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Mirror Symmetry for Perverse Schobers from Birational Geometry

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Abstract: Perverse schobers are categorical analogs of perverse sheaves. Examples arise from varieties admitting flops, determined by diagrams of derived categories of coherent sheaves associated to the flop: in this paper we construct mirror partners to such schobers, determined by diagrams of Fukaya categories with stops, for examples in dimensions 2 and 3. Interpreting these schobers as supported on loci in mirror moduli spaces, we prove homological mirror symmetry equivalences between them. Our construction uses the coherent–constructible correspondence and a recent result of Ganatra et al. (Microlocal morse theory of wrapped fukaya categories. arXiv:1809.08807) to relate the schobers to certain categories of constructible sheaves. As an application, we obtain new mirror symmetry proofs for singular varieties associated to our examples, by evaluating the categorified cohomology operators of Bondal et al. (Selecta Math **24**(1):85–143, 2018) on our mirror schobers.

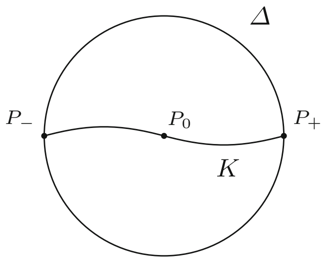
1. Introduction

Mirror symmetry is a collection of mysterious conjectural relationships between complex and symplectic geometry, inspired by the physics of superstring theory. For certain pairs of a complex geometry X and a symplectic geometry Y, key predictions are:

1. that a stringy Kähler moduli space $\mathcal{M}_{\text{Käh}}$ for X is isomorphic to a complex structure moduli space $\mathcal{M}_{\text{CS}}$ for Y, and
2. an equivalence between a derived category of coherent sheaves on X and a Fukaya category for Y, called *homological mirror symmetry*.

Unifying these, mirror symmetry predicts:

3. an equivalence between a locally constant family of derived categories on $\mathcal{M}_{\text{Käh}}$, and a corresponding family of Fukaya categories on $\mathcal{M}_{\text{CS}}$.



$$\mathcal{P}_- \hookrightarrow \mathcal{P}_0 \hookleftarrow \mathcal{P}_+$$

Fig. 1. Data determining perverse sheaf, and spherical pair

We interpret a 'locally constant family of categories' here as an appropriate categorical analog of a locally constant sheaf. Such sheaves arise as solution sheaves for ordinary differential equations: when these equations have singularities, it is natural to study them using certain generalized objects called *perverse sheaves*.

Kapranov–Schechtman have suggested categorical analogs of perverse sheaves [29], named *perverse schobers*, or simply *schobers*. As perverse sheaves may be thought of as singular versions of locally constant sheaves, schobers give a notion of a locally constant families of categories with singular behaviour.

In the mirror symmetry situation, the locally constant family of categories on $\mathcal{M}_{\text{Käh}}$ is expected to have singular behaviour at certain boundary points. We focus here on boundary points known as *conifold points*. Recent research suggests that one can extend the family as a schober over such points by using categories from birational geometry. By mirror symmetry, this schober should then have a mirror partner. Namely, we have:

Questions 1. *For the locally constant family of Fukaya categories on $\mathcal{M}_{\text{CS}}$,*

i. does this family extend to a perverse schober?
ii. does this schober satisfy a mirror symmetry equivalence?

In this paper, we give affirmative answers to these questions for some examples, working to a mathematical standard of rigour. We show that appropriate extensions of families may be elegantly formulated using Fukaya categories with stops. We then apply this to give a new proof of homological mirror symmetry for singularities associated to these examples, by evaluating a categorified cohomology operator on the resulting schober of Fukaya categories.

1.1. Background. We give background on perverse schobers and mirror symmetry, before explaining our results in Sect. 1.2.

Stringy Kähler moduli heuristic In this paper, we take X to be a resolution of a surface or 3-fold quadric cone. As a heuristic for these examples, we take $\mathcal{M}_{\text{Käh}}$ to be a punctured disk $\Delta - p$, and p to be a conifold point. This should be thought of as a local slice of the full stringy Kähler moduli: for further physical discussion in the 3-fold case, see Aspinwall [1, Sect. 4]. We therefore construct schobers on Δ, singular at p, to answer Question (i): this will mean constructing a *spherical pair*, as follows.

Spherical pairs Kapranov and Schechtman give different categorifications of a perverse sheaf P on Δ, singular at p, for different *skeletons* $K \subset \Delta$ [29]. Take K a path which passes through 0 as in Fig. 1. Then the sheaf of local cohomology of P with support in K is concentrated in some fixed degree (this important property is known as *purity*), and is constructible. Furthermore, because P is singular only at p, this sheaf has only 3 distinct stalks ($P_\pm$ and P_0). These stalks, along with natural maps between them, turn out to determine P.

This motivates the definition of a spherical pair, illustrated in Fig. 1, as the data of triangulated categories $\mathcal{P}_\pm$ and $\mathcal{P}_0$, along with embeddings satisfying certain conditions, and further conditions on orthogonals to these embeddings: for details, see Sect. 3.1.

Examples of spherical pairs arise from birational geometry. Letting $X_\pm$ be the two sides of a 3-fold flop, or more generally certain flops of families of curves, Bodzenta and Bondal [5] construct a spherical pair given by the following data.

$$D(X_-) \hookrightarrow \mathcal{Q}_0 \hookleftarrow D(X_+)$$

Here $\mathcal{Q}_0$ is an appropriate quotient of the derived category $D(X_B)$ of the fibre product of $X_\pm$ over their common contraction. In this paper we give an alternative construction using a subcategory of $D(X_B)$, in Theorem A below.

Flobers Bondal, Kapranov, and Schechtman give related constructions for certain webs of flops, under the name of *flobers*, which may be viewed as categorified perverse sheaves on $\mathbb{C}^n$, singular along a real hyperplane arrangement [7]. The prototype of a flober (on $\mathbb{C}$, singular at 0) is determined by data as follows.

$$D(X_-) \hookrightarrow D(X_B) \hookleftarrow D(X_+) \qquad (*)$$

A flober is defined by strictly weaker conditions than a spherical pair: in this case, these conditions amount to the usual Fourier–Mukai functors between $D(X_-)$ and $D(X_+)$ being equivalences.

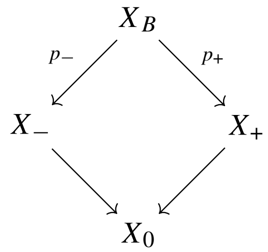
Mirror symmetry equivalences Our version of homological mirror symmetry combines work of the second author [36] and Ganatra, Pardon, and Shende [21]. For a large class of toric stacks X, this gives an equivalence between the bounded derived category of coherent sheaves $D(X)$ and a certain Fukaya category. Namely, we take a torus T obtained from the (dual of the) toric data for X, and consider the *wrapped Fukaya category* $\mathcal{W}_{\Lambda^\infty}(\Omega_T)$ with *stop* Λ^∞, a locus in contact infinity of the cotangent bundle Ω_T which Lagrangians in the category must avoid, also determined by the toric data of X. Further details are given in Sect. 1.3.

1.2. Results. Take X_0 to be one of the following singularities, along with two (stacky) resolutions $X_\pm$ as described.

Example 1. Take X_0 the quotient of $\mathbb{C}^2$ by $\mathbb{Z}/2\mathbb{Z}$ acting by ± 1. Then let X_- be the associated Deligne–Mumford quotient stack, and X_+ be the minimal resolution of the singularity X_0.

Example 2. Take X_0 the conifold singularity $\{xy - zw = 0\}$, and $X_\pm$ two small resolutions related by an Atiyah flop.

We denote the fibre product X_B of $X_\pm$ over the singularity X_0, and associated morphisms, as follows: see Sect. 4 for an explicit construction of X_B.



Spherical pairs Following our heuristic above, we take $\mathcal{M}_{\mathrm{CS}}$ and $\mathcal{M}_{\mathrm{Käh}}$ to be punctured disks $\Delta - p$. The following is our main theorem. It constructs perverse schobers on partial compactifications $\overline{\mathcal{M}}_{\mathrm{CS}}$ and $\overline{\mathcal{M}}_{\mathrm{Käh}}$, both identified with the disk Δ. We give these schobers in the form of spherical pairs, and show they are equivalent, answering Questions (i) and (ii) for our examples.

Theorem A (Theorems 6, 7). *For Examples 1 and 2 , we have*

i. *schobers given by data below, and*
ii. *an equivalence of these schobers via homological mirror symmetry, explained in Sect. 1.3.*

Symplectic side: *Let T be a torus and $\Lambda^\infty_\pm$ be loci in contact infinity of Ω_T, all determined by the toric data of $X_\pm$ in Sect. 4. Then take data as follows, with embeddings given in the course of the proof.*

$$\mathcal{W}_{\Lambda^\infty_-}(\Omega_T) \hookrightarrow \mathcal{W}_{\Lambda^\infty_- \cup \Lambda^\infty_+}(\Omega_T) \hookleftarrow \mathcal{W}_{\Lambda^\infty_+}(\Omega_T) \tag{A}$$

Complex side: *Let $\mathcal{P}_0$ be the subcategory of $D(X_B)$ generated by the images of the embeddings of $D(X_\pm)$ in $(*)$. Then take data as follows.*

$$D(X_-) \hookrightarrow \mathcal{P}_0 \hookleftarrow D(X_+) \tag{B}$$

Remark 1. The mirror operation to the 3-fold flop in Example 2 is given by Fan, Hong, Lau, and Yau [15]. It would be interesting to relate this to the locally constant family of Fukaya categories on the punctured disk $\Delta - p$ which arises from Theorem A.

Remark 2. It would be interesting to try to use the schobers appearing in Theorem A to categorify perverse sheaves extending the A-model and B-model local systems of cohomology on $\mathcal{M}_{\mathrm{CS}}$ and $\mathcal{M}_{\mathrm{Käh}}$, in our examples and more generally. In particular, this could lead to an *a priori* reason why the derived equivalences associated to a flop may be organized into a categorified perverse sheaf, as asked by Bondal–Kapranov–Schechtman [7, Sect. 0A].

Symplectic flobers Some of the techniques used to prove Theorem A then yield the following, giving a counterpart to the flober $(*)$ on the symplectic side.

Theorem B (Theorem 8). *For Examples 1 and 2, the flober $(*)$ is equivalent to a flober as follows, where Λ^∞_B is a locus in contact infinity of Ω_T, determined by the toric data of X_B.*

$$\mathcal{W}_{\Lambda^\infty_-}(\Omega_T) \hookrightarrow \mathcal{W}_{\Lambda^\infty_B}(\Omega_T) \hookleftarrow \mathcal{W}_{\Lambda^\infty_+}(\Omega_T)$$

Application Perverse sheaves admit certain cohomology operators which yield vectors spaces: these are used, for instance, to define intersection cohomology. By analogy, Bondal, Kapranov, and Schechtman [7] define certain categorified cohomology operators on perverse schobers which yield categories. In particular, they define the 2nd cohomology with compact support $\mathbb{H}^2_c$ to be a homotopy push-out of the diagram defining the flober. They prove that $\mathbb{H}^2_c$ for (∗) is $D(X_0)$, with X_0 the singular base given above. We prove a symplectic analog of this result, as follow.

Proposition C (Proposition 13). *The 2nd cohomology with compact support* $\mathbb{H}^2_c$ *for the symplectic flober in Theorem* B *is given by the category*

$$\mathcal{W}_{\Lambda^\infty_- \cap \Lambda^\infty_+}(\Omega_T).$$

We immediately obtain the following statement of homological mirror symmetry for the singular space X_0.

Corollary D (Corollary 4). *The equivalence of Theorem* B *induces an equivalence*

$$D(X_0) \cong \mathcal{W}_{\Lambda^\infty_- \cap \Lambda^\infty_+}(\Omega_T).$$

An equivalence of the categories in Corollary D follows from work of the second author in [36], as the methods there apply in singular cases. The corollary provides a new alternatively proof of this, which uses the results of [36] only for smooth cases, along with the categorified cohomology operator $\mathbb{H}^2_c$.

Remark 3. Bondal–Kapranov–Schechtman also define 1st cohomology $\mathbb{H}^1$ of a flober [7, Sect. 2D]: for the flober (∗) this is by definition a quotient of $D(X_B)$ by the triangulated subcategory generated by the embeddings of $D(X_\pm)$. In this language, the spherical pairs of Theorem A are special in the sense that their $\mathbb{H}^1$ is zero, whereas the flober (∗) and the flober in Theorem B have non-zero $\mathbb{H}^1$.

1.3. Method of proof. Let X be one of the toric varieties or stacks appearing above, and let T be the corresponding dual torus: namely, if X is described by a toric fan Σ in $N_\mathbb{R}$ where $N = \mathrm{Hom}(M, \mathbb{Z})$ for a lattice M, then we take $T = M_\mathbb{R}/M$. Write $\mathbf{Sh}^w_\Lambda(T)$ for a category of *wrapped constructible sheaves* on T, with microsupport Λ in Ω_T, where Λ is the *FLTZ skeleton* for the toric data. Furthermore let Λ^∞ be the associated locus at infinity in Ω_T, where full details are given in Sect. 3.2. Then the homological mirror symmetry equivalences used above are obtained by composition as follows.

$$\mathcal{W}_{\Lambda^\infty}(\Omega_T) \underset{(1)}{\cong} \mathbf{Sh}^w_\Lambda(T)^{\mathrm{op}} \underset{(2)}{\cong} D(X)^{\mathrm{op}} \underset{(3)}{\cong} D(X)$$

1. This is proved in recent work of Ganatra, Pardon, and Shende [21], refining a result of Nadler–Zaslow [42].
2. This is a version of the coherent–constructible correspondence, as proved by the second author in [36].
3. Here we take the derived dual $\mathbb{R}\mathrm{Hom}(-, \mathcal{O}_X)$ to give a covariant composition.

We prove Theorem A by showing that the diagrams (A) and (B) appearing there are equivalent to a diagram as follows (where, for convenience, we drop 'op' from the notation).

$$\mathbf{Sh}^w_{\Lambda_-}(T) \hookrightarrow \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) \hookleftarrow \mathbf{Sh}^w_{\Lambda_+}(T) \qquad \text{(C)}$$

For the symplectic side diagram (A), this equivalence follows immediately from (1) and by construction of the diagrams. For the complex side (B), we proceed as follows. We first apply the equivalence (2) to the spaces which appear in the flober ($*$). From these we deduce that there is an embedding

$$\kappa\colon \mathcal{P}_0 \hookrightarrow \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T).$$

We prove that κ is an equivalence by constructing and comparing certain *semiorthogonal decompositions* of these categories, as follows. To illustrate our proof, and for interest, these decompositions are presented in Sect. 2 for the surface case.

- A decomposition of $\mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T)$ with a component $\mathbf{Sh}^w_{\Lambda_\pm}(T)$ arises from an explicit presentation of the skeleton $\Lambda_- \cup \Lambda_+$.
- A decomposition of $\mathcal{P}_0$ with a component $D(X_\pm)$ is standard in the 3-fold case. For the surface case, we construct one by studying derived categories of appropriate GIT quotients.

We thus establish that κ is an equivalence. It follows from the decompositions of $\mathcal{P}_0$ that the diagram (B) gives a spherical pair, and we thence deduce that (C) gives a spherical pair. That the diagram (A) gives a spherical pair then follows by composing with the equivalence (1).

Theorem B is proved by a similar, but simpler, argument using a diagram of wrapped constructible categories as follows.

$$\mathbf{Sh}^w_{\Lambda_-}(T) \hookrightarrow \mathbf{Sh}^w_{\Lambda_B}(T) \hookleftarrow \mathbf{Sh}^w_{\Lambda_+}(T)$$

To prove Proposition C we adapt a method of Bondal, Kapranov, and Schechtman to show that $\mathbb{H}^2_c$ applied to the the diagram above gives $\mathbf{Sh}^w_{\Lambda_0}(T)$, using explicit presentation of skeleta. The result follows again by composing with the equivalence (1).

Remark 4. In the course of the proof of Proposition C we obtain a new proof of an instance of the coherent constructible correspondence, namely

$$D(X_0) \cong \mathbf{Sh}^w_{\Lambda_0}(T).$$

At the end of paper, we present a conjectural picture about how this method could be extended to prove more general such results.

1.4. Related work. Nadler has also discussed mirror equivalences of schobers on the disk [41]. In this case a different skeleton K is used so that the schober takes the form of a *spherical functor*. He proves a homological mirror symmetry statement relating a certain Landau–Ginzburg A-model to the B-model for the higher-dimensional pair of pants [41, Corollary 1.5] by deducing it from a mirror equivalence of such schobers [41, Theorem 1.4]. The A-model schober in this case is over a disk in the space of values of the (complex) superpotential.

Harder and Katzarkov have used schobers to given a new proof of a mirror equivalence for the projective space $\mathbb{P}^3$ [25].

The first author previously constructed schobers on the complex side of mirror symmetry, associated to wall crossings in GIT [12]. In subsequent work he applied these to study schobers associated to standard flops, which led to a discussion of the Questions (i) and (ii) [13, Sect. 1.2]. In further work [14], the first author and M. Wemyss give a mathematical treatment of the stringy Kähler moduli for general 3-fold flops of irreducible curves.

1.5. Structure of paper. In Sect. 2 we explain some details of our surface example informally. In Sect. 3 we give preliminaries. In Sect. 4 we construct the data of spherical pairs and flobers, and in Sect. 5 we prove that they satisfy the required properties, and give homological mirror symmetry equivalences between them, proving Theorems A and B. Finally, in Sect. 6 we prove Proposition C, deduce Corollary D, and finish with some remarks and conjectures on further applications of flobers to the coherent–constructible correspondence.

1.6. Categories. We denote categories as follows: for details, see Sect. 3.2.

D	bounded derived category of coherent sheaves
$\mathbf{Sh}^c$	constructible sheaves
$\mathbf{Sh}^\diamond$	quasi-constructible sheaves
$\mathbf{Sh}^w$	wrapped constructible sheaves [40]
$\mathcal{W}_{\Lambda^\infty}$	wrapped Fukaya category with stop Λ^∞ [20,46]

Our categories are dg-categories (or A_∞-categories). However, in our proofs, the only point where we use enhancements is for existence of homotopy pushouts in Sect. 6.

1.7. Conventions. We often consider maps $p\colon X_{\Sigma'} \to X_\Sigma$ of toric varieties induced by a refinement Σ' of a fan Σ. In that case we have a functors

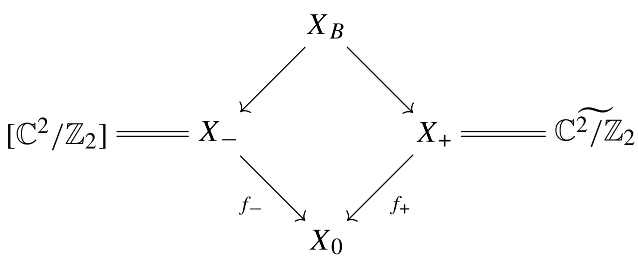
$$p^*\colon D(X_\Sigma) \to D(X_{\Sigma'}) \qquad \text{and} \qquad q^*\colon \mathbf{Sh}^\diamond_{\Lambda_\Sigma}(T) \to \mathbf{Sh}^\diamond_{\Lambda_{\Sigma'}}(T).$$

For the construction of the latter one, see Sect. 3.3, in particular (3). The notation here is chosen because these functors correspond under mirror symmetry, and the mirror reflection of the letter p is q. Functors between other categories of constructible sheaves, and between Fukaya categories, are denoted similarly. We write adjoints $p_! \dashv p^* \dashv p_* \dashv p^!$ and, by analogy, adjoints $q_! \dashv q^* \dashv q_* \dashv q^!$.

2. Surface Example

In this section we present semiorthogonal decompositions for the surface case, to illustrate our proof, and for interest.

2.1. Setting. We take the singularity $X_0 = \mathbb{C}^2/\mathbb{Z}_2$, and take (stacky) resolutions $X_\pm$ given by the corresponding Deligne–Mumford stack X_- and the minimal resolution X_+, denoted as below. The fibre product X_B is a further Deligne–Mumford stack, given explicitly in Sect. 4.1.



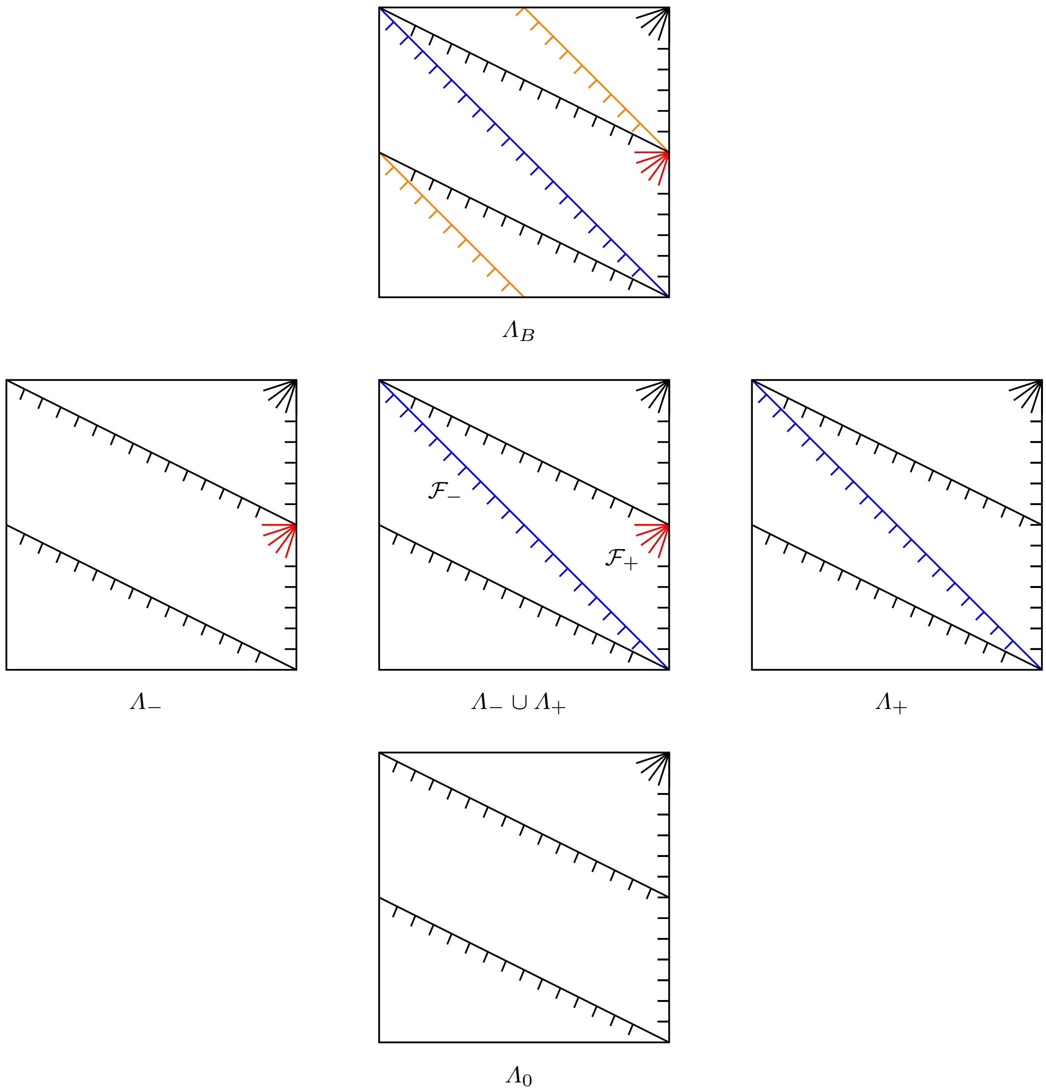


Fig. 2. Skeleta for symplectic side schober

The fibres of f_- and f_+ over the singularity 0 are the stacky point $[0/\mathbb{Z}_2]$, and a projective line $\mathbb{P}^1$. Let $\mathcal{E}_-$ and $\mathcal{E}_+$ be the structure sheaves of these fibres, tensored by the non-trivial irreducible representation of $\mathbb{Z}_2$, and the twisting sheaf $\mathcal{O}(-1)$, respectively. The fibre product X_B contains, by definition, the product E of these fibres and so we may regard $\mathcal{E}_\pm$ as sheaves on E via pullback, and thence on X_B via pushforward.

Now take skeleta Λ_0, $\Lambda_\pm$, and Λ_B determined by the toric data of X_0, $X_\pm$, and X_B, as illustrated in Fig. 2, which we explain now. The construction of these skeleta is discussed in Sects. 4 and 5.2. The squares in the figure show a fundamental domain for the quotient $T = \mathbb{R}^2/\mathbb{Z}^2$. The 'hairs' denote cotangent directions in Ω_T, and thereby indicate conical loci in Ω_T corresponding to the skeleton Λ. We let $\mathcal{F}_-$ and $\mathcal{F}_+$ be microlocal skyscraper sheaves [40] in $\mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T)$ corresponding to the cotangent directions marked.

Proposition D (Propositions 5, 8). *The category $\mathcal{P}_0$ appearing in Theorem A has semiorthogonal decompositions as follows.*

$$\langle D(X_-), \mathcal{E}_- \rangle = \mathcal{P}_0 = \langle D(X_+), \mathcal{E}_+ \rangle$$

The category $\mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T)$ has decompositions as follows.

$$\left\langle \mathbf{Sh}^w_{\Lambda_-}(T), \mathcal{F}_- \right\rangle = \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) = \left\langle \mathbf{Sh}^w_{\Lambda_+}(T), \mathcal{F}_+ \right\rangle$$

In the course of the proof we see that these categories, and the components in the above decompositions, all correspond under the coherent–constructible correspondence.

The embeddings of the $D(X_\pm)$ are the pullbacks $p_{\pm *}$, and it follows that the equivalences between $D(X_-)$ and $D(X_+)$ required in the definition of a spherical pair follow from simple cases of results of Bridgeland–King–Reid [10].

Remark 5. We get decompositions of Fukaya categories

$$\left\langle \mathcal{G}_-, \mathcal{W}_{\Lambda_-^\infty}(\Omega_T) \right\rangle = \mathcal{W}_{\Lambda_-^\infty \cup \Lambda_+^\infty}(\Omega_T) = \left\langle \mathcal{G}_+, \mathcal{W}_{\Lambda_+^\infty}(\Omega_T) \right\rangle$$

for certain objects $\mathcal{G}_\pm$ of $\mathcal{W}_{\Lambda_-^\infty \cup \Lambda_+^\infty}(\Omega_T)$, by applying the equivalence of Ganatra, Pardon, and Shende. (That the categories $\mathcal{W}$ appear in the decompositions here on the right, whereas the categories $\mathbf{Sh}^w$ appear on the left in the proposition above, is due to the fact that this equivalence is contravariant.)

3. Preliminaries

3.1. Perverse schobers. We work with *spherical pairs*, as follows. Recall that a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{C}, \mathcal{D} \rangle$, is determined by embeddings

$$\gamma : \mathcal{C} \longrightarrow \mathcal{T} \quad \text{and} \quad \delta : \mathcal{D} \longrightarrow \mathcal{T},$$

and induces projection functors given by adjoints as follows.

$$\gamma^{\mathsf{LA}} : \mathcal{T} \longrightarrow \mathcal{C} \quad \text{and} \quad \delta^{\mathsf{RA}} : \mathcal{T} \longrightarrow \mathcal{D}.$$

We may then make the following definition.

Definition 1. [29, Sect. 3C] A *spherical pair* $\mathcal{P}$ is a triangulated category $\mathcal{P}_0$ with admissible subcategories $\mathcal{P}_\pm$ and semi-orthogonal decompositions

$$\left\langle \mathcal{Q}_-, \mathcal{P}_- \right\rangle = \mathcal{P}_0 = \left\langle \mathcal{Q}_+, \mathcal{P}_+ \right\rangle,$$

such that compositions of the embeddings and projections above

$$\mathcal{Q}_- \longleftrightarrow \mathcal{Q}_+ \quad \text{and} \quad \mathcal{P}_- \longleftrightarrow \mathcal{P}_+,$$

are equivalences.

Remark 6. As indicated in Sect. 1.1, this data should be thought of as a categorification of vector space data determining a perverse sheaf. Further discussion is given by Kapranov–Schechtman in [28, Sect. 9A] and [29].

Definition 2. An isomorphism between spherical pairs $\mathcal{P}$ and $\mathcal{P}'$ consists of equivalences $\mathcal{P}_\bullet \simeq \mathcal{P}'_\bullet$ and $\mathcal{Q}_\bullet \simeq \mathcal{Q}'_\bullet$ intertwining the embeddings.

A weaker notion is discussed by Bondal, Kapranov, and Schechtman [7] omitting the condition on orthogonals $\mathcal{Q}_\bullet$.

Definition 3 [7, Sect. 1B]. A *flober* or *weak spherical pair* $\mathcal{P}$ is a triangulated category $\mathcal{P}_0$ with admissible subcategories $\mathcal{P}_\pm$ with embeddings $\delta_\pm$ such that the compositions of $\delta_\pm$ and $\delta_\pm^{\mathsf{RA}}$ as follows

$$\mathcal{P}_- \longleftrightarrow \mathcal{P}_+$$

are equivalences.

A spherical pair yields a flober in the obvious way.

Example 1. Canonical examples of spherical pairs may be obtained from sphere bundles [29, Example 3.9]. They also arise from 3-fold flops and more general flops of curves, see [5, Sect. 5.2]: for a discussion of weak spherical pairs in the same setting, see [7, Sect. 1B, 1C].

Bondal–Kapranov–Schechtman make following definition.

Definition 4 [7, Sect. 2E]. For a weak spherical pair $\mathcal{P}$, the homology with compact support $\mathbb{H}^2_c(\Delta, \mathcal{P})$ is defined as the homotopy push-out in the Morita model of the following.

$$\begin{array}{ccccc} & & \mathcal{P}_0 & & \\ & \swarrow^{\delta_-^{\mathsf{RA}}} & & \searrow^{\delta_+^{\mathsf{RA}}} & \\ \mathcal{P}_- & & & & \mathcal{P}_+ \end{array}$$

We will often say, for brevity, that a spherical pair $\mathcal{P}$ is determined by data written as follows, sometimes omitting the adjoint functors.

$$\mathcal{P}_- \underset{\delta_-^{\mathsf{RA}}}{\overset{\delta_-}{\rightleftarrows}} \mathcal{P}_0 \underset{\delta_+^{\mathsf{RA}}}{\overset{\delta_+}{\leftrightarrows}} \mathcal{P}_+ \tag{1}$$

As we will study anti-equivalences, it will also be convenient to say that a spherical pair is determined by the *opposite* of categorical data shown in (1). We explain our conventions for this now. For the functors $\delta_\pm \colon \mathcal{P}_\pm \to \mathcal{P}_0$ we use the same letter for the opposite functors, namely $\delta_\pm \colon \mathcal{P}_\pm^{\mathrm{op}} \to \mathcal{P}_0^{\mathrm{op}}$. Noting that taking opposites reverses the direction of adjoints, the opposite of (1) is written as follows.

$$\mathcal{P}_-^{\mathrm{op}} \underset{\delta_-^{\mathsf{LA}}}{\overset{\delta_-}{\rightleftarrows}} \mathcal{P}_0^{\mathrm{op}} \underset{\delta_+^{\mathsf{LA}}}{\overset{\delta_+}{\leftrightarrows}} \mathcal{P}_+^{\mathrm{op}} \tag{2}$$

In particular, given data as in (2) we have semi-orthogonal decompositions

$$\langle \mathcal{P}_-^{\mathrm{op}}, \mathcal{Q}_-^{\mathrm{op}} \rangle = \mathcal{P}_0^{\mathrm{op}} = \langle \mathcal{P}_+^{\mathrm{op}}, \mathcal{Q}_+^{\mathrm{op}} \rangle,$$

and $\mathbb{H}^2_c(\Delta, \mathcal{P})$ is given by a push-out of the following.

$$\begin{array}{ccccc} & & \mathcal{P}_0^{\mathrm{op}} & & \\ & \swarrow^{\delta_-^{\mathsf{LA}}} & & \searrow^{\delta_+^{\mathsf{LA}}} & \\ \mathcal{P}_-^{\mathrm{op}} & & & & \mathcal{P}_+^{\mathrm{op}} \end{array}$$

3.2. Fukaya category and microlocal sheaf theory. In this subsection, we give a brief introduction to a relationship between the Fukaya category and microlocal sheaf theory. Let Z be a real analytic manifold and k_Z be the constant sheaf valued in a field k. We denote the bounded derived category of k_Z-modules by $\mathbf{Mod}(k_Z)$. We define microsupport (also known as 'singular support'), one of the most important notion in microlocal sheaf theory.

Definition 5. The microsupport $SS(\mathcal{E})$ of $\mathcal{E} \in \mathbf{Mod}(k_Z)$ is a subset of the cotangent bundle Ω_Z defined by its complement as follows: $(x, \xi) \in \Omega_Z$ (where $x \in Z$ and $\xi \in \Omega_{Z,x}$) is not contained in $SS(\mathcal{E})$ if there exists an open neighbourhood U of (x, ξ) such that

$$\left(\mathbb{R}\Gamma_{\{z|\psi(z)\geq\psi(y)\}}\,\mathcal{E}\right)_y \simeq 0$$

for any point $y \in Z$ and any smooth function ψ with $\mathrm{Graph}(d\psi) \subset U$.

We view Ω_Z as a symplectic manifold with its standard exact symplectic structure. The microsupport $SS(\mathcal{E})$ is a Lagrangian subset of Ω_Z if $\mathcal{E}$ is constructible:

Definition 6. A sheaf $\mathcal{E}$ is constructible (respectively quasi-constructible) if there exists a real analytic Whitney stratification $\mathcal{S}$ of Z such that the restriction of $\mathcal{E}$ to each stratum $S \in \mathcal{S}$ is a locally constant sheaf of finite rank (respectively locally constant sheaf possibly of infinite rank). For the definition of such a stratification, we refer to [30].

Theorem 1 (Involutivity theorem [30, Theorem 8.4.2]). *An object $\mathcal{E} \in \mathbf{Mod}(k_Z)$ is quasi-constructible if and only if* $SS(\mathcal{E})$ *is Lagrangian.*

Remark 7. Here, and elsewhere, we allow Lagrangians to be singular.

We write $\mathbf{Sh}^c(Z)$ (respectively $\mathbf{Sh}^\diamondsuit(Z)$) for the (dg-)category of bounded complexes of constructible sheaves (respectively unbounded complexes of quasi-constructible sheaves) over Z. For a subset $\Lambda \subset \Omega_Z$, the full subcategory of $\mathbf{Sh}^c(Z)$ (respectively $\mathbf{Sh}^\diamondsuit(Z)$) spanned by objects with microsupport living inside Λ is denoted by $\mathbf{Sh}^c_\Lambda(Z)$ (respectively $\mathbf{Sh}^\diamondsuit_\Lambda(Z)$).

Before stating a relationship with Fukaya category, we introduce one more category.

Definition 7. Let $\Lambda \subset \Omega_Z$ be a subset. The full subcategory $\mathbf{Sh}^w_\Lambda(Z)$ of $\mathbf{Sh}^\diamondsuit_\Lambda(Z)$ is defined by the following: $\mathcal{E} \in \mathbf{Sh}^w_\Lambda(Z)$ if and only if

$$\mathrm{Hom}_{\mathbf{Sh}^\diamondsuit_\Lambda(Z)}(\mathcal{E}, -)\colon \mathbf{Sh}^\diamondsuit_\Lambda(Z) \to \mathbf{Vect}$$

commutes with any direct sums. We call an object of $\mathbf{Sh}^w_\Lambda(Z)$ a wrapped constructible sheaf.

A relationship between microlocal sheaf theory and the Fukaya category was first clearly stated by Nadler–Zaslow by using an 'infinitesimally wrapped Fukaya category'. Our setting is a more recent variant of Nadler–Zaslow established by Ganatra–Pardon–Shende [21]: Let Z be a real analytic manifold and Ω_Z be the cotangent bundle of Z. Let $\{x_i\}$ be local coordinates of Z and $\{\xi_i\}$ be the corresponding cotangent coordinates. Then Ω_Z has an exact symplectic structure locally written as $\sum_i d\xi_i \wedge dx_i$. Let us fix g a Riemannian metric over Z. Then we define the cosphere bundle by

$$S^*Z_a := \{(x, \xi) \in \Omega_Z \mid g_x(\xi, \xi) = a\}\,.$$

The symplectic structure induces a contact structure on S^*Z_a. Since the S^*Z_a for various a are contactoisomorphic to each other, we consider $a \to \infty$ virtually and call this abstract contact manifold *contact infinity* and denote it by Ω_Z^∞. Let Λ^∞ be a subanalytic Legendrian of Ω_Z^∞. Then one can define an A_∞-category $\mathcal{W}_{\Lambda^\infty}(\Omega_Z)$, the wrapped Fukaya category of Ω_Z with the stop Λ^∞. Roughly, this category consists of

1. Objects: (possibly noncompact) Lagrangian submanifold whose noncompact ends live away from Λ^∞.
2. Morphisms: wrapped Floer cohomology.

For details, see Ganatra, Pardon, and Shende [20, Sects. 1.1, 2] and Sylvan [46].

Given a Legendrian Λ^∞ as above, we obtain a locus in Ω_Z as follows.

Definition 8. Let $\Lambda \subset \Omega_Z$ be given by $\Lambda = (\mathbb{R}_{>0} \cdot \Lambda^\infty) \cup Z$ where $\mathbb{R}_{>0}$ acts by scaling the cotangent fibers, and Z is the zero section in Ω_Z.

We can also go the other way around: namely, for a given conic (i.e. invariant under $\mathbb{R}_{>0}$) Lagrangian Λ in $\Omega_Z \backslash Z$ we have the following.

Definition 9. We obtain a Legendrian $\Lambda^\infty \subset \Omega_Z^\infty$ as follows. First, for any a we obtain a Legendrian in S^*Z_a by $\Lambda_a = S^*Z_a \cap \Lambda$. The conicness implies that Λ_a is preserved under the isomorphism $S^*Z_a \cong S^*Z_b$ for any a, b. Hence we get a Legendrian Λ^∞ in the abstract contact manifold Ω_Z^∞.

We may now state the following theorem.

Theorem 2 (Ganatra–Pardon–Shende [21]). *There exists an equivalence*

$$\mathbf{Sh}^w_\Lambda(Z)^{\mathrm{op}} \simeq \mathcal{W}_{\Lambda^\infty}(\Omega_Z)$$

of A_∞-categories, where op *denotes the opposite category.*

Remark 8. We may remove the op in the above, but at the expense of negating the Liouville form, or performing a similar operation on Λ^∞ [21, after Theorem 1.1].

It is expected that this theorem generalizes to the case of Weinstein manifolds instead of Ω_Z. This is known as Kontsevich's conjecture [38]. Progress towards a proof has been made by many people, see in particular work of Nadler [39].

A point for us is that a Landau–Ginzburg model gives an isotropic subset Λ^∞, hence a partially wrapped Fukaya category. We sometimes call this category *Fukaya–Seidel category of the Landau–Ginzburg model* [20]. In general, this category is not generated by Lefschetz thimbles emanating from critical points:

Example 2. We take $W(z) = z$ on $\mathbb{C}^*$. Then the set of Lefschetz thimbles is empty but the partially wrapped Fukaya category is equivalent to the derived category of coherent sheaves over $\mathbb{A}^1$.

Hence, to study mirror symmetry, the partially wrapped Fukaya category is appropriate.

The Lagrangian skeletons for mirror Landau–Ginzburg models of toric varieties can be combinatorially defined. Such skeletons were first proposed by Fang, Liu, Treumann, and Zaslow [17, Sect. 5.5]. This followed earlier work of Fang [16], and pioneering work of Bondal [4].

Notation 1. *We use standard notation in the toric setting, as follows.*

M	*a rank n free abelian group*
N	*the dual of M*
Σ	*a rational finite fan in* $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$
X_Σ	*the toric variety of* Σ
T	*real torus* $M_{\mathbb{R}}/M$
π	*projection* $M_{\mathbb{R}} \to T$

For a subset $\sigma \subset N_{\mathbb{R}}$*, we set*

$$\sigma^{\perp} := \{m \in M_{\mathbb{R}} \mid n(m) = 0 \text{ for any } n \in \sigma\}.$$

We then have the following.

Definition 10 (FLTZ skeleton).

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma} \pi(\sigma^{\perp}) \times (-\sigma) \subset T \times N_{\mathbb{R}} \cong \Omega_T.$$

The following result is due to Gammage–Shende [19], Zhou [47], and Ganatra–Pardon–Shende [21].

Theorem 3. *The FLTZ skeleton at infinity* Λ_Σ^∞ *is a Weinstein skeleton of a generic fiber of the Hori–Vafa mirror potential. If moreover* X_Σ *is Fano, then* $\mathcal{W}_{\Lambda_\Sigma^\infty}(\Omega_T)$*, and hence* $\mathbf{Sh}^w_{\Lambda_\Sigma}(T)$*, is equivalent to the Fukaya–Seidel category of W.*

We will not define 'Weinstein skeleton' here. The main point for us is that $\mathcal{W}_{\Lambda_\Sigma^\infty}(\Omega_T)$ in general can be considered as a generalization of a Fukaya–Seidel category.

Since mirror symmetry predicts an equivalence between the B-model on X_Σ and the mirror Landau–Ginzburg A-model, we can interpret homological mirror symmetry as an equivalence between the derived category of coherent sheaves over X_Σ and $\mathbf{Sh}^w_{\Lambda_\Sigma}(T)$. This is the content of the next subsection.

3.3. Coherent–constructible correspondence. In this subsection, we review the result in [17] and [36] for the smooth variety case. We now take the field k to be $\mathbb{C}$. Let Λ_Σ and X_Σ be as in the previous section. For $\sigma \in \Sigma$, we have the corresponding affine coordinate $i_\sigma \colon U_\sigma \subset X_\Sigma$. Letting **Qcoh** denote the unbounded derived category of quasicoherent sheaves, there exists a unique functor

$$\kappa_\Sigma \colon \mathbf{Qcoh}(X_\Sigma) \to \mathbf{Sh}^{\diamond}_{\Lambda_\Sigma}(X_\Sigma)$$

which maps $\Theta'(\sigma) := i_{\sigma *}\mathcal{O}_{U_\sigma}$ to $\Theta(\sigma) := \pi_! \mathbb{C}_{\mathrm{Int}(\sigma^\vee)}$ where Int is the interior and $\sigma^\vee$ is the polar dual of σ.

Theorem 4. [36] *The restriction of* κ_Σ *to* $D(X_\Sigma)$ *gives an equivalence*

$$D(X_\Sigma) \xrightarrow{\sim} \mathbf{Sh}^w_{\Lambda_\Sigma}(T).$$

As discussed in the last part of the previous section, this equivalence is an instance of homological mirror symmetry.

According to [36], we redefine the equivalence functor as

$$K_\Sigma := \kappa_\Sigma(- \otimes \omega_\Sigma^{-1})$$

where ω_Σ is the canonical sheaf. Since ω_Σ is invertible, we have:

Corollary 1. [36] *The functor K_Σ gives an equivalence*

$$D(X_\Sigma) \xrightarrow{\sim} \mathbf{Sh}^w_{\Lambda_\Sigma}(T).$$

The reason why we use this modified functor is to get a commutativity with push-forwards on the B-side, as appeared in the proof of Theorem 5.

As a corollary of the above theorem and Theorem 2, we get a homological mirror symmetry between coherent sheaves and a Fukaya category:

Corollary 2. *We have an equivalence*

$$D(X_\Sigma) \xrightarrow{\sim} \mathcal{W}_{\Lambda^\infty_\Sigma}(\Omega_T)^{\mathrm{op}}.$$

Next we would like to discuss the functoriality. Let Σ' be a refinement of Σ. We denote the corresponding map by $p\colon X_{\Sigma'} \to X_\Sigma$. Then there exists an inclusion relation $\Lambda_\Sigma \subset \Lambda_{\Sigma'}$, which induces an inclusion

$$\mathfrak{q}^*\colon \mathbf{Sh}^\diamond_{\Lambda_\Sigma}(T) \hookrightarrow \mathbf{Sh}^\diamond_{\Lambda_{\Sigma'}}(T). \tag{3}$$

Proposition 1 (Fang–Liu–Treumann–Zaslow [17, Theorem 3.8]). *There is natural isomorphism of triangulated functors*

$$\kappa_{\Sigma'} \circ p^* \simeq \mathfrak{q}^* \circ \kappa_\Sigma.$$

We remark that we do not need a dg-level statement for this natural isomorphism (see also Sect. 1.6 for discussion on dg-categories).

For a general inclusion of Lagrangians $\Lambda \subset \Lambda'$, the inclusion

$$\mathbf{Sh}^\diamond_\Lambda(T) \hookrightarrow \mathbf{Sh}^\diamond_{\Lambda'}(T)$$

takes compact objects to compacts objects, and therefore restricts to a functor $\mathbf{Sh}^c_\Lambda(T) \hookrightarrow \mathbf{Sh}^c_{\Lambda'}(T)$. It does not, however, give a functor $\mathbf{Sh}^w_\Lambda(T) \hookrightarrow \mathbf{Sh}^w_{\Lambda'}(T)$. Nevertheless, in our case, we have:

Proposition 2. *If X_Σ is smooth, the functor* (3) *above restricts to a functor*

$$\mathfrak{q}^*\colon \mathbf{Sh}^w_{\Lambda_\Sigma}(T) \hookrightarrow \mathbf{Sh}^w_{\Lambda_{\Sigma'}}(T).$$

Proof. By Proposition 1 and Theorem 4, this is equivalent to saying that the pullback $p^*\colon \mathbf{Qcoh}(X_\Sigma) \to \mathbf{Qcoh}(X_{\Sigma'})$ restricts to a functor between bounded derived categories, which is true under the assumption. □

We also note the following for later use.

Proposition 3. *For a general inclusion of Lagrangians $\Lambda \subset \Lambda'$, a left adjoint ι^{LA} of the natural embedding $\iota : \mathbf{Sh}^\diamond_\Lambda(T) \hookrightarrow \mathbf{Sh}^\diamond_{\Lambda'}(T)$ restricts to an essentially surjective functor*

$$\iota^w : \mathbf{Sh}^w_{\Lambda'}(T) \to \mathbf{Sh}^w_\Lambda(T).$$

Proof. This is by general nonsense, see Nadler [40, end of Sect. 3.6]. For the reader's convenience, we indicate a sketch of the proof. For a smooth point $(x, \xi) \in \Lambda$, consider the local cohomology functor measuring the existence of microsupport at (x, ξ). Provided that Λ is Lagrangian, one can explicitly represent the functor by an object in $\mathbf{Sh}^c(T)$ (not necessarily with support contained in Λ). This implies that the functor is cocontinuous. By Brown representability, one gets an object in $\mathbf{Sh}^w_\Lambda(T)$ which represents the functor. The object is called a microlocal skyscraper sheaf at (x, ξ).

By the discussion of Nadler [40], such objects generate $\mathbf{Sh}^w_\Lambda(T)$. The same holds for Λ'. Moreover, for $(x, \xi) \in \Lambda \subset \Lambda'$, the microlocal skyscraper sheaf on (x, ξ) in $\mathbf{Sh}^w_{\Lambda'}(T)$ is mapped to the one on (x, ξ) in $\mathbf{Sh}^w_\Lambda(T)$ under ι^w. Hence a set of generating objects of $\mathbf{Sh}^w_{\Lambda'}(T)$ is mapped to that of $\mathbf{Sh}^w_\Lambda(T)$. This completes the proof. □

3.4. Toric stacks. Here we recall a definition of toric stacks, following Gerashenko–Satriano [22]. Let L and N be free abelian groups and

$$f : L \to N$$

be a morphism. We can associate a morphism between algebraic groups

$$f \otimes_{\mathbb{Z}} \mathbb{C}^* : L \otimes_{\mathbb{Z}} \mathbb{C}^* \to N \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

Let Σ be a fan in L and X_Σ be the associated toric variety. Then the toric stack associated to the data (Σ, f) is defined by the quotient stack

$$[X_\Sigma / \ker(f \otimes_{\mathbb{Z}} \mathbb{C}^*)]$$

where $\ker(f \otimes_{\mathbb{Z}} \mathbb{C}^*)$ acts on X_Σ via the action of $L \otimes_{\mathbb{Z}} \mathbb{C}^*$ on X_Σ.

4. Schober Constructions

We give the required description of the geometry of our examples, before constructing schobers on the A-side and B-side of mirror symmetry.

4.1. Surface example. Consider the A_1 quotient singularity $X_0 = \mathbb{C}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \{\pm 1\}$. This has resolutions a Deligne–Mumford quotient stack, and a minimal resolution, which we denote as follows.

$$X_- = [\mathbb{C}^2/\mathbb{Z}_2] \qquad X_+ = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$$

The minimal resolution may be realised as the total space of a line bundle.

$$X_+ = \operatorname{Tot} \mathcal{O}_{\mathbb{P}^1}(-2)$$

The fibre product of $X_\pm$ over X_0 is a further Deligne–Mumford stack, as follows.

$$X_B = [\operatorname{Tot} \mathcal{O}_{\mathbb{P}^1}(-1)/\mathbb{Z}_2].$$

Here $\mathbb{Z}_2$ acts as ± 1 on the fibres of the bundle. The associated morphisms

$$p_\pm : X_B \to X_\pm$$

are described as follows.

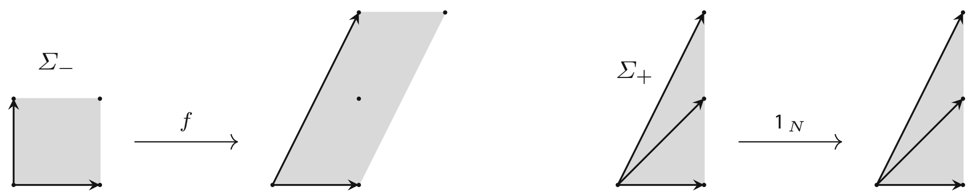


Fig. 3. Toric data for $X_{\pm}$

– The map p_- is the blowup of $[0/\mathbb{Z}_2] \subset X_-$, in other words the blowup of $0 \in \mathbb{C}^2$, noting that this is equivariant with respect to the $\mathbb{Z}_2$-actions.
– The map p_+ is given by the root stack construction along $\mathbb{P}^1$ (for a general discussion, see [18]), namely a family version of the following morphism.

$$[\mathbb{C}/\mathbb{Z}_2] \to \mathbb{C}\colon z \mapsto z^2$$

Toric description Let N be a rank 2 lattice with basis e_1, e_2, and let M be its dual. The faces of the cone $\mathrm{Cone}(e_1, e_1+2e_2) \subset N_{\mathbb{R}}$ give a fan Σ_0 representing the singularity X_0. Then the small resolution X_+ is represented by a refinement Σ_+ of Σ_0 given by the faces of the following cones.

$$\mathrm{Cone}(e_1, e_1+e_2),\ \mathrm{Cone}(e_1+e_2, e_1+2e_2) \subset N_{\mathbb{R}}$$

To obtain toric data for the stack X_-, in the sense of Sect. 3.4, take a further rank 2 lattice L with basis g_1, g_2 and consider the standard fan representing $\mathbb{C}^2$, generated by faces of $\mathrm{Cone}(g_1, g_2) \subset L_{\mathbb{R}}$, and denote it Σ_-. The lattice map

$$\begin{aligned} f\colon L &\to N \\ g_1 &\mapsto e_1 \\ g_2 &\mapsto e_1+2e_2 \end{aligned}$$

then induces a map between the corresponding algebraic tori

$$T_L \to T_N\colon (a,b) \mapsto (ab, b^2).$$

The kernel of this map is $\mathbb{Z}_2 = \{\pm 1\}$. It follows that the toric stack corresponding to the data of Σ_L and f is the toric stack $[\mathbb{C}^2/\mathbb{Z}_2] = X_-$, the stacky resolution of the A_1-singularity, see Fig. 3. The variety X_+ is realized as a toric stack by taking the data Σ_+ and $\mathbb{1}_N$.

For the toric data corresponding to the fibre product X_B, take the fan Σ_B given by faces of the following cones

$$\mathrm{Cone}(e_1, e_1+e_2),\ \mathrm{Cone}(e_1+e_2, e_2) \subset L_{\mathbb{R}}.$$

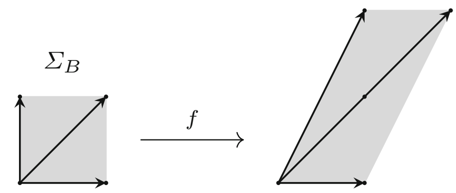


Fig. 4. Toric data for X_B

Note that this fan refines Σ_-, and corresponds to Tot $\mathcal{O}_{\mathbb{P}^1}(-1)$. Then we take data Σ_B and f to obtain X_B, see Fig. 4. Note this is not a toric Deligne–Mumford stack in the sense of Borisov–Chen–Smith [8].

Now the projections $p_\pm$ from X_B are induced by appropriate commutative squares of lattices, as shown.

$$\begin{array}{ccc} X_+ & \qquad & N \xrightarrow{\mathbb{1}_N} N \\ \uparrow{\scriptstyle p_+} & & {\scriptstyle f}\uparrow \qquad \| \\ X_B & & L \xrightarrow{\ f\ } N \\ \downarrow{\scriptstyle p_-} & & \| \qquad\ \ \| \\ X_- & & L \xrightarrow{\ f\ } N \end{array}$$

4.2. Threefold example. We recall the geometry of the Atiyah flop local model. Namely, take X_0 to be the conifold singularity $(xy - zw = 0)$ in $\mathbb{C}^4$. This has crepant resolutions $X_\pm$ with exceptional curves $E_\pm \cong \mathbb{P}^1$ such that

$$X_\pm \cong \operatorname{Tot} \mathcal{O}_{E_\pm}(-1)^{\oplus 2},$$

related by an Atiyah flop as follows.

$$\begin{array}{ccc} X_- & \leftarrow\!\dashrightarrow & X_+ \\ & \searrow \quad \swarrow & \\ & X_0 & \end{array}$$

Let X_B be the fibre product of this diagram. This is isomorphic to the blowup of $X_\pm$ along $E_\pm$. Write E for the common exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$.

Toric data Let N be a rank 3 lattice with basis e_i, and dual M. The set of faces of the cone $\operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subset N_\mathbb{R}$ gives a fan Σ_0 representing the conifold. The small resolutions $X_\pm$ come from

$$\Sigma_+ := \Sigma_0 \cup \{\operatorname{Cone}(e_2, e_1 + e_3)\}$$
$$\Sigma_- := \Sigma_0 \cup \{\operatorname{Cone}(e_1, e_2 + e_3)\}$$

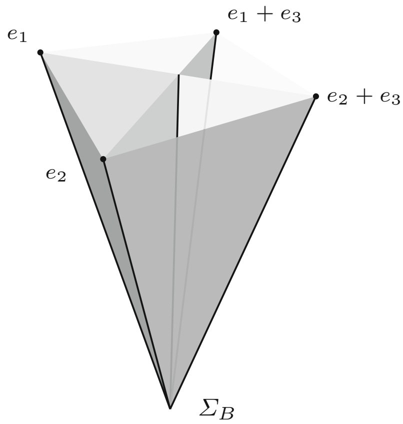


Fig. 5. Threefold toric fan in $N_{\mathbb{R}}$

and the blow-up X_B from

$$\Sigma_B := \Sigma_+ \cup \Sigma_- \cup \{\mathbb{R}_{\geq 0} \cdot (e_1 + e_2 + e_3)\}.$$

This data is sketched in Fig. 5. Note that the fan Σ_B refines $\Sigma_\pm$ giving the morphisms

$$p_\pm \colon X_B \to X_\pm.$$

4.3. B-side constructions. Here we obtain schobers involving the categories $D(X_\pm)$. To do this, we verify that the diagram (B) from the introduction gives a schober, by an analysis of semiorthogonal decompositions. For the 3-fold case which we study first, our schober should be equivalent to the one constructed, in a more general setting, by Bodzenta–Bondal in [5]. We here use a different method to theirs, which also applies to the stacky surface case.

Threefold case First note we have $p_\pm^* \colon D(X_\pm) \to D(X_B)$ with left adjoints $p_{\pm!}$, where we take $p_! = p_*(\omega_p[\dim p] \otimes -)$. The compositions

$$p_{+*} p_-^* \qquad \text{and} \qquad p_{-*} p_+^*$$

are equivalences by Bondal–Orlov [6, Theorem 3.6, and remark following]: see also an account of the proof of fully faithfulness in Huybrechts [26, Proposition 11.23], and an explanation of why the equivalence property follows in [26, Remark 11.22 and Proposition 1.54]. By taking left adjoints, the compositions

$$p_{-!} p_+^* \qquad \text{and} \qquad p_{+!} p_-^*$$

are equivalences.

Definition 11. Let $\mathcal{P}_0$ be full triangulated subcategory of $D(X_B)$ generated by the images of p_+^* and p_-^*.

Then we have $p_\pm^*\colon D(X_\pm) \to \mathcal{P}_0$ with left adjoints $p_{\pm!}$ obtained by restriction, because $\mathcal{P}_0$ is a full subcategory of $D(X_B)$, and we deduce that the compositions $D(X_-) \leftrightarrow D(X_+)$ in the following diagram, namely $p_{-!}p_+^*$ and $p_{+!}p_-^*$, are equivalences.

$$D(X_-) \underset{p_{-!}}{\overset{p_-^*}{\rightleftarrows}} \mathcal{P}_0 \underset{p_{+!}}{\overset{p_+^*}{\leftrightarrows}} D(X_+) \tag{4}$$

To show that this gives (the opposite of) a spherical pair, and for use in our equivalence proof in Sect. 5, we construct semi-orthogonal decompositions of $\mathcal{P}_0$. Set notation for the blowup of $X_\pm$ in $E_\pm$ as follows.

$$\begin{array}{ccc} X_\pm & \xleftarrow{\;p_\pm\;} & X_B \\ {\scriptstyle j_\pm}\uparrow & & \uparrow{\scriptstyle i} \\ E_\pm & \xleftarrow[\;q_\pm\;]{} & E \end{array} \tag{5}$$

Proposition 4. *The data of* (4) *yields the opposite of a spherical pair. In particular, we have semi-orthogonal decompositions*

$$\mathcal{P}_0 = \left\langle p_\pm^* D(X_\pm), i_* q_\pm^* \mathcal{O}_{E_\pm}(-1) \right\rangle.$$

Proof. Recall that by a result of Orlov [44], see for instance [37, Theorem 2.6], we have

$$D(X_B) = \left\langle p_+^* D(X_+), i_* q_+^* D(E_+) \right\rangle.$$

To obtain a semi-orthogonal decomposition of $\mathcal{P}_0 \subset D(X_B)$, we therefore calculate the image of $p_-^* D(X_-) \subset \mathcal{P}_0$ under the projection to $D(E_+)$: the projection functor is $q_{+*}i^!$. Recalling that $i^! = \omega_i[\dim i] \otimes i^*$, it thence suffices to calculate the image of the following functor on $D(X_-)$.

$$q_{+*}\big(\omega_i \otimes i^* p_-^*(-)\big) \tag{6}$$

Now $\omega_i = \mathcal{N}_{E|X}$, and it is straightforward to show that

$$\mathcal{N}_{E|X} = q_+^* \mathcal{O}_{E_+}(-1) \otimes q_-^* \mathcal{O}_{E_-}(-1), \tag{7}$$

see [26, Sect. 11.3] for instance. The functor (6) is then given as follows.

$$\begin{aligned} q_{+*}\big(\omega_i \otimes i^* p_-^*(-)\big) &\cong q_{+*}\big(\omega_i \otimes q_-^* j_-^*(-)\big) && \text{(commutativity of (5))} \\ &\cong q_{+*}\big(q_+^*\mathcal{O}_{E_+}(-1) \otimes q_-^*\mathcal{O}_{E_-}(-1) \otimes q_-^* j_-^*(-)\big) \\ &\cong \mathcal{O}_{E_+}(-1) \otimes q_{+*}\big(q_-^*\mathcal{O}_{E_-}(-1) \otimes q_-^* j_-^*(-)\big) \\ &\cong \mathcal{O}_{E_+}(-1) \otimes q_{+*}q_-^*\big(\mathcal{O}_{E_-}(-1) \otimes j_-^*(-)\big) \end{aligned}$$

Here the third line is obtained by the projection formula. Now, writing $r_\pm\colon E_\pm \to \mathrm{pt}$, we have $q_{+*}q_-^* \cong r_+^* r_{-*}$ by flat base change. The functor (6) is therefore isomorphic to the composition of

$$r_{-*}\big(\mathcal{O}_{E_-}(-1) \otimes j_-^*(-)\big) \qquad \text{then} \qquad \mathcal{O}_{E_+}(-1) \otimes r_+^*(-).$$

The first functor is essentially surjective onto $D(\mathrm{pt})$: to see this, apply it to $s_-^*\mathcal{O}_{E_-}(+1)$ where s_- is the bundle projection of X_-, and note that $j_-^* s_-^* \cong \mathrm{id}$. The image of (6) is thence the image of the second functor, namely the subcategory of $D(E_+)$ generated by $\mathcal{O}_{E_+}(-1)$. We deduce a semi-orthogonal decomposition as in the statement, with the other sign following by symmetry.

Finally we show the schober conditions. The required equivalences between $D(X_-)$ and $D(X_+)$ are explained above (4). For the equivalences between the orthogonals, these are generated by single objects, and therefore by symmetry it suffice[s] to prove the following lemma. □

Writing $\mathcal{E}_\pm = i_* q_\pm^* \mathcal{O}_{E_\pm}(-1)$, we have the following.

Lemma 1. *The image of $\mathcal{E}_-$ under the projection to $D(E_+)$ is $\mathcal{E}_+[-2]$.*

Proof. The projection to $D(E_+)$ is $q_{+*}i^!$. Noting that i is the embedding of a divisor we have a triangle of functors

$$\mathrm{id} \to i^! i_* \to (- \otimes \mathcal{N}_{E|X})[-1] \to .$$

Applying this to $q_-^*\mathcal{O}_{E_-}(-1)$ and using the expression (7) for $\mathcal{N}_{E|X}$ gives

$$q_-^*\mathcal{O}_{E_-}(-1) \to i^! i_* q_-^*\mathcal{O}_{E_-}(-1) \to q_+^*\mathcal{O}_{E_+}(-1) \otimes q_-^*\mathcal{O}_{E_-}(-2)[-1] \to .$$

Applying q_{+*} to the lefthand term gives zero, where we use flat base change $q_{+*}q_-^* \cong r_+^* r_{-*}$. Applying to the righthand term gives $\mathcal{F}_+[-2]$ because $r_{-*}\mathcal{O}_{E_-}(-2) = \mathcal{O}_{\mathrm{pt}}[-1]$, and the claim follows. □

Surface case We apply the same approach to the surface case, constructing the required semiorthogonal decompositions using variation of GIT. For the surfaces $X_\pm$, we have fibre product diagrams as in (5) as follows.

$$\begin{array}{ccc} X_\pm & \xleftarrow{p_\pm} & X_B \\ {\scriptstyle j_\pm}\uparrow & & \uparrow{\scriptstyle i} \\ E_\pm & \xleftarrow[q_\pm]{} & E \end{array} \qquad (8)$$

Here we take

$$E_- = [0/\mathbb{Z}_2] \qquad \text{and} \qquad E_+ = \mathbb{P}^1,$$

with j_- the obvious embedding, and j_+ the embedding of the zero section of the bundle X_+. The common fibre product E is isomorphic to $[\mathbb{P}^1/\mathbb{Z}_2]$ with trivial $\mathbb{Z}_2$-action.

We have $\mathcal{P}_0$ a full subcategory of $D(X_B)$ defined as in the 3-fold case.

Proposition 5. *For the surfaces $X_\pm$, the data shown in (4) yields the opposite of a spherical pair. In particular, have semi-orthogonal decompositions*

$$\mathcal{P}_0 = \left\langle p_\pm^* D(X_\pm), i_* q_\pm^* \mathcal{O}_{E_\pm}(-1) \right\rangle .$$

Here $\mathcal{O}_{E_-}(-1)$ is used to denote the sheaf on the stack $E_- = [0/\mathbb{Z}_2]$ corresponding to the non-trivial irreducible representation of $\mathbb{Z}_2$.

Remark 9. There is literature on semiorthogonal decompositions for 'stacky blowups': this term is convenient to describe both blowups of stacks and root stack construction (see for instance [3, Introduction]), as both arise naturally in birational geometry of Deligne–Mumford stacks. Derived categories of root stacks have been considered in [27, Theorem 1.6], and these results generalized in [3] and [45]. Kawamata has also constructed semiorthogonal decompositions in this situation, for discussion see in particular [31, Theorem 3.4] with proofs found in [32–34]. Although the results below are likely covered by this work, we found it convenient to apply the variation of GIT method herein.

Proof. We use a general construction of Coates, Iritani, Jiang, and Segal [11] to realize the blowup p_- as a variation of GIT. We then use standard technology which relates derived categories under variation of GIT [2,23] to obtain a semi-orthogonal decomposition. The semi-orthogonal decomposition for p_+ follows by an appropriate adaptation of this argument.
Case$(-)$: First recall that p_- is the blowup in $E_- = [0/\mathbb{Z}_2] \subset X_-$. We choose a bundle $\mathcal{F}$ on X_- with a section σ cutting out E_-. Following [11, Sect. 5.2], we consider the total space $\mathrm{Tot}(\mathcal{F} \oplus \mathcal{O})$ with a $\mathbb{C}^*$-action of weight -1 on the fibres of $\mathcal{F}$, and weight $+1$ on the fibres of $\mathcal{O}$. Writing (v, z) for fibre coordinates, and π for the projection to X_-, we take the substack

$$[M/\mathbb{C}^*] := \{vz = \pi^*\sigma\} \subset \mathrm{Tot}(\mathcal{F} \oplus \mathcal{O}).$$

The stack $[M/\mathbb{C}^*]$ has GIT quotients

$$X_- \quad \text{and} \quad \mathrm{Bl}_{E_-} X_- \cong X_B,$$

and fixed locus $M^{\mathbb{C}^*} \cong E_-$. Furthermore, we obtain a semiorthogonal decomposition as follows.

$$D(X_B) = \left\langle p_-^* D(X_-), i_* q_-^* D(E_-) \right\rangle \tag{9}$$

We explain some of the details, so that we can adapt them to the case of p_+ below. By general theory, the GIT quotients are derived equivalent to certain 'windows' in $D[M/\mathbb{C}^*]$ of the form

$$\mathcal{C}_d = \left\{ \mathcal{E} \in D[M/\mathbb{C}^*] \,\middle|\, \mathcal{H}^\bullet L i_Z^* \mathcal{E} \text{ have weights in } [0, d) \right\}.$$

Namely, the restriction functors

$$\mathcal{C}_{\mathrm{rk}\,\mathcal{F}} \to D(X_B) \qquad \mathcal{C}_1 \to D(X_-)$$

are equivalences. Furthermore, there is a semiorthogonal decomposition of with $D(X_B)$ with components $D(X_-)$ and $D(E_-)$, the latter appearing with multiplicity $\mathrm{rk}\,\mathcal{F} - 1 = 1$. In particular, [11, Lemma 5.2(1)] is used to show that $D(X_-)$ embeds via p_-^*, and the embedding of the orthogonal follows from [24, Lemma 2.3].

We then obtain the required semiorthogonal decomposition of $\mathcal{P}_0$ with component $D(X_-)$ from (9) by following the argument for the 3-fold case. Note in particular that the expression (7) for $\mathcal{N}_{E|X}$ holds verbatim.
Case (+): We argue similarly, considering $E_+ = \mathbb{P}^1 \subset X_+$, and taking a line bundle $\mathcal{G}$ on X_+ with a section σ cutting out E_+. For this case we take $\mathrm{Tot}(\mathcal{G} \oplus \mathcal{O})$ with a $\mathbb{C}^*$-action

of weight -2 on the fibres of $\mathcal{G}$, denote fibre coordinates as above, and consider the substack

$$[N/\mathbb{C}^*] := \{vz^2 = \pi^*\sigma\} \subset \mathrm{Tot}(\mathcal{G} \oplus \mathcal{O}).$$

By the arguments of [11, Sect. 5.2], we see that $[N/\mathbb{C}^*]$ has a GIT quotient X_+. We claim furthermore that it has X_B as a GIT quotient.

For this, first note that X_+ may be described as a $\mathbb{C}^*$-quotient of $\mathbb{C}^3$ with weights $\begin{pmatrix}1 & 1 & -2\end{pmatrix}$, coordinates (x_1, x_2, y), and semistables $\{(x_1, x_2) \neq 0\}$. Under this description $[N/\mathbb{C}^*]$ is the locus $\{(x_1, x_2) \neq 0,\ vz^2 = y\}$ in the stack $[\mathbb{C}^5/\mathbb{C}^{*2}]$ given as follows.

$$\begin{matrix} x_1 & x_2 & y & v & z \end{matrix}$$
$$\begin{pmatrix} 1 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{pmatrix}$$

By projecting away from y, this is isomorphic to the left-hand stack $[\mathbb{C}^4/\mathbb{C}^{*2}]$ below. By row and column operations, this is isomorphic to the right-hand one, which can be seen to have a GIT quotient $X_B = [\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-1)/\mathbb{Z}_2]$. By inspection, the variation of GIT for $[N/\mathbb{C}^*]$ induces the map $p_+ \colon X_B \to X_+$.

$$\begin{matrix} x_1 & x_2 & v & z \end{matrix} \qquad\qquad \begin{matrix} x_1 & x_2 & z & v \end{matrix}$$
$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

We then again use the argument of [11, Sect. 5.2] to obtain a semi-orthogonal decomposition of $\mathcal{P}_0$ with component $D(X_+)$. This proceeds as above, except that $\mathcal{C}_{2\cdot \mathrm{rk}\,\mathcal{G}}$ takes the place of $\mathcal{C}_{\mathrm{rk}\,\mathcal{F}}$. In particular, the multiplicity of components $D(E_+)$ in the semiorthogonal decomposition is now given by $2 \cdot \mathrm{rk}\,\mathcal{G} - 1 = 2 \cdot 1 - 1 = 1$, as required.

Finally we show the schober conditions. The required equivalences between $D(X_-)$ and $D(X_+)$ are an instance of the main theorem of Bridgeland–King–Reid [10]. The proof of the equivalences between orthogonals goes through as for the 3-fold case, using the expression (7) for $\mathcal{N}_{E|X}$ as before, which applies verbatim. □

4.4. A-side constructions. Here we construct and study the functors in the diagram (C) of categories of wrapped constructible sheaves from the introduction. This is used, in the next section, to prove our homological mirror symmetry statement in Theorem A.

For brevity we use notation as follows.

$$\Lambda_0 := \Lambda_{\Sigma_0}$$
$$\Lambda_\pm := \Lambda_{\Sigma_\pm}$$
$$\Lambda_B := \Lambda_{\Sigma_B}$$

Note that Σ_B is a refinement of $\Sigma_\pm$, so there exist an inclusion $\Lambda_\pm \subset \Lambda_B$, and thence

$$\Lambda_- \cup \Lambda_+ \subset \Lambda_B.$$

Remark 10. The skeleton $\Lambda_- \cup \Lambda_+$ is not represented as Λ_Σ for some Σ. It would be interesting to study whether it naturally arises in some other way, for instance from a Landau–Ginzburg model.

Using that $\Lambda_\pm \subset \Lambda_- \cup \Lambda_+$, there exist natural embeddings:

$$\mathbf{Sh}^{\diamond}_{\Lambda_\pm}(T) \hookrightarrow \mathbf{Sh}^{\diamond}_{\Lambda_- \cup \Lambda_+}(T) \tag{10}$$

We want to define such a functor on the level of wrapped constructible sheaf categories. For this, recall the following.

i. Using smoothness of $X_\pm$ and X_B, Proposition 2 gives functors

$$\mathbf{Sh}^{w}_{\Lambda_\pm}(T) \hookrightarrow \mathbf{Sh}^{w}_{\Lambda_B}(T).$$

ii. Using the inclusion $\Lambda_- \cup \Lambda_+ \subset \Lambda_B$, Proposition 3 gives a functor

$$\iota^w \colon \mathbf{Sh}^{w}_{\Lambda_B}(T) \to \mathbf{Sh}^{w}_{\Lambda_- \cup \Lambda_+}(T).$$

Definition 12. Composing (i) and (ii) immediately above, we write

$$\mathfrak{p}^*_\pm \colon \mathbf{Sh}^{w}_{\Lambda_\pm}(T) \to \mathbf{Sh}^{w}_{\Lambda_- \cup \Lambda_+}(T).$$

Using Proposition 3, we also have functors

$$\mathfrak{p}_{\pm !} \colon \mathbf{Sh}^{w}_{\Lambda_- \cup \Lambda_+}(T) \to \mathbf{Sh}^{w}_{\Lambda_\pm}(T).$$

Remark 11. Following the convention in Sect. 1.7, we use the notations $\mathfrak{p}^*_\pm$ and $\mathfrak{p}_{\pm !}$ because these functors will turn out to be mirror to $p^*_\pm$ and $p_{\pm !}$ between $D(X_\pm)$ and $\mathcal{P}_0$.

Proposition 6. *The functor $\mathfrak{p}^*_\pm$ is isomorphic to the restriction of the embedding (10), namely*

$$\mathbf{Sh}^{\diamond}_{\Lambda_\pm}(T) \hookrightarrow \mathbf{Sh}^{\diamond}_{\Lambda_- \cup \Lambda_+}(T),$$

to the full subcategory $\mathbf{Sh}^{w}_{\Lambda_\pm}(T)$. In particular, $\mathfrak{p}^*_\pm$ is an embedding.

Proof. By functoriality, composing (10) with the embedding

$$\iota \colon \mathbf{Sh}^{\diamond}_{\Lambda_- \cup \Lambda_+}(T) \hookrightarrow \mathbf{Sh}^{\diamond}_{\Lambda_B}(T)$$

gives the embeddings $\iota_\pm \colon \mathbf{Sh}^{\diamond}_{\Lambda_\pm}(T) \hookrightarrow \mathbf{Sh}^{\diamond}_{\Lambda_B}(T)$. Writing ι^{LA} for a left adjoint, we have $\iota^{\mathsf{LA}} \circ \iota \simeq \mathrm{id}$. Therefore (10) is isomorphic to the composition of $\iota_\pm$ with ι^{LA}. Restricting the latter composition to $\mathbf{Sh}^{w}_{\Lambda_\pm}(T)$ gives $\mathfrak{p}^*_\pm$ by definition. □

Claim. Embeddings $\mathfrak{p}^*_\pm$ give a spherical pair, which is mirror to the B-side spherical pair from Sect. 4.3.

We will prove this claim by using mirror symmetry in the next section.

5. Mirror Equivalences

In this section we prove homological mirror symmetry statements for the A-side and B-side schobers of the previous section, obtaining proofs of Theorems A and B.

5.1. Threefold proof. Theorem 4 gives equivalences

$$\kappa_{\Sigma_B} : D(X_B) \xrightarrow{\sim} \mathbf{Sh}^w_{\Lambda_B}(T) \qquad \kappa_{\Sigma_\pm} : D(X_\pm) \xrightarrow{\sim} \mathbf{Sh}^w_{\Lambda_\pm}(T)$$

which we write as κ_B and $\kappa_\pm$ for brevity.

Mirror symmetry at the conifold point Let us consider the composition

$$\kappa_{\mathcal{P}} : \mathcal{P}_0 \hookrightarrow D(X_B) \xrightarrow{\kappa_B} \mathbf{Sh}^w_{\Lambda_B}(T) \xrightarrow{\iota^w} \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T).$$

Proposition 7. *$\kappa_{\mathcal{P}}$ is fully faithful.*

Proof. Let us consider the precomposition of the inclusion $D(X_\pm) \hookrightarrow \mathcal{P}_0$ with $\kappa_{\mathcal{P}}$ and denote it $\kappa_{\mathcal{P}\pm}$. Then by the functoriality in Proposition 1, this is the same as the composition of $\kappa_\pm$ and the functor $\mathfrak{q}^*_\pm : \mathbf{Sh}^w_{\Lambda_\pm}(T) \hookrightarrow \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T)$ constructed in Sect. 4.4. Hence $\kappa_{\mathcal{P}\pm}$ is fully faithful. Since $D(X_\pm)$ jointly generate $\mathcal{P}_0$ by definition, the functor $\kappa_{\mathcal{P}}$ is also fully faithful. □

Take a point $(x, \xi) \in \Omega_T$ which lies in $\Lambda_- \cup \Lambda_+$, but not in Λ_-. Let $\mathcal{F}_{x,\xi}$ be a microlocal skyscraper sheaf.

Remark 12. Skyscraper sheaves are exceptional objects. However microlocal skyscraper sheaves are not exceptional in general.

This becomes clearer if one recalls that a microlocal skyscraper sheaf depends on the ambient category $\mathbf{Sh}^w_\Lambda(T)$, i.e. it is defined as a representing object of a local cohomology functor (a *microstalk* functor). Hence the correct analogue is a stalk functor, but not a skyscraper sheaf. Then the stalk functor on $\mathbf{Sh}^w_\Lambda(T)$ is also not represented by an exceptional object.

However, in our case,

Lemma 2. *$\mathcal{F}_{x,\xi}$ is exceptional.*

Proof. Let us consider the locally closed subset D in $M_\mathbb{R}$, as shown in Fig. 6, consisting of those $m \in M_\mathbb{R}$ such that the following hold.

$$\langle m, e_1 + e_3 \rangle \le -1 \quad \langle m, e_2 \rangle \ge 0 \quad \langle m, e_1 \rangle > -1 \quad \langle m, -e_2 + e_3 \rangle > -1$$

The 1-cell Δ of D cut out by

$$\langle m, e_1 + e_3 \rangle = -1 \quad \langle m, e_2 \rangle = 0$$

has conormal $(\Lambda_- \cup \Lambda_+) \setminus \Lambda_-$, and it follows that the sheaf $\pi_! \mathbb{C}_D$ gives $\mathcal{F}_{x,\xi}$ by non-characteristic deformation as in [35]. Since $\mathbb{R}\mathrm{Hom}(\pi_! \mathbb{C}_D, \pi_! \mathbb{C}_D)$ is the microstalk of $\pi_! \mathbb{C}_D$, which is rank 1 and degree 0 from the picture below, the claim follows. □

Proposition 8.

$$\mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) \simeq \left\langle \mathbf{Sh}^w_{\Lambda_-}(T), \mathcal{F}_{x,\xi} \right\rangle.$$

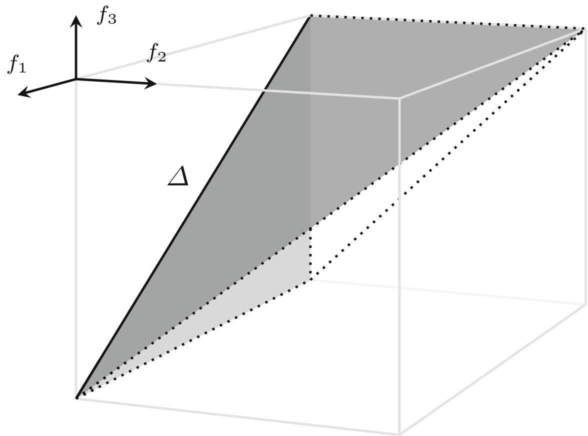


Fig. 6. The region D in $M_{\mathbb{R}}$: the basis $\{f_i\}$ is dual to $\{e_i\}$, and shaded faces are included in D

Proof. Since $\mathcal{F}_{x,\xi}$ is exceptional, we have a semi-orthogonal decomposition

$$\mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T) \simeq \langle \mathcal{C}, \mathcal{F}_{x,\xi} \rangle$$

where $\mathcal{C}$ is the left orthogonal of $\mathcal{F}_{x,\xi}$. By the definition of microlocal skyscraper, $\mathcal{C} \subset \mathbf{Sh}^{\diamondsuit}_{\Lambda_-}(T)$.

Since we now have an inclusion $\mathbf{Sh}^w_{\Lambda_-}(T) \subset \mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T)$, we also have $j\colon \mathbf{Sh}^w_{\Lambda_-}(T) \hookrightarrow \mathcal{C}$. So it suffices to show that this latter inclusion is essentially surjective.

By the definition of semi-orthogonal decomposition, we have a left adjoint

$$i^{\mathsf{LA}}\colon \mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T) \to \mathcal{C}$$

of $i\colon \mathcal{C} \hookrightarrow \mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T)$. On the other hand, we also have a composition of the inclusion $j\colon \mathbf{Sh}^w_{\Lambda_-}(T) \hookrightarrow \mathcal{C}$ and

$$\iota^w\colon \mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T) \to \mathbf{Sh}^w_{\Lambda_-}(T)$$

which is the left adjoint of $\iota\colon \mathbf{Sh}^w_{\Lambda_-}(T) \hookrightarrow \mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T)$. Then we have, for any $\mathcal{E} \in \mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T)$ and $\mathcal{F} \in \mathcal{C}$,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(j \circ \iota^w(\mathcal{E}), \mathcal{F}) &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda_-}}(\iota^w(\mathcal{E}), \mathcal{F}) \\
&\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda_-\cup\Lambda_+}}(\mathcal{E}, \mathcal{F}) \\
&\simeq \mathrm{Hom}_{\mathbf{Sh}^{w}_{\Lambda_-\cup\Lambda_+}}(\mathcal{E}, i(\mathcal{F})) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(i^L(\mathcal{E}), \mathcal{F}).
\end{aligned}$$

Hence we have $i^{\mathsf{LA}} \simeq j \circ \iota^w$. Since i^{LA} is essentially surjective, we can conclude that j is also essentially surjective. This completes the proof. □

Remark 13. We also have a semi-orthogonal decomposition of the form

$$\mathbf{Sh}^w_{\Lambda_-\cup\Lambda_+}(T) \simeq \langle \mathbf{Sh}^w_{\Lambda_+}(T), \mathcal{F}_{x,\xi} \rangle$$

where $(x, \xi) \in (\Lambda_- \cup \Lambda_+)\backslash\Lambda_-$. In this case, $F_{x,\xi}$ is $\pi_!\mathbb{C}_{D'}$ where D' is the image of D under the reflection $e_1 \leftrightarrow e_2$.

Remark 14. We expect these semi-orthogonal decompositions to be related to those obtained by Ganatra–Pardon–Shende using their 'forward stopped' criterion [20, Sect. 1.7], but did not investigate further.

We also have a semiorthogonal decomposition $\mathcal{P}_0 \simeq \langle D(X_-), D(\mathrm{pt})\rangle$ from Proposition 4. Note that the functor κ_B restricts to the equivalence κ_-.

Proposition 9. *$\kappa_{\mathcal{P}}$ is an equivalence.*

Proof. We have already shown that $\kappa_{\mathcal{P}}$ is fully faithful, and that its restricts on $D(X_-)$ to an equivalence $D(X_-) \to \mathbf{Sh}^w_{\Lambda_-}(T)$. Using the semiorthogonal decompositions above we deduce that the restriction of $\kappa_{\mathcal{P}}$ to the orthogonal $D(\mathrm{pt})$ of $D(X_-)$ gives a functor

$$\kappa_{\mathrm{pt}} \colon D(\mathrm{pt}) \to \left\langle \mathcal{F}_{x,\xi} \right\rangle.$$

This is fully faithful, therefore via the equivalence $\left\langle \mathcal{F}_{x,\xi} \right\rangle \simeq D(\mathrm{pt})$ it is isomorphic to a homological shift $[s]$. Hence κ_{pt} is essentially surjective, and so the same holds for $\kappa_{\mathcal{P}}$, completing the proof. □

Relating mirror schobers We use Proposition 9 above to conclude equivalences of schobers.

We now have an opposite spherical pair

$$D(X_-) \underset{p_{-!}}{\overset{p_-^*}{\rightleftarrows}} \mathcal{P}_0 \underset{p_{+!}}{\overset{p_+^*}{\leftrightarrows}} D(X_+)$$

Let us set $\mathcal{P}_0' := \mathcal{P}_0 \otimes \omega_{X_B} \subset D(X_B)$. By tensoring with the canonical sheaves on each category of the above schober, we get an equivalent opposite spherical pair.

$$D(X_-) \underset{p_{-*}}{\overset{p_-^!}{\rightleftarrows}} \mathcal{P}_0' \underset{p_{+*}}{\overset{p_+^!}{\leftrightarrows}} D(X_+)$$

To show this we use the isomorphism

$$p_\pm^! \cong \omega_{X_B} \otimes p_\pm^*(\omega_{X_\pm}^{-1} \otimes -),$$

which holds because the relative dimension of $p_\pm$ is zero, and its left adjoint.

Theorem 5. *The opposite spherical pair*

$$D(X_-) \underset{p_{-*}}{\overset{p_-^!}{\rightleftarrows}} \mathcal{P}_0' \underset{p_{+*}}{\overset{p_+^!}{\leftrightarrows}} D(X_+)$$

is equivalent via the mirror symmetry functor $K_{\Sigma_\pm}$ to an opposite spherical pair as follows.

$$\mathbf{Sh}^w_{\Lambda_-}(T) \underset{q_{-!}}{\overset{q_-^*}{\rightleftarrows}} \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) \underset{q_{+!}}{\overset{q_+^*}{\leftrightarrows}} \mathbf{Sh}^w_{\Lambda_+}(T)$$

Proof. By the argument above, in particular using Theorem 4, Proposition 1, and Proposition 9, we have commutative squares as follows

$$\begin{array}{ccccc} D(X_-) & \xrightarrow{p_-^*} & \mathcal{P}_0 & \xleftarrow{p_+^*} & D(X_+) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathbf{Sh}^w_{\Lambda_-}(T) & \xrightarrow{q_-^*} & \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) & \xleftarrow{q_+^*} & \mathbf{Sh}^w_{\Lambda_+}(T) \end{array}$$

where the vertical arrows are the functors κ.

Taking left adjoints, we furthermore obtain the following diagram.

$$\begin{array}{ccccc} D(X_-) & \xleftarrow{p_{-!}} & \mathcal{P}_0 & \xrightarrow{p_{+!}} & D(X_+) \\ \uparrow \wr & & \uparrow \wr & & \uparrow \wr \\ \mathbf{Sh}^w_{\Lambda_-}(T) & \xleftarrow{q_{-!}} & \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) & \xrightarrow{q_{+!}} & \mathbf{Sh}^w_{\Lambda_+}(T) \end{array}$$

Since the upper lines of the diagrams give the data of a weak spherical pair and satisfy the condition to be a spherical pair, we deduce that the lower lines give the data of an equivalent spherical pair. Note that the associated semi-orthogonal decompositions are given in Proposition 8 and Remark after the proposition.

Composing the tensor of the inverse of canonical sheaves with the above schober equivalence, we have

$$\begin{array}{ccccc} D(X_-) & \xrightarrow{p_-^!} & \mathcal{P}_0' & \xleftarrow{p_+^!} & D(X_+) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathbf{Sh}^w_{\Lambda_-}(T) & \xrightarrow{q_-^*} & \mathbf{Sh}^w_{\Lambda_- \cup \Lambda_+}(T) & \xleftarrow{q_+^*} & \mathbf{Sh}^w_{\Lambda_+}(T) \end{array}$$

where the vertical arrows are $\kappa \circ (- \otimes \omega^{-1}) = K$. □

Let $\mathbb{D} := \mathcal{H}om(-, \omega)$ be the Grothendieck duality. Then

$$\begin{aligned} \mathbb{D} \circ p^! \circ \mathbb{D} &\simeq \mathbb{D} \circ \left(p^* \mathcal{H}om(-, \omega) \otimes \omega \otimes p^* \omega^{-1}\right) \\ &\simeq \mathbb{D} \circ \left(p^*(-)^\vee \otimes p^* \omega \otimes \omega \otimes p^* \omega^{-1}\right) \\ &\simeq \mathbb{D} \circ \left(p^*(-)^\vee \otimes \omega\right) \\ &\simeq \mathcal{H}om\left(p^*(-)^\vee \otimes \omega, \omega\right) \\ &\simeq \left(p^*(-)^\vee\right)^\vee \\ &\simeq p^*. \end{aligned}$$

Therefore applying $\mathbb{D}$ to the opposite spherical pair given by the top line in the diagram below

$$\begin{array}{ccccc} D(X_-) & \xrightarrow{p_-^!} & \mathcal{P}_0' & \xleftarrow{p_+^!} & D(X_+) \\ \downarrow \mathbb{D} & & \downarrow \mathbb{D} & & \downarrow \mathbb{D} \\ D(X_-) & \xrightarrow{p_-^*} & \mathbb{D}\mathcal{P}_0' & \xleftarrow{p_+^*} & D(X_+) \end{array}$$

we get an anti-equivalent spherical pair. By the construction, $\mathbb{D}\mathcal{P}_0'$ is the image of the functors $p_\pm^*$ and hence $\mathbb{D}\mathcal{P}_0' = \mathcal{P}_0$. Consequently, we have a spherical pair as follows.

$$D(X_-) \underset{p_{-*}}{\overset{p_-^*}{\rightleftarrows}} \mathcal{P}_0 \underset{p_{+*}}{\overset{p_+^*}{\leftrightarrows}} D(X_+)$$

Combining this with Theorem 2 we obtain the following.

Theorem 6. *In the* 3*-fold setting of Sect.* 4.2*, there exists a spherical pair*

$$\mathcal{W}_{\Lambda_-^\infty}(\Omega_T) \underset{q_{-*}}{\overset{q_-^*}{\rightleftarrows}} \mathcal{W}_{\Lambda_-^\infty \cup \Lambda_+^\infty}(\Omega_T) \underset{q_{+*}}{\overset{q_+^*}{\leftrightarrows}} \mathcal{W}_{\Lambda_+^\infty}(\Omega_T)$$

which is equivalent via mirror symmetry to the spherical pair as follows.

$$D(X_-) \underset{p_{-*}}{\overset{p_-^*}{\rightleftarrows}} \mathcal{P}_0 \underset{p_{+*}}{\overset{p_+^*}{\leftrightarrows}} D(X_+)$$

Proof. Note that $\Lambda_-^\infty \cup \Lambda_+^\infty = (\Lambda_- \cup \Lambda_+)^\infty$ by the definition. We take the functors between Fukaya categories as the compositions of the functor in Theorem 2 with the functors between wrapped constructible sheaves. The commutativity and equivalences follow from this description. □

Remark 15. The functors $q_{\pm *}$ between Fukaya categories are isomorphic to stop removal functors as given in [21]. The functors $q_\pm^*$ are a bit more subtle, as we used the smoothness of mirrors to construct them. A Fukaya-categorical description may be as follows: the image of a Lagrangian submanifold is its result under the Reeb flow until it stops. However, making this claim precise is a problem.

5.2. Surface proof. We return to the setting of Sect. 4.1, in particular we use notation as follows.

$$X_- = [\mathbb{C}^2/\mathbb{Z}_2] \qquad X_+ = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$$

For a toric Deligne–Mumford stack, we have to generalize the definition of Λ_Σ. We only describe the result:

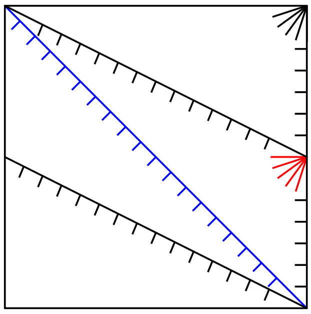
$$\Lambda_- := \Lambda_0 \cup (\Lambda_0 + \tfrac{1}{2} \cdot [e_1^\vee])$$
$$\Lambda_B := \Lambda_+ \cup (\Lambda_+ + \tfrac{1}{2} \cdot [e_1^\vee])$$

where $+\frac{1}{2} \cdot [e_1^\vee]$ means the translation by the class $\frac{1}{2} \cdot [e_1^\vee] \in T = M_\mathbb{R}/M$.

As before, let us take a point $(x, \xi) \in \Omega_T$ which lies in $\Lambda_- \cup \Lambda_+$, but not in Λ_-. Let $\mathcal{F}_{x,\xi}$ be a microlocal skyscraper sheaf. Then the following is the analogue of Lemma 2.

Lemma 3. *$\mathcal{F}_{x,\xi}$ is exceptional.*

Proof. Again, we have an explicit description.



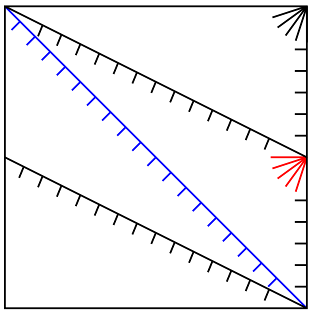
We take the constant sheaf on the shaded triangle, where the blue side is included, and other sides are not. □

By the same argument as in the conifold case, we have the following analogue of Proposition 8.

Proposition 10.

$$\mathbf{Sh}^{w}_{\Lambda_- \cup \Lambda_+}(T) \simeq \left\langle \mathbf{Sh}^{w}_{\Lambda_-}(T), \mathcal{F}_{x,\xi} \right\rangle.$$

Remark 16. The skeleton Λ_+ is the usual FLTZ skeleton, as shown in Fig. 2. Again, we have a microlocal skyscraper on a point in $(\Lambda_- \cup \Lambda_+)\backslash\Lambda_+$ as follows:



We can prove a semi-orthogonal decomposition for this case:

$$\mathbf{Sh}^{w}_{\Lambda_- \cup \Lambda_+}(T) \simeq \langle \mathbf{Sh}^{w}_{\Lambda_+}(T), \mathcal{F}_{x',\xi'} \rangle.$$

In the same way as in the conifold case, we can get a mirror equivalence between schobers, as follows.

Theorem 7. *In the surface setting of Sect. 4.1, the analog of Theorem 6 holds.*

5.3. Result for flober. The coherent–constructible correspondence and an argument similar to the one presented in the last two subsections also gives a mirror equivalence of flobers. Let us only describe the result. We have an opposite flober

$$D(X_-) \underset{p_{-*}}{\overset{p_-^!}{\rightleftarrows}} D(X_B) \underset{p_{+*}}{\overset{p_+^!}{\leftrightarrows}} D(X_+)$$

which is anti-equivalent to a flober as follows.

$$D(X_-) \underset{p_{-*}}{\overset{p_-^*}{\rightleftarrows}} D(X_B) \underset{p_{+*}}{\overset{p_+^*}{\leftrightarrows}} D(X_+)$$

We denote the latter one by $\mathcal{P}_B$.

On the other hand, the coherent–constructible correspondence K takes the former opposite flober to another flober

$$\mathbf{Sh}^w_{\Lambda_-}(T) \underset{q_{-!}}{\overset{q_-^*}{\rightleftarrows}} \mathbf{Sh}^w_{\Lambda_B}(T) \underset{q_{-!}}{\overset{q_+^*}{\leftrightarrows}} \mathbf{Sh}^w_{\Lambda_+}(T)$$

and then Theorem 2 takes this to an anti-equivalent flober as follows.

$$\mathcal{W}_{\Lambda_-^\infty}(\Omega_T) \underset{q_{-*}}{\overset{q_-^*}{\rightleftarrows}} \mathcal{W}_{\Lambda_B^\infty}(\Omega_T) \underset{q_{-*}}{\overset{q_+^*}{\leftrightarrows}} \mathcal{W}_{\Lambda_+^\infty}(\Omega_T)$$

We denote this by $\mathcal{P}_A$. We then have the following.

Theorem 8. *The two flobers $\mathcal{P}_A$ and $\mathcal{P}_B$ are equivalent via mirror symmetry.*

6. Applications to Singularities

We first explain work of Bondal, Kapranov, and Schechtman calculating cohomology of the flober $\mathcal{P}_B$ in the 3-fold case, and explain how their method applies also to the surface case. We then give an analogous calculation for the flober $\mathcal{P}_A$, proving Proposition C and Corollary D.

6.1. B-side calculation. Bondal–Kapranov–Schechtman showed the following in our 3-fold setting, and for general 3-fold flops:

Proposition 11. [7, Proposition 2.12] *The flober $\mathcal{P}_B$ has the 2nd compact cohomology*

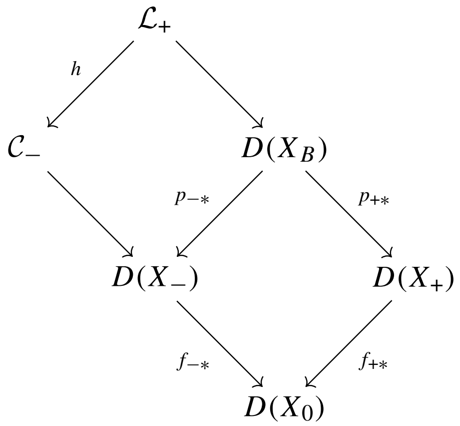
$$\mathbb{H}^2_c(\Delta, \mathcal{P}_B) \simeq D(X_0),$$

meaning that the diagram

$$\begin{array}{ccccc} & & D(X_B) & & \\ & \swarrow^{p_{-*}} & & \searrow^{p_{+*}} & \\ D(X_-) & & & & D(X_+) \\ & \searrow_{f_{-*}} & & \swarrow_{f_{+*}} & \\ & & D(X_0) & & \end{array} \tag{11}$$

is a push-out in the Morita model category of dg-categories, where the $f_\pm$ are the resolutions $X_\pm \to X_0$.

Let us recall their logic. First we extend the diagram to the following, defining $\mathcal{L}_+$ (respectively $\mathcal{C}_-$) to be the kernel of p_{+*} (respectively f_{-*}).

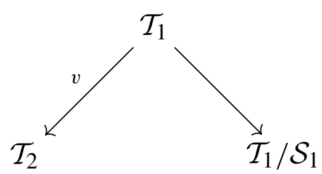


Lemma 4 [7, Theorem 2.14]. *In this situation, we have $D(X_B)/\mathcal{L}_+ \simeq D(X_+)$ and $D(X_-)/\mathcal{C}_- \simeq D(X_0)$.*

$$D(X_B)/\mathcal{L}_+ \simeq D(X_+) \quad \text{and} \quad D(X_-)/\mathcal{C}_- \simeq D(X_0).$$

We moreover have the following general proposition, which is a slight modification of [7, Lemma 2.17].

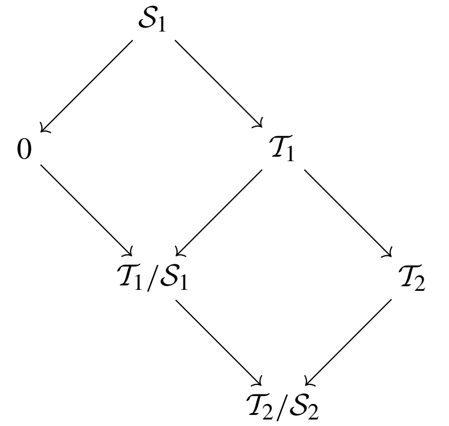
Proposition 12. *Let $u\colon \mathcal{S}_1 \to \mathcal{T}_1$ be a fully faithful dg-functor between two Karoubian pre-triangulated dg-categories and $v\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a dg-functor. Let $\mathcal{S}_2$ be the thick triangulated hull of objects of $v(\mathcal{S}_1)$. Then the push-out of*



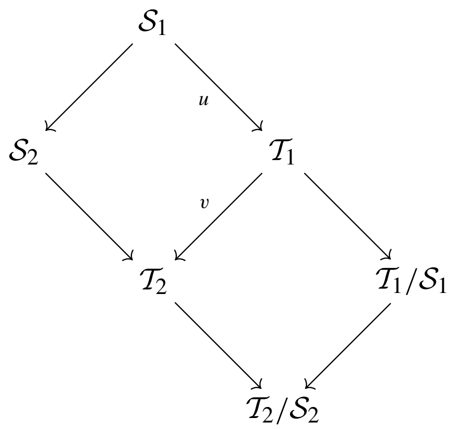
is Morita-equivalent to $\mathcal{T}_2/\mathcal{S}_2$.

Proof. Let $\mathcal{S}_2'$ be the full sub dg-category of $\mathcal{T}_2$ spanned by objects of $v(\mathcal{S}_1)$. Then [7, Lemma 2.17] says the desired push-out is quasi-equivalent to $\mathcal{T}_2/\mathcal{S}_2'$. Since $\mathcal{T}_2/\mathcal{S}_2'$ is Morita-equivalent to $\mathcal{T}_2/\mathcal{S}_2$ (for example, one can deduce it from [7, Lemma 2.16]), we complete the proof. □

Remark 17. An alternative short explanation of the above proposition was suggested by the referee. We have a commutative diagram as follows: the left-hand square and the rectangle are push-outs, so the right-hand square is also.



The situation of the above proposition is:



To apply this proposition to our situation, the following is sufficient.

Lemma 5 [7, Lemma 2.18]. *The functor* $h\colon \mathcal{L}_+ \to \mathcal{C}_-$ *is a split generation.*

This completes the logic to prove $D^b(X_0) \simeq$ push-out of (11) in the 3-fold case. We now explain how this method gives the same result for the surface case.

The argument for [7, Theorem 2.14], which is quoted as Lemma 4 above, is quite general, and applies in the setting of Deligne–Mumford stacks. It requires certain t-structures on X_B and X_- which are compatible with the contractions p_+ and f_- respectively. In the 3-fold case, we may take one of the t-structures ${}^p\mathrm{Per}$ of Bridgeland [9, Sect. 3]: for the surface case we may translate the same notion to the stacky setting.

The split generation claim for h, quoted in Lemma 5 above, may be checked explicitly in the surface case: using the notation in the commutative diagram (8), we have that $j_{-*}\mathcal{O}_{E_-}(-1)$ is a split generator of $\mathcal{C}_-$, and is the image of $i_* q_-^* \mathcal{O}_{E_-}(-1)$ under h.

6.2. A-side calculation. Let us now turn to the A-side. We would like to prove the following:

Proposition 13. *The A-model flober $\mathcal{P}_A$ has 2nd compact cohomology*

$$\mathbb{H}^2_c(\Delta, \mathcal{P}_A) \simeq \mathcal{W}_{\Lambda^\infty_- \cap \Lambda^\infty_+}(\Omega_T).$$

We would first like to set up a somewhat general situation. Let Z be a real analytic manifold. Let Λ be a conic Lagrangian in Ω_Z and Λ' be a closed conic Lagrangian subset of Λ. Then there exists a canonical inclusion

$$\iota\colon \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z) \to \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)$$

and a left adjoint $\iota^{\mathsf{LA}}\colon \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z) \to \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)$.

Notation 2. *Let $S_\Lambda(\Lambda \backslash \Lambda')$ be the set of microlocal skyscraper sheaves in $\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)$ over $\Lambda \backslash \Lambda'$ and $S_\Lambda(\Lambda')$ be the set of microlocal skyscraper sheaves in $\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)$ over Λ'.*

For brevity, we set

$$S := S_\Lambda(\Lambda \backslash \Lambda') \cup S_\Lambda(\Lambda') \quad \text{and} \quad R := S_\Lambda(\Lambda \backslash \Lambda').$$

Then ι^{LA} maps R to 0 and $S \backslash R$ to a set of generators of $\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)$. Then $\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z) = \langle S \rangle$ by Nadler's generation result [40]. By Thomason's localization theorem [43, Theorem 1.14], we have

$$\langle R \rangle^{\text{idem}} = \langle R \rangle \cap \mathbf{Sh}^{w}_{\Lambda}(Z)$$

where $\langle R \rangle^{\text{idem}}$ is the smallest thick subcategory containing R and

$$\left(\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R \rangle\right)^c \simeq \mathbf{Sh}^{w}_{\Lambda}(Z)/\langle R \rangle \cap \mathbf{Sh}^{w}_{\Lambda}(Z).$$

where $(-)^c$ is the full subcategory spanned by compact objects (more precisely $\aleph_0$-compact). Combining these, we have the following.

$$\left(\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R \rangle\right)^c \simeq \mathbf{Sh}^{w}_{\Lambda}(Z)/\langle R \rangle^{\text{idem}}$$

To prove the next lemma, we recall some facts.

Definition 13 (*Bousfield localization*). Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ be a thick subcategory. The Bousfield localization functor for the pair $(\mathcal{T}, \mathcal{S})$ is a right adjoint of the quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$.

Theorem 9 [43, Theorem 9.1.16]. *The Bousfield localization functor is fully faithful.*

Now we would like to prove the following:

Lemma 6.

$$\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R \rangle \simeq \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z).$$

Proof. It suffices to show this on the level of homotopy categories. Again, general nonsense tells us that there exists a right adjoint π_1^{RA} of the quotient $\pi_1 \colon \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z) \to \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle$ (cf. [43, Example 8.4.5]). Hence this is a Bousfield localization and hence fully faithful. For an object $\mathcal{E} \in \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle$ and an $r \in R$, we have

$$\mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)}(r, \pi_1^{\mathsf{RA}}(\mathcal{E})) \simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle}(\pi_1(r), \mathcal{E}) \simeq 0.$$

Hence $\mathrm{SS}(\pi_1^{\mathsf{RA}}(\mathcal{E})) \subset \Lambda'$ by the definition of microlocal skyscraper sheaves. So we have a fully faithful functor $\widetilde{\pi_1^{\mathsf{RA}}} \colon \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle \to \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)$ such that $\pi_1^{\mathsf{RA}} = \iota \circ \widetilde{\pi_1^{\mathsf{RA}}}$. We would like to see that this is essentially surjective.

Let us take $\mathcal{F} \in \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)$. It is enough to prove $\widetilde{\pi_1^{\mathsf{RA}}} \circ \pi_1 \circ \iota(\mathcal{F}) \simeq \mathcal{F}$ for our purpose. Letting π_2 be the functor $\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle \to \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)$ induced by ι^{LA}, which satisfies $\pi_2 \circ \pi_1 = \iota^{\mathsf{LA}}$.

First, let us see π_2 is the left adjoint of $\pi_1 \circ \iota$. Let us consider an object $\pi_1(\mathcal{G}) \in \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle$ which is represented by $\mathcal{G} \in \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)$. We have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}(\pi_2 \circ \pi_1(\mathcal{G}), \mathcal{F}) &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}(\iota^{\mathsf{LA}}(\mathcal{G}), \mathcal{F})\\ &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)}(\mathcal{G}, \iota(\mathcal{F}))\\ &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle}(\pi_1(\mathcal{G}), \pi_1 \circ \iota(\mathcal{F})).\end{aligned}$$

For the last equality, we used the fact that $\langle R\rangle$ is in the left orthogonal of $\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)$. Next, we can see $\pi_1 \circ \iota$ is fully faithful because of the following.

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle}(\pi_1 \circ \iota(\mathcal{F}'), \pi_1 \circ \iota(\mathcal{F})) &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}(\pi_2 \circ \pi_1 \circ \iota(\mathcal{F}'), \mathcal{F})\\ &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}(\iota^{\mathsf{LA}} \circ \iota(\mathcal{F}'), \mathcal{F})\\ &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}(\mathcal{F}', \mathcal{F})\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}\left(\mathcal{F}', \widetilde{\pi_1^{\mathsf{RA}}} \circ \pi_1 \circ \iota(\mathcal{F})\right) &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)}(\iota(\mathcal{F}'), \pi_1^{\mathsf{RA}} \circ \pi_1 \circ \iota(\mathcal{F}))\\ &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle}(\pi_1 \circ \iota(\mathcal{F}'), \pi_1 \circ \iota(\mathcal{F}))\\ &\simeq \mathrm{Hom}_{\mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z)}(\mathcal{F}', \mathcal{F}).\end{aligned}$$

Yoneda then completes the proof. □

Lemma 7. *The functor π_2 is an equivalence.*

Proof. From the proof above, we also see that $\pi_1 \circ \iota \colon \mathbf{Sh}^{\diamondsuit}_{\Lambda'}(Z) \to \mathbf{Sh}^{\diamondsuit}_{\Lambda}(Z)/\langle R\rangle$ is an equivalence. Since π_2 is the left adjoint of $\pi_1 \circ \iota$, we actually have

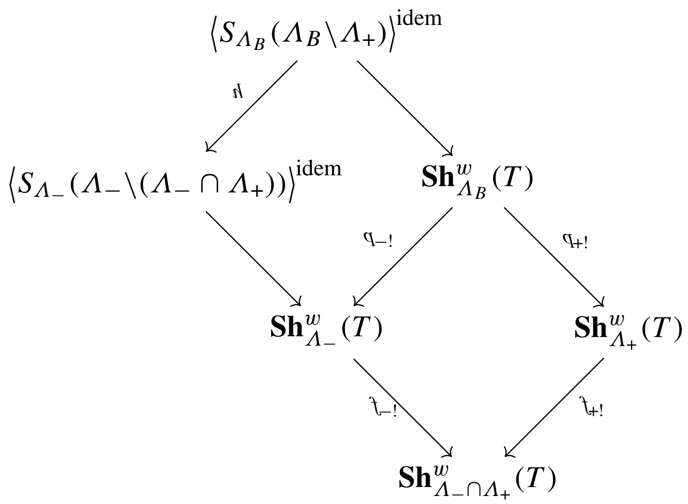
$$\pi_2 = \widetilde{\pi_1^{\mathsf{RA}}}.$$

$\pi_2 = \widetilde{\pi_1^{\mathsf{RA}}}$. □

Corollary 3. *The functor π_2 induces an equivalence as follows.*

$$\mathbf{Sh}^w_{\Lambda'}(Z) \simeq \mathbf{Sh}^w_{\Lambda}(Z)/\langle R\rangle^{\text{idem}}$$

Let us go back to our situation. We now have a diagram.

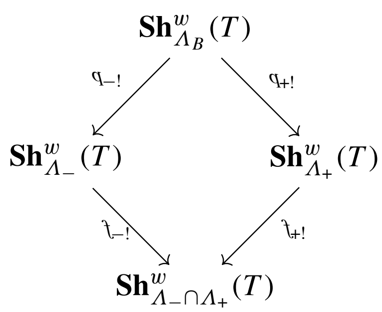


The two sequences from upper left to lower right are Verdier–Drinfeld quotients by Corollary 6.5. The functor $\mathfrak{s}$ is the restriction of $\mathfrak{q}_{-!}$ to the uppermost category. The functors $\mathfrak{l}_{\pm!}$ are the restrictions of the left adjoints of the inclusions $\mathbf{Sh}^w_{\Lambda_-\cap\Lambda_+}(T) \subset \mathbf{Sh}^w_{\Lambda_\pm}(T)$.

Lemma 8. *The category $\langle\Lambda_-\backslash(\Lambda_-\cap\Lambda_+)\rangle^{\text{idem}}$ is split-generated by the image of $\mathfrak{s}$.*

Proof. Note that $\mathfrak{q}_{-!}$ takes a microlocal skyscraper sheaf in $\mathbf{Sh}^w_{\Lambda_B}(T)$ over a point in Λ_- to a microlocal skyscraper sheaf in $\mathbf{Sh}^w_{\Lambda_-}(T)$ over the same point in Λ_- [40]. Note also that $\mathfrak{q}_{-!}$ takes a microlocal skyscraper sheaf in $\mathbf{Sh}^w_{\Lambda_B}(T)$ over a point in $\Lambda_B\backslash\Lambda_-$ to zero [40]. These imply the well-definedness of $\mathfrak{s}$. Since $\Lambda_B\backslash\Lambda_+ \supset \Lambda_-\backslash(\Lambda_-\cap\Lambda_+)$, these also imply the surjectivity of $\mathfrak{s}$ on the split generators. □

Repeating the logic presented in the beginning of the section, we can conclude



is a homotopy push-out in the Morita model. Noting that $(\Lambda_-\cap\Lambda_+)^\infty = \Lambda_-^\infty\cap\Lambda_+^\infty$ by the definition of taking infinity, Theorem 2 takes this diagram to another push-out diagram.

$$
\begin{array}{ccccc}
 & & \mathcal{W}_{\Lambda_B^\infty}(\Omega_T) & & \\
 & {\scriptstyle q_{-*}} \swarrow & & \searrow {\scriptstyle q_{+*}} & \\
\mathcal{W}_{\Lambda_-^\infty}(\Omega_T) & & & & \mathcal{W}_{\Lambda_+^\infty}(\Omega_T) \\
 & {\scriptstyle \iota_{-*}} \searrow & & \swarrow {\scriptstyle \iota_{+*}} & \\
 & & \mathcal{W}_{\Lambda_-^\infty \cap \Lambda_+^\infty}(\Omega_T) & &
\end{array}
$$

Here $\iota_{\pm *}$ is defined as the composition of the functor in Theorem 2 and $\iota_{\pm !}$. This implies Proposition 13.

On the other hand, Theorem 8 implies that $\mathbb{H}_c^2(\Delta, \mathcal{P}_A) \simeq \mathbb{H}_c^2(\Delta, \mathcal{P}_B)$. Combining with Proposition 13, we have homological mirror symmetry for singular varieties

Corollary 4.

$$D(X_0) \simeq \mathcal{W}_{\Lambda_-^\infty \cap \Lambda_+^\infty}(\Omega_T).$$

Remark 18. In this section, we used flobers to study the compact cohomology rather than spherical pairs. If one instead uses the spherical pairs from Theorem A, then one will arrive at the same conclusion as presented here, i.e. the flobers and spherical pairs have the same 2nd compact cohomology.

Remark 19. The method presented here might be generalized to another general proof of the coherent–constructible correspondence for singular toric varieties, as proved by the second author in [36].

Let X be a singular toric variety. Let S_X be the category consisting of

1. Objects: smooth toric Deligne–Mumford stacks refining X.
2. Morphisms: morphisms of stacks corresponding to refinement.

There exists a functor from S_X to the category of dg-categories defined by $X' \mapsto D(X')$, where morphisms are mapped to push-forwards along them.

Conjecture 1. The universality morphism gives an equivalence

$$\varinjlim_{X' \in S_X} D(X') \simeq D(X).$$

One can define the A-model counterpart, where morphisms are mapped to left adjoints of functors supplied by Proposition 2, as above, and conjecture as follows.

Conjecture 2. The universality morphism gives an equivalence

$$\varinjlim_{X' \in S_X} \mathbf{Sh}^w_{\Lambda_{X'}}(T) \simeq \mathbf{Sh}^w_{\Lambda_X}(T).$$

These two conjectures would allow us to conclude an equivalence $D(X) \simeq \mathbf{Sh}^w_{\Lambda_X}(T)$ from the smooth coherent–constructible correspondence.

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