UNFOLDED SEIBERG-WITTEN FLOER SPECTRA, II: RELATIVE INVARIANTS AND THE GLUING THEOREM

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We use the construction of unfolded Seiberg–Witten Floer spectrum of general 3-manifolds defined in our previous paper to extend the notion of relative Bauer–Furuta invariants to general 4-manifolds with boundary. One of the main purposes of this paper is to give a detailed proof of the gluing theorem for the relative invariants.

Contents

1. Introduction

Bauer–Furuta invariant, which was introduced in [2], can be regarded a stable homotopy refinement of the Seiberg–Witten invariants [15] for closed 4-manifolds. The invariant takes value in equivariant stable cohomotopy group of spheres and can give interesting application in 4-manifold theory, such as the 10/8-theorem [5]. On the other hand, Seiberg–Witten Floer spectrum, which was first introduced by Manolescu for rational homology 3-spheres [11], can be regarded as a stable homotopy refinement of monopole Floer homology [9]. Using this Seiberg–Witten Floer spectrum, Manolescu extended the notion of Bauer–Furuta invariant to 4-manifolds whose boundary are rational homology spheres. This "relative" invariant takes value in stable cohomotopy group of Seiberg–Witten Floer spectrum of the boundary manifold.

In the previous paper [7], we have constructed the "unfolded" version of Seiberg–Witten Floer spectrum for general 3-manifolds. It is then natural to extend Manolescu's construction of relative Bauer–Furuta invariant to arbitrary 4-manifold with boundary. Recall that the unfolded spectrum comes with two variations: type-A and type-R. Consequently, the unfolded relative Bauer–Furuta invariant will also come with type-A and type-R variations.

Let X be a compact, connected, oriented, 4–manifold with nonempty boundary $\partial X := Y$ not necessarily connected. It is, in fact, more convenient to consider X as a cobordism, i.e. we label each connected component of Y as either incoming or outgoing so that $Y = -Y_{\text{in}} \sqcup Y_{\text{out}}$. We often denote such a cobordism by $X\colon Y_{\text{in}} \to Y_{\text{out}}$. We equip X with a Riemannian metric $\hat{g}$, a spin^c structure $\hat{\mathfrak{s}}$ and a spin^c connection $\hat{A}_0$. Denote the restriction of $\mathfrak{s}, \hat{A}_0, \hat{g}$ to Y_{in} (resp. Y_{in}) by $\mathfrak{s}_{\text{in}}$, A_{in} and g_{in} (resp. $\mathfrak{s}_{\text{out}}$, A_{out} and g_{out}).

Theorem 1.1. *For a spin^c cobordism $X\colon Y_{in} \to Y_{out}$, the type-A unfolded relative Bauer–Furuta invariant of X can be constructed as a morphism in the stable category $\mathfrak{S}$*

$$\underline{\mathrm{bf}}^A(X, \hat{\mathfrak{s}}; S^1)\colon$$
$$\Sigma^{-(V_X^+ \oplus V_{in})} T(X, \hat{\mathfrak{s}}; S^1) \wedge \underline{\mathrm{swf}}^A(Y_{in}, \mathfrak{s}_{in}, A_{in}, g_{in}; S^1) \to \underline{\mathrm{swf}}^A(Y_{out}, \mathfrak{s}_{out}, A_{out}, g_{out}; S^1).$$

The type-R unfolded relative Bauer–Furuta invariant of X can be constructed analogously as a morphism in the stable category $\mathfrak{S}^*$

$$\underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}; S^1)\colon$$
$$\Sigma^{-(V_X^+ \oplus V_{out})} T(X, \hat{\mathfrak{s}}; S^1) \wedge \underline{\mathrm{swf}}^R(Y_{in}, \mathfrak{s}_{in}, A_{in}, g_{in}; S^1) \to \underline{\mathrm{swf}}^R(Y_{out}, \mathfrak{s}_{out}, A_{out}, g_{out}; S^1).$$

The object $T(X, \hat{\mathfrak{s}}; S^1)$ is the Thom spectrum of virtual index bundle associated to the Dirac operators.

Theorem 1.2. *As one varies $(\hat{g}, \hat{A}_0)$, both domain and target of $\underline{\mathrm{bf}}^A(X, \hat{\mathfrak{s}}; S^1)$ are changed by suspensding or desuspending same number of copies of $\mathbb{C}$; the morphism $\underline{\mathrm{bf}}^A(X, \hat{\mathfrak{s}}; S^1)$ is invariant as a stable homotopy class. Same result holds for $\underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}; S^1)$. Moreover, when $c_1(\mathfrak{s}|_Y)$ is torsion, one can construct further normalizations:*

$$\underline{\mathrm{BF}}^A(X, \hat{\mathfrak{s}}; S^1)\colon \Sigma^{-(V_X^+ \oplus V_{in})} T(X, \hat{\mathfrak{s}}; S^1) \wedge \underline{\mathrm{SWF}}^A(Y_{in}, \mathfrak{s}_{in}; S^1) \to \underline{\mathrm{SWF}}^A(Y_{out}, \mathfrak{s}_{out}; S^1).$$

$$\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}; S^1) \colon \Sigma^{-(V_X^+ \oplus V_{in})} T(X, \hat{\mathfrak{s}}; S^1) \wedge \underline{\mathrm{SWF}}^R(Y_{in}, \mathfrak{s}_{in}; S^1) \to \underline{\mathrm{SWF}}^R(Y_{out}, \mathfrak{s}_{out}; S^1),$$

which are completely metric/base-connection independent.

Remark. First, we emphasize that our unfolded relative invariant is defined over the *relative* Picard torus

$$\mathrm{Pic}^0(X, Y) \cong \ker(H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R})) / \ker(H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z})).$$

Secondly, the choice of labeling each boundary component corresponds to which side its unfolded spectrum will appear in the morphism. Essentially, $\underline{\mathrm{swf}}^A(Y)$ is the Spanier–Whitehead dual of $\underline{\mathrm{swf}}^R(-Y)$ and $\underline{\mathrm{bf}}^A(X)$ is the same as $\underline{\mathrm{bf}}^R(X^\dagger)$ where $X^\dagger \colon -Y_{\mathrm{out}} \to -Y_{\mathrm{in}}$ is the adjoint cobordism of $X \colon Y_{\mathrm{in}} \to Y_{\mathrm{out}}$. Finally, both $\underline{\mathrm{bf}}^A$ and $\underline{\mathrm{bf}}^R$ agree with Manolescu's construction when $b_1(Y) = 0$.

One of the main goals of the paper is to prove the gluing theorem for unfolded relative Bauer–Furuta invariants. When decomposing a 4-manifold X to two pieces along a 3-manifold Y, the gluing theorem can express the (relative) Bauer–Furuta invariant of X in term of a "product" of relative invariants of the two pieces. The case when $Y = S^3$ was first proved by Bauer [1] using only invariants of closed 4-manifolds and positive scalar curvature metric of S^3. The case when Y is a homology 3-sphere was proved by Manolescu [12]. Our setup and argument closely follow and generalize those of Manolescu.

Generally, our gluing theorem works when Y is any 3-manifold. Some mild homological assumptions will be made. These conditions are not essential in the sense that they can be removed under more generalized notion of category and unfolded spectrum (see the upcoming remark for more explanation). We now state the gluing theorem which will reappear in Section 6.1 with more details.

Theorem 1.3. *Let $X_0 \colon Y_0 \to Y_2$ and $X_1 \colon Y_1 \to -Y_2$ be connected, oriented cobordisms and $X \colon Y_0 \sqcup Y_1 \to \emptyset$ be the glued cobordism along Y_2. If the following conditions hold*

(i) Y_2 *is connected,*
(ii) $b_1(Y_0) = b_1(Y_1) = 0$,
(iii) $\mathrm{im}(H^1(X_0; \mathbb{R}) \to H^1(Y_2; \mathbb{R})) \subset \mathrm{im}(H^1(X_1; \mathbb{R}) \to H^1(Y_2; \mathbb{R}))$,

then, under the natural identification between domains and targets, one has

$$\mathrm{BF}(X)|_{\mathrm{Pic}^0(X, Y_2)} = \tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1)), \tag{1}$$

where $\tilde{\boldsymbol{\epsilon}}(\cdot, \cdot)$ is the Spanier-Whitehead duality operation defined in Section 4.3.

Remark. The main limitation of unfolded construction is that one can recover only the partial Bauer–Furuta invariant of X on the relative Picard torus $\mathrm{Pic}^0(X, Y_2)$ rather than the full Picard torus. Regarding the hypotheses of the theorem,

- Condition (ii) is to avoid dealing with type-A and type-R of $\underline{\mathrm{swf}}(Y_0)$, $\underline{\mathrm{swf}}(Y_1)$, and $\mathrm{BF}(X)$. If one tries to extend this direction, a category containing more general kinds of diagrams in $\mathfrak{C}$ will be required
- Condition (iii) is to control harmonic action of the relative gauge groups on Y_2. Otherwise, more generalized version of unfolded spectrum such as mixture of type-A and type-R will be needed.

Application of the gluing theorem will be focused on our subsequent paper [6]. Here we mention some examples.

- If a closed 4-manifold X contains an embedded sphere which is essential, framed with nonzero self-intersection, then $BF(X) = 0$
- Surgery on a loop on a closed 4-manifold does not change fiberwise Bauer–Furuta invariant.
- Computation of unfolded spectra of connected sum of 3-manifolds.

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2. Summary of constructions and proofs

Most required backgrounds in Conley theory are contained in Section 3. Backgrounds for our stable categories and Spanier–Whitehead duality are contained in Section 4. We summarize major constructions here.

2.1. **Unfolded Seiberg–Witten Floer spectra.** Here we will recall construction and definition of the unfolded Seiberg–Witten Floer spectrum [7]. Let Y be a closed spin^c 3-manifold (not necessarily connected) with a spinor bundle S_Y.

We always work on a Coulomb slice $Coul(Y) = \{(a, \phi) \in i\Omega^1(Y) \oplus \Gamma(S_Y) \mid d^*a = 0\}$ with Sobolev completion. With a basepoint chosen on each connected component, we identify residual gauge group with the based harmonic gauge group $\mathcal{G}^{h,o}_Y \cong H^1(Y; \mathbb{Z})$ acting on $Coul(Y)$. We consider a strip of balls in $Coul(Y)$ translated by this action

$$Str(R) = \{x \in Coul(Y) \mid \exists h \in \mathcal{G}^{h,o}_Y \text{ s.t. } \|h \cdot x\|_{L^2_k} \leq R\}. \tag{2}$$

The boundedness result for 3-manifolds states that all finite-type trajectories are contained in $Str(R)$ for R sufficiently large.

The basic idea of unfolded construction is to consider increasing sequences of bounded regions in the Coulomb slice. These regions are obtained from cutting $Str(R)$ by level sets of certian functions so that their boundaries are transverse to Seiberg–Witten flow in specific direction. Let $g_{j,\pm}$ be functions on $Coul(Y)$ which keep track of the $\mathcal{G}^{h,o}_Y$-action. Define bounded region

$$J^{\pm}_m := Str(\tilde{R}) \cap \bigcap_{1 \leq j \leq b_1} g^{-1}_{j,\pm}(-\infty, \theta + m], \tag{3}$$

where θ is some generic real number. Pick a sequence of finite-dimensional subspaces $V^{\mu_n}_{\lambda_n}$ coming from eigenspaces and define $J^{n,\pm}_m := J^{\pm}_m \cap V^{\mu_n}_{\lambda_n}$.

The main point is that $J^{n,\pm}_m$ becomes an isolating neighborhood with respect to the approximated Seiberg–Witten flow φ^n on $V^{\mu_n}_{\lambda_n}$ when n is large relative to m. We can now define desuspended Conley indices

$$\begin{aligned} I^{n,+}_m &= \Sigma^{-\bar{V}^0_{\lambda_n}} I(\mathrm{inv}(J^{n,+}_m), \varphi_n), \\ I^{n,-}_m &= \Sigma^{-V^0_{\lambda_n}} I(\mathrm{inv}(J^{n,-}_m), \varphi_n) \end{aligned} \tag{4}$$

as objects in the stable category $\mathfrak{C}$ (see Section 4). Here $\bar{V}^0_{\lambda_n}$ is the orthogonal complement of the space harmonic 1-forms in $V^0_{\lambda_n}$. Note that these objects does not depend on n up to canonical isomorphism of the form

$$\tilde{\rho}^{n,\pm}_m \colon I^{n,\pm}_m(Y) \to I^{n+1,\pm}_{m_1}(Y). \tag{5}$$

The unfolded Seiberg-Witten Floer spectra are represented by direct and inverse systems in the stable category $\mathfrak{C}$ as follows

$$\begin{aligned} &\underline{\mathrm{swf}}^A(Y) : I^+_1 \xrightarrow{j_1} I^+_2 \xrightarrow{j_2} \cdots \\ &\underline{\mathrm{swf}}^R(Y) : I^-_1 \xleftarrow{\bar{j}_1} I^-_2 \xleftarrow{\bar{j}_2} \cdots . \end{aligned} \tag{6}$$

Connecting morphisms in the diagram for $\underline{\mathrm{swf}}^A(Y)$ are induced by attractor relation while morphisms in $\underline{\mathrm{swf}}^R(Y)$ are induced by repeller relation. More precisely, we have morphisms between desuspended Conley indices

$$\tilde{i}^{n,+}_m \colon I^{n,+}_m(Y) \to I^{n,+}_{m+1}(Y) \text{ and } \tilde{i}^{n,-}_{m-1} \colon I^{n,-}_m(Y) \to I^{n,-}_{m-1}(Y).$$

Then, the morphisms $j_m, \bar{j}_m$ in (6) are given by composition of $\tilde{\rho}^{n,\pm}_m$'s and $\tilde{i}^{n,\pm}_m$ appropriately.

2.2. **Unfolded Relative Bauer–Furuta invariants.** Let X be a compact, connected, oriented, Riemannian 4–manifold with boundary $Y = -Y_{\mathrm{in}} \sqcup Y_{\mathrm{out}}$. We pick homological data which corresponds to a choice of basis of $H^1(X;\mathbb{R})$ and keeps track of both kernel and image of $\iota^* \colon H^1(X;\mathbb{R}) \to H^1(Y;\mathbb{R})$.

In this construction, we use the double Coulomb slice $Coul^{CC}(X)$ as a gauge fixing. The main idea is to find suitable finite-dimensional approximation for the Seiberg–Witten map together with the restriction map

$$(SW, r) \colon Coul^{CC}(X) \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \oplus Coul(Y). \tag{7}$$

Note that there is an action of $H^1(X;\mathbb{Z})$ on both sides with restriction on $Coul(Y)$. Compactness of solutions can only be achieved modulo this action. However, the construction of the unfolded spectra does not behave well under the action of $H^1(X;\mathbb{Z})$ on $Coul(Y)$. This is essentially reason we can define the unfolded relative invariant only on the relative Picard torus induced from $\ker \iota^*$. As one can see in the basic boundedness result (Theorem 5.9), we need a priori bound on the $\operatorname{im} i^*$-part quantified by the projection $\hat{p}_\beta$.

We will focus on type-A relative invariant $\underline{\mathrm{bf}}^A(X)$. Although it is formulated as a morphism from $\underline{\mathrm{swf}}^A(Y_{\mathrm{in}})$ to $\underline{\mathrm{swf}}^A(Y_{\mathrm{out}})$, the main part of the construction is to obtain maps of the form

$$B(W_{n,\beta})/S(W_{n,\beta}) \to (B(U_n)/S(U_n)) \wedge I(\mathrm{inv}(J^{n,-}_{m_0}(-Y_{\mathrm{in}}))) \wedge I(\mathrm{inv}(J^{n,+}_{m_1}(Y_{\mathrm{out}}))). \tag{8}$$

The left hand side is the Thom space of a finite-dimensional subbundle and $B(U_n)/S(U_n)$ is a sphere. We point out that the right hand side is intuitively $\underline{\mathrm{swf}}^R(-Y_{\mathrm{in}}) \wedge \underline{\mathrm{swf}}^A(Y_{\mathrm{out}})$, which may be viewed as a 'mixed'-type unfolded spectrum of Y. It is possible to formally consider this in a larger category containing both $\mathfrak{S}$ and $\mathfrak{S}^*$, but we will not pursue in this paper. Another remark is that $W_{n,\beta}$ has extra constraint $\hat{p}_{\beta,\mathrm{out}} = 0$ to control the

$\operatorname{im} i^*$-part mentioned earlier. The reason we only need the part on Y_{out} is because we start with a fixed m_0 and then choose sufficiently large m_1. The order of dependency of parameters is established at the beginning of Section 5.4.

A notion of pre-index pair (see Section 3.2) is also required to define the map (8). This part closely resembles original Manolescu's construction [11] in the case $b_1(Y) = 0$. The last step to to apply Spanier–Whitehead duality (see Section 4.4) between $\underline{\mathrm{swf}}^R(-Y_{\mathrm{in}})$ and $\underline{\mathrm{swf}}^A(Y_{\mathrm{in}})$ and define the relative invariant as a morphism in $\mathfrak{S}$.

2.3. **The Gluing theorem.** Let $X_0\colon Y_0 \to Y_2$ and $X_1\colon Y_1 \to -Y_2$ be connected, oriented cobordisms. We consider the composite cobordism $X = X_0 \cup_{Y_2} X_1$ glued along Y_2 from $Y_0 \sqcup Y_1$ to the empty manifold.

The main technical difficulty of the proof of the gluing theorem is that two different kinds of index pairs arise in the construction. On one hand, to define the relative invariant, we require an index pair (N_1, N_2) to contain a certain pre-index pair (K_1, K_2). On the other hand, we need a manifold isolating block when dealing with duality morphisms. In general, a canonical homotopy equivalence between index pairs can be given by flow maps (Theorem 3.4), but the formula can sometimes be inconvenient to work with and the common squeeze time T can be arbitrary.

This is the reason we introduce the concept of T-tameness, which is a quantitative refinement of notions in Conley theory (see Section 3.2 and 3.4). The flow maps from T-tame index pairs can be simplified (Lemma 3.13). Most boundedness results in this paper are stated for trajectories with finite length. As a result, the time parameter T, which also corresponds to the length of a cylinder, has a uniform bound during the construction.

The proof of the gluing theorem can be divided to two major parts. The first part, contained in Section 6.2, involves simplifying the flow maps and duality morphisms. We carefully set up all the parameters needed to explicitly write down $\tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1))$. For instance, we can represent Conley index part of the map as a composition of smash product of flow maps and Spanier–Whitehead duality map

$$\tilde{\boldsymbol{\epsilon}}(\iota_0, \iota_1)\colon K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^+ \wedge \tilde{N}_1/\tilde{N}_1^+ \wedge B^+(V_n^2, \bar{\epsilon})$$

given by formula (65). After two steps, we deform the formula to the one given in Proposition 6.9.

The second part of the proof of the gluing theorem, contained in Section 6.4, is to deform Seiberg–Witten maps on X_0 and X_1 to the Seiberg–Witten map on X. Many of the arguments here will be similar to Manolescu's proof [12] when $b_1(Y_2) = 0$. The crucial part is to deform gauge fixing with boundary conditions and harmonic gauge groups on X_0 and X_1 to those on X. For clarity, we subdivide deformation to seven steps. One of the most recurring technique is to move between maps and conditions on the domain (Lemma 6.12). Other ingredients such as stably c-homotopic pairs are contained in Section 6.3.

3. Conley Index

3.1. **Conley theory: definition and basic properties.** In this section, we recall basic facts regarding the Conley index theory. See [4] and [14] for more details. Note that all the results can be adapted to the G-equivariant theory.

Let Ω be a finite dimensional manifold and φ be a smooth flow on Ω, i.e. a C^∞-map $\varphi\colon \Omega \times \mathbb{R} \to \Omega$ such that $\varphi(x, 0) = x$ and $\varphi(x, s + t) = \varphi(\varphi(x, s), t)$ for any $x \in \Omega$ and $s, t \in \mathbb{R}$. We often denote by $\varphi(x, I) := \{\varphi(x, t) \mid t \in I\}$ for a subset $I \subset \mathbb{R}$.

Definition 3.1. Let A be a compact subset of Ω.

(1) The *maximal invariant subset* of A is given by $\operatorname{inv}(\varphi, A) := \{x \in A \mid \varphi(x, \mathbb{R}) \subset A\}$. We simply write $\operatorname{inv}(A)$ when the flow is clear from the context.
(2) A is called an *isolating neighborhood* if $\operatorname{inv}(A)$ is contained in $\operatorname{int}(A)$, the interior of A.
(3) A compact subset S of Ω is called an *isolated invariant set* if there is an isolating neighborhood $\tilde{A}$ such that $\operatorname{inv}(\tilde{A}) = S$. In this situation, we also say that A is an isolating neighborhood of S.

A central idea in Conley index theory is a notion of index pairs.

Definition 3.2. For an isolated invariant set S, a pair (N, L) of compact sets $L \subset N$ is called an *index pair* of S if the following conditions hold:

(i) $\operatorname{inv}(N \setminus L) = S \subset \operatorname{int}(N \setminus L)$;
(ii) L is an exit set for N, i.e. for any $x \in N$ and $t > 0$ such that $\varphi(x, t) \notin N$, there exists $\tau \in [0, t)$ with $\varphi(x, \tau) \in L$;
(iii) L is positively invariant in N, i.e. for $x \in L$ and $t > 0$, if we know $\varphi(x, [0, t]) \subset N$, then we have $\varphi(x, [0, t]) \subset L$.

We state two fundamental facts regarding index pairs:

- For an isolated invariant set S with an isolating neighborhood A, there always exists an index pair (N, L) of S such that $L \subset N \subset A$.
- For any two index pairs (N, L) and (N', L') of S, there is a natural homotopy equivalence $N/L \to N'/L'$.

These lead to definition of the Conley index.

Definition 3.3. Given an isolated invariant set S of a flow φ with an index pair (N, L), we denote by $I(\varphi, S, N, L)$ the space N/L with $[L]$ as the basepoint. The *Conley index* $I(\varphi, S)$ can be defined as a collection of pointed spaces $I(\varphi, S, N, L)$ together with natural homotopy equivalences between them. We sometimes write $I(S)$ when the flow is clear from the context.

Given two index pairs, precise formulation of natural homotopy equivalence is given by Salamon.

Theorem 3.4 ([14, Lemma 4.7])**.** *If (N, L) and (N', L') are two index pairs for the same isolated invariant set S, then there exists $\bar{T} > 0$ such that*

- $\varphi(x, [-\bar{T}, \bar{T}]) \subset N' \setminus L'$ *implies* $x \in N \setminus L$*;*
- $\varphi(x, [-\bar{T}, \bar{T}]) \subset N \setminus L$ *implies* $x \in N' \setminus L'$*.*

Moreover, for any $T \geq \bar{T}$, the map $s_T : N/L \to N'/L'$ given by

$$s_T([x]) := \begin{cases} [\varphi(x, 3T)] & \text{if } \varphi(x, [0, 2T]) \subset N \setminus L \text{ and } \varphi(x, [T, 3T]) \subset N' \setminus L' \\ [L'] & \text{otherwise} \end{cases}$$

is well-defined and continuous. For different $T \geq \bar{T}$, the maps s_T are all homotopic to each other and they give natural homotopy equivalence between N/L and N'/L'. We call s_T the flow map at time T.

Next, we consider a situation when an isolated invariant set can be decomposed to smaller isolated invariant sets.

Definition 3.5.
(1) For a subset A, we define
$$\alpha(A) = \bigcap_{t<0} \overline{\varphi(A, (-\infty, t])} \quad \text{and} \quad \omega(A) = \bigcap_{t>0} \overline{\varphi(A, [t, \infty))}.$$
(2) Let S be an isolated invariant set. A compact subset $T \subset S$ is called an *attractor* (resp. *repeller*) if there exists a neighborhood U of T in S such that $\omega(U) = T$ (resp. $\alpha(U) = T$).
(3) When T is an attractor in S, we define the set $T^* := \{x \in S \mid \omega(x) \cap T = \emptyset\}$, which is a repeller in S. We call (T, T^*) an *attractor-repeller pair* in S.

Note that an attractor and a repeller are isolated invariant sets. We state an important result relating Conley indices of an attractor-repeller pair.

Proposition 3.6 ([14, Theorem 5.7])**.** *Let S be an isolated invariant set with an isolating neighborhood A and (T, T^*) be an attractor-repeller pair in S. Then there exist compact sets $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ such that the pairs $(\tilde{N}_2, \tilde{N}_3), (\tilde{N}_1, \tilde{N}_3), (\tilde{N}_1, \tilde{N}_2)$ are index pairs for T, S and T^* respectively. The maps induced by inclusions give a natural coexact sequence of Conley indices*
$$I(\varphi, T) \xrightarrow{i} I(\varphi, S) \xrightarrow{r} I(\varphi, T^*) \to \Sigma I(\varphi, T) \to \Sigma I(\varphi, S) \to \cdots .$$
We call the triple $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ an index triple for the pair (T, T^*) and call the maps i and r the attractor map and the repeller map respectively.

3.2. **T-tame pre-index pair and T-tame index pair.** Let us introduce the following notation: For a set A and $I \subset \mathbb{R}$, we define
$$A^I := \{x \in \Omega \mid \varphi(x, I) \subset A\}.$$
We also write $A^{[0,\infty)}$ and $A^{(-\infty,0]}$ as A^+ and A^- respectively.

The following notion was introduced by Manolescu [11].

Definition 3.7. A pair (K_1, K_2) of compact subsets of an isolating neighborhood A is called *a pre-index pair* in A if
(i) For any $x \in K_1 \cap A^+$, we have $\varphi(x, [0, \infty)) \subset \mathrm{int}(A)$;
(ii) $K_2 \cap A^+ = \emptyset$.

We have two basic results regarding pre-index pairs.

Theorem 3.8 ([11, Theorem 4])**.** *For any pre-index pair (K_1, K_2) in an isolating neighborhood A, there exists an index pair (N, L) satisfying*
$$K_1 \subset N \subset A, \ K_2 \subset L. \tag{9}$$

Theorem 3.9 ([8, Proposition A.5])**.** *Let (K_1, K_2) be a pre-index pair and (N_1, L_2), (N_2, L_2) be two index pairs containing (K_1, K_2). Denote by $\iota_j \colon K_1/K_2 \to N_j/L_j$ the map induced by inclusion. Let $s_T \colon N_1/L_1 \to N_2/L_2$ be the flow map for some large T. Then, the composition $s_T \circ \iota_1$ is homotopic to ι_2.*

Consequently, when (K_1, K_2) is a pre-index pair in an isolating neighborhood A, we have a *canonical map* to Conley index

$$\iota \colon K_1/K_2 \to I(S), \tag{10}$$

where $S = \mathrm{inv}(A)$ and the map is induced by inclusion.

Next, we discuss the quantitative refinement of Theorem 3.8, which will be especially useful in many situations. Let us introduce the following definition.

Definition 3.10. Let A be an isolating neighborhood. For a positive real number T, a pair (K_1, K_2) of compact subsets of A is called a *T-tame pre-index pair* in A if it satisfies the following conditions:

(i) There exists a compact set $A' \subset \mathrm{int}(A)$ containing $A^{[-T,T]}$ such that, if $x \in K_1 \cap A^{[0,T']}$ for some $T' \geq T$, then $\varphi(x, [0, T' - T]) \subset A'$.
(ii) $K_2 \cap A^{[0,T]} = \emptyset$.

It is easy to see that a T-tame pre-index pair in A is pre-index pair in A. The converse also holds.

Lemma 3.11. *Let (K_1, K_2) be a pre-index pair in an isolating neighborhood A. Then, there exists $\bar{T} > 0$ such that (K_1, K_2) is a T-tame pre-index pair in A for any $T \geq \bar{T}$.*

Proof. It is easy to see that $K_2 \cap A^{[0,+\infty)} = \emptyset$ implies $K_2 \cap A^{[0,T]} = \emptyset$ for a sufficiently large $T > 0$. We are left with checking that condition (i) of Definition 3.10 holds for a sufficiently large $T > 0$.

Suppose that the condition does not hold for $T_j > 0$. Then we can find sequences $\{x_{j,k}\}$, $\{T'_{j,k}\}$ and $\{T''_{j,k}\}$ where $x \in K_1 \cap A^{[0,T'_{j,k}]}$ and $0 \leq T''_{j,k} \leq T'_{j,k} - T_j$ such that $\{\varphi(x_{j,k}, T''_{j,k})\}$ converges to a point on ∂A as $k \to \infty$. Now assume that there is a sequence of such $\{T_j\}$ with $T_j \to \infty$. Passing to a subsequence, one can find a sequence $\{k_j\}$ such that $x_{j,k_j} \to x_\infty \in K_1 \cap A^+$ and $\varphi(x_{j,k_j}, T''_{j,k_j}) \to y \in \partial A$. If $T''_{j,k_j} \to T''$, we see that $\varphi(x_\infty, T'') = y$. This contradicts with definition of the pre-index pair (K_1, K_2). On the other hand, we observe that $\varphi(x_{j,k_j}, T''_{j,k_j}) \in A^{[-T''_{j,k_j}, T_j]}$. If $\{T''_{j,k_j}\}$ goes to infinity, we obtain that $y \in \mathrm{inv}(A)$. This is a contradiction because A is an isolating neighborhood, i.e. $\mathrm{inv}(A) \cap \partial A = \emptyset$.

∎

We next consider the T-tame version of index pairs.

Definition 3.12. For a positive real number T, an index pair (N, L) contained in an isolating neighborhood A is called a *T-tame index pair* in A if it satisfies the following conditions:

(i) Both N, L are positively invariant in A;

(ii) $A^{[-T,T]} \subset N$;
(iii) $A^{[0,T]} \cap L = \emptyset$.

One important reason why we are interested in T-tame index pairs is that the definition of the flow maps can be simplified when one of the index pairs is T-tame.

Lemma 3.13. *Let (N, L) and (N', L') be two index pairs in an isolating neighborhood A. Let T be a sufficiently large number so that the flow map $s_T : N/L \to N'/L'$ is well-defined. If the index pair (N, L) is T-tame, then flow map s_T can be given by a formula*

$$s_T([x]) = \begin{cases} [\varphi(x, 3T)] & \text{if } \varphi(x, [0, 3T]) \subset A \text{ and } \varphi(x, [T, 3T]) \subset N' \setminus L', \\ [L'] & \text{otherwise.} \end{cases}$$

Proof. We only need to show that the following two conditions are equivalent for $x \in N$.
(1) $\varphi(x, [0, 3T]) \subset A$ and $\varphi(x, [T, 3T]) \subset N' \setminus L'$;
(2) $\varphi(x, [0, 2T]) \subset N \setminus L$ and $\varphi(x, [T, 3T]) \subset N' \setminus L'$.
It is easy to see that (2) implies (1) since $N \subset A$. Let us suppose that $\varphi(x, [0, 3T]) \subset A$. Since N is positively invariant in A, we have $\varphi(x, [0, 3T]) \subset N$. Since By the property of T-tame index pair, we have $\varphi(x, [0, 2T]) \cap L = \emptyset$ and we are done.

□

We now show a quantitative refinement of [11, Theorem 4].

Theorem 3.14. *For any $T > 1$, let A be a $(T-1)$-tame isolating neighborhood and (K_1, K_2) be a $(T-1)$-tame pre-index pair in A. Then, there exists a T-tame index pair in A which contains (K_1, K_2).*

Proof. The proof is an adaption of arguments in [11] to T-tame setting. Let us introduce the following notation beforehand: for a subset B, we define the set

$$P_A(B) := \{\varphi(x, t) \mid x \in B,\ t \geq 0 \text{ and } \varphi(x, [0, t]) \subset A\}.$$

Denote by $\tilde{K}_1 = K_1 \cup A^{[-T,T]}$. We claim that $(\tilde{K}_1, K_2)$ is a pre-index pair in A. Since (K_1, K_2) is also a pre-index pair in A, it suffices to check that $\varphi(y, [0, \infty)) \subset \mathrm{int}(A)$ for any $y \in A^{[-T,T]} \cap A^+ = A^{[-T,\infty]}$. This is straightforward since A is $(T-1)$-tame.

By Theorem 3.8, there exists an index pair containing $(\tilde{K}_1, K_2)$. More specifically, one picks a compact subset $C \subset A$ and chooses an open neighborhood V of C such that the following conditions hold:

(I) C is a compact neighborhood of $A^+ \cap \partial A$ in A;
(II) $C \cap A^- = \emptyset$;
(III) $C \cap P_A(\tilde{K}_1) = \emptyset$;
(IV) V is an open neighborhood of A^+ in A;
(V) $\overline{V \setminus C} \subset \mathrm{int}(A)$;
(VI) $K_2 \cap V = \emptyset$.

Let us say that a pair (C, V) is good if it satisfies all of the above conditions. After specifying a good pair (C, V), a compact subset B can be chosen so that

$$(N, L) = (P_A(B) \cup P_A(A \setminus V), P_A(A \setminus V))$$

is an index pair containing $(\tilde{K}_1, K_2)$. We will carefully choose a good pair (C, V) so that (N, L) is also a T-tame index pair in A which contains (K_1, K_2).

Since (K_1, K_2) is a T-tame pre-index pair (as $T > T-1$), we can take a compact set A' in A satisfying condition (i) of Definition 3.10. Fix a compact set A'' in A such that

$$A' \subset \operatorname{int}(A''),\ A'' \subset \operatorname{int}(A)$$

and pick a real number $T' \in (T-1, T)$. Consider a pair

$$(C_0, V_0) = ((A \setminus \operatorname{int}(A'')) \cap A^{[0,T']}, A^{[0,T']})$$

Note that V_0 is closed. We have the following observations:

- $A^+ \cap \partial A \subset C_0$; This is obvious as $A'' \subset \operatorname{int}(A)$ and $A^+ \subset A^{[0,T']}$.
- The distance between C_0 and A^- is positive; Observe that
$$C_0 \cap A^- = (A \setminus \operatorname{int}(A'')) \cap A^{(-\infty,T']} \subset (A \setminus \operatorname{int}(A'')) \cap A^{[-T+1,T-1]} = \emptyset,$$
where we have used the fact that $A^{[-T+1,T-1]} \subset A' \subset \operatorname{int}(A'')$. Since C_0 and A^- are compact, the distance between them is positive.
- The distance between C_0 and $P_A(\tilde{K}_1)$ is positive; Suppose that this is not true. Since C_0 is compact, there would be a sequence $\{x_j\}$ of points in $\tilde{K}_1$ and a sequence of nonnegative number $\{t_j\}$ such that $\varphi(x_j, [0, t_j]) \subset A$ and $y_j = \varphi(x_j, t_j)$ converges to a point y in C_0.
  If $t_j \to \infty$, we would have $\varphi(y, (-\infty, 0]) \subset A$, which means that $y \in A^-$. This is a contradiction since $C_0 \cap A^- = \emptyset$.
  After passing to a subsequence, we now assume that $(x_j, t_j) \to (x, t)$ a point in $\tilde{K}_1 \times [0, \infty)$. If $x \in K_1$, then $x \in K_1 \cap A^{[0,t+T']}$ because $\varphi(x, [0, t]) \subset A$ and $y = \varphi(x, t) \in C_0 \subset A^{[0,T']}$. By the property of A', we have
$$\varphi(x, [0, t + T' - (T-1)]) \subset A',$$
which implies that $y \in A'$. This is a contradiction since $C_0 \cap A' = \emptyset$. If $x \in A^{[-T,T]}$, then $y \in A^{[-T-t,T']} \subset A^{[-T+1,T-1]}$. This is also a contradiction since $C_0 \cap A^{[-T+1,T-1]} = \emptyset$.
- $A^+ \subset V_0$; This is clear from the definition of V_0.
- $\overline{V_0 \setminus C_0} \subset \operatorname{int}(A)$; We will actually prove that $\overline{V_0 \setminus C_0} \subset A''$. Since A'' is closed, it is sufficient to show that $V_0 \setminus C_0 \subset A''$. It is then straightforward to see that $V_0 \setminus C_0 = A^{[0,T']} \cap \operatorname{int}(A'') \subset A''$.
- The distance between K_2 and V_0 is positive; Since (K_1, K_2) is $(T-1)$-tame, we have $K_2 \cap A^{[0,T-1]} = \emptyset$, and consequently $K_2 \cap V_0 = \emptyset$. Since K_2 and V_0 are compact, the distance between them is positive.

For a sufficiently small positive number d, we define

$$C := \{x \in A | \operatorname{dist}(x, C_0) \le d\},\ V := \{x \in A | \operatorname{dist}(x, \tilde{V}_0) < d\}.$$

From the above observations, one can check that (C, V) is a good pair.

We finally check that $(N, L) = (P_A(B) \cup P_A(A \setminus V), P_A(A \setminus V))$ is T-tame.

(i) Notice that $P_A(S)$ is positively invariant for any subset $S \subset A$ and that the union of two positively invariant sets in A is again positively invariant in A. Thus, N, L are positively invariant in A

(ii) From our construction, we have $A^{[-T,T]} \subset \tilde{K}_1 \subset N$.

(iii) We are left to show that $A^{[0,T]} \cap L = \emptyset$. Suppose that there is an element $x \in A^{[0,T]} \cap L$. From the definition, we obtain $y \in A \setminus V$ and $t \geq 0$ such that $\varphi(y, [0,t]) \subset A$ and $x = \varphi(y,t)$. It follows that $y \in A^{[0,T+t]}$. On the other hand, we have $A^{[0,T+t]} \subset A^{[0,T']} = V_0 \subset V$. This is a contradiction since $y \notin \tilde{V}$.

□

3.3. **The attractor-repeller pair arising from a strong Morse decomposition.** In many situations, we obtain an attractor-repeller pair by decomposing an isolating neighborhood to two parts. Sometimes, a decomposition satisfies the following definition.

Definition 3.15. Let (A_1, A_2) be a pair of compact subsets of an isolating neighborhood A. We say that (A_1, A_2) is a *strong Morse decomposition of* A if

- $A = A_1 \cup A_2$;
- For any $x \in A_1 \cap A_2$, there exists $\epsilon > 0$ such that

$$\varphi(x, (0, \epsilon)) \cap A_1 = \emptyset \text{ and } \varphi(x, (-\epsilon, 0)) \cap A_2 = \emptyset. \tag{11}$$

Simply speaking, the flow leaves A_1 immediately and enters A_2 immediately at any point on $A_1 \cap A_2$ (see Figure 1). A strong Morse decomposition naturally occurs when we split A by a level set of some function transverse to the flow. Let us summarize some basic properties of a strong Morse decomposition in the following lemma. The proof is straightforward and we omit it.

Figure 1. A strong Morse decomposition

Lemma 3.16. *Let (A_1, A_2) be a strong Morse decomposition of an isolating neighborhood A. Then, we have the following results.*

(1) A_1 (resp. A_2) is negatively (resp. positively) invariant in A;
(2) $A_1 \cap A_2 = \partial A_1 \cap \partial A_2$ and $\partial A_i = (\partial A \cap A_i) \cup (A_1 \cap A_2)$ for $i = 1, 2$;
(3) A_1 and A_2 are isolating neighborhoods;
(4) $(\operatorname{inv}(A_2), \operatorname{inv}(A_1))$ is an attractor-repeller pair in $\operatorname{inv}(A)$.

One can make extra assumption for an index triple of an attractor-repeller pair arise from a strong Morse decomposition as follows.

Lemma 3.17. *Let (A_1, A_2) be a strong Morse decomposition of A. Suppose that $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ is an index triple for $(\operatorname{inv}(A_2), \operatorname{inv}(A_1))$ and denote by $\tilde{N}_2' = \tilde{N}_2 \cup (\tilde{N}_1 \cap A_2)$. Then, $(\tilde{N}_3, \tilde{N}_2', \tilde{N}_1)$ is again an index triple for $(\operatorname{inv}(A_2), \operatorname{inv}(A_1))$. In particular, we can always assume that an index triple $(\tilde{N}_3, \tilde{N}_2, \tilde{N}_1)$ of $(\operatorname{inv}(A_2), \operatorname{inv}(A_1))$ satisfies $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$.*

Proof. We simply check each condition of index pairs one by one.

- $\tilde{N}_2'$ is positively invariant in $\tilde{N}_1$; Since A_2 is positively invariant in A, $A_2 \cap \tilde{N}_1$ is positively invariant in $\tilde{N}_1$. The set $\tilde{N}_2$ is also positively invariant in $\tilde{N}_1$ Because $(\tilde{N}_1, \tilde{N}_2)$ is an index pair. It is easy to see that the union of two positively invariant set is a positively invariant set.
- $\tilde{N}_2'$ is an exit set for $\tilde{N}_1$ because $\tilde{N}_2'$ contains $\tilde{N}_2$, which is an exit set for $\tilde{N}_1$.
- $\operatorname{inv}(A_1) = \operatorname{inv}(\tilde{N}_1 \setminus \tilde{N}_2') \subset \operatorname{int}(\tilde{N}_1 \setminus \tilde{N}_2')$; Consider an element $x \in \operatorname{inv}(\tilde{N}_1 \setminus \tilde{N}_2) = \operatorname{inv}(A_1)$. Then, $\varphi(x, (-\infty, \infty))$ is contained in $(\tilde{N}_1 \setminus \tilde{N}_2) \cap \operatorname{int}(A_1)$. Since $\operatorname{int}(A_1) \cap A_2 = \emptyset$, we see that $\varphi(x, (-\infty, \infty)) \subset \tilde{N}_1 \setminus (\tilde{N}_2 \cup (\tilde{N}_1 \cap A_2))$. Thus, $x \in \operatorname{inv}(\tilde{N}_1 \setminus \tilde{N}_2')$ and $\operatorname{inv}(\tilde{N}_1 \setminus \tilde{N}_2) \subset \operatorname{int}(\tilde{N}_1 \setminus \tilde{N}_2')$. Since $\tilde{N}_1 \setminus \tilde{N}_2' \subset \tilde{N}_1 \setminus \tilde{N}_2$, we have $\operatorname{inv}(\tilde{N}_1 \setminus \tilde{N}_2') = \operatorname{inv}(\tilde{N}_1 \setminus \tilde{N}_2) = \operatorname{inv}(A_1)$. Note that $\operatorname{inv}(A_1) \subset \operatorname{int}(\tilde{N}_1 \setminus \tilde{N}_2')$ because $\operatorname{inv}(A_1) \subset \operatorname{int}(\tilde{N}_1 \setminus \tilde{N}_2)$ and $\operatorname{inv}(A_1) \cap A_2 = \emptyset$.
- $\tilde{N}_3$ is positively invariant in $\tilde{N}_2'$ because $\tilde{N}_3$ is positively invariant in $\tilde{N}_1$, which contains $\tilde{N}_2'$.
- $\tilde{N}_3$ is an exit set for $\tilde{N}_2'$; We only have to check that $\tilde{N}_3$ is an exit set for $\tilde{N}_1 \cap A_2$. Suppose that $x \in \tilde{N}_1 \cap A_2$ but $\varphi(x, t) \notin \tilde{N}_1 \cap A_2$ for some $t > 0$. Notice that a flow cannot go from A_2 to A_1 since (A_1, A_2) is a strong Morse decomposition. If $\varphi(x, t) \in \tilde{N}_1$, we would have $\varphi(x, t) \notin A_2$ which implies $\varphi(x, t) \notin A$ a contradiction. When $\varphi(x, t) \notin \tilde{N}_1$, we can use the fact that $\tilde{N}_3$ is an exit set for $\tilde{N}_1$.
- $\operatorname{inv}(A_2) = \operatorname{inv}(\tilde{N}_2' \setminus \tilde{N}_3) \subset \operatorname{int}(\tilde{N}_2' \setminus \tilde{N}_3)$; Suppose that we have $x \in \operatorname{inv}(\tilde{N}_2' \setminus \tilde{N}_3)$ such that $\varphi(x, t) \notin \tilde{N}_2 \setminus \tilde{N}_3$ for some $t \in \mathbb{R}$. Since $\varphi(x, (-\infty, \infty))$ does not intersect $\tilde{N}_3$, which is an exit set for both $\tilde{N}_2$ and $\tilde{N}_1 \cap A_2$, one can deduce that $\varphi(x, (-\infty, \infty)) \subset \tilde{N}_1 \cap A_2$. This implies $x \in \operatorname{inv}(A_2) = \operatorname{inv}(\tilde{N}_2 \setminus \tilde{N}_3)$ which is a contradiction. Therefore, $\operatorname{inv}(\tilde{N}_2' \setminus \tilde{N}_3) \subset \operatorname{inv}(\tilde{N}_2 \setminus \tilde{N}_3)$ while the converse is trivial. Consequently, $\operatorname{inv}(\tilde{N}_2' \setminus \tilde{N}_3) = \operatorname{inv}(A_2)$ is contained in $\operatorname{int}(\tilde{N}_2 \setminus \tilde{N}_3) \subset \operatorname{int}(\tilde{N}_2' \setminus \tilde{N}_3)$.

□

It turns out pre-index pairs behave nicely with attractor-repeller pairs arise from a strong Morse decomposition. More precisely, we will show that the canonical maps are compatible with the attractor and repeller maps in this situation.

Proposition 3.18. *Let (A_1, A_2) be a strong Morse decomposition of A and let (K_1, K_2) be a pre-index pair in A_2. Then, we have the following:*

(1) (K_1, K_2) is also a pre-index pair in A;
(2) We have a commutative diagram

$$\begin{tikzcd} K_1/K_1 \arrow[r, "\iota_2"] \arrow[dr, "\iota"'] & I(\operatorname{inv}(A_2)) \arrow[d, "i"] \\ & I(\operatorname{inv}(A)) \end{tikzcd}$$

where ι, ι_2 are the canonical maps and $i\colon I(\operatorname{inv}(A_2)) \to I(\operatorname{inv}(A))$ is the attractor map.

Proof.

(1) Consider $x \in K_1$ satisfying $\varphi(x, [0, \infty)) \subset A$. Since A_2 is positively invariant in A and $x \in K_1 \subset A_2$, we have $\varphi(x, [0, \infty)) \subset A_2$. Consequently, we see that $\varphi(x, [0, \infty)) \subset \operatorname{int}(A_2) \subset \operatorname{int}(A)$ because (K_1, K_2) is an pre-index pair in A_2. Now, consider $x \in K_2 \cap A^+$. Again, since A_2 is positively invariant in A, we have $\varphi(x, [0, \infty)) \subset A_2$. This is impossible because $K_2 \cap A_2^+ = \emptyset$.

(2) Let $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ be an index triple for $(\operatorname{inv}(A_2), \operatorname{inv}(A_1))$ such that $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$ (cf. Lemma 3.17) and let $L \subset N \subset A$ (resp. $L_2 \subset N_2 \subset A_2$) be an index pair for $\operatorname{inv}(A)$ (resp. $\operatorname{inv}(A_2)$) that contains (K_1, K_2). By Theorem 3.14, we may also assume that both (N, L) and (N_2, L_2) are T-tame. By possibly increasing T, we also assume that we have flow maps $s_T \colon N/L \to \tilde{N}_1/\tilde{N}_3$ and $s'_T \colon N_2/L_2 \to \tilde{N}_2/\tilde{N}_3$. Then, the map $i \circ \iota$ is represented by a composition

$$K_1/K_2 \xrightarrow{\iota_2} N_2/L_2 \xrightarrow{s'_T} \tilde{N}_2/\tilde{N}_3 \xrightarrow{i} \tilde{N}_1/\tilde{N}_3$$

while the map ι is represented by the composition

$$K_1/K_2 \xrightarrow{\iota} N/L \xrightarrow{s_T} \tilde{N}_1/\tilde{N}_3.$$

We will show that these two compositions are in fact the same map.

Applying Lemma 3.13, one can check that $i \circ s'_T \circ \iota_2$ sends $[x]$ to $[\varphi(x, 3T)]$ if

$$\varphi(x, [0, 3T]) \subset A_2, \ \varphi(x, [T, 3T]) \subset \tilde{N}_2 \setminus \tilde{N}_3 \tag{12}$$

and to the basepoint otherwise. On the other hand, $s_T \circ \iota$ sends $[x]$ to $[\varphi(x, 3T)]$ if

$$\varphi(x, [0, 3T]) \subset A, \ \varphi(x, [T, 3T]) \subset \tilde{N}_1 \setminus \tilde{N}_3 \tag{13}$$

and to the basepoint otherwise. It is obvious that condition (12) implies (13). On the other hand, condition (13) implies (12) for $x \in K_1 \subset A_2$ simply because A_2 is positively invariant in A and $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$.

□

Proposition 3.19. *Let (A_1, A_2) be a strong Morse decomposition of A and let (K_3, K_4) be a pre-index pair in A. Consider a pair $(K'_3, K'_4) := (K_3 \cap A_1, (K_4 \cap A_1) \cup (K_3 \cap A_1 \cap A_2))$. Then, we have the followings:*

(1) The pair (K'_3, K'_4) is a pre-index pair in A_1;

(2) A map $q \colon K_3/K_4 \to K'_3/K'_4$ given by

$$q([x]) = \begin{cases} [x] & \text{if } x \in K'_3, \\ [K'_4] & \text{otherwise,} \end{cases}$$

is well-defined and continuous;

(3) We have a commutative diagram

$$\begin{array}{ccc} K_3/K_4 & \xrightarrow{\iota} & I(\operatorname{inv}(A)) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ K'_3/K'_4 & \xrightarrow{\iota'} & I(\operatorname{inv}(A_1)) \end{array}$$

where ι, ι' are the canonical maps and $r \colon I(\operatorname{inv}(A)) \to I(\operatorname{inv}(A_1))$ is the repeller map.

Proof.

(1) We will check the two conditions of pre-index pair directly. Suppose that $x \in K_3'$ and $\varphi(x,[0,\infty)) \subset A_1$. It is clear that $\varphi(x,[0,\infty)) \cap (A_1 \cap A_2) = \emptyset$ from the property (11) of strong Morse decomposition. Since (K_3, K_4) is a pre-index pair in A and $x \in K_3 \cap A^+$ we have $\varphi(x,[0,\infty)) \cap \partial A = \emptyset$. Consequently, we can deduce that $\varphi(x,[0,+\infty)) \cap \partial A_1 = \emptyset$ because $\partial A_1 = (\partial A \cap A_1) \cup (A_1 \cap A_2)$.

Since (K_3, K_4) is a pre-index pair in A, we have $K_4 \cap A^+ = \emptyset$. It follows directly that $(K_4 \cap A_1) \cap A_1^+ = \emptyset$. On the other hand, we can see that $(K_3 \cap A_1 \cap A_2) \cap A_1^+ = \emptyset$ as a point on $A_1 \cap A_2$ leaves A_1 immediately. Therefore, K_4' has empty intersection with A_1^+.

(2) Note that q is continuous because $(\overline{K_3 \setminus K_3'}) \cap K_3' = K_3 \cap A_1 \cap A_2 \subset K_4'$. For $x \in K_4 \cap K_3' \subset K_4 \cap A_1 \subset K_4'$, we see that q is well-defined.

(3) As in the proof of Proposition 3.18, let $\tilde{N}_3 \subset \tilde{N}_2 \subset \tilde{N}_1 \subset A$ be an index triple for $(\mathrm{inv}(A_2), \mathrm{inv}(A_1))$ with $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$ and let $L \subset N \subset A$ (resp. $L_1 \subset N_1 \subset A_1$) be an index pair for A (resp. for A_1) that contains (K_3, K_4) (resp. (K_3', K_4')). By Theorem 3.14, we can assume that (N, L) and (N_1, L_1) are both T-tame. By possibly increasing T, we also assume that we have flow maps $s_T \colon N/L \to \tilde{N}_1/\tilde{N}_3$ and $s_T' \colon N_1/L_1 \to \tilde{N}_1/\tilde{N}_2$. Then, the map $q \circ \iota'$ is represented by

$$K_3/K_4 \xrightarrow{q} K_3'/K_4' \xrightarrow{\iota'} N_1/L_1 \xrightarrow{s_T'} \tilde{N}_1/\tilde{N}_2,$$

and the map $r \circ \iota$ is represented by

$$K_3/K_4 \xrightarrow{\iota} N/L \xrightarrow{s_T} \tilde{N}_1/\tilde{N}_3 \xrightarrow{r} \tilde{N}_1/\tilde{N}_2.$$

We will show that these two compositions are in fact the same maps.

Applying Lemma 3.13, one can check that $s_T' \circ \iota' \circ q$ sends $[x]$ to $[\varphi(x, 3T)]$ if

$$\varphi(x,[0,3T]) \subset A_1 \text{ and } \varphi(x,[T,3T]) \subset \tilde{N}_1 \setminus \tilde{N}_2 \tag{14}$$

and to the basepoint otherwise. On the other hand, $r \circ s_T \circ \iota$ sends $[x]$ to $[\varphi(x,3T)]$ if

$$\varphi(x,[0,3T]) \subset A, \ \varphi(x,[T,3T]) \subset \tilde{N}_1 \setminus \tilde{N}_3 \text{ and } \varphi(x,3T) \notin \tilde{N}_2 \tag{15}$$

and to the basepoint otherwise. Clearly, condition (14) implies condition (15). We will check that the two conditions are the same. Consider an element $x \in K_3$ satisfying (15). We see that $\varphi(x,3T) \in \tilde{N}_1 \setminus \tilde{N}_2 \subset A_1$ because $\tilde{N}_1 \cap A_2 \subset \tilde{N}_2$. Since A_1 is negatively invariant in A, we have $\varphi(x,[0,3T]) \subset A_1$. Moreover, the facts $\varphi(x,3T) \notin \tilde{N}_2$ and $\varphi(x,[T,3T]) \cap \tilde{N}_3 = \emptyset$ imply that $\varphi(x,[T,3T]) \cap \tilde{N}_2 = \emptyset$ since $\tilde{N}_3$ is an exit set for $\tilde{N}_2$. We have proved that x satisfies condition (14).

□

3.4. *T*-tame manifold isolating block for Seiberg-Witten flow.

Definition 3.20. For a compact set N in Ω, we consider the following subsets of its boundary:

$$n^+(N) := \{x \in \partial N | \exists \epsilon > 0 \text{ s.t. } \varphi(-\epsilon, 0) \cap N = \emptyset\},$$
$$n^-(N) := \{x \in \partial N | \exists \epsilon > 0 \text{ s.t. } \varphi(0, \epsilon) \cap N = \emptyset\}.$$

A compact set N is called an *isolating block* if $\partial N = n^+(N) \cup n^-(N)$.

It is easy to verify that an isolating block N is an isolating neighborhood and that $(N, n^-(N))$ is an index pair.

Definition 3.21. If N is a compact submanifold of Ω and is also an isolating block, we call N a *manifold isolating block*.

In [3], it is proved that, for any isolating neighborhood A, we can always find a manifold isolating block N of $\operatorname{inv} A$ with $N \subset A$. We also introduce a notion of tameness for an isolating block as quantitative refinement as in Section 3.2.

Definition 3.22. Let A be an isolating neighborhood and T be a positive number. An isolating block N in A is called T-tame if $A^{[-T,T]} \subset \operatorname{int}(N)$.

We turn into special situation involving construction of the spectrum invariants a 3-manifold Y: $\underline{\text{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$ and $\underline{\text{swf}}^R(Y, \mathfrak{s}, A_0, g; S^1)$. Let R_0 be the universal constant from [7, Theorem 3.2]. Take a positive number $\tilde{R}$ with $\tilde{R} > R_0$, sequences $\lambda_n \to -\infty$, $\mu_n \to \infty$ and functions $g_{j,\pm} \colon V \to \mathbb{R}$. Put

$$J_m^{\pm} := Str(\tilde{R}) \cap \bigcap_{1 \le j \le b_1} g_{j,\pm}^{-1}(-\infty, \theta + m],$$
$$J_m^{n,\pm} := J_m^{\pm} \cap V_{\lambda_n}^{\mu_n},$$

where V is a certain Hilbert space and $V_{\lambda_n}^{\mu_n}$ is its finite-dimensional subspace (see [7, Section 5.1] for more details).

Lemma 3.23. *For each positive integer m, there is a positive number T_m independent of n such that*

$$(J_m^{n,+})^{[-2T,2T]} \subset \operatorname{int}\left\{(J_m^{n,+})^{[-T,T]}\right\},$$

for all $T > T_m$ and n sufficiently large. In particular, $(J_m^{n,+})^{[-2T,2T]} \subset \operatorname{int}(J_m^{n,+})$. Similar results hold for $J_m^{n,-}$.

Proof. If the statement is not true, we have a sequence $T_n \to \infty$ such that we can take elements

$$x_n \in (J_m^{n,+})^{[-2T_n,2T_n]} \cap \partial\left\{(J_m^{n,+})^{[-T_n,T_n]}\right\}.$$

In particular, we would have

$$\varphi_m^n(x_n, [-2T_n, 2T_n]) \subset J_m^{n,+} \text{ and } \varphi_m^n(x_n, t_n) \in \partial J_m^{n,+} \text{ for some } t_n \in [-T_n, T_n]$$

which implies $\varphi_m^n(x_n, t_n) \in (J_m^{n,+})^{[-T_n,T_n]}$. On the other hand, by [7, Lemma 5.4], we must have $\varphi_m^n(x_n, t_n) \in \partial Str(\tilde{R})$. This is a contradiction to [7, Lemma 5.5 (a)].

□

We now state the main result of this section.

Proposition 3.24. *Let T_m be the constant from Lemma 3.23. When $T > 4T_m$ and n is sufficiently large, we can always find a T-tame manifold isolating block $N_m^{n,+}$ of* $\operatorname{inv}(J_m^{n,+})$ *with $N_m^{n,+} \subset J_m^{n,+}$. Similar result holds for $J_m^{n,-}$.*

Proof. Fix m and suppose that n is sufficiently large so that the statement of Lemma 3.23 holds. Take a positive number T with $T > 4T_m$. By Lemma 3.23, we have

$$(J_m^{n,+})^{[-T,T]} \subset \operatorname{int}\left\{(J_m^{n,+})^{[-T/2,T/2]}\right\} \text{ and } (J_m^{n,+})^{[-T/2,T/2]} \subset \operatorname{int}\left\{(J_m^{n,+})^{[-T/4,T/4]}\right\}.$$

We can take a smooth function $\tau : V_{\lambda_n}^{\mu_n} \to [0,1]$ such that

$$\tau = 0 \text{ on } (J_m^{n,+})^{[-T,T]}, \text{ and } \tau = 1 \text{ on } V_{\lambda_n}^{\mu_n} \setminus (J_m^{n,+})^{[-T/2,T/2]}.$$

Let $\tilde{\varphi}_m^n$ be the flow on $V_{\lambda_n}^{\mu_n}$ generated by $\tau \cdot \iota_m \cdot (l + p_{\lambda_n}^{\mu_n} \circ c)$, where ι_m is the bump function as in [7, p.21]. We will prove that $J_m^{n,+}$ is an isolating neighborhood of $\operatorname{inv}(\tilde{\varphi}_m^n, J_m^{n,+})$. If this is not true, we can take

$$x \in \partial J_m^{n,+} \cap \operatorname{inv}(\tilde{\varphi}_m^n, J_m^{n,+}).$$

Put

$$P^+(x) := \{\varphi_m^n(x,t) | t \geq 0, \varphi_m^n(x,[0,t]) \subset J_m^{n,+}\},$$
$$P^-(x) := \{\varphi_m^n(x,t) | t \leq 0, \varphi_m^n(x,[t,0]) \subset J_m^{n,+}\}.$$

Suppose that $P^+(x) \cap (J_m^{n,+})^{[-T/2,T/2]} = \emptyset$. This means a forward φ_m^n-trajectory of x inside $J_m^{n,+}$ lie outside $(J_m^{n,+})^{[-T/2,T/2]}$, so that a forward φ_m^n-trajectory agrees with a forward $\tilde{\varphi}_m^n$-trajectory. Consequently, we have $\varphi_m^n(x,[0,\infty)) = \tilde{\varphi}_m^n(x,[0,\infty)) \subset J_m^{n,+}$. Hence $\varphi_m^n(x,T/2) \in P^+(x)$ and $\varphi_m^n(x,T/2) \in (J_m^{n,+})^{[-T/2,T/2]}$ which is a contradiction. We can now conclude that $P^+(x) \cap (J_m^{n,+})^{[-T/2,T/2]} \neq \emptyset$ and, in particular, $x \in (J_m^{n,+})^{[0,T/2]}$.

Similarly we can deduce that $x \in (J_m^{n,+})^{[-T/2,0]}$. These facts imply that

$$x \in (J_m^{n,+})^{[-T/2,T/2]} \cap \partial J_m^{n,+},$$

which is a contradiction because

$$(J_m^{n,+})^{[-T/2,T/2]} \subset \operatorname{int}\left\{(J_m^{n,+})^{[-T/4,T/4]}\right\} \subset \operatorname{int}(J_m^{n,+}).$$

Therefore $J_m^{n,+}$ is an isolating neighborhood of $\operatorname{inv}(\tilde{\varphi}_m^n, J_m^{n,+})$. By the result of Conley and Easton [3], we can find a manifold isolating block $N_m^{n,+}$ of $\operatorname{inv}(\tilde{\varphi}_m^n, J_m^{n,+})$ with $N_m^{n,+} \subset J_m^{n,+}$. Note that

$$(J_m^{n,+})^{[-T,T]} \subset \operatorname{inv}(\tilde{\varphi}_m^n, J_m^{n,+}) \subset \operatorname{int} N_m^{n,+}.$$

Since the directions of the flows φ_m^n and $\tilde{\varphi}_m^n$ coincide on $\partial N_m^{n,+} \subset J_m^{n,+} \setminus \tau^{-1}(0)$, we see that $N_m^{n,+}$ is also a manifold isolating block of $\operatorname{inv}(\varphi_m^n, J_m^{n,+})$. Thus $N_m^{n,+}$ is a T-tame manifold isolating block of $\operatorname{inv}(\varphi_m^n, J_m^{n,+})$ in $J_m^{n,+}$.

□

4. Stable homotopy categories

4.1. **Summary.** In this section, we will discuss the stable homotopy categories $\mathfrak{C}$, $\mathfrak{S}$, $\mathfrak{S}^*$. The discussion in this section will be needed to construct the gluing formula in Theorem 6.1.

First let us briefly recall the definition of the categories. (See [7] for the details.) An object of $\mathfrak{C}$ is a triple (A, m, n), where A is a pointed topological space with S^1-action which is S^1-homotopy equivalent to a finite S^1-CW complex, m is an integer and n is a rational number. The set of morphisms between (A_1, m_1, n_1) and (A_2, m_2, n_2) is given by

$$\mathrm{mor}_{\mathfrak{C}}((A_1, m_1, n_1), (A_2, m_2, n_2)) = \lim_{u,v\to\infty}[(\mathbb{R}^u\oplus\mathbb{C}^v)^+\wedge A_1, (\mathbb{R}^{u+m_1-m_2}\oplus\mathbb{C}^{v+n_1-n_2})^+\wedge A_2]_{S^1}$$

if $n_1 - n_2$ is an integer, and we define $\mathrm{mor}_{\mathfrak{C}}((A_1, m_1, n_1), (A_2, m_2, n_2))$ to be the empty set if $n_1 - n_2$ is not an integer. Here $[\cdot,\cdot]_{S^1}$ is the set of pointed S^1-homotopy classes, $\mathbb{R}$ is the one dimensional trivial representation of S^1 and $\mathbb{C}$ is the standard two dimensional representation of S^1. The category $\mathfrak{S}$ is the category of direct systems

$$Z : Z_1 \overset{j_1}{\to} Z_2 \overset{j_2}{\to} \cdots$$

in $\mathfrak{C}$. Here Z_m and j_m are an object and morphism in $\mathfrak{C}$ respectively. For objects Z, Z' in $\mathfrak{S}$, the set morphism is defined by

$$\mathrm{mor}_{\mathfrak{S}}(Z, Z') = \lim_{\infty\leftarrow m}\lim_{n\to\infty}\mathrm{mor}_{\mathfrak{C}}(Z_m, Z'_n).$$

The category $\mathfrak{S}^*$ is the category of inverse systems

$$\bar{Z} : \bar{Z}_1 \overset{\bar{j}_1}{\leftarrow} \bar{Z}_2 \overset{\bar{j}_2}{\leftarrow} \cdots$$

in $\mathfrak{C}$. Here $\bar{Z}_m$ and $\bar{j}_m$ are an object and morphism in $\mathfrak{C}$ respectively. For objects $\bar{Z}$, $\bar{Z}'$ in $\mathfrak{S}^*$, the set of morphisms is defined by

$$\mathrm{mor}_{\mathfrak{S}^*}(\bar{Z}, \bar{Z}') = \lim_{\infty\leftarrow n}\lim_{m\to\infty}\mathrm{mor}_{\mathfrak{C}}(\bar{Z}_m, \bar{Z}'_n).$$

In Section 4.2, we will define the smash product in the category $\mathfrak{C}$ and prove that $\mathfrak{C}$ is a symmetric, monoidal category (Lemma 4.1). In Section 4.3, we will introduce the notion of the S^1-equivariant Spanier-Whitehead duality between the categories $\mathfrak{S}$ and $\mathfrak{S}^*$. We will say that $Z \in \mathrm{ob}\,\mathfrak{S}$ and $\bar{Z} \in \mathrm{ob}\,\mathfrak{S}^*$ are S^1-equivariant Spanier-Whitehead dual to each other if there are elements

$$\epsilon \in \lim_{\infty\leftarrow m}\lim_{n\to\infty}\mathrm{mor}_{\mathfrak{C}}(\bar{Z}_n \wedge Z_m, S),\ \eta \in \lim_{\infty\leftarrow n}\lim_{m\to\infty}(S, Z_m \wedge \bar{Z}_n),$$

which satisfy certain conditions (Definition 4.3). Here $S = (S^0, 0, 0) \in \mathfrak{C}$. The elements ϵ, η are called duality morphisms. In Section 4.4, we will prove that the Seiberg-Witten Floer stable spectra $\underline{\mathrm{swf}}^A(Y) \in \mathrm{ob}\,\mathfrak{S}$ and $\underline{\mathrm{swf}}^R(-Y) \in \mathrm{ob}\,\mathfrak{S}^*$ are S^1-equivariant Spanier-Whitehead dual to each other (Proposition 4.11). We will construct natural duality morphisms for $\underline{\mathrm{swf}}^A(Y)$ and $\underline{\mathrm{swf}}^R(-Y)$ which will be needed for the gluing formula of the Bauer-Furuta invariants (Theorem 6.1).

We will focus on the S^1-equivariant stable homotopy categories. But the statements can be proved for the $Pin(2)$-equivariant stable homotopy categories in a similar way.

4.2. **Smash product.** In this subsection, we establish the symmetric monoidal structure on the category $\mathfrak{C}$. To do this, we will define the smash product as a bifunctor $\wedge : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$. First we define the smash product of two objects $(A_1, m_1, n_1), (A_2, m_2, n_2) \in \mathfrak{C}$. Here A_i is an S^1-topological space, $m_i \in 2\mathbb{Z}$, $n_i \in \mathbb{Q}$. We define the smash product by

$$(A_1, m_1, n_1) \wedge (A_2, m_2, n_2) := (A_1 \wedge A_2, m_1 + m_2, n_1 + n_2),$$

where $A_1 \wedge A_2$ denotes the classical smash product on pointed topological spaces. Next we define the smash product of morphisms. Suppose that for $i = 1, 2$ a map

$$f_i : (\mathbb{R}^{k_i} \oplus \mathbb{C}^{l_i})^+ \wedge A_i \to (\mathbb{R}^{k_i + m_i - m_i'} \oplus \mathbb{C}^{l_i + n_i - n_i'})^+ \wedge A_i'$$

represents a morphism $[f_i] \in \mathrm{mor}_{\mathfrak{C}}((A_i, m_i, n_i), (A_i', m_i', n_i'))$. We may suppose that k_i is even. We define a map

$$\begin{gathered} f_1 \wedge f_2 : (\mathbb{R}^{k_1} \oplus \mathbb{R}^{k_2} \oplus \mathbb{C}^{l_1} \oplus \mathbb{C}^{l_2})^+ \wedge A_1 \wedge A_2 \to \\ (\mathbb{R}^{k_1 + m_1 - m_1'} \oplus \mathbb{R}^{k_2 + m_2 - m_2'} \oplus \mathbb{C}^{l_1 + n_1 - n_1'} \oplus \mathbb{C}^{l_2 + n_2 - n_2'})^+ \wedge A_1' \wedge A_2' \end{gathered}$$

by putting the suspension indices for f_1 on the left and those for f_2 on the right. We define $[f_1] \wedge [f_2]$ to be the morphism represented by $f_1 \wedge f_2$. To prove that this operation is well defined, we need to check that for $a, b \in \mathbb{Z}_{>0}$, we have

$$\Sigma^{(\mathbb{R}^a \oplus \mathbb{C}^b)^+}(f_1 \wedge f_2) \cong (\Sigma^{(\mathbb{R}^a \oplus \mathbb{C}^b)^+} f_1) \wedge f_2 \cong f_1 \wedge (\Sigma^{(\mathbb{R}^a \oplus \mathbb{C}^b)^+} f_2),$$

where $\cong$ means S^1-equivariant stably homotopic. The first equivalence is obvious. The second equivalence follows from the fact that the following diagram is commutative up to homotopy for $u_1 = k_1, k_1 + m_1 - m_1'$, $u_2 = k_2, k_2 + m_2 - m_2'$, $v_1 = l_1, l_1 + n_1 - n_1'$, $v_2 = l_2, l_2 + n_2 - n_2'$:

$$\begin{tikzcd}
(\mathbb{R}^a \oplus \mathbb{R}^{u_1} \oplus \mathbb{R}^{u_2})^+ \wedge (\mathbb{C}^b \oplus \mathbb{C}^{v_1} \oplus \mathbb{C}^{v_2})^+ \arrow[dr, "\mathrm{id}"] \arrow[dd, "(\gamma_{\mathbb{R}^a, \mathbb{R}^{u_1}} \oplus \mathrm{id}_{\mathbb{R}^{u_2}})^+ \wedge (\gamma_{\mathbb{C}^b, \mathbb{C}^{v_1}} \oplus \mathrm{id}_{\mathbb{C}^{v_2}})^+"'] & \\
& (\mathbb{R}^{a + u_1 + u_2})^+ \wedge (\mathbb{C}^{b + v_1 + v_2})^+ \\
(\mathbb{R}^{u_1} \oplus \mathbb{R}^a \oplus \mathbb{R}^{u_2})^+ \wedge (\mathbb{C}^{v_1} \oplus \mathbb{C}^b \oplus \mathbb{C}^{v_2})^+ \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

Here $\gamma_{\mathbb{R}^a, \mathbb{R}^{u_1}}$ is the map which interchange $\mathbb{R}^a$ and $\mathbb{R}^{u_1}$. Similarly for $\gamma_{\mathbb{C}^b, \mathbb{C}^{v_1}}$. Note that $u_1 \in 2\mathbb{Z}$ by the assumption on k_1, m_1, m_1'.

There is an isomorphism

$$\gamma_{(A_1, m_1, n_1), (A_2, m_2, n_2)} : (A_1, m_1, n_1) \wedge (A_2, m_2, n_2) \to (A_2, m_2, n_2) \wedge (A_1, m_1, n_1)$$

represented by the obvious homeomorphism $A_1 \wedge A_2 \to A_2 \wedge A_1$. It is not difficult to see that γ is natural in (A_i, m_i, n_i). That is, the following diagrams are commutative for

$f_i \in \mathrm{mor}_{\mathfrak{C}}((A_i, m_i, n_i), (A'_i, m'_i, n'_i))$:

$$\begin{array}{ccc}
(A_1, m_1, n_1) \wedge (A_2, m_2, n_2) & \xrightarrow{\gamma} & (A_2, m_2, n_2) \wedge (A_1, m_1, n_1) \\
{\scriptstyle f_1 \wedge f_2} \downarrow & & \downarrow {\scriptstyle f_2 \wedge f_1} \\
(A'_1, m'_1, n'_1) \wedge (A'_2, m'_2, n'_2) & \xrightarrow[\gamma]{} & (A'_2, m'_2, n'_2) \wedge (A'_1, m'_1, n'_1).
\end{array}$$

(Again, we need the assumption that m_i is even here.) Once the well-definedness of $\wedge$ and the naturality are established we can prove the following lemma easily by checking the axioms at the level of topological spaces.

Lemma 4.1. *The category $\mathfrak{C}$ equipped with $\wedge$ and γ is a symmetric monoidal category with unit $S = (S^0, 0, 0)$.*

We briefly mention the $Pin(2)$-case. The smash product $\wedge$ and the interchanging operation γ can be defined on the category $\mathfrak{C}_{Pin(2)}$ in exactly the same way as before. As a result, the category $\mathfrak{C}_{Pin(2)}$ is also an symmetric monoidal category.

4.3. **Equivariant Spanier-Whitehead duality.** In this subsection we will set up the equivariant Spanier-Whitehead duality between the categories $\mathfrak{S}$ and $\mathfrak{S}^*$. Although we will mostly focus on the S^1-case for simplicity, all definitions and proofs can be easily adapted to the $Pin(2)$-case. As a result, a duality between $\mathfrak{S}_{Pin(2)}$ and $\mathfrak{S}^*_{Pin(2)}$ can also be set up in a similar way.

The following definition is motivated by [10, Chapter III] and [13, Chapter XVI Section 7].

Definition 4.2. Let U, W be objects of $\mathfrak{C}$ and put $S = (S^0, 0, 0) \in \mathrm{ob}\,\mathfrak{C}$. Suppose that there exist morphisms

$$\epsilon : W \wedge U \to S,\ \eta : S \to U \wedge W$$

such that the compositions

$$U \cong S \wedge U \xrightarrow{\eta \wedge \mathrm{id}} U \wedge W \wedge U \xrightarrow{\mathrm{id} \wedge \epsilon} U \wedge S \cong U$$

and

$$W \cong W \wedge S \xrightarrow{\mathrm{id} \wedge \eta} W \wedge U \wedge W \xrightarrow{\epsilon \wedge \mathrm{id}} S \wedge W \cong W$$

are equal to the identity morphisms respectively. Then we say that U and W are Spanier-Whitehead dual to each other and call ϵ and η duality morphisms.

We generalize this definition to the duality between $\mathfrak{S}$ and $\mathfrak{S}^*$.

Definition 4.3. Let

$$Z : Z_1 \to Z_2 \to Z_3 \to \cdots$$

be an object of $\mathfrak{S}$ and

$$\bar{Z} : \bar{Z}_1 \leftarrow \bar{Z}_2 \leftarrow \bar{Z}_3 \leftarrow \cdots$$

be an object of $\mathfrak{S}^*$. Suppose that we have an element

$$\epsilon \in \lim_{\infty \leftarrow m} \lim_{n \to \infty} \mathrm{mor}_{\mathfrak{C}}(\bar{Z}_n \wedge Z_m, S)$$

represented by a collection $\{\epsilon_{m,n} : \bar{Z}_n \wedge Z_m \to S\}_{m>0, n \gg m}$ and an element

$$\eta \in \lim_{\infty \leftarrow n} \lim_{m \to \infty} \operatorname{mor}_{\mathfrak{C}}(S, Z_m \wedge \bar{Z}_n)$$

represented by a collection $\{\eta_{m,n} : S \to Z_m \wedge \bar{Z}_n\}_{n>0, m \gg n}$ which satisfy the following conditions:

(i) For any $m > 0$ there exists n large enough relative to m and m' large enough relative to n such that the composition

$$Z_m \cong S \wedge Z_m \xrightarrow{\eta_{m',n} \wedge \mathrm{id}} Z_{m'} \wedge \bar{Z}_n \wedge Z_m \xrightarrow{\mathrm{id} \wedge \epsilon_{m,n}} Z_{m'} \wedge S \cong Z_{m'}$$

is equal to the connecting morphism $Z_m \to Z_{m'}$ of the inductive system Z.

(ii) For any $n > 0$, there exists m large enough relative n and n' large enough to m such that the composition

$$\bar{Z}_{n'} \cong \bar{Z}_{n'} \wedge S \xrightarrow{\mathrm{id} \wedge \eta_{m,n}} \bar{Z}_{n'} \wedge Z_m \wedge \bar{Z}_n \xrightarrow{\epsilon_{m,n'} \wedge \mathrm{id}} S \wedge \bar{Z}_n \cong \bar{Z}_n$$

is equal to the connecting morphism $\bar{Z}_{n'} \to \bar{Z}_n$ of the projective system $\bar{Z}$.

Then we say that Z and $\bar{Z}$ are S^1-equivariant Spanier-Whitehead dual to each other and we call ϵ and η duality morphisms.

We end this subsection with introducing a smashing operation $\tilde{\epsilon}$, which will be used to give the statement of the gluing theorem for the Bauer-Furuta invariant.

Definition 4.4. Let $Z \in \operatorname{ob} \mathfrak{S}$ and $\bar{Z} \in \operatorname{ob} \mathfrak{S}^*$ be objects that are S^1-equivariant Spanier-Whitehead dual to each other with duality morphisms ϵ, η. Suppose that we have objects $W \in \operatorname{ob} \mathfrak{C}$ ($\subset \operatorname{ob} \mathfrak{S}$), $\bar{W} \in \operatorname{ob} \mathfrak{C}$ ($\subset \operatorname{ob} \mathfrak{S}^*$) and morphisms

$$\rho \in \operatorname{mor}_{\mathfrak{S}}(W, Z), \ \ \bar{\rho} \in \operatorname{mor}_{\mathfrak{S}^*}(\bar{W}, \bar{Z}).$$

Choose a morphism $\rho_m : W \to Z_m$ which represents ρ and let $\{\bar{\rho}_n : \bar{W} \to \bar{Z}_n\}_{n>0}$ be the collection which represents $\bar{\rho}$. We define the morphism $\tilde{\epsilon}(\rho, \bar{\rho}) \in \operatorname{mor}_{\mathfrak{C}}(W \wedge \bar{W}, S)$ by the composition

$$\bar{W} \wedge W \xrightarrow{\bar{\rho}_n \wedge \rho_m} \bar{Z}_n \wedge Z_m \xrightarrow{\epsilon_{m,n}} S.$$

It can be proved that $\tilde{\epsilon}(\rho, \bar{\rho})$ does not depend on the choices of m, n and ρ_m. (Note that $\bar{\rho}_n$ is determined by n and $\bar{\rho}$.)

4.4. **Spanier-Whitehead duality of the unfolded Seiberg-Witten Floer spectra.** Let Y be a closed, oriented 3-manifold with a Riemannian metric g and spin^c structure $\mathfrak{s}$, and let $-Y$ be Y with opposite orientation. As in Section 2.1, the unfolded Seiberg-Witten Floer spectrum $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1) \in \operatorname{ob} \mathfrak{S}$ is represented by

$$\underline{\mathrm{swf}}^A(Y) : I_1 \xrightarrow{j_1} I_2 \xrightarrow{j_2} \cdots$$

with $I_n := \Sigma^{-V^0_{\lambda_n}} I(\operatorname{inv}(V^{\mu_n}_{\lambda_n} \cap J^+_n), \varphi_n)$. It is not hard to see that the unfolded spectrum $\underline{\mathrm{swf}}^R(-Y, \mathfrak{s}, A_0, g; S^1) \in \operatorname{ob} \mathfrak{S}^*$ can be represented by

$$\underline{\mathrm{swf}}^R(-Y) : \bar{I}_1 \xleftarrow{\bar{j}_1} \bar{I}_2 \xleftarrow{\bar{j}_2} \cdots ,$$

where $\bar{I}_n := \Sigma^{-V_0^{\mu_n}} I(\mathrm{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+), \overline{\varphi}_n)$ and $\overline{\varphi}_n$ is the reverse flow of φ_n. For integers m, n with $m < n$ we also write $j_{m,n}$, $\bar{j}_{m,n}$ for the compositions

$$I_m \xrightarrow{j_m} I_{m+1} \xrightarrow{j_{m+1}} \cdots \xrightarrow{j_{n-1}} I_n,$$
$$\bar{I}_n \xrightarrow{\bar{j}_{n-1}} \bar{I}_{n-1} \xrightarrow{\bar{j}_{n-2}} \cdots \xrightarrow{\bar{j}_m} \bar{I}_m.$$

We will define duality morphisms ϵ and η between $\underline{\mathrm{swf}}^A(Y, \mathfrak{s}, A_0, g; S^1)$ and $\underline{\mathrm{swf}}^R(-Y, \mathfrak{s}, A_0, g; S^1)$. as follows. Take a manifold isolating block N_n for $\mathrm{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+)$. That is, N_n is a compact submanifold of $V_{\lambda_n}^{\mu_n}$ of codimension 0 and there are submanifolds $L_n, \overline{L}_n$ of ∂N_n of codimension 0 such that

$$L_n \cup \overline{L}_n = \partial N_n,\ \partial L_n = \partial \overline{L}_n = L_n \cap \overline{L}_n$$

and that (N_n, L_n), $(N_n, \overline{L}_n)$ are index pairs for $\mathrm{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+, \varphi_n)$, $\mathrm{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+, \overline{\varphi}_n)$ respectively. Fix a small positive number $\delta > 0$. For a subset $P \subset V_{\lambda_n}^{\mu_n}$ we write $\nu_\delta(P)$ for

$$\{x \in V_{\lambda_n}^{\mu_n} \,|\, \mathrm{dist}(x, P) \leq \delta\}.$$

Choose S^1-equivariant homotopy equivalences

$$a_n : N_n \to N_n \setminus \nu_\delta(\overline{L}_n),\ b_n : N_n \to N_n \setminus \nu_\delta(L_n)$$

such that

$$\begin{aligned} &\|a_n(x) - x\| < 2\delta \text{ for } x \in N_n,\ a_n(L_n) \subset L_n,\ a_n(x) = x \text{ for } x \in N_n \setminus \nu_{3\delta}(\partial N_n), \\ &\|b_n(y) - y\| < 2\delta \text{ for } y \in N_n,\ b_n(\overline{L}_n) \subset \overline{L}_n,\ b_n(y) = y \text{ for } y \in N_n \setminus \nu_{3\delta}(\partial N_n). \end{aligned} \tag{16}$$

Put $B_\delta = \{x \in V_{\lambda_n}^{\mu_n} | \|x\| \leq \delta\}$ and $S_\delta = \partial B_\delta$. Define

$$\hat{\epsilon}_{n,n} : (N_n/\overline{L}_n) \wedge (N_n/L_n) \to (V_{\lambda_n}^{\mu_n})^+ = B_\delta/S_\delta$$

by the formula

$$\hat{\epsilon}_{n,n}([y] \wedge [x]) = \begin{cases} [b_n(y) - a_n(x)] & \text{if } \|b_n(y) - a_n(x)\| < \delta \\ * & \text{otherwise.} \end{cases}$$

It is easy to see that $\hat{\epsilon}_n$ is a well-defined, continuous S^1-equivariat map. Taking the desuspension by $V_{\lambda_n}^{\mu_n}$ we get a morphism

$$\epsilon_{n,n} : \bar{I}_n \wedge I_n \to S.$$

For m, n with $m < n$, we define a morphism $\epsilon_{m,n} : \bar{Z}_n \wedge Z_m \to S$ to be the composition

$$\bar{I}_n \wedge I_m \xrightarrow{\mathrm{id} \wedge j_{m,n}} \bar{I}_n \wedge I_n \xrightarrow{\epsilon_{n,n}} S.$$

Lemma 4.5. *With the above notation, the morphism $\epsilon_{m,n} \in \mathrm{mor}_{\mathfrak{C}}(\bar{I}_n \wedge I_m, S)$ is independent of the choices of N_n, a_n, b_n and δ.*

Proof. The proof of the independence from δ is easy. We prove the independence from N_n, a_n and b_n. Fix an isolating neighborhood $A(\subset V_{\lambda_n}^{\mu_n} \cap J_n^+)$ of $\mathrm{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+)$. Take two manifold isolating blocks N_n, N_n' for $\mathrm{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+)$ included in $\mathrm{int}\, A$. Then we get two maps

$$\hat{\epsilon}_{n,n} : (N_n/\overline{L}_n) \wedge (N_n/L_n) \to B_\delta/S_\delta,\ \hat{\epsilon}'_{n,n} : (N_n'/\overline{L}_n') \wedge (N_n'/L_n') \to B_\delta/S_\delta.$$

It is sufficient to show that the following diagram is commutative up to S^1-equivariant homotopy:

$$\begin{tikzcd} (N_n/\overline{L}_n) \wedge (N_n/L_n) \arrow[r, "\hat{\epsilon}_{n,n}"] \arrow[d, "\bar{s}\wedge s"'] & B_\delta/S_\delta \\ (N_n'/\overline{L}_n') \wedge (N_n'/L_n') \arrow[ur, "\hat{\epsilon}'_{n,n}"'] & \end{tikzcd}$$

Here $s = s_T : N_n/L_n \to N_n'/L_n'$, $\bar{s} = \bar{s}_T : N_n/\overline{L}_n \to N_n'/\overline{L}_n'$ are the flow maps with large $T > 0$:

$$s([x]) = \begin{cases} [\varphi(x, 3T)] & \text{if } \varphi(x, [0, 2T]) \subset N_n \setminus L_n, \varphi(x, [T, 3T]) \subset N_n' \setminus L_n', \\ * & \text{otherwise.} \end{cases}$$

$$\bar{s}([y]) = \begin{cases} [\varphi(y, -3T)] & \text{if } \varphi(y, [-2T, 0]) \subset N_n \setminus \overline{L}_n, \varphi(y, [-3T, -T]) \subset N_n' \setminus \overline{L}_n', \\ * & \text{otherwise.} \end{cases}$$

The proof can be reduced to the case $N_n' \subset \mathrm{int}\, N_n$ since we can find a manifold isolating block N_n'' with $N_n'' \subset \mathrm{int}\, N_n, \mathrm{int}\, N_n'$. Assume that $N_n' \subset \mathrm{int}\, N_n$. Taking sufficiently large $T > 0$ we have

$$A^{[-T,T]} \subset (N_n' \setminus \nu_{3\delta}(\partial N_n')) \subset (N_n \setminus \nu_{3\delta}(\partial N_n)). \tag{17}$$

It is easy to see that $\hat{\epsilon}_{n,n}$ is homotopic to a map $\hat{\epsilon}^{(0)}_{n,n} : (N_n/\overline{L}_n) \wedge (N_n/L_n) \to B_\delta/S_\delta$ defined by

$$\hat{\epsilon}^{(0)}_{n,n}([y]\wedge[x]) = \begin{cases} [b_n(\varphi(y, -3T)) - a_n(\varphi(x, 3T))] & \text{if} \begin{cases} \varphi(x, [0, 3T]) \subset N_n \setminus L_n, \\ \varphi(y, [-3T, 0]) \subset N_n \setminus \overline{L}_n, \\ \| b_n(\varphi(y, -3T)) - a_n(\varphi(x, 3T)) \| < \delta \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Suppose that $\epsilon^{(0)}([y] \wedge [x]) \neq *$. Then

$$\varphi(x, 3T) \in N_n^{[-3T,0]}, \varphi(y, -3T) \in N_n^{[0,3T]},\ \| \varphi(y, -3T) - \varphi(x, 3T) \| < 5\delta.$$

Taking small $\delta > 0$ and the using the fact that $N_n \subset \mathrm{int}\, A$, we may suppose that

$$\varphi(x, 3T), \varphi(y, -3T) \in A^{[-3T,3T]},$$

which implies

$$a_n(\varphi(x, 3T)) = \varphi(x, 3T),\ b_n(\varphi(y, -3T)) = \varphi(y, -3T).$$

Here we have used (16) and (17). We can assume that δ is independent of x, y since N_n is compact. So we have

$$\hat{\epsilon}^{(0)}_{n,n}([y]\wedge[x]) = \begin{cases} [\varphi(y,-3T)-\varphi(x,3T)] & \text{if} \begin{cases} \varphi(x,[0,3T]) \subset N_n \setminus L_n, \\ \varphi(y,[-3T,0]) \subset N_n \setminus \overline{L}_n, \\ \|\varphi(y,-3T)-\varphi(x,3T)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

On the hand, we can write

$$\hat{\epsilon}'_{n,n} \circ (s \wedge \bar{s})([y]\wedge[x]) =$$
$$\begin{cases} [b'_n(\varphi(y,-3T)) - a'_n(\varphi(x,3T))] & \text{if} \begin{cases} \varphi(x,[0,2T]) \subset N_n \setminus L_n, \\ \varphi(x,[T,3T]) \subset N'_n \setminus L'_n, \\ \varphi(y,[-2T,0]) \subset N_n \setminus \overline{L}_n, \\ \varphi(y,[-3T,-T]) \subset N'_n \setminus \overline{L}'_n, \\ \|b'_n(\varphi(y,-3T)) - a'_n(\varphi(x,3T))\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

As before, if $\hat{\epsilon}'_{n,n} \circ (s \wedge \bar{s})([y]\wedge[x]) \neq *$ we have

$$\varphi(x,3T), \varphi(y,-3T) \in A^{[-3T,3T]}$$

and we can write

$$\hat{\epsilon}'_{n,n} \circ (s \wedge \bar{s})([y]\wedge[x]) = \begin{cases} [\varphi(y,-3T) - \varphi(x,3T)] & \text{if} \begin{cases} \varphi(x,[0,2T]) \subset N_n \setminus L_n, \\ \varphi(x,[T,3T]) \subset N'_n \setminus L'_n, \\ \varphi(y,[-2T,0]) \subset N_n \setminus \overline{L}_n, \\ \varphi(y,[-3T,-T]) \subset N'_n \setminus \overline{L}'_n, \\ \|\varphi(y,-3T)) - \varphi(x,3T)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

We will show that $\hat{\epsilon}^{(0)}_{n,n} = \hat{\epsilon}'_{n,n} \circ (s \wedge \bar{s})$. It is sufficient to prove that $\hat{\epsilon}^{(0)}_{n,n}([y]\wedge[x]) \neq *$ if and only if $\hat{\epsilon}'_{n,n} \circ (s \wedge \bar{s})([y]\wedge[x]) \neq *$. It is easy to see that if $\hat{\epsilon}'_{n,n} \circ (s \wedge \bar{s})([y]\wedge[x]) \neq *$ then $\hat{\epsilon}^{(0)}_{n,n}([y]\wedge[x]) \neq *$ using the assumption that $N'_n \subset \operatorname{int} N_n$. Conversely, suppose that $\hat{\epsilon}^{(0)}_{n,n}([y]\wedge[x]) \neq *$. Then $\varphi(x,3T), \varphi(y,-3T) \in A^{[-3T,3T]}$ and we have

$$\begin{aligned} \varphi(x,[2T,3T]) &= \varphi(\varphi(x,3T),[-T,0]) \subset A^{[-2T,2T]} \subset \operatorname{int} N_n, \\ \varphi(y,[-3T,2T]) &= \varphi(\varphi(y,-3T),[0,T]) \subset A^{[-2T,2T]} \subset \operatorname{int} N_n. \end{aligned}$$

This implies that $\hat{\epsilon}^{(0)}_{n,n}([y]\wedge[x]) \neq *$.

□

A calculation similar to that in the proof of Lemma 4.5 proves the following:

Lemma 4.6. *Suppose that $\lambda < \lambda_n$, $\mu > \mu_n$. Take manifolds isolating blocks N_n, N'_n for $\operatorname{inv}(V^{\mu_n}_{\lambda_n} \cap J^+_n)$, $\operatorname{inv}(V^{\mu}_{\lambda} \cap J^+_n)$. Note that we have canonical homotopy equivalences*

$$\Sigma^{V^{\lambda_n}_{\lambda}}(N_n/L_n) \cong N'_n/L'_n, \ \Sigma^{V^{\mu}_{\mu_n}}(N_n/\overline{L}_n) \cong N'_n/\overline{L}'_n.$$

See Proposition 5.6 of [7]*. The following diagram is commutative up to S^1-equivariant homotopy:*

$$\begin{tikzcd} \Sigma^{V^{\mu}_{\mu_n}}(N_n/\overline{L}_n) \wedge \Sigma^{V^{\lambda_n}_{\lambda}}(N_n/L_n) \arrow[r, "\Sigma^W \hat{\epsilon}_{n,n}"] \arrow[d, "\cong"'] & (V^{\mu}_{\lambda})^+ \\ (N'_n/\overline{L}'_n) \wedge (N'_n/L'_n) \arrow[ur, "\hat{\epsilon}'_{n,n}"'] & \end{tikzcd}$$

Here $W = V^{\lambda_n}_{\lambda} \oplus V^{\mu}_{\mu_n}$.

This lemma implies that the morpshim $\epsilon_{n,n}$ (and hence $\epsilon_{m,n}$) is independent of the choices of λ_n, μ_n.

We have obtained a collection $\{\epsilon_{m,n} : \bar{I}_n \wedge I_m \to S\}_{n \geq m}$ of morphisms. Since $j_{m,n} = j_{m+1,n} \circ j_{m,m+1}$, the following diagram is commutative:

$$\begin{tikzcd} \bar{I}_n \wedge I_m \arrow[r, "\epsilon_{m,n}"] \arrow[d, "\mathrm{id} \wedge j_{m,m+1}"'] & S \\ \bar{I}_n \wedge I_{m+1} \arrow[ur, "\epsilon_{m+1,n}"'] & \end{tikzcd} \tag{18}$$

Lemma 4.7. *For $m < n$, the following diagram is commutative:*

$$\begin{tikzcd} \bar{I}_n \wedge I_m \arrow[r, "\epsilon_{m,n}"] & S \\ \bar{I}_{n+1} \wedge I_m \arrow[u, "\bar{j}_{n,n+1} \wedge \mathrm{id}"] \arrow[ur, "\epsilon_{m,n+1}"'] & \end{tikzcd} \tag{19}$$

Proof. We have to prove that the following diagram is commutative up to S^1-equivariant homotopy:

$$\begin{tikzcd} (N_n/\overline{L}_n) \wedge (N_m/L_m) \arrow[r, "\hat{\epsilon}_{n,m}"] & B_\delta/S_\delta \\ (N_{n+1}/\overline{L}_{n+1}) \wedge (N_m/L_m) \arrow[u, "\bar{\imath}_{n,n+1} \wedge \mathrm{id}"] \arrow[ur, "\hat{\epsilon}_{m,n+1}"'] & \end{tikzcd} \tag{20}$$

By Lemma 4.5, we can use the following specific manifold isolating blocks (with corners). First take a manifold isolating block N_{n+1} for $\mathrm{inv}(V^{\mu_{n+1}}_{\lambda_{n+1}} \cap J^+_{n+1})$. We have compact submanifolds $L_{n+1}, \overline{L}_{n+1}$ in ∂N_{n+1} with

$$\partial N_{n+1} = L_{n+1} \cup \overline{L}_{n+1}, \ \partial L_{n+1} = \partial \overline{L}_{n+1} = L_{n+1} \cap \overline{L}_{n+1}.$$

Moreover (N_{n+1}, L_{n+1}) is an index pair for $(\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_{n+1}^+), \varphi_{n+1})$ and $(N_{n+1}, \overline{L}_{n+1})$ is an index pair for $(\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_{n+1}^+), \overline{\varphi}_{n+1})$, where $\overline{\varphi}_{n+1}$ is the reverse flow of φ_{n+1}. Put

$$\begin{aligned}
N_m &:= N_{n+1} \cap J_m^+ = N_{n+1} \cap \bigcap_{j=1}^{b_1} g_{j,+}^{-1}((-\infty, m+\theta]), \\
L_m &:= L_{n+1} \cap N_m, \\
\overline{L}_m &:= (\overline{L}_{n+1} \cap N_m) \cup \bigcup_{j=1}^{b_1} N_m \cap g_{j,+}^{-1}(m+\theta), \\
N_n &:= N_{n+1} \cap J_n^+ = N_{n+1} \cap \bigcap_{j=1}^{b_1} g_{j,+}^{-1}((-\infty, n+\theta]), \\
L_n &:= L_{n+1} \cap N_n, \\
\overline{L}_n &:= (\overline{L}_{n+1} \cap N_n) \cup \bigcup_{j=1}^{b_1} N_n \cap g_{j,+}^{-1}(n+\theta)
\end{aligned}$$

Then N_m, N_n are isolating blocks for $\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_m^+)$, $\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_n^+)$ and N_m, N_n, L_m, $\overline{L}_m$, L_n, $\overline{L}_n$ are manifolds with corners (for generic θ). Moreover (N_m, L_m), $(N_m, \overline{L}_m)$, (N_n, L_n), $(N_n, \overline{L}_n)$ are index pairs for $(\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_m^+), \varphi_{n+1})$, $(\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_m^+), \overline{\varphi}_{n+1})$, $(\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_n), \varphi_{n+1})$, $(\mathrm{inv}(V_{\lambda_{n+1}}^{\mu_{n+1}} \cap J_n), \overline{\varphi}_{n+1})$ respectively. Also we have

$$\begin{aligned}
&L_m \cup \overline{L}_m = \partial N_m, \ \partial L_m = \partial \overline{L}_m = L_m \cap \overline{L}_m, \\
&L_n \cup \overline{L}_n = \partial N_n, \ \partial L_n = \partial \overline{L}_n = L_m \cap \overline{L}_n.
\end{aligned}$$

The connecting morphisms $j_{m,n} : I_m \to I_n$, $j_{m,n+1} : I_m \to I_{n+1}$ and $\bar{j}_{n,n+1} : \bar{I}_{n+1} \to \bar{I}_n$ are induced by the inclusions

$$i_{m,n} : N_m/L_m \to N_n/L_n, \ i_{m,n+1} : N_m/L_m \to N_{n+1}/L_{n+1}$$

and projection

$$\bar{i}_{n,n+1} : N_{n+1}/\overline{L}_{n+1} \to N_{n+1} \Bigg/ \left\{ \overline{L}_{n+1} \cup \bigcup_j \Big(N_{n+1} \cap g_{j,+}^{-1}([n+\theta, \infty)) \Big) \right\} = N_n/\overline{L}_n.$$

With the index pairs we have taken above, for $x \in N_m, y \in N_{n+1}$ we can write

$$\hat{\epsilon}_{m,n+1}([y] \wedge [x]) = \begin{cases} [b_{n+1}(y) - a_{n+1}(x)] & \text{if } \|b_{n+1}(y) - a_{n+1}(x)\| < \delta, \\ * & \text{otherwise.} \end{cases}$$

Also we have

$$\hat{\epsilon}_{m,n} \circ (\bar{i}_{n,n+1} \wedge \mathrm{id})([y] \wedge [x]) = \begin{cases} [b_n(y) - a_n(x)] & \text{if } y \in N_n, \ \|b_n(y) - a_n(x)\| < \delta, \\ * & \text{otherwise.} \end{cases}$$

We may suppose that $a_n(x) = a_{n+1}(x)$ for $x \in N_m$. Note that if $\hat{\epsilon}_{m,n+1}([y] \wedge [x]) \neq *$ or $\hat{\epsilon}_{m,n} \circ (\bar{i}_{n,n+1} \wedge \mathrm{id})([y] \wedge [x]) \neq *$ we have $y \in \nu_{5\delta}(N_m)$. For small $\delta > 0$ we can suppose that $\nu_{5\delta}(N_m) \cap N_{n+1} \subset N_n$ and that $b_{n+1}(y) = b_n(y)$ for $y \in \nu_{5\delta}(N_m) \cap N_{n+1}$. This implies that (20) commutes.

$\square$

The commutativity of the diagrams (18) and (19) means that the collection $\{\epsilon_{m,n}\}_{m,n}$ defines an element ϵ of $\lim\limits_{\infty \leftarrow m} \lim\limits_{n \to \infty} \operatorname{mor}_{\mathfrak{C}}(\bar{I}_n \wedge I_m, S)$.

Next we will define $\eta \in \lim\limits_{\infty \leftarrow n} \lim\limits_{m \to \infty} \operatorname{mor}_{\mathfrak{C}}(S, I_m \wedge \bar{I}_n)$. Take a manifold isolating block $N_n (\subset V_{\lambda_n}^{\mu_n})$ of $\operatorname{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+)$. As usual we have compact submanifolds $L_n, \overline{L}_n$ of ∂N_n such that

$$\partial N_n = L_n \cup \overline{L}_n, \ \partial L_n = \partial \overline{L}_n = L_n \cap \overline{L}_n$$

and that (N_n, L_n), $(N_n, \overline{L}_n)$ are index pairs for $(\operatorname{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+), \varphi_n)$, $(\operatorname{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+), \overline{\varphi}_n)$ respectively. Taking a large positive number $R > 0$ we may suppose that $N_n \subset B_{R/2}$, where $B_{R/2} = \{x \in V_{\lambda_n}^{\mu_n} | \|x\| \leq R/2\}$. We define

$$\hat{\eta}_{n,n} : (V_{\lambda_n}^{\mu_n})^+ = B_R/S_R \to (N_n/L_n) \wedge (N_n/\overline{L}_n)$$

by

$$\hat{\eta}_{n,n}([x]) = \begin{cases} [x] \wedge [x] & \text{if } x \in N_n, \\ * & \text{otherwise.} \end{cases}$$

We can see that $\hat{\eta}_{n,n}$ is a well-defined continuous map and induces a morphism

$$\eta_{n,n} : S \to I_n \wedge \bar{I}_n.$$

For $m > n$, we define $\eta_{m,n} : S \to I_m \wedge \bar{I}_n$ to be the composition

$$S \xrightarrow{\eta_{n,n}} I_n \wedge \bar{I}_n \xrightarrow{j_{n,m} \wedge \mathrm{id}} I_m \wedge \bar{I}_n.$$

Lemma 4.8. *The morphism $\eta_{m,n} \in \operatorname{mor}_{\mathfrak{S}}(S, I_m \wedge \bar{I}_n)$ is independent of the choices of R and N_n.*

Proof. The independence from R is easy. We prove the independence from the choice of N_n. Take another manifold isolating block N_n' of $\operatorname{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+)$. We may assume that $N_n, N_n' \subset A$ for an isolating neighborhood A of $\operatorname{inv}(V_{\lambda_n}^{\mu_n} \cap J_n^+)$. It is sufficient to show that the following diagram is commutative up to S^1-equivariant homotopy:

$$\begin{tikzcd} B_R/S_R \arrow[r, "\hat{\eta}_{n,n}"] \arrow[dr, "\hat{\eta}'_{n,n}"'] & (N_n/L_n) \wedge (N_n/\overline{L}_n) \arrow[d, "s \wedge \bar{s}"] \\ & (N_n'/L_n') \wedge (N_n'/\overline{L}_n') \end{tikzcd}$$

Here $s = s_T, \bar{s} = \bar{s}_T$ are the flow maps with $T \gg 0$. For $x \in B_R$ we have

$(s \wedge \bar{s}) \circ \hat{\eta}_{n,n}([x]) =$

$$\begin{cases} [\varphi(x, 3T)] \wedge [\varphi(x, -3T)] & \text{if} \begin{cases} \varphi(x, [0, 2T]) \subset N_n \setminus L_n, \ \varphi(x, [T, 3T]) \subset N_n' \setminus L_n', \\ \varphi(x, [-2T, 0]) \subset N_n \setminus \overline{L}_n, \ \varphi(x, [-3T, -T]) \subset N_n' \setminus \overline{L}_n', \end{cases} \\ * & \text{otherwise} \end{cases}$$

and

$$\hat{\eta}'_{n,n}([x]) = \begin{cases} [x] \wedge [x] & \text{if } x \in \operatorname{int} N'_n, \\ * & \text{otherwise.} \end{cases}$$

We can reduce the proof to the case $N_n \subset \operatorname{int} N'_n$. Suppose $N_n \subset \operatorname{int} N'_n$. Also we may assume that $A^{[-T,T]} \subset \operatorname{int} N_n$, choosing a sufficiently large T. If $(s \wedge \bar{s}) \circ \hat{\eta}_{n,n}([x]) \neq *$, we have

$$\varphi(x, [-3T, 3T]) \subset \operatorname{int} N'_n.$$

Conversely, suppose that $\varphi(x, [-3T, 3T]) \subset \operatorname{int} N'_n$. Then we have $x \in A^{[-3T,3T]}$. Hence

$$\varphi(x, [-2T, 2T]) \subset A^{[-T,T]} \subset \operatorname{int} N_n.$$

Therefore $\varphi(x, [0, 2T]) \subset N_n \setminus L_n, \varphi(x, [-2T, 0]) \subset N_n \setminus \overline{L}_n$. Thus $(s \wedge \bar{s}) \circ \hat{\eta}_{n,n}([x]) \neq *$. We have obtained:

$$(s \wedge \bar{s}) \circ \hat{\eta}_{n,n}([x]) = \begin{cases} [\varphi(x, 3T)] \wedge [\varphi(x, -3T)] & \text{if } \varphi(x, [-3T, 3T]) \subset \operatorname{int} N'_n, \\ * & \text{otherwise} \end{cases}$$

This is homotopic to $\hat{\eta}'_{n,n}$ through a homotopy H defined by

$$H([x], s) =$$
$$\begin{cases} [\varphi(x, 3(1-s)T)] \wedge [\varphi(x, -3(1-s)T)] & \text{if } \varphi(x, [-3(1-s)T, 3(1-s)T]) \subset \operatorname{int} N'_n, \\ * & \text{otherwise.} \end{cases}$$

□

Lemma 4.9. *Let $\lambda < \lambda_n, \mu > \mu_n$. Take manifolds index pairs N_n, N'_n for* $\operatorname{inv}(J_n \cap V_{\lambda_n}^{\mu_n})$, $\operatorname{inv}(J_n \cap V_\lambda^\mu)$. *Then we have the canonical S^1-equivariant homotopy equivalence:*

$$\Sigma^{V_\lambda^{\lambda_n}}(N_n/L_n) \cong N'_n/L'_n,\ \Sigma^{V_{\mu_n}^{\mu}}(N_n/\overline{L}_n) \cong N'_n/\overline{L}'_n.$$

See Proposition 5.6 of [7]. *The following diagram is commutative up to S^1-equivariant homotopy:*

$$\begin{array}{ccc} (V_\lambda^\mu)^+ & \xrightarrow{\Sigma^W \hat{\eta}_{n,n}} & \Sigma^W(N_n/L_n) \wedge (N_n/\overline{L}_n) \\ & \searrow^{\hat{\eta}'_{n,n}} & \downarrow \\ & & (N'_n/L'_n) \wedge (N'_n/\overline{L}'_n) \end{array}$$

Here $W = V_\lambda^{\lambda_n} \oplus V_{\mu_n}^{\mu}$.

This lemma implies that $\eta_{n,n}$ (and hence $\eta_{m,n}$) is independent of the choice of λ_n, μ_n. Since $j_{n,m+1} = j_{m,m+1} \circ j_{n,m}$ for $m \geq n$, the following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{\eta_{m,n}} & I_m \wedge \bar{I}_n \\ & \searrow^{\eta_{m+1,n}} & \downarrow{\scriptstyle j_{m,m+1} \wedge \mathrm{id}} \\ & & I_{m+1} \wedge \bar{I}_n \end{array} \tag{21}$$

Lemma 4.10. *For $m \geq n+1$, the following diagram is commutative:*

$$\begin{tikzcd} S \arrow[r, "\eta_{m,n}"] \arrow[dr, "\eta_{m,n+1}"'] & I_m \wedge \bar{I}_n \\ & I_m \wedge \bar{I}_{n+1} \arrow[u, "\mathrm{id} \wedge \bar{j}_{n,n+1}"'] \end{tikzcd} \tag{22}$$

Proof. Let $m \geq n+1$. We have to show that the following diagram is commutative up to S^1-equivariant homotopy:

$$\begin{tikzcd} B_R/S_R \arrow[r, "\hat{\eta}_{m,n}"] \arrow[dr, "\hat{\eta}_{m,n+1}"'] & (N_m/L_m) \wedge (N_n/\overline{L}_n) \\ & (N_m/L_m) \wedge (N_{n+1}/\overline{L}_{n+1}). \arrow[u, "\mathrm{id} \wedge \bar{i}_{n,n+1}"'] \end{tikzcd} \tag{23}$$

By Lemma 4.8, we can use the following specific manifold isolating blocks N_m, N_n, N_{n+1} (with corners). Fix a manifold isolating block N_m for $\operatorname{inv}(V_{\lambda_m}^{\mu_m} \cap J_m^+)$. Then we have compact submanifolds $L_m, \overline{L}_m$ in ∂N_m such that

$$\partial N_m = L_m \cap \overline{L}_m,\ \partial L_m = \partial \overline{L}_m = L_m \cap \overline{L}_m.$$

Moreover (N_m, L_m) is an index pair for $(\operatorname{inv}(V_{\lambda_m}^{\mu_m} \cap J_m^+), \varphi_m)$ and $(N_m, \overline{L}_m)$ is an index pair for $(\operatorname{inv}(V_{\lambda_m}^{\mu_m} \cap J_m^+), \overline{\varphi}_m)$. Put

$$\begin{aligned} N_{n+1} &:= N_m \cap J_{n+1}^+ = N_m \cap \bigcap g_{j,+}^{-1}((-\infty, n+1+\theta]), \\ L_{n+1} &:= N_{n+1} \cap L_m, \\ \overline{L}_{n+1} &:= (\overline{L}_m \cap N_{n+1}) \cup \bigcup_{j=1}^{b_1} (N_{n+1} \cap g_{j,+}^{-1}(n+1+\theta)). \end{aligned}$$

Then N_{n+1}, L_{n+1} and $\overline{L}_{n+1}$ are manifolds with corners (for generic θ), and (N_{n+1}, L_{n+1}), $(N_{n+1}, \overline{L}_{n+1})$ are index pairs for $(\operatorname{inv}(V_{\lambda_m}^{\mu_m} \cap J_{n+1}^+), \varphi_m)$, $(\operatorname{inv}(V_{\lambda_m}^{\mu_m} \cap J_{n+1}^+), \overline{\varphi}_m)$ respectively. We define $N_n, L_n, \overline{L}_n$ similarly.

The attractor maps $i_{n,m} : N_n/L_n \to N_m/L_m$, $i_{n+1,m} : N_{n+1}/L_{n+1} \to N_m/L_m$ are the inclusions. The repeller map $\bar{i}_{n,n+1} : N_{n+1}/\overline{L}_{n+1} \to N_n/\overline{L}_n$ is the projection:

$$N_{n+1}/\overline{L}_{n+1} \to N_{n+1} \Bigg/ \left\{ \overline{L}_{n+1} \cup \bigcup_{j=1}^{b_1} (N_{n+1} \cap g_{j,+}^{-1}([n+\theta, \infty))) \right\} = N_n/\overline{L}_n.$$

With these index pairs, for $x \in B_R$ we can write

$$\hat{\eta}_{m,n}([x]) = \begin{cases} [x] \wedge [x] & \text{if } x \in N_n, \\ * & \text{otherwise,} \end{cases}$$

and

$$(\mathrm{id} \wedge \bar{i}_{n,n+1}) \circ \hat{\eta}_{m,n+1}([x]) \begin{cases} [x] \wedge [x] & \text{if } x \in N_n, \\ * & \text{otherwise.} \end{cases}$$

Thus the diagram (23) is commutative. $\square$

The commutativity of the diagrams (21), (22) implies that the collection $\{\eta_{m,n}\}_{m,n}$ defines an element $\eta \in \lim\limits_{\infty \leftarrow n} \lim\limits_{m \to \infty} \mathrm{mor}_{\mathfrak{C}}(S, I_m \wedge \bar{I}_n)$.

Proposition 4.11. *The morphisms ϵ and η are duality morphisms between $\underline{\mathrm{swf}}^A(Y)$ and $\underline{\mathrm{swf}}^R(-Y)$.*

Proof. Fix positive numbers R, δ with $0 < \delta \ll 1 \ll R$. Let $\pi : B_R/S_R \to B_\delta/S_\delta$ be the projection

$$B_R/S_R \to B_R/(B_R \setminus \mathrm{int}\, B_\delta) = B_\delta/S_\delta,$$

which is a homotopy equivalence. We have to prove that the diagrams (24) below is commutative for $m \ll n \ll m'$ and that the diagram (25) below is commutative up to S^1-equivariant homotopy for $n \ll m \ll n'$. (See Lemma 3.5 of [10].)

$$\begin{tikzcd}
(B_R/S_R) \wedge (N_m/L_m) \arrow[r, "\hat{\eta}_{m',n} \wedge \mathrm{id}"] \arrow[dr, "\gamma \circ (\pi \wedge i_{m,m'})"'] & (N_{m'}/L_{m'}) \wedge (N_n/\bar{L}_n) \wedge (N_m/L_m) \arrow[d, "\mathrm{id} \wedge \hat{\epsilon}_{m,n}"] \\
 & (N_{m'}/L_{m'}) \wedge (B_\delta/S_\delta)
\end{tikzcd} \tag{24}$$

Here $B_R = B(V_{\lambda_{m'}}^{\mu_{m'}}, R)$, $S_R = \partial B(V_{\lambda_{m'}}^{\mu_{m'}}, R)$, $N_m, N_n, N_{m'}$ are isolating blocks for $\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_m^+)$, $\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_n^+)$, $\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_{m'}^+)$ and γ is the interchanging map $(B_\delta/S_\delta) \wedge (N_{m'}/L_{m'}) \to (N_{m'}/L_{m'}) \wedge (B_\delta/S_\delta)$.

$$\begin{tikzcd}
(N_{n'}/\bar{L}_{n'}) \wedge (B_R/S_R) \arrow[r, "\mathrm{id} \wedge \hat{\eta}_{m,n}"] \arrow[d, "\gamma \circ (\bar{i}_{n,n'} \wedge \pi)"'] & (N_{n'}/\bar{L}_{n'}) \wedge (N_m/L_m) \wedge (N_n/\bar{L}_n) \arrow[d, "\hat{\epsilon}_{m,n'} \wedge \mathrm{id}"] \\
(B_\delta/S_\delta) \wedge (N_n/\overline{L}_n) \arrow[r, "\sigma \wedge \mathrm{id}"'] & (B_\delta/S_\delta) \wedge (N_n/\bar{L}_n)
\end{tikzcd} \tag{25}$$

Here $B_R = B(V_{\lambda_{n'}}^{\mu_{n'}}, R)$, $S_R = \partial B(V_{\lambda_{n'}}^{\mu_{n'}}, R)$, $N_m, N_n, N_{n'}$ are isolating blocks for $\mathrm{inv}(V_{\lambda_{n'}}^{\mu_{n'}} \cap J_m^+)$, $\mathrm{inv}(V_{\lambda_{n'}}^{\mu_{n'}} \cap J_n^+)$, $\mathrm{inv}(V_{\lambda_{n'}}^{\mu_{n'}} \cap J_{n'}^+)$, γ is the interchanging map $(N_n/L_n) \wedge (B_\delta/S_\delta) \to (B_\delta/S_\delta) \wedge (N_n/L_n)$ and $\sigma : B_\delta/S_\delta \to B_\delta/S_\delta$ is defined by $\sigma(v) = -v$.

First we consider (24). Let $m \ll n \ll m'$. Take a manifold isolating block $N_{m'}$ for $\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_{m'}^+)$. As in the proof of Lemma 4.7, from $N_{m'}$ and the functions $g_{j,+}$, we get index pairs

$$(N_n, L_n),\ (N_n, \overline{L}_n),\ (N_m, L_m),\ (N_m, \overline{L}_m)$$

for

$$(\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_n^+), \varphi_{m'}),\ (\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_n^+), \overline{\varphi}_{m'}),\ (\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_m^+), \varphi_{m'}),\ (\mathrm{inv}(V_{\lambda_{m'}}^{\mu_{m'}} \cap J_m^+), \overline{\varphi}_{m'}).$$

The attractor map

$$i_{m,n} : N_m/L_m \to N_n/L_n,\ i_{n,m'} : N_n/L_n \to N_{m'}/L_{m'}$$

are the injections, and the repeller maps

$$\bar{i}_{n,m'} : N_{m'}/\overline{L}_{m'} \to N_n/\overline{L}_n,\ \bar{i}_{m,n} : N_n/\overline{L}_n \to N_m/\overline{L}_m$$

are the projections.

For $x \in N_m$ and $y \in B_R(= B(V^{\mu_{m'}}_{\lambda_{m'}}, R))$, we can write

$$(\mathrm{id} \wedge \hat{\epsilon}_{m,n}) \circ (\hat{\eta}_{m',n} \wedge \mathrm{id})([y] \wedge [x]) = \begin{cases} [y] \wedge [b_n(y) - a_n(x)] & \text{if} \begin{cases} y \in N_n, \\ \|b_n(y) - a_n(x)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Note that if $\|b_n(y) - a_n(x)\| < \delta$ for some $x \in N_m$ we have $y \in \nu_{5\delta}(N_m)$. Fix an S^1-equivariant homotopy equivalence

$$r : \nu_{5\delta}(N_m) \to N_m$$

which is close to the identity such that

$$r(\nu_{5\delta}(L_n) \cap \nu_{5\delta}(N_m)) \subset L_m,\ r(\nu_{5\delta}(L_m)) \subset L_m.$$

Then $(\mathrm{id} \wedge \hat{\epsilon}_{m,n}) \circ (\hat{\eta}_{m',n} \wedge \mathrm{id})$ is homotopic to a map

$$f : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_\delta/S_\delta)$$

defined by

$$f([y] \wedge [x]) = \begin{cases} [r(y)] \wedge [b_n(y) - a_n(x)] & \text{if} \begin{cases} x \in N_m, y \in N_n \cap \nu_{5\delta}(N_m), \\ \|b_n(y) - a_n(x)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Define

$$H : (B_R/S_R) \wedge (N_m/L_m) \times [0,1] \to (N_{m'}/L_{m'}) \wedge (B_\delta/S_\delta)$$

by

$$\begin{aligned} & H([y] \wedge [x], s) \\ & = \begin{cases} [r((1-s)y + sx)] \wedge [b_n(y) - a_n(x)] & \text{if} \begin{cases} x \in N_m, y \in N_n \cap \nu_{5\delta}(N_m), \\ \|b_n(y) - a_n(x)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases} \end{aligned}$$

We can easily see that H is well-defined. We will show that H is continuous. It is sufficient to show that if we have a sequence (x_j, y_j, s_j) in $N_m \times N_n \times [0,1]$ with $y_j \to y \in \partial N_n = L_n \cup \overline{L}_n$ we have $H([y_j] \wedge [x_j], s_j) \to *$. If $y \in \overline{L}_n$ we have $\|b_n(y_j) - a_n(x_j)\| \geq \delta$ for large j. Hence $H([y_j] \wedge [x_j], s_j) \to *$. Consider the case $y \in L_n$. Assume that $\lim_{j \to \infty} H([y_j] \wedge [x_j], s_j) \neq *$. After passing to a subsequence we may suppose that $H([y_j] \wedge [x_j], s_j) \neq *$ for all j. Then $\|y_j - x_j\| < 5\delta$ for all j. For large j we have $(1 - s_j)y_j + s_j x_j \in \nu_{5\delta}(L_n) \cap \nu_{5\delta}(N_m)$. Hence $r((1 - s_j)y_j + s_j x_j) \in L_m \subset L_{m'}$, which implies $H([y_j] \wedge [x_j], s_j) = *$. This is a contradiction. Therefore H is continuous.

We have $H(\cdot, 0) = f$ and

$$H([y] \wedge [x], 1) = \begin{cases} [r(x)] \wedge [b_n(y) - a_n(x)] & \text{if} \begin{cases} x \in N_m, y \in N_n, \\ \|b_n(y) - a_n(x)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Fix a positive number $\delta' > 0$ with $0 \ll \delta' \ll \delta$. Take an S^1-equivaraint continuous map $a_n' : N_n \to N_n$ such that

$$\|a_n'(x) - a_n(x)\| < 2\delta', \ a_n'(N_n) \subset N_n \setminus \nu_{\delta'}(\partial N_n).$$

Through the homotopy equivalence

$$B_\delta/S_\delta = V^{\mu_{m'}}_{\lambda_{m'}}/(V^{\mu_{m'}}_{\lambda_{m'}} - \operatorname{int} B_\delta) \to V^{\mu_{m'}}_{\lambda_{m'}}/(V^{\mu_{m'}}_{\lambda_{m'}} - \operatorname{int} B_{\delta'}) = B_{\delta'}/S_{\delta'},$$

$H(\cdot, 1)$ is homotopic to a map

$$f' : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$$

defined by

$$f'([y] \wedge [x]) = \begin{cases} [r(x)] \wedge [b_n(y) - a_n'(x)] & \text{if} \begin{cases} x \in N_m, y \in N_n, \\ \|b_n(y) - a_n'(x)\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

There is a homotopy $h : N_n \times [0,1] \to N_n$ from b_n to the identity such that

$$h(\overline{L}_n, s) \subset \overline{L}_n, \ \|h(y,s) - y\| < 2\delta$$

for all $y \in N_n$ and $s \in [0,1]$. Then h naturally induces a homotopy

$$H' : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$$

defined by

$$H'([y] \wedge [x], s) = \begin{cases} [r(x)] \wedge [h(y,s) - a_n'(x)] & \text{if} \begin{cases} x \in N_m, y \in N_n, \\ \|b_n(y) - a_n'(x)\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

It is easy to see that H' is well-defined. To show that H' is continuous, it is sufficient to prove that if we have a sequence (x_j, y_j, s_j) in $N_m \times N_n \times [0,1]$ with $y_j \to y \in \partial N_n = L_n \cup \overline{L}_n$ then $H'([y_j] \wedge [x_j], s_j) \to *$. Suppose that $y \in \overline{L}_n$. Then for large j we have $\|h(y_j, s) - a_n'(x)\| \geq \delta'$. Thus $H([y_j] \wedge [x_j], s_j) \to *$. Suppose that $y \in L_n$. If $\lim_{j\to\infty} H'([y_j] \wedge [x_j], s_j) \neq *$, after passing to a subsequence, we may assume that $H([y_j] \wedge [x_j], s_j) \neq *$ for all j, which implies that $\|y_j - x_j\| < 5\delta$. So we have $x_j \in \nu_{5\delta}(L_n) \cap \nu_{5\delta}(N_m)$ for large j. Hence $r(x_j) \in N_m$, which means $H'([y_j] \wedge [x_j], s_j) = *$. This is contradiction. Therefore H' is continuous.

We can see that H' is a homotopy from f' to a map $f'' : (B_R/S_R) \wedge (N_m/L_m) \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$ defined by

$$f''([y] \wedge [x]) = \begin{cases} [r(x)] \wedge [y - a_n'(x)] & \text{if} \begin{cases} x \in N_m, y \in B_R, \\ \|y - a_n'(x)\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Note that for $y \in B_R \setminus N_n$ we have $f''([y] \wedge [x]) = *$ since $\|y - a_n'(x)\| \geq \delta'$. Define

$$H'' : (B_R/S_R) \wedge (N_m/L_m) \times [0,1] \to (N_{m'}/L_{m'}) \wedge (B_{\delta'}/S_{\delta'})$$

by

$$H''([y]\wedge[x],s) = \begin{cases} [r(x)]\wedge[y-(1-s)a'_n(x)] & \text{if} \begin{cases} x\in N_m, y\in B_R, \\ \|y-(1-s)a'_n(x)\|<\delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

It is easy to see that H'' is well-defined and continuous. We have

$$H''([y]\wedge[x],1) = \begin{cases} [r(x)]\wedge[y] & \text{if} \begin{cases} x\in N_m, y\in B_R, \\ \|y\|<\delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

We can easily show that $H''(\cdot,1)$ is homotopic to $\gamma\circ(\pi\wedge i_{m,m'})$.

We have proved that $(\mathrm{id}\wedge\hat{\epsilon}_{m,n})\circ(\hat{\eta}_{m',n}\wedge\mathrm{id})$ is S^1-equivriantly homotopic to $\gamma\circ(\pi\wedge i_{m,m'})$, which implies that the diagram (24) is commutative up to S^1-equivariant homotopy.

Let us consider (25). We have to prove that for $n\ll m\ll n'$ the composition

$$(N_{n'}/\overline{L}_{n'})\wedge(B_R/S_R)\xrightarrow{\mathrm{id}\wedge\hat{\eta}_{m,n}}(N_{n'}/\overline{L}_{n'})\wedge(N_m/L_m)\wedge(N_n/\overline{L}_n)\xrightarrow{\hat{\epsilon}_{m,n'}\wedge\mathrm{id}}(B_\delta/S_\delta)\wedge(N_n/\overline{L}_n)$$

is S^1-equivarilantly homotopic to $(\sigma\wedge\mathrm{id})\circ\gamma\circ(\bar{i}_{n',n}\wedge\pi)$.

For $x\in B_R=B(V_{\lambda_{n'}}^{\mu_{n'}},R)$, $y\in N_{n'}$ we have

$$\begin{gathered}(\hat{\epsilon}_{m,n'}\wedge\mathrm{id})\circ(\mathrm{id}\wedge\hat{\eta}_{m,n})([y]\wedge[x]) = \\ \begin{cases} [b_{n'}(y)-a_{n'}(x)]\wedge[x] & \text{if} \begin{cases} x\in N_n, \\ \|b_{n'}(y)-a_{n'}(x)\|<\delta, \end{cases} \\ * & \text{otherwise.} \end{cases}\end{gathered}$$

Take a homotopy equivalence $\bar{r}:\nu_{5\delta}(N_n)\to N_n$ which is a close to the indentiy such that

$$\bar{r}(\nu_{5\delta}(\overline{L}_{n'})\cap\nu_{5\delta}(N_n))\subset\overline{L}_n,\ \ \bar{r}(\nu_{5\delta}(\overline{L}_n))\subset\overline{L}_n. \tag{26}$$

Note that if $\|b_{n'}(y)-a_{n'}(x)\|<\delta$ for some $x\in N_n$ we have $y\in\nu_{5\delta}(N_n)$.

It is easy to see that $(\hat{\epsilon}_{m,n'}\wedge\mathrm{id})\circ(\mathrm{id}\wedge\hat{\eta}_{m,n})$ is homotopic to a map

$$f:(N_{n'}/\overline{L}_{n'})\wedge(B_R/S_R)\to(B_\delta/S_\delta)\wedge(N_n/L_n)$$

defined by

$$f([y]\wedge[x]) = \begin{cases} [b_{n'}(y)-a_{n'}(x)]\wedge[\bar{r}(x)] & \text{if} \begin{cases} x\in N_n, y\in N_{n'}\cap\nu_{5\delta}(N_n), \\ \|b_{n'}(y)-a_{n'}(x)\|<\delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Define a homotopy $H:(N_{n'}/\overline{L}_{n'})\wedge(B_R/S_R)\times[0,1]\to(B_\delta/S_\delta)\wedge(N_n/\overline{L}_n)$ by

$$\begin{gathered}H([y]\wedge[x],s) = \\ \begin{cases} [b_{n'}(y)-a_{n'}(x)]\wedge[\bar{r}((1-s)x+sy)] & \text{if} \begin{cases} x\in N_n, y\in N_{n'}\cap\nu_{5\delta}(N_n), \\ \|b_{n'}(y)-a_{n'}(x)\|<\delta \end{cases} \\ * & \text{otherwise.} \end{cases}\end{gathered}$$

Then H is a well-defined and continuous homotopy from $(\mathrm{id} \wedge \hat{\epsilon}_{m,n'}) \circ (\mathrm{id} \wedge \hat{\eta}_{m,n})$ to $H(\cdot, 1)$. We have

$$H([y] \wedge [x], 1) = \begin{cases} [b_{n'}(y) - a_{n'}(x)] \wedge [\bar{r}(y)] & \text{if} \begin{cases} x \in N_n, y \in N_{n'} \cap \nu_{5\delta}(N_n), \\ \|b_{n'}(y) - a_{n'}(x)\| < \delta, \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Fix a positive number δ' with $0 < \delta' \ll \delta$. Take a continuous map $b'_{n'} : N_n \to N_n$ such that

$$b'_{n'}(N_n) \subset N_n \setminus \nu_{\delta'}(\partial N_n), \ \|b'_{n'}(y) - b_{n'}(y)\| < 2\delta' \quad (\text{for } x \in N_n).$$

Then though the homotopy equivariance $B_\delta / S_\delta \to B_{\delta'}/S_{\delta'}$, f is homotopic to a map

$$f' : (N_{n'}/\overline{L}_{n'}) \wedge (B_R/S_R) \to (B_{\delta'}/S_{\delta'}) \wedge (N_n/\overline{L}_n)$$

defined by

$$f'([y] \wedge [x]) = \begin{cases} [b'_{n'}(y) - a_{n'}(x)] \wedge [\bar{r}(y)] & \text{if} \begin{cases} x \in N_n, y \in N_{n'} \cap \nu_{5\delta}(N_n), \\ \|b'_{n'}(y) - a_{n'}(x)\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

There is a homotopy $h : N_{n'} \times [0, 1] \to N_{n'}$ from $a_{n'}$ to the identity such that that

$$h(L_{n'}) \subset L_{n'}, \ \|h(y, s) - y\| < 2\delta.$$

We can see that h induces a homotopy H' from f' to a map f'' defined by

$$f''([y] \wedge [x]) = \begin{cases} [b'_{n'}(y) - x] \wedge [\bar{r}(y)] & \text{if} \begin{cases} x \in B_R, y \in N_{n'} \cap \nu_{5\delta}(N_n), \\ \|b'_{n'}(y) - x\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Note that for $x \in B_R \setminus \operatorname{int} N_n$ we have $f'([y] \wedge [x]) = *$ since $\|b'_{n'}(y) - x\| \geq \delta'$. Define a homotopy $H'' : (N_{n'}/\overline{L}_{n'}) \wedge (B_R/S_R) \times [0, 1] \to (B_{\delta'}/S_{\delta'}) \wedge (N_n/\overline{L}_n)$ by

$$H''([y] \wedge [x], s) := \begin{cases} [(1-s)b'_{n'}(y) - x] \wedge [\bar{r}(y)] & \text{if} \begin{cases} x \in B_R, y \in N_{n'} \cap \nu_{5\delta}(N_n), \\ \|(1-s)b'_{n'}(y) - x\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

Then H'' is well-defined and continuous, and we have

$$H''([y] \wedge [x], 1) := \begin{cases} [-x] \wedge [\bar{r}(y)] & \text{if} \begin{cases} x \in B_R, y \in N_{n'} \cap \nu_{5\delta}(N_n), \\ \|x\| < \delta', \end{cases} \\ * & \text{otherwise.} \end{cases}$$

It is easy to see that $H''(\cdot, 1)$ is homotopic to $(\sigma \wedge \mathrm{id}) \circ \gamma \circ (\bar{\imath}_{m,n} \wedge \pi)$. Thus the diagram (25) is commutative up S^1-equivariant homotopy.

$\square$

5. Relative Bauer-Furuta invariants for 4-manifolds

5.1. **Setup.** Let X be a compact, connected, oriented, Riemannian 4–manifold with nonempty boundary $\partial X := Y$ not necessarily connected. Equip X with a spin^c structure $\hat{\mathfrak{s}}$ which induces a spin^c structure $\mathfrak{s}$ on Y. Denote by $S_X = S^+ \oplus S^-$ the spinor bundle of X and denote by $\hat{\rho}$ the Clifford multiplication. Choose a metric $\hat{g}$ on X so that a neighborhood of the boundary is isometric to the cylinder $[-3, 0] \times Y$ with the product metric and ∂X identified with $\{0\} \times Y$. To make some distinction, we will often decorate notations associated to X with hat. For instance, let g be the Riemannian metric on Y restricted from $\hat{g}$ on X. Let S_Y be the associated spinor bundle on Y and $\rho\colon TY \to \mathrm{End}(S_Y)$ be the Clifford multiplication.

We write $Y = \coprod Y_j$ as a union of connected component. From now on, we will treat X as a spin^c cobordism, i.e. we label each connected component of Y as either incoming or outgoing satisfying $Y = -Y_{\mathrm{in}} \sqcup Y_{\mathrm{out}}$. We sometimes write this cobordism as $X\colon Y_{\mathrm{in}} \to Y_{\mathrm{out}}$. Denote by $\iota\colon Y \hookrightarrow X$ the inclusion map. We also choose the following auxiliary data when defining our invariants

- A basepoint $\hat{o} \in X$.
- A set of loops $\{\alpha_1, \dots, \alpha_{b_{1,\alpha}}\}$ in X representing a basis of cokernel of the induced map $\iota_*\colon H_1(Y; \mathbb{R}) \to H_1(X; \mathbb{R})$.
- A set of loops $\{\beta_1, \dots, \beta_{b_{\mathrm{in}}}\}$ in Y_{in} representing a basis of a subspace complementary to kernel of the induced map $\iota_*\colon H_1(Y_{\mathrm{in}}; \mathbb{R}) \to H_1(X; \mathbb{R})$.
- A set of loops $\{\beta_{b_{\mathrm{in}}+1}, \dots, \beta_{b_{1,\beta}}\}$ in Y_{out} such that $\{\beta_1, \dots, \beta_{b_{1,\beta}}\}$ represents a basis of a subspace complementary to kernel of the induced map $\iota_*\colon H_1(Y; \mathbb{R}) \to H_1(X; \mathbb{R})$.
- A *based path data* $[\vec{\eta}]$, whose definition is given below.

Definition 5.1. A based path data is an equivalent class of paths $(\eta_1, \eta_2, \dots, \eta_{b_0(Y)})$, where each η_j is a path from $\hat{o}$ to a point in Y_j. We say that $(\eta_1, \dots, \eta_{b_0(Y)})$ and $(\eta'_1, \dots, \eta'_{b_0(Y)})$ are equivalent if the composed path $\eta'_j * (-\eta_j)$ represents the zero class in $H_1(X, Y; \mathbb{R})$ for each $j = 1, \dots, b_0(Y)$.

Remark. (i) The set of loops $\{\alpha_1, \dots, \alpha_{b_{1,\alpha}}\}$ corresponds to a dual basis of kernel of $\iota^*\colon H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R})$.
(ii) The set of loops $\{\beta_1, \dots, \beta_{b_{1,\beta}}\}$ corresponds to a dual basis of image of $\iota^*\colon H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R})$.
(iii) It follows that $b_{1,\alpha} = \dim \ker \iota^*$, $b_{1,\beta} = \dim \mathrm{im}\, \iota^*$, and $b_{1,\alpha} + b_{1,\beta} = b_1(X)$.

As usual, we will set up the Seiberg–Witten equations on a particular slice of the configuration space. For the manifold with boundary X, we will consider the double Coulomb condition introduced by the first author [8] rather than the classical Coulomb–Neumann condition. Let us briefly recall the definition.

Definition 5.2. For a 1-form $\hat{a}$ on X, we have a decomposition $\hat{a}|_Y = \mathbf{t}\hat{a} + \mathbf{n}\hat{a}$ on the boundary, where $\mathbf{t}\hat{a}$ and $\mathbf{n}\hat{a}$ are the tangential part and the normal part respectively. When $Y = \coprod Y_i$ has several connected components, we denote by $\mathbf{t}_i\hat{a}$ and $\mathbf{n}_i\hat{a}$ the corresponding parts of $\hat{a}|_{Y_i}$. We say that a 1-form $\hat{a}$ satisfies the double Coulomb condition if:

(1) $\hat{a}$ is coclosed, i.e. $d^*\hat{a} = 0$;
(2) Its restriction to the boundary is coclosed, i.e. $d^*(\mathbf{t}\hat{a}) = 0$;
(3) For each j, we have $\int_{Y_j} \mathbf{t_j}(*\hat{a}) = 0$.

We denote by $\Omega^1_{CC}(X)$ the space of 1-forms satisfying the double Coulomb condition.

As a consequence of [8, Proposition 2.2], we can identify $H^1(X;\mathbb{R})$ with a space of harmonic 1-forms satisfying double Coulomb condition

$$H^1(X;\mathbb{R}) \cong \mathcal{H}^1_{CC}(X) := \{\hat{a} \in \Omega^1_{CC}(X) \mid d\hat{a} = 0\}.$$

Since X is connected, we observe that the cohomology long exact sequence of the pair (X,Y) gives rise to a short exact sequence

$$0 \to \mathbb{R}^{b_0(Y)-1} \to H^1(X,Y;\mathbb{R}) \to \ker \iota^* \to 0.$$

By classical Hodge Theorem, the relative cohomology group $H^1(X,Y;\mathbb{R})$ is represented with harmonic 1-forms with Dirichlet boundary condition. Since condition (3) from Definition 5.2 is of codimension $b_0(Y) - 1$, we can conclude that a space of harmonic 1-forms satisfying both Dirichlet boundary condition and condition (3) from Definition 5.2 is isomorphic to $\ker \iota^*$. Notice that such 1-forms trivially satisfy other double Coulomb conditions. Hence, we make an identification

$$\ker \iota^* \cong \mathcal{H}^1_{DC}(X) := \{\hat{a} \in \Omega^1_{CC}(X) \mid d\hat{a} = 0,\ \mathbf{t}\hat{a} = 0\}. \tag{27}$$

The double Coulomb slice $Coul^{CC}(X)$ is defined as

$$Coul^{CC}(X) := L^2_{k+1/2}(i\Omega^1_{CC}(X) \oplus \Gamma(S^+)), \tag{28}$$

where k is an integer greater than 4 fixed throughout the paper. Next, we introduce projections from $Coul^{CC}(X)$ related to the loops $\{\alpha_1, \dots, \alpha_{b_{1,\alpha}}\}$ and $\{\beta_1, \dots, \beta_{b_{1,\beta}}\}$. We define a (nonorthogonal) projection

$$\hat{p}_\alpha \colon Coul^{CC}(X) \to \mathcal{H}^1_{DC}(X) \tag{29}$$

by sending $(\hat{a}, \hat{\phi})$ to the unique element in $\mathcal{H}^1_{DC}(X)$ satisfying

$$\int_{\alpha_j} \hat{a} = i \int_{\alpha_j} \hat{p}_\alpha(\hat{a}, \hat{\phi}) \text{ for every } j = 1, 2, \dots, b_{1,\alpha}.$$

On the other hand, we define a map

$$\begin{aligned} \hat{p}_\beta \colon Coul^{CC}(X) &\to \mathbb{R}^{b_{1,\beta}} \\ (\hat{a}, \hat{\phi}) &\mapsto (-i\int_{\beta_1} \mathbf{t}\hat{a},\ \dots\ , -i\int_{\beta_{b_{1,\beta}}} \mathbf{t}\hat{a}). \end{aligned} \tag{30}$$

Note that $\hat{p}_\alpha$ and $\hat{p}_\beta$ together keep track of the $H^1(X;\mathbb{R})$-component of $(\hat{a}, \hat{\phi})$. We have a decomposition

$$\hat{p}_\beta = \hat{p}_{\beta,\mathrm{in}} \oplus \hat{p}_{\beta,\mathrm{out}},$$

where

$$\hat{p}_{\beta,\mathrm{in}}(\hat{a},\hat{\phi}) = (-i\int_{\beta_1} \mathbf{t}\hat{a},\ldots,-i\int_{\beta_{b_{\mathrm{in}}}} \mathbf{t}\hat{a}),$$

$$\hat{p}_{\beta,\mathrm{out}}(\hat{a},\hat{\phi}) = (-i\int_{\beta_{b_{\mathrm{in}}+1}} \mathbf{t}\hat{a},\ldots,-i\int_{\beta_{b_{1,\beta}}} \mathbf{t}\hat{a}).$$

We now proceed to describe the group of gauge transformations. Denote by $\mathcal{G}_X$ the $L^2_{k+3/2}$-completion of $\mathrm{Map}(X,S^1)$. The action of an element $\hat{u}\in \mathrm{Map}(X,S^1)$ is given by

$$\hat{u}\cdot(\hat{a},\hat{\phi}) = (\hat{a}-\hat{u}^{-1}d\hat{u},\hat{u}\hat{\phi}).$$

The proof of the following lemma is a slight adaption of [8, Proposition 2.2] and we omit it.

Lemma 5.3. *Inside each connected component of $\mathcal{G}_X$, there is a unique element $\hat{u}: X\to S^1$ satisfying*

$$\hat{u}(\hat{o})=1,\ u^{-1}du\in i\Omega^1_{CC}(X).$$

These elements form a subgroup, denoted by $\mathcal{G}^{h,\hat{o}}_X$, of harmonic gauge transformation with double Coulomb condition.

Consequently, there is a natural isomorphism

$$\mathcal{G}^{h,\hat{o}}_X \cong \pi_0(\mathcal{G}_X)\cong H^1(X;\mathbb{Z}). \tag{31}$$

We also denote by $\mathcal{G}^{h,\hat{o}}_{X,Y}$ the subgroup of $\mathcal{G}^{h,\hat{o}}_X$ that corresponds to the subgroup $\ker(H^1(X;\mathbb{Z})\to H^1(Y;\mathbb{Z}))$ of $H^1(X;\mathbb{Z})$. Observe that each element in $\mathcal{G}^{h,\hat{o}}_{X,Y}$ restricts to a constant function on each component of Y.

Now we define the relative Picard torus

$$\begin{aligned}\mathrm{Pic}^0(X,Y) :&= \mathcal{H}^1_{DC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}\\ &\cong \ker(H^1(X;\mathbb{R})\to H^1(Y;\mathbb{R}))/\ker(H^1(X;\mathbb{Z})\to H^1(Y;\mathbb{Z})).\end{aligned} \tag{32}$$

This is a torus of dimension $b_{1,\alpha}$. The double Coulomb slice $Coul^{CC}(X)$ is preserved by $\mathcal{G}^{h,\hat{o}}_X$ and thus $\mathcal{G}^{h,\hat{o}}_{X,Y}$.

Our main object of interest will be the quotient space $Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}$ regarded as a Hilbert bundle over $\mathrm{Pic}^0(X,Y)$ with bundle structure induced by the projection $\hat{p}_\alpha$. The bundle will be denoted by

$$\mathcal{W}_X := Coul^{CC}(X)/\mathcal{G}^{h,\hat{o}}_{X,Y}.$$

Remark. A different Hilbert bundle structure of $\mathcal{W}_X$ can be induced by the orthogonal projection

$$\hat{p}_\perp : Coul^{CC}(X)\to \mathcal{H}^1_{DC}(X).$$

However, we prefer $\hat{p}_\alpha$ because $\hat{p}_\alpha$ behaves better than $\hat{p}_\perp$ and simplifies our argument in the proof of gluing theorem for relative Bauer-Furuta invariants.

Definition 5.4. For a pair $(\hat{a},\hat{\phi})\in Coul^{CC}(X)$, we denote by $[\hat{a},\hat{\phi}]$ the corresponding element in the Hilbert bundle $\mathcal{W}_X$. We write $\|\cdot\|_F$ for the fiber-direction norm on $\mathcal{W}_X$.

Note that the norm $\|\cdot\|_F$ is not directly given by the restriction of the $L^2_{k+1/2}$-norm on $Coul^{CC}(X)$ because the latter is not invariant under $\mathcal{G}^{h,\hat{o}}_{X,Y}$. However, we can construct $\|\cdot\|_F$ using a partition of unity and the compactness of $\mathrm{Pic}^0(X,Y)$.

Let us fix a fundamental $\mathbf{D} \subset \mathcal{H}^1_{DC}(X)$ through out this section. We only state equivalence of the norms on $\mathbf{D}$ below without the proof.

Lemma 5.5. *There exists a positive constant C such that for any $(\hat{a},\hat{\phi}) \in Coul^{CC}(X)$ such that $\hat{p}_\alpha(\hat{a},\hat{\phi}) \in \mathbf{D}$, we have*

$$\frac{\|[\hat{a},\hat{\phi}]\|_F}{C} \leq \|(\hat{a},\hat{\phi})\|_{L^2_{k+1/2}} \leq C\cdot(\|[\hat{a},\hat{\phi}]\|_F + 1).$$

Lastly, we will consider some restriction maps on the bundle. Recall that the Coulomb slice on 3-manifolds is given by

$$Coul(Y) := \{(a,\phi) \in L^2_k\left(i\Omega^1(Y)\oplus\Gamma(S_Y)\right) \mid d^*a = 0\}.$$

From the definition of double Coulomb slice, we obtain a natural restriction map

$$\begin{array}{rcl} r\colon Coul^{CC}(X) & \to & Coul(Y) \\ (\hat{a},\hat{\phi}) & \mapsto & (\mathbf{t}\hat{a}, \hat{\phi}|_Y). \end{array} \tag{33}$$

We would want to also define a restriction map from $\mathcal{W}_X$ to $Coul(Y)$. Notice that $r(\hat{u}\cdot(\hat{a},\hat{\phi}))$ might not be equal to $r(\hat{a},\hat{\phi})$ even if $\hat{u} \in \mathcal{G}^{h,\hat{o}}_{X,Y}$ because $\hat{u}|_Y \neq 1$ in general. This is where we use the based path data $[\vec{\eta}]$ to define a "twisted" restriction map

$$\begin{aligned} r' = r'_{[\vec{\eta}]}\colon Coul^{CC}(X) \to \prod_{j=1}^{b_0(Y)} Coul(Y_j) = Coul(Y) \\ (\hat{a},\hat{\phi}) \mapsto \prod_{j=1}^{b_0(Y)} (\mathbf{t}_j\hat{a}, e^{i\int_{\eta_j}\hat{p}_\alpha(\hat{a},\hat{\phi})}\cdot\hat{\phi}|_{Y_j}). \end{aligned} \tag{34}$$

The following result can be verified by a simple calculation.

Lemma 5.6. *For each $\hat{u} \in \mathcal{G}^{h,\hat{o}}_{X,Y}$, we have $r'(\hat{u}\cdot(\hat{a},\hat{\phi})) = r'(\hat{a},\hat{\phi})$. Moreover, the twisted restriction map r' does not depend on the choice of the representative $\vec{\eta}$ in its equivalent class.*

As a result, we can define the induced twisted restriction map

$$\tilde{r} = \tilde{r}_{[\vec{\eta}]}\colon \mathcal{W}_X \to Coul(Y). \tag{35}$$

Note that $\tilde{r}$ is fiberwise linear since $\hat{p}_\alpha(\hat{a},\hat{\phi})$ is constant on each fiber. Moreover, there is a decomposition $(\tilde{r}_{\mathrm{in}},\tilde{r}_{\mathrm{out}})\colon \mathcal{W}_X \to Coul(-Y_{\mathrm{in}})\times Coul(Y_{\mathrm{out}})$

5.2. **Seiberg–Witten maps and finite-dimensional approximation.** On the boundary 3-manifold Y, we fix a base spin^c connection A_0. We require that the induced curvature $F_{A_0^t}$ on $\det(S_Y)$ equals $2\pi i\nu_0$, where ν_0 is the harmonic 2-form representing $-c_1(\mathfrak{s})$. Furthermore, we pick a good perturbation $f = (\bar{f}, \delta)$ where $\bar{f}$ is an extended cylinder function and δ is a real number (see [7, Definition 2.3] for details). Auxiliary choices in the construction of the unfolded spectrum $\underline{\mathrm{SWF}}(Y, \mathfrak{s})$ will be made but not mentioned at this point.

On the 4-manifold X, we fix a base spin^c connection $\hat{A}_0$ such that $\nabla_{\hat{A}_0} = \frac{d}{dt} + \nabla_{A_0}$ on $[-3, 0] \times Y$. As usual, the space of spin^c connections on S_X can be identified with $i\Omega^1(X)$ via the correspondence $\hat{A} \mapsto \hat{A} - \hat{A}_0$. For a 1-form $\hat{a} \in i\Omega^1(X)$, we let $\not{D}^+_{\hat{a}} \colon \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator associated to the connection $\hat{A}_0 + \hat{a}$. We also denote by $\not{D}^+ := \not{D}^+_0$ the Dirac operator corresponding to the base connection $\hat{A}_0$, so we can write $\not{D}^+_{\hat{a}} = \not{D}^+ + \hat{\rho}(\hat{a})$. On Y, we denote by $\not{\partial}_{A_0+a}$ the Dirac operator associated to the connection $A_0 + a$ where $a \in i\Omega^1(Y)$ and denote by $\not{\partial} := \not{\partial}_{A_0}$

Furthermore, we perturb the Seiberg–Witten map by choosing the following data

- Pick a closed 2-form $\omega_0 \in i\Omega^2(X)$ such that $\omega_0|_{[-3,0]\times Y} = \pi i\nu_0$.
- Pick a bump-function $\iota \colon [-3, 0] \to [0, 1]$ satisfying $\iota \equiv 0$ on $[-3, -2]$ and $\iota \equiv 1$ on $[-1, 0]$ and $0 \le \iota'(x) \le 2$. For $t \in [-3, 0]$, denote by a_t the pull back of $\hat{a}$ by the inclusion $\{t\} \times Y \to X$ and let $\phi_t = \hat{\phi}|_{\{t\}\times Y}$. We define a perturbation on X supported in the collar neighborhood of Y by

$$\hat{q}(\hat{a}, \hat{\phi}) := \iota(t)((dt \wedge \operatorname{grad}^1 f(a_t, \phi_t) + * \operatorname{grad}^1 f(a_t, \phi_t)), \operatorname{grad}^2 f(a_t, \phi_t)). \tag{36}$$

The (perturbed) Seiberg-Witten map is then given by

$$\begin{aligned} SW \colon Coul^{CC}(X) &\to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \\ (\hat{a}, \hat{\phi}) &\mapsto (d^+\hat{a}, \not{D}^+\hat{\phi}) + (\frac{1}{2}F^+_{\hat{A}_0^t} - \hat{\rho}^{-1}(\hat{\phi}\hat{\phi}^*)_0 - \omega_0^+, \hat{\rho}(\hat{a})\hat{\phi}) + \hat{q}(\hat{a}, \hat{\phi}), \end{aligned} \tag{37}$$

where $(\hat{\phi}\hat{\phi}^*)_0$ denotes the trace-free part of $\hat{\phi}\hat{\phi}^* \in \Gamma(\mathrm{End}(S^+_X))$. We consider a decomposition

$$SW = L + Q \tag{38}$$

where

$$L(\hat{a}, \hat{\phi}) = (d^+\hat{a}, \not{D}^+_{\hat{p}_\alpha(\hat{a})}\hat{\phi}) \text{ and } Q = SW - L.$$

By similar computation, making use of the tameness condition on $\operatorname{grad} f$ (see [9, Definition 10.5.1]), we can deduce the following lemma:

Lemma 5.7. *For any number $j \ge 2$, if a subset $U \subset Coul^{CC}(X)$ is bounded in L^2_j, then the set $Q(U)$ is also bounded in L^2_j.*

We will next consider Seiberg–Witten maps on to the Hilbert bundle $\mathcal{W}_X$. Notice that the map

$$(SW, \hat{p}_\alpha) \colon Coul^{CC}(X) \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \times \mathcal{H}^1_{DC}(X) \tag{39}$$

is equivariant under the action of $\mathcal{G}^{h,\hat{o}}_{X,Y}$, where the action on the target space is given by

$$\hat{u}\cdot((\omega,\hat{\phi}),\hat{h}) := ((\omega,\hat{u}\hat{\phi}),\hat{h}-\hat{u}^{-1}d\hat{u}).$$

Consequently, $(SW,\hat{p}_\alpha)$ induces a bundle map over $\mathrm{Pic}^0(X,Y)$ denoted by

$$\overline{SW}\colon \mathcal{W}_X \to (L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X))\times\mathcal{H}^1_{DC}(X))/\mathcal{G}^{h,\hat{o}}_{X,Y}.$$

By Kuiper's theorem, the Hilbert bundle $(L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X))\times\mathcal{H}^1_{DC}(X))/\mathcal{G}^{h,\hat{o}}_{X,Y}$ can be trivialized. We fix a trivialization and consider the induced projection from this bundle to its fiber $L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X))$. Composing the map $\overline{SW}$ with this projection, we obtain a map

$$\widetilde{SW}\colon \mathcal{W}_X \to L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X)).$$

As the map $(L,\hat{p}_\alpha)$ is also equivariant under the action of $\mathcal{G}^{h,\hat{o}}_{X,Y}$, the decomposition (38) induces a decomposition

$$\widetilde{SW} = \tilde{L}+\tilde{Q},$$

where $\tilde{L}$ is a fiberwise linear map.

On the 3-dimensional Coulomb slice $Coul(Y)$, a Seiberg–Witten trajectory is a trajectory $\gamma\colon I\to Coul(Y)$ on some interval $I\subset\mathbb{R}$ satisfying an equation

$$-\frac{d\gamma(t)}{dt} = (l+c)(\gamma(t)),$$

where $l+c$ comes from gradient of the perturbed Chern–Simons–Dirac functional $CSD_{\nu_0,f}$ (cf. [7, Section 2]). Recall that $l=(*d,\not{D})$ and c has nice compactness properties.

Let $V^\mu_\lambda\subset Coul(Y)$ be the span of eigenspaces of l with eigenvalues in the interval $(\lambda,\mu]$ and let p^μ_λ be the L^2-orthogonal projection onto V^μ_λ. An approximated Seiberg–Witten trajectory is a trajectory on a finite-dimensional subspace $\gamma\colon I\to V^\mu_\lambda$ satisfying an equation

$$-\frac{d\gamma(t)}{dt} = (l+p^\mu_\lambda\circ c)(\gamma(t)).$$

From now on, let us fix a decreasing sequence of negative real numbers $\{\lambda_n\}$ and an increasing sequence of positive real numbers $\{\mu_n\}$ such that $-\lambda_n,\mu_n\to\infty$. As a consequence of [8, Proposition 3.1], the linear part

$$(\tilde{L},p^{\mu_n}_{-\infty}\circ\tilde{r})\colon \mathcal{W}_X\to L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X))\oplus V^{\mu_n}_{-\infty} \tag{40}$$

is fiberwise Fredholm. Now we choose an increasing sequence $\{U_n\}$ of finite-dimensional subspaces of $L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X))$ with the following two properties:

(i) As $n\to\infty$, the orthogonal projection $P_{U_n}: L^2_{k-1/2}(i\Omega^2_+(X)\oplus\Gamma(S^-_X))\to U_n$ converges to the identity map pointwisely.
(ii) For any point $p\in\mathrm{Pic}^0(X,Y)$ and any n, the restriction of $(\tilde{L},p^{\mu_n}_{-\infty}\circ\tilde{r})$ to the fiber over p is transverse to U_n.

Note that $\hat{p}_\alpha(\hat{a}) = 0$ on ∂X and hence the family of the Dirac operators $\not{D}^+_{\hat{p}_\alpha(\hat{a})}$ has no spectral flow. Consequently, we see that

$$W_n := (\tilde{L}, p^{\mu_n}_{-\infty} \circ \tilde{r})^{-1}(U_n \times V^{\mu_n}_{\lambda_n}) \tag{41}$$

is a finite-dimensional vector bundle over the Picard torus $\mathrm{Pic}^0(X, Y)$. We define an approximated Seiberg-Witten map as

$$\widetilde{SW}_n := \tilde{L} + P_{U_n} \circ \tilde{Q} \colon W_n \to U_n. \tag{42}$$

5.3. **Boundedness results.** In this section, we will establish analytical results needed to set up our construction of the relative Bauer–Furuta invariants. Uniform boundedness of the following objects and their approximated analogues will be our main focus here.

Definition 5.8. A finite type X-trajectory is a pair $(\tilde{x}, \gamma)$ such that

- $\tilde{x} \in \mathcal{W}_X$ satisfying $\widetilde{SW}(\tilde{x}) = 0$;
- $\gamma \colon [0, \infty) \to Coul(Y)$ is a finite type Seiberg–Witten trajectory;
- $\tilde{r}(\tilde{x}) = \gamma(0)$.

Recall that a smooth path in $Coul(Y)$ is called *finite type* if it is contained in a fixed bounded set (in the L^2_k-norm).

With a basepoint chosen on each connected component Y_j, we recall that we can define the based harmonic gauge group $\mathcal{G}^{h,o}_Y \cong H^1(Y; \mathbb{Z})$. The group $\mathcal{G}^{h,o}_Y$ has a residual action on $Coul(Y)$. Then we consider a strip of balls in $Coul(Y)$ translated by this action

$$Str(R) = \{x \in Coul(Y) \mid \exists h \in \mathcal{G}^{h,o}_Y \text{ s.t. } \|h \cdot x\|_{L^2_k} \leq R\}. \tag{43}$$

Loosely speaking, a finite type X-trajectory corresponds to a Seiberg–Witten solution on $X^* := X \cup ([0, \infty) \times Y)$. The following result resembles [8, Corollary 4.3] but we give a more direct proof without relying on broken trajectories and regular perturbations.

Theorem 5.9. *For any $M > 0$, there exists a constant $R_0(M) > 0$ such that for any finite type X-trajectory $(\tilde{x}, \gamma)$ satisfying*

$$\hat{p}_\beta(\tilde{x}) \in [-M, M]^{b_{1,\beta}} \tag{44}$$

we have

$$\|\tilde{x}\|_F < R_0(M) \text{ and } \gamma([0, \infty)) \subset \mathrm{int}(Str(R_0(M))).$$

Proof. Let $\{(\tilde{x}_n, \gamma_n)\}$ be a sequence of finite type X-trajectories satisfying (44). Without loss of generality, we may pick a representative $\tilde{x}_n = [(\hat{a}_n, \hat{\phi}_n)]$ such that

$$\hat{p}_\alpha(\hat{a}_n, \hat{\phi}_n) \in \mathbf{D} \tag{45}$$

where $\mathbf{D}$ is the fundamental domain fixed before Lemma 5.5.

Since γ_n is finite type, we see that the energy of $\gamma_n|_{[t-1,t+1]}$ goes to 0 as $t \to \infty$ for any n. In particular, the energy of $\gamma_n|_{[t-1,t+1]}$ is bounded above by 1 for any n and any t large enough compared to n. Then, it is not hard to show that there is a constant R' such that $\gamma_n(t) \in \mathrm{int}(Str(R'))$ for any n and any t large enough compared to n. Since $CSD_{\nu_0,f}$ is bounded on $\mathrm{int}(Str(R'))$ and $CSD_{\nu_0,f}$ is decreasing along γ_n, we can obtain a uniform lower bound C_1 of $CSD_{\nu_0,f}(\gamma_n(t))$ for any $n \in \mathbb{N}, t \geq 0$.

We now consider solutions on $X' := X \cup ([0,1] \times Y)$ obtained by gluing together $(\hat{a}_n, \hat{\phi}_n)$ and $\gamma_n|_{[0,1]}$. Remark that the condition $\tilde{r}(\tilde{x}) = \gamma(0)$ from the twisted restriction is slightly different from the setup in [8, Corollary 4.3]. However, we can still glue in a controlled manner since we control $\hat{p}_\alpha(\hat{a}_n, \hat{\phi}_n)$ in (45). The uniform lower bound C_1 of $CSD_{\nu_0,f}(\gamma_n(t))$ implied that the energy of these solutions on X' (see [9, (4.21),(24.25)] for definition) has a uniform upper bound. We now apply compactness theorem [9, Theorem 24.5.2] adapted to the balanced situation; after passing to a subsequence and applying suitable gauge transformations, the solution on X' converges in C^∞ on the interior domain X. In particular, we can find $\hat{u}_n \in \mathcal{G}^{h,\hat{o}}_X$ such that $\hat{u}_n \cdot (\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$ to some $(\hat{a}_\infty, \hat{\phi}_\infty) \in Coul^{CC}(X)$.

By (44) and (45), we have controlled values of $\hat{p}_\alpha$ and $\hat{p}_\beta$ of $(\hat{a}_n, \hat{\phi}_n)$. This implies that $\{\hat{u}_n\}$ takes only finitely many values in $\mathcal{G}^{h,\hat{o}}_X$. After passing to a subsequence, we can assume that $\hat{u}_n$ does not depend on n and $(\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$.

On the collar neighborhood $[-1,0] \times Y$ of X, the solution $(\hat{a}_n, \hat{\phi}_n)$ can be transformed to a Seiberg–Witten trajectory in a controlled manner. We subsequently glue this part together with γ_n to obtain a Seiberg–Witten trajectory

$$\gamma'_n \colon [-1,\infty) \to Coul(Y).$$

Since $(\hat{a}_n, \hat{\phi}_n)$ converges in $L^2_{k+1/2}$, we have a uniform upper bound C_2 of $CSD_{\nu_0,f}(\gamma'_n(-1))$. As a result, the energy of a trajectory $\gamma'_n|_{[t-1,t+1]}$ is bounded above by $C_2 - C_1$ for any $t \geq 0$ and $n \in \mathbb{N}$. We can then conclude that there is a constant R'' such that $\gamma_n(t) \in \operatorname{int}(Str(R''))$ for any $t \geq 0$ and $n \in \mathbb{N}$. This finishes the proof.

□

Corollary 5.10. *There exists a uniform constant R_1 such that for any finite type X-trajectory $(\tilde{x}, \gamma)$, we have $\gamma(t) \in Str(R_1)$ for any $t \in [0,\infty)$.*

Proof. By looking at the lattice induced by the chosen basis on $\operatorname{im} \iota^*$, there is a constant C such that, for any $\tilde{x} \in \mathcal{W}_X$, one can find a gauge transformation $\hat{u} \in \mathcal{G}^{h,\hat{o}}_X$ satisfying $\hat{p}_\beta(\hat{u} \cdot \tilde{x}) \in [-C, C]^{b_{1,\beta}}$.

Let $(\tilde{x}, \gamma)$ be an arbitrary finite type X-trajectory. We then apply Theorem 5.9 to $(\hat{u} \cdot \tilde{x}, (\hat{u}|_Y) \cdot \gamma)$ with $M = C$ and $\hat{u}$ chosen as in the previous paragraph. Consequently, we may set $R_1 = R_0(C)$ so that $(\hat{u}|_Y) \cdot \gamma(t) \in \operatorname{int}(Str(R_1)$ for any $t \in [0,\infty)$. This implies $\gamma(t) \in \operatorname{int}(Str(R_1))$ for any $t \in [0,\infty)$.

□

Now we consider an approximated version of X-trajectories.

Definition 5.11. For $n \in \mathbb{N}$, $\epsilon \in [0,\infty)$, and $T \in (0,\infty]$, a finite type (n,ϵ)-approximated X-trajectory of length T is a pair $(\tilde{x}, \gamma)$ such that

- $\tilde{x} \in W_n$ satisfies $\|\widetilde{SW_n}(\tilde{x})\|_{L^2_{k-1/2}} \leq \epsilon$;
- $\gamma : [0,T) \to V^{\mu_n}_{\lambda_n}$ is a finite type trajectory satisfying $-\frac{d\gamma(t)}{dt} = (l + p^{\mu_n}_{\lambda_n} \circ c)(\gamma(t))$;
- $\gamma(0) = p^{\mu_n}_{-\infty} \circ \tilde{r}(\tilde{x})$.

Note that $p^{\lambda_n}_{-\infty} \circ \tilde{r}(\tilde{x})$ always belongs to $V^{\mu_n}_{\lambda_n}$ from the definition of W_n.

The proof of the following convergence of approximated trajectories is a slight adaption of [8, Lemma 4.4] and we omit it.

Lemma 5.12. *Let $\tilde{S}, S$ be bounded subsets of $\mathcal{W}_X$ and $Coul(Y)$ respectively. Let $\{(\tilde{x}_j, \gamma_j)\}$ be a sequence of finite type (n_j, ϵ_j)-approximated X-trajectory of length T_j such that $\tilde{x}_j \in \tilde{S}, \gamma_j \subset S$ for any j and $(n_j, \epsilon_j, T_j) \to (\infty, 0, \infty)$. Then there exists a finite type X-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that, after passing to a subsequence, we have*

- *$\tilde{x}_j$ converges to $\tilde{x}_\infty$ in $\mathcal{W}_X$;*
- *γ_j converges to γ_∞ uniformly in L^2_k on any compact subset of $[0, \infty)$.*

As a result, we can deduce boundedness for approximated X-trajectories.

Proposition 5.13. *Let $M \geq 0$ be a fixed number. For any bounded subsets $\tilde{S} \subset \mathcal{W}_X$ and $S \subset Coul(Y)$, there exist $\epsilon_0, N, \bar{T} \in (0, \infty)$ such that: for any finite type (n, ϵ)-approximated X-trajectory $(\tilde{x}, \gamma)$ of length $T \geq \bar{T}$ satisfying*

$$n \geq N,\ \epsilon \leq \epsilon_0,\ \tilde{x} \in \tilde{S}, \gamma \subset \tilde{S} \text{ and } \hat{p}_\beta(\tilde{x}) \in [-M, M]^{b_{1,\beta}},$$

we have $\|\tilde{x}\|_F < R_0(M)$ where $R_0(M)$ is the constant from Theorem 5.9.

Proof. Suppose the result is not true for some $\tilde{S}, S$. There would be a sequence $\{(\tilde{x}_j, \gamma_j)\}$ of finite type (n_j, ϵ_j)-approximated X-trajectory of length T_j with $\tilde{x}_j \in \tilde{S}, \gamma_j \subset S$ and $(n_j, \epsilon_j, T_j) \to (\infty, 0, \infty)$ such that $\|\tilde{x}_j\|_F \geq R_0(M)$ and $\hat{p}_\beta(\tilde{x}) \in [-M, M]^{b_{1,\beta}}$.

By Lemma 5.12, after passing to a subsequence, we can find a finite type X-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that $\tilde{x}_j \to \tilde{x}_\infty$ in $\mathcal{W}_X$. In particular, this implies

$$\|\tilde{x}_\infty\|_F = \lim_{j\to\infty} \|\tilde{x}_j\|_F \geq R_0(M) \text{ and } \tilde{p}_\beta(x_\infty) = \lim_{j\to\infty} \tilde{p}_\beta(\tilde{x}_j) \in [-M, M]^{b_{1,\beta}},$$

which is a contradiction with Theorem 5.9. □

Proposition 5.14. *There exists a constant R_2 with the following significance: for any bounded subsets $\tilde{S} \subset \mathcal{W}_X$ and $S \subset Coul(Y)$, there exist $\epsilon_0, N, \bar{T} \in (0, +\infty)$ such that for any finite type (n, ϵ)-approximated X-trajectory $(\tilde{x}, \gamma)$ of length $T \geq \bar{T}$ satisfying*

$$n \geq N,\ \epsilon \leq \epsilon_0,\ \tilde{x} \in \tilde{S} \text{ and } \gamma \subset S$$

We have $\gamma|_{[0, T-\bar{T}]} \subset Str(R_2)$.

Proof. Recall that there is a universal constant R_0 such that any sufficiently approximated Seiberg–Witten trajectory $\gamma' : [-T, T] \to V^\mu_\lambda$ with sufficiently long length T and with $\gamma' \subset S$ must satisfy $\gamma(0) \in \mathrm{Str}(R_0)$ (cf. the constant R_0 from [7, Corollary 3.8]). We pick $R_2 = \max\{R_0, R_1\}$ where R_1 is the constant from Corollary 5.10.

Suppose the result is not true for some $\tilde{S}, S$. Then we can find sequences $n_j, \epsilon_j, \bar{T}_j, T_j$ with $n_j \to \infty, \bar{T}_j \leq T_j, \bar{T}_j \to \infty$ such that there is a sequence $\{(\tilde{x}_j, \gamma_j)\}$ of finite type (n_j, ϵ_j)-approximated X-trajectory of length T_j with $\tilde{x}_j \subset \tilde{S}, \gamma_j \subset S$ and with $\gamma_j([0, T_j - \bar{T}_j]) \not\subset Str(R_2)$.

We have a number $t_j \in [0, T_j - \bar{T}_j]$ such that $\gamma_j(t_j) \notin Str(R_2)$. The property of R_0 forces t_j to converge to a finite number t_∞ after passing to a subsequence.

By Lemma 5.12, there exists an finite type X-trajectory $(\tilde{x}_\infty, \gamma_\infty)$ such that, after passing to a subsequence, γ_j converges to γ_∞ uniformly in L^2_k on any compact subset of $[0,\infty)$. In particular, $\gamma_j(t_j) \to \gamma_\infty(t_\infty)$ which contradicts with Proposition 5.10.

□

5.4. **Construction.** Majority of this section, in fact, is dedicated to construction of type-A unfolded relative invariant. The construction of type-R invairant can be obtained almost immediately after applying duality argument.

Let us pick $\tilde{R}$ a number greater than the constant R_2 from Proposition 5.14. Recall that the unfolded spectra $\underline{\mathrm{swf}}^A(Y_{\mathrm{out}})$ and $\underline{\mathrm{swf}}^R(-Y_{\mathrm{in}})$ are obtained by cutting the unbounded set $Str_Y(\tilde{R})$ into bounded subsets and applying finite dimensional approximations. With a choice of cutting functions, we obtain increasing sequences of bounded sets $\{J^-_m(-Y_{\mathrm{in}})\}$ contained in $Str_{Y_{in}}(\tilde{R})$ and $\{J^+_m(Y_{\mathrm{out}})\}$ contained in $Str_{Y_{out}}(\tilde{R})$ for each positive integer n. See Section 2.1 for brief summary.

For a normed vector bundle V, we will denote by $B(V,r)$ the disk bundle of radius r and denote by $S(V,r)$ the sphere bundle of radius r. We will consider a subbundle of $\mathcal{W}_X$ given by

$$\mathcal{W}_{X,\beta} := \{\tilde{x} \in \mathcal{W}_X \mid \hat{p}_{\beta,\mathrm{out}}(\tilde{x}) = 0\}.$$

We also denote $W_{n,\beta} = W_n \cap \mathcal{W}_{X,\beta}$ and let $\widetilde{SW}_{n,\beta}$ be the restriction of $\widetilde{SW}_n$ on $W_{n,\beta}$.

For a fixed positive integer m_0, since $\{J^-_{m_0}(-Y_{\mathrm{in}})\}$ is bounded, we can find a number $M(m_0)$ such that $|\int_{\beta_j} ia| \leq M(m_0)$ for all $(a,\phi) \in J^-_{m_0}(-Y_{\mathrm{in}})$ and $j = 1, \dots, b_{\mathrm{in}}$. We then choose a number R greater than $R_0(M(m_0))$ the constant from Theorem 5.9. Since $\tilde{r}_{\mathrm{out}}(B(\mathcal{W}_X, R))$ is bounded, we can find a positive integer m_1 such that

$$\tilde{r}_{\mathrm{out}}(B(\mathcal{W}_X, R)) \cap Str_{Y_{\mathrm{out}}}(\tilde{R}) \subset J^+_{m_1}(Y_{\mathrm{out}}). \tag{46}$$

For $\epsilon > 0,\ n \in \mathbb{N}$, we consider the following subsets of $V^{\mu_n}_{\lambda_n}$;

$$\begin{aligned}
&K_1(n, m_0, R, \epsilon) = \\
&\quad p^{\mu_n}_{-\infty} \circ \tilde{r}\left(\widetilde{SW}^{-1}_{n,\beta}(B(U_n,\epsilon)) \cap B(W_{n,\beta}, R)\right) \cap \left(J^{n,-}_{m_0}(-Y_{\mathrm{in}}) \times Str_{Y_{\mathrm{out}}}(\tilde{R})\right); \\
&K_2(n, m_0, R, \epsilon) = \\
&\quad \left(p^{\mu_n}_{-\infty} \circ \tilde{r}\left(\widetilde{SW}^{-1}_{n,\beta}(B(U_n,\epsilon)) \cap S(W_{n,\beta}, R)\right) \cap \left(J^{n,-}_{m_0}(-Y_{\mathrm{in}}) \times Str_{Y_{\mathrm{out}}}(\tilde{R})\right)\right) \\
&\quad \cup \left(p^{\mu_n}_{-\infty} \circ \tilde{r}\left(\widetilde{SW}^{-1}_{n,\beta}(B(U_n,\epsilon)) \cap B(W_{n,\beta}, R)\right) \cap \partial\left(J^{n,-}_{m_0}(-Y_{\mathrm{in}}) \times Str_{Y_{\mathrm{out}}}(\tilde{R})\right)\right).
\end{aligned} \tag{47}$$

Notice that $K_1(n, m_0, R, \epsilon) \subset J^{n,-}_{m_0}(-Y_{\mathrm{in}}) \times J^{n,+}_{m_1}(Y_{\mathrm{out}})$ from our choice of m_1 and $K_2(n, m_0, R, \epsilon)$ plays a role of a boundary of $K_1(n, m_0, R, \epsilon)$.

The following is the key result of this section (cf. [8, Proposition 4.5]).

Proposition 5.15. *For a choice of m_0, m_1 and R chosen above, there exist $N \in \mathbb{N}$ and $\bar{T}, \epsilon_0 > 0$ such that, for any $n \geq N$ and $\epsilon \leq \epsilon_0$, the pair $(K_1(n, m_0, R, \epsilon), K_2(n, m_0, R, \epsilon))$ is a $\bar{T}$-tame pre-index pair in an isolating neighborhood $J^{n,-}_{m_0}(-Y_{in}) \times J^{n,+}_{m_1}(Y_{out})$.*

Proof. We choose numbers $(N, \bar{T}, \epsilon_0)$ which satisfy both Proposition 5.13 and Proposition 5.14 with $\tilde{S} = B(\mathcal{W}_X, R)$, $S = J^-_{m_0}(-Y_{\text{in}}) \times J^+_{m_1}(Y_{\text{out}})$, and $M = M(m_0)$. Moreover, we may pick a larger N so that $J^{n,-}_{m_0}(-Y_{\text{in}}) \times J^{n,+}_{m_1}(Y_{\text{out}})$ is an isolating neighborhood for all $n > N$ (cf. [7, Lemma 5.5 and Proposition 5.8]). We will check directly that $(K_1(n, m_0, R, \epsilon), K_2(n, m_0, R, \epsilon))$ is a $\bar{T}$-tame pre-index pair.

Suppose that $y \in K_1(n, m_0, R, \epsilon)$ and $\varphi_n(y, [0, T]) \subset J^{n,-}_{m_0}(-Y_{\text{in}}) \times J^{n,+}_{m_1}(Y_{\text{out}})$ with $T \geq \bar{T}$. From definition, there is $\tilde{x} \in W_{n,\beta}$ such that $\|\widetilde{SW}_n(\tilde{x})\| \leq \epsilon$ and $p^{\mu_n}_{-\infty} \circ \tilde{r}(\tilde{x}) = y$. These give rise to a finite type (n, ϵ)-approximated X-trajectory $(\tilde{x}, \gamma)$ of length T. By Proposition 5.14, we have $\varphi_n(y, [0, T - \bar{T}]) \subset Str(R_2) \subset \text{int}(Str(\tilde{R}))$. From our choices of $J^-_{m_0}, J^+_{m_1}$, it is not hard to check that $\varphi_n(y, [0, T - \bar{T}])$ lies in some compact subset inside the interior of $J^{n,-}_{m_0}(-Y_{\text{in}}) \times J^{n,+}_{m_1}(Y_{\text{out}})$.

For the second pre-index pair condition, let us assume that $y \in K_2(n, m_0, R, \epsilon)$ and $\varphi_n(y, [0, \bar{T}]) \subset J^{n,-}_{m_0}(-Y_{\text{in}}) \times J^{n,+}_{m_1}(Y_{\text{out}})$. This also gives rise to a finite type (n, ϵ)-approximated X-trajectory $(\tilde{x}, \gamma)$ of length $\bar{T}$. Since $p^{\mu_n}_{-\infty} \circ \tilde{r}_{\text{in}}(\tilde{x}) \in J^{n,-}_{m_0}(-Y_{\text{in}})$ and $\tilde{x} \in \mathcal{W}_{X,\beta}$, we can see that $\hat{p}_\beta(\tilde{x}) \in [-M(m_0), M(m_0)]^{b_{1,\beta}}$.

By Proposition 5.13, we have $\|\tilde{x}\|_F < R_0(M) < R$, which implies that

$$y \in \partial \left(J^{n,-}_{m_0}(-Y_{\text{in}}) \times Str_{Y_{\text{out}}}(\tilde{R}) \right).$$

Again, from Proposition 5.14, we must have

$$y \in \left\{ \partial J^{n,-}_{m_0}(-Y_{\text{in}}) \setminus \partial Str_{Y_{\text{in}}}(\tilde{R}) \right\} \times Str_{Y_{\text{out}}}(\tilde{R}).$$

This is impossible because the approximated trajectories on $\partial J^{n,-}_{m_0}(-Y_{\text{in}}) \setminus \partial Str_{Y_{\text{in}}}(\tilde{R})$ immediately leave $J^{n,-}_{m_0}(-Y_{in})$.

□

The proposition allows us to consider a map

$$\begin{aligned} \upsilon(n, m_0, R, \epsilon) \colon & B(W_{n,\beta}, R)/S(W_{n,\beta}, R) \\ & \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (K_1(n, m_0, R, \epsilon)/K_2(n, m_0, R, \epsilon)) \end{aligned} \tag{48}$$

given by

$$\upsilon(n, m_0, R, \epsilon)(x) := \begin{cases} (\widetilde{SW}_{n,\beta}(x), p^{\mu_n}_{-\infty} \circ \tilde{r}(x)) & \text{if } \|\widetilde{SW}_{n,\beta}(x)\|_{L^2_{k-1/2}} \leq \epsilon \text{ and} \\ & p^{\mu_n}_{-\infty} \circ \tilde{r}(x) \in J^{n,-}_{m_0}(-Y_{\text{in}}) \times Str_{Y_{\text{out}}}(\tilde{R})), \\ * & \text{otherwise.} \end{cases}$$

It follows from our construction that this map is well-defined and continuous. By Proposition 5.15 and Theorem 3.8, we have a canonical map from $K_1(n, m_0, R, \epsilon)/K_2(n, m_0, R, \epsilon)$ to the Conley index of $J^{n,-}_{m_0}(-Y_{\text{in}}) \times J^{n,+}_{m_1}(Y_{\text{out}})$. This gives a map

$$\begin{aligned} \tilde{\upsilon}(n, m_0, R, \epsilon) \colon & B(W_{n,\beta}, R)/S(W_{n,\beta}, R) \\ & \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge I(\text{inv}(J^{n,-}_{m_0}(-Y_{\text{in}}))) \wedge I(\text{inv}(J^{n,+}_{m_1}(Y_{\text{out}}))). \end{aligned} \tag{49}$$

It is a standard argument to check that $\tilde{\upsilon}(n, m_0, R, \epsilon)$ does not depend on R or ϵ as long as they satisfy all the requirements to define $\upsilon(n, m_0, R, \epsilon)$.

Before proceeding, let us describe the Thom space $B(W_{n,\beta}, R)/S(W_{n,\beta}, R)$ in term of index bundle. Consider a family of Dirac operators

$$\begin{aligned}\mathbf{D}\colon L^2_{k+1/2}(S^+_X) \times \mathcal{H}^1_{DC}(X) &\to L^2_{k-1/2}(S^-_X) \times H^-_{Dir} \times \mathcal{H}^1_{DC}(X)\\ (\hat{\phi}, h) &\mapsto (\not{D}^+_h \hat{\phi}, \Pi^-_{Dir}(\hat{\phi}|_Y), h),\end{aligned}$$

where H^-_{Dir} is the closure in $L^2_k(\Gamma(S_Y))$ of the subspace spanned by the eigenvectors of $\not{\partial}$ with nonpositive eigenvalues and let Π^-_{Dir} be the orthogonal projection. As in Section 5.2, this map is equivariant under an action by $\mathcal{G}^{h,\hat{o}}_{X,Y}$. We then take the quotient to obtain a map between Hilbert bundles over $\mathrm{Pic}^0(X, Y)$ and trivialize the right hand side so that we have

$$\widetilde{\mathbf{D}}\colon (L^2_{k+1/2}(S^+_X) \times \mathcal{H}^1_{DC}(X))/\mathcal{G}^{h,\hat{o}}_{X,Y} \to L^2_{k-1/2}(S^-_X) \times H^-_{Dir}.$$

Since $\widetilde{\mathbf{D}}$ is fiberwise Fredholm, the preimage $\widetilde{\mathbf{D}}^{-1}(U)$ is a finite-dimensional subbundle for a finite-dimensional subspace $U \subset L^2_{k-1/2}(S^-_X) \times H^-_{Dir}$ transverse to the image of the restriction of $\widetilde{\mathbf{D}}$ to any fiber. Here we use the fact that the rank of $\widetilde{\mathbf{D}}^{-1}(U)$ is constant because $h|_Y = 0$ and there is no spectral flow.

We consider desuspension $\Sigma^{-U} B(\widetilde{\mathbf{D}}^{-1}(U), R)/S(\widetilde{\mathbf{D}}^{-1}(U), R)$ of the Thom space in the stable category $\mathfrak{C}$. The following lemma follows from standard homotopy argument.

Lemma 5.16. *The object $\Sigma^{-U} B(\widetilde{\mathbf{D}}^{-1}(U), R)/S(\widetilde{\mathbf{D}}^{-1}(U), R)$ does not depend on any choice in the construction given that $\hat{g}|_Y = g$ and $\hat{A}_0|_Y = A_0$. We will call this object Thom spectrum of virtual index bundle associated to the Dirac operators, denoted by $T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1)$.*

Remark. For different choices of base points, one can construct an isomorphism by choosing a path between them. However, isomorphisms given by different pathes are different unless they are homotopic relative to Y.

Recall from Section 2.1 that we have desuspended Conley indices

$$\begin{aligned}I^{n,-}_{m_0}(-Y_{\mathrm{in}}) &= \Sigma^{-V^0_{\lambda_n}(-Y_{\mathrm{in}})} I(\mathrm{inv}(J^{n,-}_{m_0}(-Y_{\mathrm{in}}))),\\ I^{n,+}_{m_1}(Y_{\mathrm{out}}) &= \Sigma^{-\bar{V}^0_{\lambda_n}(Y_{\mathrm{out}})} I(\mathrm{inv}(J^{n,+}_{m_1}(Y_{\mathrm{out}}))).\end{aligned} \tag{50}$$

We see that if we desuspend the map $\tilde{v}(n, m_0, R, \epsilon)$ by $V^0_{\lambda_n}(-Y_{\mathrm{in}}) \oplus \bar{V}^0_{\lambda_n}(Y_{\mathrm{out}}) \oplus U_n$, the right hand side will become $I^{n,-}_{m_0}(-Y_{\mathrm{in}}) \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})$. As a consequence of Lemma 5.16, we can also identify the left hand side after desuspension as follows

Lemma 5.17. *Let V^+_X be a maximal positive subspace of* $\mathrm{Im}(H^2(X, \partial X; \mathbb{R}) \to H^2(X; \mathbb{R}))$ *with respect to the intersection form and let V_{in} be the cokernel of $\iota^*\colon H^1(X; \mathbb{R}) \to H^1(Y_{in}; \mathbb{R})$. Then, we have*

$$\Sigma^{-(V^0_{\lambda_n}(-Y_{in}) \oplus \bar{V}^0_{\lambda_n}(Y_{out}) \oplus U_n)} B(W_{n,\beta}, R)/S(W_{n,\beta}, R) \cong \Sigma^{-(V^+_X \oplus V_{in})} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1).$$

Proof. This is a bundle version of index computation in [8, Proposition 3.1]. From there, we are only left to keep track of $H^1(X; \mathbb{R})$ and $H^1(Y; \mathbb{R})$ as we pass to bundle and subspace, i.e. the base of the bundle is the torus of dimension $b_{1,\alpha}$ and we take a slice of codimension $b_{1,\beta}-b_{\text{in}}$. Note that we desuspend by $\bar{V}^0_{\lambda_n}(Y_{\text{out}})$, the orthogonal complement of $H^1(Y_{\text{out}};\mathbb{R})$ in $V^0_{\lambda_n}(Y_{\text{out}})$. One may compute the rank of the Thom space of the index bundle of the real part of $(\tilde{L}, p^0\circ\tilde{r})$ suspended by $H^1(Y_{\text{out}};\mathbb{R})$ as follows

$$b_1(X)-b^+(X)-b_1(Y)-b_{1,\alpha}-(b_{1,\beta}-b_{\text{in}})+b_1(Y_{\text{out}}) = -b^+(X)-(b_1(Y_{\text{in}})-b_{\text{in}}).$$

The desired isomorphism follows in the same manner.

□

Consequently, we obtain a morphism

$$\psi^n_{m_0,m_1}\colon \Sigma^{-(V^+_X\oplus V_{\text{in}})}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1)\to I^{n,-}_{m_0}(-Y_{\text{in}})\wedge I^{n,+}_{m_1}(Y_{\text{out}}) \tag{51}$$

in the stable category $\mathfrak{C}$. Note that such a morphism is defined for any positive integer m_0 with m_1 large relative to m_0 and n large relative to m_0, m_1.

Recall that, to define unfolded spectra $\underline{\text{swf}}^A(Y_{\text{out}})$ and $\underline{\text{swf}}^R(-Y_{\text{in}})$, we have canonical isomorphisms

$$\tilde{\rho}^{n,-}_{m_0}(-Y_{\text{in}})\colon I^{n,-}_{m_0}(-Y_{\text{in}})\to I^{n+1,-}_{m_0}(-Y_{\text{in}}) \text{ and } \tilde{\rho}^{n,+}_{m_1}(Y_{\text{out}})\colon I^{n,+}_{m_1}(Y_{\text{out}})\to I^{n+1,+}_{m_1}(Y_{\text{out}})$$

and also morphisms

$$\tilde{i}^{n,-}_{m_0-1}\colon I^{n,-}_{m_0}(-Y_{\text{in}})\to I^{n,-}_{m_0-1}(-Y_{\text{in}}) \text{ and } \tilde{i}^{n,+}_{m_1}\colon I^{n,+}_{m_1}(Y_{\text{out}})\to I^{n,+}_{m_1+1}(Y_{\text{out}})$$

induced by repeller and attractor respectively. To have a morphism to the unfolded spectra, we have to to check that the maps $\{\psi^n_{m_0,m_1}\}$ are compatible with all such morphisms.

Lemma 5.18. *When n is large enough relative to m_0, m_1, we have the following:*

(1) $(\tilde{\rho}^{n,-}_{m_0}(-Y_{in})\wedge\tilde{\rho}^{n,+}_{m_1}(Y_{out}))\circ\psi^n_{m_0,m_1}=\psi^{n+1}_{m_0,m_1}$;
(2) $(\tilde{i}^{n,-}_{m_0-1}\wedge \mathrm{id}_{I^{n,+}_{m_1}(Y_{out})})\circ\psi^n_{m_0,m_1}=\psi^n_{m_0-1,m_1}$;
(3) $(\mathrm{id}_{I^{n,-}_{m_0}(-Y_{in})}\wedge\tilde{i}^{n,+}_{m_1})\circ\psi^n_{m_0,m_1}=\psi^n_{m_0,m_1+1}$.

Proof. The proof of (1) can be given by standard homotopy arguments similar to [11, Section 9] and [7, Proposition 5.6]. Whereas (2) and (3) follow from Proposition 3.18 and 3.19 respectively. □

The last step is to apply Spanier–Whitehead duality between $I^{n,-}_{m_0}(-Y_{\text{in}})$ and $I^{n,+}_{m_0}(Y_{\text{in}})$ (see Section 4.3 and 4.4 for details). As a result, we can turn the morphism $\psi^n_{m_0,m_1}$ to a morphism

$$\widetilde{\psi}^n_{m_0,m_1}\colon \Sigma^{-(V^+_X\oplus V_{\text{in}})}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1)\wedge I^{n,+}_{m_0}(Y_{\text{in}})\to I^{n,+}_{m_1}(Y_{\text{out}}), \tag{52}$$

which will define the relative Bauer–Furuta invariant.

Definition 5.19. For the cobordism $X\colon Y_{\text{in}}\to Y_{\text{out}}$, the collection of morphisms $\{\widetilde{\psi}^n_{m_0,m_1}\mid m_0\in\mathbb{N},\ m_1\gg m_0, n\gg m_0,m_1\}$ in $\mathfrak{C}$ gives rise to a morphism

$\underline{\text{bf}}^A(X,\hat{\mathfrak{s}},A_0,g,\hat{o},[\vec{\eta}];S^1)\colon$

$$\Sigma^{-(V^+_X\oplus V_{\text{in}})}T(X,\hat{\mathfrak{s}},A_0,g,\hat{o};S^1)\wedge\underline{\text{swf}}^A(Y_{\text{in}},\mathfrak{s}_{\text{in}},A_{\text{in}},g_{\text{in}};S^1)\to\underline{\text{swf}}^A(Y_{\text{out}},\mathfrak{s}_{\text{out}},A_{\text{out}},g_{\text{out}};S^1)$$

in $\mathfrak{S}$. This will be called the type-A unfolded relative Bauer–Furuta invariant of X.

Note that Lemma 5.18 and compatibility of the dual maps ensure that $\{\widetilde{\psi}^n_{m_0,m_1}\}$ are compatible with the direct systems. When $\mathfrak{s} = \hat{\mathfrak{s}}|_Y$ is torsion, we can also define the normalized relative Bauer–Furuta invariant. In this torsion case, let us define the normalized Thom spectrum

$$\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1) := (T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1), 0, n(Y, \mathfrak{s}, A_0, g)),$$

where $n(Y, \mathfrak{s}, A_0, g)$ is given by $\frac{1}{2}\left(\eta(\not{D}) - \dim_{\mathbb{C}}(\ker \not{D}) + \frac{\eta_{\text{sign}}}{4}\right)$ (see (21) of [7]).

Definition 5.20. When $\mathfrak{s} = \hat{\mathfrak{s}}|_Y$ is torsion, the normalized type-A unfolded relative Bauer–Furuta invariant of X

$$\underline{\mathrm{BF}}^A(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1) \colon \Sigma^{-(V_X^+ \oplus V_{\mathrm{in}})}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1) \wedge \underline{\mathrm{SWF}}^A(Y_{\mathrm{in}}, \mathfrak{s}_{\mathrm{in}}; S^1) \to \underline{\mathrm{SWF}}^A(Y_{\mathrm{out}}, \mathfrak{s}_{\mathrm{out}}; S^1)$$

is given by desuspending $\underline{\mathrm{bf}}^A(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1)$ by $n(Y, \mathfrak{s}, A_0, g)$.

We then define the type-R invariant by simply considering the dual of type-A invariant of the adjoint cobordism $X^\dagger \colon -Y_{\mathrm{out}} \to -Y_{\mathrm{in}}$. In particular, the dual of the morphism

$$\widetilde{\psi}^n_{m_1,m_0}(X^\dagger) \colon \Sigma^{-(V^+_{X^\dagger} \oplus V_{\mathrm{in}}(X^\dagger))} T(X^\dagger, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1) \wedge I^{n,+}_{m_1}(-Y_{\mathrm{out}}) \to I^{n,+}_{m_0}(-Y_{\mathrm{in}}),$$

gives a morphism

$$\breve{\psi}^n_{m_0,m_1} \colon \Sigma^{-(V_X^+ \oplus V_{\mathrm{out}})} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1) \wedge I^{n,-}_{m_0}(Y_{\mathrm{in}}) \to I^{n,-}_{m_1}(Y_{\mathrm{out}}).$$

Note that $V_{\mathrm{in}}(X^\dagger)$ is the cokernel of $\iota^* \colon H^1(X; \mathbb{R}) \to H^1(Y_{\mathrm{out}}; \mathbb{R})$ and we denote by V_{out} and such a morphism is defined for any positive integer m_1 with m_0 large relative to m_1 and n large relative to m_0, m_1. We can now give a definition in a similar fashion.

Definition 5.21. For the cobordism $X \colon Y_{\mathrm{in}} \to Y_{\mathrm{out}}$, the type-R unfolded relative Bauer–Furuta invariant of X is a morphism

$$\underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1) \colon$$
$$\Sigma^{-(V_X^+ \oplus V_{\mathrm{out}})} T(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}; S^1) \wedge \underline{\mathrm{swf}}^R(Y_{\mathrm{in}}, \mathfrak{s}_{\mathrm{in}}, A_{\mathrm{in}}, g_{\mathrm{in}}; S^1) \to \underline{\mathrm{swf}}^R(Y_{\mathrm{out}}, \mathfrak{s}_{\mathrm{out}}, A_{\mathrm{out}}, g_{\mathrm{out}}; S^1)$$

in $\mathfrak{S}^*$ given by the collection of morphisms $\{\breve{\psi}^n_{m_0,m_1} \mid m_1 \in \mathbb{N},\ m_0 \gg m_1, n \gg m_0, m_1\}$. When $\mathfrak{s} = \hat{\mathfrak{s}}|_Y$ is torsion, one can also desuspend $\underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1)$ by $n(Y, \mathfrak{s}, A_0, g)$ to obtain the normalized type-R unfolded relative Bauer–Furuta invariant of X

$$\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1) \colon \Sigma^{-(V_X^+ \oplus V_{\mathrm{out}})}\tilde{T}(X, \hat{\mathfrak{s}}, \hat{o}; S^1) \wedge \underline{\mathrm{SWF}}^R(Y_{\mathrm{in}}, \mathfrak{s}_{\mathrm{in}}; S^1) \to \underline{\mathrm{SWF}}^R(Y_{\mathrm{out}}, \mathfrak{s}_{\mathrm{out}}; S^1).$$

Remark. One can also construct the maps $\breve{\psi}^n_{m_0,m_1}$ directly by replacing $(-Y_{\mathrm{in}}, Y_{\mathrm{out}})$ with $(Y_{\mathrm{out}}, -Y_{\mathrm{in}})$ in the construction through out this section.

5.5. **Invariance of the relative invariants.** In this subsection, we will show that the morphism $\underline{\mathrm{bf}}^A = \underline{\mathrm{bf}}^A(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1)$ and $\underline{\mathrm{bf}}^R = \underline{\mathrm{bf}}^R(X, \hat{\mathfrak{s}}, A_0, g, \hat{o}, [\vec{\eta}]; S^1)$ depends only on $A_0, g, \hat{o}, [\vec{\eta}]$. We have to check that they are independent of the choices of

(i) cutting function $\bar{g}$, cutting value θ, harmonic 1-forms $\{h_j\}_{j=1}^{b_1}$ representing generators of $\mathrm{im}(H^1(Y; \mathbb{Z}) \to H^1(Y; \mathbb{R}))$,
(ii) Riemann metric $\hat{g}$, connection $\hat{A}_0$ on X with $\hat{g}|_Y = g, \hat{A}_0|_Y = A_0$,
(iii) perturbation $f : Coul(Y) \to \mathbb{R}$.

Moreover when $c_1(\mathfrak{s})$ is torsion, we will show that $\underline{\mathrm{BF}}^A(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1)$ and $\underline{\mathrm{BF}}^R(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}]; S^1)$ are independent of A_0, g too.

Choose two cutting functions $\bar{g}$, $\bar{g}'$, cutting values θ, θ' and sets of harmonic 1-forms $\{h_j\}_{j=1}^{b_1}$, $\{h'_j\}_{j=1}^{b_1}$ representing generators of $2\pi i \operatorname{im}(H^1(Y;\mathbb{Z}) \to H^1(Y;\mathbb{R}))$. We get two inductive systems

$$\underline{\mathrm{swf}}^A(Y, \{h_j\}_j, \bar{g}, \theta) = (I_1 \to I_2 \to \cdots),$$
$$\underline{\mathrm{swf}}^A(Y, \{h'_j\}_j, \bar{g}', \theta') = (\tilde{I}_1 \to \tilde{I}_2 \to \cdots)$$

in $\mathfrak{C}$. Here I_m, $\tilde{I}_m$ are the desuspension of the Conley indices $I_{S^1}(\varphi^n, \operatorname{inv}(J_m^{n,+}))$, $I_{S^1}(\varphi^n, \operatorname{inv}(\tilde{J}_m^{n,+}))$ for $n \gg m$ by $V_{\lambda_n}^0$, and $J_m^{n,+}$, $\tilde{J}_m^{n,+}$ are the bounded sets in $Str(\tilde{R})$ defined by using $(\{h_j\}_j, \bar{g}, \theta)$, $(\{h'_j\}_j, \bar{g}', \theta')$.

Choosing integers $m_j \ll \tilde{m}_j \ll m_{j+1}$, we can assume that $\operatorname{inv}(J_{m_j}^{n,+})$ is an attractor in $\operatorname{inv}(\tilde{J}_{\tilde{m}_j}^{n,+})$ and we have the attractor map

$$I_{S^1}(\operatorname{inv}(J_{m_j}^{n,+})) \to I_{S^1}(\operatorname{inv}(\tilde{J}_{\tilde{m}_j}^{n,+}))$$

which induces a morphism

$$I_{m_j} \to \tilde{I}_{\tilde{m}_j}.$$

Similarly we have a morphism

$$\tilde{I}_{\tilde{m}_j} \to I_{m_{j+1}}.$$

These morphisms induce an isomorphism between $\underline{\mathrm{swf}}^A(Y, \{h_j\}_j, \bar{g}, \theta)$ and $\underline{\mathrm{swf}}^A(Y, \{h'_j\}_j, \bar{g}', \theta')$. The isomorphism between $\underline{\mathrm{swf}}^R(Y, \{h_j\}_j, \bar{g}, \theta), \underline{\mathrm{swf}}^R(Y, \{h'_j\}_j, \bar{g}, \theta')$ is obtained similarly. The morphisms in (51) inducing the relative invariants $\underline{\mathrm{bf}}^A, \underline{\mathrm{bf}}^R$ are compatible with the attractor maps and repeller maps as in stated in Lemma 5.18. It means that $\underline{\mathrm{bf}}^A$, $\underline{\mathrm{bf}}^R$ are independent of the choices of $\{h_j\}_j$, $\bar{g}$, θ up to the canonical isomorphisms.

Choose connections $\hat{A}_0, \hat{A}'_0$ on X with $\hat{A}_0|_Y = \hat{A}'_0|_Y = A_0$ and Riemannian metrics $\hat{g}, \hat{g}'$ on X with $\hat{g}|_Y = \hat{g}'|_Y = g$. Then the homotopies

$$\hat{A}_0(s) = (1-s)\hat{A}_0 + s\hat{A}'_0, \ \hat{g}(s) = (1-s)\hat{g} + s\hat{g}'$$

naturally induce the homotopy between the maps v, v' defined in (48) associated with $(\hat{g}_0, \hat{A}_0), (\hat{g}', \hat{A}'_0)$. Hence $\underline{\mathrm{bf}}^A, \underline{\mathrm{bf}}^R$ are independent of $\hat{A}_0, \hat{g}$.

Take sequences $\lambda_n, \lambda'_n, \mu_n, \mu'_n$ with $-\lambda_n, -\lambda'_n, \mu_n, \mu'_n \to \infty$. Then we get objects

$$I_{m_0}^{n,-}(-Y_{\mathrm{in}}), I_{m_1}^{n,+}(Y_{\mathrm{out}}), \tilde{I}_{m_0}^{n,-}(-Y_{\mathrm{in}}), \tilde{I}_{m_1}^{n,+}(Y_{\mathrm{out}}).$$

We have canonical isomorphisms

$$I_{m_0}^{n,-}(-Y_{\mathrm{in}}) \cong I_{m_0}^{n,-}(-Y_{\mathrm{in}}), \ I_{m_1}^{n,+}(Y_{\mathrm{out}}) \cong I_{m_1}^{n,+}(Y_{\mathrm{out}})$$

for n large relative to m_0, m_1. The morphisms ψ_{m_0,m_1}^n are compatible with these isomorphisms as stated in Lemma 5.18. Therefore $\underline{\mathrm{bf}}^A$, $\underline{\mathrm{bf}}^R$ is independent of λ_n, μ_n up to canonical isomorphisms.

Let us consider the invariance of $\underline{\mathrm{bf}}^A, \underline{\mathrm{bf}}^R$ with respect to the perturbation f. Take two perturbations $f_1, f_2 : Coul(Y) \to \mathbb{R}$. Then we obtain two inductive systems

$$\underline{\mathrm{swf}}^A(Y, f_1) = (I_1 \to I_2 \to \cdots)$$
$$\underline{\mathrm{swf}}^A(Y, f_2) = (\tilde{I}_1 \to \tilde{I}_2 \to \cdots)$$

in the category $\mathfrak{C}$, which are isomorphic to each other. Let us recall how to get the isomorphism briefly. (See Section 6.3 of [7] for the details.) The perturbations f_1, f_2 define the functionals $\mathcal{L}_1$, $\mathcal{L}_2$, which induce the flows

$$\varphi^n(\mathcal{L}_1), \varphi^n(\mathcal{L}_2) : V_{\lambda_n}^{\mu_n} \times \mathbb{R} \to V_{\lambda_n}^{\mu_n}.$$

The objects $I_m, \tilde{I}_m$ are the desuspensions by $V_{\lambda_n}^0$ of the Conley indices

$$I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{inv}(J_m^{n,+})),\ \ I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{inv}(\tilde{J}_m^{n,+})).$$

Choose integers $k_m, \tilde{k}_m$ with $0 \ll k_m \ll \tilde{k}_m \ll k_{m+1}$. Then we have

$$J_{k_m}^+ \subset p_{\mathcal{H}}^{-1}([-e_m+1, e_m-1]^{b_1}) \cap Str(\tilde{R}) \subset p_{\mathcal{H}}^{-1}([-e_m, e_m]^{b_1}) \cap Str(\tilde{R}) \subset \tilde{J}_{\tilde{k}_m}^+$$

for some large positive number e_m. We have a map

$$\bar{i}_m^n : I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{inv}(J_{k_m}^{n,+})) \to I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{inv}(\tilde{J}_{\tilde{k}_m}^{n,+})),$$

which induces the isomorphism between $\underline{\mathrm{swf}}^A(Y, f_1)$ and $\underline{\mathrm{swf}}^A(Y, f_2)$. The map $\bar{i}_m^n$ is the composition $\rho_1 \circ \rho_2$ of

$$\rho_1 : I_{S^1}(\varphi^n(\mathcal{L}_{e_m}^0), \mathrm{inv}(\tilde{J}_{\tilde{k}_m}^n)) \to I_{S^1}(\varphi^n(\mathcal{L}_2), \mathrm{inv}(\tilde{J}_{\tilde{k}_m}^{n,+}))$$

and

$$\rho_2 : I_{S^1}(\varphi^n(\mathcal{L}_1), \mathrm{inv}(J_{k_m}^{n,+})) \to I_{S^1}(\varphi^n(\mathcal{L}_{e_m}^0), \mathrm{inv}(\tilde{J}_{\tilde{k}_m}^{n,+})).$$

Here $\mathcal{L}_{e_m}^0$ is a functional on $Coul(Y)$ such that

$$\mathcal{L}_{e_m}^0 = \mathcal{L}_1 \text{ on } p_{\mathcal{H}}^{-1}([-e_m+1, e_m-1]^{b_1}),$$
$$\mathcal{L}_{e_m}^0 = \mathcal{L}_2 \text{ on } p_{\mathcal{H}}^{-1}(\mathbb{R}^{b_1} \setminus [-e_m, e_m]^{b_1})\ .$$

The map ρ_1 is the homotopy equivalence induced by a homotopy $\{\varphi(\mathcal{L}_{e_m}^s)\}_{0 \le s \le 1}$, where $\mathcal{L}_{e_m}^s = s\mathcal{L}_1 + (1-s)\mathcal{L}_{e_m}^0$. Note that $\mathrm{inv}(J_{k_m}^{n,+}, \varphi^n(\mathcal{L}_{e_m}^0))(= \mathrm{inv}(J_{k_m}^{n,+}, \varphi^n(\mathcal{L}_1)))$ is an attractor in $\mathrm{inv}(\tilde{J}_{\tilde{k}_m}^{n,+}, \varphi^n(\mathcal{L}_{e_m}^0))$. The map ρ_2 is the attractor map.

Similarly the isomorphism between $\underline{\mathrm{swf}}^R(Y, f_1)$ and $\underline{\mathrm{swf}}^R(Y, f_2)$ is induced by the composition of the repeller map and the homotopy equivalence induced by the homotopy of the flows.

To prove the invariance of $\underline{\mathrm{bf}}^A, \underline{\mathrm{bf}}^R$ with respect to perturbation f, we need to show that the morphisms (51) are compatible with the attractor maps, the repeller maps and the homotopy equivalence induced by the homotopy of the flows. The compatibility with the attractor maps and the repeller maps is already stated in Lemma 5.18. We will show the compatibility with the homotopy equivalence induced by the homotopy of the flows.

Take perturbations $f_0, f_1 : Coul(-Y_{in}) \coprod Coul(Y_{out}) \to \mathbb{R}$. Let us consider the flow

$$\widetilde{\varphi}_n : V_{\lambda_n}^{\mu_n} \times [0,1] \times \mathbb{R} \to V_{\lambda_n}^{\mu_n} \times [0,1]$$

on $V^{\mu_n}_{\lambda_n} \times [0,1]$, induced by the homotopy

$$\mathcal{L}^s_{Y_{\mathrm{in}},e_{m_0}} \coprod \mathcal{L}^s_{Y_{\mathrm{out}},e_{m_1}} : Coul(-Y_{\mathrm{in}}) \coprod Coul(Y_{\mathrm{out}}) \to \mathbb{R} \quad (0 \le s \le 1). \tag{53}$$

We also have the Seiberg-Witten map on X induced by the homotopy:

$$Coul^{CC}(X) \times [0,1] \to L^2_{k-1}(i\Omega^+(X) \oplus S^-_X) \oplus V^{\mu_n}_{-\infty} \times [0,1].$$

Using the flow and the Seiberg-Witten map, for a small positive number $\epsilon > 0$, we define

$$\widetilde{K}_1 = \widetilde{K}_1(n, m_0, \epsilon),\ \widetilde{K}_2 = \widetilde{K}_2(n, m_0, \epsilon) \subset B(V^{\mu_n}_{\lambda_n}, \widetilde{R}) \times [0,1]$$

as in (47). As before we can show that $(\widetilde{K}_1, \widetilde{K}_2)$ is a pre-index pair and can find an index pair $(\widetilde{N}, \widetilde{L})$ such that

$$\widetilde{K}_1(n, m_0, \epsilon) \subset \widetilde{N},\ \widetilde{K}_2(n, m_0, \epsilon) \subset \widetilde{L}.$$

For $s \in [0,1]$, put

$$\begin{aligned} K_{1,s}(n, m_0, \epsilon) &:= \widetilde{K}_1(n, m_0, \epsilon) \cap (V^{\mu_n}_{\lambda_n} \times \{s\}), \\ K_{2,s}(n, m_0, \epsilon) &:= \widetilde{K}_2(n, m_0, \epsilon) \cap (V^{\mu_n}_{\lambda_n} \times \{s\}), \\ N_s &:= \widetilde{N} \cap (V^{\mu_n}_{\lambda_n} \times \{s\}), \\ L_s &:= \widetilde{L} \cap (V^{\mu_n}_{\lambda_n} \times \{s\}). \end{aligned}$$

We get the map

$$\begin{aligned} v_s : B(W_{n,\beta}, R)/S(W_{n,\beta}, R) &\to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (K_{1,s}(n, m_0, \epsilon)/K_{2,s}(n, m_0, \epsilon)) \\ &\hookrightarrow (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (N_s/L_s). \end{aligned}$$

The maps v_0, v_1 induce morphisms

$$\begin{aligned} \psi_0 &: \Sigma^{-(V^+_X \oplus V_{\mathrm{in}})} T \to I^{n,-}_{m_0}(-Y_{\mathrm{in}})_0 \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})_0, \\ \psi_1 &: \Sigma^{-(V^+_X \oplus V_{\mathrm{in}})} T \to I^{n,-}_{m_0}(-Y_{\mathrm{in}})_1 \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})_1 \end{aligned}$$

for $0 \ll m_0 \ll m_1 \ll n$ as before. We have to check that the following diagram is commutative:

$$\begin{tikzcd} \Sigma^{-(V^+_X \oplus V_{\mathrm{in}})} T \arrow[r, "\psi_0"] \arrow[dr, "\psi_1"'] & I^{n,-}_{m_0}(-Y_{\mathrm{in}})_0 \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})_0 \arrow[d, "\cong"] \\ & I^{n,-}_{m_0}(-Y_{\mathrm{in}})_1 \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})_1 \end{tikzcd} \tag{54}$$

Here $I^{n,-}_{m_0}(-Y_{\mathrm{in}})_0 \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})_0 \cong I^{n,-}_{m_0}(-Y_{\mathrm{in}})_1 \wedge I^{n,+}_{m_1}(Y_{\mathrm{out}})_1$ is the isomorphism induced by the homotopy (53). Consider the inclusion

$$i_s : N_s/L_s \hookrightarrow \widetilde{N}/\widetilde{L}$$

for $s \in [0,1]$. By Theorem 6.7 and Corollary 6.8 of [14], i_s is a homotopy equivalence and the following diagram is commutative up to homotopy:

$$\begin{tikzcd} N_0/L_0 \arrow[r, "i_0"] \arrow[d, "\cong"'] & \widetilde{N}/\widetilde{L} \\ N_1/L_1 \arrow[ur, "i_1"'] & \end{tikzcd} \tag{55}$$

Here $N_0/L_0 \cong N_1/L_1$ is the homotopy equivalence induced by the homotopy (53). With the homotopy

$$i_s \circ v_s : B(W_{n,\beta}, R)/S(W_{n,\beta}) \to (B(U_n, \epsilon)/S(U_n, \epsilon)) \wedge (\widetilde{N}/\widetilde{L})$$

between $i_0 \circ v_0$ and $i_1 \circ v_1$ and the commutativity of the diagram (55), we can see that the diagram (54) is commutative. The invariance of $\underline{\mathrm{bf}}^A, \underline{\mathrm{bf}}^R$ with respect to perturbation f has been proved.

Assume that $c_1(\mathfrak{s})$ is torsion. We will prove that the normalized invariants $\underline{\mathrm{BF}}^A, \underline{\mathrm{BF}}^R$ are independent of Riemann metric g and base connection A_0 on Y. Take Riemann metrics g, g' and connections A_0, A_0' on Y. Let us consider the homotopy

$$A_0(s) = (1-s)A_0 + sA_0', \ g(s) = (1-s)g + sg' \ (s \in [0,1]).$$

Choose continuous families of Riemann metrics $\hat{g}(s)$ and connections $\hat{A}_0(s)$ on X with $\hat{g}(s)|_Y = g(s), \hat{A}_0(s)|_Y = A_0(s)$. Splitting the interval $[0,1]$ into small intervals $[0,1] = [0,t_1] \cup \cdots \cup [t_{N-1}, t_N]$, the discussion is reduced to the case when λ_n, μ_n (for some fixed, large number n) are not an eigenvalue of the Dirac operators D_s on Y associated to $g(s), A(s)$. In this case, the dimension of $W_{n,\beta}(s)$ is constant, where

$$W_{n,\beta}(s) := (\tilde{L}_s, p_\infty^{\mu_n})^{-1}(U_n \times V_{\lambda_n}^{\mu_n}(s)) \cap \mathcal{W}_{X,\beta}(s).$$

Then we can mimic the discussion about the invariance with respect to perturbation f to get a homotopy v_s between v_0 and v_1 which are the maps in (48) associated $(\hat{g}, \hat{A}_0), (\hat{g}', \hat{A}_0')$. Therefore the morphisms $\psi^n_{m_0,m_1}$ associated with $(\hat{g}_0, \hat{A}_0)$ and $(\hat{g}_1, \hat{A}_1)$ are the same. Note that the objects $(V^0_{\lambda_n}(s) \oplus \mathbb{C}^{n(Y, g_s, A_s)})^+$ of $\mathfrak{C}$ for $s = 0, 1$ are isomorphic to each other. Taking the desuspension by $V^0_{\lambda_n}(s) \oplus \mathbb{C}^{n(Y, g_s, A_s)}$, we conclude that $\underline{\mathrm{BF}}^A, \underline{\mathrm{BF}}^R$ are independent of g, A_0 up to canonical isomorphisms.

6. The gluing theorem

6.1. **Statement and setup of the gluing theorem.** In this section, let $X_0 \colon Y_0 \to Y_2$ and $X_1 \colon Y_1 \to -Y_2$ be connected, oriented cobordisms with the following properties:

- Y_2 is connected;
- Y_0, Y_1 may not be connected but $b_1(Y_0) = b_1(Y_1) = 0$.

By gluing the two cobordisms along Y_2, we obtain a cobordism $X \colon Y_0 \cup Y_1 \to \emptyset$. As in Section 5, we choose the following data when defining the relative Bauer–Furuta invariants:

- A spin^c structure $\hat{\mathfrak{s}}$ on X.
- A Riemannian metric $\hat{g}$ on X, we require it equals the product metric near Y_i.

- A base connection $\hat{A}^0$ on X;
- A base point $\hat{o} \in Y_2$ and a based path data $[\vec{\eta}_i]$ on X_i for $i = 0, 1$. The path from $\hat{o}$ to Y_2 is chosen to be the constant path. By patching $[\vec{\eta}_1]$ and $[\vec{\eta}_2]$ together in the obvious way, we get a based path data $[\vec{\eta}]$ on X;
- Denote the restriction of $\hat{\mathfrak{s}}$ (resp. $\hat{g}$ and $\hat{A}^0$) to X_i by $\hat{\mathfrak{s}}_i$ (resp. $\hat{g}_i$ and $\hat{A}^0_i$) and the restriction to Y_j by $\mathfrak{s}_j$ (resp. g_j and A^0_j).

With the above data chosen, we obtain the invariants $\underline{\mathrm{bf}}^A(X_0, \hat{\mathfrak{s}}_0, \hat{A}^0_0, \hat{g}_0, \hat{o}, [\vec{\eta}_0]; S^1)$ and $\underline{\mathrm{bf}}^R(X_1, \hat{\mathfrak{s}}_1, \hat{A}^1_0, \hat{g}_1, \hat{o}, [\vec{\eta}_1]; S^1)$ and $\mathrm{BF}(X, \hat{\mathfrak{s}}, \hat{o}, [\vec{\eta}])$. For shorthand, we write them as $\underline{\mathrm{bf}}^A(X_0)$, $\underline{\mathrm{bf}}^R(X_1)$ and $\mathrm{BF}(X)$ respectively throughout this section.

Theorem 6.1. *If the following condition holds*

$$\mathrm{im}(H^1(X_0; \mathbb{R}) \to H^1(Y_2; \mathbb{R})) \subset \mathrm{im}(H^1(X_1; \mathbb{R}) \to H^1(Y_2; \mathbb{R})), \tag{56}$$

then, under the natural identification between domains and targets, one has

$$\mathrm{BF}(X)|_{\mathrm{Pic}^0(X,Y_2)} = \tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1)),$$

where $\tilde{\boldsymbol{\epsilon}}(\cdot, \cdot)$ is the Spanier-Whitehead duality operation defined in Section 4.3.

Corollary 6.2. *When the map $H^1(X_0; \mathbb{R}) \to H^1(Y_2; \mathbb{R})$ is trivial, one has*

$$\mathrm{BF}(X) = \tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1)).$$

Corollary 6.3. *When $\mathfrak{s}_2$ is torsion and (56) is satisfied, one has*

$$\mathrm{BF}(X)|_{\mathrm{Pic}^0(X,Y_2)} = \tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{BF}}^A(X_0), \underline{\mathrm{BF}}^R(X_1)).$$

We begin by setting up some notations. Let $\iota_i \colon Y_2 \to X_i$ be the inclusion map. We pick a set of loops $\{\alpha^0_1, \cdots \alpha^0_{b^0_{1,\alpha}}\}, \{\alpha^1_1, \cdots, \alpha^1_{b^1_{1,\alpha}}\}, \{\beta_1, \cdots \beta_{b_{1,\beta}}\}$ with the following properties:

- For $i = 0, 1$, the set $\{\alpha^i_1, \cdots \alpha^i_{b^i_{1,\alpha}}\}$ is contained in the interior of X_i and represents a basis of cokernel of the induced map $(\iota_i)_* \colon H_1(Y_2; \mathbb{R}) \to H_1(X_i; \mathbb{R})$.
- $\{\beta_1, \cdots \beta_{b_{1,\beta}}\} \subset Y_2$ represents a basis for a subspace complementary to the kernel of $(\iota_0)_* \colon H_1(Y_2; \mathbb{R}) \to H_1(X_0; \mathbb{R})$.

Under the assumption (56), the above properties further imply the following two properties:

- $\{\alpha^0_1, \cdots \alpha^0_{b^0_{1,\alpha}}\} \cup \{\alpha^1_1, \cdots, \alpha^1_{b^1_{1,\alpha}}\} \cup \{\beta_1, \cdots \beta_{b_{1,\beta}}\}$ represent a basis of $H_1(X; \mathbb{R})$;
- $\{\alpha^0_1, \cdots \alpha^0_{b^0_{1,\alpha}}\} \cup \{\alpha^1_1, \cdots, \alpha^1_{b^1_{1,\alpha}}\}$ represent a basis of $H_1(X, Y_2; \mathbb{R})$.

As before, we use $\mathcal{G}^{h,\hat{o}}$ to denote the group of harmonic gauge transformations u on X_i such that $u(\hat{o}) = 1$ and $u^{-1}du \in i\Omega^1_{CC}(X_i)$, and let $\mathcal{G}^{h,\hat{o}}_{X_i,\partial X_i}$ be the subgroup of $\mathcal{G}^{h,\hat{o}}_{X_i}$ corresponding to $\ker(H^1(X_i; \mathbb{Z}) \to H^1(\partial X_i; \mathbb{Z}))$. We have $\mathcal{G}^{h,\hat{o}}_{X_i,\partial X_i} \cong H^1(X_i, Y_2; \mathbb{Z})$. Recall that $b_1(Y_0) = b_1(Y_1) = 0$.

For $i = 0, 1$, consider the bundles

$$\mathcal{W}_{X_i} = Coul^{CC}(X_i)/\mathcal{G}^{h,\hat{o}}_{X_i,\partial X_i},$$

over $\mathrm{Pic}^0(X_i, \partial X_i)$ and the subbundle

$$\mathcal{W}_{X_0,\beta} := \{x \in \mathcal{W}_{X_0} \mid \hat{p}_\beta(x) = 0\},$$

where the projection $\hat{p}_\beta \colon Coul^{CC}(X_0) \to \mathbb{R}^{b_{1,\beta}}$ is given by

$$\hat{p}_\beta(\hat{a}, \hat{\phi}) = (-i \int_{\beta_1} \mathbf{t}\hat{a}, \dots, -i \int_{\beta_{b_{1,\beta}}} \mathbf{t}\hat{a})$$

as in Section 5.

We have the following boundedness result:

Proposition 6.4. *There exists a constant R_3 with the following significance: For any tuple $(\tilde{x}_0, \tilde{x}_1, \gamma_0, \gamma_1, \gamma_2, T)$ satisfying the following conditions*

- $(\tilde{x}_0, \tilde{x}_1) \in \mathcal{W}_{X_0,\beta} \times \mathcal{W}_{X_1}$ *satisfies* $\widetilde{SW}(\tilde{x}_j) = 0$*;*
- $\gamma_i \colon (-\infty, 0] \to Coul(Y_i)$ $(i = 0, 1)$ *and* $\gamma_2 \colon [-T, T] \to Coul(Y_2)$ *are finite type Seiberg-Witten trajectories;*
- $\tilde{r}_0(\tilde{x}_0) = \gamma_0(0)$, $\tilde{r}_2(\tilde{x}_0) = \gamma_2(-T)$, $\tilde{r}_2(\tilde{x}_1) = \gamma_2(T)$ *and* $\tilde{r}_1(\tilde{x}_1) = \gamma_1(0)$, *where $\tilde{r}_j$ denotes the twisted restriction map to $Coul(Y_j)$;*

one has $\|\tilde{x}_i\|_F \leq R_3$ for $i = 0, 1$ and $\gamma_j \subset Str_{Y_j}(R_3)$ for $j = 0, 1, 2$.

Proof. Suppose there exists a sequence not satisfying such uniform bounds. We also assume that $T \to +\infty$ as the case when T is uniformly bounded is trivial. From the condition $\hat{p}_\beta(x_0) = 0$, the norm of γ_0 and the norm $\|\tilde{x}_0\|_F$ is controlled by Theorem 5.9. Notice that the solutions converge to a broken trajectory on the Y_2-neck, which is contained in $Str_{Y_2}(R)$ for some universal constant R by [7, Theorem 3.2]. As in the construction, of $\underline{\mathrm{swf}}(Y_2)$, we consider a bounded subset J_m^+ of $Str_{Y_2}(R)$ (cf. [7, Definition 5.3]). We cut $Str_{Y_2}(\tilde{R_2})$ into $\cup J_m^+(Y_2)$. Since $\|\tilde{x}_0\|_F$ is uniformly bounded, $\tilde{r}_2(\tilde{x}_0)$ is contained in $J_m^+(Y_2)$ for some fixed m. From the fact that $J_m^+(Y_2)$ is an attractor with respect to the Seiberg–Witten flow, we see that the whole broken trajectory is contained in $J_m^+(Y_2)$. In particular, $\tilde{r}_2(\tilde{x}_1)$ also belongs to $J_m^+(Y_2)$. We then apply Theorem 5.9 again on X_1 to control $\|\tilde{x}_1\|_F$ and the norm of γ_1. $\square$

Following Section 5.4, we will start to consider finite-dimensional approximation of the Seiberg–Witten map on both X_0 and X_1. Let us fix an increasing sequence of positive real numbers $\{\mu_n\}$ such that $\mu_n \to \infty$. For $i = 0, 1, 2$, let $V_n^i \subset Coul(Y_i)$ be the span of eigenspaces with respect to $(*d, \not{D})$ with eigenvalues in the interval $[-\mu_n, \mu_n]$. For $i = 0, 1$, we choose appropriate finite-dimensional subspaces $U_n^i \subset L^2_{k-1/2}(i\Omega_2^+(X_i) \oplus \Gamma(S^-_{X_i}))$. The preimages of $U_n^i \times V_n^i \times V_n^2$ under $(\tilde{L}, p^{\mu_n}_{-\infty} \circ \tilde{r})$ give rise to finite-dimensional subbundles $W^0_{n,\beta} \subset \mathcal{W}_{X_0,\beta}$ and $W^1_n \subset \mathcal{W}_{X_1}$.

We now state the boundedness result for approximated solutions.

Proposition 6.5. *For any $R > 0$ and $L \geq 0$ and any bounded subsets S_i of $Coul(Y_i)$ $(i = 0, 1, 2)$, there exist constants $\epsilon, N, \bar{T} > 0$ with the following significance: For any tuple $(\tilde{x}_0, \tilde{x}_1, \gamma_0, \gamma_1, \gamma_2, n, T, T')$ satisfying the following conditions:*

- $n > N$, $T' > \bar{T}$, *and* $T \leq L$.
- $(\tilde{x}_0, \tilde{x}_1) \in B(W^0_{n,\beta}, R) \times B(W^1_n, R)$ *such that* $\|\widetilde{SW}_n(\tilde{x}_j)\|_{L^2_{k-1/2}} < \epsilon$ $(j = 0, 1)$*;*

- $\gamma_i\colon (-T',0] \to V_n^i \cap S_i$ $(i=0,1)$ *and* $\gamma_2\colon [-T,T] \to V_n^2 \cap S_2$ *are finite type approxiamted Seiberg-Witten trajectories;*
- $p^{\mu_n}_{-\infty} \circ \tilde{r}_0(\tilde{x}_0) = \gamma_0(0)$, $p^{\mu_n}_{-\infty} \circ \tilde{r}_2(\tilde{x}_0) = \gamma_2(-T)$, $p^{\mu_n}_{-\infty} \circ \tilde{r}_2(\tilde{x}_1) = \gamma_2(T)$ *and* $p^{\mu_n}_{-\infty} \circ \tilde{r}_1(\tilde{x}_1) = \gamma_1(0)$*;*

one has the following estimate

- $\|\tilde{x}_i\|_F \leq R_3 + 1$ *for* $i = 0,1$*;*
- $\gamma_2 \subset Str_{Y_2}(R_3+1)$*;*
- $\gamma_i|_{[-(T'-\bar{T}),0]} \subset B_{Y_i}(R_3+1)$ *for* $i=0,1$*.*

Here R_3 *is the constant from Proposition 6.4.*

Proof. The proof is analogous to that of Proposition 5.13 and Proposition 5.14 where one applies Proposition 6.4 instead. □

For $i = 0,1$, the manifold Y_i is a rational homology sphere and a sufficiently large ball $B_{Y_i}(\tilde{R}_i)$ in the Coulomb slice contains all finite type Seiberg-Witten trajectories (cf.[11]). On Y_2, an unbounded subset $Str_{Y_2}(\tilde{R}_2)$ contains all finite type Seiberg-Witten trajectories when $\tilde{R}_2$ is sufficiently large. With a choice of cutting functions, we obtain increasing sequences of bounded sets $\{J^+_m(Y_2)\}$ contained in $Str_{Y_2}(\tilde{R}_2)$. Note that we can identify $J^{n,-}_m(-Y_2) = J^{n,+}_m(Y_2)$.

Throughout the rest of the section, we will fix the following parameters carefully step by step in term of dependency.

(i) Pick $\hat{R}_0 > R_3$ such that any finite type X_0-trajectories (x,γ) with $x \in \mathcal{W}_{X_0,\beta}$ satisfies $\|x\|_F \leq \hat{R}_0$ (cf. Theorem 5.9).
(ii) Pick $\tilde{R}_0, \tilde{R}_2 > R_3+2$ such that $\tilde{r}(B(\mathcal{W}_{X_0,\beta},\hat{R}_0)) \subset B_{Y_0}(\tilde{R}_0) \times Str_{Y_2}(\tilde{R}_2)$ and also $B_{Y_0}(\tilde{R}_0-1) \times Str_{Y_2}(\tilde{R}_2-1)$ contains all finite type Seiberg-Witten trajectories.
(iii) Choose a positive integer m such that $\tilde{r}_2(B(\mathcal{W}_{X_0,\beta},\hat{R}_0)) \subset J^+_{m-1}(Y_2)$.
(iv) Pick $\hat{R}_1 > R_3+1$ such that any finite type X_1-trajectory (x,γ) with $\tilde{r}_2(x) \in J^+_m(Y_2)$, one has $\|x\|_F < \hat{R}_1$.
(v) Choose a positive number $\tilde{R}_1$ such that $\tilde{r}_2(B(\mathcal{W}_{X_1},\hat{R}_1)) \subset B_{Y_1}(\tilde{R}_1)$ and $B_{Y_1}(\tilde{R}_1-1)$ contains all finite type Seiberg-Witten trajectory on Y_1.

6.2. **Deformation of the duality pairing.** In this section, we will focus on describing the right hand side $\tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1))$ and its deformation. As in Section 5.4, we will consider subsets of the following forms in order to define $\underline{\mathrm{bf}}^A(X_0)$ and $\underline{\mathrm{bf}}^R(X_1)$:

$$K_0 = p^{\mu_n}_{-\infty} \circ \tilde{r}(\widetilde{SW}_n^{-1}(B(U^0_n,\epsilon)) \cap B(W^0_{n,\beta},\hat{R}_0)),$$

$$S_0 = p^{\mu_n}_{-\infty} \circ \tilde{r}(\widetilde{SW}_n^{-1}(B(U^0_n,\epsilon)) \cap S(W^0_{n,\beta},\hat{R}_0)),$$

$$K_1 = p^{\mu_n}_{-\infty} \circ \tilde{r}(\widetilde{SW}_n^{-1}(B(U^1_n,\epsilon)) \cap B(W^1_n,\hat{R}_1)) \cap (V^1_n \times J^{n,-}_m(-Y_2)),$$

$$S_1 = \{p^{\mu_n}_{-\infty} \circ \tilde{r}(\widetilde{SW}_n^{-1}(B(U^1_n,\epsilon)) \cap S(W^1_n,\hat{R}_1)) \cap (V^1_n \times J^{n,-}_m(-Y_2))\} \cup \{K_1 \cap (V^1_n \times \partial J^{n,-}_m(Y_2))\}.$$

Note that some of the subsets are simplified because $b_1(Y_0) = b_1(Y_1) = 0$. The parameters $(\hat{R}_0, \hat{R}_1, \tilde{R}_0, \tilde{R}_1, \tilde{R}_2, m)$ are selected earlier. We will also consider a large number L_0 with the following property and then proceed to pick n and ϵ.

(vi) Choose a positive number L_0 such that, for any large n and small ϵ, one has
   (a) (K_0, S_0) and (K_1, S_1) are L_0-tame pre-index pairs. This follows from Proposition 5.15 applying to X_0 and X_1;
   (b) The pair (K^0, S^0), as defined below
   $$K^0 = \{(y_0, y_1) \mid (y_0, y) \times (y_1, y) \in K_0 \times K_1 \text{ for some } y\}$$
   $$S^0 = \{(y_0, y_1) \mid (y_0, y) \times (y_1, y) \in S_0 \times K_1 \cup K_0 \times S_1 \text{ for some } y\},$$
   is an L_0-tame pre-index pair for $B(V_n^0, \tilde{R}_0) \times B(V_n^1, \tilde{R}_1)$. This follows from Proposition 6.5 with $L = 0$.
   (c) Pick a slightly smaller closed subset $J'_m \subset \operatorname{int}(J_m^+(Y_2))$ such that for any approximated trajectory $\gamma\colon [-L_0, L_0] \to B(V_n^0, \tilde{R}_0) \times B(V_n^1, \tilde{R}_1) \times J_m^{n,+}(Y_2)$, one has $\gamma(0) \in B(V_n^0, \tilde{R}_0 - 1) \times B(V_n^1, \tilde{R}_1 - 1) \times J'_m$ (cf. [7, Lemma 5.5]).
   (d) $L_0 > 4T_m(j)$ where $T_m(j)$ is the constant which appeared in Lemma 3.23 applying to the manifold Y_j.

(vii) Finally, we pick a large positive integer n and a small positive real number ϵ so that
   (a) The above assertions for L_0 holds;
   (b) Proposition 6.5 holds for $L = 3L_0$, $R = \max(\hat{R}_0, \hat{R}_1)$, $S_0 = B_{Y_0}(\tilde{R}_0)$, $S_1 = B_{Y_1}(\tilde{R}_1)$ and $S_2 = J_m^+(Y_2)$.

With all the above parameters fixed, we have canonical maps to Conley indices
$$\iota_0\colon K_0/S_0 \to I(B(V_n^0, \tilde{R}_0)) \wedge I(J_m^{n,+}(Y_2)),$$
$$\iota_1\colon K_1/S_1 \to I(B(V_n^1, \tilde{R}_1)) \wedge I(J_m^{n,-}(-Y_2)).$$
For simplicity, we will write $A_j = B(V_n^j, \tilde{R}_j)$ and $A'_j = B(V_n^j, \tilde{R}_j - 1)$ for $j = 0, 1$. We also let A_2 denote $J_m^{n,+}(Y_2)$ and let A'_2 be a closed subset satisfying
$$(Str_{Y_2}(\tilde{R}_2 - 1) \cap J_{m-1}^{n,+}(Y_2)) \cup (J'_m \cap V_n^2) \subset \operatorname{int}(A'_2) \subset A'_2 \subset \operatorname{int}(A_2).$$
By our choice of L_0 and Proposition 3.24, there exists a manifold isolating block $\tilde{N}_j$ satisfying
$$A_j^{[-L_0, L_0]} \subset \operatorname{int}(\tilde{N}_j) \subset \tilde{N}_j \subset A'_j. \tag{57}$$
Let φ^j be the approximated Seiverg–Witten flow on A_j. Denote by $\tilde{N}_j^+$ (resp. $\tilde{N}_j^-$) be the submanifold of $\partial\tilde{N}_j$ where φ^j points outward (resp. inward).

By the choice of L_0 and Lemma 3.13 and Theorem 3.14, we can express the smash product of canonical maps
$$\iota_0 \wedge \iota_1\colon K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^+ \wedge \tilde{N}_2/\tilde{N}_2^+ \wedge \tilde{N}_1/\tilde{N}_1^+ \wedge \tilde{N}_2/\tilde{N}_2^- \tag{58}$$
as a map sending (y_0, y_2, y_1, y'_2) to $(\varphi^0(y_0, 3L_0), \varphi^2(y_2, 3L_0), \varphi^1(y_1, 3L_0), \varphi^2(y_2, -3L_0))$ when the following conditions are all satisfied
$$\varphi^j(y_j, [0, 3L_0]) \subset A_j \text{ and } \varphi^j(y_j, [L_0, 3L_0]) \subset \tilde{N}_j \setminus \tilde{N}_j^+ \text{ for } j = 0, 1; \tag{59}$$

$$\varphi^2(y_2,[0,3L_0])\subset A_2 \text{ and } \varphi^2(y_2',[-3L_0,0])\subset A_2; \tag{60}$$

$$\varphi^2(y_2,[L_0,3L_0])\subset \tilde{N}_2\setminus \tilde{N}_2^+ \text{ and } \varphi^2(y_2',[-L_0,-3L_0])\subset \tilde{N}_2\setminus \tilde{N}_2^-. \tag{61}$$

Otherwise, it will be sent to the base point. From here on, we will sometimes not mention the part which is sent to the basepoint. We will see that some of the above conditions can be simplified in specific setup.

Lemma 6.6. *There exists a positive constant $\bar{\epsilon}_0$ such that one can find a closed subset $B_0\subset \operatorname{int}(\tilde{N}_2)$ with the following property: For any (y_2,y_2') satisfying (60) and*

$$\|\varphi^j(y_2,3L_0)-\varphi(y_2',-3L_0)\|\leq 5\bar{\epsilon}_0,$$

one has

$$\varphi^2(y_2,[L_0,3L_0])\subset B_0 \text{ and } \varphi^2(y_2',[-L_0,-3L_0])\subset B_0. \tag{62}$$

In particular, (y_2,y_2') will satisfy (61).

Proof. From (57), we see that one can choose $B_0=A_2^{[-L_0,L_0]}$ if we consider the case $\bar{\epsilon}_0=0$. For positive $\bar{\epsilon}_0$, we pick B_0 to be a slightly larger closed subset containing $A_2^{[-L_0,L_0]}$ and then apply continuity argument. □

To deform our maps, we also consider a variation of the above lemma.

Lemma 6.7. *There exists a positive constant $\bar{\epsilon}_1$ such that for any $L\in[0,L_0]$ and any $(y_0,y_2,y_2',y_1)\in K_0\times K_1$ satisfying (59) and*

$$\varphi^2(y_2,[0,3L])\subset A_2 \text{ and } \varphi^2(y_2',[-3L,0])\subset A_2, \tag{63}$$

$$\|\varphi^2(y_2,3L)-\varphi^2(y_2',-3L)\|\leq \bar{\epsilon}_1, \tag{64}$$

we have

$$\varphi^2(y_2,[0,3L])\subset A_2' \text{ and } \varphi^2(y_2',[-3L,0])\subset A_2'.$$

Proof. We first consider the case $\bar{\epsilon}_1=0$. Then, by Proposition 6.5 and our choice of (n,ϵ), we have $\varphi^2(y_2,[0,6L])\subset Str_{Y_2}(\tilde{R}_2-1)$. From our choice, we also have $y_2\in J^{n,+}_{m-1}(Y_2)\subset V^2_n$. Since $J^{n,+}_{m-1}(Y_2)$ is an attractor in $J^{n,+}_m(Y_2)$, we have $\varphi^2(y_2,[0,6L])\subset J^{n,+}_{m-1}(Y_2)$. Thus

$$\varphi^2(y_2,[0,6L])\subset (Str_{Y_2}(\tilde{R}_2-1)\cap J^{n,+}_{m-1}(Y_2))\subset \operatorname{int}(A_2').$$

The general case follows from continuity argument. □

We will also consider the following subsets enlarging (K^0,S^0)

$K^{\bar{\epsilon}}:=\{(y_0,y_1)\mid (y_0,y_2)\times(y_1,y_2')\in K_0\times K_1 \text{ for some } y_2,y_2' \text{ with } \|y_2-y_2'\|\leq\bar{\epsilon}\}$,

$S^{\bar{\epsilon}}:=\{(y_0,y_1)\mid (y_0,y_2)\times(y_1,y_2')\in (S_0\times K_1)\cup(K_0\times S_1) \text{ for some } y_2,y_2' \text{ with } \|y_2-y_2'\|\leq\bar{\epsilon}\}$.

Since (K^0,S^0) is an L_0-tame pre-index pair, the following can be obtained by continuity argument.

Lemma 6.8. *There exists a positive constant $\bar{\epsilon}_2$ such that the pair $(K^{\bar{\epsilon}},S^{\bar{\epsilon}})$ is an L_0-tame pre-index pair for any $0\leq\bar{\epsilon}\leq\bar{\epsilon}_2$.*

For a vector space or a vector bundle, denote by $B^+(V,R)$ the sphere $B(V,R)/S(V,R)$. Recall that the Spanier-Whitehead duality map (see Section 4.4)

$$\boldsymbol{\epsilon}\colon \tilde{N}_2/\tilde{N}_2^+ \wedge \tilde{N}_2/\tilde{N}_2^- \to B^+(V_n^2,\bar{\epsilon})$$

can be given by

$$\boldsymbol{\epsilon}(y_2,y_2') = \begin{cases} \eta_+(y_2)-\eta_-(y_2') & \text{if } \|\eta_+(y_2)-\eta_-(y_2')\| \le \bar{\epsilon}, \\ * & \text{otherwise.} \end{cases}$$

Here we pick $\bar{\epsilon} < \min\{\bar{\epsilon}_0,\bar{\epsilon}_1,\bar{\epsilon}_2\}$ and $\eta_\pm\colon \tilde{N}_2 \to \tilde{N}_2$ are deformation retractions which are identity on $B_0 \subset \operatorname{int}(\tilde{N}_2)$ and satisfy $\|\eta_\pm(x)-x\| \le 2\bar{\epsilon}$ for any x. Here B_0 is the closed set in Lemma 6.6.

Consequently we can write down the composition of $\iota_0 \wedge \iota_1$ and $\boldsymbol{\epsilon}$ as a map

$$\tilde{\boldsymbol{\epsilon}}(\iota_0,\iota_1)\colon K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^+ \wedge \tilde{N}_1/\tilde{N}_1^+ \wedge B^+(V_n^2,\bar{\epsilon})$$

given by

$$(y_0,y_2,y_1,y_2') \mapsto (\varphi^0(y_0,3L_0),\varphi^1(y_1,3L_0),\varphi^2(y_2,3L_0)-\varphi^2(y_2',-3L_0)) \tag{65}$$

if (59) and (60) and

$$\|\varphi^2(y_2,3L_0)-\varphi^2(y_2',-3L_0)\| \le \bar{\epsilon} \tag{66}$$

are satisfied. This follows from Lemma 6.6 and our choice of $\bar{\epsilon}$ and $\eta_\pm$.

We now begin to deform the map $\tilde{\boldsymbol{\epsilon}}(\iota_0,\iota_1)$.

Step 1. We will deform the map so that L_0 in the last factor of (65) goes from L_0 to 0. To achieve this, we consider a family of maps

$$\begin{gathered} K_0/S_0 \wedge K_1/S_1 \to \tilde{N}_0/\tilde{N}_0^+ \wedge \tilde{N}_1/\tilde{N}_1^+ \wedge B^+(V_n^2,\bar{\epsilon}) \\ (y_0,y_2,y_1,y_2') \mapsto (\varphi^0(y_0,3L_0),\varphi^1(y_1,3L_0),\varphi^0(y_2,3L)-\varphi^2(y_2',-3L)) \end{gathered}$$

if (59) together with

$$\varphi^2(y_2,[0,3L]) \subset A_2,\ \varphi^2(y_2',[-3L,0]) \subset A_2 \text{ and } \|\varphi^2(y_2,3L)-\varphi^2(y_2',-3L)\| \le \bar{\epsilon}$$

are satisfied. Lemma 6.7 guarantees that this is a continuous family. Thus, $\tilde{\boldsymbol{\epsilon}}(\iota_0,\iota_1)$ is homotopic to the map $\tilde{\boldsymbol{\epsilon}}_0(\iota_0,\iota_1)$ at $L=0$, which is given by

$$(y_0,y_2,y_2',y_1) \mapsto (\varphi^0(y_0,3L_0),\varphi^1(y_1,3L_0),y_2-y_2') \tag{67}$$

if (59) and $\|y_2-y_2'\| \le \bar{\epsilon}$ are satisfied.

Step 2. By Lemma 6.8, $(K^{\bar{\epsilon}},S^{\bar{\epsilon}})$ is an L_0-tame pre-index pair and we have a canonical map

$$\iota^{\bar{\epsilon}}\colon K^{\bar{\epsilon}}/S^{\bar{\epsilon}} \to I(B(V_n^0,\tilde{R}_0)) \wedge I(B(V_n^1,\tilde{R}_1)).$$

It is not hard to check that a map given by

$$K_0/S_0 \wedge K_1/S_1 \to I(B(V_n^0,\tilde{R}_0)) \wedge I(B(V_n^1,\tilde{R}_1)) \wedge (V_n^2)^+. \tag{68}$$

$$(y_0,y_2,y_1,y_2') \mapsto \begin{cases} (\iota^{\bar{\epsilon}}(y_0,y_1),y_2-y_2') & \text{if } \|y_2-y_2'\| \le \bar{\epsilon}, \\ * & \text{otherwise} \end{cases}$$

is well-defined and continuous. From Lemma 3.13, we can represent $\iota^{\bar{\epsilon}}$ by a map

$$\begin{aligned} K^{\bar{\epsilon}}/S^{\bar{\epsilon}} &\to \tilde{N}_0/\tilde{N}_0^+ \wedge \tilde{N}_1/\tilde{N}_1^+ \\ (y_0, y_1) &\mapsto (\varphi^0(y_0, 3L_0), \varphi^1(y_1, 3L_0)), \end{aligned}$$

if (59) is satisfied. Consequently, the map $\tilde{\boldsymbol{\epsilon}}_0(\iota_1, \iota_2)$ can be written as the map (68).

Recall that $\underline{\mathrm{bf}}^A(X_0)$ is obtained from composition of a map

$$B^+(W^0_{n,\beta}, \hat{R}_0) \to B^+(U^0_n, \epsilon) \wedge K_0/S_0$$

and the canonical map ι_0. The map $\underline{\mathrm{bf}}^R(X_1)$ is obtained similarly. Then, $\tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1))$ is given by applying Spanier–Whitehead dual map to their smash product. From previous paragraphs, we can conclude the following result

Proposition 6.9. *The morphism $\tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1))$ can be represented by suitable desuspension of the map*

$$\begin{aligned} B^+(W^0_{n,\beta}, \hat{R}_0) \wedge B^+(W^1_n, \hat{R}_1) &\to B^+(U^0_n, \epsilon) \wedge B^+(U^1_n, \epsilon) \wedge B^+(V^2_n, \bar{\epsilon}) \wedge I^n(-Y_0) \wedge I^n(-Y_1) \\ (\tilde{x}_0, \tilde{x}_1) &\mapsto (\widetilde{SW}_n(\tilde{x}_0), \widetilde{SW}_n(\tilde{x}_1), r_2(\tilde{x}_0) - r_2(\tilde{x}_1), \iota^{\bar{\epsilon}}(r_0(\tilde{x}_0), r_1(\tilde{x}_1)) \end{aligned}$$

if $\|\widetilde{SW}_n(\tilde{x}_i)\| \leq \epsilon$ and $\|r_2(\tilde{x}_0) - r_2(\tilde{x}_1)\| \leq \bar{\epsilon}$ and sending $(\tilde{x}_0, \tilde{x}_1)$ to the base point otherwise. Here $I^n(-Y_i)$ denotes $I(B(V^i_n, \tilde{R}_i))$ for $i = 0, 1$.

6.3. **Stably c-homotopic pairs.** In this subsection, we recall notions of stably c-homotopy and SWC triples which were introduced by Manolescu [12]. These provide a convenient framework when deforming stable homotopy maps coming from construction of Bauer–Furuta invariants. Although most of the definitions are covered in [12], we rephrase them in a slightly more general setting which is easier to apply in our situation. We also give some details for completeness and concreteness.

Let $p_i : E_i \to B$ $(i = 1, 2)$ be Hilbert bundles over some compact space B. We denote by $\| \cdot \|_i$ the fiber-direction norm of E_i. Let $\bar{E}_1$ be the fiberwise completion of E_1 using a weaker norm, which we denote by $| \cdot |_1$. We also assume that for any bounded sequence $\{x_n\}$ in E_1, there exist $x_\infty \in E_1$ such that after passing to a subsequence, we have

- $\{x_n\}$ converge to x_∞ weakly in E_1.
- $\{x_n\}$ converge to x_∞ strongly in $\bar{E}_1$.

Definition 6.10. A pair $l, c \colon E_1 \to E_2$ of bounded continuous bundle maps is called an admissible pair if it satisfies the following conditions:

- l is a fiberwise linear map;
- c extends to a continuous map $\bar{c} \colon \bar{E}_1 \to E_2$.

At this point, we will specialize on the context of gluing theorem as in Section 6.1. Let $V = Coul(Y_0) \times Coul(Y_1)$ and recall that we assumed $b_1(Y_0) = b_1(Y_1) = 0$. There is a Seiberg–Witten flow on V given by negative gradient flow of the Chern-Simons-Dirac functional. All critical points and finite types flow lines are contained in a sufficiently large ball. As before, denote by V^μ_λ the subspace spanned by the eigenvectors of $(*d, \not{D})$ with eigenvalue in $(\lambda, \mu]$ and denote the projection $V \to V^\mu_\lambda$ by p^μ_λ. Motivated by the Seiberg-Witten map on 4-manifolds with boundary, we give the following definition.

Definition 6.11. Let (l, c) be an admissible pair from E_1 to E_2 and let $r\colon E_1 \to V$ be a continuous map which is linear on each fiber. We call (l, c, r) an SWC-triple (which stands for Seiberg-Witten-Conley) if the following conditions are satisfied:

(1) The map $l \oplus (p^0_{-\infty} \circ r)\colon E_1 \to E_2 \oplus V^0_{-\infty}$ is fiberwise Freedholm.
(2) There exists $M' > 0$ such that for any pair of $x \in E_1$ satisfying $(l + c)(x) = 0$ and a half-trajectory of finite type $\gamma\colon (-\infty, 0] \to V$ with $r(x) = \gamma(0)$, we have $\|x\|_1 < M'$ and $\|\gamma(t)\| < M'$ for any $t \geq 0$.

Two SWC-triples (l_i, c_i, r_i) $(i = 0, 1)$ (with the same domain and targets) are called c-homotopic if there is a homotopy between them through a continuous family of SWC triples with a uniform constant M'.

Two SWC-triples (l_i, c_i, r_i) $(i = 0, 1)$ (with possibly different domain and targets) are called stably c-homotopic if there exists Hilbert bundles E_3, E_4 such that $((l_1 \oplus \mathrm{id}_{E_3}, c_1 \oplus 0_{E_3}), r_1 \oplus 0_{E_3})$ is c-homotopic to $((l_2 \oplus \mathrm{id}_{E_4}, c_2 \oplus 0_{E_4}), r_2 \oplus 0_{E_4})$.

For any SWC triple (l, c, r), we can define a relative Bauer–Furuta type invariant as a pointed stable homotopy class

$$BF(l, c, r) \in \{\Sigma^{n\mathbb{C}}\mathbf{T}(\mathrm{ind}(l, p^0_{-\infty} \circ r)), \mathrm{SWF}(-Y_0) \wedge \mathrm{SWF}(-Y_1)\},$$

where $n = n(Y_0, \mathfrak{s}_{Y_0}, g_{Y_0}) + n(Y_1, \mathfrak{s}_{Y_1}, g_{Y_1})$ by "SWC-construction" analogous to construction in Section 5 described below.

Let us pick a trivialization $q\colon E_2 \to F_2$, an increasing sequence of $\lambda_n \to \infty$ and a sequence of increasing finite-dimensional subspaces $\{F_2^n\}$ of F_2 such that the projections $p_n\colon F_2 \to F_2^n$ converge pointwisely to the identity map and $q^{-1}(F_2^n) \times V^{\lambda_n}_{-\lambda_n} \subset E_2 \times V^{\lambda_n}_{-\infty}$ is transverse to the image of $(l, p^{\lambda_n}_{-\infty} \circ r)$ on each fiber. Let E_1^n be the preimage $(l, p^{\lambda_n}_{-\infty} \circ r)^{-1}(q^{-1}(F_2^n) \times V^{\lambda_n}_{-\lambda_n})$ which is a finite rank subbundle.

Consider the approximated map

$$f_n = p_n \circ q \circ (l + c)\colon E_1^n \to F_2^n.$$

From the definition of the SWC triple, one can deduce the following in the same manner as the construction of relative invariants for Seiberg–Witten maps: for any $R', R \gg 0$ satisfying $r(B(E_1, R)) \subset B(V, R')$, there exist N, ϵ_0 such that for any $n \geq N$ and $\epsilon < \epsilon_0$, the pair of subsets

$$(p^{\lambda_n}_{-\infty} \circ r(f_n^{-1}(B(F_2^n, \epsilon)) \cap B(E_1, R)), p^{\lambda_n}_{-\infty} \circ r(f_n^{-1}(B(F_2^n, \epsilon)) \cap S(E_1, R))))$$

is a pre-index pair in the isolating neighborhood $B(V^{\lambda_n}_{-\lambda_n}, R')$.

From this, we can find an index pair (N, L) containing the above pre-index pair, which allows us to define an induced map $B(E_1^n, R)/S(E_1^n, R) \to B(F_2^n, \epsilon)/S(F_2^n, \epsilon) \wedge N/L$. After desuspension, we obtain a stable map

$$h\colon \Sigma^{n\mathbb{C}}\mathbf{T}(\mathrm{ind}(l, p^0_{-\infty} \circ r)) \to \mathrm{SWF}(-Y_0) \wedge \mathrm{SWF}(-Y_1).$$

By standard homotopy arguments, the stable homotopy class $[h]$ does not depend on the parameter we chose. We define the stable homotopy class $[h]$ to be the relative invariant $BF(l, c, r)$ for this SWC triple.

It is straightforward to prove that two stably c-homotopic SWC triples give the same stable homotopy class. This is the main point of introducing SWC construction. We end with a very useful lemma which is proved in [12] and allows us to move between maps and conditions on the domain.

Lemma 6.12. *Let (l, c) be an admissible pairs from E_1 to E_2 and let $r\colon E_1 \to V$ be a continuous map which is linear on each fiber. Suppose that we have a surjective bundle map $g\colon E_1 \to E_3$. Then the triple $(l \oplus g, c \oplus 0_{E_e}, r)$ is an SWC triple if and only if the triple $(l|_{\ker g}, c|_{\ker g}, r|_{\ker g})$ is an SWC triple. In the case that such two triples are SWC triples, they are stably c-homotopic to each other.*

6.4. **Deformation of the Seiberg-Witten map.** Throughout this section, we will denote by

$$G = H^1(X, Y_2; \mathbb{Z}) \cong H^1(X_0, Y_2; \mathbb{R}) \times H^1(X_1, Y_2; \mathbb{Z})$$

and fix such identification . Furthermore, we introduce the notation

$$\Omega^1(X_1, Y_1, \alpha^1) := \{\hat{a} \in \Omega^1(X_1) \mid d^*\mathbf{t}_{Y_1}(\hat{a}) = 0,\ \int_{Y_1^j} (*\hat{a}) = 0,\ \int_{\alpha_k^1} \hat{a} = 0,\ \forall j, k\}$$

and define $\Omega^1(X_0, Y_0, \alpha^0 \cup \beta)$ and $\Omega^1(X, Y_0 \cup Y_1, \alpha^0 \cup \alpha^1 \cup \beta)$ similarly. Consider all the following Hilbert spaces

- $V_{X_0} := L^2_{k+1/2}(i\Omega^1(X_0, Y_0, \alpha^0 \cup \beta) \oplus \Gamma(S^+_{X_0}))$;
- $V_{X_1} := L^2_{k+1/2}(i\Omega^1(X_1, Y_1, \alpha^1) \oplus \Gamma(S^+_{X_1}))$;
- $V_X := L^2_{k+1/2}(i\Omega^1(X, Y_0 \cup Y_1, \alpha^0 \cup \alpha^1 \cup \beta) \oplus \Gamma(S^+_X))$;
- $V := Coul(Y_0) \times Coul(Y_1)$;
- $U_{X_i} := L^2_{k-1/2}(i\Omega^0(X_i) \oplus i\Omega^2_+(X_i) \oplus \Gamma(S^-_{X_i}))$ for $i = 0, 1$;
- $U_X := L^2_{k-1/2}(i\Omega^0_0(X) \oplus i\Omega^2_+(X) \oplus \Gamma(S^-_X))$;
- $H^1(X_\bullet, Y_2; \mathbb{R})$, where $X_\bullet$ stands for X_0, X_1 or X.

Here $\Omega^0_0(X)$ denotes the space of functions on X which integrate to zero. Note that G acts on all these spaces as following:

- On differential forms, the action is trivial.
- On spinors, we use the identification
$$G \cong \mathcal{G}^{h,\hat{o}}_{X,Y_2}, \tag{69}$$
where $\mathcal{G}^{h,o}_{X,Y_2}$ denotes the group of harmonic gauge transformations u on X such that $u^{-1}du \in i\Omega^1_{CC}(X)$ and $u|_{Y_2} = e^f$ with $f(\hat{o}) = 0$. The action is by gauge transformation. Note that we will use the restriction of $\mathcal{G}^{h,\hat{o}}_{X,Y_2}$ on X_0 and X_1 instead of the harmonic gauge transformation satisfying boundary condition on X_0 or X_1.
- On the homology $H^1(X_\bullet, Y_2; \mathbb{R})$, the action is given by *negative* translation.

We consider Hilbert bundles

$$\tilde{V}_X = (V_X \times H^1(X, Y_2; \mathbb{R}))/G,$$
$$\tilde{U}_X = (U_X \times H^1(X, Y_2; \mathbb{R}))/G$$

over $\mathrm{Pic}^0(X, Y_2)$ and a pair of maps

$$l_X, c_X \colon V_X \times H^1(X, Y_2; \mathbb{R}) \to L^2_{k-1/2}(i\Omega^2_+(X) \oplus \Gamma(S^-_X)) \times H^1(X, Y_2; \mathbb{R})$$

given by

$$l_X(\hat{a}, \phi, h) := (d^+\hat{a}, \not{D}^+_{\hat{A}_0 + i\tau(h)}\phi, h),\ c_X := (F^+_{\hat{A}^t_0} - \rho^{-1}(\phi\phi^*)_0, \rho(\hat{a})\phi, h),$$

where $\tau(h)$ is the unique harmonic 1-form u on X representing h such that $\mathbf{t}_{Y_2}(\tau(h))$ is exact and $\tau(h) \in i\Omega^1_{CC}(X)$. It is straightforward to see that l_X and c_X are equivariant under the G-action. Thus, we can take the quotient and obtain bundle maps

$$(d^* \oplus \tilde{l}_X), (0 \oplus \tilde{c}_X) \colon \tilde{V}_X \to \tilde{U}_X.$$

Let us recall the construction of Bauer–Furuta invariant $\mathrm{BF}(X)$ (cf. Section 5 or [8]). We can see that the suitable double Coulomb slice in the construction is given by

$$\{(\hat{a}, \hat{\phi}) \in V_X \mid d^*(\hat{a}) = 0\}.$$

Let $\tilde{r}_i : \tilde{V}_X \to Coul(Y_i)$ denotes the twisted restriction map as in Section 5. It follows that $(\tilde{l}_X|_{\ker d^*}, \tilde{c}_X|_{\ker d^*}, (\tilde{r}_0, \tilde{r}_1)|_{\ker d^*})$ is a SWC-triple and $\mathrm{BF}(X)|_{\mathrm{Pic}^0(X,Y_2)}$ is precisely obtained from the SWC-construction of this triple.

The goal of this section is to deform $\mathrm{BF}(X)|_{\mathrm{Pic}^0(X,Y_2)}$ to the map $\tilde{\boldsymbol{\epsilon}}(\underline{\mathrm{bf}}^A(X_0), \underline{\mathrm{bf}}^R(X_1))$ represented in Proposition 6.9. There will be several steps.

Step 1. We move the gauge fixing condition $d^* = 0$ to stably c-homotopic maps. Since

$$d^* : i\Omega^1(X, Y_0 \cup Y_1, \alpha^0 \cup \alpha^1 \cup \beta) \to i\Omega^0_0(X)$$

is surjective, we directly apply Lemma 6.12 and obtain the following:

Lemma 6.13. *The relative Bauer-Furuta invariant* $\mathrm{BF}(X)|_{\mathrm{Pic}^0(X,Y_2)}$ *is obtained by the SWC construction on the triple* $(d^* \oplus \tilde{l}_X, 0 \oplus \tilde{c}_X, (\tilde{r}_0, \tilde{r}_1))$*, where*

$$\tilde{r}_i : \tilde{V}_X \to Coul(Y_i)$$

denotes the twisted restriction map to boundary Y_i*.*

Step 2. We begin to glue configurations on X_0 and X_1 to a configuration on X. Let us consider a Sobolev space of configurations on the boundary

$$V^{k-m}_{Y_2} := L^2_{k-m}(i\Omega^1(Y_2) \oplus i\Omega^0(Y_2) \oplus \Gamma(S_{Y_2})).$$

for $0 \le m \le k$.

For any 1-form $\hat{b}$ on X, we can combine the Levi-Civita connection on $\Lambda^* T^*(X_i)$ and the spin^c connection $\hat{A}_0|_{X_i} + \hat{b}$ to obtain a connection on $\Lambda^* T^*(X_i) \oplus S_{X_i}$. We use $\nabla^{\hat{b}}$ to denote the corresponding covariant derivative. Consider a map

$$\begin{aligned} D^{(m)}\colon V_{X_0} \times V_{X_1} \times H^1(X, Y_2; \mathbb{R}) &\to V_{Y_2}^{k-m} \times H^1(X, Y_2; \mathbb{R}) \\ (x_0, x_1, h) &\mapsto ((\nabla_{\vec{n}}^{\tau(h)|_{X_0}})^m x_0)|_{Y_2} - ((\nabla_{\vec{n}}^{\tau(h)|_{X_1}})^m x_1)|_{Y_2}, h), \end{aligned}$$

where $\vec{n}$ is the outward normal direction of $Y_2 \subset X_0$. Here we applied obvious bundle isomorphisms $T^*(X_i)|_{Y_2} \cong T^* Y_2 \oplus \underline{\mathbb{R}}$ and $S_{X_i}^+|_{Y_2} \cong S_{Y_2}$.

It is clear that the map $D^{(m)}$ is equivariant under the action of G. As a result, we can take quotient and obtain a map

$$\tilde{D}^{(m)}\colon \tilde{V}_{X_0, X_1} \to \tilde{V}_{Y_2}^{k-m},$$

where we set

$$\begin{aligned} \tilde{V}_{X_0, X_1} &:= (V_{X_0} \times V_{X_1} \times H^1(X, Y_2; \mathbb{R}))/G \\ \tilde{V}_{Y_2}^{k-m} &:= (V_{Y_2}^{k-m} \times H^1(X, Y_2; \mathbb{R}))/G. \end{aligned}$$

We state the gluing result of these spaces, which is a variation of the gluing result [12, Lemma 3]. The proof is only local near Y_2 and can be adapted without change.

Lemma 6.14. *The bundle map*

$$(\tilde{D}^{(k)}, \cdots, \tilde{D}^{(0)}) : \tilde{V}_{X_0, X_1} \to \bigoplus_{m=0}^{k} \tilde{V}_{Y_2}^{k-m}$$

is fiberwise surjective and the kernel can be identified with the bundle $\tilde{V}_X$.

Analogous to the maps $d^* \oplus l_X$ and $0 \oplus c_X$, we define

$$l_{X_0, X_1}\colon V_{X_0} \times V_{X_1} \times H^1(X, Y_2; \mathbb{R}) \to U_{X_0} \times U_{X_1} \times H^1(X, Y_2; \mathbb{R}) \tag{70}$$

$$((\hat{a}_0, \phi_0), (\hat{a}_1, \phi_1), h) \mapsto ((d^* \hat{a}_0, d^+ \hat{a}_0, \not{D}^+_{(\hat{A}_0 + i\tau(h))|_{X_0}} \phi_0), (d^* \hat{a}_1, d^+ \hat{a}_1, \not{D}^+_{(\hat{A}_0 + i\tau(h))|_{X_1}} \phi_1), h),$$

$$c_{X_0, X_1}\colon V_{X_0} \times V_{X_1} \times H^1(X, Y_2; \mathbb{R}) \to U_{X_0} \times U_{X_1} \times H^1(X, Y_2; \mathbb{R})$$

$$((\hat{a}_0, \phi_0), (\hat{a}_1, \phi_1), h) \mapsto ((0, F^+_{\hat{A}_0^t}|_{X_0} - \rho^{-1}(\phi_0 \phi_0^*)_0, \rho(\hat{a}_0)\phi_0), (0, F^+_{\hat{A}_0^t}|_{X_1} - \rho^{-1}(\phi_1 \phi_1^*)_1, \rho(\hat{a}_1)\phi_1), h).$$

Then, by taking quotient, we get bundle maps

$$\tilde{l}_{X_0, X_1}, \tilde{c}_{X_0, X_1}\colon \tilde{V}_{X_0, X_1} \to \tilde{U}_{X_0, X_1},$$

where $\tilde{U}_{X_0, X_1} := (U_{X_0} \times U_{X_1} \times H^1(X, Y_2; \mathbb{R}))/G$. By gluing of Sobolev spaces, the bundle $\tilde{U}_X$ can be identified as a subbundle of $\tilde{U}_{X_0, X_1}$. Let pj be the orthogonal projection to this subbundle. The following result is then a consequence of Lemma 6.14 and Lemma 6.12.

Lemma 6.15. *The triple*

$$((\mathrm{pj} \circ \tilde{l}_{X_0, X_1}, \tilde{D}^{(k)}, \cdots, \tilde{D}^{(0)}), (\mathrm{pj} \circ \tilde{c}_{X_0, X_1}, 0, \cdots, 0), (\tilde{r}_0, \tilde{r}_1)) \tag{71}$$

is a SWC-triple and is stably c-homotopic to $(d^* \oplus \tilde{l}_X, 0 \oplus \tilde{c}_X, (\tilde{r}_0, \tilde{r}_1))$.

Step 3. Next, we will glue the Sobolev spaces of the target. Let us consider a map

$$E^{(m)}\colon U_{X_0}\times U_{X_1}\times H^1(X,Y_2;\mathbb{R})\to V_{Y_2}^{k-1-m}\times H^1(X,Y_2;\mathbb{R})$$
$$(y_0,y_1,h)\mapsto (((\nabla_{\vec{n}}^{\tau(h)|_{X_0}})^m y_0)|_{Y_2}-((\nabla_{\vec{n}}^{\tau(h)|_{X_1}})^m y_1)|_{Y_2},h),$$

where we apply the standard bundle isomorphisms $\Lambda^2_+(X_i)|_{Y_2}\cong T^*Y_2,\ S^-_{X_i}|_{Y_2}\cong S_{Y_2}$. By taking quotient with respect to the action of G, we obtain bundle maps

$$\tilde{E}^{(m)}\colon \tilde{U}_{X_0,X_1}\to \tilde{V}_{Y_2}^{k-1-m}.$$

Proposition 6.16. *The triple*

$$\begin{aligned}&((\mathrm{pj}\circ\tilde{l}_{X_0,X_1},\ \tilde{E}^{(k-1)}\circ\tilde{l}_{X_0,X_1},\ \cdots,\tilde{E}^{(0)}\circ\tilde{l}_{X_0,X_1},\ \tilde{D}^{(0)}),\\ &(\mathrm{pj}\circ\tilde{c}_{X_0,X_1},\ \tilde{E}^{(k-1)}\circ\tilde{c}_{X_0,X_1},\ \cdots,\tilde{E}^{(0)}\circ\tilde{c}_{X_0,X_1},\ 0),(\tilde{r}_0,\tilde{r}_1))\end{aligned}\tag{72}$$

is a SWC-triple and is c-homotopic to the triple (71).

Proof. We simply consider a linear c-homotopy between them as follows: For $1\le m\le k$ and $0\le t\le 1$, define a map

$$\tilde{D}_t^{(m)}=(1-t)\cdot\tilde{D}^{(m)}+t\cdot\tilde{E}^{(m-1)}\circ\tilde{l}_{X_0,X_1}$$

and the following maps from $\tilde{V}_{X_0,X_1}$ to $\tilde{U}_X\oplus\left(\oplus_{m=0}^k\tilde{V}_{Y_2}^{k-m}\right)$

$$\begin{aligned}l_t&:=(\mathrm{pj}\circ\tilde{l}_{X_0,X_1},\ \tilde{D}_t^{(k)},\cdots,\tilde{D}_t^{(1)},\tilde{D}^{(0)}),\\ c_t&:=(\mathrm{pj}\circ\tilde{c}_{X_0,X_1},\ t\cdot\tilde{E}^{(k-1)}\circ\tilde{c}_{X_0,X_1},\ \cdots,t\cdot\tilde{E}^{(0)}\circ\tilde{c}_{X_0,X_1},\ 0).\end{aligned}$$

This will give a c-homotopy as a result of the following lemma. □

Lemma 6.17. *For any $0\le t\le 1$, the map*

$$(l_t,p^0_{-\infty}\circ(\tilde{r}_0,\tilde{r}_1))\colon \tilde{V}_{X_0,X_1}\to\tilde{U}_X\oplus(\bigoplus_{m=0}^{k}\tilde{V}_{Y_2}^{k-m})\oplus V^0_{-\infty}(-Y_0\cup -Y_1)$$

is fiberwise Fredholm. Moreover, the zero set $(l_t+c_t)^{-1}(0)\subset\tilde{V}_{X_0,X_1}$ is independent of t and can be described as

$$\{[(\hat{a},\phi,h)]\in\tilde{V}_X\mid d^*\hat{a}=0\ \textit{and}\ (\hat{A}_0+i\tau(h)+\hat{a},\phi)\ \textit{is a Seiberg-Witten solution}\}.$$

Proof. The key observation is that $E^{(m)}\circ l_{X_1,X_2}-\tilde{D}^{(m+1)}$ contains at most m-th derivative in the normal direction. Then, one can prove inductively that

$$(\tilde{D}_t^{(k)},\cdots,\tilde{D}_t^{(1)},\tilde{D}^{(0)})(x_0,x_1)=0\implies(\tilde{D}^{(k)},\cdots,\tilde{D}^{(0)})(x_0,x_1)=0,$$

so that the kernel of l_t does not depend on t. Similarly, one can show that $(\tilde{D}_t^{(k)},\cdots,\tilde{D}_t^{(1)},\tilde{D}^{(0)})$ is fiberwise surjective for all t. Since $t=0$ is the map from Lemma 6.15, the map $(l_t,p^0_{-\infty}\circ(\tilde{r}_0,\tilde{r}_1))$ is fiberwise Fredholm for all t.

This second part is essentially proved in [12, Section 4.11] using similar inductive argument.

□

Step 4. We now make the following identification:

Lemma 6.18. *The bundle map (over* $\mathrm{Pic}^0(X, Y_2)$*)*

$$(\mathrm{pj}, \tilde{E}^{(k-1)} \cdots \tilde{E}^{(0)}, \xi) \colon \tilde{U}_{X_0,X_1} \to \tilde{U}_X \oplus (\bigoplus_{m=0}^{k-1} \tilde{V}^{k-1-m}_{Y_2}) \oplus \underline{\mathbb{R}}$$

is an isomorphism. The map ξ *is given by* $\xi(x_1, x_2, h) = \int_{X_0} f_0 + \int_{X_1} f_1$*, where* f_i *is the 0-form component of* x_i*.*

Proof. This also follows from gluing result of Sobolev spaces [12, Lemma 3]. The only difference here is that the 0-form component $\tilde{U}_X$ consists of functions which integrate to 0. From standard decomposition $\Omega^0(X) = \Omega^0_0(X) \oplus \mathbb{R}$, we can see that the projection onto $\mathbb{R}$ is given by ξ. □

On the other hand, we decompose $\tilde{D}^{(0)}$ from the following decomposition of the Hilbert space:

$$V^k_{Y_2} = Coul(Y_2) \oplus H \oplus \mathbb{R} \text{ with } H = L^2_k(i(d\Omega^0(Y_2) \oplus \Omega^0_0(Y_2))). \tag{73}$$

We denote the corresponding components of $D^{(0)}$ (resp. $\tilde{D}^{(0)}$) by D_{Y_2}, D_H and $D_{\mathbb{R}}$ (resp. $\tilde{D}_{Y_2}, \tilde{D}_H$ and $\tilde{D}_{\mathbb{R}}$).

We make an observation that the SWC-triple (72) in Proposition 6.16 arises from a composition

$$\tilde{V}_{X_0,X_1} \to \tilde{U}_{X_0,X_1} \oplus Coul(Y_2) \oplus H \to \tilde{U}_X \oplus (\bigoplus_{m=0}^{k-1} \tilde{V}^{k-1-m}_{Y_2}) \oplus \underline{\mathbb{R}} \oplus Coul(Y_2) \oplus H,$$

where the first arrow is $(\tilde{l}_{X_0,X_1} + \tilde{c}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_H)$ and the second arrow is the isomorphism $(\mathrm{pj}, \tilde{E}^{(k-1)} \cdots \tilde{E}^{(0)}, \xi, id, id)$. The only thing we need to check is that $\tilde{D}_{\mathbb{R}} = \xi \circ \tilde{l}_{X_0,X_1}$ on the 1-form component, which follows from Green-Stokes formula

$$\int_{Y_2} \mathbf{t}(*\hat{a}_0) - \int_{Y_2} \mathbf{t}(*\hat{a}_1) = \int_{X_0} d^*\hat{a}_0 + \int_{X_1} d^*\hat{a}_1.$$

Thus we conclude

Lemma 6.19. *The SWC-triple (72) can be identified with the triple*

$$((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_H), (\tilde{c}_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1)). \tag{74}$$

Step 5. In this step, we focus on deforming the $\tilde{D}_H$-component which corresponds to boundary conditions for gauge fixing. We sometimes omit spinors from expressions in this step.

For $\hat{a}_j \in i\Omega^1(X_j)$, we have a Hodge decomposition $\mathbf{t}_{Y_2}(\hat{a}_j) = a_j + b_j$ on Y_2 with $a_j \in \ker d^*$ and $b_j \in \operatorname{im} d$. We also denote by $e_j := c_j - \frac{\int c_j d\mathrm{vol}}{\mathrm{vol}(Y_2)} \in i\Omega^0_0(Y_2)$, where $\hat{a}_j|_{Y_2} = \mathbf{t}_{Y_2}(\hat{a}_j) + c_j dt$. With this formulation, we see that $D_H(\hat{a}_0, \hat{a}_1) = (b_0 - b_1, e_0 - e_1)$.

Let us consider an isomorphism

$$\bar{d} : L^2_k(i\Omega^0_0(Y_2)) \to L^2_k(id\Omega^0(Y_2))$$

defined by $\bar{d}f := \lambda^{-1}df$ for any $f \in i\Omega^0_0(Y_2)$ with $d^*df = \lambda^2 f$ with $\lambda > 0$ using the spectral decomposition of d^*d. We let

$$\bar{d}^* : L^2_k(id\Omega^0(Y_2)) \to L^2_k(i\Omega^0_0(Y_2))$$

be its formal adjoint. Note that $\bar{d}^*$ can also be obtained directly by $\bar{d}^*\alpha := \lambda f$ for $\alpha = df$ satisfying $dd^*\alpha = \lambda^2\alpha$ with $\lambda > 0$ and $\int_{Y_2} f = 0$. We then define a family of maps

$$D_{H,t} \colon V_{X_0} \times V_{X_1} \to H$$

given by

$$D_{H,t}(\hat{a}_0, \hat{a}_1) := (b_0 - b_1, t \cdot \bar{d}^*(b_0 + b_1) + (1-t) \cdot (e_0 - e_1)).$$

The main point here is to establish that the gauge fixing conditions $D_{H,t} = 0$ are isomorphic and vary continuously. In particular, we will find a harmonic gauge transformation from the identity component relating them. For coclosed $(\hat{a}_0, \hat{a}_1) \in \Omega^1(X_0, Y_0, \alpha^0 \cup \beta) \times \Omega^1(X_1, Y_1, \alpha^1)$ with $b_0 = b_1$, it amounts to solve for functions $(f_0, f_1) \in \Omega^0(X_0) \times \Omega^0(X_1)$ such that

$$2t \cdot \bar{d}^*d(f_0|_{Y_2}) + (1-t)(\partial_{\vec{n}} f_0|_{Y_2} - \partial_{\vec{n}} f_1|_{Y_2}) = 2t \cdot \bar{d}^*(b_0) + (1-t)(e_0 - e_1)$$

satisfying other gauge fixing conditions. We have the following existence and uniqueness result.

Lemma 6.20. *Let $W \subset L^2_{k+3/2}(X_0;\mathbb{R}) \times L^2_{k+3/2}(X_1;\mathbb{R})$ be the subspace containing all functions (f_0, f_1) satisfying the following conditions:*

(1) $\Delta f_i = 0$;
(2) $f_i(\hat{o}) = 0$;
(3) $f_0|_{Y_2} = f_1|_{Y_2}$;
(4) f_i is a constant on each component of Y_i, $i = 0, 1$;
(5) $\partial_{\vec{n}} f_i$ integrates to zero on each component of Y_i, $i = 0, 1$.

Then the map $\rho_t \colon W \to L^2_k(\Omega^0_0(Y_2))$ defined by

$$\rho_t(f_0, f_1) = 2t \cdot \bar{d}^*d(f_0|_{Y_2}) + (1-t)(\partial_{\vec{n}} f_0|_{Y_2} - \partial_{\vec{n}} f_1|_{Y_2})$$

is an isomorphism.

Proof. We first show that ρ_t is an isomorphism when $t = 1$. For $\xi \in L^2_k(i\Omega^0_0(Y_2))$, we want to find f_i such that $f_i|_{Y_2} = \frac{\xi}{2} - \frac{\xi(\hat{o})}{2}$ and satisfies the other conditions. The existence and uniqueness of such functions follow from the same argument as in the double Coulomb condition (cf. [8, Proposition 2.2]).

Since each ρ_t corresponds to Laplace equation with mixed Dirichlet and Neumann boundary condition, it is Fredholm with index zero (from $t = 1$). Thus, for $t < 1$, we are left to show that ρ_t is injective. Suppose $\rho_t(f_0, f_1) = 0$. Then by Green's formula, we have

$$(1-t)\left(\int_{X_0} \langle df_0, df_0 \rangle + \int_{X_1} \langle df_1, df_1 \rangle\right) = (1-t)\int_{Y_2} f_0(\partial_{\vec{n}} f_0 - \partial_{\vec{n}} f_1) = -2t \int_{Y_2} f_0 \cdot (\bar{d}^*d(f_0|_{Y_2}))$$

The first expression is nonnegative but $\int_{Y_2} f_0(\bar{d}^* d(f_0|_{Y_2})) = \int_{Y_2}(f_0)^2 - \frac{1}{volY_2}(\int_{Y_2} f_0)^2$ is also nonnegative by Cauchy–Schwartz inequality. Hence both f_0 and f_1 must be constant and are in fact identically zero because $f_i(\hat{o}) = 0$.

□

As $D_{H,t}$ is equivariant, we can form bundle maps $\tilde{D}_{H,t}$ and obtain a c-homotopy.

Proposition 6.21. *For any $t \in [0,1]$, the triple $((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t}), (\tilde{c}_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1))$ is an SWC-triple. Consequently, this provides a c-homotopy between the triples.*

Proof. The statement for $t = 0$ follows from Lemma 6.19. For each element in the kernel of $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ there is a unique gauge transformation to an element in the kernel of $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,0})$ as a result of Lemma 6.20. This provides a linear bijection, so the kernel of $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ is also finite-dimensional.

The map $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ differs from the map $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,0})$ only at the $\Omega^0_0(Y_2)$-component. By Lemma 6.20, the map ρ_t is surjective, so the map $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ is surjective on the $\Omega^0_0(Y_2)$-component. This implies that the cokernels at each t are in fact the same. Therefore, $(\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t})$ are Fredholm.

Applying Lemma 6.20 again, one can see that there is a unique gauge transformation from a solution of $((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_H), (\tilde{c}_{X_0,X_1}, 0, 0))$ to a solution of $((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,t}), (\tilde{c}_{X_0,X_1}, 0, 0))$ which depends continuously. This provides a homeomorphism between them. Then the boundedness result follows from the case $t = 0$ and compactness of $[0,1]$.

□

Step 6. Here, we will basically change the action of G by identifying it with a group of harmonic gauge transformations but different boundary conditions. Recall from our setup that $\tau(h)$ for $h \in H^1(X, Y_2; \mathbb{R})$ is the unique harmonic 1-form on X representing h such that $\mathbf{t}_{Y_2}(\tau(h))$ is exact and $\tau(h) \in i\Omega^1_{CC}(X)$. Note that for $t \in [0,1]$,

$$D_{H,t}(\tau(h)|_{X_0}, \tau(h)|_{X_1}) = (0, 2t\bar{d}^*(\mathbf{t}_{Y_2}(\tau(h)))).$$

We put

$$(\xi_{0,t}(h), \xi_{1,t}(h)) := \rho_t^{-1}(2t\bar{d}^*(\mathbf{t}_{Y_2}(\tau(h)))).$$

We then apply gauge transformation to $\tau(h)$ and define

$$\begin{aligned}\tau_t = (\tau_{X_0,t}, \tau_{X_1,t})\colon H^1(X, Y_2; \mathbb{R}) &\to \Omega^1_h(X_0) \times \Omega^1_h(X_1)\\ h &\mapsto (\tau(h)|_{X_0} - d\xi_{0,t}(h), \tau(h)|_{X_1} - d\xi_{1,t}(h)).\end{aligned}$$

From our construction, we have $D_{H,t}(\tau_t(h)) = 0$ and $d\xi_{i,0} = 0$.

We will consider harmonic gauge transformations corresponding to boundary condition $D_{H,t} = 0$. For $h \in G$, we define $u_t(h) := (u_{X_0,t}(h), u_{X_1,t}(h))$ such that $u_{X_i,t}(h)$ is the unique gauge transformation on X_i satisfying

$$u_{X_i,t}(h)(\hat{o}) = 1,\ u^{-1}_{X_i,t} du_{X_i,t} = \tau_{X_i,t}(h).$$

Notice that for $u_{X_i,0}$ is the restriction of $u \in \mathcal{G}^{h,o}_{X,Y_2}$ and $u_{X_i,t}(h) = e^{-\xi_{i,t}(h)} u_{X_i,0}(h)$.

Consider a new action φ_t of G on the spaces V_{X_i}, U_{X_i}, $H^1(X_i, Y_2; \mathbb{R})$, $Coul(Y_i)$ and H such that the action on spinors is given by the gauge transformations $(u_{X_0,t}(h), u_{X_1,t}(h))$ instead of restriction of $u \in \mathcal{G}^{h,o}_{X,Y_2}$ as earlier. We also consider a map

$$l^t_{X_0,X_1}, c^t_{X_0,X_1} \colon V_{X_0} \times V_{X_1} \times H^1(X, Y_2; \mathbb{R}) \to U_{X_0} \times U_{X_1} \times H^1(X, Y_2; \mathbb{R})$$

by replacing the term $\tau(h)|_{X_i}$ in the definition (cf. (70)) with $\tau_{X_i,t}(h)$.

It is not hard to check that the maps $l^t_{X_0,X_1}$, $c^t_{X_0,X_1}$, $D_{Y_2} \times \mathrm{id}_{H^1(X,Y_2;\mathbb{R})}$ and $D_{H,t} \times \mathrm{id}_{H^1(X,Y_2;\mathbb{R})}$ are all equivariant under the action φ_t. By taking quotient, we obtain bundles

$$\tilde{V}^t_{X_0,X_1} := (V_{X_0} \times V_{X_1} \times H^1(X, Y_2; \mathbb{R}))/(G, \varphi_t);$$

$$\tilde{U}^t_{X_0,X_1} := (U_{X_0} \times U_{X_1} \times H^1(X, Y_2; \mathbb{R}))/(G, \varphi_t)$$

and bundle maps $\tilde{l}^t_{X_0,X_1}$, $\tilde{c}^t_{X_0,X_1}, \tilde{D}_{Y_2,t}, \tilde{D}_{H,t}$. We can consider an obvious bundle isomorphism from $\tilde{V}_{X_0,X_1}$ (resp. $\tilde{U}_{X_0,X_1}$) to $\tilde{V}^t_{X_0,X_1}$ (resp. $\tilde{U}_{X_0,X_1}$) by sending (a_i, ϕ_i, h) to $(a_i, e^{\xi_{i,t}(h)}\phi_i, h)$. All of the above maps fit in a commutative diagram.

$$\begin{tikzcd} \tilde{V}_{X_0,X_1} \arrow[r] \arrow[d] & \tilde{U}_{X_0,X_1} \arrow[d] \\ \tilde{V}^t_{X_0,X_1} \arrow[r] & \tilde{U}^t_{X_0,X_1} \end{tikzcd}$$

We can conclude:

Lemma 6.22. *The triple* $((\tilde{l}^1_{X_0,X_1}, \tilde{D}^1_{Y_2}, \tilde{D}^1_H), (\tilde{c}^1_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1))$ *is an SWC triple and is c-homotopic to* $((\tilde{l}_{X_0,X_1}, \tilde{D}_{Y_2}, \tilde{D}_{H,1}), (\tilde{c}_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1))$.

Let us take a closer look at the SWC triple

$$((\tilde{l}^1_{X_0,X_1}, \tilde{D}^1_{Y_2}, \tilde{D}^1_H), (\tilde{c}^1_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1)).$$

Observe that the boundary condition $b_0 - b_1 = 0$ and $\bar{d}^*(b_0 + b_1) = 0$ implies $b_0 = b_1 = 0$. This allows us to recover double Coulomb condition on X_i.

Lemma 6.23. *The operator*

$$(d^*_{X_0}, d^*_{X_1}, D_{H,1}) \colon V_{X_1} \times V_{X_2} \to L^2_{k-1/2}(i\Omega^0(X_0) \oplus i\Omega^0(X_1)) \oplus H$$

is surjective and its kernel can be written as

$$L^2_{k+1/2}(i\Omega^1_{CC}(X_0, \alpha^0 \cup \beta) \oplus \Gamma(S^+_{X_0})) \times L^2_{k+1/2}(i\Omega^1_{CC}(X_1, \alpha^1) \oplus \Gamma(S^+_{X_1})).$$

Proof. We consider exact forms (df_0, df_1). Then, surjectivity reduces to finding a solution of Poisson's equation with Dirichlet boundary condition on X_0 and X_1.

□

Note that we can make an identification

$$L^2_{k+1/2}(i\Omega^1_{CC}(X_0, \alpha^0 \cup \beta) \oplus \Gamma(S^+_{X_0})) \times H^1(X_0, Y_2; \mathbb{R}) \cong Coul^{CC}(X_0, \beta) \tag{75}$$

by sending $((\hat{a}_0, \phi), h)$ to $(\hat{a}_0 + \hat{a}_h, \phi)$, where $\hat{a}_h$ is the element in $\mathcal{H}^1_{DC}(X_0)$ corresponding to h (cf. (27) from Section 5). Under this identification, the natural projection to $H^1(X_0, Y_2; \mathbb{R})$ becomes the map $\hat{p}_{\alpha,X_0}$ (cf. (29)). Similarly, we have an isomorphism

$$L^2_{k+1/2}(i\Omega^1_{CC}(X_1, \alpha^1) \oplus \Gamma(S^+_{X_0})) \times H^1(X_1, Y_2; \mathbb{R}) \cong Coul^{CC}(X_1). \tag{76}$$

Consequently, the action φ^1 provides an action o fon $Coul^{CC}(X_0, \beta) \times Coul^{CC}(X_1)$ via an identification

$$G = H^1(X_0, Y_2) \times H^1(X_1, Y_2) \cong \mathcal{G}^{h,\hat{o}}_{X_0,\partial X_0} \times \mathcal{G}^{h,\hat{o}}_{X_1,\partial X_1}.$$

This holds because Y_0 and Y_1 are homology spheres.

As in Section 5, we have Seiberg–Witten maps

$$\overline{SW}_{X_0} = \bar{L}_{X_0} + \bar{Q}_{X_0} \colon Coul^{CC}(X_0, \beta)/\mathcal{G}^{h,\hat{o}}_{X_0,\partial X_0} \to (L^2_{k-1/2}(i\Omega^+_2(X_0) \oplus \Gamma(S^-_{X_0})) \times \mathcal{H}^1_{DC}(X_0))/\mathcal{G}^{h,\hat{o}}_{X_0,\partial X_0},$$
$$\overline{SW}_{X_1} = \bar{L}_{X_1} + \bar{Q}_{X_1} \colon Coul^{CC}(X_1)/\mathcal{G}^{h,\hat{o}}_{X_1,\partial X_1} \to (L^2_{k-1/2}(i\Omega^+_2(X_1) \oplus \Gamma(S^-_{X_1})) \times \mathcal{H}^1_{DC}(X_1))/\mathcal{G}^{h,\hat{o}}_{X_1,\partial X_1}.$$

Since an element of $\mathcal{G}^{h,\hat{o}}_{X_i,\partial X_i}$ takes value 1 on Y_2, there are well-defined restriction maps r_2 from $Coul^{CC}(X_0, \beta)/\mathcal{G}^{h,\hat{o}}_{X_0,\partial X_0}$ and $Coul^{CC}(X_1)/\mathcal{G}^{h,\hat{o}}_{X_1,\partial X_1}$ to $Coul(Y_2)$. We then consider a map

$$\begin{aligned}\bar{D}_{Y_2} \colon Coul^{CC}(X_0, \beta)/\mathcal{G}^{h,\hat{o}}_{X_0,\partial X_0} \times Coul^{CC}(X_1)/\mathcal{G}^{h,\hat{o}}_{X_1,\partial X_1} &\to Coul(Y_2)\\ (x_0, x_1) &\mapsto r_2(x_0) - r_2(x_1).\end{aligned}$$

With this setup, we can identify the previous SWC triple with maps which almost represent relative Bauer–Furuta invariants on X_0 and X_1.

Corollary 6.24. *The triple* $((\bar{L}_{X_0}, \bar{L}_{X_1}, \bar{D}_{Y_2}), (\bar{Q}_{X_0}, \bar{Q}_{X_1}, 0), (\tilde{r}_0, \tilde{r}_1))$ *is an SWC triple stably c-homotopic to* $((\tilde{l}^1_{X_0,X_1}, \tilde{D}^1_{Y_2}, \tilde{D}^1_H), (\tilde{c}^1_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1))$.

Proof. This follows by applying Lemma 6.12 to the triple $((\tilde{l}^1_{X_0,X_1}, \tilde{D}^1_{Y_2}, \tilde{D}^1_H), (\tilde{c}^1_{X_0,X_1}, 0, 0), (\tilde{r}_0, \tilde{r}_1))$ with $g = (d^*_{X_0}, d^*_{X_1}, D_{H,1})$ as in Lemma 6.23. □

Step 7. This is the final step. Recall from Section 6.1 that we chose finite-dimensional subspaces U^i_n of $L^2_{k-1/2}(i\Omega^+_2(X_i) \oplus \Gamma(S^-_{X_i}))$ and eigenspaces V^i_n of $Coul(Y_i)$. In the SWC construction of the triple $((\bar{L}_{X_0}, \bar{L}_{X_1}, \bar{D}_{Y_2}), (\bar{Q}_{X_0}, \bar{Q}_{X_1}, 0), (\tilde{r}_0, \tilde{r}_1))$, the subbundles involved are preimages of the map $(\bar{L}_{X_0}, \bar{L}_{X_1}, \bar{D}_{Y_2}, p^{\mu_n}_{-\infty} \circ \tilde{r}_0, p^{\mu_n}_{-\infty} \circ \tilde{r}_1)$ rather than preimages of the product map $(\bar{L}_{X_0}, p^{\mu_n}_{-\infty} \circ \tilde{r}_0, p^{\mu_n}_{-\infty} \circ \tilde{r}_2) \times (\bar{L}_{X_1}, p^{\mu_n}_{-\infty} \circ \tilde{r}_1, p^{\infty}_{-\mu_n} \circ \tilde{r}_2)$ in the construction of relative Bauer–Furuta invariants. Note that there is a choice of trivialization but we do not emphasize here.

Using spectral decomposition, we see that $r_2(x_0) - r_2(x_1) \in V^{\mu_n}_{-\mu_n}$ if and only if

$$\begin{aligned}p^{\infty}_{\mu_n} \circ r_2(x_0) &= p^{\infty}_{\mu_n} \circ r_2(x_1),\\ p^{-\mu_n}_{-\infty} \circ r_2(x_1) &= p^{-\mu_n}_{-\infty} \circ r_2(x_0).\end{aligned}$$

We introduce a family of subbundles by 'rotating' the above condition: for $\theta \in [0, \frac{\pi}{4}]$,

$$\begin{aligned} W^{n,\theta}_{X_0,X_1} := \{(x_0,x_1) \in &(Coul^{CC}(X_0,\beta)/\mathcal{G}^{h,\hat{o}}_{X_0,\partial X_0}) \times (Coul^{CC}(X_1)/\mathcal{G}^{h,\hat{o}}_{X_1,\partial X_1}) \mid \\ &p^{\mu_n}_{-\infty}\tilde{r}_i(x_i) \in V^i_n,\ \bar{L}_{X_i}(x_i) \in U^i_n, \\ &p^{\infty}_{\mu_n} \circ r_2(x_0) = \tan\theta \cdot p^{\infty}_{\mu_n} \circ r_2(x_1), \\ &p^{-\mu_n}_{-\infty} \circ r_2(x_1) = \tan\theta \cdot p^{-\mu_n}_{-\infty} \circ r_2(x_0)\}. \end{aligned}$$

We have boundedness result for this family.

Lemma 6.25. *For any $R > 0$, there exist N, ϵ_0 with the following significance: For any $n > N, \theta \in [0, \frac{\pi}{4}]$, $(x_0, x_1) \in B^+(W^{n,\theta}_{X_0,X_1}, R)$ and $\gamma_i : (-\infty, 0] \to B(V^{\lambda_n}_{-\lambda_n}(Y_i), R)$ where $i = 0, 1$ satisfying the following conditions:*

- $\|p^{\mu_n}_{-\mu_n}(r_2(x_0) - r_2(x_1))\|_{L^2_k} \leq \epsilon$;
- $\|p_{U^i_n} \circ \overline{SW}_{X_i}(x_i)\|_{L^2_{k-1/2}} \leq \epsilon$;
- *γ_i is an approximated trajectory with $\gamma_i(0) = p^{\mu_n}_{-\mu_n} \circ \tilde{r}_i(x_i)$,*

one has $\|x_i\|_F \leq R_3+1$ and $\|\gamma_i(t)\|_{L^2_k} \leq R_3+1$, where R_3 is the constant in Proposition 6.4.

Proof. The proof is essentially identical to Proposition 6.5 by using [12, Lemma 1] to control $\|p^{\infty}_{\mu_n} \circ r_2(x_0)\|_{L^2_k}$ (resp. $\|p^{-\mu_n}_{-\infty} \circ r_2(x_1)\|_{L^2_k}$) in terms of $\|\bar{L}_{X_0}(x_0)\|_{L^2_{k-1/2}}$ (resp. $\|\bar{L}_{X_1}(x_1)\|_{L^2_{k-1/2}}$). □

As a result, one can apply the construction of the relative invariants, which should be familiar by now, to define a stable homotopy class

$$[B^+(W^{n,\theta}_{X_0,X_1}, R), B^+(U^0_n, \epsilon) \wedge B^+(U^1_n, \epsilon) \wedge B^+(V^2_n, \epsilon) \wedge I^n(-Y_0) \wedge I^n(-Y_1)]$$

from the map $(\overline{SW}_{X_0}, \overline{SW}_{X_1}, \bar{D}_{Y_2}, \tilde{r}_0, \tilde{r}_1)$. When $\theta = \frac{\pi}{4}$, this is the same as SWC construction for the original triple. Finally, we see that, at $\theta = 0$, we have

$$W^{n,0}_{X_0,X_1} = W^0_{n,\beta} \times W^1_n$$

and we recover the homotopy class in Proposition 6.9. The proof of the gluing theorem is finished.

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