

Fixed subgroups of automorphisms of hyperbolic 3-manifold groups

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Abstract

For fixed subgroups $Fix(\phi)$ of automorphisms ϕ on hyperbolic 3-manifold groups $\pi_1(M)$, we observed that $rk(Fix(\phi)) < 2rk(\pi_1(M))$ and the constant 2 in the inequality is sharp; we also classify all possible groups $Fix(\phi)$.

1 Introduction

For a group G and an automorphism $\phi : G \rightarrow G$, we define $Fix(\phi) = \{\omega \in G | \phi(\omega) = \omega\}$, which is a subgroup of G , and use $rk(G)$ to denote the rank of G .

The so called Scott conjecture proved 20 years ago in a celebrate work of M. Bestvina and M. Handel [BH] states that:

Theorem 1.1. *For each automorphism ϕ on a free group $G = F_n$,*

$$rk(Fix(\phi)) \leq rk(G).$$

In a recent paper by B.J. Jiang, S. D. Wang and Q. Zhang [JWZ], it is proved that

Theorem 1.2. *For each automorphisms ϕ on a compact surface group $G = \pi_1(S)$,*

$$rk(Fix(\phi)) \leq rk(G).$$

It is obvious that the bounds given in Theorem 1.1 and Theorem 1.2 are sharp and can be achieved by the identity maps.

In this note, we will address the similar problem for hyperbolic 3-manifold groups. We call a compact 3-manifold M is *hyperbolic*, if M is orientable, and the interior of M admits a complete hyperbolic structure of finite volume (then M is either closed or ∂M is a union of tori). Therefore $G = \pi_1(M)$ is isomorphic a cofinite volume torsion free Kleinian group. A main observation in this paper is the following

Theorem 1.3. *For each automorphism ϕ on a hyperbolic 3-manifold group $G = \pi_1(M)$,*

$$rk(Fix(\phi)) < 2rk(G),$$

and the upper bound is sharp when G runs over all hyperbolic 3-manifold groups.

Theorem 1.3 is a conclusion of the following Theorems 1.4, and 1.5.

Theorem 1.4. *There exist a sequences automorphisms $\phi_n : \pi_1(M_n) \rightarrow \pi_1(M_n)$ on closed hyperbolic 3-manifolds M_n such that $Fix(\phi_n)$ is the group of a closed surface, and*

$$\frac{rk(Fix(\phi_n))}{rk(\pi_1(M_n))} > 2 - \epsilon \text{ as } n \rightarrow \infty$$

for any $\epsilon > 0$.

Theorem 1.5. *Suppose ϕ is an automorphism on $G = \pi_1(M)$, where M is a hyperbolic 3-manifold. Then $rk(Fix(\phi)) < 2rk(G)$.*

The proof of Theorem 1.4 is self-contained up to some primary (and elegant) facts on hyperbolic geometry and on combinatoric topology and group theory. Roughly speaking each (M_i, ϕ_i) in Theorem 1.4 is constructed as follows: We first construct the hyperbolic 3-manifold P_i with connected totally geodesic boundary. Then we double two copies of P_i along their boundaries to get the closed hyperbolic 3-manifold M_i . The reflection of M_i along ∂P_i will induce an automorphism $\phi_i : \pi_1(M_i) \rightarrow \pi_1(M_i)$ with $Fix(\phi) = \pi_1(\partial P_i)$. In this process all involved ranks are carefully controlled, we get the inequality in Theorem 1.4.

To prove Theorem 1.5, besides some combinatoric arguments on topology and on group theory, we need the following Theorem 1.6 which classify all possible groups $Fix(\phi)$ for automorphisms ϕ on hyperbolic 3-manifold groups. Recall that each automorphism ϕ on $\pi_1(M)$ can be realized by an isometry f on M according to Mostow rigidity theorem.

Theorem 1.6. *Suppose $G = \pi_1(M)$, where M is a hyperbolic 3-manifold, and ϕ is a automorphism of G . Then $Fix(\phi)$ is one of the following types: the whole group G ; the trivial group $\{e\}$; \mathbb{Z} ; $\mathbb{Z} \oplus \mathbb{Z}$; the surfaces group $\pi_1(S)$, where S can be orientable or not, and closed or not. More precisely*

- (1) *Suppose ϕ is induced by an orientation preserving isometry.*
 - (i) *$Fix(\phi)$ is either \mathbb{Z} , or $\mathbb{Z} \oplus \mathbb{Z}$, or G , or $\{e\}$; moreover*
 - (ii) *if M is closed, then $Fix(\phi)$ is either \mathbb{Z} or G ;*
- (2) *Suppose ϕ is induced by an orientation reversing isometry f .*
 - (i) *If $\phi^2 \neq id$, then $Fix(\phi)$ is either \mathbb{Z} or $\{e\}$;*
 - (ii) *if $\phi^2 = id$, then $Fix(\phi)$ is either $\{e\}$, or the surface group $\pi_1(S)$, where the surface S is pointwisely fixed by f .*

Theorem 1.6 is proved by using the algebraic version Mostow Rigidity theorem, as well as some hyperbolic geometry and covering space argument.

The paper is organized as follows: In Section 2 we will prove Theorem 1.4, and we also generalize the examples from closed hyperbolic 3-manifolds to hyperbolic 3-manifolds with cusps. Theorem 1.6 and Theorem 1.5 will be proved in Section 3 and Section 4 respectively.

Suppose a compact 3-manifold M is hyperbolic and S is a proper embedded surface in M . We say S totally geodesic surface implies that S° , the interior of S , is totally geodesic, and call ∂S the boundary of S° . Below we will use the same $M(S)$ to present the interior of $M(S)$.

For terminologies not defined, see [He1] and [Th1] for geometry and topology of 3-manifolds, and see [SW] for group theory.

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2 Proof of Theorem 1.4

In this section, we construct examples stated in Theorem 1.4. Roughly speaking those examples are constructed as follow: we first construct a hyperbolic 3-manifold P with totally geodesic boundary. Then we double it to get a closed hyperbolic 3 manifold DP . Now if we choose the base point on the boundary of P , the reflection along ∂P will induce ϕ on the fundamental group of DP , and this automorphism ϕ will have the property we desired.

There are different approaches to construct hyperbolic manifolds with totally geodesic boundaries. We will use the most original and the most direct one due to Thurston. (For another approach see Remark 2.6) .

In Thurston's Lecture Notes (Section 3.2 of [Th1]), there is a very concrete and beautiful construction of hyperbolic 3-manifolds with totally geodesic boundaries involving primary hyperbolic geometry only.

In 3-dimensional hyperbolic space H^3 , there is a one-parameter family of truncated hyperbolic tetrahedron as in Figure 1: Each of its 8 faces is totally geodesic; each of its 18 edges is geodesic line segment. There are 4 triangle faces and 4 hexagon faces. The 12 edges of the 4 triangle faces have the same length, and the remain 6 edges, we call them "inner edge", also have the same length. The triangle faces are perpendicular to the hexagon faces. The angles between hexagon faces are all equal and can be arbitrary angles between $(0^\circ, 60^\circ)$.

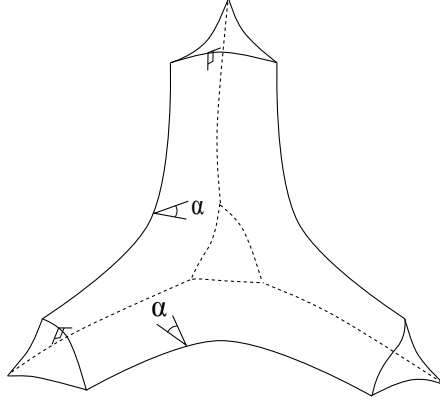


Figure 1

We will use those simplices to construct a hyperbolic manifold with totally geodesic boundaries. Suppose we have some copies of tetrahedron. We pair the faces of tetrahedron and gluing them together (therefore some edges and vertexes are also glued together). After gluing, if we remove a neighborhood of the vertex, we will get a topological manifold P . A tetrahedron with its vertex neighborhood removed is homeomorphic to the truncated simplex mention above. Suppose every k edges of the tetrahedron are glued together ($k > 6$). We can set the face angle α of the truncated simplex to be $\frac{2\pi}{k}$. Then the hyperbolic structure of the truncated simplex fix together to give the hyperbolic structure of P , and the triangle faces of the truncated simplex are matched together to form the totally geodesic ∂P . It is easy to see that the number of vertex of tetrahedron (after gluing) equals the number of the boundary component.

Moreover, if we remove the neighborhood of the inner edges in P . We will get a handlebody H . To see this, we remove the neighborhood of the 6 edges of a tetrahedron. Topologically, it is homeomorphic to D^3 and the 4 tetrahedron faces are 4 disjoint disks on ∂D^3 . Then, we glue them together. If we glue some 3 balls alone disks on their boundary, we get a handlebody. So P can be obtained by attaching m 2-handles on a handlebody of genus $n + 1$. It is easy to see that m is the number of inner edges after gluing and n is the number of tetrahedron.

Now we double P along its boundary to get a closed hyperbolic manifold DP . We have to control the rank of $\pi_1(DP)$. This is done in the following lemma.

Lemma 2.1. *Suppose P is obtained by attaching l -handles to a handlebody of genus k . Then $rk(\pi_1(DP)) \leq k + l$ (DP is the double of P).*

Proof. Suppose P is obtained from a handlebody H of genus k by attaching l 2-handles h_1, h_2, \dots, h_l with attaching curves $\gamma_1, \gamma_2, \dots, \gamma_l$, where $\gamma_1, \gamma_2, \dots, \gamma_l$ are disjoint simple closed curves on ∂H , and for each 2-handle $D^2 \times I$, $\partial D^2 \times I$ is identified with the attaching region $N(\gamma_i)$, the regular neighborhood of γ_i , for some i . Then we have

$$P = H \bigcup_{\{N(\gamma_i)\}} \{h_i\}.$$

Note in the doubling DP , the two copies of handlebody H and H' are glued together along $\partial H - \bigcup_i N(\gamma_i)$, and each two copies $D^2 \times I$ of the 2-handle h_i are glued along the $D^2 \times \partial I$ to get a solid torus S_i , which is attached to $H \bigcup_{\partial H - \bigcup_i N(\gamma_i)} H'$ along the torus boundary formed by two copies of $N(\gamma_i)$. So we have

$$DP = (H \bigcup_{\partial H - \bigcup_i N(\gamma_i)} H') \bigcup_i S_i.$$

Because attaching solid torus along the torus boundary of $H \bigcup_{\partial H - \bigcup_i N(\gamma_i)} H'$ does not increase the rank of the fundamental group, we just need to control $\text{rk}(\pi_1(H \bigcup_{\partial H - \bigcup_i N(\gamma_i)} H'))$. We consider the two skeleton of this space. The two skeleton of the handlebody H consists of a surface of genus k and k copies of compressing disks. So the two-skeletons of $H \bigcup_{\partial H - \bigcup_i N(\gamma_i)} H'$ can be obtained as follows: starting from a surface S_k of genus k , we glue two copies of S_k along $S_k - \bigcup_i N(\gamma_i)$. This is equally to attach l copies of annulus along $\partial N(\gamma_i), i = 1, 2, \dots, l$. Then we glue k compressing disks on both side.

Compared to the two-skeleton of H , we see that only l new generator are involved by the attached annulus and the new attaching disk does not increase the rank of fundamental group. So $\text{rk}(\pi_1(H \bigcup_{\partial H - \bigcup_i N(\gamma_i)} H')) \leq k + l$. ■

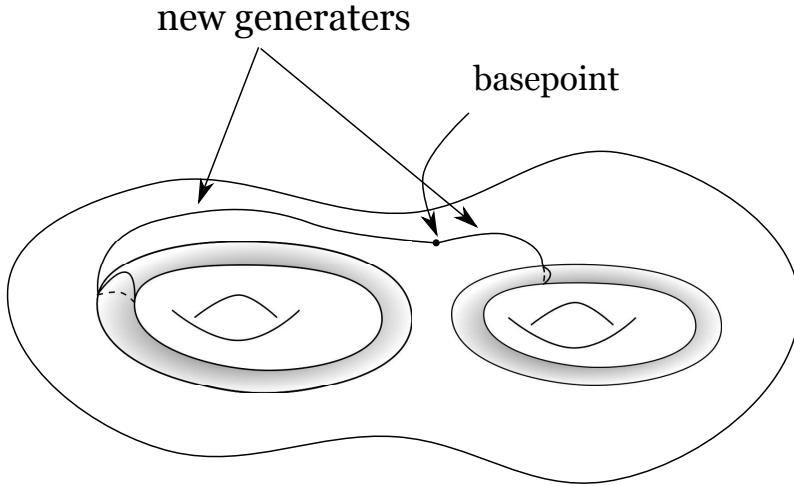


Figure 2

Now we can construct our examples. We start from n ($n > 3, 3 \nmid n$) copies of the tetrahedron indicated in Figure 3, where the edges are marked. We represent the faces by

the edges around it. Each tetrahedron T_i has 4 faces $(1, 3, 2)_i, (4, 5, 3)_i, (2, 6, 4)_i, (5, 1, 6)_i$. Then we group the $4n$ faces into $2n$ pairs:

$$[(1, 3, 2)_i, (4, 5, 3)_{i+1}]; [(2, 6, 4)_i, (5, 1, 6)_{i+1}], \quad i = 1, 2, \dots, n, \quad \text{and } n + 1 \equiv 1.$$

The two faces in each pair are glued together, and the orders of the edges are preserved. (It's easy to see that the arrows on the edges are preserved too.) Then we get a simplex X .

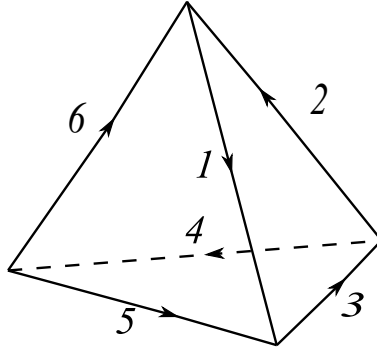


Figure 3

We write $a_i \leftrightarrow b_j$ to indicate that the edge a in T_i is glued together with b in T_j . With this notation, we have

$$1_k \leftrightarrow 4_{k+1} \leftrightarrow 6_{k+2} \leftrightarrow 1_{k+3}; \quad 2_k \leftrightarrow 5_{k+1} \leftrightarrow 3_k \leftrightarrow 2_{k-1}.$$

We first count the number of the edges after the gluing: Since $3 \nmid n$, we see that the $3n$ edges $1_*, 4_*, 6_*$ are glued together, and the $3n$ edges $2_*, 5_*, 3_*$ are glued together, so there are two edges in X .

Then we count the number of the vertices after the gluing: If we denote the initial point and the terminal point of the directed edge i_k by $I(i_k)$ and $E(i_k)$, then we have:

$$E(1_{k+1}) \leftrightarrow I(3_{k+1}) \leftrightarrow I(2_k) \leftrightarrow I(4_k) \leftrightarrow I(1_{k-1}) \leftrightarrow E(2_{k-1}) \quad (2.1)$$

The first, third and fifth identifications are shown in Figure 3. The second and fourth identifications follow from that respectively 3_{k+1} and $2_k, 4_k$ and 1_{k-1} are glued together as direct edges. Since the two edges $[1_*], [2_*]$ after the gluing. (2.1) implies that all ends of $[1_*], [2_*]$ are identified to a point. Hence there is only one vertex in the simplex X .

Finally we check the orientation: If we use the right hand coordinate system, the face $(1, 3, 2)$ and $(5, 1, 6)$ correspond to outward normal vectors while $(2, 6, 4)$ and $(4, 5, 3)$ correspond to inward ones. Since each inward face is glued with an outward one, the orientations are matched after gluing.

Now if we remove a regular neighborhood of the unique vertex, by the discussion at the begin of this section, we get an orientable hyperbolic three manifold P with connected, totally geodesic boundary.

As we have discussed, P can be constructed by attaching two two-handles on a handle body of genus $n + 1$. The genus of ∂P is $n - 1$. Now double P along its boundary to get a closed 3-manifold DP . If we choose a point $p \in \partial P$ as base point, the reflection f on $D(M)$ along ∂P will induce an automorphism $\phi : \pi_1(DP) \rightarrow \pi_1(DP)$.

Lemma 2.2. *In the construction above, $Fix(\phi) = Im(i_*(\pi_1(\partial M)))$.*

Proof. Note first that P is boundary incompressible and $\pi_1(DP)$ is a free product of two copies of $\pi_1(M)$ amalgamated over their subgroup $\pi_1(\partial P)$, that is

$$\pi_1(DP) = \pi_1(M)_{\pi_1(\partial P)} * \pi_1(M),$$

and the amalgamation is induced from the doubling.

We will apply the standard form of elements in free product of groups with amalgamations to prove the lemma. For convenient we denote $\pi_1(M)_{\pi_1(\partial P)} * \pi_1(M)$ by $G *_H G'$, where G and G' are two identical copies of $\pi_1(M)$ and H is the $\pi_1(\partial P)$. For each $g \in G$, denote $\phi(g) = g'$ (therefore $\phi(g') = g$) and clearly $\phi(h) = h$ for each $h \in H$.

For each right coset $g_i H$ of H in G , fix its representative g_i . We choose the unit 1 as the representative for the right coset H itself. Then $\{g'_i H\}$ give the right coset decomposition of H in G' and fix representative g'_i for $g'_i H$.

According to [SW, Theorem 1.7], each element γ in $G *_H G'$ can be written uniquely in a form $\gamma = a_1 b'_1 a_2 b'_2 \dots a_n b'_n h$, where $h \in H$, a_i is some representative g_j and b_i is some representative g_k ; moreover $a_i = 1$ implies $i = 1$ and $b_i = 1$ implies $i = n$. Then

$$\phi(\gamma) = a'_1 b_1 a'_2 b_2 \dots a'_n b_n h.$$

By uniqueness of the standard form, it is direct to see that if $\phi(\gamma) = \gamma$, then $\gamma = h$. hence only element in H can be fixed by ϕ . ■

By Lemma 2.1, $rk(\pi_1 DP) \leq n + 3$. Since ∂P has genus $n - 1$, by Lemma 2.2 $Fix(\phi) = Im(i_*(\pi_1(\partial P))) \cong \pi_1(\partial P)$ has rank $2n - 2$. For each $n > 2$, construct such pair (DP, ϕ) , and denoted as (M_n, ϕ_n) . Then

$$\frac{rk(Fix(\phi_n))}{rk(\pi_1(M_n))} \geq \frac{2n - 2}{n + 3} > 2 - \epsilon, \text{ as } n \rightarrow \infty$$

for any $\epsilon > 0$. Hence we finished the proof of Theorem 1.4. ■

The construction in Theorem 1.4 for closed hyperbolic 3-manifold can be modified to the case of hyperbolic 3-manifold with cusps. Precisely

Proposition 2.3. *There exist a sequences of hyperbolic 3-manifolds M_n with cusps and automorphisms $\phi_n : \pi_1(M_n) \rightarrow \pi_1(M_n)$, such that $Fix(\phi_n)$ is a free group, and*

$$\frac{rk(Fix(\phi_n))}{rk(\pi_1(M_n))} > 2 - \epsilon \text{ as } n \rightarrow \infty.$$

for any $\epsilon > 0$.

We give some theorems to prove Proposition 2.3.

Theorem 2.4. *[Th2] Suppose M is a hyperbolic 3 manifold with finite volume and f is a involution of M . Then M admit a hyperbolic structure with finite volume such that f is an isometric with respect to this structure.*

Theorem 2.5. *[Ko][Zh] Suppose M is a hyperbolic 3-manifold with finite volume and α is a simple closed geodesic in M . Then $M - \alpha$ admit a hyperbolic structure with finite volume.*

Proof. In the proof of Theorem 1.4, the hyperbolic 3-manifold P with connected totally geodesic boundary is obtained by attaching two 2-handles to a handlebody H along the attaching curve γ_1 and γ_2 . Now we choose a simple non-separating closed geodesic α in ∂P such that $\alpha \subset \partial H \setminus N(\gamma_1) \cup N(\gamma_2)$. Then α remains a geodesic in $D(P)$. Remove α from the closed hyperbolic manifold DP , we get a new hyperbolic manifold with a cusp by Theorem 2.5, denoted by DP' . The reflection f on $D(P)$ along ∂P defines a restriction on DP' , which is still an involution f' . By Theorem 2.6, DP' admit a hyperbolic structure so that f' is a isometry under this hyperbolic structure. So as the fixed point set of an isometry, the non-closed surface $\partial P - \alpha$ must be totally geodesic, and therefore incompressible.

If we pick the base point on $\partial P - \alpha$ and consider the automorphism ϕ induced by f' , the same combinational group theory argument as before shows that $Fix(\phi) = \pi_1(\partial P - \alpha)$. Because ∂P has genus $n - 1$, n is the same as in the proof of Theorem 1.4, $\partial P - \alpha$ is two punctured surface of genus $n - 2$, hence $\pi_1(\partial P - \alpha)$ is a free group of rank $2(n - 2) + 1 = 2n - 3$.

Now we control $rk(\pi_1(DP'))$ via the same technique in the proof of lemma 2.1: DP' consist of two parts: the first part is two copies of the handlebody H glued along $\partial H \setminus N(\gamma_1) \cup N(\gamma_2) \cup N(\alpha)$; the second part is two solid torus resulting from the doubling of the 2-handles. So the same argument as the proof of Lemma 2.1 shows that $rk(\pi_1(DP')) < n + 4$, and we have

$$\frac{rk(Fix(\phi))}{rk(\pi_1(DP'))} \geq \frac{2n - 3}{n + 4} > 2 - \epsilon, \text{ as } n \rightarrow \infty,$$

for any $\epsilon > 0$. ■

Remark 2.6. *There is another way to find hyperbolic 3-manifold with totally geodesic boundary, which is based on a most profound result in the 3-manifold theory and a result on Heegaard splitting:*

Theorem 2.7. [Th2] Suppose M is a compact 3-manifold M with non-empty boundary and infinite π_1 . If M contains no essential surface of genus smaller than 2, Then M admits a hyperbolic structure with totally geodesic boundary.

Theorem 2.8. [He] Let M be a closed oriented 3-manifold which is Seifert fibered or which contains an essential torus. Then any splitting of M is a Heegaard distance ≤ 2 splitting.

Theorem 2.8 in [He] is stated for closed 3-manifolds, but the argument there can be used to prove the similar theorem for non-closed case. Combine Theorem 2.8 and Theorem 2.7 we can conclude that: If we attach some two handles to a handlebody so that the distances in curve complex between the attaching curves of the two-handles and the boundaries of the compressing disks of the handlebody are larger than 3, then we will get a hyperbolic 3 manifold with totally geodesic boundary.

3 Proof of Theorem 1.6

In this section, we will classify all the possible fixed subgroups of automorphisms of cofinite volume klein groups. We use $Iso\mathbb{H}^3$ (resp. $Iso_+\mathbb{H}^3$) to denote the group of (resp. orientation preserving) isometries of the 3-dimensional hyperbolic 3-space.

The most important tool is the following algebraic version of Mostow rigidity theorem. Most topologists know the geometric version of Mostow rigidity: Any homotopy equivalence between finite volume hyperbolic 3 manifolds can be homotoped to an isometry. The following algebraic version appears in [MR], which is equivalent to the geometric version.

Theorem 3.1. Let Γ_1 and Γ_2 be two cofinite volume klein groups, and $\phi : \Gamma_1 \rightarrow \Gamma_2$ be an isomorphism between them. Then there exist $\gamma \in Iso(\mathbb{H}^3)$ (γ may be orientation reversing) such that for any $\alpha \in \Gamma_1, \phi(\alpha) = \gamma\alpha\gamma^{-1}$.

Now let's prove Theorem 1.6.

Proof. Since $G = \pi_1(M)$ is a hyperbolic 3-manifold group, G can be considered as cofinite volume torsion free Kleinian group in $Iso_+\mathbb{H}^3$. Now G acts on \mathbb{H}^3 as the deck transformation group for the covering $\pi : \mathbb{H}^3 \rightarrow \mathbb{H}^3/G \cong M$. Then by Theorem 3.1, there exist $\gamma \in Iso(\mathbb{H}^3)$ such that for any $\alpha \in G, \phi(\alpha) = \gamma\alpha\gamma^{-1}$. Then

$$Fix(\phi) = \{\alpha \in G \mid \alpha\gamma = \gamma\alpha\}. \quad (3.1)$$

Because $\gamma G \gamma^{-1} = G$, γ induces an isometry f of M , such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{H}^3 & \xrightarrow{\gamma} & \mathbb{H}^3 \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array} \quad (3.2)$$

The verification of Theorem 1.6 will be based on (3.1) and the verification will be divided into two cases according to if γ in (3.1) is orientation preserving or not.

Case (1) γ is orientation preserving.

(i) If $\gamma = e$ then clearly $Fix(\phi) = G$.

Below we assume that γ is nontrivial. It is well-known that each element in G is either hyperbolic or parabolic; moreover two nontrivial elements α, β in $Iso_+(\mathbb{H}^3)$ commute if and only if in one of the following cases happen:

(a) Both α and β are parabolic elements and they share the same fixed point in the infinite sphere S^∞ ;

(b) Both α and β are non-parabolic elements (elliptic or hyperbolic) and they share the same axis.

(c) Both α and β are elliptic elements with rotation angle π and their axis are perpendicular to each other.

Since elements in G (therefore in $Fix(\phi)$) can not be elliptic, we just need to consider case(a) and case (b). In these two cases, if α, β, γ are all nontrivial, α commutes with β , β commutes with γ , then α commute with γ . We see that $Fix(\phi)$ is a torsion free abelian group. As we know, the fundamental group of a hyperbolic 3-manifold can contain torsion free abelian subgroups of ranks at most 2. So we have proved that: if ϕ is induced by an orientation preserving map, $Fix(\phi)$ can only be $e, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}$, or G .

(ii) If we further assume that M is closed, then we have more restrictions.

First $\pi_1(M)$ contains no subgroup $\mathbb{Z} \oplus \mathbb{Z}$ for a closed hyperbolic 3-manifold M .

Also we claim that $Fix(\phi) \neq \{e\}$. In fact the self-isometry of a closed hyperbolic 3 manifold is always periodic. So there exists positive integer n such that $f^n = id$. By commuting diagram (3.2), γ^n induces the identity on M , therefore $\gamma^n \in G$. Then clearly $\gamma^n \in Fix(\phi)$. If $\gamma^n \neq e$, then $Fix(\phi)$ is not trivial. If $\gamma^n = e$, then γ is an elliptic element, so there is an axis l pointwise fixed by γ . By the commuting diagram (3.2), $\pi(l)$ is pointwise fixed by the isometry f . Since M is closed, the fixed point set of a orientation preserving isometry can only be closed geodesics. So $\pi(l)$ is a closed geodesic. This means that there is a hyperbolic covering transformation $\alpha \in G$ sharing the axis l with γ , so $\alpha\gamma = \gamma\alpha$, and therefore $\alpha \in Fix(\phi)$ by (3.1).

We have actually proved that if ϕ is induced by a orientation preserving map and M is closed. Then $Fix(\phi)$ can be either \mathbb{Z} or G .

Case (2) γ is orientation reversing.

Note $Fix(\phi) \subseteq Fix(\phi^2)$ and ϕ^2 is induced by an orientation preserving map. There are two subcases now:

(i) $\phi^2 \neq id$. Then $Fix(\phi^2)$ can only be e, \mathbb{Z} or $\mathbb{Z} \oplus \mathbb{Z}$ by Case (1) (i) and its proof, therefore $Fix(\phi)$ can only be e, \mathbb{Z} or $\mathbb{Z} \oplus \mathbb{Z}$.

But in fact the situation $\mathbb{Z} \oplus \mathbb{Z}$ never happens. Because in this situation $Fix(\phi^2) = \mathbb{Z} \oplus \mathbb{Z}$, which was generated by two parabolic elements β_1 and β_2 sharing the same fixed point p on the infinite sphere. Since we assume that $\phi^2 \neq id$, γ^2 is also a parabolic element

with the the fixed point p . Since $Fix(\gamma) \subset Fix(\gamma^2)$, one can derived that γ has the unique fixed point p in $\mathbb{H}^3 \cup S^\infty$, and in the upper-half model of \mathbb{H}^3 (we set $p = \infty$), $\gamma^2, \beta_1, \beta_2$ are translations along some directions v_1, v_2, v' respectively.

Consider their extended action on the plane $z = 0$. Then γ acts as a conformal (orientation reversing) map on this plane (in order to see this, we can just compose γ with an arbitrary reflection r' which fixes p to get an orientation preserving isometry. By classical fact, both $r' \circ \gamma$ and r' act conformally on plane $z = 0$, then so does γ). Because γ^2 is a translation, γ must act as an orientation reversing isometry on the plane $z = 0$. So $\gamma|_{z=0} = r \circ h$, where h is a translation along the direction v' and r a reflection along an invariant line of h . Then it is a direct verification that γ commutes with β_i if and only if the directions of v' and v_i are either the same or opposite. But the directions of v_1 and v_2 are neither the same nor opposite, so γ can not commute with both two generators of $\mathbb{Z} \oplus \mathbb{Z}$.

We have proved that in this subcase $Fix(\phi)$ is either e or \mathbb{Z} .

(ii) $\phi^2 = id$. Then γ^2 commute with the whole group G . So $\gamma^2 = e$, γ has order 2. An order 2 orientation reversing isometry of H^3 can only be the reflection along a single point or reflection along a geodesic plane.

If γ is the reflection along a point $p \in \mathbb{H}^3$, then p is the only fixed point of γ . For any $\alpha \in Fix(\phi)$, we have $\gamma\alpha = \alpha\gamma$ by (3.1). Hence $\gamma\alpha(p) = \alpha\gamma(p) = \alpha(p)$, that is γ also has fixed point $\alpha(p)$, hence $p = \alpha(p)$. Because α is a covering transformation, we msut have $\alpha = e$. We have proved that if γ is a reflection along a single point, then $Fix(\phi)$ is trivial.

If γ is the reflection along a totally geodesic plane P . Then P is pointwise fixed by γ . Because of the commuting diagram (3.2), $\pi(P)$ is pointwise fixed by f . We know that the fixed point set of an orientation reversing isometry of a hyperbolic 3-manifold must be totally geodesic surfaces if it is dimension 2. So $\pi(P) \simeq S$, S is a totally geodesic surface in M . (S may be non-orientable although M is orientable. And if M has cusps, S may have cusps too).

For any $\alpha \in Fix(\phi)$, $\gamma\alpha = \alpha\gamma$. Then for each $x \in P$, $\gamma\alpha(x) = \alpha\gamma(x) = \alpha(x)$, that is $\alpha(x) \in P$. It follows that P is invariant under α .

Conversely, suppose a covering transformation $\alpha \in G$ such that $\alpha(P) = P$. Then it is easy to see that $\gamma\alpha = \alpha\gamma$ and therefore $\alpha \in Fix(\phi)$. So $Fix(\phi)$ is exactly the covering transformations of the universal covering map $P \xrightarrow{\pi|_P} S$. We have proved that $Fix(\phi) \cong \pi_1(S)$. ■

4 Proof of Theorem 1.5

By Theorem 1.6, the situation $rk(Fix(\phi)) > rk(\pi_1(M))$ can appear only if Case (2) (ii) in Theorem 1.6 happens, and if Case (2) (ii) in Theorem 1.6 happens, then $Fix(\phi) = \pi_1(S)$ for some surface S which is pointwise fixed by an orientation reversing isometry f of order 2 on M . So the proof of the Theorem 1.5 will be completed by the following

Proposition 4.1. *Suppose M is a hyperbolic 3-manifold and S is a proper embedded surface in M . If there is an orientation reversing isometry f of order 2 on M fixing S pointwisly. Then*

$$\text{rk}(\pi_1(S)) < 2\text{rk}(\pi_1(M)).$$

To prove Proposition 4.1, we need the following lemma which contains several elementary facts:

Lemma 4.2. (1) *Suppose G is a group with subgroup H of index n . Then*

$$\text{rk}(G) \geq \frac{\text{rk}(H) + n - 1}{n}.$$

(2) *Suppose S is a boundary component of the compact 3-manifold M , and $D_S(M)$ is the doubling of two copies of M along S . Then*

$$\text{rk}(\pi_1(D_S(M))) \geq \text{rk}(\pi_1(M)).$$

(3) *(Half die half alive Lemma) Suppose M is a compact orientable 3-manifold. Then*

$$\dim\{\text{image } i^* : H^1(\partial M, \mathbb{Q}) \rightarrow H^1(M, \mathbb{Q})\} = \frac{\dim H^1(\partial M, \mathbb{Q})}{2}$$

where i^* is induced by the inclusion $i : \partial M \rightarrow M$.

(4) *Suppose M is a compact orientable 3-manifold and S is an incompressible boundary component of M . If the homomorphism $\pi_1(S) \rightarrow \pi_1(M)$ induced by the inclusion is not a surjection, then there is a finite covering $p : \tilde{M} \rightarrow M$ such that $p^{-1}(S)$ contains more than one component.*

Proof. (1) If $\text{rk}(G) = k$, we can find a 2-dimensional CW complex X with fundamental group G so that in X has only one vertex and k edges. Let \tilde{X} be the n -sheet covering space corresponding to the subgroup H . Then there n vertex and nk edge in \tilde{X} , and this lifted CW complex of \tilde{X} provides a presentation of H with $n(k - 1) + 1$ generators. So $\text{rk}(H) \leq n(k - 1) + 1$, that is

$$\text{rk}(G) = k \geq \frac{\text{rk}(H) + n - 1}{n}.$$

(2) It is clear that there is a reflection f about S on $D_S(M)$, which provides a folding map $D_S(M) \rightarrow D_S(M)/f \cong M$, and which is obviously induce a epimorphism between fundamental groups. Hence

$$\text{rk}(\pi_1(M)) \geq \text{rk}(\pi_1(M_1)).$$

(3) See [Mo, Section 23]

(4) By the assumptions, $\pi_1(S)$ is a proper subgroup of $\pi_1(M)$ (we pick a base point x on S for both $\pi_1(S)$ and $\pi_1(M)$). Pick an non-zero element α of $\pi_1(M)$ but not in $\pi_1(S)$.

By [LN, Theorem 1] (peripheral subgroups are separable), there is a finite index subgroup H of $\pi_1(M)$ which contains $\pi_1(S)$ but does not contain α . Consider the finite covering $p : \tilde{M} \rightarrow M$ corresponding to H , then there is a component \tilde{S} of $p^{-1}(S)$ homeomorphic to S (since $\pi_1(S) \subset H$), and $p^{-1}(S)$ has more than one component (since α does not in H , the lift $\tilde{\alpha}$ of α with one end in \tilde{S} must has another end in another component of $p^{-1}(S)$).

■

Lemma 4.3. *Suppose M is a compact orientable 3-manifold and S is an incompressible boundary component of M with genus g . If the homomorphism $\pi_1(S) \rightarrow \pi_1(M)$ induced by the inclusion is not a surjection, then $\text{rk}(\pi_1(M)) > g$.*

Proof. Let $p : \tilde{M} \rightarrow M$ be the n sheet covering provided by the proof of Lemma 4.2 (4). Then the preimage of S has $m > 1$ component. Now we can compute the sum of the genus of these m boundary components. Because S has euler number $2 - 2g$, the sum of the euler number of the preimage of S is $n(2 - 2g)$. Therefore, the sum of their genus is $n(g - 1) + m$. Since $H^1(\partial M', \mathbb{Q})$ is a direct sum of the homology of the boundary components, it contains at least $2n(g - 1) + 2m$ copies of \mathbb{Q} .

By Lemma 4.2 (3), we have $\text{rk}(H^1(M', \mathbb{Q})) \geq n(g - 1) + m$. Since $H^1(M', \mathbb{Q})$ is a quotient group of $\pi_1(M')$, we have $\text{rk}(\pi_1(M')) \geq n(g - 1) + m$. Since $\pi_1(M')$ is subgroup of $\pi_1(M)$ of index n , by Lemma 4.1 (1) we have

$$\text{rk}(\pi_1(M)) \geq \frac{n(g - 1) + m + n - 1}{n} = g + \frac{m - 1}{n}.$$

Since $m > 1$. Then Lemma 4.3 is proved. ■

Remark 4.4. *Suppose M is a hyperbolic 3-manifold and S is a closed embedded surface in M which is pointwisely fixed by an orientation reversing involution on M . Cutting M along S , we get a compact 3-manifold M' (may be not connected) with a boundary component S . In the proof of Theorem 4.1 we will apply Lemma 4.3 to $S \subset M'$ directly by the following reason:*

Using the fact that S is a proper embedded surface in M which is pointwisely fixed by an orientation reversing involution f on M and M contain no essential spheres and essential tori, it follows that (1) S must be incompressible (otherwise M would contain essential spheres); (2) the homomorphism $\pi_1(S) \rightarrow \pi_1(M')$ induced by the inclusion is not a surjection, (otherwise $M' = S \times [0, 1]$ by a result of J. Stallings [He1, 10.2 Theorem], and then M would contain essential tori).

Now we start to prove Proposition 4.1.

Proof. There are several cases to be considered. The surface may be either separating or not, either orientable or not, either closed or not. The proofs of all those cases are similar, but some subtle differences may appear. So we write all the details for each case.

Suppose the surface S has k boundary components, denoted by c_1, c_2, \dots, c_k . Different c_i may be contained in the same torus component, but a torus component can contain at most 2 of such c_i . In fact, suppose $c_i, c_j \subset T$, a torus component of ∂M . Now $f|_T$ is an orientation reversing involution on T . c_i and c_j are two parallel circles on T , fixed pointwisely by f . It's easy to see that f interchange the two connected components of $T - c_i \cup c_j$, so T can not contain any other component c_l other than c_i and c_j .

Without loss of generality, we can assume that among the boundary components of S . c_{2i-1} and c_{2i} are in the same torus boundary component, $i = 1, 2, \dots, m$. and $c_{2m+j}, j = 1, 2, \dots, l$ are contained in other l different boundary components.

Case (1) S is orientable surface of genus g . Note

$$\text{rk}(\pi_1(S)) = 2g + k - 1 \text{ if } k > 0 \text{ and } \text{rk}(\pi_1(S)) = 2g \text{ if } k = 0. \quad (4.1)$$

We will divided the discussion into two subcases according to if S is separating or not.

(i) S is separating. Then it is easy to see that $k = 2m$ and $l = 0$. Cutting M along the surface S , we get two homeomorphic components M_1, M_2 , and f interchanges them. Now each pair c_{2i-1}, c_{2i} bounds an annulus in M_1 connecting S , which increases the genus of S by 1. So we obtained a boundary component of M_1 with genus $(g + \frac{k}{2})$.

If $k > 0$, then

$$2\text{rk}(\pi_1(M)) \geq 2\text{rk}(\pi_1(M_1)) \geq 2\text{rk}(H_1(M_1, Q)) \geq 2(g + \frac{k}{2}) > 2g + k - 1 = \text{rk}(\pi_1(S)).$$

The first and the third inequalities and the last equality are based on Lemma 4.2 (2), (3) and (4.1) respectively.

If $k = 0$, then M_1 is a hyperbolic 3-manifold with a totally geodesic boundary component S , which is incompressible. Then

$$2\text{rk}(\pi_1(M)) \geq 2\text{rk}(\pi_1(M_1)) > 2g = \text{rk}(\pi_1(S)).$$

Those two inequalities and one equality are based on Lemma 4.2 (2), Lemma 4.3 (also Remark 4.4) and (4.1) respectively.

(ii) S is non-separating. Cutting M along S we get a new connected manifold M' with two copies of S , denoted by S_1 and S_2 , in $\partial M'$.

Suppose S_1, S_2 are contained in the same boundary component S' of M . Then $k > 0$ and S' consist of S_1, S_2 and $2m + l$ annulus, which is clearly closed and orientable, and

$$g(\partial M') \geq g(S') = 2g + 2m + l - 1 = 2g + k - 1 \quad (4.2).$$

As before, by Lemma 4.2 (2), (3) and (4.2) we get

$$\text{rk}(\pi_1(M')) \geq \text{rk}(H_1(M', Q)) \geq \frac{\text{rk}(H_1(S', Q))}{2} = 2g + k - 1 \quad (4.3).$$

Now $f' = f|_{M-S}$ is an involution on M' , which keeps the boundary component S' invariant and interchanges S_1 and S_2 . Now let's take two copies of M' , denote them by M'_1 and

M'_2 , and glue them on S_1 and S_2 via the identity to get a new manifold \widetilde{M} . Let r be the reflection on \widetilde{M} about $S_1 \cup S_2$. Then we have a free involution \tilde{f} on \widetilde{M} defined as

$$\tilde{f}|_{M'_1} = f' \circ r \text{ and } \tilde{f}|_{M'_2} = r \circ f'.$$

It is easy to verify that $\pi : \widetilde{M} \rightarrow \widetilde{M}/\tilde{f} = M$ is a two fold covering.

Applying Lemma 4.2 (1) for $n = 2$, Lemma 4.2 (2), (4.3) and (4.1) (recall that in this case $k > 0$), we have

$$2\text{rk}(\pi_1(M)) \geq \text{rk}(\pi_1(\widetilde{M})) + 1 \geq \text{rk}(\pi_1(M')) + 1 \geq 2g + k > \text{rk}(\pi_1(S)).$$

Suppose S_1 and S_2 belong to two different components of $\partial M'$. Then $l = 0$ and each component consists of one S_i and m annuli, hence

$$g(\partial M') \geq g(S_1) + m + g(S_2) + m = 2g + k \quad (4.4).$$

Doubling two copies of M' along S_1 and S_2 and constructing a 2-fold covering $\widetilde{M} \rightarrow M$ using the involution f as before, apply (4.4) we can prove similarly that:

$$2\text{rk}(\pi_1(M)) \geq 2g + k + 1 > \text{rk}(\pi_1(S)).$$

(2) S is non-orientable surface of genus g (connected sum of g real projective planes).
Note

$$\text{rk}(\pi_1(S)) = g + k - 1 \text{ if } k > 0 \text{ and } \text{rk}(\pi_1(S)) = g \text{ if } k = 0. \quad (4.5)$$

In this case S is non-separating. As before, we cut M along S to get a new manifold M' with one boundary component S' consisting of the orientable double cover \widetilde{S} of S and $2m + l$ annulus, and we have

$$g(\partial M') \geq g(S') = g - 1 + 2m + l = k + g - 1 \quad (4.6)$$

The involution $f' = f|_{M-S}$ provides a covering transformation of $\widetilde{S} \rightarrow S$. Again, we glue two copies of M' along \widetilde{S} to get the manifold \widetilde{M} and a double covering $\widetilde{M} \rightarrow M$.

If $k > 0$, as before, applying Lemma 4.2 (1) for $n = 2$, Lemma 4.2 (2), (3), (4.6) and (4.5) in order, we have

$$2\text{rk}(\pi_1(M)) \geq \text{rk}(\pi_1(\widetilde{M})) + 1 \geq \text{rk}(\pi_1(M')) + 1 \geq \text{rk}(H_1(M', Q)) + 1 \geq k + g > \text{rk}(\pi_1(S)).$$

If $k = 0$, applying Lemma 4.2 (1) for $n = 2$, Lemma 4.2 (2), Lemma 4.3 (also Remark 4.4), (4.6) and (4.5) in order, we have

$$2\text{rk}(\pi_1(M)) \geq \text{rk}(\pi_1(\widetilde{M})) + 1 \geq \text{rk}(\pi_1(M')) + 1 > g - 1 + 1 = \text{rk}(\pi_1(S)).$$

We finished the proof of Proposition 4.1. ■

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