{\mathfrak{p}}}(K_{\mathfrak{p}}, K_{\mathfrak{p}}) = 0$, and so $K_{\mathfrak{p}}$ is free, which is a contradiction. Hence such a $\mathfrak{p}$ with height two cannot exist, and so $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{d-3}\,\mathfrak{R}$.

The last statement is 4.7. $\square$

Remark 4.9. Neither condition (1) or (2) in 4.8 is guaranteed if $\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = d-2$; see 4.12 below and also [W, 4.18].

Continuing to work complete locally, we obtain the following tilting consequence.

Proposition 4.10. *In the setup 4.1, suppose further that R is complete locally a hypersurface. If the equivalent conditions of 4.8 hold, then I_{A} is a tilting* A*-module.*

Proof. By 4.8 there are only two cases, $t = 2$ and $t = 3$. When $t = 2$, the fact that I_{A} is tilting is just [W, A.3, A.5]. When $t = 3$, the argument is identical to [DW1, 5.10(1)], since although in that proof $d = 3$, the fact that by 4.5 $\operatorname{Ext}^1_R(K,K) = 0$ means that [DW1, (5.F)] is still exact, so exactly the same proof works. $\square$

It is the global version of 4.10 that interests us the most. The fact that $\mathrm{A}_{\mathrm{con}}$ can be relatively spherical at all closed points, but that the value of the spherical parameter can vary, is problematic; similarly the projective dimension of I_{A} can vary over the maximal ideals. Hence we do not seek an autoequivalence condition globally in terms of one parameter. Instead, in condition (3) below we ask for Λ_{con} to belong to $\mathrm{CM}_{\mathcal{S}}\, R$, which gives the required flexibility. The following is one of our main results.

Theorem 4.11. *In the setup 4.1, suppose further that R is complete locally a hypersurface. Then the following are equivalent.*

(1) $\mathrm{A}_{\mathrm{con},z}$ *is a relatively spherical* A*-module for all* $z \in \operatorname{Max} R$.
(2) $\operatorname{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con},z} < \infty$ *and* $\mathrm{A}_{\mathrm{con},z} \in \mathrm{CM}_{\mathcal{S}}\,\mathfrak{R}$ *for all* $z \in \operatorname{Max} R$.
(3) $\operatorname{pd}_{\Lambda} \Lambda_{\mathrm{con}} < \infty$ *and* $\Lambda_{\mathrm{con}} \in \mathrm{CM}_{\mathcal{S}}\, R$.

If any of these conditions hold, and they are automatic if $\dim \operatorname{Sing} R \leq d-3$, *then* I_{con} *is a tilting* Λ*-module, and* $-\otimes^{\mathbf{L}}_{\Lambda} I_{\mathrm{con}}$ *preserves* $\mathrm{D^b}(\operatorname{mod}\Lambda)$.

Proof. (1)⇔(2) is 4.8, since $d-t = d-(d-\dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}}) = \dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}}$.
(2)⇔(3) just follows since finite projective dimension can be checked complete locally at maximal ideals, and $\Lambda_{\mathrm{con}} \in \mathrm{CM}_{\mathcal{S}} R$ is again defined locally.

The statement about the assumptions being automatic is 4.7. The fact I_{con} is a tilting module then follows since being a tilting module can be checked complete locally (see e.g. the proof of [DW1, 6.2]), and the complete local statement is 4.10. $\square$

Example 4.12. Consider $M := (u,x)\oplus(u,x^2)$ for $R = \mathbb{C}[u,v,x,y]/(uv-x^2y)$. Then $\Lambda := \operatorname{End}_R(R\oplus M)$ is an NCCR of R. Complete locally at the origin, since $\Omega M \ncong M$, as in 4.8 it follows that $\mathrm{A}_{\mathrm{con}} \notin \mathrm{CM}_{\mathcal{S}} \mathfrak{R}$. In fact, this can be seen directly, since the minimal projective resolution of $\mathrm{A}_{\mathrm{con}}$ has the form

$$0 \to P_2^{\oplus 2} \to P_0^{\oplus 3}\oplus P_1 \to P_0^{\oplus 4} \to P_1 \oplus P_2 \to \mathrm{A}_{\mathrm{con}} \to 0$$

so, by inspection, $\mathrm{A}_{\mathrm{con}}$ is not relatively spherical. Note that since $\mathrm{A}_{\mathrm{con}} \notin \mathrm{CM}_{\mathcal{S}} \mathfrak{R}$, it follows that $\Lambda_{\mathrm{con}} \notin \mathrm{CM}_{\mathcal{S}} R$. However, the other condition in 4.11(3), namely $\operatorname{pd}_{\Lambda} \Lambda_{\mathrm{con}} < \infty$, does hold since Λ is an NCCR.

A more conceptual geometric explanation of the above example is given in 6.4 below.

5. Global Twist Functors

In this section, under our most general d-fold contraction assumptions of 2.3, where additionally f is crepant, we produce an endofunctor Twist_X of $\mathrm{D^b}(\operatorname{coh} X)$. After restricting singularities, we then give a criterion for Twist_X to be an autoequivalence, before giving our first application. Further applications will appear in §6.

5.1. **Twist Construction.**

By the definition of $\mathcal{D}$ in 2.12, there is an exact sequence of $\mathcal{A}$-bimodules

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{D} \to 0. \tag{5.A}$$

The bimodule $\mathcal{I}$ induces a functor

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I},-)\colon \mathrm{D}(\operatorname{Mod}\mathcal{A}) \to \mathrm{D}(\operatorname{Mod}\mathcal{E}nd_{\mathcal{A}}\mathcal{I}),$$

and the following lemma, which is simply the global version of [DW1, 6.1], ensures that it furthermore yields an endofunctor of $\mathrm{D}(\operatorname{Mod}\mathcal{A})$.

Lemma 5.1. *Under the general assumptions of 2.3, suppose further that f is crepant.*

1. *There is an isomorphism of sheaves of algebras $\mathcal{A} \cong \mathcal{E}nd_{\mathcal{A}}\mathcal{I}$.*
2. *Under this isomorphism the $(\mathcal{E}nd_{\mathcal{A}}\mathcal{I},\mathcal{A})$-bimodule structure on $\mathcal{I}$ coincides with the natural $\mathcal{A}$-bimodule structure.*

Proof. (1) There is a canonical morphism

$$\mathcal{A} \to \mathcal{E}nd_{\mathcal{A}}\mathcal{I} \tag{5.B}$$

given on any open subset V of Y by

$$\begin{aligned}\mathcal{A}(V) &\to \operatorname{Hom}_{\mathcal{A}(V)}(\mathcal{I}(V),\mathcal{I}(V))\\ \lambda &\mapsto \alpha_\lambda\end{aligned}$$

with $\alpha_\lambda\colon i \mapsto \lambda i$. By our assumptions 2.3, $\dim X \geq 2$, so since f is crepant the proof of [DW1, 6.1(1)] shows that this is an isomorphism for affine V, where 2.10 ensures that $\mathcal{I}(V)$ is the ideal considered in [DW1]. It follows that (5.B) is an isomorphism.
(2) This is a formal consequence of (1), as in [DW1, 6.1(2)]. $\square$

Composing the above endofunctor with the equivalences

$$\mathrm{D}(\mathrm{Mod}\,\mathcal{A}) \underset{G^{\mathrm{RA}}=\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{V},-)}{\overset{G=f^{-1}(-)\otimes^{\mathbf{L}}_{f^{-1}\mathcal{A}}\mathcal{V}}{\rightleftarrows}} \mathrm{D}(\mathrm{Qcoh}\,X) \tag{5.C}$$

leads to the following definition.

Definition 5.2. *Under the general assumptions of 2.3, suspose further that f is crepant. The* twist *and* dual twist *endofunctors are defined to be*

$$\mathsf{Twist}_X, \mathsf{Twist}^*_X\colon \mathrm{D}(\mathrm{Qcoh}\,X) \to \mathrm{D}(\mathrm{Qcoh}\,X)$$

$$\mathsf{Twist}_X = G \circ \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I},-) \circ G^{\mathrm{RA}},$$
$$\mathsf{Twist}^*_X = G \circ (- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}) \circ G^{\mathrm{RA}}.$$

We will show that Twist_X preserves $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ in 5.5(2) below, under the condition that $\mathcal{D}$ is a perfect $\mathcal{A}$-module, or equivalently $\mathcal{E}$ from 3.1 is an object in $\mathrm{Perf}(X)$. To do this, we first describe some additional structure on $\mathcal{E}$.

By definition $\mathcal{E} = f^{-1}\mathcal{D} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{A}} \mathcal{V}$ is a complex of sheaves on X. Computing this expression by resolving the second factor we see that $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, f^{-1}\mathcal{D}^{\mathrm{op}})$.

Next, consider the following adjoint functors

$$\mathrm{D}(\mathrm{Mod}\,\mathcal{D}) \underset{\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{D},-)}{\overset{-\otimes^{\mathbf{L}}_{\mathcal{D}}\mathcal{D}}{\rightleftarrows}} \mathrm{D}(\mathrm{Mod}\,\mathcal{A}) \underset{G^{\mathrm{RA}}=\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{V},-)}{\overset{G=f^{-1}(-)\otimes^{\mathbf{L}}_{f^{-1}\mathcal{A}}\mathcal{V}}{\rightleftarrows}} \mathrm{D}(\mathrm{Qcoh}\,X). \tag{5.D}$$

By construction and assumption, $\mathcal{D}$ and $\mathcal{V}$ are sheaves. We will write F for the composition of the top functors, and F^{RA} for the composition of the bottom functors. Note that F and F^{RA} can be expressed easily as

$$\begin{aligned} F &= f^{-1}(- \otimes^{\mathbf{L}}_{\mathcal{D}} \mathcal{D}) \otimes^{\mathbf{L}}_{f^{-1}\mathcal{A}} \mathcal{V} \\ &\cong f^{-1}(-) \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}} f^{-1}\mathcal{D} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{A}} \mathcal{V} && \text{(since } f^{-1} \text{ distributes over tensor)} \\ &\cong f^{-1}(-) \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}} \mathcal{E}, \end{aligned}$$

$$\begin{aligned} F^{\mathrm{RA}} &= \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{D}, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{V},-)) \\ &\cong \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{f^{-1}\mathcal{A}}(f^{-1}\mathcal{D}, \mathbf{R}\mathcal{H}om_X(\mathcal{V},-)) && \text{(by adjunction, c.f. [KS, (18.3.2)])} \\ &\cong \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(f^{-1}\mathcal{D} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{A}} \mathcal{V}, -) && \text{(by adjunction)} \\ &= \mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{E},-). \end{aligned}$$

Remark 5.3. Regardless of whether $\mathcal{E}$ is a sheaf or a complex, below and throughout when we write $\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{E},-)$ we will mean the functor F^{RA}, which takes values in $\mathrm{D}(\mathrm{Mod}\,\mathcal{D})$. In particular, $f^{-1}\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{E},-)$ takes values in $\mathrm{D}(\mathrm{Mod}\, f^{-1}\mathcal{D})$.

Proposition 5.4. *Under the general assumptions of 2.3, suppose further that f is crepant. Then Twist_X fits into a functorial triangle*

$$f^{-1}\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(\mathcal{E},-) \otimes^{\mathbf{L}}_{f^{-1}\mathcal{D}} \mathcal{E} \to \mathrm{Id} \to \mathsf{Twist}_X \to .$$

Proof. For any object $a \in \mathrm{D}(\mathrm{Mod}\,\mathcal{A})$, simply applying $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(-,a)$ to the sequence (5.A) gives a functorial triangle in $\mathrm{D}(\mathrm{Mod}\,\mathcal{A})$

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{D},a) \to a \to \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I},a) \to . \tag{5.E}$$

We may reinterpret the left-hand term as $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{D},a) \otimes^{\mathbf{L}}_{\mathcal{D}} \mathcal{D}_{\mathcal{A}}$, in other words, it is given by a composition of the left-hand adjoint pair in the diagram (5.D).

Hence precomposing (5.E) with G^{RA} from (5.C), and postcomposing with G, gives a functorial triangle

$$(F \circ F^{\mathrm{RA}})(a) \to a \to \mathsf{Twist}_X(a) \to .$$

Using the expressions for F and F^{RA} above, the result follows. ∎

Proposition 5.5. *Under the general assumptions of 2.3, suppose further that f is crepant. If $\mathcal{D} \in \operatorname{Perf}(\mathcal{A})$, or equivalently $\mathcal{E} \in \operatorname{Perf}(X)$, then*

(1) $\mathbf{R}\mathcal{H}om_{\mathcal{I}}(\mathcal{D}, -)$ *preserves* $\mathrm{D^b}(\operatorname{mod} \mathcal{A})$.
(2) Twist_X *preserves* $\mathrm{D^b}(\operatorname{coh} X)$.

Proof. (1) Since $\mathcal{D} \in \operatorname{Perf}(\mathcal{A})$, the functor $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{D}, -)$ preserves bounded complexes. It clearly preserves coherence, and so the result follows using the two-out-of-three property for the triangles (5.E).
(2) Again, since $\mathcal{D} \in \operatorname{Perf}(\mathcal{A})$, the functor $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{D}, -)$ in (5.D) preserves bounded coherent complexes, and all the other functors in (5.D) also preserve bounded coherence. Hence F and F^{RA} preserve bounded coherence, thus so does Twist_X by 5.4, again using the two-out-of-three property. □

5.2. **Conditions for Equivalence.** This subsection uses the Zariski local tilting result 4.11 to give a condition for when Twist_X is an equivalence globally. Recall that $\mathcal{D}$ is a *Cohen–Macaulay sheaf* if it is Cohen–Macaulay at each closed point, as defined in §4.1.

The following notion, a translation of 4.3, will be used.

Definition 5.6. *We say that $\mathcal{D}$ is t-*relatively spherical *for a closed point $z \in Z$ if*

$$\operatorname{Ext}^j_{\widehat{\mathcal{A}}_z}(\widehat{\mathcal{D}}_z, T) \cong \begin{cases} \Bbbk & \text{if } j = 0, t \\ 0 & \text{else,} \end{cases}$$

for all simple $\widehat{\mathcal{D}}_z$-modules T. Here $\widehat{\mathcal{D}}_z$ is the completion of the stalk of $\mathcal{D}$ at z.

To set notation, choose an affine open cover $Y = \bigcup V_i$, and for any $V = V_i$ consider

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}|_V}(\mathcal{I}|_V, -)\colon\ \mathrm{D}(\operatorname{Mod} \mathcal{A}|_V) \to \mathrm{D}(\operatorname{Mod} \mathcal{A}|_V).$$

As V is affine, say $V = \operatorname{Spec} R$, we may use setup 4.1, where now $\mathcal{A}|_V$ corresponds to Λ, and $\mathcal{I}|_V$ to the Λ-bimodule I_{con} by 2.10. It follows that the above functor is simply

$$\mathbf{R}\mathrm{Hom}_{\Lambda}(I_{\mathrm{con}}, -)\colon\ \mathrm{D}(\operatorname{Mod} \Lambda) \to \mathrm{D}(\operatorname{Mod} \Lambda).$$

The following is the main theorem of this section.

Theorem 5.7. *Under the general assumptions of 2.3, assume that f is crepant, and $\widehat{\mathcal{O}}_{Y,z}$ are hypersurfaces for all closed points $z \in Z$. Then the following are equivalent.*

(1) *$\mathcal{D}$ is a Cohen–Macaulay sheaf on Y, and $\mathcal{E}$ is a perfect complex on X.*
(2) *$\mathcal{D}$ is relatively spherical for all closed points $z \in Z$.*

If these conditions hold, and they are automatic provided that $\dim Z \leq d - 3$, then the functor Twist_X is an autoequivalence of $\mathrm{D^b}(\operatorname{coh} X)$.

Proof. Since being Gorenstein is an open condition [M, 24.6], Y is Gorenstein in a neighbourhood N of Z.

Conditions (1) and (2) can both be checked locally. Since $\mathcal{D}$ is supported on Z by 2.16, and recalling from 3.1 that $\mathcal{E}$ is defined as the image of $\mathcal{D}$ under an equivalence, it suffices to show that they are equivalent after restricting to N. But there, the required result is 4.11, which also shows that the conditions are automatic provided that $\dim Z \leq d - 3$.

For the final statement, since G and G^{RA} are equivalences by definition, 5.2, it suffices to prove that

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)\colon\ \mathrm{D^b}(\operatorname{mod} \mathcal{A}) \to \mathrm{D^b}(\operatorname{mod} \mathcal{A})$$

is an equivalence. We write η and $\tilde{\eta}$ respectively for the counits of the following adjunctions

$$- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I} \dashv \mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)$$
$$- \otimes^{\mathbf{L}}_{\Lambda} I_{\mathrm{con}} \dashv \mathbf{R}\mathrm{Hom}_{\Lambda}(I_{\mathrm{con}}, -).$$

Consider, for $a \in \mathrm{D}(\operatorname{Mod} \mathcal{A})$, the distinguished triangle

$$\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, a) \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I} \xrightarrow{\eta_a} a \longrightarrow \operatorname{Cone}(\eta_a) \longrightarrow .$$

This restricts to each $V = V_i$ along the inclusion $j\colon V \hookrightarrow Y$, to give

$$\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, j^*a) \otimes^{\mathbf{L}}_\Lambda I_{\mathrm{con}} \xrightarrow{j^*\eta_a} j^*a \longrightarrow j^*\,\mathrm{Cone}(\eta_a) \longrightarrow .$$

By inspection, the counit is defined locally, so $j^*\eta_a = \tilde{\eta}_{j^*a}$.

We may take our affine open cover $Y = \bigcup V_i$ to be the union of a cover of $N \supset Z$, and a cover of $Y \backslash Z$. If $V \subseteq Y\backslash Z$ then $I_{\mathrm{con}} = \Lambda$. If, on the other hand, $V \subseteq N$ then we have that V is Gorenstein, and then $\mathbf{R}\mathrm{Hom}_\Lambda(I_{\mathrm{con}}, -)$ is an equivalence by 4.11, condition (3). In either case, it follows that $j^*\eta_a$ is an isomorphism. Hence $j^*\,\mathrm{Cone}(\eta_a) = 0$ for all V, so $\mathrm{Cone}(\eta_a) = 0$ for all a, and thence η is an isomorphism.

The argument for the unit ϵ is similar, since it too by inspection is defined locally, and so it follows from standard adjoint functor results that

$$\begin{tikzcd} \mathrm{D}(\mathrm{Mod}\,\mathcal{A}) \arrow[r, shift left, "- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}"] & \mathrm{D}(\mathrm{Mod}\,\mathcal{A}) \arrow[l, shift left, "\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I},-)"] \end{tikzcd}$$

is an equivalence.

By 5.5(1) we already know that $\mathbf{R}\mathcal{H}om_{\mathcal{A}}(\mathcal{I}, -)$ preserves $\mathrm{D^b}(\mathrm{mod}\,\mathcal{A})$, so it suffices to prove that $- \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}$ also has this property.

Consider $a \in \mathrm{D^b}(\mathrm{mod}\,\mathcal{A})$, and $a \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}$. Restricting to $V \subseteq N$, we see that

$$j^*(a \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}) \cong j^*a \otimes^{\mathbf{L}}_\Lambda I_{\mathrm{con}},$$

which belongs to $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ since j^*a does and $- \otimes^{\mathbf{L}}_\Lambda I_{\mathrm{con}}$ preserves $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ by 4.11. On the other hand, for $V \subseteq Y\backslash Z$, we have $I_{\mathrm{con}} = \Lambda$, and so $- \otimes^{\mathbf{L}}_\Lambda I_{\mathrm{con}}$ certainly preserves $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$. Since we may take the cover to be finite, and the restriction of $a \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}$ to each piece is bounded coherent, it follows that $a \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{I}$ is bounded coherent. □

5.3. **Application to Springer Resolutions.** The following is a corollary of 5.7.

Corollary 5.8. *Consider the Springer resolution $X \to Y$ of the variety of singular $n \times n$ matrices. Then Twist_X is an autoequivalence of X.*

Proof. Y is a hypersurface cut out by the determinant. It is well known that Y is smooth in codimension two [BLV, §5.1], and it is one of the main results of [BLV] that X admits a tilting bundle with trivial summand [BLV, C]. □

6. One-Dimensional Fibre Applications

6.1. **Relative Spherical via CM Sheaves.** The first half of this subsection is general. If X is a Gorenstein variety with tilting bundle $\mathcal{T}$, necessarily $\Lambda := \mathrm{End}_X(\mathcal{T})$ is an Iwanaga–Gorenstein ring, namely it is noetherian with finite injective dimension on both the left and right [KIWY, 4.4]. It follows that Λ with its natural bimodule structure gives rise to a duality functor

$$\mathbf{R}\mathrm{Hom}_\Lambda(-, \Lambda)\colon\ \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda^{\mathrm{op}}).$$

On the other hand, since X is Gorenstein, it too has a good duality functor given by $\mathbf{R}\mathcal{H}om_X(-, \mathcal{O}_X)$, and so we can consider the following diagram.

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{coh}\,X) \arrow[r, "\mathbf{R}\mathrm{Hom}_X(\mathcal{T},-)", "\sim"'] \arrow[d, "\mathbf{R}\mathcal{H}om_X(-,\mathcal{O}_X)"'] & \mathrm{D^b}(\mathrm{mod}\,\Lambda) \arrow[d, "\mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda)"] \\ \mathrm{D^b}(\mathrm{coh}\,X) \arrow[r, "\mathbf{R}\mathrm{Hom}_X(\mathcal{T}^*,-)", "\sim"'] & \mathrm{D^b}(\mathrm{mod}\,\Lambda^{\mathrm{op}}) \end{tikzcd}$$

Proposition 6.1. *Suppose that X is a Gorenstein variety with tilting bundle $\mathcal{T}$. Then for all $a \in \mathrm{D^b}(\mathrm{coh}\,X)$, there is an isomorphism*

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{T}^*, \mathbf{R}\mathcal{H}om_X(a, \mathcal{O}_X)) \cong \mathbf{R}\mathrm{Hom}_\Lambda(\mathbf{R}\mathrm{Hom}_X(\mathcal{T}, a), \Lambda).$$

Proof. Since $\mathbf{R}\mathrm{Hom}_X(\mathcal{T},-)$ is a dg equivalence, there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_X(a,b) \cong \mathbf{R}\mathrm{Hom}_\Lambda(\mathbf{R}\mathrm{Hom}_X(\mathcal{T},a), \mathbf{R}\mathrm{Hom}_X(\mathcal{T},b))$$

for all $a, b \in \mathrm{D^b}(\mathrm{coh}\, X)$. Hence setting $b = \mathcal{T}$ gives an isomorphism

$$\mathbf{R}\mathrm{Hom}_X(a,\mathcal{T}) \cong \mathbf{R}\mathrm{Hom}_\Lambda(\mathbf{R}\mathrm{Hom}_X(\mathcal{T},a), \Lambda) \tag{6.A}$$

for all $a \in \mathrm{D^b}(\mathrm{coh}\, X)$.

Next, since X is Gorenstein $(-)^\vee := \mathbf{R}\mathcal{H}om_X(-,\mathcal{O}_X)$ is a duality on $\mathrm{D^b}(\mathrm{coh}\, X)$, and further since $\mathcal{T}^* = \mathcal{T}^\vee$, there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_X(a,\mathcal{T}) \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{T}^*, a^\vee) \tag{6.B}$$

for all $a \in \mathrm{D^b}(\mathrm{coh}\, X)$. Combining (6.A) and (6.B) yields the result. □

The following is the main result of this subsection, and is specific to the setting of one-dimensional fibres. It relates a homological condition about $\mathcal{D}$ on the base Y to a homological condition about $\mathcal{E}$ at closed points of X.

Theorem 6.2. *In the setting of 2.5(2), assume that Y is Gorenstein in a neighbourhood of Z, and f is crepant. Then the following conditions are equivalent.*

(1) $\mathcal{D} \in \mathrm{CM}_\mathcal{S}\, Y$.
(2) *For all $y \in Z$, we have $\mathcal{D}_y \in \mathrm{CM}_{\dim Z_y}\, \mathcal{O}_{Y,y}$.*
(3) *For all $y \in Z$, and all $x \in f^{-1}(y)$, we have $\mathcal{E}_x \in \mathrm{CM}_{\dim Z_y + 1}\, \mathcal{O}_{X,x}$.*

In particular, if these conditions hold, then above every $y \in Z$, the exceptional locus is equidimensional of dimension $\dim Z_y + 1$.

Proof. (1)⇔(2) This is the definition of $\mathrm{CM}_\mathcal{S}\, Y$, together with the fact that $\dim Z_y = \dim \mathcal{D}_y$ by the contraction theorem 2.16.
(2)⇔(3) For $y \in Z$, choose a Gorenstein affine neighbourhood $V = \mathrm{Spec}\, R$ of y. To simplify notation, write $\Lambda = \Lambda_V$, $k = \dim Z_y$ and $d = \dim R_y$.

Since f is crepant, Λ_y is a maximal CM R_y-module [IW3, 4.14], so necessarily it is singular Calabi–Yau [IW2, 2.22(2)]. Hence

$$\mathrm{Ext}^i_{R_y}(M, R_y) \cong \mathrm{Ext}^i_{\Lambda_y}(M, \Lambda_y) \tag{6.C}$$

by [IR, 3.4], for all $M \in \mathrm{mod}\, \Lambda_y$ and all $i \geq 0$. Thus,

$$\begin{aligned}
\mathcal{D}_y \in \mathrm{CM}_k\, \mathcal{O}_{Y,y} &\overset{2.15}{\iff} (\Lambda_{\mathrm{con}})_y \in \mathrm{CM}_k\, R_y \\
&\iff \mathbf{R}\mathrm{Hom}_{R_y}((\Lambda_{\mathrm{con}})_y, R_y) \text{ is concentrated in degree } d-k \\
&\iff \mathbf{R}\mathrm{Hom}_{\Lambda_y}((\Lambda_{\mathrm{con}})_y, \Lambda_y) \text{ is concentrated in degree } d-k.
\end{aligned}$$

Now there is a chain of isomorphisms

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{\Lambda_y}((\Lambda_{\mathrm{con}})_y, \Lambda_y) &\cong \mathbf{R}\mathrm{Hom}_{\Lambda_y}(\mathbf{R}\mathrm{Hom}_{X_y}(\mathcal{V}|_{X_y}, \mathcal{E}|_{X_y}), \Lambda_y) \\
&\cong \mathbf{R}\mathrm{Hom}_{X_y}(\mathcal{V}|_{X_y}^*, \mathbf{R}\mathcal{H}om_{X_y}(\mathcal{E}|_{X_y}, \mathcal{O}_{X_y})),
\end{aligned}$$

where the first follows since Λ_{con} corresponds across the equivalence to $\mathcal{E}$, and the second follows by 6.1 since X_y is Gorenstein (since $\mathrm{Spec}\, R_y$ is), and $\mathcal{V}|_{X_y}$ is a progenerator of ${}^0\mathcal{P}er(X_y, R_y)$, so it is tilting on X_y, with endomorphism ring Λ_y (e.g. [IW3, 4.3(2)]). Combining the above, we see that $\mathcal{D}_y \in \mathrm{CM}_k\, \mathcal{O}_{Y,y}$ if and only if

$$\mathbf{R}\mathrm{Hom}_{X_y}(\mathcal{V}|_{X_y}^*, \mathbf{R}\mathcal{H}om_{X_y}(\mathcal{E}|_{X_y}, \mathcal{O}_{X_y})[d-k]) \text{ is concentrated in degree } 0.$$

But now $\mathcal{V}|_{X_y}$ progenerates ${}^0\mathcal{P}er(X_y, R_y)$, and its dual progenerates ${}^{-1}\mathcal{P}er(X_y, R_y)$, and hence the above condition holds if and only if

$$\mathbf{R}\mathcal{H}om_{X_y}(\mathcal{E}|_{X_y}, \mathcal{O}_{X_y})[d-k] \in {}^{-1}\mathcal{P}er(X_y, R_y).$$

But by crepancy and Grothendieck duality

$$\begin{aligned}\mathbf{R}f_*\mathbf{R}\mathcal{H}om_{X_y}(\mathcal{E}|_{X_y},\mathcal{O}_{X_y}) &\cong \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{X_y}(\mathcal{E}|_{X_y},f^!\mathcal{O}_{R_y})\\ &\cong \mathbf{R}\mathrm{Hom}_{R_y}(\mathbf{R}f_*\mathcal{E}|_{X_y},\mathcal{O}_{R_y}) \overset{3.2}{=} 0,\end{aligned}$$

so it follows from 3.4(1) that $\mathcal{D}_y \in \mathrm{CM}_k\,\mathcal{O}_{Y,y}$ if and only if

$$\mathbf{R}\mathcal{H}om_{X_y}(\mathcal{E}|_{X_y},\mathcal{O}_{X_y})[d-(k+1)] \text{ is concentrated in degree } 0,$$

which holds if and only if $\mathcal{E}_x \in \mathrm{CM}_{k+1}\,\mathcal{O}_{X_y,x}$ for all $x \in f^{-1}(y)$. Since $\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x}$, the result follows.

Since $\mathrm{Supp}_X\,\mathcal{E}$ equals the exceptional locus by 3.7, it follows for all $x \in f^{-1}(y)$ that $\dim \mathcal{E}_x$ equals the dimension of the exceptional locus at the point x. Hence the condition $\mathcal{E}_x \in \mathrm{CM}_{\dim Z_y+1}\,\mathcal{O}_{X,x}$ forces the exceptional locus to have dimension $\dim Z_y + 1$ at all points x above y, and so the last claim follows. □

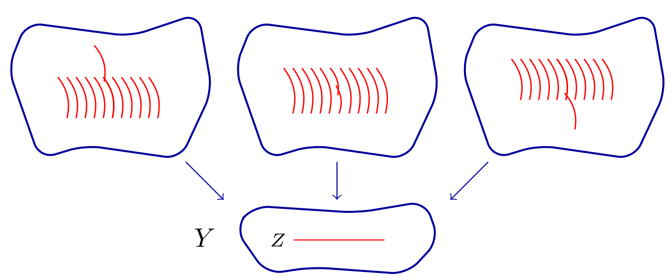
Corollary 6.3. *In the one-dimensional fibre setting of 2.5(2), assume that f is crepant, and $\widehat{\mathcal{O}}_{Y,z}$ are hypersurfaces for all closed points $z \in Z$. Then the following are equivalent.*

1. *$\mathcal{D}$ is relatively spherical for all closed points $z \in Z$.*
2. *$\mathcal{D} \in \mathrm{Perf}(\mathcal{A})$ and $\mathcal{D} \in \mathrm{CM}_{\mathcal{S}}\,Y$.*
3. *$\mathcal{E} \in \mathrm{Perf}(X)$ and $\mathcal{D} \in \mathrm{CM}_{\mathcal{S}}\,Y$.*
4. *$\mathcal{E} \in \mathrm{Perf}(X)$, and for all $y \in Z$ and $x \in f^{-1}(y)$, we have $\mathcal{E}_x \in \mathrm{CM}_{\dim Z_y+1}\,\mathcal{O}_{X,x}$.*

If these conditions hold, and they are automatic provided that $\dim Z \le d-3$, then the functor Twist_X is an autoequivalence of $\mathrm{D^b}(\mathrm{coh}\,X)$.

Proof. The assumptions force Y to be Gorenstein in a neighbourhood of Z [M, 24.6]. Recalling from 3.1 that $\mathcal{E}$ is defined as the image of $\mathcal{D}$ under an equivalence, the equivalence of (1)–(4) follows from 6.2. The rest follows from 5.7. □

Example 6.4. Consider $R = \mathbb{C}[u,v,x,y]/(uv - x^2y)$, and $Y = \mathrm{Spec}\,R$. Then Y has three crepant resolutions, sketched below.



Each gives a thickening of Z, which we write $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, respectively. Above the origin, the exceptional locus of the outer two resolutions is not equidimensional of dimension two, since in both cases there is a curve poking out of the surface. Thus, by 6.2 and 6.3, $\mathcal{D}_1$ and $\mathcal{D}_3$ are not relatively spherical at the origin. Note that Λ in 4.12 is derived equivalent to the left-hand resolution, and so the failure of the exceptional locus to be equidimensional explains geometrically why $\mathcal{D}_1 = \Lambda_{\mathrm{con}} \notin \mathrm{CM}_{\mathcal{S}}\,R$ in 4.12.

The exceptional locus of the middle resolution is equidimensional, but this does not guarantee that $\mathcal{D}_2 \in \mathrm{CM}_{\mathcal{S}}\,Y$. However, to see that this indeed holds, note that the middle resolution is derived equivalent to $\Lambda_2 = \mathrm{End}_R(R \oplus N)$, where $N = (u,x) \oplus (u,xy)$. Since $\Omega N \cong N$, applying $\mathrm{Hom}_R(N,-)$ to the exact sequence

$$0 \to N \to R^4 \to N \to 0$$

gives an exact sequence

$$0 \to P_1 \to P_0^{\oplus 4} \to P_1 \to \mathcal{D}_2 \to 0.$$

Completing the above at every closed point of Z, we see that $\mathcal{D}_2$ is 2-relatively spherical at each closed point, so as in 4.11, $\mathrm{pd}_{\Lambda_2} \mathcal{D}_2 = 2$ and $\mathcal{D}_2 \in \mathrm{CM}_{\mathcal{S}} R$. Thus Twist_X is an autoequivalence on the derived category of the middle resolution.

6.2. **The Single Curve Fibre Case.** In the case when there is a single curve in each fibre, and X is smooth, we show here that the assumptions in 6.3 hold. This can arise in the setting of moduli of simple sheaves.

The following result covers both divisor-to-curve contractions and flops.

Theorem 6.5. *In the one-dimensional fibre setting of 2.5(2), suppose that $d = 3$, X is smooth, Y is Gorenstein, and f is crepant such that every reduced fibre above a closed point in Z contains precisely one irreducible curve. Then*

(1) *$\mathcal{D}_z \in \mathrm{CM}_{\mathcal{S}} \mathcal{O}_{Y,z}$ for all closed points $z \in Z$.*
(2) *$\mathcal{D} \in \mathrm{CM}_{\mathcal{S}} Y$.*
(3) *Twist_X is an autoequivalence of X.*

Proof. Since there is only one curve above each point $z \in Z$, each $\mathrm{A}_{\mathrm{con},z}$ is local and so by 2.19 every $\widehat{\mathcal{O}}_{Y,z}$ is a hypersurface.
(1) The assertion can be checked at the completion, thus we can assume that $Y = \mathrm{Spec}\,\mathfrak{R}$, with maximal ideal $\mathfrak{m}$ lying in Z. We just need to check that $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{\mathcal{S}} \mathfrak{R}$. But since X is smooth, $\mathrm{pd}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}} < \infty$, and also by assumption $\dim \mathrm{Spec}\,\mathfrak{R} = 3$, hence it follows by e.g. [IW2, 6.19(4)] that $\mathrm{inj.dim}_{\mathrm{A}_{\mathrm{con}}} \mathrm{A}_{\mathrm{con}} \leq 1$. This being the case, since further $\mathrm{A}_{\mathrm{con}}$ is local, by Ramras [R, 2.15] there is a chain of equalities

$$\mathrm{depth}_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = \dim_{\mathfrak{R}} \mathrm{A}_{\mathrm{con}} = \mathrm{inj.dim}_{\mathrm{A}_{\mathrm{con}}} \mathrm{A}_{\mathrm{con}}$$

and hence $\mathrm{A}_{\mathrm{con}} \in \mathrm{CM}_{\mathcal{S}} \mathfrak{R}$.
(2) Since the support of $\mathcal{D}$ equals Z by 2.16, this is an immediate corollary of (1).
(3) Since X is smooth, automatically $\mathcal{E} \in \mathrm{Perf}(X)$. Hence the result follows by combining (2) and 6.3(3). ☐

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