Quantum sheaf cohomology on Grassmannians

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In this paper we study the quantum sheaf cohomology of Grassmannians with deformations of the tangent bundle. Quantum sheaf cohomology is a (0,2) deformation of the ordinary quantum cohomology ring, realized as the OPE ring in A/2-twisted theories. Quantum sheaf cohomology has previously been computed for abelian gauged linear sigma models (GLSMs); here, we study (0,2) deformations of nonabelian GLSMs, for which previous methods have been intractable. Combined with the classical result, the quantum ring structure is derived from the one-loop effective potential. We also utilize recent advances in supersymmetric localization to compute A/2 correlation functions and check the general result in examples. In this paper we focus on physics derivations and examples; in a companion paper, we will provide a mathematically rigorous derivation of the classical sheaf cohomology ring.

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1 Introduction

Computing nonperturbative corrections to charged matter couplings in heterotic string compactifications is one of the outstanding problems in string compactifications. On the (2,2) locus, when the gauge connection is determined by the spin connection, charged matter couplings such as the $\overline{\mathbf{27}}^3$ and $\mathbf{27}^3$ in compactifications to four dimensions are computed by the A and B model topological field theories, and their values are by now well-understood via mirror symmetry. Off the (2,2) locus, much less is known.

In principle, charged matter couplings off the (2,2) locus can be computed by the A/2 and B/2 pseudo-topological field theories, and work has been done in that direction, starting with [1] (motivated by the mirror symmetry analysis of [2]). These twists of a (0,2) nonlinear sigma model on a space X with bundle $\mathcal{E}$ exist when

$$\det \mathcal{E} \;\cong\; K_X^{\pm 1}, \;\; \mathrm{ch}_2(\mathcal{E}) \;=\; \mathrm{ch}_2(TX).$$

OPE's in these pseudo-topological theories define 'quantum sheaf cohomology' rings, generalizing ordinary quantum cohomology rings. For example, in A/2 theories, the chiral states are of the form

$$\oplus H^\bullet \left(X, \wedge^\bullet \mathcal{E}^* \right),$$

and quantum sheaf cohomology encodes nonperturbative quantum corrections to the product structure on the sheaf cohomology above. This work continued in *e.g.* [3–18], and culminated in a description of quantum sheaf cohomology rings on toric varieties with gauge bundles given by deformations of the tangent bundle, as described physically in GLSMs in [19, 20] and mathematically in [21, 22]. (See also [23–29] for more recent discussions, and [30] for a recent discussion of perturbative contributions to Yukawa couplings.)

Although those results are an important step, computing nonperturbative corrections and quantum sheaf cohomology for compact Calabi-Yau's with bundles that are not deformations of tangent bundles remains an open question.

As a stepping-stone towards that goal, we have been considering quantum sheaf cohomology on Grassmannians. These have technical complications beyond those of toric varieties, yet also have enough symmetries to make one hope that a tractable solution exists. In terms of GLSMs, this involves understanding nonabelian cases, whereas all previous work in quantum sheaf cohomology in (0,2) models has been in abelian GLSMs. In terms of the underlying mathematics, this becomes a story about nontrivial sheaves on Quot schemes, a technical step beyond toric cases, for which the pertinent moduli spaces are again toric varieties and induced sheaves are locally-free.

In this paper, we will present the results of that program, namely quantum sheaf cohomology rings on Grassmannians with deformations of the tangent bundle. Specifically, we will use one-loop effective action and supersymmetric localization [29, 31–34] arguments to

derive and discuss the form of the quantum sheaf cohomology ring. For generic deformations off the (2,2) locus, we will argue that the quantum sheaf cohomology ring can be expressed as

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)}, \right.$$
$$\left. R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \right\rangle,$$

where

$$D_m = \det\left(\sigma_{(1+j-i)}\right)_{1 \leq i,j \leq m},$$

$$R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i,$$

for I_i the coefficients of the characteristic polynomial of a matrix B defining the deformation. We discuss how the ring above encodes the ordinary quantum cohomology ring (and the classical cohomology ring) of the Grassmannian as special cases. We also discuss two sets of non-generic loci where the expression above fails to hold. Along one, the 'discriminant locus', both the deformed bundle and the corresponding physical theory degenerate. We give explicit expressions for this locus. In addition, there is a second locus of interest, where the additive part of the cohomology ring jumps. We derive an expression for this second 'jumping' locus.

In a companion paper [35], we will give a mathematical proof of the classical sheaf cohomology ring corresponding to the $q \to 0$ limit of our quantum sheaf cohomology ring. A purely mathematical proof of the form of the quantum sheaf cohomology ring on Grassmannians with deformations of the tangent bundle, an exercise in sheaf theory on Quot schemes, is left for the future.

The paper is organized as follows. In section 2, we discuss general issues regarding (0,2)-deformations on the Grassmannian, its one-loop effective potential on the Coulomb branch and supersymmetric localization. In section 3, we give a representation for the quantum sheaf cohomology by making use of our result in [35] and the one-loop effective potential. We also discuss its applicability: our formula will be valid for generic deformations, but will break down on certain codimension one subvarieties, which we discuss in detail. We also check that our quantum sheaf cohomology ring correctly specializes to both the classical and quantum (ordinary) cohomology rings, as expected on the (2,2) locus where the bundle is just the tangent bundle. In section 4, we check the given quantum sheaf cohomology ring in examples. We compute correlation functions using supersymmetric localization, yielding analogues of Jeffrey-Kirwan-Grothendieck residues on the Coulomb branch, as in [29]. We also explicitly discuss the codimension-one subvarieties along which our description of the quantum sheaf cohomology ring breaks down. In appendix A, we give a mathematical description of the classical sheaf cohomology ring, outlining the approach and results of [35]. In appendix B, we outline how the product structures can be understood as an exercise in homological algebra, leading to a speculation that 'quantum sheaf cohomology' might

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be understood as 'quantum homological algebra.' Finally, in appendix C, we outline the arguments that will appear in [35] in the special case of a deformation defined by $B \propto I$, for which the resulting bundle is isomorphic to the tangent bundle, and the deformation trivial.

In passing, the reader should note that when we speak of "(0,2) chiral rings" or OPE rings, we are referring to a finite-dimensional truncation of the infinite-dimensional chiral ring of a (0,2) theory, a truncation which reduces on the (2,2) locus to the ordinary (2,2) chiral ring. It was argued in [7] that the OPE algebra of this truncation closes into itself, so it is consistent to refer to this as an OPE ring. In any event, this is the ring in the twisted theory that physically defines quantum sheaf cohomology.

2   Nonabelian A/2 models

2.1   (0,2) deformation

The gauged linear sigma model can be used to implement various geometric settings. On the (2,2) locus, the Grassmannian $G(k,n)$ is described by a two-dimensional supersymmetric $U(k)$ gauge theory with n chirals in the fundamental representation. Let's denote these chiral fields by $\Phi_\alpha^i, \alpha = 1, \cdots, k, i = 1, \cdots, n$. The (2,2) vector supermultiplet decomposes into a (0,2) vector multiplet V and a chiral multiplet Σ. The bosonic component of Σ is an adjoint valued scalar σ. The chiral supermultiplet, Φ_a^i, decomposes into a (0,2) chiral multiplet $\Phi_\alpha^i = (\phi_\alpha^i, \psi_{+\alpha}^i)$ and a (0,2) Fermi multiplet $\Lambda_\alpha^i = (\psi_{-\alpha}^i, F_\alpha^i)$, obeying

$$\overline{D}_+ \Lambda_\alpha^i \;=\; \sigma_\alpha^\beta \Phi_\beta^i. \tag{1}$$

In a (0,2) theory, the covariant derivative of the Fermi superfield can be any function annihilated by the covariant derivative, i.e., (1) is generalized to

$$\overline{D}_+ \Lambda_\alpha^i \;=\; E_\alpha^i, \tag{2}$$

where E is a holomorphic function of the chiral superfields satisfying

$$\overline{D}_+ E = 0.$$

In particular, we can deform off the (2,2) locus by taking

$$\overline{D}_+ \Lambda_\alpha^i \;=\; A_j^i \sigma_\alpha^\beta \Phi_\beta^j \;+\; B_j^i (\mathrm{Tr}\,\sigma) \Phi_\alpha^j,$$

where A and B are n by n matrices. For simplicity, in this paper we will assume A is invertible, which will guarantee our models can be deformed to the (2,2) locus. In principle, one could also imagine nonlinear deformations, functions of say

$$\epsilon^{\alpha_1 \cdots \alpha_k} \Phi_{\alpha_1}^{i_1} \cdots \Phi_{\alpha_k}^{i_k},$$

but as conjectured in [4] and later demonstrated in [21, 22, 25, 29], the A/2 model correlation functions and quantum sheaf cohomology ring relations are independent of nonlinear deformations, so we only consider linear deformations.

The left moving fermion is now a section of the vector bundle $\phi^*\mathcal{E}$, where $\mathcal{E}$ is a vector bundle on $G(k, n)$ defined by the short exact sequence

$$0 \to \mathcal{S} \otimes \mathcal{S}^* \xrightarrow{g} \mathcal{V} \otimes \mathcal{S}^* \to \mathcal{E} \to 0, \tag{3}$$

where g can be represented as

$$\omega_\alpha^\beta \mapsto A_j^i \omega_\alpha^\beta x_\beta^j + \omega_\beta^\beta B_j^i x_\alpha^j.$$

The dual of (3) is

$$0 \to \mathcal{E}^* \xrightarrow{i} \mathcal{V}^* \otimes \mathcal{S} \xrightarrow{f} \mathcal{S}^* \otimes \mathcal{S} \to 0, \tag{4}$$

where f can be represented as

$$t_i^\alpha \mapsto t_i^\alpha f_\beta^i = t_i^\alpha A_j^i x_\beta^j + \delta_\beta^\alpha t_i^\gamma B_j^i x_\gamma^j.$$

Our goal is to study the quantum sheaf cohomology ring

$$\bigoplus_{r \geqslant 0} H^r(G(k, n), \wedge^r \mathcal{E}^*).$$

The number of bundle moduli is equal to $h^1(X, \operatorname{End} TX)$. In the case at hand, $X = G(k, n)$, and $TX = \mathcal{S}^* \otimes \mathcal{Q}$, where $\mathcal{S}$ is the universal vector bundle and $\mathcal{Q}$ is the universal quotient bundle. Applying the Borel-Weil-Bott theorem, one can compute

$$h^1(G(k, n), \operatorname{End} TG(k, n)) = \begin{cases} n^2 - 1 & 1 < k < n - 1, \\ 0 & \text{else.} \end{cases}$$

In other words, projective spaces have no tangent bundle moduli, but other Grassmannians do. Let us see how this number emerges from our description of the deformation.

Our description above encodes moduli in the two $n \times n$ matrices A, B. The invertible matrix A can be transformed into the identity matrix using a $GL(n)$ field redefinition, so that in effect only one matrix (B, or rather BA^{-1}) encodes the moduli. However, the overall trace in B is trivial, and does not define any bundle deformations, which we can see as follows. Without loss of generality, take A to be the identity. Denote by i the imbedding of $\mathcal{S}$ in $\mathcal{V}$. Given a local section of $\mathcal{V}^* \otimes \mathcal{S} = \mathcal{H}om(\mathcal{V}, \mathcal{S})$, denoted by t, $f(t)$ can be written as $ti + \operatorname{Tr}(tBi)I_{k \times k}$, where $I_{k \times k}$ is the $k \times k$ identity matrix. If t is in the kernel of f and $B = \varepsilon I_{n \times n}$, then

$$ti + \varepsilon \operatorname{Tr}(ti) I_{k \times k} = 0. \tag{5}$$

Taking the trace, we get

$$(1 + \varepsilon k) \operatorname{Tr}(ti) = 0.$$

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For generic ε, this implies $\mathrm{Tr}(ti) = 0$, but then $ti = 0$ by (5). This means t is in the kernel of f_0 (f with $B = 0$). The converse is also true. We conclude that $\mathcal{E}^* \cong \Omega$, the holomorphic cotangent bundle, when $B = \varepsilon I_{n \times n}$. Thus, we see the number of nontrivial deformations is $n^2 - 1$, encoded in B (or BA^{-1} if A is nontrivial), modulo an overall trace.

Not all $n \times n$ matrices define a vector bundle through equation (3). In fact, in [35], we show that a B-deformation fails to give rise to a vector bundle on $G(k, n)$ if and only if there exist k eigenvalues of B (or BA^{-1}, if A is nontrivial) that sum to -1. Physically, if this condition is satisfied, then the GLSM develops a noncompact branch, independent of the value of the Fayet-Iliopoulos parameter. In any event, this criterion gives us the discriminant locus along which the $A/2$ correlation functions diverge.

2.2   One-loop effective potential

We will derive the quantum sheaf cohomology ring relations from the one-loop effective potential on the Coulomb branch, which we review in this section.

For the GLSM corresponding to $G(k, n)$, the gauge group is $U(k)$, which is generically[1] broken to $U(1)^k$ along the Coulomb branch. For σ the adjoint-valued field in the (2,2) vector multiplet, Take $\sigma_a, a = 1, \cdots, k$, to be the components of σ in the Cartan subalgebra. These will act as coordinates along the Coulomb branch. On this branch, the charge for Φ_a^i is δ_a^b under the b-th $U(1)$. Notice that all the Φ_a^i's with the same a have the same charges under all the $U(1)$'s. For fixed a, we can rewrite (2) as

$$\overline{\mathcal{D}}_+ \Lambda_a^i = E_j^i(\sigma_a)\Phi_a^j,$$

where the $n \times n$ matrix E_j^i is given by

$$E_j^i(\sigma_a) = \sigma_a A_j^i + \mathrm{Tr}(\sigma)B_j^i$$

for general A, or for A taken to be the identity,

$$E_j^i(\sigma_a) = \sigma_a \delta_j^i + \mathrm{Tr}(\sigma)B_j^i.$$

According to [1], the one-loop effective J function is

$$\tilde{J}_a = -\ln\left[-q^{-1}\det(E_a)\right]. \tag{6}$$

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[1] For our computations, we will be able to essentially ignore loci with enhanced gauge symmetry. For example, in supersymmetric localization computations, residues vanish along such loci, because of $e.g.$ factors of the form

$$\prod_{a \neq b}(\sigma_a - \sigma_b)$$

in the numerator of the integrand. As a result, such loci do not contribute to our computations, and will be ignored in this paper.

(Here a minus sign is inserted to comply with the convention in mathematical literature, this corresponds to an overall shift in the theta angle.) The equations of motion are $\tilde{J}_a = 0$ for each a, or more simply, for each a,

$$\det(E(\sigma_a)) = \det(\sigma_a A + \mathrm{Tr}(\sigma) B) = -q \tag{7}$$

for general matrices A.

2.3 Supersymmetric localization

We shall check the predictions for the quantum sheaf cohomology ring by computing correlation functions in examples, using supersymmetric localization. Now, it is not known how to apply supersymmetric localization to an untwisted (0,2) theory, but in this paper we are concerned with a twist of the (0,2) theory, known as the A/2 model. In a (0,2) nonlinear sigma model on a space X with bundle $\mathcal{E}$, we can understand the A/2 twist as follows. Before the twist, the right moving fermion ψ_+ is a section of $K^{1/2} \otimes \phi^* TX$, and the left moving fermion λ_- is a section of $\overline{K}^{1/2} \otimes \phi^* \mathcal{E}^*$, where K is the canonical line bundle of the worldsheet. In an A/2 twisted nonlinear sigma model [1], for example, we have

$$\psi_+^i \in \Gamma(\phi^* T^{1,0} X),$$
$$\psi_+^{\bar{\imath}} \in \Gamma(K \otimes \phi^* T^{0,1} X),$$
$$\lambda_-^a \in \Gamma(\overline{K} \otimes \phi^* \mathcal{E}^*),$$
$$\lambda_-^{\bar{a}} \in \Gamma(\phi^* \overline{\mathcal{E}}^*),$$

with the chiral ring being isomorphic to

$$\bigoplus_{r \geqslant 0} H^r(X, \wedge^r \mathcal{E}^*).$$

In the UV GLSM for the Grassmannian $G(k, n)$, the gauge-invariant chiral ring operators are of the form $\mathrm{Tr}\,\sigma^k$ for integers k and σ the bosonic field of the chiral multiplet in the adjoint representation. We will express these in terms of symmetric polynomials in commuting elements forming a basis along the Coulomb branch, denoted $\sigma_a = \sigma_1, \sigma_2, \cdots, \sigma_k$. A/2 correlation functions of symmetric polynomials in the σ_a then suffice to determine the quantum sheaf cohomology associated with the chiral ring. In terms of these commuting elements, the bosonic potential becomes of the form

$$\sum_{i,a} |A_j^i \sigma_a \phi_a^j \; + \; B_j^i (\mathrm{Tr}\,\sigma) \phi_a^j|^2$$
$$= \; \sum_{i,a} \overline{\phi}_a^j \phi_a^k \left(A_j^i \sigma_a \; + \; B_j^i (\mathrm{Tr}\,\sigma) \right)^* \left(A_k^i \sigma_a \; + \; B_k^i (\mathrm{Tr}\,\sigma) \right)$$
$$= \; \sum_{i,a} \overline{\phi}_a^j \phi_a^k (E_{j,a}^i)^* E_{k,a}^i$$

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where
$$E^i_{j,a} = A^i_j \sigma_a + B^i_j (\mathrm{Tr}\, \sigma)$$

The Yukawa couplings have the form
$$-\overline{\psi}^j_{-a} \psi^i_{+a} E^i_{j,a} + \mathrm{c.c.}.$$

These couplings – the bosonic potential and Yukawa couplings – define what amount to σ-dependent masses that play a crucial role in the one-loop partition function in supersymmetric localization.

Supersymmetric localization in the A/2 model for (0,2) theories given by deformations of (2,2) theories was recently discussed in [29]. From the results there,

$$Z^{1-\mathrm{loop}} = \prod_{a=1}^{k} \left(\frac{1}{\det \tilde{E}(\sigma_a)} \right)$$

where

$$\tilde{E}^i_j(\sigma_a) = A^i_j \sigma_a + B^i_j \left(\sum_b \sigma_b \right)$$

This implies that for any polynomial f in $\sigma_a, a = 1, \cdots, k$, the correlation functions off the (2,2) locus should have the form

$$\langle f(\sigma) \rangle =$$
$$\frac{1}{k!} \sum_{\mathfrak{m}_1, \cdots, \mathfrak{m}_k \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res} \left\{ (-1)^{(n-1) \sum \mathfrak{m}_i} q^{\sum \mathfrak{m}_i} \left(\prod_{a \neq b} (\sigma_a - \sigma_b) \right) \prod_{a=1}^{k} \left(\frac{1}{\det \tilde{E}(\sigma_a)} \right)^{\mathfrak{m}_i + 1} f(\sigma) \right\},$$
$$(8)$$

where 'JKG' denotes the Jeffrey-Kirwan-Grothendieck residues defined in [29].

In principle, given the $A/2$ correlation functions, the quantum sheaf cohomology ring is defined in the same way as the ordinary quantum cohomology. If we take a basis e_i for $\bigoplus_{r \geqslant 0} H^r(G(k,n), \wedge^r \mathcal{E}^*)$ as a vector space, and a dual basis $\hat{e}_i$ in the sense that

$$\langle e_i \hat{e}_j \rangle = \delta_{ij},$$

the generating relations read

$$\sigma = \sum_i \langle \sigma e_i \rangle \hat{e}_i$$

for any σ. More to the point, the quantum (sheaf) cohomology ring relations define identities in the correlation functions: if in the ring, some quantity R is set to zero, then any correlation function containing R should vanish. We will use localization to check the ring structure in examples in section 4.

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3 Ring structure of quantum sheaf cohomology

The quantum sheaf cohomology ring is the OPE ring of an A/2-twisted theory, just as the ordinary quantum cohomology ring is the OPE ring of an A-twisted theory – quantum sheaf cohomology is the (0,2) generalization of ordinary quantum cohomology. In this section we will describe it for Grassmannians with deformations of the tangent bundle, and give a physics-based derivation.

Also, so far we have given results for general deformation matrices A and B, but as previously observed, the matrix A is redundant. In the rest of this paper, we will assume without loss of generality that A is the identity. The general case can be reconstructed by replacing B (in results derived for $A = I$) with BA^{-1}.

3.1 Gauge-invariant operators

The Coulomb branch arguments given in the last section, both one-loop effective actions and supersymmetric localization, involve for a $U(k)$ gauge theory a set of k mutually commuting fields $\sigma_1, \cdots, \sigma_k$ which act as local coordinates on the Coulomb branch. However, these individually are not quite invariant under $U(k)$, as there is still a residual Weyl group action.

The complete group invariants are symmetric polynomials in $\sigma_1, \cdots, \sigma_k$, and these can be naturally associated to Young diagrams, via what are known as Schur polynomials (see [36][chapter 6] or [37][appendix B] for an introduction). For example, if $k = 2$, then

$$
\begin{aligned}
\sigma_{\square} &= \sigma_1 + \sigma_2, \\
\sigma_{\square\square} &= \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2, \\
\sigma_{\square\square\square} &= \sigma_1^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 + \sigma_2^3, \\
\sigma_{\substack{\square \\ \square}} &= \sigma_1\sigma_2, \\
\sigma_{\substack{\square\square \\ \square}} &= \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2, \\
\sigma_{\substack{\square\square \\ \square\square}} &= \sigma_1^2\sigma_2^2,
\end{aligned}
$$

and so forth. Each polynomial is homogeneous, of degree equal to the number of boxes in the Young diagram.

As we will see in detail in appendix A and [35], Young diagrams as above correspond mathematically to elements of sheaf cohomology groups

$$ H^\bullet \left(G(k,n), \wedge^\bullet \mathcal{E}^* \right), $$

for $\mathcal{E}$ the pertinent tangent bundle deformation, which arise in nonlinear-sigma-model-based analyses. For example, there is a well-known correspondence between generators of cohomology of the Grassmannian $G(k,n)$ of fixed degree, and Young diagrams that fit inside

a $k \times (n-k)$ box. (Young diagrams that extend outside of that box would correspond mathematically to cohomology classes of too-high degree, which classically vanish.)

Now, for the purposes of describing the ring, including all the Young diagrams is redundant, as there are relations between their products. For example, in the $k = 2$ case above,
$$\sigma_\square^2 \;=\; \sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2 \;=\; \sigma_{\square\square} + \sigma_{\substack{\square\\\square}},$$
and so $\sigma_{\substack{\square\\\square}}$ is determined algebraically by $\sigma_\square^2$ and $\sigma_{\square\square}$. More generally, the symmetric polynomials corresponding to any Young diagram that extends past the first row can be expressed algebraically in terms of Young diagrams that run along the first row only. This is known as the Giambelli formula (see $e.g.$ [36][section 9.4]), which for a Young diagram λ, reads
$$\sigma_\lambda \;=\; \det\left(\sigma_{(\lambda_i + j - i)}\right)_{1 \leq i,j \leq r}$$
for r the number of boxes in λ, λ_i the number of boxes in the ith row, and $\sigma_{(n)}$ corresponding to a Young diagram with one horizontal row of n boxes, $e.g.$
$$\sigma_{(1)} \;=\; \sigma_\square, \quad \sigma_{(2)} \;=\; \sigma_{\square\square}, \quad \sigma_{(3)} \;=\; \sigma_{\square\square\square},$$
and so forth, in conventions in which $\sigma_{(m)} = 0$ for $m < 0$, and is 1 if $m = 0$. For example, the Giambelli formula says
$$\sigma_{\substack{\square\\\square}} \;=\; \det \begin{bmatrix} \sigma_\square & \sigma_{\square\square} \\ 1 & \sigma_\square \end{bmatrix} \;=\; \sigma_\square^2 - \sigma_{\square\square},$$
which we verified explicitly above. For another example,
$$\sigma_{\substack{\square\square\\\square}} \;=\; \det \begin{bmatrix} \sigma_{\square\square} & \sigma_{\square\square\square} & \sigma_{\square\square\square\square} \\ 1 & \sigma_\square & \sigma_{\square\square} \\ 0 & 0 & 1 \end{bmatrix} \;=\; \sigma_\square \sigma_{\square\square} - \sigma_{\square\square\square},$$
which is easily checked.

Altogether, the classical cohomology ring of the Grassmannian $G(k,n)$ can be expressed in terms of generators corresponding to Young diagrams with only a single horizontal row, as [38–43]
$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, \cdots, D_n \right\rangle, \tag{9}$$
where
$$D_m \;=\; \det\left(\sigma_{(1+j-i)}\right)_{1 \leq i,j \leq m},$$
in conventions in which $\sigma_{(m)} = 0$ if $m < 0$ or $m > n - k$, as each D_m should only be constructed from the available generators.

It should be mentioned that the classical cohomology ring can also be written in the presentation
$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(k)}\right] / \left\langle D_{n-k+1}, \cdots, D_n \right\rangle.$$

These two presentations define equivalent rings. When describing the ordinary cohomology ring of the Grassmannian, it is often convenient to think of the generators as Chern classes: Chern classes of the universal quotient bundle in (9, and Chern classes of the universal subbundle above (see *e.g.* [42]). For the ordinary cohomology ring, they can be related by transposing Young diagrams, though that description is not symmetric in the quantum case. In any event, in this paper we will primarily refer back to presentation (9).

Now, we are interested in computing both quantum and (0,2) modifications to the classical Grassmannian cohomology ring structure, so *a priori*, it might happen that Young diagrams extending past the first row are needed. Nevertheless, it turns out that they are not, the quantum sheaf cohomology ring can be determined solely by Young diagrams along the first row only.

As a special case, the standard result for the ordinary quantum cohomology ring of $G(k,n)$ is (*e.g.* [38–43])

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, \cdots, D_{n-1}, D_n - (-)^{n-k-1}q\right\rangle,$$

or, in terms of the other presentation of the classical cohomology ring,

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(k)}\right] / \left\langle D_{n-k+1}, \cdots, D_n - (-)^{k-1}q\right\rangle.$$

3.2 Quantum sheaf cohomology ring

We will see that the quantum sheaf cohomology ring (the OPE ring of the A/2 twist) of a (0,2) deformation of the Grassmannian $G(k,n)$ is given *generically* by

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \Big\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, \cdots, R_{(n-1)}, \\ R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, R_{(n+2)} + q\sigma_{(2)}, \cdots \Big\rangle, \tag{10}$$

specializing for $k = 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_{k+1}, D_{k+2}, \cdots, R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, \cdots\right\rangle,$$

and for $k = n - 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_n, D_{n+1}, \cdots, R_{(2)}, \cdots, R_{(n-1)}, R_{(n)} + q, R_{(n+1)} + q\sigma_{(1)}, \cdots\right\rangle,$$

where

$$D_m = \det\left(\sigma_{(1+j-i)}\right)_{1 \le i,j \le m},$$

$$R_{(r)} = \sum_{i=0}^{\min(r,n)} I_i \sigma_{(r-i)} \sigma_{(1)}^i,$$

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for I_i the coefficients of the characteristic polynomial of B, given by

$$\det(tI + B) = \sum_{i=0}^{n} I_{n-i} t^i$$

For example, $I_0 = 1$, independent of B, but the other I_i depend upon B. In particular,

$$I_1 = \operatorname{tr} B, \quad I_n = \det B.$$

In passing, it will sometimes be helpful to define a generalization of $R_{(r)}$. For a Young diagram μ, we define R_μ to be

$$R_\mu = \det \begin{bmatrix} R_{(\mu_1)} & R_{(\mu_1+1)} & R_{(\mu_1+2)} & \cdots & R_{(\mu_1+k-1)} \\ \sigma_{(\mu_2-1)} & \sigma_{(\mu_2)} & \sigma_{(\mu_2+1)} & \cdots & \sigma_{(\mu_2+k-2)} \\ \sigma_{(\mu_3-2)} & \sigma_{(\mu_3-1)} & \sigma_{(\mu_3)} & \cdots & \sigma_{(\mu_3+k-3)} \\ \vdots & & & & \ddots \\ \sigma_{(\mu_k-k+1)} & \sigma_{(\mu_k-k+2)} & \sigma_{(\mu_k-k+3)} & \cdots & \sigma_{(\mu_k)} \end{bmatrix}.$$

In the special case that the Young diagram μ consists of a single horizontal row of r boxes, which we would label (r), note that

$$R_\mu = R_{(r)},$$

and it is in this sense that R_μ generalizes $R_{(r)}$.

The description of the ring above holds generically in the space of tangent bundle deformations, but does break down along certain loci. Specifically, the description of the classical sheaf cohomology ring, described by the limit $q \to 0$, breaks down along

$$X \cup V_{n-k+1} \cup V_{n-k+2} \cup \cdots,$$

where X is the discriminant locus of the tangent bundle deformation (meaning, the locus where the bundle degenerates), and V_m is a locus defined by R_m, as follows. First, for every Young diagram μ of size $|\mu| = m$, such that μ_1, the number of boxes in the first row, is greater than $n - k$, and no column has more than k boxes, expand the determinant below in a sum of Schur polynomials for Young diagrams of the same size:

$$R_\mu = \sum_\nu C_{\mu\nu}^m \sigma_\nu,$$

where $|\nu| = m = |\mu|$. In this fashion, we define a matrix $(C_{\mu\nu}^m)$. Then, we define V_m to be the locus where the rank of the (not necessarily square) matrix $(C_{\mu\nu}^m)$ drops.

Along the V_m for

$$n - k + 1 \leq m \leq k(n - k),$$

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the V_m define loci where the dimensions of the sheaf cohomology groups may jump. For $m > k(n - k)$, the V_m merely define loci where the presentation breaks down, where the given relations may not suffice, but the dimensions of the sheaf cohomology groups do not jump.

A derivation of this locus is outlined in appendix A. It is useful to note that the locus above is codimension at least one, and so the presentation of the classical sheaf cohomology ring is pertinent for generic tangent bundle deformations.

It is also important to notice that the locus above does not intersect the (2,2) locus. On the (2,2) locus, where $B = 0$ and $R_{(n)} = \sigma_{(n)}$, from the Giambelli formula $R_\mu = \sigma_\mu$, and so we see that along the (2,2) locus, $C_{\mu\nu}^m = \delta_{\mu\nu}$, whose rank does not drop, and so V_m is the empty set. Thus, V_m never intersects the (2,2) locus, and neither does the discriminant.

In passing, note that the result above is consistent with claims of [7] that in a sufficiently small neighborhood of the (2,2) locus, the OPE's can be consistently defined within the topological subsector.

We conjecture that for $m > k(n - k)$, the loci V_m are all identical to one another and to the discriminant locus, so that the total number of components of the locus where the quantum sheaf cohomology ring relations break down in some fashion is finite. We will see this in examples later, though we do not yet have a general proof for all cases.

So far we have described the loci along which the presentation of the classical sheaf cohomology ring degenerates. The degeneration loci of the presentation of the quantum sheaf cohomology ring are not completely understood by us at present, though we conjecture that the same loci V_m are involved, as we shall see in examples later.

In the remainder of this section, we will check that the ansatz above correctly specializes to the ordinary classical and quantum cohomology rings. We will derive the quantum sheaf cohomology ring above from the one-loop effective action later in section 3.3.

3.2.1 Specialization to ordinary classical cohomology

First, as one extreme, let us reduce to the classical cohomology ring of $G(k, n)$. Here, $B = 0$ and $q = 0$. In this case,

$$R_{(r)} = \sigma_{(r)},$$

and the quantum sheaf cohomology ring above becomes

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_{k+1}, D_{k+2}, \cdots, \sigma_{(n-k+1)}, \sigma_{(n-k+2)}, \cdots\right\rangle,$$

or more simply,

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, D_{k+2}, \cdots\right\rangle.$$

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This is almost identical to the presentation of the ordinary cohomology ring of the Grassmannian given in equation (9), except that here the relations involve all D's of degree greater than n, rather than going only to D_n. However, we can establish that the two sets of relations are equivalent, as follows.

We will show that D_{n+1} and all higher relations are linear combinations of the relations $\{D_{k+1}, \cdots, D_n\}$, so that the ring presented above is equivalent to (9). To do this, we expand down the first column of the determinant in the Giambelli formula to derive the recursion relation

$$\sigma_{(\ell,1,\cdots,1)} = \sigma_{(\ell)} D_m - \sigma_{(\ell+1,1,\cdots,1)}$$

where $\sigma_{(\ell,1,\cdots,1)}$ denotes the Schur polynomial associated to a Young tableau with ℓ boxes in the first row and 1 box in the next m rows, and $\sigma_{(\ell+1,1,\cdots,1)}$ denotes a similar Schur polynomial, albeit associated to a Young diagram with $m-1$ rows with one box. Applying this recursively, one can quickly show

$$D_{n+1} = \sigma_{(1)} D_n - \sigma_{(2)} D_{n-1} + \cdots + (-)^{n-k+1} \sigma_{(n-k)} D_{k+1} + (-)^{n-k} \sigma_{(n-k+1,1,\cdots,1)}. \quad (11)$$

However, from the Giambelli formula, $\sigma_{(n-k+1,1,\cdots,1)}$ is given by a determinant whose first row vanishes (since it involves σ's all of which are outside the range of the generators), hence $\sigma_{(n-k+1,1,\cdots,1)} = 0$. Thus, we see that D_{n+1} is a linear combination of the relations $\{D_{k+1}, \cdots, D_n\}$, and one can similarly demonstrate the same result for all D_m for $m > n$. In this fashion, we see that the ring above is isomorphic to the ordinary cohomology ring of the Grassmannian given in equation (9).

3.2.2 Specialization to ordinary quantum cohomology

Next, let us verify that the quantum sheaf cohomology ring (10) reduces to the ordinary quantum cohomology ring of the Grassmannian $G(k,n)$ along the (2,2) locus. This is the case $B = 0$, but $q \neq 0$. As before,

$$R_{(r)} = \sigma_{(r)},$$

and so the quantum sheaf cohomology ring above becomes

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_{k+1}, D_{k+2}, \cdots, \sigma_{(n-k+1)}, \cdots, \sigma_{(n-1)}, \right. \\ \left. \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \right\rangle, \quad (12)$$

specializing for $k = 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_2, D_3, \cdots, \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \right\rangle,$$

and for $k = n - 1$ to

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_n, D_{n+1}, \cdots, \sigma_{(2)}, \cdots, \sigma_{(n-1)}, \sigma_{(n)} + q, \sigma_{(n+1)} + q\sigma_{(1)}, \cdots \right\rangle.$$

The expression above for the quantum cohomology ring is not yet in a standard form, and can be simplified to such a form. First, we show that, for $\ell \geqslant 1$,

$$D_{n+\ell} = 0, \tag{13}$$

in the sense that it is redundant, defining no new relations, and hence can be removed from the presentation above. This can be proved by induction on ℓ. First, we will need a small identity. By expanding the determinant in the definition of D_m across the first row, (and then expanding the determinant of each submatrix along the first column), we find

$$D_m = \sigma_{(1)}D_{m-1} - \sigma_{(2)}D_{m-2} + \cdots + (-)^m \sigma_{(m-1)}D_1 + (-)^{m+1}\sigma_{(m)}.$$

Now, we proceed with the induction. For $\ell = 1$, we have

$$
\begin{aligned}
D_{n+1} &= \sigma_{(1)}D_n + \cdots + (-)^{n-k+1}\sigma_{(n-k)}D_{k+1} + (-)^{n-k+2}\sigma_{(n-k+1)}D_k \\
&\quad + \cdots + (-)^n \sigma_{(n-1)}D_2 + (-)^{n+1}\sigma_{(n)}D_1 + (-)^{n+2}\sigma_{(n+1)}, \\
&= (-)^{n+1}(\sigma_{(n)}D_1 - \sigma_{(n+1)}), \\
&= (-)^n q(D_1 - \sigma_{(1)}) = 0,
\end{aligned}
$$

where we have used the ring relations

$$D_{k+1} = D_{k+2} = \cdots = 0, \quad \sigma_{(n-k+1)} = \cdots = \sigma_{(n-1)} = 0.$$

Next, assume that (13) is true for all $\ell \leqslant m$. When $m < k$, we have

$$
\begin{aligned}
D_{n+m+1} &= \sigma_{(1)}D_{n+m} - \sigma_{(2)}D_{n+m-1} + \cdots + (-)^{n+m-k+1}\sigma_{(n+m-k)}D_{k+1} + \\
&\quad (-)^{n+m-k+2}\sigma_{(n+m-k+1)}D_k + \cdots + (-)^{n+1}\sigma_{(n)}D_{m+1} + \cdots + (-)^{n+m+2}\sigma_{(n+m+1)}, \\
&= (-)^{n+1}\sigma_{(n)}D_{m+1} + \cdots + (-)^{n+m+2}\sigma_{(n+m+1)}, \\
&= (-)^n q\left(D_{m+1} - \sigma_{(1)}D_m + \cdots + (-)^{m+1}\sigma_{(m+1)}\right) = 0.
\end{aligned}
$$

When $m \geqslant k$, we have

$$
\begin{aligned}
D_{n+m+1} &= \sigma_{(1)}D_{n+m} - \sigma_{(2)}D_{n+m-1} + \cdots + (-)^n \sigma_{(n-1)}D_{m+2} \\
&\quad + (-)^{n+1}\sigma_{(n)}D_{m+1} + \cdots + (-)^{n+m+2}\sigma_{(n+m+1)}, \\
&= (-)^{n+1}\sigma_{(n)}D_{m+1} + \cdots + (-)^{n+m+2}\sigma_{(n+m+1)}, \\
&= (-)^n q(D_{m+1} - \sigma_{(1)}D_m + \cdots + (-)^{m+1}\sigma_{(m+1)}) = 0.
\end{aligned}
$$

Thus, we have shown (13).

Next, using the relations

$$\sigma_{(n+\ell)} = -q\sigma_{(\ell)},$$

we can express the $\sigma_{(i)}$'s with $i > n - k$ as polynomials of q and $\sigma_{(i)}$'s with $i \leqslant n - k$, and so we can rewrite the ring in terms of generators

$$\sigma_{(1)}, \cdots, \sigma_{(n-k)}.$$

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Finally, we derive an expression for D_n. Starting with

$$\begin{aligned} D_n &= \sigma_{(1)}D_{n-1} - \sigma_{(2)}D_{n-2} + \cdots + (-)^{n-k}\sigma_{(n-k-1)}D_{k+1} + (-)^{n-k-1}\sigma_{(n-k)}D_k \\ &\quad (-)^{n-k+2}\sigma_{(n-k+1)}D_{k-1} + \cdots + (-)^n\sigma_{(n-1)}D_1 + (-)^{n+1}\sigma_{(n)}, \end{aligned}$$

we use the ring relations

$$D_{k+1} = \cdots = D_{k(n-k)} = 0, \quad \sigma_{(n-k+1)} = \cdots = \sigma_{(n-1)} = 0$$

and the fact that

$$n - 1 \leq k(n-k)$$

to simplify D_n to

$$D_n = (-)^{n-k-1}\sigma_{(n-k)}D_k + (-)^{n+1}\sigma_{(n)},$$

which using further ring relations can be written as

$$D_n = (-)^{n-k-1}\sigma_{(n-k)}D_k + (-)^n q.$$

Finally, let us simplify the ring presentation (12). We have shown that inside that quotient ring, D_i is redundant for $i > n$, and given our expression for D_n above, it is straightforward to see that the ring (12) can be reduced to

$$\mathbb{C}\left[\sigma_{(1)}, \cdots, \sigma_{(n-k)}\right] / \left\langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n q\right\rangle, \tag{14}$$

where the "D_n" above is the 'classical' D_n, namely

$$(-)^{n-k-1}\sigma_{(n-k)}D_k$$

in the notation of (12). This new presentation is a standard representation of the quantum cohomology ring of $G(k,n)$ (see $e.g.$ [38–43]).

3.3   Derivation from one-loop effective action

In this section we will describe how the quantum sheaf cohomology ring relations can be computed from the one-loop effective action, and in the next section we will check our results against A/2 correlation functions computed via supersymmetric localization. (The classical sheaf cohomology ring relations will be derived mathematically in the companion paper [35]; a purely mathematical derivation of the quantum sheaf cohomology ring relations here is left for future work.)

Before computing quantum sheaf cohomology for general bundle deformations, we shall begin by deriving the ordinary quantum cohomology, along the (2,2) locus, from the one-loop effective action, as a warm-up exercise.

As before, let us take the diagonal elements of the σ field to be $\sigma_i, i = 1, \cdots, k$. On the (2,2) locus, where $B = 0$, we see from the one-loop effective potential (6) that the σ_i obey equation (7), or more simply

$$\sigma_i^n = -q, \quad i = 1, \cdots, k.$$

Because these equations are of order n, all relations with order lower than n are not affected, i.e.

$$\sigma_{(i)} = 0, \quad i = n - k + 1, \cdots, n - 1.$$

Then, for example, from

$$\sigma_{(n-1)}\sigma_1 = \sum_{\substack{\alpha_1 + \cdots + \alpha_k = n \\ \alpha_1 \neq 0}} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k} = 0,$$

we get

$$\sigma_{(n)} = \sum_{\alpha_1 + \cdots + \alpha_k = n} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k} = \sum_{\alpha_2 + \cdots + \alpha_k = n} \sigma_2^{\alpha_2} \cdots \sigma_k^{\alpha_k}.$$

Similarly, from $\sigma_{(n-2)}\sigma_1 = 0$, we have

$$\sigma_{(n-1)} = \sum_{\alpha_2 + \cdots + \alpha_k = n-1} \sigma_2^{\alpha_2} \cdots \sigma_k^{\alpha_k}.$$

Then $\sigma_{(n-1)}\sigma_2 = 0$ shows that

$$\sigma_{(n)} = \sum_{\alpha_3 + \cdots + \alpha_k = n} \sigma_3^{\alpha_3} \cdots \sigma_k^{\alpha_k}.$$

This procedure stops in $k - 1$ steps, and we get our first quantum corrected relation

$$\sigma_{(n)} = \sigma_k^n = -q. \tag{15}$$

To derive the equation above, we arbitrarily made use of $\sigma_1, \cdots, \sigma_{k-1}$; by picking a different set of $k - 1$ σ_i's, we arrive at equation (15) for each value of k.

We can use the same method to deduce the higher order relations, for example

$$\sigma_{(n)}\sigma_1 = -q\sigma_1 = \sum_{\substack{\alpha_1 + \cdots + \alpha_k = n+1 \\ \alpha_1 \neq 0}} \sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k},$$

which implies

$$\sigma_{(n+1)} = \left(\sum_{\alpha_2 + \cdots + \alpha_k = n+1} \sigma_2^{\alpha_2} \cdots \sigma_k^{\alpha_k} \right) - q\sigma_1.$$

Repeatedly using lower order relations leads to

$$\sigma_{(n+1)} = -q\sigma_1 - q\sigma_2 - \cdots - q\sigma_k = -q\sigma_{(1)}.$$

Similarly, one can compute

$$\sigma_{(n+l)} = -q\sigma_{(l)}, \ l \geqslant 1.$$

In this fashion, we find that the one-loop effective action implies a quantum cohomology ring of the form (12), which we show in section 3.2.2 matches standard presentations of the ordinary quantum cohomology ring.

So far we have shown how one-loop effective action arguments can be used to derive the ordinary quantum cohomology ring along the (2,2) locus. Next we shall leave the (2,2) locus and consider more general (0,2) cases by turning on a nonzero B deformation.

First, we note that the relations

$$D_{k+1} \ = \ D_{k+2} \ = \ \cdots = 0$$

are trivially satisfied for all $\sigma_{(k)}$ constructed as Schur polynomials in k variables $\sigma_1, \cdots, \sigma_k$. It remains to derive the relations

$$R_{(n-k+1)} \ = \ \cdots \ = \ R_{(n-1)} \ = \ 0, \ \ R_{(n)} \ = \ -q, \ \ R_{(n+1)} \ = \ -q\sigma_{(1)}, \ \ \cdots$$

For any $n \times n$ matrix B, the quantum corrected relations are encoded in

$$\det(E(\sigma_\alpha)) = \det(\sigma_\alpha I + B\sigma_{(1)}) = -q \tag{16}$$

due to the one-loop effective potential (6). Note that, by definition of I_i we have

$$\det(E(\sigma_a)) = \sum_{i=0}^{n} I_i \sigma_{(1)}^i \sigma_a^{n-i}. \tag{17}$$

Again, the relations with dimension smaller than n do not receive quantum corrections, *i.e.* the relations

$$R_{(n-k+1)} = R_{(n-k+2)} = \cdots = R_{(n-1)} = 0 \tag{18}$$

still hold in the quantum case. Now let's compute the relation at order n. First, note

$$R_{(n-1)}\sigma_1 = \sum_{i=0}^{n-1} I_i \sigma_{(n-i-1)}\sigma_{(1)}^i \sigma_1 = \sum_{i=0}^{n-1} I_i \left(\sigma_{(n-i)} - \sum_{\substack{|\alpha|=n-i \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i = 0,$$

where $\sigma^{[\alpha]}$ denotes $\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\cdots\sigma_k^{\alpha_k}$, α is the corresponding multi-index, and we have used the relation $R_{(n-1)} = 0$. This implies that

$$R_{(n)} = \sum_{i=0}^{n} I_i \sigma_{(n-i)}\sigma_{(1)}^i = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n-i \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i.$$

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Similarly, $R_{(n-2)}\sigma_1 = 0$ implies

$$R_{(n-1)} = \sum_{i=0}^{n-1} I_i \left(\sum_{\substack{|\alpha|=n-i-1 \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i,$$

and $R_{(n-1)}\sigma_2 = 0$ leads to

$$R_{(n)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n-i \\ \alpha_1=\alpha_2=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i.$$

Because we have $k-1$ relations in (18), induction shows that this procedure allows us to eliminate σ_1 through σ_{k-1} in the expression of R_n, $i.e.$

$$R_{(n)} = \sum_{i=0}^{n} I_i \sigma_k^{n-i} \sigma_{(1)}^i,$$

which is equal to $-q$ due to (16) and (17). Thus the relation at order n is

$$R_{(n)} + q = 0. \tag{19}$$

We can follow the same procedure to compute the quantum correction to $R_{(n+1)}$. Since now there are k relations at hand including (19), all the σ_i dependence can be eliminated except those proportional to q. One can compute

$$R_{(n+1)} + q\sigma_{(1)} = 0. \tag{20}$$

In general, we can show

$$R_{(n+\ell)} = -q\sigma_{(\ell-1)}P_1 + q\sigma_{(\ell-2)}P_2 - q\sigma_{(\ell-3)}P_3 + \cdots + (-)^k q\sigma_{(\ell-k)}P_k, \tag{21}$$

where $\ell > 0$ and P_i is the elementary symmetric polynomial of order i in $\sigma_1, \cdots, \sigma_k$. Again, we define $\sigma_{(s)} = 0$ when $s < 0$ in the above formula. From the fact that

$$\prod_{i=1}^{k}(1+\sigma_i t)^{-1} = \sum_{j=0}^{\infty}(-)^j \sigma_{(j)} t^j$$

$$\prod_{i=1}^{k}(1+\sigma_i t) = \sum_{j=0}^{k} P_j t^j,$$

we have

$$\sigma_{(\ell)} - \sigma_{(\ell-1)}P_1 + \sigma_{(\ell-2)}P_2 + \cdots + (-)^k \sigma_{(\ell-k)}P_k = 0,$$

which implies

$$R_{(n+\ell)} + q\sigma_{(\ell)} = 0, \quad \ell \geqslant 0. \tag{22}$$

Actually, (22), or equivalently (21), can be proved by induction with the same method as for $R_{(n)}$ and $R_{(n+1)}$. Indeed, if we assume (22) is true for l replaced with any positive integer smaller than l, we get

$$R_{(n+\ell-1)}\sigma_1 = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-i \\ \alpha_1 \neq 0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i,$$

must match

$$-q\sigma_{(\ell-1)}\sigma_1,$$

(by the inductive assumption) and hence, if we define $E_{s,t}$ to be the elementary polynomial of order t in $\sigma_1, \cdots, \sigma_s$, for $0 \leqslant s \leqslant k$ and $t \leqslant s$, then

$$R_{(n+\ell)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-i \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i - q\sigma_{(\ell-1)}E_{1,1}.$$

Let's suppose that

$$R_{(n+\ell-s)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-s-i \\ \alpha_1=\cdots=\alpha_{u-s}=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i - q\sigma_{(\ell-s-1)}E_{u-s,1} + q\sigma_{(\ell-s-2)}E_{u-s,2} +$$
$$\cdots + (-)^{u-s}q\sigma_{(\ell-t)}E_{u-s,u-s} \tag{23}$$

for any $u \leqslant t < k$ and $0 \leqslant s \leqslant u$ (we have seen this is true for $t = 1$). Starting with

$$R_{(n+\ell-t)} = \sum_{i=0}^{n} I_i \left(\sum_{\substack{|\alpha|=n+\ell-t-i \\ \alpha_1=0}} \sigma^{[\alpha]} \right) \sigma_{(1)}^i - q\sigma_{(\ell-t-1)}\sigma_1,$$

which is obtained from $R_{(n+\ell-t-1)}\sigma_1 = -q\sigma_{(\ell-t-1)}\sigma_1$, induction on s shows (23) is valid for $u \leqslant t+1$ and $s \leqslant u$. Thus we can take $u = k$ and $s = 0$ in (23) to get

$$R_{(n+\ell)} = -q\sigma_{(\ell-1)}E_{k,1} + q\sigma_{(\ell-2)}E_{k,2} + \cdots + (-)^k q\sigma_{(\ell-k)}E_{k,k},$$

which is exactly (21), hence proving (22).

4   Examples

In this section, we will perform consistency tests on the quantum sheaf cohomolgy ring (10) by using supersymmetric localization to compute A/2 correlation functions in examples, and check that the predictions of the quantum sheaf cohomology ring are consistent with those correlation functions.

In each example, we will begin by describing correlation functions and quantum cohomology along the (2,2) locus, and will generalize to (0,2). Furthermore, in all our (0,2) examples, we will take B to be diagonal:
$$B = \text{diag}(b_1, \cdots, b_n)$$
on $G(k, n)$. The methods of this paper apply to general B; however, the resulting formulas for general B are rather complex, and it suffices to consider the special case of B diagonal for the purposes of illustrative examples.

We will begin by looking at examples of projective spaces as special cases of the construction described here, and then will turn to Grassmannians which are not projective spaces.

4.1 $G(1,3)$

The Grassmannian $G(1,3)$ is the projective space $\mathbb{P}^2$, so any tangent bundle deformation is trivial – the tangent bundle is rigid. Nevertheless, this example and $G(2,3)$ will serve as simple prototypes for later results.

Let us begin by computing correlation functions on the (2,2) locus. Since $G(1,3)$ is described by a $U(1)$ gauge theory, there is only a single σ field. Here, the localization formula (8) implies that classical (two-point) correlation functions are given by

$$\langle f(\sigma) \rangle = \text{JKG} - \text{Res}\left\{ \frac{1}{\sigma_1^3} f(\sigma) \right\},$$

which trivially reduces to the ordinary one-dimensional residue. The only nonvanishing classical correlation function is given by

$$\langle \sigma^2 \rangle = 1 = \langle \sigma_{\square\square} \rangle$$

Similarly, the one-instanton contributions to correlation functions are given by

$$\langle f(\sigma) \rangle = \text{JKG} - \text{Res}\left\{ \frac{q}{\sigma_1^6} f(\sigma) \right\},$$

which is again an ordinary contour integral about $\sigma = 0$. Clearly the only nonvanishing correlation function in the one-instanton sector is

$$\langle \sigma^5 \rangle = q,$$

and so we read off the quantum cohomology relation

$$\sigma^3 = \sigma_{\square\square\square} = q.$$

This reproduces the quantum cohomology ring of $\mathbb{P}^2$, given by

$$\mathbb{C}[x]/(x^3 - q).$$

After the (trivial) (0,2) deformation, classical correlation functions take the form

$$\langle f(\sigma) \rangle = \text{JKG} - \text{Res} \left\{ \left(\frac{1}{\det \tilde{E}(\sigma)} \right) f(\sigma) \right\},$$

where the JKG residue is trivially an ordinary residue, and

$$\tilde{E}(\sigma) = (I + B)(\sigma)$$

for the case we shall consider, hence

$$\det \tilde{E}(\sigma) = \sigma^3 \det(I + B).$$

The only nonzero classical correlation function is

$$\langle \sigma^2 \rangle = \langle \sigma_{\square\square} \rangle = \frac{1}{\det(I + B)}.$$

Similarly, in the one-instanton sector,

$$\langle f(\sigma) \rangle = \text{JKG} - \text{Res} \left\{ q \left(\frac{1}{\det \tilde{E}(\sigma)} \right)^2 f(\sigma) \right\},$$

where again the JKG residue is an ordinary residue at $\sigma = 0$, and the only nontrivial correlation function is

$$\langle \sigma^5 \rangle = \frac{1}{(\det(I + B))^2}.$$

From the structure of these correlation functions, as well as the one-loop effective action, one can read off that the quantum ring relation is given by

$$\det \tilde{E}(\sigma) = q$$

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or equivalently

$$\sigma^3 = \sigma_{\square\square\square}\det(I + B) = q.$$

By a simple redefinition of q, the resulting ring is identical to that on the (2,2) locus, as expected for a trivial bundle deformation.

Now, let us compare to the prediction of the quantum sheaf cohomology ring (10). In this case, the ring should be given by

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_2, \cdots, R_{(3)} + q, R_{(4)} + q\sigma_{(1)}, \cdots \right\rangle = \mathbb{C}\left[\sigma_{(1)}\right] / \left\langle R_{(3)} + q \right\rangle.$$

In writing the above, we have used the fact that

$$D_2 = \sigma_{(1)}^2 - \sigma_{(2)},$$

hence $\sigma_{(2)}$ (and, similarly, higher $\sigma_{(i)}$) are redundant, and also the consequence

$$R_{(3+\ell)} + q\sigma_{(\ell)} = \sigma_{(\ell)}\left(R_{(3)} + q\right),$$

which makes the higher $R_{(n)}$ relations redundant. Finally, note that

$$
\begin{aligned}
R_{(3)} &= \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma^i, \\
&= \left(\sum_{i=0}^{3} I_i\right) \sigma^3, \\
&= (\det(I + B)) \sigma^3,
\end{aligned}
$$

Clearly, this ring precisely matches the relation derived above.

Let us conclude by computing the locus on which the sheaf cohomology jumps, defined by

$$V_3 \cup V_4 \cup V_5 \cup \cdots$$

as described in section 3.2. Since $G(1,3)$ is a projective space, and projective spaces admit no tangent bundle deformations, we should recover no more than the discriminant locus, but this will be both a good consistency test as well as an explicit demonstration of the V's.

We begin by computing $V_{2+\ell}$. There is only one Young diagram with $2+\ell$ boxes, at least two along the first row, and none in later rows, namely $(2+\ell)$, and trivially

$$R_{(2+\ell)} = (1 + I_1 + I_2 + I_3)\sigma_{(2+\ell)} = \sigma_{(2+\ell)} \det(1 + B).$$

Thus, for any $V_{2+\ell}$, the matrix $C_{\mu\nu}$ is a 1×1 matrix, with single component

$$\det(1 + B),$$

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and so we see that for all $\ell \geq 1$, the locus $V_{2+\ell}$ coincides with the discriminant locus. Thus, there are no new components, as expected from the fact that (for B not in the discriminant locus) the tangent bundle deformations are all trivial.

It is easy to see that a nearly identical argument holds for $G(1,n)$, that all of the loci $V_{n-k+\ell}$ for such Grassmannians are copies of the discriminant locus.

4.2 $G(2,3)$

Before moving on to Grassmannians with nontrivial bundle deformations, let us look at a different presentation of $\mathbb{P}^2$, as a $U(2)$ gauge theory rather than a $U(1)$ gauge theory.

As before, let us first examine the (2,2) locus. In this theory, the classical (two-point) functions are given by

$$\langle f(\sigma) \rangle \;=\; \frac{1}{2!} \mathrm{JKG} - \mathrm{Res}\left\{ (-)(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^3} \frac{1}{\sigma_2^3} f(\sigma) \right\},$$

or explicitly,

$$\langle \sigma_1^2 \rangle \;=\; -\frac{1}{2!},$$
$$\langle \sigma_1 \sigma_2 \rangle \;=\; +\frac{2}{2!},$$
$$\langle \sigma_2^2 \rangle \;=\; -\frac{1}{2!},$$

using the fact that the JKG residue in this case is just iterated ordinary residues at $\sigma_2 = 0$ and $\sigma_1 = 0$. We interpret the σ_i's on the (2,2) locus as Chern roots of the universal subbundle S. In that language, the degree two cohomology is generated by

$$\sigma_\square \;=\; \sigma_1 + \sigma_2,$$

corresponding to a (1,1) form on $\mathbb{P}^2$, and the degree four cohomology is generated by

$$\sigma_{\Box\!\Box} \;=\; \sigma_1 \sigma_2,$$

corresponding to a (2,2) form on $\mathbb{P}^2$. As a consistency check, note that

$$\langle \sigma_{\Box\!\Box} \rangle \;=\; \langle \sigma_1 \sigma_2 \rangle \;\neq\; 0,$$

as expected since $\sigma_{\Box\!\Box}$ should correspond to a top-form.

Furthermore, as all the contributing Schubert cells are associated with subdiagrams of the $k \times (n-k)$ box

$$\boxed{}\!\!\vphantom{x}$$

there is a relation

$$\sigma_{\square\square} = 0$$

which we can check explicitly. In terms of Schur polynomials,

$$\sigma_{\square\square} = \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2$$

and it is immediate that

$$\langle \sigma_1^2 \rangle + \langle \sigma_1\sigma_2 \rangle + \langle \sigma_2^2 \rangle = 0$$

Note in passing that with the relation $\sigma_{\square\square} = 0$, we have that $\sigma_\square^2 = \sigma_{\square\square}$, as expected for the cohomology ring of $\mathbb{P}^2$.

Thus, in this fashion we can interpret the $\sigma_{1,2}$ and see the cohomology of $G(2,3) = \mathbb{P}^2$ in the correlation functions above.

Next, let us turn to the formal (0,2) deformations of this theory. Now, ultimately because the tangent bundle of $\mathbb{P}^2$ has no nontrivial deformations, we should get isomorphic results, but this is a good warm-up exercise for nontrivial examples later.

Here, classical correlation functions have the form

$$\langle f(\sigma) \rangle = \frac{1}{2!} \text{JKG} - \text{Res} \left\{ (-)(\sigma_1 - \sigma_2)^2 \left(\frac{1}{\det \tilde{E}(\sigma_1)} \right) \left(\frac{1}{\det \tilde{E}(\sigma_2)} \right) f(\sigma) \right\}$$

where

$$\tilde{E}(x) = Ix + B(\sigma_1 + \sigma_2).$$

The JKG residue above is computed as iterated ordinary residues, by first summing the residues about

$$\sigma_1 = -\sigma_2 \frac{b_1}{1+b_1}, \quad -\sigma_2 \frac{b_2}{1+b_2}, \quad -\sigma_2 \frac{b_3}{1+b_3},$$

corresponding to the zeroes of $\det \tilde{E}(\sigma_1)$, and then taking the residue about $\sigma_2 = 0$.

Define

$$\begin{aligned}
\Delta &= 2 \prod_{i<j} (1 + b_i + b_j), \\
&= 2 \left(1 + 2I_1 + I_1^2 + I_2 + I_1 I_2 - I_3 \right),
\end{aligned}$$

then

$$\langle \sigma_1^2 \rangle \;=\; \Delta^{-1}\left[-1 - 2I_2 - 2I_1\right],$$

$$\langle \sigma_1 \sigma_2 \rangle \;=\; \Delta^{-1}\left[2 + 2I_2 + 2I_1\right],$$

$$\langle \sigma_2^2 \rangle \;=\; \Delta^{-1}\left[-1 - 2I_2 - 2I_1\right] \;=\; \langle \sigma_1^2 \rangle,$$

or equivalently

$$\langle \sigma_{\square\square} \rangle \;=\; \langle \sigma_1^2 \rangle + \langle \sigma_1 \sigma_2 \rangle + \langle \sigma_2^2 \rangle,$$
$$=\; \Delta^{-1}\left[-2I_2 - 2I_1\right],$$

$$\langle \sigma_{\square}^2 \rangle \;=\; \langle \sigma_1^2 \rangle + 2\langle \sigma_1 \sigma_2 \rangle + \langle \sigma_2^2 \rangle,$$
$$=\; 2\Delta^{-1},$$

$$\langle \sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \rangle \;=\; \langle \sigma_1 \sigma_2 \rangle,$$
$$=\; \Delta^{-1}\left[2 + 2I_2 + 2I_1\right],$$

where

$$I_3 \;=\; b_1 b_2 b_3 \;=\; \det B,$$
$$I_2 \;=\; \sum_{i<j} b_i b_j,$$
$$I_1 \;=\; \sum_i b_i \;=\; \operatorname{tr} B.$$

Now, the quantum sheaf cohomology ring predicted by (10) is of the form

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_3, D_4, \cdots, R_{(2)}, R_{(3)} + q, R_{(4)} + q\sigma_{(1)}, \cdots \right\rangle.$$

The relations

$$R_{(2)} \;=\; D_3 \;=\; D_4 \;=\; \cdots \;=\; 0$$

allow one to write $\sigma_{(2)}$ and all higher $\sigma_{(r)}$ as linear combinations of powers of $\sigma_{(1)}$, so that there is, in effect, only one generator. As the tangent bundle of $G(2,3) = \mathbb{P}^2$ is rigid, the quantum sheaf cohomology ring should be, for nondegenerate cases, isomorphic to the ordinary quantum cohomology ring of $\mathbb{P}^2$, so indeed only one generator is expected.

Now,

$$R_{(2)} \;=\; \sum_{i=0}^{2} I_i \sigma_{(2-i)} \sigma_{(1)}^i,$$
$$=\; \sigma_{(2)} + (I_1 + I_2)\sigma_{(1)}^2,$$

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and

$$R_{(3)} = \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma_{(1)}^i,$$

$$= \sigma_{(3)} + I_1 \sigma_{(2)} \sigma_{(1)} + (I_2 + I_3) \sigma_{(1)}^3.$$

Note that, with a bit of algebra, the relations

$$D_3 = 0 = R_{(2)} = R_{(3)} + q$$

can be solved for $\sigma_{(2)}$ and $\sigma_{(3)}$ to derive the relation

$$(1 + 2I_1 + I_2 - I_3 + I_1(I_1 + I_2)) \sigma_{(1)}^3 = -q. \tag{24}$$

Thus, for generic tangent bundle degenerations, we recover the ordinary quantum cohomology ring relation for $\mathbb{P}^2$, up to an irrelevant scaling. The coefficient of $\sigma_{(1)}^3$ vanishes on the discriminant locus $\{\Delta = 0\}$ (which matches the V_3 locus, as we shall see later).

We can see the ring relations in the correlation functions above as follows. First, note that already in the classical correlation functions,

$$\langle R_{(2)} \rangle = \langle \sigma_{\boxed{}} \rangle + (I_1 + I_2) \langle \sigma_{\boxed{}}^2 \rangle = \Delta^{-1} \left[-2(I_1 + I_2) + (I_1 + I_2)(2) \right] = 0,$$

as one would expect from the ring relations above. On the (2,2) locus, $R_{(2)}$ specializes to $\sigma_{\boxed{}}$, and so this becomes the relation $\sigma_{\boxed{}} = 0$ discussed earlier.

Now, let us compute the V loci described in section 3.2, where the dimensions of the classical sheaf cohomology groups jump and our description of the quantum sheaf cohomology ring breaks down. As $G(2,3)$ is a projective space, and projective spaces admit no nontrivial tangent bundle deformations, we should find that the V loci contain nothing more than the discriminant locus.

First, we compute V_2. For this, there is only one Young diagram with two boxes total and more than one box in the first row, namely (2). We write

$$R_{(2)} = R_{\boxed{}} = \sigma_{(2)} + (I_1 + I_2)\sigma_{(1)}^2 = \sigma_{(2)}(1 + I_1 + I_2) + \sigma_{(1,1)}(I_1 + I_2).$$

Define the matrix

$$\left(C_{\mu\nu}^2 \right) = [1 + I_1 + I_2, I_1 + I_2],$$

so that

$$\left[R_{(2)} \right] = \left(C_{\mu\nu}^2 \right) \begin{bmatrix} \sigma_{(2)} \\ \sigma_{(1,1)} \end{bmatrix},$$

then the locus V_2 is defined to be the locus on which $\left(C_{\mu\nu}^2 \right)$ drops rank. However, that would require

$$1 + I_1 + I_2 = 0 = I_1 + I_2,$$

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which has no solutions, therefore V_2 is the empty set.

Next, we compute V_3. Here, there are two Young diagrams with three boxes total and more than one box in the first row, namely (3) and $(2,1)$. We compute

$$R_{(3)} \; = \; R_{\boxed{}} \; = \; \sigma_{(3)} + I_1 \sigma_{(2)} \sigma_{(1)} + (I_2 + I_3) \sigma_{(1)}^3,$$
$$= \; \sigma_{(3)} \left(1 + I_1 + I_2 + I_3 \right) \; + \; \sigma_{(2,1)} \left(I_1 + 2 I_2 + 2 I_3 \right),$$

and

$$R_{(2,1)} \; = \; R_{\boxminus} \; = \; \det \begin{bmatrix} R_{(2)} & R_{(3)} \\ 1 & \sigma_{(1)} \end{bmatrix} \; = \; \sigma_{(1)} R_{(2)} - R_{(3)},$$
$$= \; \sigma_{(3)} \left(-I_3 \right) \; + \; \sigma_{(2,1)} \left(1 + I_1 - 2 I_3 \right).$$

Thus, the matrix $(C_{\mu\nu}^3)$ is given by

$$(C_{\mu\nu}^3) \; = \; \begin{bmatrix} 1 + I_1 + I_2 + I_3 & I_1 + 2 I_2 + 2 I_3 \\ -I_3 & 1 + I_1 - 2 I_3 \end{bmatrix},$$

, and the locus V_3 is then defined as

$$V_3 \; = \; \{ \det(C_{\mu\nu}^3) = 0 \}.$$

It is straightforward to check that this locus matches the discriminant locus $X = \{\Delta = 0\}$.

With a little algebra, it is straightforward to verify that the ring relation (24) can be rewritten as

$$(\det C_{\mu\nu}^3) \sigma_{(1)}^3 \; = \; -q,$$

so that we see the V_3 locus is interpreted in this case as the locus where the quantum ring relations become ill-defined (which is also the discriminant locus).

Now, let us compute the locus V_4. Proceeding as above, we find

$$\begin{bmatrix} R_{(4)} \\ R_{(3,1)} \\ R_{(2,2)} \end{bmatrix} \; = \; (C_{\mu\nu}^4) \begin{bmatrix} \sigma_{(4)} \\ \sigma_{(3,1)} \\ \sigma_{(2,2)} \end{bmatrix},$$

where

$$(C_{\mu\nu}^4) \; = \; \begin{bmatrix} 1 + I_1 + I_2 + I_3 & I_1 + I_2 + I_3 & I_2 + 2 I_3 \\ 0 & 1 & I_1 + I_2 \\ -I_3 & -1 - I_3 & 1 - I_2 - 2 I_3 \end{bmatrix}$$

and in particular we find

$$\det C_{\mu\nu}^4 \; = \; \det C_{\mu\nu}^3,$$

so that the locus

$$V_4 \; = \; V_3 \; = \; \{\Delta = 0\}.$$

As the discriminant locus already appears to exhaust the ways in which the quantum cohomology ring can degenerate, this result should not be surprising.

4.3 $G(2, 4)$

4.3.1 (2,2) theory

Let's first compute the correlation functions for the (2,2) theory engineering $G(2, 4)$. By computing the correlation functions, we want to explicitly show that

$$R_{(3)} = \sigma_{\boxed{}} = 0, \quad R_{(4)} + q = \sigma_{\boxed{}} + q = 0,$$

as implied by our general result. The four-point correlation functions in the theory are given by

$$\langle f(\sigma) \rangle = \frac{1}{2!} \text{JKG} - \text{Res} \left\{ -(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^4} \frac{1}{\sigma_2^4} f(\sigma) \right\}$$

The Jeffrey-Kirwan-Grothendieck residues in this case are merely iterated ordinary contour integrals about $\sigma_1 = 0$ and $\sigma_2 = 0$, *i.e.*

$$\langle f(\sigma) \rangle = \frac{1}{2!} \oint d\sigma_2 \oint d\sigma_1 \left\{ -(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^4} \frac{1}{\sigma_2^4} f(\sigma) \right\}$$

It is straightforward to show that

$$\langle \sigma_1^4 \rangle = 0, \quad \langle \sigma_1^3 \sigma_2 \rangle = -\frac{1}{2!},$$
$$\langle \sigma_1^2 \sigma_2^2 \rangle = +\frac{2}{2!}, \quad \langle \sigma_1 \sigma_2^3 \rangle = -\frac{1}{2!},$$
$$\langle \sigma_2^4 \rangle = 0.$$

Now, let us interpret this in terms of the cohomology of $G(2, 4)$. In principle, the cohomology classes correspond to Young tableaux sitting inside the $2 \times (4 - 2)$ box

$$\boxed{\begin{array}{cc} & \\ & \end{array}}$$

and so in particular are given by

$$\begin{aligned}
\sigma_{\square} &= \sigma_1 + \sigma_2 \\
\sigma_{\square\square} &= \sigma_1^2 + \sigma_2^2 + \sigma_1\sigma_2 \\
\sigma_{\substack{\square\\\square}} &= \sigma_1\sigma_2 \\
\sigma_{\substack{\square\square\\\square}} &= \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 \\
\sigma_{\substack{\square\square\\\square\square}} &= \sigma_1^2\sigma_2^2
\end{aligned}$$

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with relations, for example

$$\sigma_{\square\square\square} \;=\; \sigma_1^3 + \sigma_2^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 \;=\; 0$$

Let us check that the relation $R_3 = \sigma_{\square\square\square} = 0$ is encoded in the correlation functions above. Since it involves third-order powers, and the correlation functions involve fourth-order powers, we need to multiply by single copies of σ_1, σ_2. In other words, we claim the following statements should be true:

$$\langle \sigma_1 \sigma_{\square\square\square} \rangle \;=\; 0 \;=\; \langle \sigma_2 \sigma_{\square\square\square} \rangle$$

Explicitly,

$$
\begin{aligned}
\langle \sigma_1 \sigma_{\square\square\square} \rangle &= \langle \sigma_1(\sigma_1^3 + \sigma_2^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2) \rangle \\
&= \langle \sigma_1^4 \rangle + \langle \sigma_1\sigma_2^3 \rangle + \langle \sigma_1^3\sigma_2 \rangle + \langle \sigma_1^2\sigma_2^2 \rangle
\end{aligned}
$$

and it is easy to check that this does indeed vanish. One can similarly verify $\langle \sigma_2 \sigma_{\square\square\square} \rangle = 0$.

Correlation functions in the one-instanton sector are of the form

$$
\begin{aligned}
\langle f(\sigma) \rangle \;=\;\; & \frac{1}{2!}\mathrm{JKG} - \mathrm{Res}\left\{ q(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^8}\frac{1}{\sigma_2^4} f(\sigma) \right\} \\
& + \frac{1}{2!}\mathrm{JKG} - \mathrm{Res}\left\{ q(\sigma_1-\sigma_2)^2 \frac{1}{\sigma_1^4}\frac{1}{\sigma_2^8} f(\sigma) \right\}
\end{aligned}
$$

(where the JKG residues again reduce to iterated ordinary contour integrals about $\sigma_1 = 0$ and $\sigma_2 = 0$), from which we compute that the nonvanishing correlation functions are

$$\langle \sigma_1^5\sigma_2^3 \rangle \;=\; q/2 \;=\; \langle \sigma_1^7\sigma_2 \rangle, \quad \langle \sigma_1^6\sigma_2^2 \rangle \;=\; -q,$$

$$\langle \sigma_1^3\sigma_2^5 \rangle \;=\; q/2 \;=\; \langle \sigma_1\sigma_2^7 \rangle, \quad \langle \sigma_1^2\sigma_2^6 \rangle \;=\; -q.$$

Using the fact that

$$\sigma_{\square\square\square\square} = \sigma_1^4 \,+\, \sigma_1^3\sigma_2 \,+\, \sigma_1^2\sigma_2^2 \,+\, \sigma_1\sigma_2^3 \,+\, \sigma_2^4,$$

we compute

$$\langle \sigma_{\square\square\square\square} \rangle \;=\; 0$$

$$\langle \sigma_1^4 \sigma_{\square\square\square\square} \rangle \;=\; 0 \;=\; \langle \sigma_2^4 \sigma_{\square\square\square\square} \rangle$$

$$\langle \sigma_1^3\sigma_2 \sigma_{\square\square\square\square} \rangle \;=\; q/2 \;=\; \langle \sigma_1\sigma_2^3 \sigma_{\square\square\square\square} \rangle$$

$$\langle \sigma_1^2\sigma_2^2 \sigma_{\square\square\square\square} \rangle \;=\; -q$$

From the expressions

$$\sigma_{\substack{\square\square\square\\\square}} = \sigma_1^3\sigma_2 \,+\, \sigma_1^2\sigma_2^2 \,+\, \sigma_1\sigma_2^3$$

$$\sigma_{\boxplus} = \sigma_1^2 \sigma_2^2$$

we get

$$\langle \sigma_{\square\square\square\square} \; \sigma_{\square\square\square\square} \rangle = 0$$

$$\langle \sigma_{\square\square\square\square} \; \sigma_{\square\square\square\text{/}\square} \rangle = 0$$

$$\langle \sigma_{\square\square\square\square} \; \sigma_{\boxplus} \rangle = -q$$

Thus we see the relation $R_{(4)}$ is valid because $\sigma_{\square\square\square\square} = \langle \sigma_{\square\square\square\square} \; \sigma_{\boxplus} \rangle \cdot 1 = -q$. The other relation at fourth order can be read off immediately,

$$\sigma_{\square\square\square\text{/}\square} = \sigma_{\square\square\square} \, \sigma_{\square} - \sigma_{\square\square\square\square} = q.$$

4.3.2 (0,2) theory

We can study the (0,2) theories following the same procedure. Again, after absorbing a sign in q, the localization formula (8) reads

$$\langle f(\sigma) \rangle =$$

$$\frac{1}{2!} \sum_{\mathtt{m}_1, \cdots, \mathtt{m}_k \in \mathbb{Z}} \text{JKG} - \text{Res} \left\{ (-q)^{\sum \mathtt{m}_i} \left((-)(\sigma_1 - \sigma_2) \right) \prod_{a=1}^{2} \left(\frac{1}{\det \tilde{E}(\sigma_a)} \right)^{\mathtt{m}_i + 1} f(\sigma) \right\},$$

where

$$\tilde{E}_j^i(\sigma_a) = \delta_j^i \sigma_a + B_j^i \left(\sum_b \sigma_b \right).$$

In this case, the JKG residue gives us the following iterated residue prescription for generic b_j:

1. First, we perform a contour integral over σ_1, summing over the residues at the four loci

$$\sigma_1 = -\sigma_2 \frac{b_j}{1 + b_j}$$

for $j \in \{1, 2, 3, 4\}$, corresponding to the roots of $\det \tilde{E}(\sigma_1)$,

2. then, we perform a contour integral over σ_2, taking the residue at $\sigma_2 = 0$.

In this case, the results for the classical correlation functions ($\mathfrak{m}_1 = 0 = \mathfrak{m}_2$) are as follows:

$$
\begin{aligned}
\langle \sigma_1^4 \rangle &= \Delta^{-1} \left[I_1 + 2I_1^2 + 4I_1 I_2 - 2I_3 + 2I_2^2 + 2I_1 I_3 - 4I_4 + 2I_2 I_3 - 2I_1 I_4 \right], \\
\langle \sigma_1^3 \sigma_2 \rangle &= \Delta^{-1} \left[-1 - 3I_1 - 2I_1^2 - 3I_2 - 4I_1 I_2 - 2I_2^2 - I_3 - 2I_1 I_3 + 4I_4 - 2I_2 I_3 + 2I_1 I_4 \right], \\
\langle \sigma_1^2 \sigma_2^2 \rangle &= \Delta^{-1} \left[2 + 4I_1 + 2I_1^2 + 4I_2 + 4I_1 I_2 + 2I_3 - 4I_4 + 2I_2^2 + 2I_1 I_3 + 2I_2 I_3 - 2I_1 I_4 \right], \\
\langle \sigma_1 \sigma_2^3 \rangle &= \langle \sigma_1^3 \sigma_2 \rangle, \\
\langle \sigma_2^4 \rangle &= \langle \sigma_1^4 \rangle,
\end{aligned}
$$

or

$$
\begin{aligned}
\langle \sigma_{\boxed{\;\;\;\;}} \rangle &= \langle \sigma_1^4 \rangle + \langle \sigma_1^3 \sigma_2 \rangle + \langle \sigma_1^2 \sigma_2^2 \rangle + \langle \sigma_1 \sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= 2\Delta^{-1} \left[-I_2 + I_1^2 + 2I_2 I_1 - 2I_3 + I_2^2 - 2I_4 + I_1 I_3 - I_1 I_4 + I_2 I_3 \right],
\end{aligned}
$$

$$
\begin{aligned}
\langle \sigma_{\boxed{\;\;}}^2 \rangle &= \langle \sigma_1^4 \rangle + 2\langle \sigma_1^3 \sigma_2 \rangle + 3\langle \sigma_1^2 \sigma_2^2 \rangle + 2\langle \sigma_1 \sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1} \left[2 + 2I_1 + 2I_1^2 + 4I_1 I_2 - 2I_3 - 4I_4 + 2I_2^2 + 2I_1 I_3 + 2I_2 I_3 - 2I_1 I_4 \right],
\end{aligned}
$$

$$
\begin{aligned}
\langle \sigma_{\boxed{\;}} \sigma_{\boxed{\;\;\;}} \rangle &= \langle \sigma_1^4 \rangle + 2\langle \sigma_1^3 \sigma_2 \rangle + 2\langle \sigma_1^2 \sigma_2^2 \rangle + 2\langle \sigma_1 \sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1} \left[-2I_1 - 4I_2 - 4I_3 \right],
\end{aligned}
$$

$$
\begin{aligned}
\langle \sigma_{\boxed{\;}}^2 \sigma_{\boxed{\;}} \rangle &= \langle \sigma_1^4 \rangle + 3\langle \sigma_1^3 \sigma_2 \rangle + 4\langle \sigma_1^2 \sigma_2^2 \rangle + 3\langle \sigma_1 \sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1} \left[2 - 2I_2 - 2I_3 \right],
\end{aligned}
$$

$$
\begin{aligned}
\langle \sigma_{\boxed{\;}}^4 \rangle &= \langle \sigma_1^4 \rangle + 4\langle \sigma_1^3 \sigma_2 \rangle + 6\langle \sigma_1^2 \sigma_2^2 \rangle + 4\langle \sigma_1 \sigma_2^3 \rangle + \langle \sigma_2^4 \rangle, \\
&= \Delta^{-1} \left[4 + 2I_1 \right],
\end{aligned}
$$

$$
\begin{aligned}
\langle \sigma_{\boxplus} \rangle &= \langle \sigma_1^3 \sigma_2 \rangle + \langle \sigma_1^2 \sigma_2^2 \rangle + \langle \sigma_1 \sigma_2^3 \rangle, \\
&= \Delta^{-1} \left[-2I_1 - 2I_2 - 2I_1^2 + 4I_4 - 2I_2^2 - 2I_3 I_1 - 2I_3 I_2 + 2I_1 I_4 - 4I_2 I_1 \right],
\end{aligned}
$$

$$
\begin{aligned}
\langle \sigma_{\boxplus} \rangle &= \langle \sigma_1^2 \sigma_2^2 \rangle, \\
&= \Delta^{-1} \left[2 + 4I_1 + 2I_1^2 + 4I_2 + 4I_1 I_2 + 2I_3 - 4I_4 + 2I_2^2 + 2I_1 I_3 + 2I_2 I_3 - 2I_1 I_4 \right],
\end{aligned}
$$

where the characteristic polynomials of B are given explicitly as

$$
\begin{aligned}
I_1 &= \sum_i b_i = \operatorname{tr} B, \\
I_2 &= \sum_{i<j} b_i b_j, \\
I_3 &= \sum_{i<j<k} b_i b_j b_k, \\
I_4 &= b_1 b_2 b_3 b_4 = \det B,
\end{aligned}
$$

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and

$$
\begin{aligned}
\Delta \;=\;& 2 \prod_{i<j} \left(1 + b_i + b_j \right), \\
=\;& 2 \left(1 + 3I_1 + 3I_1^2 + 2I_2 + I_1^3 + 4I_1 I_2 + 2I_1^2 I_2 + I_2^2 + I_1 I_3 - 4I_4 + I_1 I_2^2 \right. \\
& \left. + I_1^2 I_3 + I_1 I_2 I_3 - 4I_1 I_4 - I_1^2 I_4 - I_3^2 \right).
\end{aligned}
$$

We see the discriminant locus is given by $\Delta = 0$, this is consistent with our general result in [35], which says that the B-deformation fails to define a vector bundle on $G(k,n)$ if and only if there exits k eigenvalues of B whose sum is -1.

Now, the quantum sheaf cohomology ring for this model is predicted by (10) to be

$$
\mathbb{C} \left[\sigma_{(1)}, \sigma_{(2)}, \cdots \right] / \left\langle D_3, D_4, \cdots, R_{(3)}, R_{(4)} + q, R_{(5)} + q\sigma_{(1)}, \cdots \right\rangle,
$$

where

$$
\begin{aligned}
R_{(3)} \;=\;& \sum_{i=0}^{3} I_i \sigma_{(3-i)} \sigma_{(1)}^{i}, \\
=\;& \sigma_{(3)} + I_1 \sigma_{(2)} \sigma_{(1)} + (I_2 + I_3) \sigma_{(1)}^3, \\
R_{(4)} \;=\;& \sum_{i=0}^{4} I_i \sigma_{(r-i)} \sigma_{(1)}^{i}, \\
=\;& \sigma_{(4)} + I_1 \sigma_{(3)} \sigma_{(1)} + I_2 \sigma_{(2)} \sigma_{(1)}^2 + (I_3 + I_4) \sigma_{(1)}^4.
\end{aligned}
$$

As a consistency test, it is straightforward to check that the relations above are reflected in the correlation functions. For example, the classical correlation functions are easily demonstrated to obey

$$
\langle \sigma_\Box D_3 \rangle \;=\; \langle D_4 \rangle \;=\; \langle \sigma_\Box R_{(3)} \rangle \;=\; 0.
$$

Now, we also have the relation $R_{(4)} = -q$, which for the purely classical correlation functions implies

$$
\langle R_{(4)} \rangle \;=\; 0,
$$

which is also easily checked to be true. By including instanton sectors, one can see the full quantum-corrected relation, as we shall discuss next.

The relation $R_{(4)} = -q$ can be derived from the quantum cohomology ring relation derived from the Jeffrey-Kirwan-Grothendieck residue expression, namely,

$$
\det \tilde{E}(\sigma_1) \;=\; -q \;=\; \det \tilde{E}(\sigma_2),
$$

where

$$
\tilde{E}(x) \;=\; Ix \;+\; B(\sigma_1 + \sigma_2).
$$

Now, it is straightforward to expand

$$\det \tilde{E}(x) \;=\; x^4 \;+\; I_1(\sigma_1 + \sigma_2)x^3 \;+\; I_2(\sigma_1 + \sigma_2)^2 x^2 \;+\; I_3(\sigma_1 + \sigma_2)^3 x \;+\; I_4(\sigma_1 + \sigma_2)^4$$

so

$$
\begin{aligned}
\langle \det \tilde{E}(\sigma_1) \rangle \;&=\; \langle \sigma_1^4 \rangle (1 + I_1 + I_2 + I_3 + 2I_4) \;+\; \langle \sigma_1^3 \sigma_2 \rangle (I_1 + 2I_2 + 4I_3 + 8I_4) \\
&\quad +\; \langle \sigma_1^2 \sigma_2^2 \rangle (I_2 + 3I_3 + 6I_4), \\
&=\; \langle \det \tilde{E}(\sigma_2) \rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
2R_{(4)} - R_3 \sigma_{\square} \;=\; \sigma_{\square\square\square\square}(1 + I_1 + I_2 + I_3 + 2I_4) \;&+\; \langle \sigma_{\square\square\square} \rangle(-1 + I_2 + 3I_3 + 6I_4) \\
+\; \langle \sigma_{\square\square} \rangle(-I_1 + 2I_3 + 4I_4) \;&=\; -2q,
\end{aligned}
$$

or simply $R_{(4)} + q = 0$ as expected.

Now, let us turn to the interpretation of the loci V_m in this example. First, consider the V_3 locus. It is straightforward to compute

$$\begin{bmatrix} R_{(3)} \end{bmatrix} \;=\; \begin{bmatrix} 1 + I_1 + I_2 + I_3, \; I_1 + 2I_2 + 2I_3 \end{bmatrix} \begin{bmatrix} \sigma_{(3)} \\ \sigma_{(2,1)} \end{bmatrix},$$

hence

$$
\begin{aligned}
V_3 \;&=\; \{1 + I_1 + I_2 + I_3 = 0 \text{ and } I_1 + 2I_2 + 2I_3 = 0\}, \\
&=\; \{I_2 + I_3 = +1, \; I_1 = -2\}.
\end{aligned}
$$

Note that this locus does not intersect the (2,2) locus, as expected on general grounds, as the I_i never all become zero.

Next, we compute the V_4 locus. It is straightforward to compute

$$\begin{bmatrix} R_{(4)} \\ R_{(3,1)} \end{bmatrix} \;=\; (C^4_{\mu\nu}) \begin{bmatrix} \sigma_{(4)} \\ \sigma_{(3,1)} \\ \sigma_{(2,2)} \end{bmatrix},$$

where

$$(C^4_{\mu\nu}) \;=\; \begin{bmatrix} 1 + I_1 + I_2 + I_3 + I_4 & I_1 + 2I_2 + 3I_3 + 3I_4 & I_2 + 2I_3 + 2I_4 \\ -I_4 & 1 + I_1 + I_2 - 3I_4 & I_1 + I_2 - 2I_4 \end{bmatrix}.$$

Let M_{12}, M_{13}, M_{23} denote the three 2×2 submatrices of $(C^4_{\mu\nu})$ formed by omitting a column, then the locus V_4 is defined as

$$V_4 \;=\; \{M_{12} = 0 = M_{13} = M_{23}\}.$$

For completeness, we also list here results for V_5 and V_6. First, V_5 is computed from the relation

$$
\begin{bmatrix} R_{(5)} \\ R_{(4,1)} \\ R_{(3,2)} \end{bmatrix} = \left(C_{\mu\nu}^5 \right) \begin{bmatrix} \sigma_{(5)} \\ \sigma_{(4,1)} \\ \sigma_{(3,2)} \end{bmatrix}
$$

for

$$
\left(C_{\mu\nu}^5 \right) = \begin{bmatrix} 1 + I_1 + I_2 + I_3 + I_4 & I_1 + 2I_2 + 3I_3 + 4I_4 & I_2 + 3I_3 + 5I_4 \\ 0 & 1 + I_1 + I_2 + I_3 & I_1 + 2I_2 + 2I_3 \\ -I_4 & -I_3 - 4I_4 & 1 + I_1 - 2I_3 - 5I_4 \end{bmatrix},
$$

and V_6 is computed from the relation

$$
\begin{bmatrix} R_{(6)} \\ R_{(5,1)} \\ R_{(4,2)} \\ R_{(3,3)} \end{bmatrix} = \left(C_{\mu\nu}^6 \right) \begin{bmatrix} \sigma_{(6)} \\ \sigma_{(5,1)} \\ \sigma_{(4,2)} \\ \sigma_{(3,3)} \end{bmatrix}
$$

for $\left(C_{\mu\nu}^6 \right)$ given by

$$
\begin{bmatrix} 1 + I_1 + I_2 + I_3 + I_4 & I_1 + 2I_2 + 3I_3 + 4I_4 & I_2 + 3I_3 + 6I_4 & I_3 + 3I_4 \\ 0 & 1 + I_1 + I_2 + I_3 + I_4 & I_1 + 2I_2 + 3I_3 + 3I_4 & I_2 + 2I_3 + 2I_4 \\ 0 & -I_4 & 1 + I_1 + I_2 - 3I_4 & I_1 + I_2 - 2I_4 \\ -I_4 & -I_3 - 4I_4 & -I_2 - 3I_3 - 6I_4 & 1 - I_2 - 2I_3 - 3I_4 \end{bmatrix}.
$$

At least when B is diagonal, it is straightforward to check that

$$
\det(C_5) = \det(C_6) = \Delta,
$$

or equivalently,

$$
V_5 = V_6 = X,
$$

consistent with our expectation that for m larger than the dimension of the Grassmannian, V_m matches the discriminant locus.

4.4 $G(2,5)$

In the previous sections we described the results for the Grassmannian $G(2,4)$. Although this is not a projective space, it can be described as a hypersurface in a projective space, so in this section we give one additional nonabelian example, one which is not related or dual to an abelian GLSM, to demonstrate the results. Specifically, in this section we will consider the theory for $G(2,5)$.
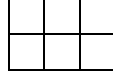
4.4.1 (2,2) theory

The (classical) six-point correlation functions are given by

$$\langle f(\sigma) \rangle \;=\; \frac{1}{2!} \text{JKG} - \text{Res}\left\{ (-)(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^5} \frac{1}{\sigma_2^5} f(\sigma) \right\}.$$

We can compute these as iterated ordinary contour integrals about $\sigma_1 = 0$ and $\sigma_2 = 0$. It is straightforward to show that

$$\langle \sigma_1^6 \rangle = 0, \quad \langle \sigma_1^5 \sigma_2 \rangle = 0,$$
$$\langle \sigma_1^4 \sigma_2^2 \rangle = -\frac{1}{2!}, \quad \langle \sigma_1^3 \sigma_2^3 \rangle = +\frac{2}{2!},$$
$$\langle \sigma_1^2 \sigma_2^4 \rangle = -\frac{1}{2!}, \quad \langle \sigma_1 \sigma_2^5 \rangle = 0,$$
$$\langle \sigma_2^6 \rangle = 0.$$

All the nonzero cohomology classes should be defined by Young diagrams fitting inside the 2×3 box

Using the correlation functions above, it is straightforward to compute the ring relations

$$\sigma_{\boxed{}} \;=\; \sigma_1^4 + \sigma_1^3 \sigma_2 + \sigma_1^2 \sigma_2^2 + \sigma_1 \sigma_2^3 + \sigma_2^4 \;=\; 0$$
$$\sigma_{\boxed{}} \;=\; \sigma_1^4 \sigma_2 + \sigma_1^3 \sigma_2^2 + \sigma_1^2 \sigma_2^3 + \sigma_1 \sigma_2^4 \;=\; 0$$
$$\sigma_{\boxed{}} \;=\; \sigma_1^4 \sigma_2^2 + \sigma_1^3 \sigma_2^3 + \sigma_1^2 \sigma_2^4 \;=\; 0$$

which matches the ring relations one expects from the cohomology theory. In each case, one multiplies in arbitrary powers of σ_1, σ_2 to get a six-point function, and in each case, the sum amounts to a scan through values that sum to zero. The top-form, described by $\sigma_{\boxed{}} = \sigma_1^3 \sigma_2^3$, has nonzero vev, as expected.

Correlation functions in the one-instanton sector are of the form

$$\langle f(\sigma) \rangle \;=\;$$
$$\frac{1}{2!} \text{JKG} - \text{Res}\left\{ q(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^{10}} \frac{1}{\sigma_2^5} f(\sigma) \right\} + \frac{1}{2!} \text{JKG} - \text{Res}\left\{ q(\sigma_1 - \sigma_2)^2 \frac{1}{\sigma_1^5} \frac{1}{\sigma_2^{10}} f(\sigma) \right\},$$

from which one can compute the nonzero correlation functions at order 11 to be

$$\langle \sigma_1^9 \sigma_2^2 \rangle = \frac{q}{2}, \quad \langle \sigma_1^8 \sigma_2^3 \rangle = -q, \quad \langle \sigma_1^7 \sigma_2^4 \rangle = \frac{q}{2},$$
$$\langle \sigma_1^4 \sigma_2^7 \rangle = \frac{q}{2}, \quad \langle \sigma_1^3 \sigma_2^8 \rangle = -q, \quad \langle \sigma_1^2 \sigma_2^9 \rangle = \frac{q}{2}.$$

Thus we see

$$\langle \sigma_{\yng(5)} \, \sigma_{\yng(3,3)} \rangle = -q, \quad \langle \sigma_{\yng(6)} \, \sigma_{\yng(2,2)} \rangle = -q,$$

which impies

$$R_{(5)} + q = 0, \quad R_{(6)} + q\sigma_{\square} = 0$$

because

$$\langle \sigma_{\yng(3,3)} \rangle = 1, \quad \langle \sigma_{\yng(2,2)} \, \sigma_{\square} \rangle = 1.$$

4.4.2 (0,2) theory

Next, let us consider the (0,2) theory defined by deformations of the tangent bundle.

In this case, classical correlation functions are given by

$$\langle f(\sigma) \rangle =$$
$$\frac{1}{2!} \mathrm{JKG} - \mathrm{Res}\left\{ (-)(\sigma_1 - \sigma_2)^2 \left(\frac{1}{\det \tilde{E}(\sigma_1)} \right) \left(\frac{1}{\det \tilde{E}(\sigma_2)} \right) f(\sigma) \right\},$$

and for B diagonal, computing much as in previous examples, we find:

$$\begin{aligned}
\langle \sigma_1^6 \rangle = \ \Delta^{-1} \big[&-I_1^2 + I_2 - 2I_1^3 + I_1 I_2 + I_3 + 3I_2^2 + 4I_1 I_3 - I_4 - 6I_1^2 I_2 - 6I_1 I_2^2 - 4I_1^2 I_3 \\
&+10I_2 I_3 - 8I_1 I_2 I_3 - 2I_2^3 + 7I_3^2 - 4I_2^2 I_3 + 5I_4 I_1 - 5I_5 + 4I_1 I_5 \\
&+8I_1^2 I_5 + 9I_4 I_2 - 2I_1 I_3^2 - 2I_2 I_3^2 - 2I_1 I_2 I_4 - 2I_2^2 I_4 + 11I_3 I_4 - 2I_2 I_3 I_4 \\
&+4I_4^2 + 2I_1 I_4^2 + I_2 I_5 + 8I_1 I_2 I_5 + 2I_2^2 I_5 - 2I_3 I_5 - 2I_4 I_5 \big],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_1^5 \sigma_2 \rangle = \ \Delta^{-1} \big[&I_1 + 3I_1^2 + 2I_1^3 + 5I_1 I_2 - 2I_3 + 6I_1^2 I_2 + 2I_2^2 + I_1 I_3 - 4I_4 + 6I_1 I_2^2 \\
&+4I_1^2 I_3 - 5I_1 I_4 - 5I_5 + 2I_2^3 - 2I_3^2 + 8I_1 I_2 I_3 - 4I_2 I_4 - 14I_1 I_5 \\
&+4I_2^2 I_3 + 2I_1 I_3^2 + 2I_1 I_2 I_4 - 6I_3 I_4 - 8I_1^2 I_5 - 6I_2 I_5 + 2I_2 I_3^2 + 2I_2^2 I_4 \\
&-4I_4^2 - 8I_1 I_2 I_5 + 2I_3 I_5 + 2I_2 I_3 I_4 - 2I_1 I_4^2 - 2I_2^2 I_5 + 2I_4 I_5 \big],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_1^4 \sigma_2^2 \rangle = \ \Delta^{-1} \big[&-1 - 4I_1 - 5I_1^2 - 4I_2 - 2I_1^3 - 10I_1 I_2 - 2I_3 - 6I_1^2 I_2 - 5I_2^2 - 6I_1 I_3 + 3I_4 \\
&-6I_1 I_2^2 - 4I_1^2 I_3 - 6I_2 I_3 + 3I_1 I_4 + 13I_5 - 2I_2^3 - 8I_1 I_2 I_3 - I_3^2 + I_2 I_4 \\
&+20I_1 I_5 - 4I_2^2 I_3 - 2I_1 I_3^2 - 2I_1 I_2 I_4 + 3I_3 I_4 + 8I_1^2 I_5 + 9I_2 I_5 - 2I_2 I_3^2 \\
&-2I_2^2 I_4 + 4I_4^2 + 8I_1 I_2 I_5 - 2I_3 I_5 - 2I_2 I_3 I_4 + 2I_1 I_4^2 + 2I_2^2 I_5 - 2I_4 I_5 \big],
\end{aligned}$$

$$\begin{aligned}
\langle \sigma_1^3 \sigma_2^3 \rangle = \ \Delta^{-1} \big[&2 + 6I_1 + 6I_1^2 + 6I_2 + 2I_1^3 + 12I_1 I_2 + 4I_3 + 6I_1^2 I_2 + 6I_2^2 - 2I_4 + 8I_1 I_3 \\
&+6I_1 I_2^2 - 16I_5 + 4I_1^2 I_3 + 8I_2 I_3 - 2I_1 I_4 + 2I_2^3 - 22I_1 I_5 + 8I_1 I_2 I_3 \\
&+2I_3^2 + 4I_2^2 I_3 + 2I_1 I_3^2 + 2I_1 I_2 I_3 - 2I_3 I_4 - 2I_1^2 I_5 - 10I_2 I_5 + 2I_2 I_3^2 \\
&+2I_2^2 I_4 - 4I_4^2 - 8I_1 I_2 I_5 + 2I_3 I_5 + 2I_2 I_3 I_4 - 2I_1 I_4^2 - 2I_2^2 I_5 + 2I_4 I_5 \big],
\end{aligned}$$

$$\langle \sigma_1^2 \sigma_2^4 \rangle = \langle \sigma_1^4 \sigma_2^2 \rangle, \quad \langle \sigma_1 \sigma_2^5 \rangle = \langle \sigma_1^5 \sigma_2 \rangle, \quad \langle \sigma_2^6 \rangle = \langle \sigma_1^6 \rangle,$$

where

$$\Delta = 2 \prod_{i<j} (1 + b_i + b_j),$$

and

$$
\begin{aligned}
I_5 &= b_1 b_2 b_3 b_4 b_5 = \det B, \\
I_4 &= \sum_{i<j<k<\ell} b_i b_j b_k b_\ell, \\
I_3 &= \sum_{i<j<k} b_i b_j b_k, \\
I_2 &= \sum_{i<j} b_i b_j, \\
I_1 &= \sum_i b_i = \operatorname{tr} B.
\end{aligned}
$$

The quantum sheaf cohomology ring (10) in this case is given by

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_3, D_4, \cdots, R_{(4)}, R_{(5)} + q, R_{(6)} + q\sigma_{(1)}, \cdots \right\rangle,$$

where

$$
\begin{aligned}
R_{(4)} &= \sum_{i=0}^{4} I_i \sigma_{(4-i)} \sigma_{(1)}^i, \\
&= \sigma_{(4)} + I_1 \sigma_{(3)} \sigma_{(1)} + I_2 \sigma_{(2)} \sigma_{(1)}^2 + (I_3 + I_4) \sigma_{(1)}^4, \\
R_{(5)} &= \sum_{i=0}^{5} I_i \sigma_{(5-i)} \sigma_{(1)}^i, \\
&= \sigma_{(5)} + I_1 \sigma_{(4)} \sigma_{(1)} + I_2 \sigma_{(3)} \sigma_{(1)}^2 + I_3 \sigma_{(2)} \sigma_{(1)}^3 + (I_4 + I_5) \sigma_{(1)}^5, \\
R_{(6)} &= \sum_{i=0}^{5} I_i \sigma_{(6-i)} \sigma_{(1)}^i, \\
&= \sigma_{(6)} + I_1 \sigma_{(5)} \sigma_{(1)} + I_2 \sigma_{(4)} \sigma_{(1)}^2 + I_3 \sigma_{(3)} \sigma_{(1)}^3 + I_4 \sigma_{(2)} \sigma_{(1)}^4 + I_5 \sigma_{(1)}^6.
\end{aligned}
$$

As a consistency test, it is straightforward to check that the relations above are reflected in the correlation functions. For example, the classical correlation functions are easily demonstrated to obey

$$\langle \sigma_\square^3 D_3 \rangle = \langle \sigma_\square^2 D_4 \rangle = \langle \sigma_\square D_5 \rangle = \langle D_6 \rangle = \langle \sigma_\square^2 R_{(4)} \rangle = 0.$$

(Other vanishings of classical correlation functions are also implied by the ring relations; for example, in the line above, one could replace any instance of $\sigma_\square^2$ with $\sigma_{\square\square}$ to get another vanishing correlation function. Our intent above is merely to list some examples, not to list every possible example.)

In addition, we can also use the classical correlation functions to check the classical limits of the relations
$$R_{(5)} \;=\; -q, \quad R_{(6)} \;=\; -q\sigma_{(1)}.$$
In particular, it is straightforward to show that the classical correlation functions obey
$$\langle \sigma_\square R_{(5)} \rangle \;=\; 0 \;=\; \langle R_{(6)} \rangle,$$
verifying the classical limit of the relations above.

One of the relations should be the classical limit of the quantum cohomology ring relation derived from the Jeffrey-Kirwan-Grothendieck residue expression, namely,
$$\det \tilde{E}(\sigma_1) \;=\; q \;=\; \det \tilde{E}(\sigma_2)$$
where
$$\tilde{E}(x) \;=\; Ix \;+\; B(\sigma_1 + \sigma_2)$$
As before, it is straightforward to expand
$$\det \tilde{E}(x) = x^5 + I_1(\sigma_1+\sigma_2)x^4 + I_2(\sigma_1+\sigma_2)^2 x^3 + I_3(\sigma_1+\sigma_2)^3 x^2 + I_4(\sigma_1+\sigma_2)^4 x + I_5(\sigma_1+\sigma_2)^5$$
so, for example,
$$\begin{aligned}
\det \tilde{E}(\sigma_1) \;=\;& \sigma_1^5(1 + I_1 + I_2 + I_3 + I_4 + I_5) \;+\; \sigma_1^4\sigma_2(I_1 + 2I_2 + 3I_3 + 4I_4 + 5I_5) \\
&+ \sigma_1^3\sigma_2^2(I_2 + 3I_3 + 6I_4 + 10I_5) \;+\; \sigma_1^2\sigma_2^3(I_3 + 4I_4 + 10I_5) \\
&+ \sigma_1\sigma_2^4(I_4 + 5I_5) \;+\; \sigma_2^5(I_5)
\end{aligned}$$

From this we derive
$$\sigma_{\square\square\square\square\square}(1 + I_1 + I_2 + I_3 + I_4 + 2I_5) \;+\; \sigma_{\substack{\square\square\square\square\\\square}}(-1 + I_2 + 2I_3 + 4I_4 + 8I_5)$$
$$+\; \sigma_{\substack{\square\square\square\\\square\square}}(-I_1 - I_2 + I_3 + 5I_4 + 10I_5) \;=\; 2q$$

It is straightforward to check, via multiplication by $\sigma_{1,2}$, that the classical limit of the relation above is indeed a property of the correlation functions given in this section.

5   Conclusions

In this paper we have presented a proposal for the quantum sheaf cohomology ring of Grassmannians with deformations of the tangent bundle. We derived this proposal from one-loop

effective actions, and checked it in examples in which correlation functions were computed with supersymmetric localization. We also discussed where this proposal is valid: on codimension one subvarieties of the space of tangent bundle deformations, not intersecting the (2,2) locus, our proposal breaks down. We discussed those loci explicitly, and computed them in examples.

There are a number of questions that remain to be addressed. One question concerns the role of duality. In an ordinary Grassmannian, $G(k, n)$ is the same space as $G(n - k, n)$, and both ordinary and quantum cohomology of Grassmannians have presentations which respect that symmetry. By contrast, for tangent bundle deformations, our presentation is not yet symmetric. For example, the ring relations for $G(1, n)$ take a very different form from those of $G(n - 1, n)$. Strictly speaking, the B deformations encoded in

$$0 \longrightarrow S^* \otimes S \longrightarrow S^* \otimes V \longrightarrow \mathcal{E} \longrightarrow 0,$$

for $\mathcal{E}$ a deformation of the tangent bundle on $G(k, n)$, dualize to deformations encoded in

$$0 \longrightarrow Q^* \otimes Q \longrightarrow Q \otimes V^* \longrightarrow \mathcal{E}' \longrightarrow 0$$

on $G(n - k, n)$, but the second sequence above does not have a simple physical realization, and it is not immediately obvious how to translate this into a parameter map.

Another open matter concerns the loci V_m. We conjecture that on $G(k, n)$, for $m > k(n - k)$, the V_m are all identical to one another and to the discriminant locus, so that there are only finitely many components of the locus on which the quantum sheaf cohomology ring relations break down, but we do not yet have a proof.

Another open matter involves mathematical derivations. A purely mathematical derivation of the classical sheaf cohomology ring will appear in [35]. An analogous mathematical derivation of the quantum sheaf cohomology ring would technically involve sheaf theory manipulations on Quot schemes, and is left for the future.

6 Acknowledgements

A   Mathematical representation

In this appendix we will summarize some of the ideas that will appear in [35], including how representatives of sheaf cohomology of deformations of the tangent bundle of $G(k, n)$ can be represented by Young diagrams.

First, in order to determine the ring relations at order r, we take the rth exterior power of (4),

$$0 \to \wedge^r \mathcal{E}^* \to \wedge^r(\mathcal{V}^* \otimes \mathcal{S}) \to \wedge^{r-1}(\mathcal{V}^* \otimes \mathcal{S}) \otimes (\mathcal{S}^* \otimes \mathcal{S}) \to \wedge^{r-2}(\mathcal{V}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^2(\mathcal{S}^* \otimes \mathcal{S})$$
$$\to \cdots \to \mathcal{V}^* \otimes \mathcal{S} \otimes \mathrm{Sym}^{r-1}(\mathcal{S}^* \otimes \mathcal{S}) \to \mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}) \to 0.$$

Breaking up the exact sequence above into r short exact sequences, we get a connecting map

$$\delta_r : \ H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})) \ \longrightarrow \ H^r(\wedge^r \mathcal{E}^*)$$

by composing the connecting maps associated with all the short exact sequences. Thus the ring relations are encoded in the kernel of δ_r.

Now, $\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})$ can be written as a direct sum in the form

$$\underset{\mu}{\oplus} \left(K_\mu \mathcal{S}^* \otimes K_\mu \mathcal{S} \right),$$

where each μ is some Young diagram standing for an irreducible representation of $U(k)$ and K_μ is the corresponding Schur functor. The direct sum ranges over all the Young diagrams with r boxes and at most k rows. Since

$$K_\mu \mathcal{S}^* \otimes K_\mu \mathcal{S} \ \cong \ \mathrm{Hom}(K_\mu \mathcal{S}, K_\mu \mathcal{S}),$$

and each $H^0(K_\mu \mathcal{S}^* \otimes K_\mu \mathcal{S})$ is one-dimensional, a basis for $H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}))$ can be taken to be

$$\{\sigma_\mu \mid \mu \text{ is a Young diagram with } r \text{ boxes and at most } k \text{ rows }\},$$

where σ_μ is the identity bundle map on $K_\mu \mathcal{S}$.

The product on

$$\underset{r \geqslant 0}{\oplus} H^0\left(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})\right)$$

is defined to be the tensor product of bundle maps. Because

$$K_\lambda \mathcal{S} \otimes K_\mu \mathcal{S} = \bigoplus_\nu N_{\lambda\mu\nu} K_\nu \mathcal{S},$$

we see

$$\sigma_\lambda \otimes \sigma_\mu = \bigoplus_\nu N_{\lambda\mu\nu} \sigma_\nu. \tag{25}$$

The numbers $N_{\lambda\mu\nu}$ are determined by the Littlewood-Richardson rule. $N_{\lambda\mu\nu}$ is the number of ways the Young diagram λ can be expanded to the Young diagram ν by a strict μ-expansion. Note that (25) would remain unchanged if one replaced each bundle map σ_μ with the Schur polynomial corresponding to μ, and the tensor product with the usual product of polynomials. This implies that the sheaf cohomology

$$\underset{r\geqslant 0}{\oplus} H^0(\mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S}))$$

is isomorphic to the ring of symmetric polynomials with k variables, which are just the diagonal elements of the σ field, as described in section 3.1. Let's denote this ring by $A(k)$.

Because the connecting map

$$\delta : \ A(k) \ \longrightarrow \ \underset{r\geqslant 0}{\oplus} H^r(\wedge^r \mathcal{E}^*)$$

(given as the sum of all δ_r's) is a surjective homomorphism of $\mathbb{C}$-algebras as proved in [35], to determine the ring structure of the latter, we only need to find $\ker(\delta)$, the kernel of the connecting map. Thus we see, the sheaf cohomology has the following representation

$$\underset{r\geqslant 0}{\oplus} H^r\left(G(k,n), \wedge^r \mathcal{E}^*\right) \cong A(k)/\ker(\delta). \tag{26}$$

As in the text, we shall denote by $\sigma_{(r)}$ the Schur polynomial corresponding to the Young diagram with r boxes in a row. In [35], we have shown that, for a generic B deformation, $\ker(\delta)$ is generated by $R_{(r)}, r = n - k + 1, \cdots$, where

$$R_{(r)} = \sum_{i=0}^{\min\{r,n\}} I_i \sigma_{(r-i)} \sigma_{(1)}^i, \tag{27}$$

and I_i is the ith characteristic polynomial of B, which is defined through

$$\det(tI + B) = \sum_{i=0}^{n} I_{n-i} t^i.$$

This gives us the classical sheaf cohomology. (In appendix C, we give a toy model of the mathematical arguments for general cases that will appear in [35], and mathematically derive ring relations for trivial bundle deformations $B \propto I$.) In section 3.3, we derive the quantum sheaf cohomology for generic deformations from the one-loop effective action. The conclusion is, when quantum corrections are taken into account, we should replace $\ker(\delta)$ with the ideal generated by

$$R_{(r)} + q\sigma_{(r-n)}, \tag{28}$$

$r = n - k + 1, \cdots$, and $\sigma_{(m)}$ is defined to be zero when $m < 0$.

In section 3.2, we describe the quantum sheaf cohomology ring explicitly. Since

$$R_\mu \in \left\langle R_{(n-k+1)}, R_{(n-k+2)}, \cdots \right\rangle,$$

for Young diagrams μ with more than $n - k$ boxes in the first row and no more than k boxes in any column, our description of the ring structure suggests that the space

$$W_m = \mathrm{Span}_{\mathbb{C}} \left\{ R_\mu \,|\, \mu_1 > n - k, |\mu| = m \right\}$$

is in $\ker(\delta_m)$ for each $m > n - k$.

If the R_μ's are linearly independent, we see

$$\dim W_m = \dim \ker(\delta_m|_{B=0}),$$

thus $W_m = \ker(\delta_m)$. Our description is complete if this is true for all $m > n - k$.

If, on the other hand, there is some m, such that the R_μ's are linearly dependent, then W_m is only a proper subspace of $\ker(\delta_m)$. This situation occurs along a locus V_m with codimension at least one in the moduli space. If we expand R_μ in terms of σ_ν, we get

$$R_\mu = \sum_{|\nu|=m} C_{\mu\nu}^m \, \sigma_\nu$$

with some constants $C_{\mu\nu}^m$ depending on B. For fixed m, $C_{\mu\nu}^m$ form a matrix. So by definition, V_m is the common zero locus of all the p minors of this matrix, where p is the number of Young tableaux whose first row has more than $n - k$ boxes and with no more than k boxes in any column. Thus, along this locus, the dimension of the ring, and the dimensions of the corresponding sheaf cohomology groups, might jump, so we exclude this locus from our discussion of the quantum sheaf cohomology ring.

Now, since V_m is defined by the kernel of δ_m, which maps into degree m cohomology of $G(k, n)$, the dimensions of the sheaf cohomology groups we are interested in can potentially jump along the loci V_m for

$$n - k + 1 \leq m \leq k(n - k),$$

as there is no cohomology of degree greater than the dimension of the Grassmannian. Along loci V_m for $m > k(n - k)$, our description of the quantum sheaf cohomology ring may be incomplete, but the dimensions of the sheaf cohomology groups cannot jump.

As a practical matter, we can phrase this in terms of two different representations of the classical sheaf cohomology ring, as follows. One way to present the classical sheaf cohomology ring is

$$\mathbb{C} \left[\sigma_{(1)}, \cdots, \sigma_{(k(n-k))} \right] / \left\langle D_{k+1}, \cdots, R_{(n-k+1)}, \cdots, R_{(k(n-k))}, S_{k(n-k)+1}(k) \right\rangle,$$

where $S_r(k)$ is the ideal of degree r terms in k variables. This presentation of the classical sheaf cohomology ring is valid on

$$\mathcal{M} - \left(X \cup V_{n-k+1} \cup \cdots \cup V_{k(n-k)}\right)$$

where $\mathcal{M}$ is the space of all B deformations, X is the discriminant locus along which any k eigenvalues of B become -1 (equivalent to the locus $\{\Delta = 0\}$ for the Δ appearing explicitly in correlation functions), and V_r is defined as above. Along the V_r for the degrees above, the dimensions of the sheaf cohomology groups may potentially jump.

A second presentation of the same classical sheaf cohomology ring is

$$\mathbb{C}\left[\sigma_{(1)}, \sigma_{(2)}, \cdots\right] / \left\langle D_{k+1}, D_{k+2}, \cdots, R_{(n-k+1)}, R_{(n-k+2)}, \cdots \right\rangle,$$

which is closer to the form we have adopted in these notes for the quantum sheaf cohomology ring. This presentation is valid on

$$\mathcal{M} - \left(X \cup V_{n-k+1} \cup \cdots \cup \cdots\right).$$

In this case, the V_r for

$$n - k + 1 \ \leq \ r \ \leq \ k(n-k)$$

define loci along which the classical sheaf cohomology groups might jump, and the V_r for $r > k(n-k)$ define loci along which the dimensions of the sheaf cohomology groups cannot jump, but, for which the presentation above may not be accurate, as additional relations may be required.

B Products via homological algebra

It is possible to see at least the classical product structure on the (2,2) locus in homological algebra, by identifying sheaf cohomology groups with Ext groups and interpreting in terms of extensions of bundles, an idea also discussed in [21, 22]. In this appendix, we will look at that structure in simple cases, first describing how products in the ordinary cohomology of the Grassmannian can be computed via homological algebra, and later outlining some of the machinery needed to do analogous computations in the (0,2) case. We will not use this machinery elsewhere in this paper, but we felt it sufficiently interesting to include here as an appendix. Furthermore, because of its existence, we speculate that perhaps 'quantum sheaf cohomology' can be understood as a 'quantum homological algebra'.

Let us begin on the (2,2) locus, and describe products in the ordinary cohomology of a Grassmannian via homological algebra. On the (2,2) locus, as is well-known, the cohomology of $G(k,n)$ is generated by Chern classes of the (dual of the) universal subbundle S. In that spirit, we can identify c_1 with an element of $\text{Ext}^1(S^*, Q^*)$, corresponding to the complex

$$0 \ \longrightarrow \ Q^* \ \longrightarrow \ V^*$$

resolving S^*, where Q is the universal quotient bundle and V is a vector space, $G(k,n) = G(k,V)$. As a consistency check, note that there is a natural map

$$\mathrm{Ext}^1(S^*, Q^*) \longrightarrow H^1(S \otimes Q^* \otimes S \otimes S^* \otimes Q \otimes Q^*) = H^1\left(\Omega^1 \otimes \mathrm{End}\, T\right),$$

which displays how to map $\mathrm{Ext}^1(S^*, Q^*)$ to the sheaf cohomology group containing the Atiyah class of the tangent bundle, and is ultimately the reason that the identification of c_1 with an element of the Ext group above is sensible.

Recall from section 3.1 the product

$$\sigma_\square^2 = \sigma_{\boxminus\text{(horizontal)}} + \sigma_{\boxminus\text{(vertical)}}.$$

We can understand this in the present language as follows. The product $\sigma_\square^2$ should be understood as

$$(c_1)^2 \in \mathrm{Ext}^2\left(S^* \otimes S^*, Q^* \otimes Q^*\right),$$

corresponding to the resolution of $S^* \otimes S^*$ below,

$$0 \longrightarrow Q^* \otimes Q^* \longrightarrow Q^* \otimes V^* + Q^* \otimes V^* \longrightarrow V^* \otimes V^* \longrightarrow S^* \otimes S^* \longrightarrow 0,$$

given by squaring the resolution of S^*. This naturally decomposes into the sum of the following two resolutions:

$$0 \longrightarrow \wedge^2 Q^* \longrightarrow Q^* \otimes V^* \longrightarrow \mathrm{Sym}^2 V^* \longrightarrow \mathrm{Sym}^2 S^* \longrightarrow 0, \tag{29}$$

and

$$0 \longrightarrow \mathrm{Sym}^2 Q^* \longrightarrow Q^* \otimes V^* \longrightarrow \wedge^2 V^* \longrightarrow \wedge^2 S^* \longrightarrow 0. \tag{30}$$

If we identify the resolution (29) with

$$\sigma_{\boxminus\text{(horizontal)}} \in \mathrm{Ext}^2\left(\mathrm{Sym}^2 S^*, \wedge^2 Q^*\right),$$

and the resolution (30) with

$$\sigma_{\boxminus\text{(vertical)}} \in \mathrm{Ext}^2\left(\wedge^2 S^*, \mathrm{Sym}^2 Q^*\right),$$

then we recover

$$\sigma_\square^2 = \sigma_{\boxminus\text{(horizontal)}} + \sigma_{\boxminus\text{(vertical)}}.$$

As a consistency check, note that in the language of Atiyah classes, the cohomology classes above should live in

$$H^2\left(\Omega^2 \otimes \mathrm{End}\, T\right) = H^2\left(\wedge^2(S \otimes Q^*) \otimes S \otimes S^* \otimes Q \otimes Q^*\right),$$

and as

$$\wedge^2(S \otimes Q^*) = \wedge^2 S \otimes \mathrm{Sym}^2 Q^* + \mathrm{Sym}^2 S \otimes \wedge^2 Q^*,$$

we see that both $\sigma_{\square\square}$ and $\sigma_{\square}$ naturally map into $H^2(\Omega^2 \otimes \mathrm{End}\, T)$, as expected.

Now, let us turn to the (0,2) case. Here, we have a deformation $\mathcal{E}$ of the tangent bundle defined by a short exact sequence

$$0 \longrightarrow \mathcal{E}^* \longrightarrow V^* \otimes S \longrightarrow S^* \otimes S \longrightarrow 0.$$

Applying $\mathrm{Hom}(\mathcal{O}, -)$ to the sequence above, one gets

$$0 \to \mathrm{Hom}(\mathcal{O}, \mathcal{S}^* \otimes \mathcal{S}) \xrightarrow{\delta} \mathrm{Ext}^1(\mathcal{O}, \mathcal{E}^*).$$

In the rest of this section, we will begin to outline some of the machinery needed to compute classical sheaf cohomology rings via homological algebra.

Note that from the construction of the connecting morphism, one finds that[2] for any $\varphi \in \mathrm{Hom}(\mathcal{O}, \mathcal{S}^* \otimes \mathcal{S})$, one has $\delta(\varphi) = [E_\varphi] \in \mathrm{Ext}^1(\mathcal{O}, \mathcal{E}^*)$, where

$$E_\varphi : 0 \to \mathcal{E}^* \to \mathcal{Z} \to \mathcal{O} \to 0$$

is constructed from the pullback diagram

$$
\begin{array}{ccc}
\mathcal{Z} & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow{\scriptstyle \varphi} \\
\mathcal{V}^* \otimes \mathcal{S} & \longrightarrow & \mathcal{S}^* \otimes \mathcal{S}
\end{array}
\quad,
$$

which fits in

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^* & \longrightarrow & \mathcal{Z} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & \mathcal{E}^* & \longrightarrow & \mathcal{V}^* \otimes \mathcal{S} & \longrightarrow & \mathcal{S}^* \otimes \mathcal{S} & \longrightarrow & 0
\end{array}
\quad .
$$

Applying the functor Sym^2 to $\mathcal{O} \xrightarrow{\varphi} \mathcal{S}^* \otimes \mathcal{S}$, we get the induced map

$$\mathcal{O} \xrightarrow{\mathrm{Sym}^2 \varphi} \mathrm{Sym}^2(\mathcal{S}^* \otimes \mathcal{S}).$$

It is easy to construct the induced extension sequence following the method in last section, and there is no difficulty to do similar things for any map $\mathcal{O} \to \mathrm{Sym}^r(\mathcal{S}^* \otimes \mathcal{S})$.

Now denote the image of $Id : K_\lambda \mathcal{S} \to K_\lambda \mathcal{S}$ in $H^0(K_\lambda \mathcal{S}^* \otimes K_\lambda \mathcal{S})$ and $\mathrm{Hom}(\mathcal{O}, K_\lambda \mathcal{S}^* \otimes K_\lambda \mathcal{S})$ by κ_λ. We strongly suspect, but have not carefully checked, that

- The multiplication $\kappa_1 \cdot \kappa_1 \cdots \cdots \kappa_1 := \mathrm{Sym}^r \kappa_1$ agrees with the ring structure of $H^\bullet(\wedge^\bullet \mathcal{E}^*)$.

---

[2] See the remark after Theorem III.5.2 of [44].

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- The multiplication $\kappa_\alpha \cdot \kappa_\beta = \Phi(\kappa_\alpha \otimes \kappa_\beta)$ agrees with the ring structure of $H^\bullet(\wedge^\bullet \mathcal{E}^*)$, where Φ is the symmetrization map

$$\mathrm{Sym}^{|\alpha|}(\mathcal{S}^* \otimes \mathcal{S}) \otimes \mathrm{Sym}^{|\beta|}(\mathcal{S}^* \otimes \mathcal{S}) \to \mathrm{Sym}^{|\alpha|+|\beta|}(\mathcal{S}^* \otimes \mathcal{S}).$$

Now, let us sketch out some computations in the case $n = 2$.

Let $\varphi_1, \varphi_2 \in \mathrm{Hom}(\mathcal{O}, \mathcal{S}^* \otimes \mathcal{S})$, $\varphi = \varphi_1 \otimes \varphi_2 \in \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^2(\mathcal{S}^* \otimes \mathcal{S}))$. We want to compare $E_{\varphi_1} \cdot E_{\varphi_2}$ with $E_\varphi^2 \in \mathrm{Ext}^2(\mathcal{O}, \wedge^2 \mathcal{E}^*)$.

Note that for $i = 1, 2$ we have

$$E_{\varphi_i} : 0 \to \mathcal{E}^* \to F_i \to \mathcal{O} \to 0$$

and tensoring with $\mathcal{E}^*$, we have

$$0 \to \mathcal{E}^* \otimes \mathcal{E}^* \to F_2 \otimes \mathcal{E}^* \to \mathcal{E}^* \to 0.$$

This gives

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^* \otimes \mathcal{E}^* & \longrightarrow & F_2 \otimes \mathcal{E}^* & \longrightarrow & \mathcal{E}^* & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
\tilde{E}_{\varphi_2} : & 0 & \longrightarrow & \wedge^2 \mathcal{E}^* & \longrightarrow & \tilde{F}_2 & \longrightarrow & \mathcal{E}^* & \longrightarrow 0
\end{array}
$$

where the first square is a push-out diagram defining $\tilde{F}_2$.

So we can define $E_{\varphi_1} \cdot E_{\varphi_2}$ as $E_{\varphi_1} \cdot \tilde{E}_{\varphi_2}$ which is represented by

$$0 \to \wedge^2 \mathcal{E}^* \to \tilde{F}_2 \to F_1 \to \mathcal{O} \to 0.$$

On the other hand, for φ we have E_φ represented by

$$0 \to \wedge^2 \mathcal{E}^* \to \wedge^2(\mathcal{V}^* \otimes \mathcal{S}) \to F \to \mathcal{O} \to 0.$$

We claim that $[E_{\varphi_1} \cdot E_{\varphi_2}] = [E_\varphi]$ in $\mathrm{Ext}^2(\mathcal{O}, \wedge^2 \mathcal{E}^*)$.

We can see this as follows. Recall that two n-extensions H_0 and H_m are equivalent iff they are connected by some morphisms of complexes: $H_0 \to H_1 \leftarrow H_2 \ldots \leftarrow H_m$, where each $H_a \to H_b$, $(a, b) = (i, i+1)$ or $(a, b) = (i+1, i)$ is of the form

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & B & \longrightarrow & E_n^a & \longrightarrow & \ldots & \longrightarrow & E_1^a & \longrightarrow & A & \longrightarrow 0 \\
& & \| & & \downarrow & & & & \downarrow & & \| & \\
0 & \longrightarrow & B & \longrightarrow & E_n^b & \longrightarrow & \ldots & \longrightarrow & E_1^b & \longrightarrow & A & \longrightarrow 0
\end{array}
\tag{31}
$$

.

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Let $\mathcal{Z} = \mathcal{V}^* \otimes \mathcal{S}$ and $\mathcal{E}nd = End(\mathcal{S}) = \mathcal{S}^* \otimes \mathcal{S}$. We need to show that there exist maps α and β such that all squares in the following diagram are commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^* \otimes \mathcal{E}^* & \longrightarrow & F_2 \otimes \mathcal{E}^* & \longrightarrow & F_1 & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & \wedge^2 \mathcal{E}^* & \longrightarrow & \tilde{F}_2 & \longrightarrow & F_1 & \xrightarrow{p_1} & \mathcal{O} & \longrightarrow & 0 \\
& & \| & & \vdots\,\beta & & \vdots\,\alpha & & \| & & \\
0 & \longrightarrow & \wedge^2 \mathcal{E}^* & \longrightarrow & \wedge^2 \mathcal{Z} & \longrightarrow & F & \xrightarrow{p_2} & \mathcal{O} & \longrightarrow & 0 \\
& & \| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^2 \mathcal{E}^* & \longrightarrow & \wedge^2 \mathcal{Z} & \longrightarrow & \mathcal{Z} \otimes \mathcal{E}nd & \longrightarrow & \mathrm{Sym}^2 \mathcal{E}nd & \longrightarrow & 0
\end{array}
\tag{32}
$$

For α, notice that we have a map $\alpha_0 : F_1 \to \mathcal{Z} \otimes \mathcal{E}nd$ constructed as in

$$
\begin{array}{ccc}
F_1 & \longrightarrow & \mathcal{O} \\
\downarrow & & \downarrow{\scriptstyle \varphi_1 \otimes 1} \\
\mathcal{Z} \otimes \mathcal{O} & \longrightarrow & \mathcal{E}nd \otimes \mathcal{O} \\
\downarrow & & \downarrow{\scriptstyle 1 \otimes \varphi_2} \\
\mathcal{Z} \otimes \mathcal{E}nd & \longrightarrow & \mathcal{E}nd \otimes \mathcal{E}nd \\
\| & & \downarrow{\scriptstyle s} \\
\mathcal{Z} \otimes \mathcal{E}nd & \longrightarrow & \mathrm{Sym}^2 \mathcal{E}nd
\end{array}
\qquad .
$$

Also, the composition of the maps of second column is exactly φ. Since the square at the lower right corner of (32) is a pull-back square, there exists α such that α_0 factors through α.

Similarly, we can construct $F_2 \otimes \mathcal{E}^* \to \wedge^2 \mathcal{Z}$ canonically via:

$$
\begin{array}{ccc}
\mathcal{E}^* \otimes \mathcal{E}^* & \longrightarrow & F_2 \otimes \mathcal{E}^* \\
\| & & \downarrow \\
\mathcal{E}^* \otimes \mathcal{E}^* & \longrightarrow & \mathcal{Z} \otimes \mathcal{E}^* \\
\| & & \downarrow \\
\mathcal{E}^* \otimes \mathcal{E}^* & \longrightarrow & \mathcal{Z} \otimes \mathcal{Z} \\
\downarrow & & \downarrow \\
\wedge^2 \mathcal{E}^* & \longrightarrow & \wedge^2 \mathcal{Z}
\end{array}
\qquad .
$$

Since the square at the upper left corner of (32) is a push-forward square, there exists β such that β_0 factors through β.

Now it suffices to prove that the central square of (32) is commutative. But we have already know that the second and third rows of (32) are exact. So the commutativity of the central square is the consequence of that of

$$
\begin{array}{ccc}
\tilde{F}_2 \longrightarrow S_1^1 \\
\downarrow \qquad \downarrow \\
\wedge^2 \mathcal{Z} \longrightarrow S_1^2
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
S_1^1 \longrightarrow F_1 \\
\downarrow \qquad \downarrow \\
S_1^2 \longrightarrow F
\end{array} \, ,
$$

where $S_1^i = Ker\ p_i, i = 1, 2$.

Next, we turn to general n.

We want to prove the following diagram is commutative:

$$
\begin{array}{ccc}
(H^0(\mathrm{Sym}^r(\mathcal{E}nd)), H^0(\mathrm{Sym}^r(\mathcal{E}nd))) \longrightarrow H^0(\mathrm{Sym}^{r+s}(\mathcal{E}nd)) \, . \\
\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \\
(H^r(\wedge^r \mathcal{E}^*), H^s(\wedge^s \mathcal{E}^*)) \longrightarrow H^{r+s}(\wedge^{r+s} \mathcal{E}^*)
\end{array}
$$

We do so by identifying it with

$$
\begin{array}{ccc}
(\mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^r(\mathcal{E}nd)), \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^s(\mathcal{E}nd))) \longrightarrow \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^{r+s}(\mathcal{E}nd)) \, . \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
(\mathrm{Ext}^r(\mathcal{O}, \wedge^r \mathcal{E}^*), \mathrm{Ext}^s(\mathcal{O}, \wedge^s \mathcal{E}^*)) \longrightarrow \mathrm{Ext}^{r+s}(\mathcal{O}, \wedge^{r+s} \mathcal{E}^*)
\end{array}
\qquad (33)
$$

We claim that the diagram (33) is commutative. We can see this as follows. Take $\varphi_1 \in \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^r(\mathcal{E}nd))$, $\varphi_2 \in \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^s(\mathcal{E}nd))$. Define the multiplication

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^r(\mathcal{E}nd)) \times \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^s(\mathcal{E}nd)) & \to & \mathrm{Hom}(\mathcal{O}, \mathrm{Sym}^{r+s}(\mathcal{E}nd)) \\
(\varphi_1, \varphi_2) & \mapsto & \varphi
\end{array} \, ,
$$

where φ is defined by the composition

$$
\mathcal{O} \xrightarrow{\varphi_1 \otimes 1} \mathrm{Sym}^r(\mathcal{E}nd) \otimes \mathcal{O} \xrightarrow{1 \otimes \varphi_2} \mathrm{Sym}^r(\mathcal{E}nd) \otimes \mathrm{Sym}^s(\mathcal{E}nd) \xrightarrow{\text{symmetrize}} \mathrm{Sym}^{r+s}(\mathcal{E}nd).
$$

Up to a minus sign, we have a sequence E_{φ_2} representing $\Delta(\varphi_2) \in \mathrm{Ext}^s(\mathcal{O}, \wedge^s \mathcal{E}^*)$

$$
\begin{array}{ccccccccc}
E_{\varphi_2}: & 0 \longrightarrow \wedge^s \mathcal{E}^* \longrightarrow & \cdots & \longrightarrow F_2 & \longrightarrow & \mathcal{O} & \longrightarrow 0 \\
& \qquad\qquad \| & & \downarrow & & \downarrow \varphi_2 & \\
& 0 \longrightarrow \wedge^s \mathcal{E}^* \longrightarrow & \cdots \longrightarrow & \mathcal{Z} \otimes \mathrm{Sym}^{s-1} \mathcal{E}nd & \longrightarrow & \mathrm{Sym}^s \mathcal{E}nd & \longrightarrow 0
\end{array}
\qquad (34)
$$

and similarly we have E_{φ_1} and E_φ for $\Delta(\varphi_1)$ and $\Delta(\varphi)$, with the pull-back sheaf F_2 replaced by F_1 and F respectively.

Tensoring (34) with $\wedge^r \mathcal{E}^*$, we get

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \wedge^s \mathcal{E}^* \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^s \mathcal{Z} \otimes \wedge^r \mathcal{E}^* & \longrightarrow & ... & \longrightarrow & F_2 \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^r \mathcal{E}^* \longrightarrow 0 \\
& & \downarrow & & \downarrow & & & & \| & & \| \\
\tilde{E}_{\varphi_1} : 0 & \longrightarrow & \wedge^{r+s} \mathcal{E}^* & \longrightarrow & \tilde{F}_2 & \longrightarrow & ... & \longrightarrow & F_2 \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^r \mathcal{E}^* \longrightarrow 0
\end{array}$$

(35)

where the first square is a push-out diagram.

To verify $[E_{\varphi_1} \cdot E_{\varphi_2}] = [E_\varphi]$ in $\mathrm{Ext}^{r+s}(\mathcal{O}, \wedge^{r+s} \mathcal{E}^*)$, we break the sequence E_φ into

$$0 \to S_r \to ... \to \mathcal{Z} \otimes \mathrm{Sym}^{r+s-1} \mathcal{E}nd \to \mathrm{Sym}^{r+s} \mathcal{E}nd \to 0$$

and

$$0 \to \wedge^{r+s} \mathcal{E}^* \to \wedge^{r+s} \mathcal{Z} \to ... \to S_r \to 0.$$

As a first step, we show that squares in the last two rows of the following diagram are commutative:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \wedge^s \mathcal{E}^* \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^s \mathcal{Z} \otimes \wedge^r \mathcal{E}^* & \longrightarrow & ... & \longrightarrow & F_2 \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^r \mathcal{E}^* \longrightarrow 0 \\
& & \downarrow & & \downarrow & & & & \| & & \| \\
0 & \longrightarrow & \wedge^{r+s} \mathcal{E}^* & \longrightarrow & \tilde{F}_2 & \longrightarrow & ... & \longrightarrow & F_2 \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^r \mathcal{E}^* \longrightarrow 0 \\
& & \| & & \vdots \, \beta_s & & & & \vdots \, \beta_1 & & \vdots \, \beta_0 \\
0 & \longrightarrow & \wedge^{r+s} \mathcal{E}^* & \longrightarrow & \wedge^{r+s} \mathcal{Z} & \longrightarrow & ... & \longrightarrow & \wedge^{r+1} \mathcal{Z} \otimes \mathrm{Sym}^{s-1} \mathcal{E}nd & \longrightarrow & S_r \longrightarrow 0
\end{array}$$

To see this, we first need to define the maps β_j, $j = 0, ..., s$. Since the upper left square is a push-out square, to define β_s it suffices to find a map $\beta_{s,0}$ such that

$$\begin{array}{ccc}
\wedge^s \mathcal{E}^* \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^s \mathcal{Z} \otimes \wedge^r \mathcal{E}^* \\
\downarrow & & \downarrow \beta_{s,0} \\
\wedge^{r+s} \mathcal{E}^* & \longrightarrow & \wedge^{r+s} \mathcal{Z}
\end{array}$$

commutes. But we have a canonical choice of $\beta_{s,0}$, namely

$$\wedge^s \mathcal{Z} \otimes \wedge^r \mathcal{E}^* \to \wedge^s \mathcal{Z} \otimes \wedge^r \mathcal{Z} \to \wedge^{r+s} \mathcal{Z}$$

with obvious maps. For $j = s-1, s-2, ..., 2$, β_j is the canonical map

$$\begin{array}{ccc}
\wedge^j \mathcal{Z} \otimes \wedge^r \mathcal{E}^* \otimes \mathrm{Sym}^{s-j} \mathcal{E}nd & \to & \wedge^{r+j} \mathcal{Z} \otimes \mathrm{Sym}^{s-j} \mathcal{E}nd \\
u \otimes v \otimes w & \mapsto & u \wedge v \otimes w
\end{array}$$

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and it is easy to see that squares with vertical maps in $\{\beta_j | j = s - 1, s - 2, ..., 2\}$ are commutative.

To see that the square containing β_s and β_{s-1} is commutative, we use the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \wedge^s \mathcal{E}^* \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^s \mathcal{Z} \otimes \wedge^r \mathcal{E}^* & \xrightarrow{d_s} & \wedge^{s-1} \mathcal{Z} \otimes \wedge^r \mathcal{E}^* \otimes \mathcal{E}nd \\
& & \downarrow{\scriptstyle \beta'_{s+1}} & & \downarrow{\scriptstyle \beta'_s} & & \| {\scriptstyle \beta'_{s-1}} \\
0 & \longrightarrow & \wedge^{r+s} \mathcal{E}^* & \longrightarrow & \tilde{F}_2 & \xrightarrow{\tilde{d}_s} & \wedge^{s-1} \mathcal{Z} \otimes \wedge^r \mathcal{E}^* \otimes \mathcal{E}nd \\
& & \| & & \downarrow{\scriptstyle \beta_s} & & \downarrow{\scriptstyle \beta_{s-1}} \\
0 & \longrightarrow & \wedge^{r+s} \mathcal{E}^* & \longrightarrow & \wedge^{r+s} \mathcal{Z} & \xrightarrow{D_s} & \wedge^{r+s-1} \mathcal{Z} \otimes \mathcal{E}nd
\end{array}
\quad . \qquad (36)
$$

By the commutativity of the lower right square and the square containing d_s and D_s, $\beta_{s-1} \circ \tilde{d}_s \circ \beta'_s = \beta_{s-1} \circ \beta'_{s-1} \circ d_s = D_s \circ \beta_s \circ \beta'_s$.

Since $\tilde{F}_2$ is a push-out, and β'_{s+1} is surjective, so is $\tilde{\beta}'_s$. Hence we have $\beta_{s-1} \circ \tilde{d}_s = D_s \circ \beta_s$. Hence the square containing β_s and β_{s-1} is commutative.

Now we define β_1, β_0. They are canonically defined by the second and third columns of the following diagram:

$$
\begin{array}{ccccc}
\wedge^2 \mathcal{Z} \otimes \mathrm{Sym}^{s-2} \mathcal{E}nd \otimes \wedge^r \mathcal{E}^* & \longrightarrow & F_2 \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \wedge^r \mathcal{E}^* \\
\downarrow & & \downarrow & & \downarrow \\
\wedge^2 \mathcal{Z} \otimes \mathrm{Sym}^{s-2} \mathcal{E}nd \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \mathcal{Z} \otimes \mathrm{Sym}^{s-1} \mathcal{E}nd \otimes \wedge^r \mathcal{E}^* & \longrightarrow & \mathrm{Sym}^s \mathcal{E}nd \otimes \wedge^r E^* \\
\downarrow & & \downarrow & & \downarrow \\
\wedge^{r+2} \mathcal{Z} \otimes \mathrm{Sym}^{s-2} \mathcal{E}nd & \longrightarrow & \wedge^{r+1} \mathcal{Z} \otimes \mathrm{Sym}^{s-1} \mathcal{E}nd & \longrightarrow & \wedge^r \mathcal{Z} \otimes \mathrm{Sym}^s \mathcal{E}nd.
\end{array}
$$

Commutativity of squares involving β_1, β_0 are obvious.

The next step is to show that squares in the first two rows of the following diagram are commutative:

$$
\begin{array}{ccccccccc}
\wedge^r \mathcal{E}^* \otimes \mathcal{O} & \hookrightarrow & \wedge^r \mathcal{Z} \otimes \mathcal{O} & \longrightarrow & \cdots & \longrightarrow & F_1 \otimes \mathcal{O} & \longrightarrow\!\!\!\!\rightarrow & \mathcal{O} \\
\downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \alpha_r} & & & & \downarrow{\scriptstyle \alpha_1} & & \| \\
S_r & \hookrightarrow & \wedge^r \mathcal{Z} \otimes \mathrm{Sym}^s \mathcal{E}nd & \longrightarrow & \cdots & \longrightarrow & F & \longrightarrow\!\!\!\!\rightarrow & \mathcal{O} \\
\| & & \| & & & & \downarrow{\scriptstyle \alpha'_1} & & \downarrow{\scriptstyle \varphi} \\
S_r & \hookrightarrow & \wedge^r \mathcal{Z} \otimes \mathrm{Sym}^s \mathcal{E}nd & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z} \otimes \mathrm{Sym}^{r+s-1} \mathcal{E}nd & \longrightarrow & \mathrm{Sym}^{r+s} \mathcal{E}nd
\end{array}
\quad .
$$

For $j = r, r-1, ..., 2$, α_j is defined by $(\wedge^j \mathcal{Z} \otimes \mathrm{Sym}^{r-j}\mathcal{E}nd) \otimes \mathcal{O} \xrightarrow{1 \otimes \varphi_2} (\wedge^j \mathcal{Z} \otimes \mathrm{Sym}^{r-j}\mathcal{E}nd) \otimes$ $\mathrm{Sym}^s \mathcal{E}nd \xrightarrow{\text{symmetrization}} \wedge^j \mathcal{Z} \otimes \mathrm{Sym}^{r+s-j}\mathcal{E}nd$, where symmetrization is the restriction of the projection $\mathcal{E}nd^{\otimes r+s-j} \to \mathrm{Sym}^{r+s-j}\mathcal{E}nd$ to $\mathrm{Sym}^{r-j}\mathcal{E}nd \otimes \mathrm{Sym}^s \mathcal{E}nd \subset \mathcal{E}nd^{\otimes r+s-j}$. The squares involving them are obviously commutative.

The following diagram and the universal property of F defines α_1.

$$
\begin{array}{ccc}
F_1 \otimes \mathcal{O} & \longrightarrow & \mathcal{O} \otimes \mathcal{O} \\
\downarrow & & \downarrow{\scriptstyle \varphi_1 \otimes 1} \\
\mathcal{Z} \otimes \mathrm{Sym}^{r-1}\mathcal{E}nd & \longrightarrow & \mathrm{Sym}^r \mathcal{E}nd \otimes \mathcal{O} \\
\downarrow & & \downarrow{\scriptstyle 1 \otimes \varphi_2} \\
\mathcal{Z} \otimes \mathrm{Sym}^{r-1}\mathcal{E}nd \otimes \mathrm{Sym}^s \mathcal{E}nd & \longrightarrow & \mathrm{Sym}^r \mathcal{E}nd \otimes \mathrm{Sym}^s \mathcal{E}nd \\
\downarrow & & \downarrow \\
\mathcal{Z} \otimes \mathrm{Sym}^{r+s-1}\mathcal{E}nd & \longrightarrow & \mathrm{Sym}^{r+s}\mathcal{E}nd
\end{array} \quad .
$$

Then an argument dual to the one regarding (36) shows that the square containing α_1 and α_2 is commutative.

Combining the two steps above, we have constructed a morphism of complexes, which shows the desired equality of extension classes in view of (31).

C   Some identities for trivial deformations

In this section we will mathematically derive ring relations in the special case of a trivial (0,2) deformation defined by a nonzero $B \propto I$. For reasons discussed in section 2, this deformation does not change the tangent bundle. This is, in essence, a toy model of the mathematical arguments for general cases that will appear in [35].

The idea is to run the relations in the ordinary cohomology ring of the Grassmannian, through the isomorphism between the 'deformed' tangent bundle and the standard presentation of the tangent bundle, to get a prototype for the classical sheaf cohomology ring relations in more general cases.

To this end, we define a map

$$
h: S^* \otimes S \longrightarrow S^* \otimes S
$$

that makes the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S^* \otimes S & \longrightarrow & S^* \otimes V & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle h} & & \| & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & S^* \otimes S & \longrightarrow & S^* \otimes V & \longrightarrow & T & \longrightarrow & 0
\end{array}
$$

commute, where $\mathcal{E}$ is, formally, the bundle corresponding to the (trivial) deformation $B = \varepsilon I$. By inspection,

$$
h(x) = x + \varepsilon \sigma_{(1)},
$$

and h extends to higher symmetric powers of $S^* \otimes S$ in the obvious way.

In the ordinary cohomology of $G(k, n)$, the only nonzero cohomology classes correspond to Young diagrams inside a $k \times (n - k)$ box. Young diagrams extending outside of that box should correspond to vanishing cohomology classes, and define relations in the cohomology ring. To that end, we will consider relations defined by Young diagrams with one row of boxes, extending outside of the $k \times (n - k)$ box.

To that end, when $B = \varepsilon I$, we claim that

$$
h(\sigma_{(r)}) = \sum_{i=0}^{r} \binom{k + r - 1}{i} \sigma_{(r-i)} \sigma_{(1)}^i \varepsilon^i. \tag{37}
$$

For notational reasons, as we will be mixing Schur polynomials σ_μ corresponding to Young diagrams and Coulomb branch basis elements σ_a, which could become confusing, in this appendix we will use the notation x_a rather than σ_a for Coulomb basis elements to help distinguish the two.

Then,

$$
\sigma_{(r)} = \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_k = r} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k},
$$

$$
h(\sigma_{(r)}) = \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_k = r} (x_1 + \varepsilon \sigma_{(1)})^{\alpha_1} \cdots (x_k + \varepsilon \sigma_{(1)})^{\alpha_k},
$$

$$
= \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_k = r} \left[\left(\sum_{i_1=0}^{\alpha_1} \binom{\alpha_1}{i_1} x_1^{\alpha_1 - i_1} \sigma_{(1)}^{i_1} \varepsilon^{i_1} \right) \cdots \left(\sum_{i_k=0}^{\alpha_k} \binom{\alpha_k}{i_k} x_k^{\alpha_k - i_k} \sigma_{(1)}^{i_k} \varepsilon^{i_k} \right) \right],
$$

$$
= \sum_{\alpha_1 + \alpha_2 + \cdots + \alpha_k = r} \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_k=0}^{\alpha_k} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_k}{i_k} x_1^{\alpha_1 - i_1} \cdots x_k^{\alpha_k - i_k} \sigma_{(1)}^{i_1 + \cdots + i_k} \varepsilon^{i_1 + \cdots + i_k}.
$$

The coefficient of $\varepsilon^i \sigma_{(1)}^i$, which we denote by g_i, is

$$g_i = \sum_{\alpha_1+\alpha_2+\cdots+\alpha_k=r} \sum_{i_1+\cdots+i_k=i} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_k}{i_k} x_1^{\alpha_1-i_1} \cdots x_k^{\alpha_k-i_k},$$

$$= \sum_{\beta_1+\beta_2+\cdots+\beta_k=r-i} \sum_{i_1+\cdots+i_k=i} \binom{\beta_1+i_1}{i_1} \cdots \binom{\beta_k+i_k}{i_k} x_1^{\beta_1} \cdots x_k^{\beta_k}.$$

From the combinatorial formula

$$\sum_{i_1+\cdots+i_k=i} \binom{\beta_1+i_1}{i_1}\binom{\beta_2+i_2}{i_2} \cdots \binom{\beta_k+i_k}{i_k} = \binom{\beta_1+\beta_2+\cdots+\beta_k+k-1+i}{i},$$

we get

$$g_i = \sum_{\beta_1+\beta_2+\cdots+\beta_k=r-i} \binom{\beta_1+\beta_2+\cdots+\beta_k+k-1+i}{i} x_1^{\beta_1} \cdots x_k^{\beta_k},$$

$$= \binom{r-i+k-1+i}{i} \sum_{\beta_1+\beta_2+\cdots+\beta_k=r-i} x_1^{\beta_1} \cdots x_k^{\beta_k},$$

$$= \binom{k+r-1}{i} \sigma_{(r-i)},$$

and

$$h(\sigma_{(r)}) = \sum_{i=0}^{r} g_i \sigma_{(1)}^i \varepsilon^i = \sum_{i=0}^{r} \binom{k+r-1}{i} \sigma_{(r-i)} \sigma_{(1)}^i \varepsilon^i.$$

With some combinatorics one can then show

$$h(\sigma_{(r)}) = \sum_{j=0}^{k+r-n-1} \sum_{i=j}^{\min\{n+j,r\}} \varepsilon^j \binom{k+r-n-1}{j} I_{i-j} \sigma_{(r-i)} \sigma_{(1)}^i. \tag{38}$$

Now, we then claim that the kernel of the connecting map

$$\bigoplus_{r=0}^{k(n-k)} H_0^0\left(G(k,n), Sym^r(S^* \otimes S)\right) \longrightarrow H^*(G(k,n), \wedge^* \mathcal{E}^\vee)$$

can be generated by $R_{(r)}, r = n-k+1, n-k+2, \cdots$, where

$$R_{(r)} = \sum_{i=0}^{\min\{r,n\}} I_i \sigma_{(r-i)} \sigma_{(1)}^i,$$

with $I_i, i = 0, 1, \cdots, n$, the characteristic polynomials of B.

To show this, first note that the Giambelli formula

$$\sigma_\lambda = \det \begin{pmatrix} \sigma_{(\lambda_1)} & \sigma_{(\lambda_1+1)} & \cdots & \sigma_{(\lambda_1+k-1)} \\ \sigma_{(\lambda_2-1)} & \sigma_{(\lambda_2)} & \cdots & \sigma_{(\lambda_2+k-2)} \\ \cdots\cdots\cdots\cdots \\ \sigma_{(\lambda_k+k-1)} & \sigma_{(\lambda_k+k)} & \cdots & \sigma_{(\lambda_k)} \end{pmatrix}$$

implies

$$h(\sigma_\lambda) = h(\sigma_{(\lambda_1)})F_1 + h(\sigma_{(\lambda_1+1)})F_2 + \cdots + h(\sigma_{(\lambda_1+k-1)})F_k, \tag{39}$$

where the F_i's are polynomials of the $\sigma_{(j)}$'s. If $h(\sigma_\lambda)$ is in the kernel of the connecting map, then $\lambda_1 \geqslant n - k + 1$. Thus, $h(\sigma_{(r)}), r = n - k + 1, \cdots, k(n - k)$ generate the kernel. Furthermore, one can write (38) as

$$h(\sigma_{(r)}) = \sum_{j=0}^{k+r-n-1} \varepsilon^j \binom{k+r-n-1}{j} \left(\sum_{i=0}^{\min\{n,r-j\}} I_i \sigma_{(r-j-i)} \sigma_{(1)}^i \right) \sigma_{(1)}^j,$$

$$= \sum_{j=0}^{k+r-n-1} \varepsilon^j \binom{k+r-n-1}{j} \sigma_{(1)}^j R_{r-j},$$

for $r = n - k + 1, \cdots, k(n - k)$, which implies that the $R_{(r)}$'s generate the kernel, as claimed.

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