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Exact Results In Two-Dimensional (2,2) Supersymmetric Gauge Theories With Boundary

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Abstract

We compute the partition function on the hemisphere of a class of two-dimensional (2,2) supersymmetric field theories including gauged linear sigma models. The result provides a general exact formula for the central charge of the D-brane placed at the boundary. It takes the form of Mellin-Barnes integral and the question of its convergence leads to the grade restriction rule concerning branes near the phase boundaries. We find expressions in various phases including the large volume formula in which a characteristic class called the Gamma class shows up. The two sphere partition function factorizes into two hemispheres glued by inverse to the annulus. The result can also be written in a form familiar in mirror symmetry, and suggests a way to find explicit mirror correspondence between branes.

Contents

1 Introduction

Localization has been a powerful tool to obtain exact results in supersymmetric quantum field theories since the very beginning [1]. Recently, the idea was applied to a part of superconformal symmetry which exists for a class of spacetime even if the theory is not conformally invariant. After the pioneering work by Pestun [2], several important results are obtained in three, four and five dimensions, and qualitatively new information of the respective theory is obtained. Along this line, the partition function on the two sphere of $(2,2)$ supersymmetric gauge theories was computed by Benini et al [3] and Doroud et al [4]. Interestingly, it was observed in some examples [5] that it computes the Kähler potential for the family of superconformal fixed points of the theory.

These developments motivate us to study the partition function on the hemisphere of two-dimensional $(2,2)$ supersymmetric gauge theories. The result will surely depend on the choice of boundary condition, or the D-brane, at the boundary of the hemisphere. There is a chance that it will tell us something non-trivial about D-branes or even about the theory itself. In this paper, we formulate $(2,2)$ supersymmetric gauge theories on the hemisphere, compute the partition function based on localization, and study some of its properties. Our main target is the class of theories called the gauged linear sigma models with the supersymmetry that admits the type of branes called the B-branes at the boundary.

One obvious question is: what does it compute? We find in general that it depends holomorphically on twisted chiral parameters and has no dependence on chiral parameters. This suggests that it is the central charge of the D-brane. Indeed, after the computation, we find that it agrees with the central charge whenever the computation can be completed on both sides. In particular, the match can be made in various phases of the theories. In the Landau-Ginzburg orbifold point, the result agrees with the formula for the central charge proposed in [6]. In the geometric phase, in which the theory reduces to the non-linear sigma model with a Kähler target space X, the large volume limit of our result is

$$Z_{D^2}(\mathcal{B}) \;=\; \int_X \widehat{\Gamma}_X\, \mathrm{e}^{B+\frac{\mathrm{i}}{2\pi}\omega}\mathrm{ch}(\mathcal{B}) \,+\, \cdots \tag{1.1}$$

where $\widehat{\Gamma}_X$ is a characteristic class of X called the Gamma class [7–10], B and ω are the B-field and the Kähler class of X respectively. This is the expected behaviour for the central charge of the brane. $+\cdots$ is the worldsheet instanton corrections, and we provide a precise form of such corrections, to all orders in the instanton number. In a class of theories, the exact expression for the central charge is known by mirror symmetry and/or by detailed instanton calculus. Our results matches with that in such cases. In other cases where the expression is not known, our result can be regarded as a prediction for the central charge.

The conjecture of [5] and our observation suggests that the two sphere partition function can be factorized into two hemispheres glued by the inverse to the annulus. We show that this is indeed the case in the geometric phase in which the formula for the annulus is known. We find though that the formula given in [3, 4] should be corrected by a shift of theta angle. This is extremely subtle becuase it is just a matter of sign in the sum over different topological sectors, and it is not always non-trivial.

The most interesting aspect of our study is that the formula for the partition function is written as an integral of some meromorphic form and the choice of contour is related to the choice of boundary condition for the vector multiplet of the gauge theory. There is no apriori rule to decide the boundary condition, and the convergence of the integral can give us some hint to find it. Deep in phases one can usually find the contour so that the integral converges for an arbitrary brane. However, near the phase boundary, a convergent contour can be found only for a very restricted class of branes. This reproduces the *grade restriction rule* found in [11] in the Abelian and Calabi-Yau cases, and generalizes it or provides a way of generalization in non-Abelian and non-Calabi-Yau cases. This is of importance to the study of analytic behaviour of the partition function, especially across the phase boundary. The integral is of the form called Mellin-Barnes integral. The present work shows that the issue of convergence of such integrals encodes a rich physics content.

Using the most famous formula for the gamma function, we can convert our result into the formula for the central charge found in [12] during the derivation of the mirror symmetry. This provides a proof of the conjecture that the hemisphere computes the central charge, in the gauge theories we study. The precise corrspondence between the original B-brane and the mirror A-brane was out of reach in the method of [12, 13]. Our results can now be used to find the correspondence, at least at the level of the Ramond-Ramond charge.

Our work may be of interest from another point of view. Supersymmetric field theories on higher dimensional spacetime with boundary can be an interesting subject of study in its own right and also in relation to the dymnamics of branes in superstring theory and M theory. We believe that some experience in two dimensions will be of some help in such investigations.

While our work was in progress but no sentence was written, we were informed by two groups of people, one by Daigo Honda and Takuya Okuda, and another by Sotaro

Sugishita and Seiji Terashima, that they are working on a possibly related subject and that they were getting to be ready or were ready to publish their papers. We asked them to wait for us to write up our results, and they kindly agreed to do so. We would like to thank them for their generosity.

2 Supersymmetry

In this section, we write down the supersymmetry transformation rule of component fields in various supermultiplets on the sphere and the hemisphere.

Notation

We follow the convention originated from Wess-Bagger [14] concerning the notation for the variational parameters as well as the component fields (ϕ, ψ, F) and (v, σ, λ, D) of the chiral and vector multiplets. Dimensional reduction [15] and a $\sqrt{2}$ rescaling yields a notation in two-dimensional Minkowski space [16]. The relation to [14] and [15] (superscripts "WB" and "W" respectively) is

$$\begin{gathered} \epsilon_\pm = \sqrt{2}\epsilon_\pm^{\mathrm{W}}, \\ \lambda_\pm = \sqrt{2}\lambda_\pm^{\mathrm{W}}, \qquad \sigma = \sigma_1 + \mathrm{i}\sigma_2 = \sqrt{2}\sigma^{\mathrm{W}} = v_1^{\mathrm{WB}} - \mathrm{i}v_2^{\mathrm{WB}}, \\ v_0 = v_0^{\mathrm{W}} = v_0^{\mathrm{WB}}, \quad v_1 = v_1^{\mathrm{W}} = v_3^{\mathrm{WB}}. \end{gathered}$$

The same relation holds for the hermitian conjugates. Other components, $\phi, \psi_\pm$, their hermitian conjugates, and D, are trivially related. Notation in the Euclidean space is obtained by Wick rotation, $x^0 \to -\mathrm{i}x^2$, $v_0 \to \mathrm{i}v_2$. We also write

$$D = \mathrm{i}D_E, \qquad F = \mathrm{i}f \overset{\mathrm{or}}{=} \mathrm{i}F_E \quad \overline{F} = \mathrm{i}\overline{f} \overset{\mathrm{or}}{=} \mathrm{i}\overline{F}_E. \tag{2.1}$$

In Euclidean signature, D_E is real and $(f, \overline{f})$ (or $(F_E, \overline{F}_E)$) is the complex conjugate pair,

$$D_E^\dagger = D_E, \qquad \overline{f} = f^\dagger \quad (\text{or } \overline{F}_E = F_E^\dagger). \tag{2.2}$$

The fermionic parameters and fields are to be regarded as sections of the spinor bundle $S = S_- \oplus S_+$ on the Euclidean space $\mathbf{R}^2$, with metric $\mathrm{d}s^2 = |\mathrm{d}z|^2$, $(g_{z\overline{z}} = 1/2)$, where $z = x^1 + \mathrm{i}x^2$. See Appendix A for conventions and facts on spinors on two-dimensional manifolds. For example,

$$\psi = \psi_-\sqrt{\mathrm{d}z} + \psi_+\sqrt{\mathrm{d}\overline{z}}, \qquad \overline{\psi} = \overline{\psi}_+\sqrt{\mathrm{d}z} + \overline{\psi}_-\sqrt{\mathrm{d}\overline{z}}, \tag{2.3}$$

and similarly for the variational parametes $\epsilon, \overline{\epsilon}$. For the fermions in vector multiplets it is more useful to write

$$\lambda = -\mathrm{i}\lambda_- \sqrt{\mathrm{d}z} + \mathrm{i}\overline{\lambda}_+ \sqrt{\mathrm{d}\overline{z}}, \qquad \overline{\lambda} = \mathrm{i}\overline{\lambda}_- \sqrt{\mathrm{d}z} - \mathrm{i}\lambda_+ \sqrt{\mathrm{d}\overline{z}}. \tag{2.4}$$

2.1 Superconformal Transformations

We first write down the $(2,2)$ superconformal transformations of various supermultiplet fields (taken from [3, 4]). They are obtained by replacing the pair $(\epsilon, \overline{\epsilon})$ of constant spinors on the Euclidean space $\mathbf{R}^2$ by a pair of (local) conformal Killing spinors on a curved surface with a spin structure (Σ, g), via the Weyl covariantization procedure [4].

Chiral multiplet (vector R-charge R):

$$\begin{aligned} \delta\phi = \langle \epsilon, \psi \rangle, &\quad \delta\overline{\phi} = -\langle \overline{\epsilon}, \overline{\psi} \rangle, \\ \delta\psi = \mathrm{i}\,\not{\partial}\phi\overline{\epsilon} + \mathrm{i}\tfrac{R}{2}\phi\,\not{\nabla}\overline{\epsilon} + \mathrm{i}f\epsilon, &\quad \delta\overline{\psi} = -\mathrm{i}\,\not{\partial}\overline{\phi}\epsilon - \mathrm{i}\tfrac{R}{2}\overline{\phi}\,\not{\nabla}\epsilon + \mathrm{i}\overline{f}\overline{\epsilon}, \\ \delta f = \langle \overline{\epsilon}, \not{\nabla}\psi \rangle - \tfrac{R}{2}\langle \not{\nabla}\overline{\epsilon}, \psi \rangle, &\quad \delta\overline{f} = \langle \epsilon, \not{\nabla}\overline{\psi} \rangle - \tfrac{R}{2}\langle \not{\nabla}\epsilon, \overline{\psi} \rangle \end{aligned} \tag{2.5}$$

Twisted chiral multiplet (axial R-charge R):

The transformation is obtained from the one for the chiral multiplet by swapping ϵ_+ and $\overline{\epsilon}_+$ while keeping ϵ_- and $\overline{\epsilon}_-$ intact. In other words, replacing $(\epsilon, \overline{\epsilon})$ by $(\widetilde{\epsilon}, \overline{\widetilde{\epsilon}})$ defined by

$$\widetilde{\epsilon} := P_-\epsilon + P_+\overline{\epsilon}, \qquad \overline{\widetilde{\epsilon}} := P_-\overline{\epsilon} + P_+\epsilon. \tag{2.6}$$

This has full information but let us anyway write down the transformations

$$\begin{aligned} \delta u = \langle \widetilde{\epsilon}, \chi \rangle, &\quad \delta\overline{u} = -\langle \overline{\widetilde{\epsilon}}, \overline{\chi} \rangle, \\ \delta\chi = \mathrm{i}\,\not{\partial}u\overline{\widetilde{\epsilon}} + \mathrm{i}\tfrac{R}{2}u\,\not{\nabla}\overline{\widetilde{\epsilon}} + \mathrm{i}g\widetilde{\epsilon}, &\quad \delta\overline{\chi} = -\mathrm{i}\,\not{\partial}\overline{u}\widetilde{\epsilon} - \mathrm{i}\tfrac{R}{2}\overline{u}\,\not{\nabla}\widetilde{\epsilon} + \mathrm{i}\overline{g}\overline{\widetilde{\epsilon}}, \\ \delta g = \langle \overline{\widetilde{\epsilon}}, \not{\nabla}\chi \rangle - \tfrac{R}{2}\langle \not{\nabla}\overline{\widetilde{\epsilon}}, \chi \rangle, &\quad \delta\overline{g} = \langle \widetilde{\epsilon}, \not{\nabla}\overline{\chi} \rangle - \tfrac{R}{2}\langle \not{\nabla}\widetilde{\epsilon}, \overline{\chi} \rangle \end{aligned} \tag{2.7}$$

Vector multiplet:

$$\begin{aligned} \delta v_\mu &= \tfrac{1}{2}\langle \widetilde{\epsilon}, \gamma_\mu\gamma_3\overline{\lambda} \rangle - \tfrac{1}{2}\langle \overline{\widetilde{\epsilon}}, \gamma_\mu\gamma_3\lambda \rangle, \\ \delta\sigma &= \langle \widetilde{\epsilon}, \lambda \rangle, \quad \delta\overline{\sigma} = -\langle \overline{\widetilde{\epsilon}}, \overline{\lambda} \rangle, \end{aligned}$$

$$\delta\lambda = \mathrm{i}\,\not{D}\sigma\bar{\tilde{\epsilon}} + \mathrm{i}\sigma\,\not{\nabla}\bar{\tilde{\epsilon}} + \mathrm{i}\left(D_E + \mathrm{i}\tfrac{v_{12}}{\sqrt{g}}\right)\tilde{\epsilon} - \tfrac{1}{2}[\sigma,\bar{\sigma}]\gamma_3\tilde{\epsilon},$$
$$\delta\bar{\lambda} = -\mathrm{i}\,\not{D}\bar{\sigma}\tilde{\epsilon} - \mathrm{i}\bar{\sigma}\,\not{\nabla}\tilde{\epsilon} + \mathrm{i}\left(D_E - \mathrm{i}\tfrac{v_{12}}{\sqrt{g}}\right)\bar{\tilde{\epsilon}} - \tfrac{1}{2}[\sigma,\bar{\sigma}]\gamma_3\bar{\tilde{\epsilon}},$$
$$\delta D_E = \tfrac{1}{2}\left(\langle\tilde{\epsilon},\not{D}\bar{\lambda} - \mathrm{i}\gamma_3[\bar{\sigma},\lambda]\rangle - \langle\not{\nabla}\tilde{\epsilon},\bar{\lambda}\rangle + \langle\bar{\tilde{\epsilon}},\not{D}\lambda - \mathrm{i}\gamma_3[\sigma,\bar{\lambda}]\rangle - \langle\not{\nabla}\bar{\tilde{\epsilon}},\lambda\rangle\right). \tag{2.8}$$

We also note

$$\delta\left(\mathrm{i}\frac{v_{12}}{\sqrt{g}}\right) = \frac{1}{2}\left(-\langle\tilde{\epsilon},\not{D}\bar{\lambda}\rangle + \langle\not{\nabla}\tilde{\epsilon},\bar{\lambda}\rangle + \langle\bar{\tilde{\epsilon}},\not{D}\lambda\rangle - \langle\not{\nabla}\bar{\tilde{\epsilon}},\lambda\rangle\right). \tag{2.9}$$

We see that $(\sigma,\lambda,D_E + \mathrm{i}\frac{v_{12}}{\sqrt{g}})$ transform as fields in a twisted chiral multiplet of axial R-charge 2, i.e. as (u,χ,g) in (2.7) with $R=2$, up to commutator terms.

<u>Charged chiral multiplet:</u>

$$\delta\phi = \langle\epsilon,\psi\rangle, \quad \delta\bar{\psi} = -\langle\bar{\epsilon},\bar{\psi}\rangle,$$
$$\delta\psi = \mathrm{i}\left(\not{D}\phi + \tfrac{R}{2}\phi\,\not{\nabla} - \mathrm{i}\sigma_1\phi\gamma_3 + \sigma_2\phi\right)\bar{\epsilon} + \mathrm{i}f\epsilon,$$
$$\delta\bar{\psi} = -\mathrm{i}\left(\not{D}\bar{\phi} + \tfrac{R}{2}\bar{\phi}\,\not{\nabla} + \mathrm{i}\bar{\phi}\sigma_1\gamma_3 + \bar{\phi}\sigma_2\right)\epsilon + \mathrm{i}\bar{f}\bar{\epsilon},$$
$$\delta f = \langle\bar{\epsilon},\not{D}\psi + \tfrac{R}{2}\overleftarrow{\not{\nabla}}\psi - \mathrm{i}\gamma_3\sigma_1\psi - \sigma_2\psi - \bar{\tilde{\lambda}}\phi\rangle,$$
$$\delta\bar{f} = \langle\not{D}\bar{\psi} + \tfrac{R}{2}\bar{\psi}\,\not{\nabla} + \mathrm{i}\gamma_3\bar{\psi}\sigma_1 - \bar{\psi}\sigma_2 - \bar{\phi}\tilde{\lambda},\epsilon\rangle. \tag{2.10}$$

Here $\tilde{\lambda} = \lambda_-\sqrt{\mathrm{d}z} + \lambda_+\sqrt{\mathrm{d}\bar{z}}$ and $\bar{\tilde{\lambda}} = \bar{\lambda}_-\sqrt{\mathrm{d}z} + \bar{\lambda}_+\sqrt{\mathrm{d}\bar{z}}$.

Commutation Relations

The above superconformal transformations form a closed algebra together with conformal and R-symmetry transformations, up to gauge transformations. Under conformal transformations, generated by vector fields $X = X^\mu\partial_\mu$ such that $\partial_{\bar{z}}X^z = \partial_z X^{\bar{z}} = 0$, all the fields transform like primary fields (in the sense of [19])

$$\delta_X^{\mathrm{conf}}\mathcal{O} = X^\mu D_\mu\mathcal{O} + (\Delta\nabla_z X^z + \tilde{\Delta}\nabla_{\bar{z}}X^{\bar{z}})\mathcal{O}, \tag{2.11}$$

except the gauge potential which transforms as

$$\delta_X^{\mathrm{conf}} v_\mu = X^\nu v_{\nu\mu}. \tag{2.12}$$

$(\Delta,\tilde{\Delta})$ are the "conformal weights" of $\mathcal{O}$ listed below together with the R-charges

$\mathcal{O}$	ϕ	ψ_-	ψ_+	f	σ	λ_-	$\bar{\lambda}_+$	D_E
$(\Delta,\tilde{\Delta})$	$(\frac{R}{4},\frac{R}{4})$	$(\frac{R}{4}+\frac{1}{2},\frac{R}{4})$	$(\frac{R}{4},\frac{R}{4}+\frac{1}{2})$	$(\frac{R}{4}+\frac{1}{2},\frac{R}{4}+\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,1)$
(F_V,F_A)	$(R,0)$	$(R-1,1)$	$(R-1,-1)$	$(R-2,0)$	$(0,2)$	$(1,1)$	$(-1,1)$	$(0,0)$

Under $(\phi, \psi_\pm, f) \to (\overline{\phi}, \overline{\psi}_\pm, \overline{f})$ and $(\sigma, \lambda_\pm) \to (\overline{\sigma}, \overline{\lambda}_\pm)$, conformal weights do not change but the R-charges changes their signs. Those for twisted chiral multiplet fields are obtained from the ones for chiral multiplet fields by the exchange of F_V and F_A (no change in $(\Delta, \widetilde{\Delta})$). It is useful to introduce right and left handed R-symmetries F_R and F_L defined by $F_V = F_R + F_L$ and $F_A = -F_R + F_L$. We write te corresponding symmetry transformations by $\delta^{\rm right}$ and $\delta^{\rm left}$.

Let δ_1 and δ_2 be the superconformal transformations with conformal Killing spinors $(\epsilon_1, \overline{\epsilon}_1)$ and $(\epsilon_2, \overline{\epsilon}_2)$. For all fields $\mathcal{O}$, we find

$$[\delta_2, \delta_1]\mathcal{O} = \delta_X^{\rm conf}\mathcal{O} + \delta_\Theta^{\rm right}\mathcal{O} + \delta_{\widetilde{\Theta}}^{\rm left}\mathcal{O} + \delta_{{\rm i}\Lambda}^{\rm gauge}\mathcal{O}. \tag{2.13}$$

where

$$\begin{aligned} X^\mu &= {\rm i}\langle \epsilon_{[1}, \gamma^\mu \overline{\epsilon}_{2]}\rangle, & (2.14)\\ \Theta &= \frac{\rm i}{2}\left(\langle \overline{\nabla}\!\!\!/\, \epsilon_{[1}, P_- \overline{\epsilon}_{2]}\rangle + \langle P_- \epsilon_{[1}, \overline{\nabla}\!\!\!/\, \overline{\epsilon}_{2]}\rangle\right), & (2.15)\\ \widetilde{\Theta} &= \frac{\rm i}{2}\left(\langle \overline{\nabla}\!\!\!/\, \epsilon_{[1}, P_+ \overline{\epsilon}_{2]}\rangle + \langle P_+ \epsilon_{[1}, \overline{\nabla}\!\!\!/\, \overline{\epsilon}_{2]}\rangle\right), & (2.16)\\ {\rm i}\Lambda &= \langle \epsilon_{[1}, (\gamma_3\sigma_1 + {\rm i}\sigma_2)\overline{\epsilon}_{2]}\rangle. & (2.17) \end{aligned}$$

Here $\langle \epsilon_{[1}, \gamma^\mu \overline{\epsilon}_{2]}\rangle := \langle \epsilon_1, \gamma^\mu \overline{\epsilon}_2\rangle - \langle \epsilon_2, \gamma^\mu \overline{\epsilon}_1\rangle$, etc. It is straightforward to find

$$X^z = 2{\rm i}\epsilon^-_{[1}\overline{\epsilon}^-_{2]}, \qquad X^{\overline{z}} = -2{\rm i}\epsilon^+_{[1}\overline{\epsilon}^+_{2]}, \tag{2.18}$$

$$\Theta = {\rm i}\epsilon^-_{[1}\partial_z\overline{\epsilon}^-_{2]} - {\rm i}\partial_z\epsilon^-_{[1}\overline{\epsilon}^-_{2]}, \qquad \widetilde{\Theta} = -{\rm i}\epsilon^+_{[1}\partial_{\overline{z}}\overline{\epsilon}^+_{2]} + {\rm i}\partial_{\overline{z}}\epsilon^+_{[1}\overline{\epsilon}^+_{2]}. \tag{2.19}$$

We see that X^z and Θ are holomorphic and $X^{\overline{z}}$ and $\widetilde{\Theta}$ are antiholomorphic, as they should be. The commutation relation (2.13) and other obvious ones form the $(2,2)$ superconformal algebra. Let us spell out a correspondence to the more standard notation. We write the superconformal transformation for the pair $(\epsilon, \overline{\epsilon})$ of conformal Killing spinors by $\delta = \delta^-_\epsilon + \delta^+_{\overline{\epsilon}}$. As a local basis of conformal Killing spinors, we use

$$\mathbf{s}_r = z^{r+\frac{1}{2}}\sqrt{\frac{\partial}{\partial z}}, \qquad \widetilde{\mathbf{s}}_r = \overline{z}^{r+\frac{1}{2}}\sqrt{\frac{\partial}{\partial \overline{z}}}. \tag{2.20}$$

Then, the correspondence is

$$\begin{aligned} L_n &= \delta^{\rm conf}_{z^{n+1}\frac{\partial}{\partial z}}, \qquad \widetilde{L}_n = \delta^{\rm conf}_{\overline{z}^{n+1}\frac{\partial}{\partial \overline{z}}}, & (2.21)\\ G^\pm_r &= {\rm e}^{-\frac{\pi {\rm i}}{4}}\delta^\pm_{\mathbf{s}_r}, \qquad \widetilde{G}^\pm_r = {\rm e}^{\frac{\pi {\rm i}}{4}}\delta^\pm_{\widetilde{\mathbf{s}}_r}, & (2.22)\\ J_n &= \delta^{\rm right}_{z^n}, \qquad \widetilde{J}_n = \delta^{\rm left}_{\overline{z}^n}. & (2.23) \end{aligned}$$

They indeed obey the standard commutation relations, such as those in [17], in which the central terms are set equal to zero.

2.2 Supersymmetry On The Sphere And The Hemisphere

We shall formulate $(2,2)$ supersymmetric field theories on the sphere and the hemisphere in such a way that a part of the $(2,2)$ superconformal symmetry is preserved. We consider the Riemann sphere with coordinates z and w, $zw = 1$, and the southern or northern hemisphere defined by $|z| \leq 1$ or $|w| \leq 1$ respectively. See Appendix A for more details on the facts on the sphere and the hemispheres. Since we consider theories which are not necessarily conformally invariant, out of the conformal generators we can at most keep isometry generators. On the round sphere, the isometry group is $O(3)$ generated by

$$\ell_3 = -z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}, \quad \ell_+ = z^2\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \ell_- = -\frac{\partial}{\partial z} - \overline{z}^2\frac{\partial}{\partial \overline{z}}. \tag{2.24}$$

Note also that only $\mathbf{s}_{\pm\frac{1}{2}}$ and $\widetilde{\mathbf{s}}_{\pm\frac{1}{2}}$ are globally defined conformal Killing spinors. On the hemisphere, only ℓ_3 can be the isometry generator, and only the linear combinations $\mathbf{s}_{\frac{1}{2}} \pm \widetilde{\mathbf{s}}_{-\frac{1}{2}}$ and $\mathbf{s}_{-\frac{1}{2}} \pm \widetilde{\mathbf{s}}_{\frac{1}{2}}$ satisfy the boundary condition at $|z| = 1$ for some spin structure. Thus, what can be included are the isometry generators,

$$\begin{aligned} \widehat{L}_3 &= \delta^{\rm conf}_{\ell_3} = -L_0 + \widetilde{L}_0, \\ \widehat{L}_+ &= \delta^{\rm conf}_{\ell_+} = L_1 + \widetilde{L}_{-1}, \\ \widehat{L}_- &= \delta^{\rm conf}_{\ell_-} = -L_{-1} - \widetilde{L}_1, \end{aligned} \tag{2.25}$$

the supercharges

$$G^+_{\pm\frac{1}{2}}, \quad G^-_{\pm\frac{1}{2}}, \quad \widetilde{G}^+_{\pm\frac{1}{2}}, \quad \widetilde{G}^-_{\pm\frac{1}{2}}, \tag{2.26}$$

and the R-symmetry generators,

$$\begin{aligned} F_V &= \delta^{\rm vector}_1 = J_0 + \widetilde{J}_0, \\ F_A &= \delta^{\rm axial}_1 = -J_0 + \widetilde{J}_0. \end{aligned} \tag{2.27}$$

We would like to find subsets of these, closed under the commutation relation, which include all the isometries of the respective geometry and a maximum number of supercharges.

On the round two-sphere, there are two possibilities

$$\begin{aligned} &\text{(A-type)} \quad && \widehat{L}_3,\ \widehat{L}_\pm,\ F_V, \\ & && Q^{A\pm}_{(+)} = \delta^\pm_{\mathbf{s}_{\pm\frac{1}{2}}} + \delta^\pm_{\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}} = e^{\frac{\pi i}{4}}(G^\pm_{\pm\frac{1}{2}} - i\widetilde{G}^\pm_{\mp\frac{1}{2}}), \\ & && Q^{A\pm}_{(-)} = \delta^\pm_{\mathbf{s}_{\mp\frac{1}{2}}} - \delta^\pm_{\widetilde{\mathbf{s}}_{\pm\frac{1}{2}}} = e^{\frac{\pi i}{4}}(G^\pm_{\mp\frac{1}{2}} + i\widetilde{G}^\pm_{\pm\frac{1}{2}}), \\ &\text{(B-type)} \quad && \widehat{L}_3,\ \widehat{L}_\pm,\ F_A, \end{aligned} \tag{2.28}$$

$$
\begin{aligned}
Q^{B\pm}_{(+)} &= \delta^{\mp}_{\mathbf{s}_{\pm\frac{1}{2}}} + \delta^{\pm}_{\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}}(G^{\mp}_{\pm\frac{1}{2}} - \mathrm{i}\widetilde{G}^{\pm}_{\mp\frac{1}{2}}),\\
Q^{B\pm}_{(-)} &= \delta^{\mp}_{\mathbf{s}_{\mp\frac{1}{2}}} - \delta^{\pm}_{\widetilde{\mathbf{s}}_{\pm\frac{1}{2}}} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}}(G^{\mp}_{\mp\frac{1}{2}} + \mathrm{i}\widetilde{G}^{\pm}_{\pm\frac{1}{2}}),
\end{aligned}
\tag{2.29}
$$

and their axial and vector R-rotations, (A$^{\beta}$-type) or (B$^{\alpha}$-type), which are obtained by the replacement $G^{\pm}_r \to \mathrm{e}^{\mp \mathrm{i}\beta} G^{\pm}_r$, $\widetilde{G}^{\pm}_r \to \mathrm{e}^{\pm \mathrm{i}\beta}\widetilde{G}^{\pm}_r$, or $G^{\pm}_r \to \mathrm{e}^{\pm \mathrm{i}\alpha} G^{\pm}_r$, $\widetilde{G}^{\pm}_r \to \mathrm{e}^{\pm \mathrm{i}\alpha}\widetilde{G}^{\pm}_r$, respectively. The generators from each set form a closed algebra which is isomorphic to $\mathfrak{osp}(2|2)$. At the south or the north pole, parts of the A-type (*resp.* B-type) supercharges define twisted chiral (*resp.* chiral) operators. For example, let us look at the A-type supercharges. At the south pole $z = 0$, operators annihilated by $Q^{A+}_{(+)} \sim \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}}\widetilde{G}^{+}_{-\frac{1}{2}}$ and $Q^{A-}_{(+)} \sim \mathrm{e}^{\frac{\pi \mathrm{i}}{4}} G^{-}_{-\frac{1}{2}}$ are twisted chiral while those annihilated by $Q^{A-}_{(-)} \sim -\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}}\widetilde{G}^{-}_{-\frac{1}{2}}$ and $Q^{A+}_{(-)} \sim \mathrm{e}^{\frac{\pi \mathrm{i}}{4}} G^{+}_{-\frac{1}{2}}$ are twisted antichiral, and things are the opposite at the north pole $z = \infty$. This is the motivation for the name “A” and “B”. In the litarature [3, 4], partition function of gauged linear sigma model preserving A-type supersymmetry is studied, and the result depends on the twisted chiral parameters but not on the chiral parameters— the Kähler parameters but not the complex structure parameters when there is a non-linear sigma model interpretation.

On the hemisphere, there are four possibilities

$$
(\mathrm{A}_{(+)}\text{-type}) \qquad \widehat{L}_3,\ Q^{A\pm}_{(+)},\ F_V, \tag{2.30}
$$

$$
(\mathrm{A}_{(-)}\text{-type}) \qquad \widehat{L}_3,\ Q^{A\pm}_{(-)},\ F_V, \tag{2.31}
$$

$$
(\mathrm{B}_{(+)}\text{-type}) \qquad \widehat{L}_3,\ Q^{B\pm}_{(+)},\ F_A, \tag{2.32}
$$

$$
(\mathrm{B}_{(-)}\text{-type}) \qquad \widehat{L}_3,\ Q^{B\pm}_{(-)},\ F_A, \tag{2.33}
$$

and their axial and vector R-rotations, (A$^{\beta}_{(\pm)}$-type) or (B$^{\alpha}_{(\pm)}$-type). The generators from each set form a closed algebra: $(\mathrm{e}^{\frac{\pi \mathrm{i}}{4}} Q, \mathrm{e}^{\frac{\pi \mathrm{i}}{4}}\overline{Q}, F) = (Q^{A\pm}_{(\pm)}, Q^{A\mp}_{(\pm)}, \pm F_V)$ or $(Q^{B\pm}_{(\pm)}, Q^{B\mp}_{(\pm)}, \pm F_A)$ obeys

$$
\begin{aligned}
&Q^2 = \overline{Q}^2 = 0,\\
&\{Q, \overline{Q}\} = -2\widehat{L}_3 + F,\\
&[\widehat{L}_3, Q] = \tfrac{1}{2} Q, \quad [\widehat{L}_3, \overline{Q}] = -\tfrac{1}{2}\overline{Q},\\
&[F, Q] = Q, \quad [F, \overline{Q}] = -\overline{Q}.
\end{aligned}
\tag{2.34}
$$

A boundary condition must be specified at the boundary $|z| = 1$. There are basically two types of subalgebra of the $(2, 2)$ supersymmetry with half the amount of supercharges that can be preserved at the boundary [18, 13] — A-type and B-type — and the boundary conditions preserving these are called A-branes and B-branes. In a superconformal field theory, the preserved generators are [18]

$$\text{A-branes:}\qquad L_n - \widetilde{L}_n,\quad G^+_r \pm \mathrm{i}\widetilde{G}^-_{-r},\quad G^-_r \pm \mathrm{i}\widetilde{G}^+_{-r},\quad J_n - \widetilde{J}_{-n}, \tag{2.35}$$

$$\text{B-branes:}\qquad L_n - \widetilde{L}_n,\quad G^-_r \pm \mathrm{i}\widetilde{G}^-_{-r},\quad G^+_r \pm \mathrm{i}\widetilde{G}^+_{-r},\quad J_n + \widetilde{J}_{-n}, \tag{2.36}$$

We see that the boundary conditions preserving $\mathrm{A}_{(\pm)}$ are B-branes while those preserving $\mathrm{B}_{(\pm)}$ are A-branes.

The sign in the parenthesis, $(\pm)$, corresponds to the choice of spin structure at the boundary — $(\pm)_0$ for the southern hemisphere D^2_0 and $(\mp)_\infty$ for the northern hemisphere D^2_∞. See Appendix A. We shall denote the partition function on the southern hemisphere D^2_0 preserving the $\mathrm{A}_{(\pm)}$-type and $\mathrm{B}_{(\pm)}$-type supersymmetry by $Z^{\mathrm{A}}_{D^2_{0(\pm)}}$ and $Z^{\mathrm{A}}_{D^2_{0(\pm)}}$ respectively, while the partition function on the northern hemisphere D^2_∞ preserving the $\mathrm{A}_{(\pm)}$-type and $\mathrm{B}_{(\pm)}$-type supersymmetry by $Z^{\mathrm{A}}_{D^2_{0(\mp)}}$ and $Z^{\mathrm{A}}_{D^2_{0(\mp)}}$ respectively. Note that the label $(\pm)$ in the partition function is correlated with the spin structure. Since there is really no difference between the southern and northern hemispheres, we have the equality

$$Z^{\mathrm{A}}_{D^2_{0(\pm)}} = Z^{\mathrm{A}}_{D^2_{\infty(\pm)}},\qquad Z^{\mathrm{B}}_{D^2_{0(\pm)}} = Z^{\mathrm{B}}_{D^2_{\infty(\pm)}}, \tag{2.37}$$

When there is no room of confusion between A and B, we shall drop the superscript.

2.3 Some Useful Formulae

For convenience in later sections, we collect some useful properties of the variational parameters for the supersymmetry transformations of each type.

We first write down the action of the Dirac operator. Let us first look at the B-type supersymmetry. The parameters of the four supercharges are $(\epsilon,\overline{\epsilon}) = (\mathbf{s}_{\frac{1}{2}}, \widetilde{\mathbf{s}}_{-\frac{1}{2}})$, $(\mathbf{s}_{-\frac{1}{2}}, -\widetilde{\mathbf{s}}_{\frac{1}{2}})$, $(-\widetilde{\mathbf{s}}_{-\frac{1}{2}}, \mathbf{s}_{\frac{1}{2}})$ and $(\widetilde{\mathbf{s}}_{\frac{1}{2}}, \mathbf{s}_{-\frac{1}{2}})$, times a constant anticommuting variational parameter. Using (A.17), we see that each satisfies $\nabla\!\!\!/\,\epsilon = -\overline{\epsilon}/r$ and $\nabla\!\!\!/\,\overline{\epsilon} = \epsilon/r$. These can also be written as $\nabla\!\!\!/\,\widetilde{\epsilon} = \gamma_3\widetilde{\epsilon}/r$, $\nabla\!\!\!/\,\overline{\widetilde{\epsilon}} = -\gamma_3\overline{\widetilde{\epsilon}}/r$ using $(\widetilde{\epsilon},\overline{\widetilde{\epsilon}})$ introduced in (2.6). The same applies to the A-type if we replace $(\epsilon,\overline{\epsilon})$ by $(\widetilde{\epsilon},\overline{\widetilde{\epsilon}})$. To summarize,

$$\begin{aligned}
&\text{(A-type)}\qquad \nabla\!\!\!/\,\widetilde{\epsilon} = -\overline{\widetilde{\epsilon}}/r,\quad \nabla\!\!\!/\,\overline{\widetilde{\epsilon}} = \widetilde{\epsilon}/r,\quad \text{or equivalently}\\
&\qquad\qquad\qquad \nabla\!\!\!/\,\epsilon = \gamma_3\epsilon/r,\quad \nabla\!\!\!/\,\overline{\epsilon} = -\gamma_3\overline{\epsilon}/r,
\end{aligned}\tag{2.38}$$

$$\begin{aligned}
&\text{(B-type)}\qquad \nabla\!\!\!/\,\epsilon = -\overline{\epsilon}/r,\quad \nabla\!\!\!/\,\overline{\epsilon} = \epsilon/r,\quad \text{or equivalently}\\
&\qquad\qquad\qquad \nabla\!\!\!/\,\widetilde{\epsilon} = \gamma_3\widetilde{\epsilon}/r,\quad \nabla\!\!\!/\,\overline{\widetilde{\epsilon}} = -\gamma_3\overline{\widetilde{\epsilon}}/r.
\end{aligned}\tag{2.39}$$

We next write down the action of $\gamma^{\widehat{n}} = g_{\mu\nu}\widehat{n}^\mu\gamma^\nu$ on the variational parameters at the boundary $|z| = 1$, where $\widehat{n}$ is the outward unit normal to the southern hemisphere D^2_0. Using (A.22), we find

$$(\mathrm{A}_{(\pm)}\text{-type}) \qquad \gamma^{\widehat{n}}\,\epsilon = \mp\epsilon, \quad \gamma^{\widehat{n}}\,\overline{\epsilon} = \mp\overline{\epsilon}, \tag{2.40}$$

$$(\mathrm{B}_{(\pm)}\text{-type}) \qquad \gamma^{\widehat{n}}\,\epsilon = \mp\overline{\epsilon}, \quad \gamma^{\widehat{n}}\,\overline{\epsilon} = \mp\epsilon. \tag{2.41}$$

3 Formulation

In this section, we formulate a class of theories on the hemisphere in such a way that some of the supersymmetry studied in the previous section are preserved. We shall first find a bulk action with appropriate boundary interaction so that the total is automatically supersymmetric, and then discuss the boundary conditions. The main target is the gauged linear sigma models with A-type supersymmetry (B-branes at the boundary), but we start with Landau-Ginzburg models with B-type supersymmetry (A-branes at the boundary) as a warm up. In view of (2.37) it is enough to consider the southern hemisphere, so we set $D^2 = D^2_0$.

3.1 Bulk Action

3.1.1 Warm Up: Landau-Ginzburg Model (B-Type Supersymmetry)

We consider the Landau-Ginzburg model of n chiral multiplets (ϕ^i, ψ^i, f^i), $i = 1, \ldots, n$, with superpotential $W(\phi) = W(\phi^1, \ldots, \phi^n)$.

Before starting, we comment on a useful fact concerning B-type supersymmetry transformation of chiral multiplets. Using (2.39) in (2.5), we find that if (ϕ, ψ, f) is a chiral multiplet of vector R-charge R, then $(\phi, \psi, f_!)$ with

$$f_! = f + \frac{R}{2r}\phi, \qquad \overline{f}_! = \overline{f} + \frac{R}{2r}\overline{\phi}, \tag{3.1}$$

transforms under the B-type supersymmetry as a chiral multiplet of vanishing vector R-charge. This remark applies equally well to A-type supersymmetry transformation of twisted chiral multiplets, as will be used in Section 3.1.2.

Kinetic term

First, let us find the kinetic term of a single chiral multiplet (ϕ, ψ, f). Let δ_1 and δ_2 be the $\mathrm{B}_{(+)}$-type supersymmetry with parameters $(\epsilon_1, \overline{\epsilon}_1)$ and $(\epsilon_2, \overline{\epsilon}_2)$. We compute $\delta_2\delta_1$ of some combination of fields and see if something like a kinetic term appears. After some try and error, we find

$$\mathrm{i}\delta_2\delta_1\left(\overline{f}_!\phi+\overline{\phi}f_!\mp\frac{1}{r}\overline{\phi}\phi\right)=\nabla_\mu J^\mu-2c_-\mathcal{L}_{\rm kin}, \tag{3.2}$$

where

$$\mathcal{L}_{\rm kin}=\partial^\mu\overline{\phi}\partial_\mu\phi+\frac{\mathrm{i}}{2}\langle\nabla\!\!\!\!/\,\overline{\psi},\psi\rangle+\frac{\mathrm{i}}{2}\langle\overline{\psi},\nabla\!\!\!\!/\,\psi\rangle+\overline{f}_!f_!, \tag{3.3}$$

and

$$\begin{aligned}J^\mu &= c_-\partial^\mu(\overline{\phi}\phi)-\langle\epsilon_1,\gamma^\mu\overline{\epsilon}_2\rangle\overline{f}_!\phi-\langle\overline{\epsilon}_1,\gamma^\mu\epsilon_2\rangle\overline{\phi}f_!\\ &\quad +c_+\left(\partial^\mu\overline{\phi}\phi-\overline{\phi}\partial^\mu\phi+\langle\overline{\psi},\gamma^\mu\psi\rangle\right)+c_{3-}\frac{\mathrm{i}}{\sqrt{g}}\epsilon^{\mu\nu}\partial_\nu(\overline{\phi}\phi),\end{aligned} \tag{3.4}$$

in which $c_\pm=\frac{1}{2}(\langle\epsilon_1,\epsilon_2\rangle\pm\langle\overline{\epsilon}_1,\overline{\epsilon}_2\rangle)$, $c_{3\pm}=\frac{1}{2}(\langle\epsilon_1,\gamma_3\epsilon_2\rangle\pm\langle\overline{\epsilon}_1,\gamma_3\overline{\epsilon}_2\rangle)$. In deriving the above we used (2.39) and some of its consequences, such as the fact that c_- and c_{3+} are constants. It is also useful to note that $c_+=c_{3-}=0$ at the equator $|z|=1$. In particular, at $|z|=1$ we have

$$\begin{aligned}\widehat{n}\cdot J &= c_-\widehat{n}^\mu\partial_\mu(\overline{\phi}\phi)-\langle\epsilon_1,\gamma^{\widehat{n}}\,\overline{\epsilon}_2\rangle\overline{f}_!\phi-\langle\overline{\epsilon}_1,\gamma^{\widehat{n}}\,\epsilon_2\rangle\overline{\phi}f_!\\ &= c_-\left(\widehat{n}^\mu\partial_\mu(\overline{\phi}\phi)\pm(\overline{f}\phi-\overline{\phi}f)\right)\end{aligned} \tag{3.5}$$

for $\mathrm{B}_{(\pm)}$-type supersymmetry, where we used (2.41) as well as (3.1). Evaluating c_- and integrating over the hemisphere, we find

$$\begin{aligned}\int_{D^2}\mathcal{L}_{\rm kin}\sqrt{g}\mathrm{d}^2x\,-\,\frac{1}{2}\int_{\partial D^2}\Big[\widehat{n}^\mu\partial_\mu(\overline{\phi}\phi)\pm(\overline{f}\phi-\overline{\phi}f)\Big]\mathrm{d}\tau\\ =\pm\frac{\mathrm{i}}{2r}\int_{D^2}Q^{B-}_{(\pm)}Q^{B+}_{(\pm)}\left(\overline{f}_!\phi+\overline{\phi}f_!\mp\frac{1}{r}\overline{\phi}\phi\right)\sqrt{g}\mathrm{d}^2x.\end{aligned} \tag{3.6}$$

Here we used a periodic coordinate $\tau\equiv\tau+2\pi r$ of the boundary ∂D^2, defined by $z=\mathrm{e}^{\mathrm{i}\tau/r}$ for $|z|=1$. Using the algebra (2.34), invariance of D^2 under the rotation ℓ_3, and the fact that $\overline{f}_!\phi+\overline{\phi}f_!-\frac{1}{r}\overline{\phi}\phi$ has vanishing axial R-charge, we find that the right hand side is $Q^{B-}_{(\pm)}$-exact as well as $Q^{B+}_{(\pm)}$-exact, and in particular, invariant under both. Thus, we can take the left hand side of (3.6) as the action we wanted. It is the usual type of kinetic term plus a particular boundary term.

With a little more hard work, we can generalize the above construction to the case of n variables with a Kähler potential K and the Kähler metric $g_{i\overline{j}}=\partial_i\partial_{\overline{j}}K$. We shall use the notation $K_i=\partial_iK$ etc. We have the relation of the form (3.2), in which we make the replacement $\overline{\phi}\phi\to K$, $\overline{f}_!\phi\to\overline{f}^{\overline{\imath}}_!K_{\overline{\imath}}-\frac{\mathrm{i}}{2}K_{\overline{\imath}\overline{\jmath}}\langle\overline{\psi}^{\overline{\imath}},\overline{\psi}^{\overline{\jmath}}\rangle$, $\overline{\phi}f_!\to K_if^i_!+\frac{\mathrm{i}}{2}K_{ij}\langle\psi^i,\psi^j\rangle$,

$\partial^\mu\overline{\phi}\phi \to \partial^\mu\overline{\phi}^{\bar\imath}K_{\bar\imath}$ and $\overline{\phi}\partial^\mu\phi \to K_i\partial^\mu\phi^i$, in the expressions for the left hand side and for J^μ, and

$$\begin{aligned}\mathcal{L}_{\rm kin} &= g_{i\bar\jmath}\partial^\mu\overline{\phi}^{\bar\jmath}\partial_\mu\phi^i + \frac{\mathrm{i}}{2}g_{i\bar\jmath}\langle \not{D}\overline{\psi}^{\bar\jmath},\psi^i\rangle + \frac{\mathrm{i}}{2}g_{i\bar\jmath}\langle\overline{\psi}^{\bar\jmath},\not{D}\psi^i\rangle + \frac{1}{4}R_{i\bar\jmath k\bar l}\langle\psi^i,\psi^k\rangle\langle\overline{\psi}^{\bar\jmath},\overline{\psi}^{\bar l}\rangle \\ &\quad + g_{i\bar\jmath}\left(\overline{f}_!^{\bar\jmath} - \frac{\mathrm{i}}{2}\Gamma^{\bar\jmath}_{\bar k\bar l}\langle\overline{\psi}^{\bar k},\overline{\psi}^{\bar l}\rangle\right)\left(f_!^i + \frac{\mathrm{i}}{2}\Gamma^i_{kl}\langle\psi^k,\psi^l\rangle\right). \end{aligned} \tag{3.7}$$

As the action, we may take

$$\int_{D^2}\mathcal{L}_{\rm kin}\sqrt{g}\mathrm{d}^2x - \frac{1}{2}\int_{\partial D^2}\left[\widehat{n}^\mu\partial_\mu K \pm\left(\overline{f}_!^{\bar\imath}K_{\bar\imath} - K_i f_!^i - \frac{\mathrm{i}}{2}K_{\bar\imath\bar\jmath}\langle\overline{\psi}^{\bar\imath},\overline{\psi}^{\bar\jmath}\rangle - \frac{\mathrm{i}}{2}K_{ij}\langle\psi^i,\psi^j\rangle\right)\right]\mathrm{d}\tau. \tag{3.8}$$

It is not only supersymmetric but also Q-exact as long as the Kähler potential is globally defined.

Superpotential term

We next turn to the superpotential term. Let us put

$$\mathcal{L}_W = \frac{\mathrm{i}}{2r}(W+\overline{W}) - \frac{\mathrm{i}}{2}f_!^i\partial_iW - \frac{\mathrm{i}}{2}\overline{f}_!^{\bar\imath}\partial_{\bar\imath}\overline{W} + \frac{1}{4}\langle\psi^i,\psi^j\rangle\partial_i\partial_jW - \frac{1}{4}\langle\overline{\psi}^{\bar\imath},\overline{\psi}^{\bar\jmath}\rangle\partial_{\bar\imath}\partial_{\bar\jmath}\overline{W}. \tag{3.9}$$

Under B-type supersummetry, it transforms as $\delta\mathcal{L}_W = \nabla_\mu J^\mu$ where

$$J^\mu = \frac{\mathrm{i}}{2}\langle\gamma^\mu\overline{\epsilon},\psi^i\rangle\partial_iW + \frac{\mathrm{i}}{2}\langle\gamma^\mu\epsilon,\overline{\psi}^{\bar\imath}\rangle\partial_{\bar\imath}\overline{W}. \tag{3.10}$$

Note that

$$\begin{aligned}\widehat{n}\cdot J &= \frac{\mathrm{i}}{2}\langle\gamma^{\widehat{n}}\overline{\epsilon},\psi^i\rangle\partial_iW + \frac{\mathrm{i}}{2}\langle\gamma^{\widehat{n}}\epsilon,\overline{\psi}^{\bar\imath}\rangle\partial_{\bar\imath}\overline{W} \\ &= \mp\frac{\mathrm{i}}{2}\left(\langle\epsilon,\psi^i\rangle\partial_iW + \langle\overline{\epsilon},\overline{\psi}^{\bar\imath}\rangle\partial_{\bar\imath}\overline{W}\right) = \mp\frac{\mathrm{i}}{2}\delta(W-\overline{W}),\end{aligned} \tag{3.11}$$

where we used (2.41). We therefore find that

$$\int_{D^2}\mathcal{L}_W\sqrt{g}\mathrm{d}^2x \pm \int_{\partial D^2}\frac{\mathrm{i}}{2}\left(W-\overline{W}\right)\mathrm{d}\tau \tag{3.12}$$

is invariant under $\mathrm{B}_{(\pm)}$-type supersymmetry. Again, for this we do not need to use any boundary condition.

When the superpotential is quasi-homogeneous, the system on the flat space has the vector $U(1)$ R-symmetry under the assignment of the R-charges so that $W(\lambda^R\phi) = \lambda^2W(\phi)$, or equivalently, $\sum_i R_i\phi^i\partial_iW = 2W$. Then, the expression (3.9) simplifies as

$$\mathcal{L}_W = -\frac{\mathrm{i}}{2}f^i\partial_iW - \frac{\mathrm{i}}{2}\overline{f}^{\bar\imath}\partial_{\bar\imath}\overline{W} + \frac{1}{4}\langle\psi^i,\psi^j\rangle\partial_i\partial_jW - \frac{1}{4}\langle\overline{\psi}^{\bar\imath},\overline{\psi}^{\bar\jmath}\rangle\partial_{\bar\imath}\partial_{\bar\jmath}\overline{W}. \tag{3.13}$$

This itself is invariant under the vector $U(1)$ R-rotation. However, the last term of the bulk kinetic term (3.3) as well as the boundary term in (3.12) violate this symmetry. Thus, the systems on the sphere and the hemisphere do not inherite the vector $U(1)$ R-symmetry.

B$^{\alpha}$-type supersymmetry

The above actions (3.6) and (3.12) can be made invariant under $\mathrm{B}^{\alpha}_{(\pm)}$-type supersymmetry provided we make the folowing changes:

(i) The shift (3.1) is modified into $f_! = f + \mathrm{e}^{2\mathrm{i}\alpha}\frac{R}{2r}\phi$ and $\overline{f}_! = \overline{f} + \mathrm{e}^{-2\mathrm{i}\alpha}\frac{R}{2r}\overline{\phi}$.

(ii) $\overline{f}\phi - \overline{\phi}f$ in the boundary term of (3.6) is changed to $\mathrm{e}^{2\mathrm{i}\alpha}\overline{f}\phi - \mathrm{e}^{-2\mathrm{i}\alpha}\overline{\phi}f$.

(iii) $W + \overline{W}$ in the expression (3.9) for $\mathcal{L}_W$ is changed to $\mathrm{e}^{2\mathrm{i}\alpha}W + \mathrm{e}^{-2\mathrm{i}\alpha}\overline{W}$.

(iv) $W - \overline{W}$ in the boundary term of (3.12) is changed to $\mathrm{e}^{2\mathrm{i}\alpha}W - \mathrm{e}^{-2\mathrm{i}\alpha}\overline{W}$.

When W is quasihomogeneous, this change is done simply by operating the vector R-symmetry transformation $\mathrm{e}^{i\alpha F_V}$ on all field variables.

3.1.2 Gauge Theory (A-Type Supersymmetry)

We consider a gauge theory with gauge group G (a complact Lie group) and a matter representation V (a unitary representation of G). We write (ϕ, ψ, f) for the chiral multiplet valued in V, and $(\sigma, v_\mu, \lambda, D_E)$ for the vector multiplet fields. We denote the superpotential by $W(\phi)$ and the twisted superpotential by $\widetilde{W}(\sigma)$. Since the A-type supersymmetry includes the vector $U(1)$ R-symmetry, $W(\phi)$ must be quasi-homogeneous and we need to assign the vector R-charges so that

$$W(\lambda^R\phi) = \lambda^2 W(\phi). \tag{3.14}$$

We assume that R commutes with the gauge symmetry. The twisted superpotential $\widetilde{W}(\sigma)$ is arbitrary at the moment, although we shall later study in detail the gauged linear sigma models in which it takes a special form

$$\widetilde{W} = -\frac{1}{2\pi}t(\sigma), \tag{3.15}$$

where $t = \zeta - \mathrm{i}\theta$ is the complex combination of Fayet-Iliopoulos and Theta parameters.

Gauge kinetic term

Recall that $(\sigma, \lambda, D_E + \mathrm{i}\frac{v_{12}}{\sqrt{g}})$ transforms like a twisted chiral multiplet with axial R charge 2. Twisted chiral multiplets transform under A-type supersymmetry in the same way as chiral multiplets do under B-type supersymmetry. Therefore, the construction of the action of the Landau-Ginzburg models with B-type supersymmetry can give us a guide to construct gauge kinetic term and the twisted superpotential term. In view of the fact that the axial R-charge of σ is fixed to be 2, it is convenient to introduce, following (3.1),

$$\mathcal{E}_{!} := \left(D_E + \mathrm{i}\frac{v_{12}}{\sqrt{g}}\right) + \frac{1}{r}\sigma, \qquad \overline{\mathcal{E}}_{!} := \left(D_E - \mathrm{i}\frac{v_{12}}{\sqrt{g}}\right) + \frac{1}{r}\overline{\sigma}. \tag{3.16}$$

Using (3.6) as a guide, we obtain the gauge kinetic term

$$\begin{aligned}\int_{D^2} \mathcal{L}^{\text{gauge}}_{\text{kin}} \sqrt{g}\,\mathrm{d}^2x - \frac{1}{4e^2}\oint_{\partial D^2} \mathrm{Tr}\left[n^\mu \partial_\mu(\overline{\sigma}\sigma) \pm 2\mathrm{i}\left(D_E\sigma_2 - \frac{v_{12}}{\sqrt{g}}\sigma_1\right)\right]\mathrm{d}\tau \\ = \pm\frac{\mathrm{i}}{4e^2 r}\int_{D^2} Q^{A-}_{(\pm)} Q^{A+}_{(\pm)} \mathrm{Tr}\left[\overline{\mathcal{E}}_{!}\sigma + \overline{\sigma}\mathcal{E}_{!} \mp \frac{1}{r}\overline{\sigma}\sigma\right]\sqrt{g}\,\mathrm{d}^2x,\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}\mathcal{L}^{\text{gauge}}_{\text{kin}} = \frac{1}{2e^2}\mathrm{Tr}\Bigg[& D^\mu\overline{\sigma}D_\mu\sigma + \frac{1}{4}[\sigma, \overline{\sigma}]^2 + \left(D_E + \frac{1}{r}\sigma_1\right)^2 + \left(\frac{v_{12}}{\sqrt{g}} + \frac{1}{r}\sigma_2\right)^2 \\ &+ \frac{\mathrm{i}}{2}\langle\overline{\lambda}, \not{D}\lambda\rangle + \frac{\mathrm{i}}{2}\langle\not{D}\overline{\lambda}, \lambda\rangle + \frac{1}{2}\langle\lambda, \gamma_3[\overline{\sigma}, \lambda]\rangle + \frac{1}{2}\langle\overline{\lambda}, \gamma_3[\sigma, \overline{\lambda}]\rangle\Bigg].\end{aligned} \tag{3.18}$$

Here "$\frac{1}{e^2}\mathrm{Tr}(XY)$" is an invariant inner product of the adjoint representation. "e^2" is a collective notation for the gauge coupling constant for each gauge group factor. The term (3.17) is not only $\mathrm{A}_{(\pm)}$-type supersymmetric but also $Q^{A+}_{(\pm)}$ and $Q^{A-}_{(\pm)}$ exact.

Twisted superpotential term

Copying (3.12), we obtain the twisted superpotential term having $\mathrm{A}_{(\pm)}$-type supersymmetry:

$$\int_{D^2} \mathcal{L}_{\widetilde{W}} \sqrt{g}\,\mathrm{d}^2x \pm \oint_{\partial D^2}\left(\frac{\mathrm{i}}{2}\widetilde{W} - \frac{\mathrm{i}}{2}\overline{\widetilde{W}}\right)\mathrm{d}\tau \tag{3.19}$$

where

$$\mathcal{L}_{\widetilde{W}} = \frac{\mathrm{i}}{2r}\left(\widetilde{W} + \overline{\widetilde{W}}\right) - \frac{\mathrm{i}}{2}\left(\mathcal{E}^a_{!}\partial_a\widetilde{W} + \overline{\mathcal{E}}^a_{!}\partial_{\bar{a}}\overline{\widetilde{W}}\right) + \frac{1}{4}\langle\lambda^a, \lambda^b\rangle\partial_a\partial_b\widetilde{W} - \frac{1}{4}\langle\overline{\lambda}^a, \overline{\lambda}^b\rangle\partial_{\bar{a}}\partial_{\bar{b}}\overline{\widetilde{W}}. \tag{3.20}$$

In the particular case (3.15), it reads

$$\int_{D^2}\left(\frac{\mathrm{i}}{2\pi}\zeta(D_E)\sqrt{g}\mathrm{d}^2x-\frac{\mathrm{i}}{2\pi}\theta(F_v)\right)\pm\int_{\partial D^2}\mathrm{Im}\left(\frac{1}{2\pi}t(\sigma)\right)\mathrm{d}\tau, \tag{3.21}$$

where $F_v=\mathrm{d}v+\frac{\mathrm{i}}{2}[v,v]$ is the curvature of the gauge potential v. We see that θ is indeed a theta parameter.

Matter kinetic term

Finding the A-type supersymmetric kinetic term for the chiral multiplet with a possibly non-trivial vector R-charge is a whole new story. However, just as we have done in finding (3.6), we compute $\delta_2\delta_1$ of some combination of fields and see whether the result looks like a kinetic term. After some try and error, we arrive at the following result:

$$\begin{aligned}&\int_{D^2}\mathcal{L}^{\text{matter}}_{\text{kin}}\sqrt{g}\,\mathrm{d}^2x\,\pm\oint_{\partial D^2}\left[\frac{\mathrm{i}}{2}\langle\overline{\psi},\psi\rangle-\overline{\phi}\sigma_2\phi\right]\mathrm{d}\tau\\ &\quad=\pm\frac{1}{2r}\int_{D^2}Q^{A-}_{(\pm)}Q^{A+}_{(\pm)}\left[\langle\overline{\psi},\gamma_3\psi\rangle+\frac{\mathrm{i}}{r}\overline{\phi}\phi+2\overline{\phi}\left(-\mathrm{i}\frac{R}{2r}+\sigma_1\right)\phi\right]\sqrt{g}\,\mathrm{d}^2x,\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}\mathcal{L}^{\text{matter}}_{\text{kin}}&=D^\mu\overline{\phi}D_\mu\phi+\overline{\phi}\left[\frac{2R-R^2}{4r^2}-\mathrm{i}D_E-\mathrm{i}\frac{R}{r}\sigma_1+(\sigma_1^2+\sigma_2^2)\right]\phi+\overline{f}f\\&\quad+\frac{\mathrm{i}}{2}\langle\overline{\psi},\not{D}\psi\rangle+\frac{\mathrm{i}}{2}\langle\not{D}\overline{\psi},\psi\rangle+\left\langle\overline{\psi},\left[\left(-\mathrm{i}\frac{R}{2r}+\sigma_1\right)\gamma_3-\mathrm{i}\sigma_2\right]\psi\right\rangle\\&\quad-\mathrm{i}\langle\overline{\psi},\overline{\tilde{\lambda}}\rangle\phi-\mathrm{i}\overline{\phi}\langle\tilde{\lambda},\psi\rangle.\end{aligned} \tag{3.23}$$

We take the left hand side of (3.22) as the matter kinetic term. This is not only $\mathrm{A}_{(\pm)}$-type supersymmetric but also $Q^{A+}_{(\pm)}$ and $Q^{A-}_{(\pm)}$ exact.

Matter superpotential

Finally, let us discuss the superpotential term. Let us put

$$\mathcal{L}_W=-\frac{\mathrm{i}}{2}f^i\partial_iW-\frac{\mathrm{i}}{2}\overline{f}^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W}+\frac{1}{4}\langle\psi^i,\psi^j\rangle\partial_i\partial_jW-\frac{1}{4}\langle\overline{\psi}^{\bar{\imath}},\overline{\psi}^{\bar{\jmath}}\rangle\partial_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}. \tag{3.24}$$

Under the condition (3.14) or equivalently $\sum_i R_i\phi^i\partial_iW=2W$, any superconformal transformation (2.5) of $\mathcal{L}_W$ can be written as $\delta\mathcal{L}_W=\nabla_\mu J^\mu$ where J^μ is the same as (3.10). Note that

$$\begin{aligned}\widehat{n}\cdot J&=\frac{\mathrm{i}}{2}\langle\gamma^{\widehat{n}}\overline{\epsilon},\psi^i\rangle\partial_iW+\frac{\mathrm{i}}{2}\langle\gamma^{\widehat{n}}\epsilon,\overline{\psi}^{\bar{\imath}}\rangle\partial_{\bar{\imath}}\overline{W}\\&=\mp\frac{\mathrm{i}}{2}\left(\langle\overline{\epsilon},\psi^i\rangle\partial_iW+\langle\epsilon,\overline{\psi}^{\bar{\imath}}\rangle\partial_{\bar{\imath}}\overline{W}\right)\end{aligned} \tag{3.25}$$

for the $\mathrm{A}_{(\pm)}$-type supersymmetry, where (2.40) is used. Unlike in (3.11), it cannot be written as a supersymmetry variation of some combination of bulk fields. Thus, we can only say

$$\delta \int_{D^2} \mathcal{L}_W \sqrt{g}\, \mathrm{d}^2 x \;=\; \mp \frac{\mathrm{i}}{2} \oint_{\partial D^2} \left[\langle \overline{\epsilon}, \psi^i \rangle \partial_i W + \langle \epsilon, \overline{\psi}^{\overline{i}} \rangle \partial_{\overline{i}} \overline{W} \right] \mathrm{d}\tau \tag{3.26}$$

The right hand side is the so called Warner term [22]. It can only be cancelled by the supersymmetry transformation of a boundary interaction on a Chan-Paton factor of rank greater than one, which we turn to next.

3.2 Chan-Paton Factor

We introduce a class of boundary interactions in the gauge theory which are important by themselves but also can be used in cancellation of the Warner term.

First, let us introduce some notations that are suited to the boundary. The most relevant ones are the fermions

$$\uppsi \;:=\; \frac{1}{\sqrt{r}} \left[z^{\frac{1}{2}} \psi_-^{\{z\}} \pm \overline{z}^{\frac{1}{2}} \psi_+^{\{z\}} \right], \qquad \overline{\uppsi} \;:=\; \frac{1}{\sqrt{r}} \left[z^{\frac{1}{2}} \overline{\psi}_-^{\{z\}} \pm \overline{z}^{\frac{1}{2}} \overline{\psi}_+^{\{z\}} \right]. \tag{3.27}$$

The superscript $\{z\}$ is there to emphasize that the field components are in the z-frame, $\sqrt{\mathrm{d}z}, \sqrt{\mathrm{d}\overline{z}}$, as in (A.4). $\uppsi$ and $\overline{\uppsi}$ can be regarded as the boundary value of ψ and $\overline{\psi}$ with respect to the natural frame at the boundary ∂D^2, $\sqrt{r\mathrm{d}z/z} \equiv \pm\sqrt{r\mathrm{d}\overline{z}/\overline{z}}$, where the sign $\pm$ corresponds to the spin structure $(\pm)_0$ which is correlated with the supersymmetry type $\mathrm{A}_{(\pm)}$. Note that $\uppsi$ and $\overline{\uppsi}$ are antiperiodic along ∂D^2. Let ε_0 and $\overline{\varepsilon}_0$ be the constant and anticommuting variational parameters for $Q^{A-}_{(\pm)}$ and $Q^{A+}_{(\pm)}$ respectively. By definition, the supersymmetry parameters ϵ and $\overline{\epsilon}$ are given by

$$\epsilon = \varepsilon_0 \left(\mathbf{s}_{\mp\frac{1}{2}} \pm \widetilde{\mathbf{s}}_{\pm\frac{1}{2}} \right), \qquad \overline{\epsilon} = \overline{\varepsilon}_0 \left(\mathbf{s}_{\pm\frac{1}{2}} \pm \widetilde{\mathbf{s}}_{\mp\frac{1}{2}} \right). \tag{3.28}$$

We now introduce a non-contant and antiperiodic variational parameters along ∂D^2:

$$\varepsilon(\tau) = \sqrt{r}\varepsilon_0\, \mathrm{e}^{\mp\mathrm{i}\frac{\tau}{2r}}, \qquad \overline{\varepsilon}(\tau) = \sqrt{r}\overline{\varepsilon}_0\, \mathrm{e}^{\pm\mathrm{i}\frac{\tau}{2r}}. \tag{3.29}$$

The supersymmetry transformation of the boundary values of the fields can now be expressed in a simple way,

$$\begin{aligned} \delta\phi \;&=\; \varepsilon\, \uppsi, & \delta\overline{\phi} \;&=\; -\overline{\varepsilon}\, \overline{\uppsi}, \\ \delta\uppsi \;&=\; 2\overline{\varepsilon} \left[D_\tau \phi \pm \mathrm{i}\frac{R}{2r}\phi \mp \sigma_1 \phi \right], & \delta\overline{\uppsi} \;&=\; 2\varepsilon \left[-D_\tau \overline{\phi} \pm \mathrm{i}\overline{\phi}\frac{R}{2r} \mp \overline{\phi}\sigma_1 \right]. \end{aligned} \tag{3.30}$$

Also, the Warner term can be written as

$$\delta \int_{D^2} \mathcal{L}_W \sqrt{g}\, \mathrm{d}^2 x \;=\; \mp \frac{\mathrm{i}}{2} \oint_{\partial D^2} \left[\overline{\varepsilon}\, \psi^i \partial_i W + \varepsilon\, \overline{\psi}^{\bar{\imath}} \partial_{\bar{\imath}} \overline{W} \right] \mathrm{d}\tau \tag{3.31}$$

A boundary interaction is specified for a choice of a homogeneous and gauge invariant matrix factorization of the superpotential [23–25, 11]. The latter consists of the following data: a $\mathbf{Z}_2$ graded hermitian Chan-Paton vector space M, a polynomial function $Q(\phi)$ of $\phi \in V$ with values in $\mathrm{End}^{od}(M)$ obeying

$$Q(\phi)^2 = \mp \mathrm{i}\, W(\phi) \cdot \mathrm{id}_M, \tag{3.32}$$

even and unitary actions of the vector R-symmetry and the gauge symmetry on M, $\lambda \mapsto \lambda^{\mathbf{r}_*}$ and $g \mapsto \rho(g)$, which commute with each other and satisfy

$$\lambda^{\mathbf{r}_*} Q(\lambda^R \phi) \lambda^{-\mathbf{r}_*} = \lambda Q(\phi), \tag{3.33}$$

$$\rho(g)^{-1} Q(g\phi)\rho(g) = Q(\phi). \tag{3.34}$$

Given such a data, we can write down the boundary interaction

$$\mathcal{A}_\tau = \rho\left(\mathrm{i} v_\tau \mp \sigma_1\right) - \frac{1}{2}\, \psi^i \partial_i Q + \frac{1}{2}\, \overline{\psi}^{\bar{\imath}} \partial_{\bar{\imath}} Q^\dagger + \frac{1}{2}\, \{Q, Q^\dagger\} \mp \frac{\mathrm{i}}{2r}\, \mathbf{r}_*, \tag{3.35}$$

which is to be placed in the Chan-Paton factor,

$$\mathrm{tr}_M \left[P \exp\left(- \oint_{\partial D^2} \mathcal{A}_\tau \mathrm{d}\tau \right) \right]. \tag{3.36}$$

In (3.35), the fermionic and anti-periodic fields ψ and $\overline{\psi}$ come with ∂Q and $\partial Q^\dagger$ which appear to be bosonic and periodic. This might look strange. However, we should note that (3.36) needs to be understood as the *graded* Chan-Paton factor where Q and $Q^\dagger$ are regarded as fermionic and anti-periodic in a specific sense. See Appenidx B for detail. Then, (3.36) makes a perfect sense.

Let us study the supersymmetry transformation of the Chan-Paton factor. We first note that the combination $(\mathrm{i} v_\tau \mp \sigma_1)$ is invariant under the $\mathrm{A}_{(\pm)}$-type supersymmetry. Thus, we have

$$\begin{aligned} \delta \mathcal{A}_\tau \;\;=\;\; & -\frac{1}{2} 2\overline{\varepsilon} \left(D_\tau \phi^i \pm \frac{\mathrm{i}}{2r} (R\phi)^i \mp (\sigma_1 \phi)^i \right) \partial_i Q \\ & + \frac{1}{2} 2\varepsilon \left(-D_\tau \overline{\phi}^{\bar{\imath}} \pm \frac{\mathrm{i}}{2r} (\overline{\phi} R)^{\bar{\imath}} \mp (\overline{\phi}\sigma_1)^{\bar{\imath}} \right) \partial_{\bar{\imath}} Q^\dagger \\ & + \frac{1}{2} \Big\{ \varepsilon \psi^i \partial_i Q, Q^\dagger \Big\} + \frac{1}{2} \Big\{ Q, -\overline{\varepsilon} \overline{\psi}^{\bar{\imath}} \partial_{\bar{\imath}} Q^\dagger \Big\}. \end{aligned} \tag{3.37}$$

Using the infinitesimal forms of (3.33) and (3.34), the first two lines are written as

$$-\overline{\varepsilon}\left(D_\tau Q \pm \frac{\mathrm{i}}{2r}Q \mp \frac{\mathrm{i}}{2r}[\mathbf{r}_*, Q] \mp [\rho(\sigma_1), Q]\right)$$
$$+\varepsilon\left(-D_\tau Q^\dagger \pm \frac{\mathrm{i}}{2r}Q^\dagger \pm \frac{\mathrm{i}}{2r}[\mathbf{r}_*, Q^\dagger] \pm [\rho(\sigma_1), Q^\dagger]\right).$$

If we use $\frac{\mathrm{d}}{\mathrm{d}\tau}\varepsilon = \mp\frac{\mathrm{i}}{2r}\varepsilon$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}\overline{\varepsilon} = \pm\frac{\mathrm{i}}{2r}\overline{\varepsilon}$ that follows from the definition, it simplifies as

$$-D_\tau(\overline{\varepsilon}Q + \varepsilon Q^\dagger) - \left[\mp\rho(\sigma_1) \mp \frac{\mathrm{i}}{2r}\mathbf{r}_*\,,\, \overline{\varepsilon}Q + \varepsilon Q^\dagger\right]. \tag{3.38}$$

If we write $\mathcal{D}_\tau(-) = \frac{\mathrm{d}}{\mathrm{d}\tau}(-) + [\mathcal{A}_\tau, (-)]$, it can be written as

$$-\mathcal{D}_\tau(\overline{\varepsilon}Q + \varepsilon Q^\dagger) + \left[-\frac{1}{2}\,\psi^i\partial_i Q + \frac{1}{2}\,\overline{\psi}^{\bar{\imath}}\partial_{\bar{\imath}}Q^\dagger + \frac{1}{2}\,\{Q, Q^\dagger\}\,,\, \overline{\varepsilon}Q + \varepsilon Q^\dagger\right]. \tag{3.39}$$

By the fermionic nature of Q and $Q^\dagger$, a part of it cancels with the third line of (3.37) and another part can be simplified as

$$[\psi^i\partial_i Q, Q] = \psi^i\partial_i QQ - Q\psi^i\partial_i Q = \psi^i(\partial_i QQ + Q\partial_i Q) = \psi^i\partial_i(Q^2). \tag{3.40}$$

Collecting all, we have

$$\begin{aligned}\delta\mathcal{A}_\tau \;&=\; -\mathcal{D}_\tau\left(\overline{\varepsilon}Q + \varepsilon Q^\dagger\right)\\ &\quad -\frac{1}{2}\,\overline{\varepsilon}\,\psi^i\partial_i Q^2 + \frac{1}{2}\,\varepsilon\,\overline{\psi}^{\bar{\imath}}\partial_{\bar{\imath}}(Q^\dagger)^2 + \frac{1}{2}\overline{\varepsilon}\Big[\,Q^\dagger, Q^2\,\Big] + \frac{1}{2}\varepsilon\Big[\,Q, Q^{\dagger 2}\,\Big].\end{aligned} \tag{3.41}$$

Finally, if we use the matrix factorization property (3.32), the last two commutator terms vanish and the two preceding terms become $\pm\frac{\mathrm{i}}{2}\left(\overline{\varepsilon}\psi^i\partial_i W + \varepsilon\overline{\psi}^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W}\right)\mathrm{id}_M$, which is equal to the Warner term (3.31) except that the sign is opposite. Note that the term of the form $\mathcal{D}_\tau(-)$ can be ignored if we consider the variation of the Chan-Paton factor (3.36). Thus, the combination

$$\exp\left(-\int_{D^2}\mathcal{L}_W\sqrt{g}\,\mathrm{d}^2x\right)\mathrm{tr}_M\left[P\exp\left(-\oint_{\partial D^2}\mathcal{A}_\tau\mathrm{d}\tau\right)\right] \tag{3.42}$$

is invariant under the $\mathrm{A}_{(\pm)}$-type supersymmetry.

3.3 Boundary Condition

Let us now discuss the boundary conditions of the field variables. Since we have constructed the action which is automatically supersymmetric, the main requirement is the supersymmetry of the boundary conditions themselves as well as compatibility with the Euler-Lagrange equations. We shall consult the analysis of [11] which studied the boundary conditions for A-branes in Landau-Ginzburg models and B-branes in gauge theories with the type of boundary interactions discussed above, in a half of the flat Minkowski space with a timelike boundary.

As in the discussion on boundary interactions, it is convenient to use the spinor components with respect to the natural frames near the boundary; $\sqrt{r\mathrm{d}z/z}$ for S_- and $\pm\sqrt{r\mathrm{d}\overline{z}/\overline{z}}$ for S_+ which are identified at the boundary in the spin structure $(\pm)_0$. We denote them in upright symbols as

$$\begin{gathered}\uppsi_- := \sqrt{\tfrac{z}{r}}\psi_-^{\{z\}}, \quad \overline{\uppsi}_- := \sqrt{\tfrac{z}{r}}\overline{\psi}_-^{\{z\}}, \quad \uppsi_+ := \pm\sqrt{\tfrac{\overline{z}}{r}}\psi_+^{\{z\}}, \quad \overline{\uppsi}_+ := \pm\sqrt{\tfrac{\overline{z}}{r}}\overline{\psi}_+^{\{z\}}, \\ \uplambda_- := \sqrt{\tfrac{z}{r}}\lambda_-^{\{z\}}, \quad \overline{\uplambda}_- := \sqrt{\tfrac{z}{r}}\overline{\lambda}_-^{\{z\}}, \quad \uplambda_+ := \pm\sqrt{\tfrac{\overline{z}}{r}}\lambda_+^{\{z\}}, \quad \overline{\uplambda}_+ := \pm\sqrt{\tfrac{\overline{z}}{r}}\overline{\lambda}_+^{\{z\}}. \end{gathered} \tag{3.43}$$

We shall use the real coordinates near the boundary, ρ and τ, which are related to the complex coordinate by $z = \exp((\rho + \mathrm{i}\tau)/r)$.

A-branes in the Landau-Ginzburg model

For concreteness, we consider the Landau-Ginzburg model of n variables with a purely quadratic Kähler potential. To study A-branes, it is convenient to use real components x^I and f_0^I $(I = 1, \dots, 2n)$ of the scalars $\phi^i = x^{2i-1} + \mathrm{i}x^{2i}$ and $f^i = \mp\mathrm{i}(f_0^{2i-1} + \mathrm{i}f_0^{2i})$. We also use linear combinations $\uppsi^I$ and $\widetilde{\uppsi}^I$ of the fermions, $\uppsi_+^i - \uppsi_-^i = \uppsi^{2i-1} + \mathrm{i}\uppsi^{2i}$, $\overline{\uppsi}_+^{\overline{\imath}} - \overline{\uppsi}_-^{\overline{\imath}} = \uppsi^{2i-1} - \mathrm{i}\uppsi^{2i}$, $\uppsi_+^i + \uppsi_-^i = \widetilde{\uppsi}^{2i-1} + \mathrm{i}\widetilde{\uppsi}^{2i}$ and $\overline{\uppsi}_+^{\overline{\imath}} + \overline{\uppsi}_-^{\overline{\imath}} = \widetilde{\uppsi}^{2i-1} - \mathrm{i}\widetilde{\uppsi}^{2i}$. We denote by $\mathcal{J}^I{}_J$ the complex structure of $\mathbf{R}^{2n}$, with non-zero entries $\mathcal{J}^{2i}_{2i-1} = -\mathcal{J}^{2i-1}_{2i} = 1$, and by g_{IJ} the flat Kähler metric. It is also convenient to use ε_1 and ε_2 defined by $\varepsilon = \mathrm{i}\varepsilon_1 - \varepsilon_2$, $\overline{\varepsilon} = -\mathrm{i}\varepsilon_1 - \varepsilon_2$, and

$$N^I := \partial_\rho x^I + \mathrm{i}f_0^I. \tag{3.44}$$

Note that there is no reality for the fermionic fields and parameters in Euclidean signature, and also that N^I are complex valued. The $\mathrm{B}_{(\pm)}$-type supersymmetry transformation at the boundary reads,

$$\begin{aligned} \delta x^I &= \mathrm{i}\varepsilon_1\uppsi^I + \mathrm{i}\varepsilon_2\mathcal{J}^I{}_J\widetilde{\uppsi}^J, \\ \delta\uppsi^I &= -2\mathrm{i}\varepsilon_1\dot{x}^I + 2\varepsilon_2\mathcal{J}^I{}_J N^J, \\ \delta\widetilde{\uppsi}^I &= -2\varepsilon_1 N^I + 2\mathrm{i}\varepsilon_2\mathcal{J}^I{}_J\dot{x}^J, \\ \delta N^I &= \varepsilon_1\left(-\dot{\widetilde{\uppsi}}^I \pm \frac{1}{2r}\mathcal{J}^I{}_J\uppsi^J\right) + \varepsilon_2\left(-\mathcal{J}^I{}_J\dot{\uppsi}^J \mp \frac{1}{2r}\widetilde{\uppsi}^I\right), \end{aligned} \tag{3.45}$$

where $\dot{\mathcal{O}} = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{O}$. Except the $1/r$ terms, this is exactly the same as the Wick rotated version of the expression for A-type supersymmetry in the flat Minkowski space [11]. An invariant set of boundary conditions is found for a *totally real* submanifold of $\mathbf{C}^n = (\mathbf{R}^{2n}, \mathcal{J})$, that is, a middle dimensnional submanifold $L \subset \mathbf{R}^{2n}$ such that the tangent space $\mathrm{T}_x L$ at each point $x \in L$ is transversal to its $\mathcal{J}_x$-image. The conditions are, at each point of the boundary,

$$x \in L, \quad \psi \in \mathrm{T}_x L \otimes \mathbf{C}, \quad \widetilde{\psi},\ N \in \mathcal{J}_x \mathrm{T}_x L \otimes \mathbf{C}. \tag{3.46}$$

The next constraint is compatibility with the Euler-Lagrange equations. Here we make a discrimination between the kinetic term and the superpotential term. We consider the superpotential term as a perturbation and take into account the Euler-Lagrange equations only from the kinetic term. This approach is suitable in the localization computation where we take the limit of large Kähler metric, $g_{IJ} \to \infty$. As analyzed in [11], the compatibility requires that at each point $x \in L$ the tangent space $\mathrm{T}_x L$ is orthogonal to its $\mathcal{J}_x$-image, or equivalently, L is a *Lagrangian* submanifold with respect to the symplectic structure $\omega_{IJ} = \mathcal{J}_I^K g_{KJ}$. Moreover, if we stick to the boundary term as in (3.6) or (3.8), only a *linear* Lagrangian subspace is allowed. We can have a more general Lagrangian submanifold by adding a boundary term which is itself Q-exact. Alternatively, we can have an arbitrary Lagrangian submanifold by simply dropping the boundary term of (3.6). In that approach, however, Q-exactness of the kinetic term is lost.

Although we consider the superpotential term as a perturbation, there is one constraint from its presence. It is that the boundary potential in the superpotential term (3.12) must be bounded below. This requires that $\mp\mathrm{Im}(W)$ is bounded below at every infinity of L. (For the $\mathrm{B}^{\alpha}_{(\pm)}$-type supersymmetry, $\mp\mathrm{Im}(\,\mathrm{e}^{2\mathrm{i}\alpha}W)$ must be bounded below.)

B-branes in the gauge theory

Let us discuss the boundary conditions for B-branes in the gauge theory. Our main interests are gauge linear sigma models where in a generic locus of the FI-parameter space, the gauge group is mostly broken and we have the theory on the Higgs branch at low energies. In such a theory, the main part of the information on the brane is expected to be carried by the Chan-Paton data $(M, Q, \rho, \mathbf{r}_*)$. This is in contrast to the A-branes discussed above where the main part of the information is carried by the choice of a Lagrangian submanifold L. Nevertheless, we need to select boundary conditions for all bulk fields in order to complete the formulation of the theory.

Before starting, we write down the essential part of the supersymmetry transformation of the fields at the the boundary. For the chiral multiplet,

$$
\begin{aligned}
&\delta\phi = \varepsilon(\psi_- + \psi_+), \qquad \delta\overline{\phi} = -\overline{\varepsilon}(\overline{\psi}_- + \overline{\psi}_+),\\
&\delta(\psi_- + \psi_-) = 2\overline{\varepsilon}D'_\tau\phi, \qquad \delta(\overline{\psi}_- + \overline{\psi}_-) = -2\varepsilon D'_\tau\overline{\phi},\\
&\delta(\psi_- - \psi_+) = 2\mathrm{i}\overline{\varepsilon}\left[D_\rho\phi \mp \sigma_2\phi\right] \mp 2\mathrm{i}\varepsilon f,\\
&\delta(\overline{\psi}_- - \overline{\psi}_+) = -2\mathrm{i}\overline{\varepsilon}\left[D_\rho\overline{\phi} \mp \overline{\phi}\sigma_2\right] \mp 2\mathrm{i}\overline{\varepsilon}\overline{f},\\
&\delta f = \overline{\varepsilon}\left[\pm D_\rho(\psi_- + \psi_+) - \sigma_2(\psi_- + \psi_+) - (\overline{\lambda}_- + \overline{\lambda}_+)\phi \pm \mathrm{i}D'_\tau(\psi_- - \psi_+)\right],\\
&\delta\overline{f} = \varepsilon\left[\pm D_\rho(\overline{\psi}_- + \overline{\psi}_+) - (\overline{\psi}_- + \overline{\psi}_+)\sigma_2 - \overline{\phi}(\overline{\lambda}_- + \overline{\lambda}_+) \pm \mathrm{i}D'_\tau(\overline{\psi}_- - \overline{\psi}_+)\right].
\end{aligned} \tag{3.47}
$$

For the vector multiplet,

$$
\begin{aligned}
\delta\sigma^a &= \mathrm{i}\varepsilon_1\lambda^a + \mathrm{i}\varepsilon_2\mathcal{J}^a_b\widetilde{\lambda}^b,\\
\delta\lambda^a &= -2\mathrm{i}\varepsilon_1 D'_\tau\sigma^a + 2\varepsilon_2\mathcal{J}^a_bN^b,\\
\delta\widetilde{\lambda}^a &= -2\varepsilon_1N^a + 2\mathrm{i}\varepsilon_2\mathcal{J}^a_bD'_\tau\sigma^b,\\
\delta N^a &= \varepsilon_1\left(-D'_\tau\widetilde{\lambda}^a \pm \frac{1}{2r}\mathcal{J}^a_b\lambda^b\right) + \varepsilon_2\left(-\mathcal{J}^a_bD'_\tau\lambda^b \mp \frac{1}{2r}\widetilde{\lambda}^b\right).
\end{aligned} \tag{3.48}
$$

In the above expressions, D'_τ is defined to be $D'_\tau\varphi = D_\tau\varphi \mp (\sigma_1 - \frac{\mathrm{i}}{2r}R)\varphi$ and $D'_\tau\overline{\varphi} = D_\tau\overline{\varphi} \pm \overline{\varphi}(\sigma_1 - \frac{\mathrm{i}}{2r}R)$ for the components of the chiral multiplet of R-charge R and

$$
D'_\tau\upsilon = D_\tau\upsilon \mp [\sigma_1, \upsilon] \tag{3.49}
$$

for the components of the vector multiplet. For other notation and for more detail, see Appendix C.

Let us first discuss the boundary conditions for the chiral multiplet. In order for the boundary interaction (3.35) to be non-trivial, we would like the boundary values of ϕ as well as the boundary values ψ and $\overline{\psi}$ of $\psi_+ + \psi_-$ and $\overline{\psi}_+ + \overline{\psi}_-$ to be as free as possible. This leaves us with no choice on the boundary conditions:

$$
\begin{aligned}
&D_\rho\phi \mp \sigma_2\phi = 0, \qquad D_\rho\overline{\phi} \mp \overline{\phi}\sigma_2 = 0\\
&\psi_+ - \psi_- = 0, \qquad \overline{\psi}_+ - \overline{\psi}_- = 0,\\
&D_\rho(\psi_+ + \psi_-) \mp \sigma_2(\psi_+ + \psi_-) \mp (\overline{\lambda}_+ + \overline{\lambda}_-)\phi = 0,\\
&D_\rho(\overline{\psi}_+ + \overline{\psi}_-) \mp (\overline{\psi}_+ + \overline{\psi}_-)\sigma_2 \mp \overline{\phi}(\lambda_+ + \lambda_-) = 0,\\
&f = 0, \qquad \overline{f} = 0.
\end{aligned} \tag{3.50}
$$

This set of boundary conditions is closed under the supersymmetry — the supersymmetry transformation of the left hand sides all vanish if we use the boundary conditions. This is so for any configuration of the vector multiplet fields. See (3.47)-(3.48) and Appendix C. The above boundary conditions are also compatible with the Euler-Lagrange equations coming from the kinetic term (3.22) which includes a particular boundary interaction. If we have the superpotential W and the matrix factorization Q, the Euler-Lagrange equation changes. However, as long as we can treat these F-terms as perturbation, we can still use (3.50) as the boundary condition. This approach is particularly suited to the localization computation in which we take $1/e^2$ and the Kähler potential for ϕ to be infinitely large.

For the vector multiplet, the boundary condition is analogous to the A-brane boundary conditions for the chiral multiplet in the Landau-Ginzburg model. Indeed, the supersymmetry transformation (3.48) is of the same form as (3.45) except that the τ-derivative is replaced by the D'_τ-derivative given in (3.49). As in the discussion there, we need to choose a Lagrangian submanifold L of the space $\mathfrak{g}_{\mathbf{C}}$ of the values of $\sigma = \sigma_1 + \mathrm{i}\sigma_2$ which is equipped with a flat Kähler metric. Because of the commutator terms $[\sigma_1, \upsilon]$ in $D'_\tau \upsilon$ for $\upsilon = \sigma_2, \widetilde{\lambda}$ and λ, we also have additional conditions

$$[\sigma_1, \sigma_2] = 0 \quad \text{on } L, \tag{3.51}$$

$$[\sigma_1, \mathrm{T}_\sigma L] \subset \mathrm{T}_\sigma L \quad \forall \sigma \in L. \tag{3.52}$$

Because we are mainly interested in gauged linear sigma models, we do not want to break the gauge symmetry at the boundary by the choice of boundary conditions on the vector multiplet fields. That is, we do not want to have any constraint on the boundary values of the gauge transformations. This requires that L is invariant under the adjoint G action,

$$GL = L. \tag{3.53}$$

Finally, we would like to require that the boundary potential is bounded below. However, the precise meaning of the boundary potential is not so clear because the vector multiplet is interacting with the chiral multiplet and also with itself. In [11], we studied the effective boundary potential on the Coulomb branch in Abelian gauged linear sigma models and obtained a general set of D-branes by choosing L to be the real locus $\mathrm{i}\mathfrak{g} \subset \mathfrak{g}_{\mathbf{C}}$ where σ_1 is free and σ_2 is zero, or its small deformations. For a general compact Lie group G, the real locus $L = \mathrm{i}\mathfrak{g}$ obviously satisfies the conditions (3.51), (3.52) and (3.53). This motivates us to take the Lagrangian to be the real locus

$$L = \mathrm{i}\mathfrak{g} \subset \mathfrak{g}_{\mathbf{C}}, \tag{3.54}$$

or its deformations satisfying (3.51), (3.52) and (3.53).

Let us determine the supersymmetric boundary conditions corresponding to the real locus $L = \mathrm{i}\mathfrak{g}$. A set of boundary conditions containing $\sigma_2 = 0$ is

$$
\begin{aligned}
& v_\rho = 0, \quad \partial_\rho v_\tau = 0 \\
& \sigma_2 = 0, \quad \partial_\rho \sigma_1 = 0, \\
& \lambda_+ + \lambda_- = 0, \quad \overline{\lambda}_+ + \overline{\lambda}_- = 0, \\
& \partial_\rho(\lambda_+ - \lambda_-) = 0, \quad \partial_\rho(\overline{\lambda}_+ - \overline{\lambda}_-) = 0, \\
& \partial_\rho D_E = 0.
\end{aligned} \tag{3.55}
$$

These are obtained from the corresponding boundary conditions in Minkowski space [11]. Because of the Wick rotation which changed the reality of the fields, a part of the conditions in [11] need to be split into the real and imaginary parts. As a consequence, these boundary conditions are not closed under the supersymmetry. The supersymmetry transformation of (3.55) generates an infinite series of new conditions, consisting of even number of normal derivatives of each, $\partial_\rho^{2k} v_\rho = 0, \ldots, \partial_\rho^{2k+1} D_E = 0$, $k = 1, 2, 3, \ldots$ This might look problematic, but we will find in Section 5.2 a reasonable space of fields on the hemisphere which satisfies all these boundary conditions.[1] By the condition $v_\rho = 0$, the gauge symmetry is broken to those $g : D^2 \to G$ satisfying the Neumann boundary condition $\partial_\rho g = 0$, but the boundary values of g are unconstrained. The boundary conditions (3.50)-(3.55) are invariant under this residual gauge symmetry. If we also require $\partial_\rho^{2k+1} g = 0$ for $k = 0, 1, 2, \ldots$, then the extended boundary conditions are also gauge invariant.

The choice of real locus (3.54) has some simplifying features. First, under (3.55) the condition for the chiral multiplet becomes the purely Neumann boundary condition,

$$
\begin{aligned}
& \partial_\rho \phi = 0, \quad \partial_\rho \overline{\phi} = 0 \\
& \psi_+ - \psi_- = 0, \quad \overline{\psi}_+ - \overline{\psi}_- = 0, \\
& \partial_\rho(\psi_+ + \psi_-) = 0, \quad \partial_\rho(\overline{\psi}_+ + \overline{\psi}_-) = 0, \\
& f = 0, \quad \overline{f} = 0.
\end{aligned} \tag{3.56}
$$

This will facilitates the analysis considerably. Second, under these conditions, the boundary terms in the gauge kinetic term (3.17) and the matter kinetic term (3.22) both vanish.

[1] The same problem existsed in (3.46) where the last condition requires that the real and imaginary parts of N, $\partial_\rho x$ and f_0, should independently belong to $\mathcal{J}_{x(p)} \mathrm{T}_{x(p)} L$. This is stronger compared to the condition in Minkowski space where $\mathrm{i}f_0$ were real and only the sum $N = \partial_\rho x + \mathrm{i}f_0$ needs to be in that real subspace. As we shall see, the same solution applies when L is a linear Lagrangian subspace.

Thus, we may simply take $\mathcal{L}_{\rm kin}^{\rm gauge}+\mathcal{L}_{\rm kin}^{\rm matter}$ as the total kinetic terms. If we had taken another Lagrangian submanifold, even if it is a small deformation of (3.54), the computation becomes suddenly very hard.

In the direct computation of the partition function, we shall take the real locus (3.54), that is, the boundary condition (3.56)-(3.55) (plus the infinite series). However, as we shall see in Section 5.6, there is a simple trick to find the result for the deformations of (3.54), once the result for (3.54) is found.

3.4 Remarks On R-Symmetry

Here we make some remarks on the vector $U(1)$ R-symmetry of the gauge theory preserving $\mathrm{A}_{(\pm)}$-type supersymmetry.

Charge integrality

An important class of theories are those in which *A-twist* is possible. It requires not only the existence of a vector $U(1)$ R-symmetry but also its *charge integrality*: The R-charges of gauge invariant operators must be integers and they reduce modulo 2 to the statistics of the operators. The quasihomogeneity (3.14) of the superpotential $W(\phi)$ only assures the existence of the symmetry. The condition for the charge integrality is

$$\mathrm{e}^{\pi i R}=J\in G, \tag{3.57}$$

that is, the linear transformation $\mathrm{e}^{\pi i R}:V\to V$ is the same as the action of an element J of G. The charge integrality is extended to the boundary sector as the following condition on the brane data $(M,Q,\rho,\mathbf{r}_*)$:

$$\mathrm{e}^{\pi i\mathbf{r}_*}\rho(J)=\begin{cases}+1 & \text{on } M^{\rm ev}\\ -1 & \text{on } M^{\rm od}.\end{cases} \tag{3.58}$$

Gauge shift of R-charges

If the gauge group G has a center Z_G with non-zero Lie algebra $\mathfrak{z}_G$, we may shift the R-charges as

$$R\to R+\mathit{\Delta} \tag{3.59}$$

for any element $\mathrm{i}\mathit{\Delta}$ of $\mathfrak{z}_G$. Indeed, if R satisfies (3.14) and commutes with G, so does $R+\mathit{\Delta}$. We shall call this "gauge shift" of the R-charges. Note that it necessarily changes the matrix factorization data $(M, Q, \rho, \mathbf{r}_*)$ as

$$\mathbf{r}_* \to \mathbf{r}_* - \rho(\mathit{\Delta}), \tag{3.60}$$

in order for the condition (3.33) to remain satisfied.

When the charge integrality is assumed, and if the element J in (3.57) belongs to the identity component of the center Z_G, then, by the above shift with $\mathit{\Delta}$ given by $J = \mathrm{e}^{-\pi \mathrm{i} \mathit{\Delta}}$, we may assume that all the bulk R-charges R_i are even integers and that all the boundary R-charges r_j (eigenvalues of $\mathbf{r}_*$) are integers which reduce modulo 2 to the $\mathbf{Z}_2$-grading of M. We shall refer to such a choice as the "R^o-frame" and denote the R-charges with the superscript "o":

$$R_i^o \in 2\mathbf{Z}, \qquad r_j^o \in \begin{cases} 2\mathbf{Z} & \text{on } M^{\mathrm{ev}} \\ 2\mathbf{Z}+1 & \text{on } M^{\mathrm{od}}. \end{cases} \tag{3.61}$$

Dressing by gauge transformation should not change any physics, and therefore, the gauge shift of the R-charges is expected to be an unphysical operation. However, that is far from obvious if we look into the R (and $\mathbf{r}_*$) dependence of the action which we have constructed. If we look more closely, however, we find that it might be possible to undo the shift by suitable change of variables. We assume the boundary conditions, (3.56) and (3.55), so that we can avoid complication coming from the boundary terms of the gauge and matter kinetic terms. The R dependence appears in the matter kinetic term (3.23). The shift (3.59) can be absorbed under the change of variables

$$\sigma_1 \;\to\; \sigma_1 + \mathrm{i}\frac{\mathit{\Delta}}{2r}, \qquad D_E \;\to\; D_E - \mathrm{i}\frac{\mathit{\Delta}}{2r^2}. \tag{3.62}$$

Note that this violates the original reality of the fields. The shift of variables (3.62) does not change the gauge kinetic term (3.18), nor the matter superpotential term plus the boundary interaction (3.35), provided we also do the gauge shift of $\mathbf{r}_*$ (3.60). However, this does change the twisted superpotential term. Therefore, the gauge shift of the R-charge is not unphysical in general. However, if $\widetilde{W}(\sigma)$ is linear in σ as in (3.15), then the change is simply a constant shift of the action:

$$\Delta S = \begin{cases} \dfrac{1}{2} t(\mathit{\Delta}) & \text{for } (+)_0 \\[2mm] \dfrac{1}{2} \overline{t}(\mathit{\Delta}) & \text{for } (-)_0. \end{cases} \tag{3.63}$$

Whether the reality violating change of variables (3.62) is allowed is a subtle question. That would be OK as long as it does not change physical observables. We shall examine the effect on the partition function when we compute it.

Range of R-charges

Under the original reality of the field variables, each term of the real part of the bulk Lagrangian is non-negative except possibly the term $\overline{\phi}\frac{2R-R^2}{4r^2}\phi$ in the matter kinetic Lagrangian (3.23). This motivates us to require $2R - R^2 \geq 0$, that is, the R-charge of each component ϕ_i must be in the range

$$0 \leq R_i \leq 2. \tag{3.64}$$

In any known models of interest, we can find R-charges in this range. Indeed, since the R-charge of the superpotential $W(\phi)$ is 2, as long as the fields ϕ_i entering into $W(\phi)$ are concerned, if we choose all R_i to be non-negative they must also satisify the upper bound $R_i \leq 2$.

4 Parameter Dependence

The kinetic terms with appropriate boundary interaction which we constrcuted in the previous section are Q-exact where Q is one or both of the two preserved supercharges. See (3.6), (3.17) and (3.22). This means that the partition function does not change if we multiply any positive number in front of these terms. For example, the result should not depend on the gauge coupling constant e. This fact is very important and will be used in a crucial way in the computation (Section 5). In this section, we study how the partition function depends on other coupling constansts — chiral parameters that enter into the superpotential and the matrix factorization and twisted chiral parameters that enter into the twisted superpotential. We will again find some kind of Q-exactness and show that it depends holomorphically (*resp.* anti-holomorphically) on the twisted chiral parameters and does not depend on the chiral parameters if the system preserves the $\mathrm{A}_{(+)}$-type (*resp.* $\mathrm{A}_{(-)}$-type) supersymmetry.

4.1 Holomorphy

Let us consider the Landau-Ginzburg model preserving the $\mathrm{B}_{(\pm)}$-type supersymmetry. For δ_1 and δ_2 as in Section 3.1.1, we have

$$\delta_2\delta_1\overline{W} = \mathrm{i}\langle\overline{\epsilon}_1, \gamma^\mu\epsilon_2\rangle\partial_\mu\overline{W} + \langle\overline{\epsilon}_1, \overline{\epsilon}_2\rangle\left(-\mathrm{i}\overline{f}_!^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W} - \frac{1}{2}\langle\overline{\psi}^{\bar{\imath}}, \overline{\psi}^{\bar{\jmath}}\rangle\partial_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}\right). \tag{4.1}$$

In what follows in this subsection, we take off the anticommuting variational parameters from $\delta_i, \epsilon_i, \overline{\epsilon}_i$ but denote the result by the same symbols. If we use (2.39) and the Fierz identity (A.6), where we should be careful that ϵ_i's are now bosonic, we find

$$\nabla_\mu \frac{\langle \overline{\epsilon}_1, \gamma^\mu \epsilon_2\rangle}{\langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle} = -\frac{1}{r}. \tag{4.2}$$

Using this we find

$$\frac{1}{\langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle}\delta_2\delta_1\overline{W} = \nabla_\mu\left(\mathrm{i}\frac{\langle \overline{\epsilon}_1, \gamma^\mu \epsilon_2\rangle}{\langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle}\overline{W}\right) + \frac{\mathrm{i}}{r}\overline{W} - \mathrm{i}\overline{f}_!^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W} - \frac{1}{2}\langle\overline{\psi}^{\bar{\imath}}, \overline{\psi}^{\bar{\jmath}}\rangle\partial_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}. \tag{4.3}$$

Integrating over $D^2 = D_0^2$ and using (2.41), we have

$$\begin{aligned}\int_{D^2}\frac{1}{2\langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle}\delta_2\delta_1\overline{W}\sqrt{g}\mathrm{d}^2x &= \mp\int_{\partial D^2}\frac{\mathrm{i}}{2}\overline{W}\,\mathrm{d}\tau \\ &+ \int_{D^2}\left(\frac{\mathrm{i}}{2r}\overline{W} - \frac{\mathrm{i}}{2}\overline{f}_!^{\bar{\imath}}\partial_{\bar{\imath}}\overline{W} - \frac{1}{4}\langle\overline{\psi}^{\bar{\imath}}, \overline{\psi}^{\bar{\jmath}}\rangle\partial_{\bar{\imath}}\partial_{\bar{\jmath}}\overline{W}\right)\sqrt{g}\mathrm{d}^2x. \end{aligned}\tag{4.4}$$

Similarly, we have

$$\begin{aligned}\int_{D^2}\frac{-1}{2\langle \epsilon_1, \epsilon_2\rangle}\delta_2\delta_1 W\sqrt{g}\mathrm{d}^2x &= \pm\int_{\partial D^2}\frac{\mathrm{i}}{2}W\,\mathrm{d}\tau \\ &+ \int_{D^2}\left(\frac{\mathrm{i}}{2r}W - \frac{\mathrm{i}}{2}f_!^{i}\partial_i W + \frac{1}{4}\langle\psi^i, \psi^j\rangle\partial_i\partial_j W\right)\sqrt{g}\mathrm{d}^2x. \end{aligned}\tag{4.5}$$

The right hand sides of (4.4) and (4.5) are precisely the $\overline{W}$ and W parts of the superpotential term (3.12). So, it appears that the entire superpotential term is Q-exact for both $\mathrm{B}_{(+)}$ and $\mathrm{B}_{(-)}$-type supersymmetry. However, note that

$$\begin{aligned}(\delta_1,\delta_2) = (Q^{B+}_{(+)}, Q^{B-}_{(+)}): &\quad \langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle = \frac{2r}{1+|z|^2}, \quad \langle \epsilon_1, \epsilon_2\rangle = \frac{-2r|z|^2}{1+|z|^2},\\ (\delta_1,\delta_2) = (Q^{B+}_{(-)}, Q^{B-}_{(-)}): &\quad \langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle = \frac{-2r|z|^2}{1+|z|^2}, \quad \langle \epsilon_1, \epsilon_2\rangle = \frac{2r}{1+|z|^2}.\end{aligned}$$

We see that division by $\langle \overline{\epsilon}_1, \overline{\epsilon}_2\rangle$ (*resp.* $\langle \epsilon_1, \epsilon_2\rangle$) is possible on D_0^2 only for $\mathrm{B}_{(+)}$-type (*resp* $\mathrm{B}_{(-)}$-type) supersymmetry. Therefore, if the system preserves the $\mathrm{B}_{(+)}$-type supersymmetry, the $\overline{W}$-part of the superpotential term (3.12) is Q-exact while the W-part is not. Hence the partition function does not depend on the anti-chiral parameters but it can depend on the chiral parameters. In other words, it depends holomorphically on the chiral parameters. If the $\mathrm{B}_{(-)}$-type supersymmetry is preserved, the partition function depends anti-holomorphically on the chiral parameters.

By the A-B exchange, that is, by the replacement $(\epsilon, \overline{\epsilon}) \to (\widetilde{\epsilon}, \overline{\widetilde{\epsilon}})$, we have also shown that the partition function of the system preserving $\mathrm{A}_{(+)}$-type (*resp.* $\mathrm{A}_{(-)}$-type) supersymmetry depends holomorphically (*resp.* anti-holomorphically) on the twisted chiral parameters.

4.2 No Dependence

Let us next study the dependence of the chiral parameters in the systems preserving $\mathrm{A}_{(\pm)}$-type supersymmetry. We note that deformation of the superpotential W and/or the matrix factorization Q is constrained by

$$Q\varDelta Q + \varDelta Q = \mp \mathrm{i} \varDelta W, \tag{4.6}$$

so that the condition (3.32) remains satisfied. In particular, any deformation of W should be accompanied by some deformation of Q, while deformation of Q for a fixed W must satisfy $\{Q, \varDelta Q\} = 0$.

Let ϵ' and $\overline{\epsilon}'$ be the variational parameters for $Q^{A-}_{(\pm)}$ and $Q^{A+}_{(\pm)}$ in which the anticommuting parameters are stripped off. (I.e., $\epsilon' = \mathbf{s}_{\mp\frac{1}{2}} \pm \widetilde{\mathbf{s}}_{\pm\frac{1}{2}}$ and $\overline{\epsilon}' = \mathbf{s}_{\pm\frac{1}{2}} \pm \widetilde{\mathbf{s}}_{\mp\frac{1}{2}}$. See (2.28).) Then we have

$$\delta\left(\langle\gamma_3\overline{\epsilon}', \psi^i\rangle \partial_i W\right) = \nabla_\mu \left(\mathrm{i}\langle\gamma_3\overline{\epsilon}', \gamma^\mu\overline{\epsilon}\rangle W\right) + \langle\gamma_3\overline{\epsilon}', \epsilon\rangle \left(\mathrm{i}f^i\partial_i W - \frac{1}{2}\langle\psi^i, \psi^j\rangle\partial_i\partial_j W\right), \tag{4.7}$$

$$\delta\left(\langle\gamma_3\epsilon', \overline{\psi}^{\bar{\imath}}\rangle \partial_{\bar{\imath}} \overline{W}\right) = \nabla_\mu \left(-\mathrm{i}\langle\gamma_3\epsilon', \gamma^\mu\epsilon\rangle W\right) + \langle\gamma_3\epsilon', \overline{\epsilon}\rangle \left(\mathrm{i}\overline{f}^{\bar{\imath}}\partial_{\bar{\imath}} \overline{W} + \frac{1}{2}\langle\overline{\psi}^{\bar{\imath}}, \overline{\psi}^{\bar{\jmath}}\rangle\partial_{\bar{\imath}}\partial_{\bar{\jmath}} \overline{W}\right), \tag{4.8}$$

where we used $\nabla_\mu\langle\gamma_3\overline{\epsilon}', \gamma^\mu\overline{\epsilon}\rangle = 0$ etc, that follows from (2.38). Note that the big parentheses on the right hand sides are parts of the superpotential term (3.24) and that the coefficient in front, $\langle\gamma_3\overline{\epsilon}', \epsilon'\rangle$, is a constant (which is $\pm 2r$). This means that the W-part of the superpotential term $\mathcal{L}_W$ given by (3.24) is $Q^{A-}_{(\pm)}$-exact while the $\overline{W}$-part is $Q^{A+}_{(\pm)}$-exact. However, this fact does not mean that the superpotential term is supersymmetric since inside the parenthesis of $\delta(\,?\,)$ on the left hand sides have non-zero R-charges. (This is another way to see that the supersymmetry variation of $\mathcal{L}_W$ is the Warner term (3.26).) But it can used to study the effect of deformation of W.

Deformation of the matrix factorization results in the following change in the Chan-Paton factor,

$$\varDelta\, \mathrm{tr}_M\left[P\,\mathrm{e}^{-\oint_{\partial D^2}\mathcal{A}}\right] = -\oint_{\partial D^2} \mathrm{tr}_M\left[\left(P\,\mathrm{e}^{-\int_\tau^{\tau+2\pi r}\mathcal{A}}\right)\varDelta\mathcal{A}_\tau(\tau)\right]\mathrm{d}\tau, \tag{4.9}$$

where

$$\varDelta\mathcal{A}_\tau = -\frac{1}{2}\,\psi^i\partial_i\varDelta Q + \frac{1}{2}\,\overline{\psi}^{\bar{\imath}}\partial_{\bar{\imath}}\varDelta Q^\dagger + \frac{1}{2}\,\{\varDelta Q, Q^\dagger\} + \frac{1}{2}\,\{Q, \varDelta Q^\dagger\}. \tag{4.10}$$

The supersymmetry transformation of expressions of the form $\mathrm{tr}_M\left[\left(P\,\mathrm{e}^{-\int_\tau^{\tau+2\pi r}\mathcal{A}}\right)\mathcal{B}(\tau)\right]$ is, due to (3.41),

$$\delta\,\mathrm{tr}_M\left[\left(P\,\mathrm{e}^{-\int_\tau^{\tau+2\pi r}\mathcal{A}}\right)\mathcal{B}(\tau)\right] = \mathrm{tr}_M\left[\left(P\,\mathrm{e}^{-\int_\tau^{\tau+2\pi r}\mathcal{A}}\right)\delta'\mathcal{B}(\tau)\right] + \cdots \tag{4.11}$$

where

$$\delta'\mathcal{B}(\tau) \ := \ \delta\mathcal{B}(\tau) - [\,\overline{\varepsilon}Q + \varepsilon Q^\dagger,\, \mathcal{B}\,](\tau), \tag{4.12}$$

and $+\cdots$ is the term that cancels the Warner term from the bulk. Let us note

$$\delta'(\varDelta Q) \ = \ \varepsilon\Big(\psi^i\partial_i\varDelta Q - \{\varDelta Q, Q^\dagger\}\Big) - \overline{\varepsilon}\{Q, \varDelta Q\}, \tag{4.13}$$

$$\delta'(\varDelta Q^\dagger) \ = \ -\overline{\varepsilon}\Big(\overline{\psi}^{\overline{\imath}}\partial_{\overline{\imath}}\varDelta Q^\dagger + \{Q, \varDelta Q^\dagger\}\Big) - \varepsilon\{Q^\dagger, \varDelta Q^\dagger\}. \tag{4.14}$$

We see that the $\varDelta Q$-part of the change (4.9) of the Chan-Paton factor is $Q^{A-}_{(\pm)}$-exact while the $\varDelta Q^\dagger$-part is $Q^{A+}_{(\pm)}$-exact.

Let us now consider the deformation preserving the supersymmetry (4.6). The effects $\varDelta\mathcal{L}_W$ and (4.9) consist of terms which are exact under either $Q^{A+}_{(\pm)}$ or $Q^{A-}_{(\pm)}$. Therefore, the partition function does not change under the deformation. It is also reassuring to note that the total variation is exact under the sum $Q_{\rm tot} = Q^{A+}_{(\pm)} + Q^{A-}_{(\pm)}$,

$$\begin{aligned}
&\varDelta\left\{ \mathrm{e}^{-\int_{D^2}\mathcal{L}_W\sqrt{g}\,\mathrm{d}^2x}\,\mathrm{tr}_M\Big[P\,\mathrm{e}^{-\oint_{\partial D^2}\mathcal{A}}\Big]\right\}\\
&\quad = Q_{\rm tot}\left\{ \mathrm{e}^{-\int_{D^2}\mathcal{L}_W\sqrt{g}\,\mathrm{d}^2x}\,\mathrm{tr}_M\Big[P\,\mathrm{e}^{-\oint_{\partial D^2}\mathcal{A}}\Big]\int_{D^2}\mathcal{C}\sqrt{g}\mathrm{d}^2x\right.\\
&\qquad\qquad \left. +\ \mathrm{e}^{-\int_{D^2}\mathcal{L}_W\sqrt{g}\,\mathrm{d}^2x}\oint_{\partial D^2}\mathrm{tr}_M\Big[\Big(P\,\mathrm{e}^{-\int_\tau^{\tau+2\pi r}\mathcal{A}}\Big)\,\mathcal{B}(\tau)\,\Big]\mathrm{d}\tau\right\},
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
\mathcal{C} &:= \pm\frac{1}{4r}\langle\gamma_3\overline{\epsilon}', \psi^i\rangle\partial_i\varDelta W \ \pm\ \frac{1}{4r}\langle\gamma_3\epsilon', \overline{\psi}^{\overline{\imath}}\rangle\partial_{\overline{\imath}}\varDelta\overline{W},\\
\mathcal{B} &:= \frac{1}{2\sqrt{r}}\,\mathrm{e}^{\pm\mathrm{i}\tau/2r}\varDelta Q \ +\ \frac{1}{2\sqrt{r}}\,\mathrm{e}^{\mp\mathrm{i}\tau/2r}\varDelta Q^\dagger.
\end{aligned}$$

To summarize, the partition function does not change under deformation of (W, Q). That is, it is independent of the chiral parameters.

Analogous statement for systems preserving $\mathrm{B}_{(\pm)}$-type supersymmetry would be that the partition function is independent of the twisted chiral parameters. For example, in the non-linear sigma model with a Kähler manifold X as the target space, the partition function does not change under deformation of the Kähler class ω of X and the necessary deformation of the A-brane data; a Lagrangian submanifold L of X and a flat bundle E on L. This seems to be difficult to prove. Even if we were able to show that the deformation of (ω, L, E) changes the action by Q-exact terms, that would not be sufficient. The path-integral measure is usually constructed using the target space metric and therefore is expected to change if the Kähler class ω is deformed. Also the deformation of L results in the change of the boundary condition whose effect needs to be analyzed. This is in sharp contrast with what we did above: The path integral meaure and the boundary conditions are defined with no reference to the data of (W, Q) and hence Q-exactness of the change in the action under the deformation was sufficient to prove the invariance of the result. A related statement in the Landau-Ginzburg model is that the partition function does not depend on the deformation of the Lagrangian submanifold L on which $\mp\mathrm{Im}(W)$ is bounded from below. As discussed in [11], Q-exact terms at the boundary generate Hamiltonian deformations of L which are general deformations as Lagrangian submanifold when L has a trivial topology. However, since the boundary condition necessarily changes, it is again difficult to prove that the result does not change. In this paper, we shall simply assume or postulate the invariance under such deformations. Based on this postulate and explicit computation in simple cases, we shall find in Section 5.5 a reasonable proposal on the general expression for the partition function.

4.3 What Does It Compute?

[1] It was conjectured in [5] that the partition function Z_{S^2} on the round two-sphere with A-type supersymmetry computes e^{-K} where K is the Kähler potential of the space of twisted chiral parameters, when there is a spacetime physics interpretation. More generally, if the theory is A-twistable, the conjecture is

$$Z_{S^2} = {}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}}. \tag{4.16}$$

$|0\rangle_{\mathrm{RR}}$ is the canonical ground state defined via the infinitely long half-cigar in which the curved region is A-twisted [20]. In other words Z_{S^2} is equal to the partition function of the infintely long cigar in which the two curved regions are A and anti-A twisted. The latter is known as a component of the tt^* metric which is known to satisfy special differential equations [20]. Although there is some attempt [21], the real understanding of the realtion between Z_{S^2} and ${}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}}$ is still missing.

Now we would like to ask what does the hemisphere partition function compute? The combination of A-type supersymmetry and B-branes, holomorphic dependence on the twisted chiral parameters, no dependence of the chiral parameters all points to one possibility: the overlaps of supersymmetric ground states and the D-brane boundary states in the Ramond-Ramond sector, as studied in [13]. If the latter are defined as

[1] What is said in this subsection holds when A and B are swapped provided 'chiral' and 'twisted chiral' are swapped at the same time.

the partition function of the infinitely long half-cigar, in which the curved region is A-twisted and B-branes are placed at the boundary, then, they are independent on the chiral parameters but depend holomorphically on twisted chiral parameters in a partucular way, so that Picard-Fuchs type equations hold [13]. When nothing is inserted at the tip of the cigar, they are the overlaps of the state $|0\rangle_{\rm RR}$ and the boundary states, called *the central charges* of the D-branes. So, we would like to ask: *Does the partition function on the round hemisphere computes the D-brane central charge?*

$$Z_{D^2_{0(+)}}(\mathfrak{B}) \stackrel{?}{=} {}_{\rm RR}\langle 0|\mathfrak{B}\rangle_{\rm RR}, \qquad Z_{D^2_{\infty(-)}}(\mathfrak{B}) \stackrel{?}{=} {}_{\rm RR}\langle \mathfrak{B}|0\rangle_{\rm RR}. \tag{4.17}$$

We shall compute the partition functions in a large classes of examples and will observe that this is indeed the case whenever the D-brane central charge is known.

5 Computation

We now compute the partition function. In the gauge theory, we perform the direct computation by choosing the simplest boundary condition for the vector multiplet in which the Lagrangian submanifold is the real locus (3.54). We also compute the partition function for A-branes in Landau-Ginzburg model, where we discuss the choice of integration measure. Using that discussion and employing the holomorphy discussed in Section 4.1, we find the expression for the gauge theory partition function for more general choice of Lagrangian submanifold.

5.1 Supersymmetric Configuration

Since the kinetic terms (3.17) and (3.22) for the vector and the chiral multiplets are Q-exact, the result of the path-integral does not depend on the gauge coupling constant e and the constant $1/\mathrm{g}^2$ which we may put in front of the matter kinetic terms. If we take the limit $e \to 0$ and $\mathrm{g} \to 0$, the path-integral localizes at the configurations in which the real parts of these kinetic terms are minimized.

Let us find the condition for the minimization. Recall that the boundary condition (3.56) and (3.55) annihilates the boundary terms of (3.17) and (3.22). Recall also that, as long as the R-charges are in the range (3.64), $0 \leq R_i \leq 2$, the real part of the bulk kinetic Lagrangian, (3.18) and (3.23), is the sum of non-negative terms. Therefore, it is minimized when each term vanishes. This condition reads

$$D_\mu \sigma \,=\, [\sigma, \overline{\sigma}] \,=\, D_E + \frac{1}{r}\sigma_1 \,=\, \frac{v_{12}}{\sqrt{g}} + \frac{1}{r}\sigma_2 \,=\, 0, \tag{5.1}$$

$$D_\mu\phi \;=\; (2R-R^2)\phi \;=\; \sigma_1\phi \;=\; \sigma_2\phi \;=\; f \;=\; 0. \tag{5.2}$$

Since σ_2 must vanish at the boundary by the boundary condition (3.55), being required to be covariantly constant, it must vanish everywhere. This then implies $v_{12}=0$, that is, the gauge field is flat. Since the hemisphere is contractible, we may set $v_\mu=0$ everywhere. Then, σ_1 and ϕ are literally constants. Note that a component ϕ_i must vanish unless its R-charge $R_i\in[0,2]$ is either 0 or 2.

Almost the same condition follows from the supersymmetry. Vanishing of the supersymmetry transformation of the gaugino, $\delta\lambda=0$ and $\delta\overline{\lambda}=0$, requires precisely the same condition as (5.1). Vanishing for the matter fermion, $\delta\psi=0$ and $\delta\overline{\psi}=0$, requires

$$D_\rho\phi-\left(x_3\frac{R}{2r}\mp\sigma_2\right)\phi \;=\; D_\tau\phi-\mathrm{i}\left(\mp\frac{R}{2r}-x_3\sigma_2\right)\phi \;=\; \sigma_1\phi \;=\; f \;=\; 0, \tag{5.3}$$

where $x_3=\frac{|z|^2-1}{|z|^2+1}$. If we put $\sigma_2=v_\mu=0$, the first and the second conditions read $\partial_\rho\phi_i-x_3\frac{R_i}{2r}\phi_i=0$ and $\partial_\tau\phi_i\pm\mathrm{i}\frac{R_i}{2r}\phi_i=0$. The first has a non-zero and regular solution only when $R_i=0$ while the second has a non-zero and single valued solution only when R_i is an even integer. That is, ϕ_i is required to vanish unless $R_i=0$. When $R_i=0$, ϕ_i must be a constant. Thus, this is stronger than the minimization condition in that ϕ_i is required to vanish when $R_i=2$.

In what follows, we shall assume that all the R-charges are in the range

$$0<R_i<2, \tag{5.4}$$

or can be made into this range by using the gauge shift (3.59) if necessary. This is certainly the case in all known examples of interest. Then, the supersymmetry requires all fields including ϕ to vanish except that $\sigma_1=-rD_E$ must have a constant value $\upsigma_1\in\mathrm{i}\mathfrak{g}$. The moduli space of supersymmetric configurations is the space of $\upsigma_1$ modulo the contant gauge transformations, that is, the quotient of $\mathrm{i}\mathfrak{g}$ by the adjoint action of G. Or equivalently,

$$\mathrm{i}\mathfrak{g}/G \;\cong\; \mathrm{i}\mathfrak{t}/W_G, \tag{5.5}$$

where $\mathfrak{t}$ is the Lie algebra of a maximal torus T_G of G and W_G is the Weyl group of G.

Let us evaluate the action at the supersymmetric background. Since the hemisphere has area $2\pi r^2$ and the boundary has length $2\pi r$, the twisted superpotential term is

$$\begin{aligned} S_0 \;&=\; 2\pi r^2\frac{\mathrm{i}}{2r}\left(\widetilde{W}(\upsigma_1)+\overline{\widetilde{W}}(\upsigma_1)\right)\pm 2\pi r\frac{\mathrm{i}}{2}\left(\widetilde{W}(\upsigma_1)-\overline{\widetilde{W}}(\upsigma_1)\right)\\ &=\; \begin{cases} 2\pi\mathrm{i}\,r\widetilde{W}(\upsigma_1) & \text{for } (+)_0\\ 2\pi\mathrm{i}\,r\overline{\widetilde{W}}(\upsigma_1) & \text{for } (-)_0\end{cases}\end{aligned} \tag{5.6}$$

$$\overset{(3.15)}{=} \begin{cases} \mathrm{i}\, rt(\sigma_1) & \text{for } (+)_0 \\ \mathrm{i}\, r\bar{t}(\sigma_1) & \text{for } (-)_0 \end{cases} \tag{5.7}$$

The superpotential term vanishes, but the boundary interaction remains, $\mathcal{A}_\tau = \mp \mathrm{i}\rho(\sigma_1) \mp \frac{\mathrm{i}}{2r}\mathbf{r}_*$. It gives the Chan-Paton factor

$$\mathrm{tr}_M\, \mathrm{e}^{\pm\pi\mathrm{i}\mathbf{r}_*}\, \mathrm{e}^{\pm 2\pi\mathrm{i} r\rho(\sigma_1)} \tag{5.8}$$

For the gauge fixing, we take the standard Lorentz gauge. The gauge fixing term is given by

$$\mathcal{L}_{\text{gauge fixing}} = \frac{1}{2e^2}\mathrm{Tr}\left[(\nabla^\mu v_\mu)^2 + \bar{c}\nabla^\mu D_\mu c\right], \tag{5.9}$$

In the $e \to 0$ and $\mathrm{g} \to 0$ limit, all the terms that is cubic or higher in the fluctuation fields become irrelevant, and we are left with the classical action computed above, plus the terms that are quadratic in the fluctuation fields. The quadratic terms are given by the sum of the following

$$\begin{aligned} S_1 &= \int_{D^2}\left\{\overline{\phi}\left[\Delta + \frac{2R-R^2}{4r^2} + \mathrm{i}\frac{1-R}{r}\sigma_1 + \sigma_1^2\right]\phi + \overline{f}f \right.\\ &\qquad \left. + \left\langle \overline{\psi}, \left[\mathrm{i}\,\not{\nabla} + \left(-\mathrm{i}\frac{R}{2r} + \sigma_1\right)\gamma_3\right]\psi\right\rangle\right\}\sqrt{g}\mathrm{d}^2x, \qquad (5.10)\\ S_2 &= \int_{D^2}\mathrm{Tr}\Big[\sigma_1'\Delta\sigma_1' + \sigma_2\left(\Delta + \sigma_1^2 + \frac{1}{r^2}\right)\sigma_2 + 2g^{z\bar{z}}v_{\bar{z}}\left(\Delta + \sigma_1^2\right)v_z + (D_E')^2 \\ &\qquad + 2\mathrm{i}g^{z\bar{z}}(\partial_z v_{\bar{z}} + \partial_{\bar{z}}v_z)\sigma_1\sigma_1' - 2\mathrm{i}g^{z\bar{z}}(\partial_z v_{\bar{z}} - \partial_{\bar{z}}v_z)\frac{1}{r}\sigma_2 \\ &\qquad + \mathrm{i}\langle\overline{\lambda}, \not{\nabla}\lambda\rangle + \frac{1}{2}\langle\lambda, \sigma_1\gamma_3\lambda\rangle + \frac{1}{2}\langle\overline{\lambda}, \sigma_1\gamma_3\overline{\lambda}\rangle\Big]\sqrt{g}\mathrm{d}^2x, \qquad (5.11)\\ S_3 &= \int_{D^2}\mathrm{Tr}[\,\bar{c}\,\Delta c\,]\,\sqrt{g}\mathrm{d}^2x. \qquad (5.12) \end{aligned}$$

In the above expressions, we have absorbed the factor of e and g by a field redefinition. σ_1' is the non-zero modes of σ_1 and $D_E' := D_E + \sigma_1/r$. Δ is the Laplace operator $\Delta = dd^\dagger + d^\dagger d$ on functions and one-forms. For a $\mathfrak{g}$-valued field $\mathcal{O}$, we denoted $[\sigma_1, \mathcal{O}]$ simply by $\sigma_1\mathcal{O}$.

5.2 Mode Expansion

We shall regard the fields on the hemisphere as the restriction of the fields on the whole sphere. All the field components on the sphere can be considered as differentiable sections of the line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$ for some n. Indeed, scalars, spinors, and vectors are sections of $\mathcal{O}(0)$, $S_\pm = \mathcal{O}(\pm 1)$, and $\mathcal{O}(\pm 2)$ respectively. We may also regard the two-sphere as the coset space $SU(2)/U(1)$, where $U(1)$ is the diagonal subgroup consisting of elements of the form $h_u = \mathrm{diag}(u, u^{-1})$ with $|u| = 1$. In this description, the sections of $\mathcal{O}(n)$ are functions on $SU(2)$ obeying the condition $F(gh_u) = u^{-n}F(g)$. If $g^{(j)}_{m,m'}$ denote matrix elements of the spin j representation[1] of $SU(2)$, we find that $g^{(j)}_{m,-\frac{n}{2}}$ satisfies this condition. Of course, we need $j - \frac{n}{2}$ to be an integer. In fact, such matrix elements span the space of global sections of $\mathcal{O}(n)$ as orthogonal basis [26],

$$\int_{S^2} \left(g^{(j)}_{m,-\frac{n}{2}}\right)^* g^{(j')}_{m',-\frac{n}{2}} \sqrt{g}\mathrm{d}^2x \;=\; 4\pi r^2 \,\frac{\delta_{j,j'}\delta_{m,m'}}{2j+1}. \tag{5.13}$$

We also notice the reality

$$(g^{(j)}_{m,m'})^* = (-1)^{2j-m-m'} g^{(j)}_{-m,-m'}. \tag{5.14}$$

The Laplace and the Dirac operator act on these elements as

$$\Delta g^{(j)}_{m,-\frac{n}{2}} = \frac{j(j+1)}{r^2} g^{(j)}_{m,-\frac{n}{2}}, \quad \text{for } n = 0, \pm 2, \tag{5.15}$$

$$\not{\nabla} g^{(j)}_{m,\frac{1}{2}} = -\frac{j+\frac{1}{2}}{r} g^{(j)}_{m,-\frac{1}{2}}, \qquad \not{\nabla} g^{(j)}_{m,-\frac{1}{2}} = \frac{j+\frac{1}{2}}{r} g^{(j)}_{m,\frac{1}{2}}. \tag{5.16}$$

Let us find the relation to the coordinate z and the frames $\sqrt{\mathrm{d}z}$, etc, which we have been using. If we write an element of $SU(2)$ as

$$g = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1, \tag{5.17}$$

then

$$z = \overline{a}/\overline{b}. \tag{5.18}$$

Also, we may identify

$$\sqrt{\mathrm{d}z} = \frac{1}{\sqrt{2r}\cdot\overline{b}}, \qquad \sqrt{\mathrm{d}\overline{z}} = \frac{1}{\sqrt{2r}\cdot b}, \tag{5.19}$$

as well as $\mathrm{d}z = 1/(2r\overline{b}^2)$ and $\mathrm{d}\overline{z} = 1/(2rb^2)$. The mode expansion of the fields takes the form

$$\begin{aligned} \phi &= \sum_{j,m} \phi_{j,m}\, g^{(j)}_{-m,0}\mu_j, \\ \psi^{\{z\}}_- &= \sqrt{2r}\sum_{j,m} \psi_{j,m}\,\overline{b}\, g^{(j)}_{-m,\frac{1}{2}}\mu_j, \end{aligned}$$

[1] It is the $2j$-th symmetic tensor power of the doublet. The orthonormal basis $\{|m\rangle\}_{m=-j}^{j}$ is the natural one in that realization so that we have the reality (5.14).

$$
\begin{aligned}
\psi_+^{\{z\}} &= \sqrt{2r}\sum_{j,m}\widetilde{\psi}_{j,m}\, b\, g^{(j)}_{-m,-\frac{1}{2}}\mu_j,\\
v_z &= 2r\sum_{j,m} v_{j,m}\,\overline{b}^2 g^{(j)}_{-m,1}\mu_j,\\
v_{\overline{z}} &= 2r\sum_{j,m}\widetilde{v}_{j,m}\, b^2 g^{(j)}_{-m,-1}\mu_j,
\end{aligned}
$$

where $\mu_j = \sqrt{(2j+1)/(2\pi r^2)}$, and similarly for $\overline{\phi}$, $\overline{\psi}_{\pm}^{\{z\}}$, $f, \overline{f}$ as well as other components of the vector multiplet fields and the ghosts. Note that σ_1' and the ghosts do not include the $j=0$ mode. We would like to find which of the terms to be kept in order for the fields to satisfy the boundary conditions discussed in Section 3.3.

For this we need some information on the matrix elements $g^{(j)}_{m,m'}$, and relevant ones can be found in standard textbooks on angular momentum such as [27]. The elements for $g = R(\alpha,\beta,\gamma) = \mathrm{e}^{-\mathrm{i}\alpha\widehat{L}_3}\,\mathrm{e}^{-\mathrm{i}\beta\widehat{L}_2}\,\mathrm{e}^{-\mathrm{i}\gamma\widehat{L}_3}$ is written as

$$
g^{(j)}_{m,m'} = \mathrm{e}^{-\mathrm{i}m\alpha} d^j_{m,m'}(\beta)\,\mathrm{e}^{-\mathrm{i}m'\gamma}. \tag{5.20}
$$

This g has

$$
a = \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}\cos\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}},\quad b = \mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\sin\frac{\beta}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}}, \tag{5.21}
$$

and hence

$$
z = \mathrm{e}^{\frac{\rho+\mathrm{i}\tau}{r}} = \mathrm{e}^{\mathrm{i}\alpha}\cot\frac{\beta}{2}. \tag{5.22}
$$

We see that $\beta = 0$, $\frac{\pi}{2}$ and π correspond to the north pole $z=\infty$, the equator $|z|=1$ and the south pole $z=0$ respectively. The hemisphere D_0^2 is in the region $\frac{\pi}{2}\le\beta\le\pi$. The functions $d^j_{m,m'}(\beta)$ satisfy some identities [27]. The most important for us is

$$
d^j_{m,m'}(\pi-\beta) = (-1)^{j+m} d^j_{m,-m'}(\beta). \tag{5.23}
$$

Note that $\beta\to\pi-\beta$ precisely correspopnds to $\rho\to-\rho$ and is nothing but the reflection with respect to the equator.

We see from (5.23) that the function $g^{(j)}_{-m,0}$ is even or odd under the reflection $\rho\to-\rho$, depending on $j-m$ is even or odd. In particular they satify Neumann or Dirichlet boundary condition at the boundary,

$$
\begin{aligned}
\partial_\rho g^{(j)}_{-m,0}\Big|_{\partial D^2} &= 0 \quad \text{if } j-m \text{ is even},\\
g^{(j)}_{-m,0}\Big|_{\partial D^2} &= 0 \quad \text{if } j-m \text{ is odd}.
\end{aligned} \tag{5.24}
$$

Out of the $(2j+1)$ spin j scalar modes, $(j+1)$ of them satisfy the Neumann boundary condition while the remaining j of them satisfy the Dirichlet boundary condition. To see the boundary conditions for the spinors and the vectors, it is best to look at the componets in the natural frames at the boundary. The spinor modes in the frames $\sqrt{r\mathrm{d}z/z}$ and $\sqrt{r\mathrm{d}\overline{z}/\overline{z}}$ are

$$\begin{aligned}\varphi^{S_-}_{j,m} &= \sqrt{2}\mu_j(\overline{a}\overline{b})^{\frac{1}{2}}g^{(j)}_{-m,\frac{1}{2}} = \mu_j\,\mathrm{e}^{\mathrm{i}m\alpha}(\sin\beta)^{\frac{1}{2}}\,d^j_{-m,\frac{1}{2}}(\beta),\\ \varphi^{S_+}_{j,m} &= \sqrt{2}\mu_j(ab)^{\frac{1}{2}}g^{(j)}_{-m,-\frac{1}{2}} = \mu_j\,\mathrm{e}^{\mathrm{i}m\alpha}(\sin\beta)^{\frac{1}{2}}\,d^j_{-m,-\frac{1}{2}}(\beta),\end{aligned}$$

and the vector modes in the frames $r\mathrm{d}z/z$ and $r\mathrm{d}\overline{z}/\overline{z}$ are

$$\begin{aligned}\varphi^{V_-}_{j,m} &= 2\mu_j\overline{a}\overline{b}\,g^{(j)}_{-m,1} = \mu_j\,\mathrm{e}^{\mathrm{i}m\alpha}\sin\beta\,d^j_{-m,1}(\beta),\\ \varphi^{V_+}_{j,m} &= 2\mu_j ab\,g^{(j)}_{-m,-1} = \mu_j\,\mathrm{e}^{\mathrm{i}m\alpha}\sin\beta\,d^j_{-m,-1}(\beta).\end{aligned}$$

We see from (5.23) that the reflection $\rho\to-\rho$ does $\varphi^{\bullet_-}_{j,m}\to(-1)^{j-m}\varphi^{\bullet_+}_{j,m}$ for both spinor and vector modes. In particular, they satisfy

$$\begin{aligned}&\left(\varphi^{\bullet_-}_{j,m}-(-1)^{j-m}\varphi^{\bullet_+}_{j,m}\right)\Big|_{\partial D^2}=0,\\ &\left(\partial_\rho\varphi^{\bullet_-}_{j,m}+(-1)^{j-m}\partial_\rho\varphi^{\bullet_+}_{j,m}\right)\Big|_{\partial D^2}=0.\end{aligned}\tag{5.25}$$

In view of (5.24)-(5.25), the boundary conditions (3.56) and (3.55) requires the following constraints on the modes. For the chiral multiplet,

$$\begin{aligned}&\phi_{j,m},\,\overline{\phi}_{j,m}\text{: } j-m \text{ even},\\ &\psi_{j,m}=\pm(-1)^{j-m}\widetilde{\psi}_{j,m},\qquad \overline{\psi}_{j,m}=\pm(-1)^{j-m}\overline{\widetilde{\psi}}_{j,m},\\ &f_{j,m},\,\overline{f}_{j,m}\text{: } j-m \text{ odd}.\end{aligned}\tag{5.26}$$

For the vector multiplet,

$$\begin{aligned}&v_{j,m}=-(-1)^{j-m}\widetilde{v}_{j,m},\\ &(\sigma_1')_{j,m},\,(D_E)_{j,m}\text{: } j-m \text{ even},\\ &(\sigma_2)_{j,m}\text{: } j-m \text{ odd},\\ &\lambda_{j,m}=\mp(-1)^{j-m}\widetilde{\lambda}_{j,m},\qquad \overline{\lambda}_{j,m}=\mp(-1)^{j-m}\overline{\widetilde{\lambda}}_{j,m}.\end{aligned}\tag{5.27}$$

Recall that we also had infinitely many conditons: even number of ρ-derivatives of (3.55). In fact, the relations (5.24) and (5.25) hold also when $g^{(j)}_{-m,0}$ and $\varphi^{\bullet\pm}_{j,m}$ are replaced by $\partial^{2k}_\rho g^{(j)}_{-m,0}$ and $\partial^{2k}_\rho\varphi^{\bullet\pm}_{j,m}$. Therefore, the vector multiplet fields with the mode expansion obeying (5.27) satisfy also these infinitely many boundary conditions. For the ghosts, we have

$$c_{j,m},\,\overline{c}_{j,m}\text{: } j-m \text{ even}.\tag{5.28}$$

Note that the reality of fields, $\overline{\phi} = \phi^\dagger$, $\overline{f} = f^\dagger$, $v_\mu = v_\mu^\dagger$ and $\mathcal{O} = \mathcal{O}^\dagger$ for $\mathcal{O} = \sigma_1'$, σ_2, D_E', yields via (5.14) the following constraints:

$$\overline{\phi}_{j,m} = (-1)^m \phi^\dagger_{j,-m}, \qquad \overline{f}_{j,n} = (-1)^m f^\dagger_{j,-m}, \tag{5.29}$$

$$\widetilde{v}_{j,m} = (-1)^{m-1} v^\dagger_{j,-m}, \qquad \mathcal{O}_{j,m} = (-1)^m \mathcal{O}^\dagger_{j,-m}. \tag{5.30}$$

Let us write down the kinetic terms. To simplify the computation, we do the following trick. Given the fields on the hemisphere $D^2 = D^2_0$ we define the fields on the other hemisphere D^2_∞ in such a way that the action on D^2_∞ is equal to the one on D^2_0. This is done as follows. First let us denote by $x \mapsto x'$ the reflection at the equator, given by $z \mapsto \overline{z}^{-1}$, or equivalently $(\tau, \rho) \to (\tau, -\rho)$, or $(\alpha, \beta) \mapsto (\alpha, \pi - \beta)$. For a scalar $\mathcal{O}_N$ or $\mathcal{O}_D$ obeying the Neumann or Dirichlet boundary condition, we define the extension by $\mathcal{O}_N(x') = \mathcal{O}_N(x)$ or $\mathcal{O}_D(x') = -\mathcal{O}_D(x)$. For the spinors, the extension is defined by $\psi_\pm(x') = \psi_\mp(x)$, $\lambda_\pm(x') = -\lambda_\mp(x)$ (and similarly for the "bared" fields). For the vectors, we define it by $v_z(x') = -(\overline{z}^2 v_{\overline{z}})(x)$ and $v_{\overline{z}}(x') = -(z^2 v_z)(x)$. Then, it is easy to see that the action on D^2_∞ is the same as the original action on D^2_0. We can also see that the fields on D^2_∞ defined this way is equal to the naïve extension of the above mode expansions, from D^2_0 to D^2_∞. Thus, we find

$$\int_{D^2} \mathcal{L}\sqrt{g}\mathrm{d}^2x = \frac{1}{2}\int_{S^2} \mathcal{L}\Big|_{\substack{\text{naive}\\ \text{extension}}} \sqrt{g}\mathrm{d}^2 x. \tag{5.31}$$

Once the action is expressed as an integral on the whole sphere, we can use the orthogonality (5.13) for the evaluation.

Let us express the quadratic part of the action, (5.10), (5.11) and (5.12), in terms of the mode variables. For computation involving $g^{z\overline{z}}\partial_{\overline{z}} v_z$ and $g^{z\overline{z}}\partial_z v_{\overline{z}}$, it is useful to note

$$g^{z\overline{z}}\partial_{\overline{z}}\left(\overline{b}^2 g^{(j)}_{-m,1}\right) = -\frac{\sqrt{j(j+1)}}{2r^2} g^{(j)}_{-m,0}, \qquad g^{z\overline{z}}\partial_z\left(b^2 g^{(j)}_{-m,-1}\right) = \frac{\sqrt{j(j+1)}}{2r^2} g^{(j)}_{-m,0}. \tag{5.32}$$

The expressions are[2]

$$\begin{aligned} S_1 &= \sum_{j-m\text{ even}} \phi^\dagger_{j,m}\left[\frac{j(j+1)}{r^2} + \frac{2R-R^2}{4r^2} + \mathrm{i}\frac{1-R}{r}\sigma_1 + \sigma_1^2\right]\phi_{j,m} + \sum_{j-m\text{ odd}} f^\dagger_{j,m} f_{j,m} \\ &\quad + 2\,\mathrm{i}\sum_{j,m}(-1)^{m+\frac{1}{2}}\overline{\psi}_{j,-m}\left[\frac{j+\frac{1}{2}}{r} \mp \mathrm{i}(-1)^{j-m}\left(\sigma_1 - \mathrm{i}\frac{R}{2r}\right)\right]\psi_{j,m}, \end{aligned} \tag{5.33}$$

[2]The bosonic variables of the vector multiplet are rescaled as follows. For the scalars $\mathcal{O} = \sigma_1', \sigma_2, D_E'$, we do $\mathcal{O}_{j,m} \to \frac{1}{\sqrt{2}}\mathcal{O}_{j,m}$ for $m \geq 1$ but keep $\mathcal{O}_{j,0}$ intact. For the vector, we do $v_{j,m} \to \frac{1}{\sqrt{8}} v_{j,m}$ for $m \geq 1$ and $v_{j,0} \to \frac{1}{\sqrt{2}} v_{j,0}$.

$$
\begin{aligned}
S_2 \;=\;& \sum_{\substack{j\geq 1,\, m\geq 0\\ j-m \text{ even}}} (\sigma_1')^\dagger_{j,m} \frac{j(j+1)}{r^2} (\sigma_1')_{j,m} + \sum_{\substack{j\geq 1,\, m\geq 0\\ j-m \text{ odd}}} (\sigma_2)^\dagger_{j,m} \left[\frac{j(j+1)}{r^2} + \mathsf{\sigma}_1^2 + \frac{1}{r^2}\right] (\sigma_2)_{j,m} \\
&+ \sum_{j\geq 1\, m\geq 0} v^\dagger_{j,m} \left[\frac{j(j+1)}{r^2} + \mathsf{\sigma}_1^2\right] v_{j,m} + \sum_{\substack{m\geq 0\\ j-m \text{ even}}} (D_E')^\dagger_{j,m} (D_E')_{j,m} \\
&- \mathrm{i} \sum_{\substack{j\geq 1,\, m\geq 0\\ j-m \text{ even}}} \left(v^\dagger_{j,m} \frac{\sqrt{j(j+1)}}{r} \mathsf{\sigma}_1 (\sigma_1')_{j,m} + v_{j,m} \frac{\sqrt{j(j+1)}}{r} \mathsf{\sigma}_1 (\sigma_1')^\dagger_{j,m} \right) \\
&- \mathrm{i} \sum_{\substack{j\geq 1,\, m\geq 0\\ j-m \text{ odd}}} \left(v^\dagger_{j,m} \frac{\sqrt{j(j+1)}}{r} \frac{1}{r} (\sigma_2)_{j,m} - v_{j,m} \frac{\sqrt{j(j+1)}}{r} \frac{1}{r} (\sigma_2)^\dagger_{j,m} \right) \\
&+ 2\,\mathrm{i} \sum_{j,m} (-1)^{m+\frac{1}{2}} \overline{\lambda}_{j,-m} \left[\frac{j+\frac{1}{2}}{r} \pm \mathrm{i}(-1)^{j-m} \mathsf{\sigma}_1 \right] \lambda_{j,m}, \qquad (5.34)
\end{aligned}
$$

$$
S_3 \;=\; \sum_{\substack{j\geq 1\\ j-m \text{ even}}} (-1)^m \overline{c}_{j,-m} \frac{j(j+1)}{r^2} c_{j,m}. \qquad (5.35)
$$

5.3 Determinants

We are now ready to compute the fluctuation determinants. We choose a maximal torus T_G of G so that that the supersymmetric background $\sigma_1 = \mathsf{\sigma}_1$ belongs to its Lie algebra $\mathfrak{t}$ times i. We choose a Weyl chamber in $\mathrm{i}\mathfrak{t}^*$ and write $\alpha > 0$ if α is a positive root with respect to that. We write d_G and l_G for the dimension and the rank of G, and put $d_V := \dim_{\mathbf{C}} V$.

Let us first consider a single chiral multiplet that has charge $+1$ under a single $U(1)$ gauge group and vector R-charge R. The big parenthesis of the first line of (5.33) factorizes as

$$
\left(\frac{j}{r} + \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)\left(\frac{j+1}{r} - \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)
$$

Thus the determinant is

$$
\begin{aligned}
\frac{\det_F}{\det_B^{\frac{1}{2}}} &= \frac{\prod_{j=\frac{1}{2}}^{\infty} \left(\frac{j+\frac{1}{2}}{r} + \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)^{j+\frac{1}{2}} \left(\frac{j+\frac{1}{2}}{r} - \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)^{j+\frac{1}{2}}}{\prod_{j=1}^{\infty} \left(\frac{j}{r} + \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)^{j+1} \left(\frac{j+1}{r} - \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)^{j+1}} \\
&= \frac{1}{\prod_{j=0}^{\infty} \left(\frac{j}{r} + \mathrm{i}\left(\mathsf{\sigma}_1 - \mathrm{i}\frac{R}{2r}\right)\right)} \qquad (5.36)
\end{aligned}
$$

For a chiral multiplet of weight Q under T_G, the result is obtained from the above by the replacement $\sigma_1 \to Q(\sigma_1)$.

Next we consider the vector multiplet. The fermionic determinant is straightforward,

$$\det{}_F = \prod_{j=\frac{1}{2}}^{\infty} \left[\left(\frac{j+\frac{1}{2}}{r} \right)^{(2j+1)l_G} \prod_{\alpha>0} \left(\left(\frac{j+\frac{1}{2}}{r} \right)^2 + \alpha(\sigma_1)^2 \right)^{2j+1} \right]. \tag{5.37}$$

The bosonic sector is complicated. We first notice that it splits into $j-m$ even part involving v and σ_1' and $j-m$ odd part involving v and σ_2. We also notice that the $m=0$ modes are real or pure imaginary while the $m \geq 1$ modes are complex. After some computation, we find

$$\det{}_B^{\frac{1}{2}} = \prod_{j=1}^{\infty} \left[\left(\frac{j(j+1)}{r^2} \right)^{(j+1)d_G + jl_G} \prod_{\alpha>0} \left(\left(\frac{j(j+1)}{r^2} + \alpha(\sigma_1)^2 \right)^2 + \frac{\alpha(\sigma_1)^2}{r^2} \right)^j \right]. \tag{5.38}$$

The ratio is

$$\frac{\det_F}{\det_B^{\frac{1}{2}}} = \frac{\prod_{j=1}^{\infty} \left[\left(\frac{j}{r} \right)^{l_G} \prod_{\alpha>0} \left(\frac{j}{r} + \mathrm{i}\alpha(\sigma_1) \right) \left(\frac{j}{r} - \mathrm{i}\alpha(\sigma_1) \right) \right]}{\prod_{j=1}^{\infty} \left(\frac{j(j+1)}{r^2} \right)^{(j+1)d_G}}. \tag{5.39}$$

Finally, the ghost determinant is

$$\det{}_{gh} = \prod_{j=1}^{\infty} \left(\frac{j(j+1)}{r^2} \right)^{(j+1)d_G}. \tag{5.40}$$

We notice that it cancels againt the denominator of (5.39).

We find two problems in the above result. One is that it is a product of infinite factors and the other is that each factor is dimensionful. The former will be dealt with by regularization and renormalization. The latter is simply because we were not careful in defining the measure, even formally. If φ is a field of canonical demension d_φ and if there is a coupling constant factor $1/g_0^2$ in front of the kinetic term, we should define the measure by

$$\mathcal{D}\varphi = \sqrt{\det \left(\frac{\Lambda_0^{D-2d_\varphi} (\varphi_n, \varphi_m)}{g_0^2} \right)} \prod_n \mathrm{d}a_n, \tag{5.41}$$

for some mode expansion $\varphi(x) = \sum_n \varphi(x) a_n$, where D is the spacetime dimension, (φ_n, φ_m) is the inner product of the modes defined by the spacetime integration, and

Λ_0 is a parameter of mass dimension which is usually taken to be the ultra-violet cut-off. In the present context, we should take $D=2$, $g_0=1$ (as e and g are absorbed into fields), and we had chosen the modes so that $(\varphi_n,\varphi_m)=\delta_{n,m}$. The net effect is to multiply Λ_0^k to each factor of length dimension k. For example, we should do the replacement

$$\left(\frac{j}{r}+\mathrm{i}\left(\sigma_1-\mathrm{i}\frac{R}{2r}\right)\right)\longrightarrow\frac{1}{\Lambda_0}\left(\frac{j}{r}+\mathrm{i}\left(\sigma_1-\mathrm{i}\frac{R}{2r}\right)\right),\tag{5.42}$$

in the denominator factor of (5.36).

Let us now discuss the regularization. We take the naïve cut off[1] where we introduce an upper bound N of the product over j. We will eventually take the $N\to\infty$ limit after a suitable renomalization of coupling constants. Since j/r corresponds to the energy scale, we may interpret $\Lambda_0:=N/r$ as the ultra-violet cut-off which we take to be the same Λ_0 in (5.41). Using the formula for the gamma function

$$\Gamma(z)=\lim_{N\to\infty}\frac{N!\,(N+1)^z}{\prod_{j=0}^{N}(j+z)},\tag{5.43}$$

together with Stirling's formula $N!\sim\sqrt{2\pi}N^{N+\frac{1}{2}}\,\mathrm{e}^{-N}$, we find

$$\prod_{j=0}^{r\Lambda_0}\frac{1}{\Lambda_0}\left(\frac{j}{r}+a\right)\;=\;\frac{1}{(r\Lambda_0)^{r\Lambda_0+1}}\prod_{j=0}^{r\Lambda_0}(j+ra)\;=\;\sqrt{2\pi}(r\Lambda_0)^{-\frac{1}{2}+ra}\,\mathrm{e}^{-r\Lambda_0}\frac{1}{\Gamma(ra)}.\tag{5.44}$$

Now the determinants make sense. The factor from the chiral multiplet is

$$\begin{aligned}Z_{\text{chiral}}\;&=\;(2\pi)^{-\frac{d_V}{2}}\times\\&\exp\left(d_Vr\Lambda_0+\sum_i\left[\frac{1-R_i}{2}-\mathrm{i}rQ_i(\sigma_1)\right]\log(r\Lambda_0)\right)\prod_i\Gamma\left(\mathrm{i}rQ_i(\sigma_1)+\frac{R_i}{2}\right).\end{aligned}\tag{5.45}$$

The factor from the vector multiplet and the ghost is

$$\begin{aligned}Z_{\text{vector+ghost}}\;&=\;\frac{(2\pi)^{\frac{d_G}{2}}\exp\left(-d_Gr\Lambda_0-\frac{d_G}{2}\log(r\Lambda_0)\right)}{\prod_{\alpha>0}\Gamma\left(\mathrm{i}r\alpha(\sigma_1)\right)\Gamma\left(-\mathrm{i}r\alpha(\sigma_1)\right)r^2\alpha(\sigma_1)^2}\\&=\;(2\pi)^{\frac{d_G}{2}}\exp\left(-d_Gr\Lambda_0-\frac{d_G}{2}\log(r\Lambda_0)\right)\prod_{\alpha>0}\frac{\sinh(\pi r\alpha(\sigma_1))}{\pi r\alpha(\sigma_1)}\end{aligned}\tag{5.46}$$

where we used $\Gamma(1+z)=z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$.

[1]This discussion is important and leads to the identification of the 2d central charge (5.81) below. This had been done on the two-sphere by Sungjay Lee as presented in conferences [28] and by the other authors of [4] [29], and also in 4d by Pestun.

5.4 The Result

Following (5.41), the zero mode measure is

$$\left(\frac{r\Lambda_0}{e}\right)^{d_G} \mathrm{d}^{d_G}\sigma_1. \tag{5.47}$$

We use the following formula that holds for an adjoint invariant function $F(\sigma_1)$,

$$\frac{1}{\mathrm{vol}(G)}\int_{\mathrm{i}\mathfrak{g}} \mathrm{d}^{d_G}\sigma_1\, F(\sigma_1) \;=\; \frac{1}{|W_G|}\int_{\mathrm{i}\mathfrak{t}} \mathrm{d}^{l_G}\sigma_1 \prod_{\alpha>0}\alpha(\sigma_1)^2 \cdot F(\sigma_1). \tag{5.48}$$

Collecting everything, for the brane data $\mathfrak{B} = (M, Q, \rho, \mathbf{r}_*)$ we find

$$\begin{aligned} Z_{D^2_{(+)}}(\mathfrak{B}) \;=\; & C\left(\frac{\Lambda_0}{e}\right)^{d_G} \exp\left((d_V - d_G)r\Lambda_0 + \frac{\widehat{c}}{2}\log(r\Lambda_0)\right) \\ & \times \int_{\mathrm{i}\mathfrak{t}} r^{l_G}\mathrm{d}^{l_G}\sigma_1 \prod_{\alpha>0} r\alpha(\sigma_1)\sinh(\pi r\alpha(\sigma_1)) \prod_i \Gamma\left(\mathrm{i}rQ_i(\sigma_1) + \frac{R_i}{2}\right) \\ & \qquad \times \exp\left(2\pi\mathrm{i}r\widetilde{W}(\sigma_1) - \mathrm{i}r\sum_i Q_i(\sigma_1)\log(r\Lambda_0)\right) \mathrm{tr}_M\left(\mathrm{e}^{\pi i \mathbf{r}_*}\,\mathrm{e}^{2\pi r\rho(\sigma_1)}\right), \end{aligned} \tag{5.49}$$

where C is a numercal factor and

$$\widehat{c} \;:=\; \sum_i (1 - R_i) - d_G. \tag{5.50}$$

$Z_{D^2_{(-)}}(\mathfrak{B})$ is the same as (5.49) except that we need to replace $\widetilde{W}$ by $\overline{\widetilde{W}}$ and invert the exponents of the Chan-Paton factor. (See (5.6) and (5.8).)

Before removing the cut-off Λ_0 we need to do a renormalization. We consider the following cut-off dependent local counter terms:

$$\text{dilaton} \;=\; \frac{\widehat{c}}{2}\log(\Lambda_0/\Lambda), \tag{5.51}$$

$$\text{boundary potential} \;=\; \frac{1}{2\pi}(d_V - d_G)\Lambda_0, \tag{5.52}$$

$$\Delta\widetilde{W}(\sigma) \;=\; \frac{1}{2\pi}\mathrm{tr}_V(\sigma)\log(\Lambda_0/\Lambda). \tag{5.53}$$

Here Λ is a finite energy scale. There is also an overall multiplicative divergence $\Lambda_0^{d_G}$ which we decide to absorb by a multiplicative change of measure, say, by replacing e in (5.47) by Λ_0. Then, we have cut-off independent expressions:

$$Z_{D^2_{(+)}}(\mathfrak{B}) \;=\; C(r\Lambda)^{\widehat{c}/2}\int_{\mathrm{i}\mathfrak{t}} r^{l_G}\mathrm{d}^{l_G}\sigma_1\prod_{\alpha>0} r\alpha(\sigma_1)\sinh(\pi r\alpha(\sigma_1))\prod_i\Gamma\Big(\mathrm{i}rQ_i(\sigma_1)+\frac{R_i}{2}\Big) \quad (5.54)$$

$$\times\exp\left(2\pi\mathrm{i}r\widetilde{W}(\sigma_1)-\mathrm{i}r\sum_i Q_i(\sigma_1)\log(r\Lambda)\right)\mathrm{tr}_M\left(\mathrm{e}^{\pi i\mathbf{r}_*}\,\mathrm{e}^{2\pi r\rho(\sigma_1)}\right),$$

$$Z_{D^2_{(-)}}(\mathfrak{B}) \;=\; C(r\Lambda)^{\widehat{c}/2}\int_{\mathrm{i}\mathfrak{t}} r^{l_G}\mathrm{d}^{l_G}\sigma_1\prod_{\alpha>0} r\alpha(\sigma_1)\sinh(\pi r\alpha(\sigma_1))\prod_i\Gamma\Big(\mathrm{i}rQ_i(\sigma_1)+\frac{R_i}{2}\Big) \quad (5.55)$$

$$\times\exp\left(2\pi\mathrm{i}r\overline{\widetilde{W}}(\sigma_1)-\mathrm{i}r\sum_i Q_i(\sigma_1)\log(r\Lambda)\right)\mathrm{tr}_M\left(\mathrm{e}^{-\pi i\mathbf{r}_*}\,\mathrm{e}^{-2\pi r\rho(\sigma_1)}\right).$$

5.5 A-Branes

Let us compute the partition function of the Landau-Ginzburg model preserving $\mathrm{B}_{(\pm)}$-type supersymmetry. We consider the model of $\mathbf{C}^n$ valued variable $\phi=(\phi^1,\ldots,\phi^n)$ with a superpotential W and a flat Kähler metric $\mathrm{g}^2\sum_i|\mathrm{d}\phi^i|^2$. We start with the case where the brane $L_\pm$ is a linear Lagrangian subspace of $\mathbf{R}^{2n}$ such that $\mp\mathrm{Im}(W)$ is bounded from below on $L_\pm$. In this case, the boundary term in the action (3.6) or (3.8) vanishes and the usual kinetic term itself is Q-exact, so that the usual localization is valid. In the limit $\mathrm{g}\to\infty$, the path-integral localizes on the supersymmetric locus,

$$\partial_\mu\phi=0,\qquad f_!=0. \tag{5.56}$$

The classical Lagrangian is

$$S_0=\begin{cases}2\pi\mathrm{i}rW(\phi) & \text{for } (+)_0,\\ 2\pi\mathrm{i}r\overline{W(\phi)} & \text{for } (-)_0.\end{cases} \tag{5.57}$$

The fluctuation determinant is independent of the location ϕ. The scalars tangent (*resp.* normal) to the brane obey Neumann (*resp.* Dirichlet) boundary condition and the fermions obey the the corresponding boundary conditon. We also need to omit the bosonic zero modes. Employing the mode expansions obatined in the gauge theory, we find

$$\frac{\det_F}{\det_B^{\frac{1}{2}}}=\left[\frac{\displaystyle\prod_{j=\frac12}^{\infty}\frac{1}{\Lambda_0}\left(\frac{j+\frac12}{r}\right)^{2j+1}}{\displaystyle\prod_{j=1}^{\infty}\frac{1}{\Lambda_0^2}\left(\frac{j(j+1)}{r^2}\right)^{\frac{j+1}{2}+\frac{j}{2}}}\right]^n=\frac{1}{(r\Lambda_0)^{n/2}}, \tag{5.58}$$

where Λ_0 is an ultra-violet cut off. The measure for the scalar zero modes is, following (5.41),

$$(r\Lambda_0)^n \mathrm{g}^n \mathrm{dvol}_{L_\pm}, \tag{5.59}$$

where $\mathrm{dvol}_{L_\pm}$ is the volume element of $L_\pm$ associated to the metric induced from the metric $\sum_i |\mathrm{d}\phi^i|^2$ of $\mathbf{C}^n$. The result has a cut-off dependence which can be renormalized by the dilaton shift $\frac{n}{2}\log(\Lambda_0/\Lambda)$. We shall also absorb the divergence as $\mathrm{g} \to \infty$ by a multiplicative change of the measure. Collecting all the elements, we find that the partition function is given by

$$Z_{D^2_{(\pm)}}(L_\pm) = (r\Lambda)^{n/2} \int_{L_\pm} \mathrm{dvol}_{L_\pm} \left\{ \begin{array}{l} \mathrm{e}^{-2\pi \mathrm{i} r W(\phi)} \\ \mathrm{e}^{-2\pi \mathrm{i} r \overline{W(\phi)}} \end{array} \right. \tag{5.60}$$

Let us next consider deforming $L_\pm$ from a Lagrangian subspace to a more general Lagrangian submanifold, while maintaining the condition that $\mp\mathrm{Im}(W)$ is bounded from below on $L_\pm$. As discussed in Section 4.2, we require that the result does not change under such a deformation. But the expression (5.60) does change if we deform $L_\pm$ and thus cannot be the correct answer. We propose that we should replace the volume element by holomorphic or antiholomorphic volume form,

$$\mathrm{dvol}_{L_+} \to \mathrm{d}^n\phi = \mathrm{d}\phi^1 \wedge \cdots \wedge \mathrm{d}\phi^n, \quad \mathrm{dvol}_{L_-} \to \mathrm{d}^n\overline{\phi} = \mathrm{d}\overline{\phi}^1 \wedge \cdots \wedge \mathrm{d}\overline{\phi}^n, \tag{5.61}$$

so that

$$Z_{D^2_{(+)}}(L_+) = (r\Lambda)^{n/2} \int_{L_+} \mathrm{d}^n\phi \, \mathrm{e}^{-2\pi \mathrm{i} r W(\phi)}, \tag{5.62}$$

$$Z_{D^2_{(-)}}(L_-) = (r\Lambda)^{n/2} \int_{L_-} \mathrm{d}^n\overline{\phi} \, \mathrm{e}^{-2\pi \mathrm{i} r \overline{W(\phi)}}. \tag{5.63}$$

Indeed, it meets the requirement of invariance under deformation of $L_\pm$ and at the same time, when $L_\pm$ is linear, it reduces to the result (5.60) up to a phase.

Let us discuss the issue of convergence of the integral (5.62)-(5.63). Thanks to the asymptotic condition that $\mp\mathrm{Im}(W)$ is bounded from below on $L_\pm$, the exponential factor does not grow at infinity. If $\mp\mathrm{Im}(W)$ grows fast enough at infinity, the integral would be absolutely convergent. Even if it does not, as long as the real part $\mathrm{Re}(W)$ changes fast enough, the integral converges due to rapid oscillation of the exponential factor. If the infinity of $L_\pm$ consists of cones of linear subspaces, the "fast enough" condition is met provided $|\partial W(\phi)|$ grows faster than a power of ϕ. See [35] for a recent explanation on the conditional convergence of the integral of this type.

Let us look at the r dependence. When $W(\phi)$ is quasi-homogeneous, $W(\lambda^R\phi) = \lambda^2 W(\phi)$, we can absorb the r in the integrand by a change of variables, $\phi \to r^{-R/2}\phi$, and we find that the r dependence is just an overall factor

$$Z_{D^2_{(\pm)}}(L_\pm) \;\sim\; r^{n/2-\mathrm{tr}(R/2)} \quad \text{for all } r. \tag{5.64}$$

This power behaviour is a characteristic feature of the partition function of a conformally invariant field theory [30], where the power must be identified with one sixth of the central charge in the case of a hemisphere. Indeed,

$$6\left(\frac{n}{2} - \mathrm{tr}\left(\frac{R}{2}\right)\right) = 3\sum_i (1 - R_i), \tag{5.65}$$

is the central charge of the superponformal field theory to which the Landau-Ginzburg model is believed to flow [31, 32]. As the extreme opposite, let us consider the case where the superpotential $W(\phi)$ is a Morse function, having isolated and non-degenerate critical points only. The theory has supersymmetric ground states with mass gaps whose wave-functions are supported at the critical points. To each critical point p, one can associate a pair of Lagrangian submanifolds $L_{p,\pm}$ passing through p, called *Lefschetz thimbles*, whose W-values are straight semi-lines emanating from $W(p)$ in the direction where $\mathrm{Im}(W)$ goes to negative/positive infinity [13]. For such Lagrangians, one can employ the saddle point approximation for large values of r, which finds

$$|Z_{D^2_{(\pm)}}(L_{p,\pm})| \;\sim\; \mathrm{e}^{-2\pi r(\mp \mathrm{Im} W(p))} \quad \text{as } r \longrightarrow \infty. \tag{5.66}$$

It is exponentially decaying or growing as a function of r, depending on whether $\mp\mathrm{Im}(W)$ is positive or negative at p. We shall consider this exponential behaviour as a signal of the vacuum with a mass gap.

If we consider the system preserving $\mathrm{B}^{\alpha}_{(\pm)}$-type supersymmetry, all we need to do is to replace W by $\mathrm{e}^{2\mathrm{i}\alpha}W$. In particular, the power behaviour (5.64) for the case of quasi-homogeneous W is independent of the parameter α, while the exponential behaviour (5.66) for Morse W is changed so that what matters is the sign of $\mp\mathrm{Im}(\mathrm{e}^{2\mathrm{i}\alpha}W(p))$.

The proposal can be extended to a more general Landau-Ginzburg model and the non-linear sigma model preserving $\mathrm{B}_{(\pm)}$-type supersymmetry. Recall that an axial $U(1)$ R-symmetry is necessary for B-type supersymmetry and the existence requires the target space X have a trivial first Chern class, $c_1(X) = 0$. In many cases, this also means that there exists a holomorphic volume form Ω. The proposal in such a case is

$$Z_{D^2_{(+)}}(L_+) = (r\Lambda)^{n/2}\int_{L_+} \Omega\, \mathrm{e}^{-2\pi \mathrm{i} r W}, \qquad Z_{D^2_{(-)}}(L_-) = (r\Lambda)^{n/2}\int_{L_-} \overline{\Omega}\, \mathrm{e}^{-2\pi \mathrm{i} r \overline{W}}. \tag{5.67}$$

When X is non-compact, the holomorphic volume form Ω is not unique and therefore we must make a choice. In the non-linear sigma model on a compact Calabi-Yau manifold X, the holomorphic volume form is unique up to constant multiplication.

The result (5.60) for linear Lagrangians as well as the proposal (5.62)-(5.63) or (5.67) for the general case are indeed the same as the formula for the central charge of the A-branes in the Landau-Ginzburg model or the non-linear sigma model [18, 13].

5.6 Deformation From The Real Locus

The above discussion allows us to propose a formula for the partition function in which the boundary condition for the vector multiplet is deformed from the one (3.55) associated to the real locus $L = \mathrm{i}\mathfrak{g}$ to a more general Lagrangian submanifold L of $\mathfrak{g}_{\mathbf{C}}$ satisfying the conditions (3.51), (3.52) and (3.53).

First, we claim that such an L is the adjoint G-orbit of a Lagrangian submanifold γ of $\mathfrak{t}_{\mathbf{C}}$,

$$L = G\gamma, \qquad \gamma \subset \mathfrak{t}_{\mathbf{C}}. \tag{5.68}$$

To prove this claim, let us take a point $\sigma = \sigma_1 + \mathrm{i}\sigma_2$ of L. By the condition $[\sigma_1, \sigma_2] = 0$ of (3.51), there is an element $g \in G$ which sends both σ_1 and σ_2 to $\mathfrak{t}$. By the condition (3.53) that L is G-invariant, we have $g(\sigma) \in L \cap \mathfrak{t}_{\mathbf{C}} =: \gamma$. Thus, we have seen $L/G \cong \gamma/W_G$. In particular, the dimension of γ is equal to the dimension of L/G which is equal to $\dim L - \dim(G/G_\sigma)$ where G_σ is the isotropy subgroup of G at $\sigma \in \gamma$. Generically, the dimension of G_σ is equal to the rank l_G of G. Thus, $\dim \gamma = \dim L - (\dim G - l_G) = l_G$. Thus, γ is a middle dimensional submanifold of $\mathfrak{t}_{\mathbf{C}}$. Let us now show that γ is a Lagrangian submanifold of $\mathfrak{t}_{\mathbf{C}}$. Since $\mathfrak{t}_{\mathbf{C}} \subset \mathfrak{g}_{\mathbf{C}}$ is a complex submanifold and $L \subset \mathfrak{g}_{\mathbf{C}}$ is a Lagrangian submanifold, at any point $\sigma \in \gamma$, $\mathcal{J}_\sigma \mathrm{T}_\sigma\gamma$ is a subspace of $\mathrm{T}_\sigma\mathfrak{t}_{\mathbf{C}}$ which is orthogonal to $\mathrm{T}_\sigma\gamma$. Since γ is middle dimensional, $\mathcal{J}_\sigma \mathrm{T}_\sigma\gamma$ is the orthocomplement of $\mathrm{T}_\sigma\gamma$ in $\mathrm{T}_\sigma\mathfrak{t}_{\mathbf{C}}$. That is, γ is a Lagrangian submanifold of $\mathfrak{t}_{\mathbf{C}}$. Finally, we show that the condition (3.52) is satisfied for such an L. By the homogeneity, we may assume $\sigma \in \gamma$. The tangent space $\mathrm{T}_\sigma L$ of $L = G\gamma$ is the direct sum of $T_\sigma\gamma$ and the orbit directions $\mathfrak{g}(\sigma)$. The former component $T_\sigma\gamma$ commutes with σ_1. The latter component $\mathfrak{g}(\sigma)$ is invariant under commutator with σ_1, since $[\sigma_1, [X, \sigma]] = [[\sigma_1, X], \sigma]$ where we used $[\sigma_1, \sigma] = 0$. This completes the proof of the claim.

The localization procedure works in the same way as the real locus and we have an integral over $\gamma = L \cap \mathfrak{t}_{\mathbf{C}}$ of the classical exponential factor times the fluctuation determinant with respect to some measure. However, the direct computation of the fluctuation determinant is very complicated for a general choice of L. At this point, we employ the holomorphy discussed in Section 4.1 and also take the lesson from the previous section concerning the measure. The integrand can be regarded as the effective partition function on the Coulomb branch in which the vector multiplet of the group T_G is fixed to be a supersymmetric background satisfying (5.1). In the background, the scalar σ takes a constant value

$$\sigma = \sigma_1 + \mathrm{i}\sigma_2, \tag{5.69}$$

and the gauge field has the boundary holonomy

$$\oint_{\partial D^2} v = -2\pi r \sigma_2. \tag{5.70}$$

In this picture, σ is a twisted chiral parameter and the integrand of $Z_{D^2_{(+)}}(\mathfrak{B})$ (*resp.* $Z_{D^2_{(-)}}(\mathfrak{B})$) must depend holomorphically (*resp.* antiholomorphically) on it. It is uniquely determined by the values at the real locus $\mathfrak{t}$, given as the integrand of (5.54) or (5.55). The (anti)holomorphic extension of the Chan-Paton factor, from $\mathrm{e}^{\pm 2\pi r\rho(\sigma_1)}$ to $\mathrm{e}^{\pm 2\pi r\rho(\sigma_1 \pm \mathrm{i}\sigma_2)}$, can be understood from (5.70), since it originates from the factor $\mathrm{e}^{-\oint_{\partial D^2}\rho(\mathrm{i}v_\tau \mp \sigma_1)\mathrm{d}\tau}$ in (3.36). The measure $\mathrm{d}^{l_G}\sigma_1$ is extended uniquely to the holomorphic or anti-holomorphic volume form of $\mathfrak{t}_{\mathbf{C}}$, denoted by $\mathrm{d}^{l_G}\sigma$ or $\mathrm{d}^{l_G}\overline{\sigma}$. In this way, we arrive at the following expressions

$$\begin{aligned} Z_{D^2_{(+)}}(\mathfrak{B}) \;=\;& C(r\Lambda)^{\widehat{c}/2}\int_{\gamma_+} r^{l_G}\mathrm{d}^{l_G}\sigma \prod_{\alpha>0} r\alpha(\sigma)\sinh(\pi r\alpha(\sigma))\prod_i \Gamma\left(\mathrm{i}rQ_i(\sigma)+\frac{R_i}{2}\right) \\ &\times \exp\left(2\pi\mathrm{i}r\widetilde{W}(\sigma) - \mathrm{i}r\sum_i Q_i(\sigma)\log(r\Lambda)\right)\mathrm{tr}_M\left(\mathrm{e}^{\pi i\mathbf{r}_*}\,\mathrm{e}^{2\pi r\rho(\sigma)}\right), \end{aligned} \tag{5.71}$$

$$\begin{aligned} Z_{D^2_{(-)}}(\mathfrak{B}) \;=\;& C(r\Lambda)^{\widehat{c}/2}\int_{\gamma_-} r^{l_G}\mathrm{d}^{l_G}\overline{\sigma} \prod_{\alpha>0} r\alpha(\overline{\sigma})\sinh(\pi r\alpha(\overline{\sigma}))\prod_i \Gamma\left(\mathrm{i}rQ_i(\overline{\sigma})+\frac{R_i}{2}\right) \\ &\times \exp\left(2\pi\mathrm{i}r\overline{\widetilde{W}(\sigma)} - \mathrm{i}r\sum_i Q_i(\overline{\sigma})\log(r\Lambda)\right)\mathrm{tr}_M\left(\mathrm{e}^{-\pi i\mathbf{r}_*}\,\mathrm{e}^{-2\pi r\rho(\overline{\sigma})}\right). \end{aligned} \tag{5.72}$$

In general, the Lagrangian submanifold $\gamma_+ \subset \mathfrak{t}_{\mathbf{C}}$ for the $\mathrm{A}_{(+)}$-type supersymmetry can be different from the one $\gamma_- \subset \mathfrak{t}_{\mathbf{C}}$ for the $\mathrm{A}_{(-)}$-type supersymmetry. As in the Landau-Ginzburg model, γ_+ and γ_- must be chosen so that the effective boundary potential is bounded from below. A concrete proposal for the right choice will be given in Section 6.

5.7 The Case Of Linear Sigma Model

Let us now restrict our attention to the gauged linear sigma models where the twisted superpotential is linear, $\widetilde{W}(\sigma) = \frac{1}{2\pi}t(\sigma)$.

In this case, the two expressions (5.71) and (5.72) are related by complex conjugation,

$$\left(Z_{D^2_{(+)}}(\mathfrak{B})\right)^* = Z_{D^2_{(-)}}(\mathfrak{B}), \tag{5.73}$$

for the real locus $\gamma_+ = \mathrm{i}\mathfrak{t} = \gamma_-$ and the relation continues to hold as long as γ_+ and γ_- are mapped to each other by the inversion $\sigma \mapsto -\sigma$. In what follows, we shall assume this relation between γ_+ and γ_- and will only mention $Z_{D^2_{(+)}}$ untill a special need of $Z_{D^2_{(-)}}$ arizes in Section 9. Hence we shall drop the subscript + from the expressions.

When $\widetilde{W}(\sigma)$ is linear, we may absorb the radius r into the integration variable as

$$\sigma' = r\sigma, \tag{5.74}$$

so that the integral (5.71) can be written as

$$\begin{aligned} Z_{D^2}(\mathfrak{B}) \;\; &= \;\; C(r\Lambda)^{\widehat{c}/2} \int\limits_{\gamma} \mathrm{d}^{l_G}\sigma' \prod_{\alpha>0} \alpha(\sigma')\sinh(\pi\alpha(\sigma')) \prod_i \Gamma\left(\mathrm{i}Q_i(\sigma') + \frac{R_i}{2}\right) \\ &\qquad\qquad \times \exp\left(\mathrm{i}t_{\mathrm{R}}(\sigma')\right) \mathrm{tr}_M\left(\mathrm{e}^{\pi i \mathbf{r}_*}\, \mathrm{e}^{2\pi\rho(\sigma')}\right). \end{aligned} \tag{5.75}$$

Here we introduce the renormalized FI parameter

$$t_{\mathrm{R}} = t - \mathrm{tr}_V \log(r\Lambda). \tag{5.76}$$

As promised, we examine the effect of the gauge shift of the R-charges (3.59)-(3.60), which reads $R_i \to R_i + Q_i(\mathit{\Delta})$ and $\mathbf{r}_* \to \mathbf{r}_* - \rho(\mathit{\Delta})$, for a generator $\mathit{\Delta}$ of the center of G. Let us shift the integration variables as

$$\sigma' \to \sigma' + \frac{\mathrm{i}}{2}\mathit{\Delta} \tag{5.77}$$

Note that $\alpha(\mathit{\Delta}) = 0$ for any root α since $\mathit{\Delta}$ is central. Also, the exponent $\widehat{c}$ may change but it is absorbed by the shift of the part $-\mathrm{i}\mathrm{tr}_V(\sigma')\log(r\Lambda)$ of $\mathrm{i}t_{\mathrm{R}}(\sigma')$. The net effect is the overall multiplication

$$Z \to \mathrm{e}^{-\frac{1}{2}t(\mathit{\Delta})}Z, \tag{5.78}$$

plus the shift of integration contour, $\gamma \to \gamma + \frac{\mathrm{i}}{2}\mathit{\Delta}$. Thus, as long as this shift does not cross any pole from the gamma functions, the change is only by the multiplication by

$e^{-\frac{1}{2}t(\Delta)}$. For the choice of real Lagrangian $\gamma = i\mathfrak{t}$, this is the case as long as the shift does not move the R-charges out of the range $0 < R_i < 2$. The shift of variables (5.77) has two-fold interpretation. One is the reality violating change of variables, as in (3.62), in which the result $Z \to e^{-\frac{1}{2}t(\Delta)}Z$ was anticipated in (3.63). The other is a change of boundary conditions which moves the Lagrangian submanifold in the σ_2-direction, $\gamma \to \gamma + \frac{i}{2}\Delta$.

Let us further specialize to the *Calabi-Yau case*:

$$G \subset SL(V). \tag{5.79}$$

Then, the trace $\mathrm{tr}_V(\sigma)$ vanishes and the FI parameter is not renormalized,

$$t_{\mathrm{R}} = t. \tag{5.80}$$

Accordingly, the number $\widehat{c}$ does not change under the gauge shift of the R-charges. The dependence on the size r of the hemisphere is only in the factor $(r\Lambda)^{\widehat{c}/2}$. As remarked in the Landau-Ginzburg model, this is precisely the form of conformal anomaly [30] in a conformal field theory of central charge

$$c \;=\; 3\,\widehat{c}. \tag{5.81}$$

With $\widehat{c}$ given by (5.50), this is indeed the central charge of the infra-red fixed point of the gauge theory obtained by identifying the conformal algebra in the $\overline{Q}_+$ chiral ring as in [33] or by a short-cut argument [34].

If the charge integrality (3.57)-(3.58) holds, the brane factor $\mathrm{tr}_M(\cdots)$ in (5.75) can be written as $\mathrm{Str}_M\rho(J^{-1}\,e^{2\pi\sigma'})$. If the gauge group G is a finite group, where the theory is a Landau-Ginzbutg orbifold, we do not have the σ integral and the result is simply

$$Z_{D^2}(B) = C(r\Lambda)^{\widehat{c}/2}\cdot \mathrm{Str}_M\rho(J^{-1}). \tag{5.82}$$

Up to the prefactor, this indeed agrees with the central charge for the B-brane $B = (M, Q, \rho, \mathbf{r}_*)$ of the Landau-Ginzburg orbifold proposed in [6]. If, instead, J is in the identity component of the center, we may gauge shift the R-charges of bulk fields from $0 < R_i < 2$ to the R^o-frame (3.61), where $R_i^o = 0$ or 2 by continuity:

$$\begin{aligned} Z_{D^2}(\mathfrak{B}) \;&=\; C(r\Lambda)^{\widehat{c}/2}\int_\gamma d^{l_G}\sigma' \prod_{\alpha>0}\alpha(\sigma')\sinh(\pi\alpha(\sigma'))\prod_i \Gamma\left(iQ_i(\sigma') + \frac{R_i^o}{2}\right) \\ &\qquad\qquad \times \exp\left(it_{\mathrm{R}}(\sigma')\right)\mathrm{Str}_M\, e^{2\pi\rho(\sigma')}. \end{aligned} \tag{5.83}$$

If the contour before the gauge shift was the real locus $i\mathfrak{t}$, then γ is such that $Q_i(\sigma')$ for $R_i^o = 0$ has a small negative imaginary part.

In what follows, we shall concentrate on the study of the hemisphere partition function (5.75) of the linear sigma model. We shall often specialize to the Calabi-Yau case (5.79) or to the case with charge integrality (3.57)-(3.58).

6 The Contour

We are left with one and the most important problem: decide which Lagrangian submanifold $L \subset \mathfrak{g}_{\mathbf{C}}$ to take for the boundary condition on the vector multiplet, or equivalently (see Section 5.6), which Lagrangian submanifold $\gamma \subset \mathfrak{t}_{\mathbf{C}}$ to take as the contour of the integration (5.75). Let us copy the integral for convenience,

$$Z_{D^2}(\mathfrak{B}) \;=\; (r\Lambda)^{\widehat{c}/2}\int\limits_{\gamma} \mathrm{d}^{l_G}\sigma' \prod_{\alpha>0}\alpha(\sigma')\sinh(\pi\alpha(\sigma'))\prod_i \Gamma\left(\mathrm{i}Q_i(\sigma')+\frac{R_i}{2}\right) \times \exp\left(\mathrm{i}t_{\mathrm{R}}(\sigma')\right)\sum_j \mathrm{e}^{\pi\mathrm{i}r_j}\,\mathrm{e}^{2\pi q_j(\sigma')}. \tag{6.1}$$

Here, we wrote the brane factor as a sum,

$$\mathrm{tr}_M\left(\mathrm{e}^{\pi\mathrm{i}\mathbf{r}_*}\,\mathrm{e}^{2\pi\rho(\sigma')}\right)=\sum_j \mathrm{e}^{\pi\mathrm{i}r_j}\,\mathrm{e}^{2\pi\rho(\sigma')},$$

where r_j and q_j are the R-charge and the T-weight of the basis element of the Chan-Paton vector space M.

6.1 A Proposal

To attack this problem, we would like to have some idea on the integrand of (6.1). In particular, we would like to know the location of singularity as well as the growth or decay rate at infinity.

The gamma function has simple poles at non-positive integers,

$$\Gamma(z)\;\sim\;\frac{(-1)^n}{n!}\frac{1}{z+n},\qquad z\sim -n. \tag{6.2}$$

Therefore, the integrand of (6.1) has poles at infinitely many hyperplanes

$$Q_i(\sigma')\;=\;\mathrm{i}\left(n_i+\frac{R_i}{2}\right),\qquad n_i=0,1,2,3,\ldots. \tag{6.3}$$

These are where $Q_i(\sigma')$ are on the positive imaginary axis if we choose $R_i>0$. In particular, the real locus $\gamma=\mathrm{i}\mathfrak{t}$ does not hit the singularity. Of course we anticipated this since the scalar ϕ_i has no zero mode when $\sigma = \sigma_1 \in \mathrm{i}\mathfrak{t}$ as long as we put the R-charges in the range (5.4). In fact, the poles (6.3) must be associated with the zero modes of the scalars ϕ_i, in the presence of the boundary term $-Q_i(\sigma_2)|\phi_i|^2$ in (3.22) which takes negative values for a positive $Q_i(\sigma_2)$.

The gamma function has the asymptotic behaviour (Stirling's formula),

$$\Gamma(z) \;\sim\; \sqrt{2\pi}\,\mathrm{e}^{-z} z^{z-\frac{1}{2}}\left(1+O(1/z)\right), \tag{6.4}$$

as $|z| \to \infty$ with $\mathrm{Arg}(z) \in (-\pi, \pi)$. We also know that

$$\sinh(z) \;\sim\; \pm\frac{1}{2}\,\mathrm{e}^{\pm z} \tag{6.5}$$

as $\mathrm{Re}(z) \to \pm\infty$. These allow us to find the asymptotic behaviour of the integrand in a generic direction in the σ'-space: The term of Chan-Paton weight q of the integrand (6.1) behaves as

$$\begin{aligned} \mathrm{integrand}_q \;&=\; \mathrm{const}\cdot\prod_{\alpha>0}\alpha(\sigma)\sinh(\pi\alpha(\sigma))\prod_i\Gamma\left(\mathrm{i}Q_i(\sigma')+\frac{R_i}{2}\right)\;\mathrm{e}^{\mathrm{i}t_R(\sigma')+2\pi q(\sigma')} \\ &\sim\; \mathrm{const}\cdot\prod_{\alpha>0}\alpha(\sigma')\prod_i Q_i(\sigma')^{R_i-\frac{1}{2}}\cdot\exp\left(-2\pi\mathrm{i}r\widetilde{W}_{\mathit{eff},q}(\sigma)\right) \end{aligned} \tag{6.6}$$

with

$$\begin{aligned} 2\pi\widetilde{W}_{\mathit{eff},q}(\sigma) \;&=\; \sum_{\alpha>0}\pm\pi\mathrm{i}\alpha(\sigma)-\sum_i Q_i(\sigma)\left(\log\left(\frac{Q_i(\sigma)}{-\mathrm{i}\Lambda}\right)-1\right) \\ &\quad -t(\sigma)+2\pi\mathrm{i}q(\sigma). \end{aligned} \tag{6.7}$$

The above is valid when $|\mathrm{Re}(\alpha(\sigma'))| \gg 1$ for all α, $|Q_i(\sigma')| \gg 1$ for all i, and $Q_i(\sigma')$ are not on the positive imaginary axis. The sign $\pm\pi\mathrm{i}\alpha(\sigma)$ is chosen when $\pm\mathrm{Re}(\alpha(\sigma'))$ is positive. The imaginary part of the logarithm is defined to have values in the open interval $(-\pi, \pi)$.

The function $\widetilde{W}_{\mathit{eff},q}(\sigma)$ is equal to the effective twisted superpotential on the Coulomb branch. We see the well-known $-\sigma(\log\sigma - 1)$ from the 1-loop integral of the matter multiplet. The term $\pm\pi\mathrm{i}\alpha(\sigma)$ may be less familiar, but it comes from the 1-loop integral of the W-boson multiplet. See [36] for the explanation based on [37]. In the bulk, or in the closed string sector, the shift of $\widetilde{W}_{\mathit{eff}}(\sigma)$ by $2\pi\mathrm{i}w(\sigma)$ does not matter for any weight w of the maximal torus T since it is just a 2π shift of the theta angle. In the presence of boundary, on the other hand, the shift does matter, since the 2π shift of the theta angle amounts to the shift of Chan-Paton weight. The last term $2\pi\mathrm{i}q(\sigma)$ is nothing but the contribution from the classical Chan-Paton factor. Also, the precise choice of the sign

$\pm\pi i\alpha(\sigma)$ and the imaginary part of the logarithm, which is irrelevant in the bulk, does matter here.

The imaginary part, $E_{\mathit{eff},q}(\sigma) = -\mathrm{Im}(\widetilde{W}_{\mathit{eff},q}(\sigma))$, may be interpreted as the effective boundary potential. For the evaluation, we use the formula

$$\mathrm{Arg}(iz) = \mathrm{sgn}\left(\mathrm{Re}(z)\right)\left(\frac{\pi}{2} + \arctan\left[\frac{\mathrm{Im}(z)}{|\mathrm{Re}(z)|}\right]\right), \tag{6.8}$$

which holds if we assume that $\mathrm{Arg}(-)$ and $\arctan(-)$ take values in the intervals $(-\pi,\pi)$ and $(-\frac{\pi}{2},\frac{\pi}{2})$ respectively — we assume this in what follows too. We find

$$\begin{aligned} E_{\mathit{eff},q}(\sigma) = &-\frac{1}{2}\sum_{\alpha>0}|\alpha(\sigma_1)| \\ &+\frac{1}{2\pi}\sum_i\left\{Q_i(\sigma_2)\left(\log\left|\frac{Q_i(\sigma)}{\Lambda}\right|-1\right)+|Q_i(\sigma_1)|\left(\frac{\pi}{2}+\arctan\left[\frac{Q_i(\sigma_2)}{|Q_i(\sigma_1)|}\right]\right)\right\} \\ &+\frac{1}{2\pi}\zeta(\sigma_2)-\left(\frac{\theta}{2\pi}+q\right)(\sigma_1). \end{aligned} \tag{6.9}$$

The second line is nothing but the effective boundary energy of the matter system, which was obtained in [11] by computing the energy density of the ground state of the canonically quantized matter system on an interval or a half line with the same boundary condition as (3.50), in which we set (σ_1,σ_2) to be a constant (σ_1,σ_2). See Section 6, Eqn (6.79) of [11]. The first line is regarded as the boundary energy of the W-boson multiplet. It would be interesting to check it directly by a computation like [11]. The third line is already there in the classical action as the classical boundary potential, see (3.21) and (3.35).

This $E_{\mathit{eff}}(\sigma)$ shows the asymptotic growth or decay of the integrand. In order to isolate the dependence on the size r of the hemisphere, it is more convenient to use the σ' variables. We have

$$\Big|\mathrm{integrand}_q\Big| \;\sim\; P(\sigma')\cdot\exp\Big(-A_q(\sigma')\Big), \tag{6.10}$$

where $P(\sigma')$ is a power of σ' and

$$\begin{aligned} A_q(\sigma') \;=\; &-\sum_{\alpha>0}\pi|\alpha(\sigma'_1)| \\ &+\sum_i\left\{Q_i(\sigma'_2)\Big(\log|Q_i(\sigma')|-1\Big)+|Q_i(\sigma'_1)|\left(\frac{\pi}{2}+\arctan\left[\frac{Q_i(\sigma'_2)}{|Q_i(\sigma'_1)|}\right]\right)\right\} \\ &+\zeta_R(\sigma'_2)-(\theta+2\pi q)(\sigma'_1). \end{aligned} \tag{6.11}$$

The r dependence is only in the renormalized FI parameter,

$$\zeta_R = \zeta - \sum_i Q_i\log(r\Lambda). \tag{6.12}$$

Notice that $A_q(\sigma')$ is essentially piecewise linear at infinity in the σ' space. One can also see from (6.7) that the oscilation part, i.e. the imaginary part of the exponent, is also essentially piecewise linear in σ'. For the absolute convergence of the integral, we need to choose the Lagrangian $\gamma \subset \mathfrak{t}_{\mathbf{C}}$ so that $A_q(\sigma')$ grows at infinity of γ. One may also allow $A_q(\sigma')$ to approach a constant at infinity, hoping for the conditional convergence due to rapid oscillation. However, the linear growth of the imaginary part makes the case very subtle (see for example [35]). This motivates us to make the following proposal:

> *The Lagrangian submanifold γ is a deformation of the real locus $i\mathfrak{t}$, avoiding the poles (6.3), so that for any Chan-Paton weight q of the brane, $A_q(\sigma')$ in (6.11) grows to infinity in every asymptotic direction of γ.*

We shall refer to the asymptotic region in which $A_q(\sigma')$ grows to infinity as the *admissible region*. Thus, γ is obtained from the real locus $i\mathfrak{t}$ by "bending" the infinity, if necessary, so that every asymptotic direction is in the admissible region, and we require that the poles (6.3) are not hit in the bending process. We shall also refer to the Lagrangian γ satisfying this condition as *admissible*.

The main question is whether there exists an admissible Lagrangian submanifold γ, and if so, whether it is unique up to deformation. In the rest of this section, we shall examine this question in several examples, and at the same time identify the deformation class of admissible Lagrangian submanifolds, when that is possible. In particular, we will find that, at some special loci in the parameter space, called windows between phase boundaries, it is not always possible to find an admissible Lagrangian submanifold for an arbitrary brane $\mathfrak{B}$. The factor $e^{2\pi q(\sigma')}$ is exponentially growing in a certain direction of the σ' space, and the other factors cannot rule this divergence for any choice of γ, if the parameter is on a window. This means that there is a severe constraint, depending on the window, on the possible range of Chan-Paton weight q of the brane $\mathfrak{B}$.

The problem of identifying a Lagrangian submanifold γ for the boundary condition on the vector multiplet was studied in [11] for the Abelian and Calabi-Yau cases, and essentially the same condition on γ was obtained: The condition of adimissible asymptotic direction of γ matches because, as we have just seen, the effective boundary potential in that case is precisely equal to $E_{eff,q}(\sigma)$. Also, there is a singularity along the entire positive imaginary axis of $Q_i(\sigma)$ due to the zero mode of ϕ_i localized near the boundary. Avoiding that is the counterpart of avoiding the poles (6.3) at discrete points along the same axis, which are associated with the zero mode of ϕ_i on the hemi-sphere. And in [11], a constraint on the possible range of Chan-Paton weight q on windows was obtained and was named the *grade restriction rule* for D-brane transport across windows.

In Abelain and Calabi-Yau cases, we will indeed reproduce the grade restriction rule. This is of course of no surprize regarding what we have just said, but the convergence of the integral (6.1) provides a somewhat sharper constraint. We shall also discuss the non Calabi-Yau and/or non-Abelian theories as well.

The integrals of the type (6.1) are known as the (multiple) Mellin-Barnes integrals and have been a subject of mathematical study, from the old time to more recent days, especially after the discovery [38] of the importance of mirror symmetry. The present discussion shows that the issue of convergence, or the problem of identifying convergent domains, of such integrals encodes a rich physical content.

6.2 $U(1)$ Theories

We first consider the theories with gauge group $G = U(1)$. As the basic class of examples, we consider the theory with matter fields P, $X_1, \ldots, X_N$ of charge $-d, 1, \ldots, 1$, and with superpotential $W = Pf(X_1, \ldots, X_N)$ where f is a homogeneous polynomial of degree d. We assume that f is generic so that the projective hypersurface $X_f = (f = 0) \subset \mathbb{CP}^{N-1}$ is smooth. The R-charge assignment is unique up to the gauge shift, $2 - d\epsilon$ to P and ϵ to $X_1, \ldots, X_N$. The bound (5.4) is ensured by $0 < \epsilon < 2/d$. The R^o-frame is obtained by the limit $\epsilon \searrow 0$. Before attacking the problem to identify the admissible contour γ, we recall some basic facts on the bulk theory [15].

The nature of the classical theory depends very much on the sign of the FI parameter ζ, which enters into the D-term equation,

$$\sum_{i=1}^{N} |x_i|^2 - d|p|^2 = \zeta. \tag{6.13}$$

When ζ is positive, this requires some x_i to have a non-zero value which breaks the $U(1)$ gauge group completely. When ζ is negative, this requires p to have a non-zero value which breaks the $U(1)$ gauge group to the cyclic subgroup $\mathbf{Z}_d$ or order d. When ζ is zero, there is a locus $x = p = 0$ in which the $U(1)$ is unbroken. The classical low energy theory is the non-linear sigma model with the target X_f for $\zeta \gg 0$ and the $\mathbf{Z}_d$ orbifold of the Landau-Ginzburg model with superpotential $W = f(X_1, \ldots, X_N)$ for $\zeta \ll 0$. The theory is said to be in the geometric phase and the Landau-Ginzburg orbifold phase respectively, when $\zeta \gg 0$ and $\zeta \ll 0$.

The nature of the quantum theory depends very much on the sign of $(N-d)$, since the FI parameter runs as $\zeta_R = \zeta - (N-d)\log(r\Lambda)$. When $d < N$ (*resp.* $d > N$), it runs from positive to negative (*resp.* negtaive to positive) as the distance scale r is increased. There are also massive vacua on the Coulomb branch where σ is non-zero: They are found by solving

$$t_{\mathit{eff}}(\sigma) := -2\pi\partial_\sigma \widetilde{W}_{\mathit{eff}} \equiv 0 \mod 2\pi\mathrm{i}\mathbf{Z}, \tag{6.14}$$

where $\widetilde{W}_{\mathit{eff}}$ is the effective twisted superpotential, which is equal to (6.7) but omitting q since it does not matter for this purpose. This equation takes the form $\sigma^{N-d} = (-d)^d\,\mathrm{e}^{-\zeta}(-\mathrm{i}\Lambda)^{N-d} := (-1)^d(-\mathrm{i}\widetilde{\Lambda})^{N-d}$ and hence has $|N-d|$ solutions. When $d = N$ (Calabi-Yau case), the FI parameter does not run, and we have a family of theories parametrized by $t = \zeta - \mathrm{i}\theta \in \mathbf{C}/2\pi\mathrm{i}\mathbf{Z}$. In this case (6.14) is a constant, $t_{\mathit{eff}} \equiv t - N\log(-N)$. The theory is singular at $t_{\mathit{eff}} \equiv 0$ due to the presence of non-compact Coulomb branch. This point, $\zeta = N\log N$ and $\theta \equiv \pi N$, is the quantum remnant of the phase boundary which 'separates' the $\zeta \gg 0$ geometric phase and the $\zeta \ll 0$ Landau-Ginzburg orbifold phase, although it does not really separate the two regimes since we can go around a complex codimension one locus.

6.2.1 Calabi-Yau Case

Let us first consider the Calabi-Yau case, $d = N$. In this case, the expression for A_q is very simple

$$A_q(\sigma') = \zeta_{\mathit{eff}}\sigma'_2 + \Big(N\pi - \mathrm{sgn}(\sigma'_1)(\theta + 2\pi q)\Big)|\sigma'_1|, \tag{6.15}$$

where $\zeta_{\mathit{eff}} = \zeta - N\log N$. When $\zeta_{\mathit{eff}} > 0$ (*resp.* $\zeta_{\mathit{eff}} < 0$), the entire region of the σ'-plane above (*resp.* below) the broken line $A_q(\sigma') = 0$ is admissible. Figure 1 depicts the

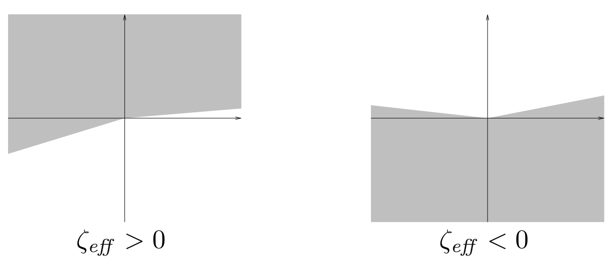


Figure 1: Admissible regions (Calabi-Yau case)

σ'-planes for two values of $(\zeta, \theta + 2\pi q)$, where the admissible regions are shaded. For any finite q, a sector of positive angle including the positive (*resp.* negative) imaginary axis is inside the admissible region. Therefore, we can take γ to be the curve obtained by bending the real line $\mathbf{R}$ toward the positive (*resp.* negative) imaginary direction, like a graph of a function which grows or decays faster than a linear function. See Figure 2-Left

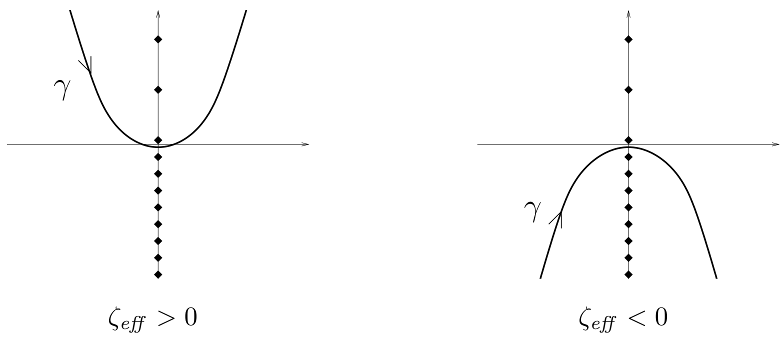


Figure 2: Admissible contours (Calabi-Yau case)

(*resp.* -Right), where the poles (6.3) are also drawn together as dots. Then, the integral (6.1) is convergent for any brane whose Chan-Paton charge q ranges over an *arbitary* but finite set of integers.

When $\zeta_{\mathit{eff}} = 0$, the situation is very different. If $\theta + 2\pi q \geq N\pi$, the entire right half plane is not admissible. There is no way to move the right end of the real line to the left half plane without hitting the poles. Therefore, an admissible contour does not exist. Similarly for the case $\theta + 2\pi q \leq -N\pi$ where the entire left half plane is not admissible. If $-N\pi < \theta + 2\pi q < N\pi$, the entire directions is admissible except infinitesimally small sectors including the imaginary axis. Therefore, the real line $\gamma = \mathbf{R}$ itself is admissible, as well as any of its deformation that keeps a non-zero angle against the imaginary axis. Thus, we have a strong constraint on the brane $\mathfrak{B} = (M, Q, \rho, \mathbf{r}_*)$:

At $\zeta_{\mathit{eff}} = 0$, all the Chan-Paton charges q of $\mathfrak{B}$ must be in the range

$$-\frac{N}{2} \;<\; \frac{\theta}{2\pi} + q \;<\; \frac{N}{2}. \tag{6.16}$$

The allowed charges form a set of N consecutive integers provided θ avoids $N\pi + 2\pi\mathbf{Z}$, which are singular values for θ at $\zeta_{\mathit{eff}} = 0$. This set does not change if θ moves inside an open interval, or a window, of length 2π of regular values. If a brane $\mathfrak{B}$ obeys the condition (6.16) for θ in such an interval, we shall call it *grade restricted with respect to the window.*

This is strange. If ζ_{eff} is positive or negative, an arbitrary brane has an admissible Lagrangian submanifold γ for the boundary condition on the vector multiplet. At $\zeta_{\mathit{eff}} = 0$, that is possible only for grade restricted branes which form a tiny subset of the set of all branes in the linear sigma model.

To illustrate the problem, let us see what happens to the partition function for a brane

$\mathfrak{B} = (M, Q, \mathbf{r}_*, \rho)$ if we move the parameter $t = \zeta - \mathrm{i}\theta$ along a path from one phase to another, say from the geometric phase $\zeta \gg 0$ to the Landau-Ginzburg orbifold phase $\zeta \ll 0$. The path must avoid the singularity $t \equiv N \log N + N\pi\mathrm{i}$ and hence must go through one of the windows at $\zeta = N \log N$. First, let us consider the case where the brane $\mathfrak{B}$ is grade restricted with respect to that window. The move of the admissible

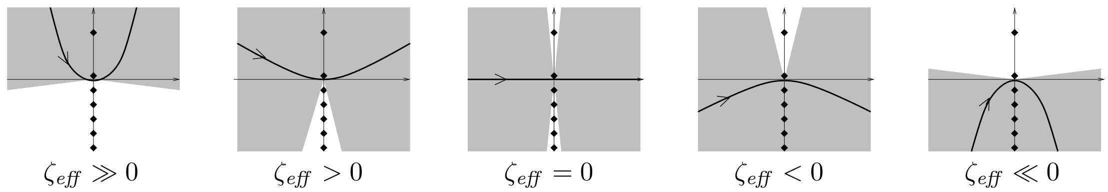


Figure 3: Grade Restricted Case

region for any Chan-Paton charge q of the brane is shown in Fig. 3. We see that one can find a continuous family of admissible contours as depicted in the same Figure. Therefore, the partition function for the brane $\mathfrak{B}$ in the phase $\zeta \gg 0$ is related to the one for the same brane $\mathfrak{B}$ in the phase $\zeta \ll 0$ by anlytic continuation along the path. Let us next consider the case where the brane $\mathfrak{B}$ is not grade restricted with respect to the window. Then, it includes a Chan-Paton charge q which is outside, say above, the set (6.16). The move of the admissible region for such a charge is depicted in Figure 4. We see that,

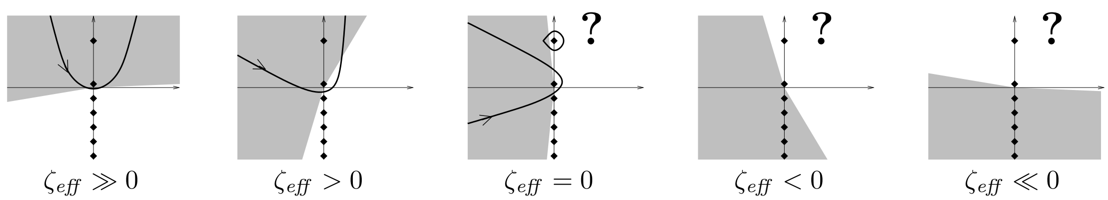


Figure 4: Not Grade Restricted Case

before ζ_{eff} approaches 0, an admissible contour is forced to hit the singularity along the positive imaginary axis. The integral must pick these infinitely many poles when ζ_{eff} goes negative, but the convergence of the infinite sum is not obvious at all. So, we do not know what happens to the partition function if we try to see it this way. The same problem arizes if there is a charge q below the set (6.16).

In a sense, only grade resticted branes can cross the window safely. This is how the grade restriction rule was stated in [11]. What does this mean? Is there a real phase boundary between the geometric phase and the Landau-Ginzburg orbifold phase across which some of the branes cannot cross? That would be strange since the points with $\zeta_{\mathit{eff}} = 0$, $\theta \not\equiv \pi N$ have no special status compared to other points in the parameter space. The answer to this problem, given in [11], is that in either phase, there is a huge equivalence relation among the branes, and each equivalence class has a grade restricted representative. This point, which we shall call the "classical grade restriction rule", will be revisited in the next section, where we show that the partition function takes the same value on branes in the same equivalence class. This will give a solution to the above problem of analytic continuation of the partition function for a brane which is not grade restricted with respect to the window.

In this paper, we shall call the constraint (6.16) itself the grade restriction rule.

6.2.2 Non Calabi-Yau Case

In the non-Calabi-Yau case, $d \neq N$, the function A_q can be written as

$$\begin{aligned} A_q(\sigma') &= (N-d)\left(\log\left|\frac{\sigma'}{r\widetilde{\Lambda}}\right| - 1\right)\sigma'_2 \\ &\quad + \left(\frac{\pi}{2}(N+d) + (N-d)\arctan\left[\frac{\sigma'_2}{|\sigma'_1|}\right] - \mathrm{sgn}(\sigma'_1)(\theta+2\pi q)\right)|\sigma'_1|. \end{aligned} \tag{6.17}$$

We see that the coefficient of σ'_2 changes its sign at the cricle $|\sigma'| = r\widetilde{\Lambda}\,\mathrm{e} = r\widetilde{\Lambda}\times 2.1718...$ — it is postive outside the circle and negative inside when $d < N$ and the other way around when $d > N$. Also, the coefficient of $|\sigma'_1|$ is positive on the real axis if $|\theta+2\pi q| < \frac{\pi}{2}(N+d)$. Fig 5 shows the σ' planes for three values of $(\zeta, \theta+2\pi q)$ for the case $d < N$. We shade

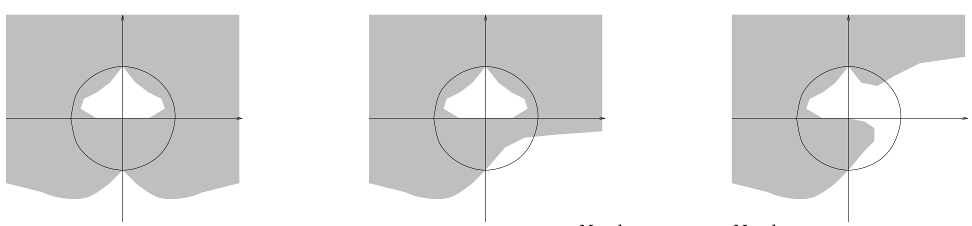


Figure 5: Regions with positive A_q (the case $d < N$).

the region with positive A_q and draw the circle at $|\sigma'| = r\widetilde{\Lambda}\,\mathrm{e}$. We assume $r\Lambda \gg 1$ so that (6.10) is a good approximation at the scale $\sigma' \sim r\widetilde{\Lambda}$. For any value of $(\zeta, \theta+2\pi q)$, the function A_q grows at least linearly in any ray direction on the upper half plane. Therefore, for any brane, the contour γ can be taken to be a curve, as in Fig. 6-Left, which comes in from and goes out to the region where $\mathrm{Im}(\sigma')$ is positive infinity. This is as in the $\zeta_{eff} > 0$ phase of the Calabi-Yau case, which may be understood by the fact that the effective FI parameter $\zeta_{eff}(\sigma) = (N-d)\log|\sigma/\widetilde{\Lambda}|$ goes to positive infinity as $|\sigma'| \to \infty$. For the case

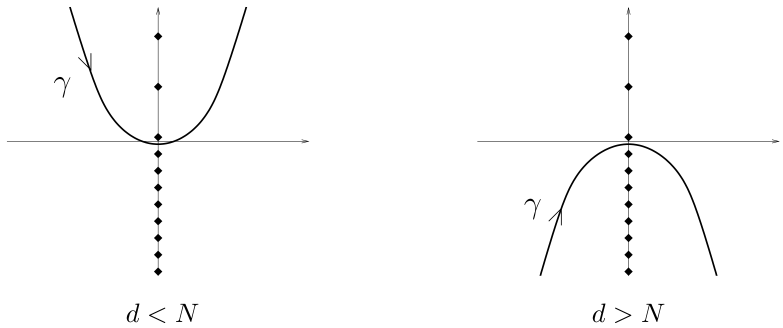


Figure 6: Admissible contours (Non Calabi-Yau cases)

$d > N$, the picture for the $A_q > 0$ region is upside down compared to Fig. 5, with d and N exchanged in the subtitles. Also, the effective FI-parameter $\zeta_{eff}(\sigma)$ goes to negative infinity as $|\sigma'| \to \infty$. Thus, the contour γ for any brane can be taken as in Fig. 6-Right, coming in from and going out to the region where $\mathrm{Im}(\sigma')$ is negative infinity.

6.2.3 More General Theories

What is said on this particular class of examples applies more generally. Let us consider the $U(1)$ theory with fields X_i of R- and gauge charge (R_i, Q_i) and with some superpotential W. We assume that each Q_i is non-zero. We put $N_\pm := \sum_{\pm Q_i > 0} |Q_i|$. In the Calabi-Yau case $N_+ = N_-$, we have a family of theories parameterized by t, with the singularity at $t \equiv -\sum_i Q_i \log Q_i$. The contour γ can be chosen as in Fig. 2 if $\zeta_{eff} = \zeta - \sum_i Q_i \log |Q_i|$ is non-zero. At $\zeta_{eff} = 0$, the brane must obey the grade restrcition rule (6.16), with N replaced by $N_+ = N_-$. In the case $N_+ > N_-$ (*resp.* $N_+ < N_-$), the FI parameter runs from positive to negative (*resp.* negtaive to positive) and there are $|N_+ - N_-|$ massive vacua on the Coulomb branch. The contour can be chosen as in Fig. 6-Left (*resp.* -Right).

6.3 Higher Rank Abelian Theories

In the rest of this paper, except when we discuss $U(1)$ theories, we write the real and imaginary parts of σ as

$$\tau = \mathrm{Re}(\sigma), \quad \upsilon = \mathrm{Im}(\sigma), \tag{6.18}$$

instead of σ_1 and σ_2, in order to avoid confusion between the index of coordinates on i$\mathfrak{t}$ and the $(1, 2)$ for the (real, imaginary) part.

In this subsection, we consider theories with Abelian gauge group G. For simplicity we take it to be a connected group so that $G = T$. We write $k := d_G = l_G$. The FI parameter ζ takes values in $\mathrm{i}\mathfrak{t}^*$. We consider a theory with a matter chiral multiplet $\phi = (\phi_i)_{i\in I}$ of R-charge $(R_i)_{i\in I}$ and gauge charge $(Q_i)_{i\in I}$. We assume some superpotential W of R-charge 2. We shall only consider the Calabi-Yau case, $\sum_i Q_i = 0$, so that we decide not to distinguish σ' from σ.

As in the $U(1)$ theory, the space of FI parameters is divided into phases. For ζ in a phase, any solution to the D-term equation

$$\sum_{i\in I} Q_i|\phi_i|^2 = \zeta \tag{6.19}$$

breaks the gauge group to a finite subgroup. An interface between two phases, called a phase boundary, is a positive linear span of $(k-1)$ independent charges from $\{Q_i\}_{i\in I}$. For ζ in such a phase boundary, there is a solution to (6.19) which breaks the gauge group to a subgroup of rank one whose Lie algebra is the common kernel of the $(k-1)$ charges. The rank of the possibly unbroken subgroup will be higher for intersection of phase boundaries. The quantum theory is parametrized by the FI-theta parameter $t = \zeta - \mathrm{i}\theta \in \mathfrak{t}^*_{\mathbb{C}}/2\pi\mathrm{i}\mathrm{P}$ where $\mathrm{P} \subset \mathrm{i}\mathfrak{t}^*$ is the weight lattice of T. The theory is singular at a hypersurface in which $t_{\mathit{eff}} := -2\pi \mathrm{d}\widetilde{W}_{\mathit{eff}} \equiv 0$, i.e.

$$t + \sum_{i\in I} Q_i \log Q_i(\sigma) \equiv 0 \qquad \mathrm{mod}\ 2\pi\mathrm{i}\mathrm{P}, \tag{6.20}$$

has a set of solutions, i.e. a non-compact Coulomb branch. There can also be additional singularity from mixed Coulomb-Higgs branches. The ζ images of the singular hypersurfaces asymptote to the phase boundaries, and the Coulomb branch approaches the one for the unbroken gauge group at each of them.

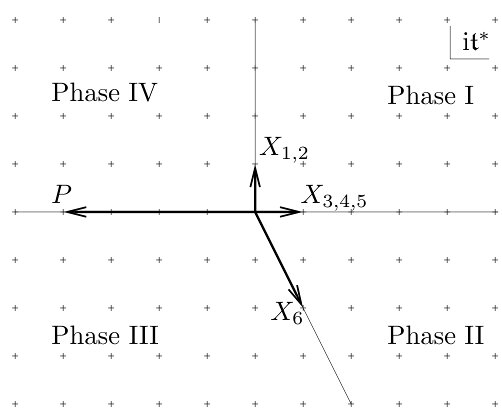


Figure 7: A two parameter model

As an illustration, let us consider the $U(1) \times U(1)$ linear sigma model familiar to physicists [39]: Fields are $X_1, \ldots, X_6$ and P whose charges are as in Fig. 7, and with superpotential $W = Pf(X)$ where $f(X)$ is a polynomial of $X_1, \ldots, X_6$ of bidegree $(0,4)$. The theory has four phases, I, II, III and IV, which are respectively the geometric, orbifold, Landau-Ginzbirg orbifold and hybrid phases. The quantum theory is singular at the two curves

$$C_1 : \qquad \mathrm{e}^{-t^1} = 4^{-4}(1-2u), \quad \mathrm{e}^{-t^2} = \frac{u^2}{(1-2u)^2}, \tag{6.21}$$

$$C_2 : \qquad \mathrm{e}^{-t^2} = 2^{-2}. \tag{6.22}$$

The curve C_1 is associated to the pure Coulomb branch with $\sigma_2/\sigma_1 = u$. (Here, σ_1 and σ_2 are not the real and the imaginary parts of σ, but the first and the second $U(1)$ components of σ. This is why we introduce the new notation (6.18) for the real and imaginary parts.) The limit points $u = 0, \frac{1}{2}, \infty$ correspond to the I-IV, II-III, III-IV boundaries with the right unbroken gauge groups, $(\sigma_1, \sigma_2) \in \mathbf{C}(1,0), \mathbf{C}(2,1), \mathbf{C}(0,1)$. The curve C_2 is associated to a mixed Higgs-Coulomb branch in which the second $U(1)$ is unbroken. It corresponds to the I-II and III-IV boundaries. For more detail of the relevant aspects of the theory, see [39, 11].

Assuming the Calabi-Yau condition, the function (6.11) can be written as

$$A_q(\upsigma) = \zeta_{\mathit{eff}}(\upsilon) - \theta_{\mathit{eff},q}(\uptau), \tag{6.23}$$

where ζ_{eff} and $\theta_{\mathit{eff},q}$ are defined by $t_{\mathit{eff},q} := -2\pi \mathrm{d}\widetilde{W}_{\mathit{eff},q}$ using (6.7). Although we hide from the notation to avoid clatter, ζ_{eff} and $\theta_{\mathit{eff},q}$ depend on σ. In fact they depend only on the direction $\widehat{\sigma} = \sigma/\|\sigma\|$. If t is on the singular hypersurface, they 'vanish', i.e., $\zeta_{\mathit{eff}} = 0$ and $\theta_{\mathit{eff}} \equiv 0 \pmod{2\pi \mathrm{i} \mathrm{P}}$, for $\widehat{\sigma}$ in the Coulomb branch direction. If ζ is deep inside a phase, $\zeta_{\mathit{eff}}(\upsilon)$ is dominated by the classical part $\zeta(\upsilon)$ for any direction $\widehat{\upsigma}$, and nearly the entire half space

$$\left\{ \upsigma = \uptau + \mathrm{i}\upsilon \,\middle|\, \zeta(\upsilon) > 0 \right\} \subset \mathfrak{t}_{\mathbf{C}}$$

is admissible. When ζ approaches a phase boundary, the quantum correction becomes comparable to the classical part and a careful analysis will be needed.

Since the contour γ is defined to be a deformation of the real locus, $\upsilon = 0$, it may be regarded as a graph of a map, $\uptau \in \mathrm{i}\mathfrak{t} \mapsto \upsilon(\uptau) \in \mathrm{i}\mathfrak{t}$. It must obey the condition that $A_q(\uptau + \mathrm{i}\upsilon(\uptau))$ grows to infinity as $|\uptau| \to \infty$ in any direction. Also, the deformation should avoid the poles at (6.3), that is, $Q_i(\uptau) = 0$ and $Q_i(\upsilon) = n_i + \frac{R_i}{2}$ with $n_i = 0, 1, 2, \ldots$. This condition is satisfied if the contour γ avoids the wedge, $Q_i(\uptau) = 0$ and $Q_i(\upsilon) > 0$, i.e., if we choose the map $\upsilon = \upsilon(\uptau)$ to avoid positive values of $Q_i(\upsilon)$ over the hyperplane $Q_i(\uptau) = 0$. Let us introduce some notations. Let $H_i \subset \mathrm{i}\mathfrak{t}$ be the hyperplane annihilated

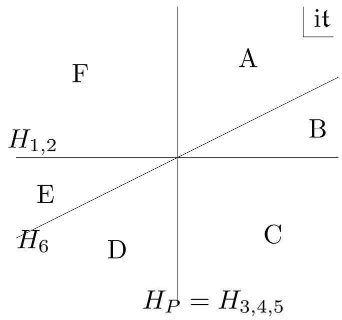


Figure 8: The chamber decomposition of $i\mathfrak{t}$ in the model of Fig. 7.

by Q_i, i.e, $H_i = \mathrm{Ker}\ Q_i$, and let $D_i^{\pm} \subset i\mathfrak{t}$ be the half space with positive values of $\pm Q_i$. The hyperplanes $\{H_i\}$ define a chamber decomposition of $i\mathfrak{t}$. See Fig. 8 for an example.

As the first step, we look for a piecewise linear map $\tau \mapsto \upsilon(\tau)$ satisfying the conditions, which is linear on each chamber. The wedge condition to avoid poles is a condition on the values at the walls of the chambers, $\upsilon(H_i) \subset \overline{D_i^-}$.

If ζ is deep inside a phase, the growth condition is satisfied if the image is deep inside the ζ-positive half space D_ζ^+ and if it is of full rank on each chamber. In particular, the image is a cone of full dimension inside the half space D_ζ^+. We shall call it the image cone of the map. Let us show examples of such maps in the two parameter model:

$$\zeta \in \text{Phase I}: \qquad (\upsilon_1, \upsilon_2) = (|\tau_1|, |\tau_2|), \tag{6.24}$$

$$\zeta \in \text{Phase II}: \qquad (\upsilon_1, \upsilon_1 - 2\upsilon_2) = (|\tau_1|, |\tau_1 - 2\tau_2|), \tag{6.25}$$

$$\zeta \in \text{Phase III}: \qquad (\upsilon_1, \upsilon_1 - 2\upsilon_2) = (-|\tau_1|, |\tau_1 - 2\tau_2|), \tag{6.26}$$

$$\zeta \in \text{Phase IV}: \qquad (\upsilon_1, \upsilon_2) = (-|\tau_1|, |\tau_2|), \tag{6.27}$$

The choice may not be unique. For example, if ζ is in the subset $\zeta^1 < 0$, $\zeta^2 < 0$ of Phase III, we may also take $(\upsilon_1, \upsilon_2) = (-|\tau_1|, -|\tau_2|)$. However, the two can be continuously connected to each other by a homotopy which stays inside the admissible region. That is, they are in the same deformation class. In Fig. 9, we show the image cones of these maps. $C_{\mathrm{I}}, \ldots, C_{\mathrm{IV}}$ are the image cones of (6.24), ..., (6.27), and $C_{\mathrm{III}'}$ is the one for the other map on a part of Phase III. We may try to generalize the examples (6.24)-(6.27). Suppose that ζ is a positive linear span of a set $\{Q_j\}_{j\in J}$ of k charges, which must be linearly independent if ζ is deep inside a phase. Then, define $\upsilon(\tau)$ by

$$Q_j(\upsilon(\tau)) = |Q_j(\tau)|, \qquad \forall j \in J. \tag{6.28}$$

It certainly satisfies the growth condition, but the question is the wedge condition to avoid poles. The latter is always satisfied when $k = 2$ and also in many other examples with

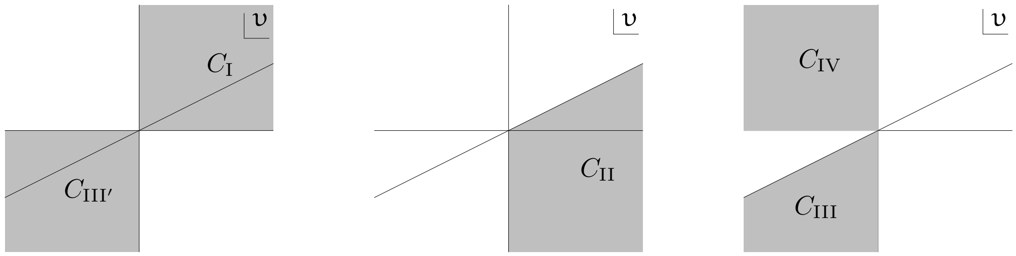


Figure 9: The image cones

higher k. However, it is easy to find counter examples with $k = 3$.

The graph of such a piecewise linear map is not always Lagrangian. For example, the maps (6.24) and (6.27) already define (piecewise) Lagrangian submanifolds with respect to $\omega = \mathrm{d}\tau_1 \wedge \mathrm{d}\upsilon_1 + \mathrm{d}\tau_2 \wedge \mathrm{d}\upsilon_2$, but the maps (6.25) and (6.25) do not. In fact we may modify the maps as

$$(\upsilon_1, \upsilon_1 - 2\upsilon_2) = (\pm f(\tau)|\tau_1|, g(\tau)|\tau_1 - 2\tau_2|) \tag{6.29}$$

for some positive valued functions $f(\tau)$ and $g(\tau)$. It is straightforward though technically involved to find such functions so that the graph is a Lagrangian. Such a modification is also useful even if the graph is already a Lagrangian. For a piecewise linear map, no matter how deep inside ζ is, if we consider a very large Chan-Paton charge q, the growth condition can be violated. However, the graph can be 'bent' by multiplying positive functions to the maps, as in (6.29). For example, in Phase I, the map (6.24) can be modified to $(\upsilon_1, \upsilon_2) = (|\tau_1|^{1+\epsilon}, |\tau_2|^{1+\epsilon})$ for some positive ϵ, say 1. Then, the growth condition is satisfied for any brane $\mathfrak{B}$ with an arbitrary set of Chan-Paton charges.

At this moment, we do not have a general proof of the existence and uniqueness of the deformation class of a map $\tau \mapsto \upsilon(\tau)$ satisfying the conditions. We leave it as a problem for a future work.

Let us now consider the region where ζ is not deep inside a phase. Since the analysis is very complicated in general, in this paper, we focus on the region near an "asymptotic phase boundary", that is, deep in the interior of the boundary between two phases. Take a phase boundary spanned by $(k-1)$ charges $\{Q_i\}_{i\in I_b}$. We denote by T^u the unbroken subgroup at the boundary and take its integral generator $e^u \in \mathrm{i}\mathfrak{t}$. We write $\xi(e^u) = \xi^u$ for $\xi \in \mathfrak{t}^*_{\mathbf{C}}$. Since e^u is the common kernel of $\{Q_i\}_{i\in I_b}$ we have $Q^u_i = 0$ for $i \in I_b$. If we choose some element $e_u \in \mathrm{i}\mathfrak{t}^*$ such that $e_u(e^u) = 1$ we can write $t = \sum_{i\in I_b} Q_i t^i + e_u t^u$. We are looking at the regime where $\zeta^i \gg 0$ for all $i \in I_b$. For any fixed $(t^i)_{i\in I_b}$ in that regime, we have an array of singular 'points' in the t^u-plane, separated by $2\pi\mathrm{i}$. (Each 'point' is in general a collection of a number of points which are very close to each other.) In any limit with $\zeta^i \to +\infty$, the singular 'points' approach the points

$$t^u = -\sum_{i\in I} Q_i^u \log Q_i^u + 2\pi i n, \quad n \in \mathbf{Z}. \tag{6.30}$$

The line $\zeta^u = -\sum_{i\in I} Q_i^u \log|Q_i^u|$ is the asymptotic phase boundary, and the open intervals of length 2π between the adjacent singular points shall be called the windows between the phases in the asymptotic regime. In the two parameter model, the asymptotic singular points of the four phase boundaries are

$$\text{I-II}: \quad t^2 \equiv 2\log 2, \quad [1], \tag{6.31}$$
$$\text{II-III}: \quad 2t^1 + t^2 \equiv 9\log 4, \quad [1], \tag{6.32}$$
$$\text{III-IV}: \quad t^2 \equiv 2\log 2. \quad [2], \tag{6.33}$$
$$\text{IV-I}: \quad t^1 \equiv 4\log 4, \quad [2]. \tag{6.34}$$

The number in the bracket shows the number of points in the collection.

Let us examine the image of the map $\tau \mapsto \upsilon(\tau)$ over the line $\tau \in \mathbf{R}e^u$ of the unbroken gauge group T^u. This line is equal to the intersection of the hyperplanes H_i for $i \in I_b$. By the wedge condition, we need $Q_i(\upsilon(\tau)) \leq 0$ for $i \in I_b$. On the other hand, by the growth condition, none of $Q_i(\upsilon(\tau))$ with $i \in I_b$ cannot go large negative since we are looking at the regime $\zeta^i \gg 0$ for all $i \in I_b$. Therefore, $Q_i(\upsilon(\tau))$ with $i \in I_b$ are frozen to be small on the line $\tau \in \mathbf{R}\,\mathrm{e}^u$. This is the incarnation of the Higgs mechanism in which $Q_i(\sigma) = 0$ is enforced by the non-vanishing value of ϕ_i at a solution to the D-term equation with $\zeta^i \gg 0$. As a consequence, on this line, $\tau = \tau_u e^u$, the function $A_q(\tau, \upsilon(\tau))$ is dominated by the one for the theory with the gauge group T^u only, that is,

$$A_q = \left(\zeta^u + \sum_{i\in I} Q_i^u \log|Q_i^u|\right)\upsilon_u(\tau) + \left(\sum_{Q_i^u>0} Q_i^u \pi - \mathrm{sgn}(\tau_u)(\theta^u + 2\pi q^u)\right)|\tau_u|. \tag{6.35}$$

If ζ^u is exactly on the asymptotic phase boundary, where the first term vanishes, we obtain a constraint

$$-\frac{1}{2}\sum_{Q_i^u>0} Q_i^u \;<\; \frac{\theta^u}{2\pi} + q^u \;<\; \frac{1}{2}\sum_{Q_i^u>0} Q_i^u. \tag{6.36}$$

This is the grade restriction rule. It is a constraint on the Chan-Paton charges with respect only to the unbroken gauge group T^u at the phase boundary. (It is called the *band restriction rule* in [11].) The set of charges satisfying this condition depends only on the window, as in the $U(1)$ theories. If ζ^u is above (*resp.* below) the asymptotic phase boundary, we may choose $\upsilon_u(\tau)$ on the line $\tau = \tau_u e^u$ to be a function that goes to postive (*resp.* negative) infinity faster than $|\tau_u|$ (*resp.* $-|\tau_u|$). Then, at least along this line, the growth condition is satisfied for any charge q^u. In fact, this behaviour is consistent with the choice of contour γ deep inside either of the two phases, provided the latter is constructed based on the map (6.28). In the phase above the boundary, as the set $\{Q_j\}_{j\in J}$ we take $\{Q_i\}_{i\in I_b} \cup \{Q_{j_+}\}$ where Q_{j_+} is one of the charges such that $Q^u_{j_+} > 0$. Then, up to a positive rescaling, we may assume $e_u = Q_{j_+}$. This means that $\upsilon_u(\tau) = |\tau_u|$ on the line (before the further bending). In the phase below the boundary, we take Q_{j_-} with $Q^u_{j_-} < 0$ instead of Q_{j_+}, and we have $\upsilon_u(\tau) = -|\tau_u|$ on the line (before the further bending).

6.4 Non-Abelian Examples

The linear sigma model with non-Abelian gauge groups is a surprisingly rich subject of study. One interesting feature is that there can be phases in which a continuous subgroup of the gauge group is totally unbroken. The low energy physics of such a strongly coupled system is usually hard to understand. Exact results obtained in this paper may provide some clue towards better understanding. In this subsection, we describe some examples with geometric phase where we can find admissible contours, as well as an example where a simple grade restriction rule can be obtained. Full exploration is beyond the scope of the present paper and will be left for future works.

The models treated are all Calabi-Yau and hence we write σ for σ'. Also reminded is the notation (6.18) for the real and imaginary parts of σ.

6.4.1 Rødland Model

The first example is the Rødland model [40, 34]. It is a $U(2)$ gauge theory with seven fundamental doublets, $X_1, \ldots, X_7$ and seven fields $P^1, \ldots, P^7$ in the $\det^{-1}$ representation. The superpotential is of the form $W = \sum_{i,j,k=1}^{7} A_k^{ij} P^k [X_i X_j]$ where $[X_i X_j]$ are the baryons $X_i^1 X_j^2 - X_i^2 X_j^1$ and A_k^{ij} are generic complex coefficients which are antisymmetric in the upper indices. The R-charge assignment is unique up to te gauge shift, $2-2\epsilon$ for P^i's and ϵ for X_i's, with $0 < \epsilon < 1$. There is one FI and one theta parameters, $\zeta \in \mathbf{R}$, $\theta \in \mathbf{R}/2\pi i\mathbf{Z}$. $\zeta \gg 0$ is the usual geometric phase where the gauge group is completely broken and the low energy theory is the non-linear sigma model on the complete intersection of seven hypersurfaces, $A_k^{ij}[x_i x_j] = 0$, $k = 1, \ldots, 7$, in the Grassmannian $G(2, 7)$. $\zeta \ll 0$ is the phase in which the $SU(2)$ subgroup is totally unbroken. Obtaining and applying some understanding of $SU(2)$ gauge theories, it is found [34] that the low energy theory is the non-linear sigma model after all, whose target space is the Pfaffian locus of $p \in \mathbb{CP}^6$ where the 7×7 antisymmetric matrix $(A^{ij}(p)) = (\sum_k A^{ij}_k p^k)$ is of rank 4. $\zeta \gg 0$ is called the Grassmannian phase while $\zeta \ll 0$ is called the Pfaffian phase. The quantum theory is parametrized by $t = \zeta - \mathrm{i}\theta$ and there are three singular points in the middle, at $\mathrm{e}^t = (1+\omega)^7, (1+\omega^2)^7, (1+\omega^3)^7$ with $\omega = \mathrm{e}^{2\pi \mathrm{i}/7}$.[1]

Let us write down the function $A_q(\sigma)$,

$$\begin{aligned} A_q(\sigma) \;&=\; \zeta(\upsilon_1+\upsilon_2) - \theta(\tau_1+\tau_2) - 2\pi q^1\tau_1 - 2\pi q^2\tau_2 - \pi|\tau_1-\tau_2| \\ &\quad +7(-\upsilon_1-\upsilon_2)\log|\sigma_1+\sigma_2| + 7|\tau_1+\tau_2|\left(\frac{\pi}{2} + \arctan\left[\frac{-\upsilon_1-\upsilon_2}{|\tau_1+\tau_2|}\right]\right) \\ &\quad +7\upsilon_1\log|\sigma_1| + 7|\tau_1|\left(\frac{\pi}{2} + \arctan\left[\frac{\upsilon_1}{|\tau_1|}\right]\right) \\ &\quad +7\upsilon_2\log|\sigma_2| + 7|\tau_2|\left(\frac{\pi}{2} + \arctan\left[\frac{\upsilon_2}{|\tau_2|}\right]\right). \end{aligned} \tag{6.37}$$

It can also be written as (6.23), i.e., $A_q(\sigma) = \zeta_{\mathit{eff}}(\upsilon) - \theta_{\mathit{eff},q}(\tau)$, where $t_{\mathit{eff}} = -2\pi \mathrm{d}\widetilde{W}_{\mathit{eff},q}$ is the effective FI-theta parameter which depends on the direction $\widehat{\sigma} = \sigma/\|\sigma\|$. When $\zeta \gg 0$ or $\zeta \ll 0$, the term $\zeta_{\mathit{eff}}(\upsilon)$ is dominated by $\zeta(\upsilon_1+\upsilon_2)$. Therefore the admissible region is the region with $(\upsilon_1+\upsilon_2) \gg 0$ or $(\upsilon_1+\upsilon_2) \ll 0$.

As in the higher rank Abelian theories, we would like to think of γ as the graph of a map $\tau \mapsto \upsilon = \upsilon(\tau)$. The wedge condition to avoid poles (6.3) is

$$\tau_1+\tau_2 = 0 \implies \upsilon_1+\upsilon_2 \geq 0, \tag{6.38}$$
$$\tau_1 = 0 \implies \upsilon_1 \leq 0, \tag{6.39}$$
$$\tau_2 = 0 \implies \upsilon_2 \leq 0. \tag{6.40}$$

In the Grassmannian phase $\zeta \gg 0$, an admissible contour is easy to find. For example, we can take

$$\upsilon_1 = (\tau_1)^2, \qquad \upsilon_2 = (\tau_2)^2. \tag{6.41}$$

It may be replaced by $\upsilon_1 = |\tau_1|^\alpha$, $\upsilon_2 = |\tau_2|^\alpha$ for any $\alpha > 1$. For such a choice, the growth condition is satisfied for any q. Therefore, this can be used for any brane $\mathfrak{B}$ with an arbitrary set of Chan-Paton representations.

[1] Compared to [34], there is a sign difference. This is the effect of the W-boson integral. The formulae in [34] should be corrected by the replacement $\mathrm{e}^{-t} \to \mathrm{e}^{-t}(-1)^{k+1}$ for $U(k)$ gauge theory. The relationship between the theta angle and the B-field mentioned in [34] is totally explainable by Morrison-Plesser mechanism [39]. The same formula as [34] is copied in v1 of [36]. That is a careless mistake.

In the Pfaffian phase $\zeta \ll 0$, on the other hand, it is hard to find any admissible contour of the above type. The growth condition $(\upsilon_1 + \upsilon_1) \ll 0$ is in conflict with the wedge condition (6.38). We may need to select the allowed set of Chan-Paton representations over the entire phase. We plan to explore this problem in the future works.

The contour choice of the type (6.41) works in the usual geometric phase in a Calabi-Yau model. For example, take a $U(k)$ gauge theory with N fundamentals $X_1, \ldots, X_N$ and a number of powers of $\det^{-1}$ representations, $P^1, \ldots, P^S$, and a gauge invariant superpotential $W = P^1 f_1(B) + \cdots + P^S f_S(B)$ with $f_i(B)$'s being polynomials of the baryons $B_{i_1 \cdots i_k} = [X_{i_1} \cdots X_{i_k}]$. $\zeta \gg 0$ is a geometric phase where the gauge group is completely broken and the low energy theory is the non-linear sigma model on the complete intersection of hypersurfaces $f_1 = \cdots = f_S = 0$ of the Grassmannian $G(k, N)$. In this phase, the contour

$$\upsilon_a = (\tau_a)^2, \qquad a = 1, \ldots, k, \tag{6.42}$$

is admissible for any brane $\mathfrak{B}$.

6.4.2 A Model With A Simple Grade Restriction Rule

The next example is the $U(2)$ gauge theory with four fundamentals, $X_1, \ldots, X_4$, and four antifundamentals, $Y^1, \ldots, Y^4$. Choice of superpotential and R-charge assignment are not relevant for the matters we would like to discuss. $\zeta \gg 0$ and $\zeta \ll 0$ are both phases where the gauge group is completely broken and X_i's and Y^i's span the Grassmannian $G(2, 4)$ respectively. The quantum theory is parametrized by $t = \zeta - \mathrm{i}\theta$ and there is a single singularity in the middle,

$$t \equiv \pi\mathrm{i} \mod 2\pi\mathrm{i}\mathbf{Z}. \tag{6.43}$$

This π shift of the theta angle comes from the single pair of the W-bosons.

The function $A_q(\sigma)$ is astonishingly simple,

$$\begin{aligned} A_q(\sigma) &= \zeta(\upsilon_1 + \upsilon_2) - \theta(\tau_1 + \tau_2) - 2\pi(q^1\tau_1 + q^2\tau_2) \\ &\quad -\pi|\tau_1 - \tau_2| + 4\pi|\tau_1| + 4\pi|\tau_2|. \end{aligned} \tag{6.44}$$

The wedge condition to avoid poles is

$$\tau_1 = 0 \;\Rightarrow\; \upsilon_1 = 0, \qquad \tau_2 = 0 \;\Rightarrow\; \upsilon_2 = 0. \tag{6.45}$$

In the $\zeta \gg 0$ phase, as an admissible contour, we can take

$$\upsilon_1 = (\tau_1)^2, \qquad \upsilon_2 = (\tau_2)^2. \tag{6.46}$$

In the $\zeta \ll 0$ phase, as an admissible contour, we can take

$$\upsilon_1 = -(\tau_1)^2, \quad \upsilon_2 = -(\tau_2)^2. \tag{6.47}$$

At the phase boundary, $\zeta = 0$, the υ dependence disappears and the choice of graph $\upsilon = \upsilon(\tau)$ does not matter. The growth condition simply requires that

$$A_q(\sigma) \;=\; -\theta(\tau_1+\tau_2) - 2\pi(q^1\tau_1 + q^2\tau_2) - \pi|\tau_1 - \tau_2| + 4\pi|\tau_1| + 4\pi|\tau_2|$$

goes to positive infinity as $|\tau| \to \infty$ in any direction. After some elementary exercise, we find that this condition is equivalent to

$$-\frac{3}{2} < \frac{\theta}{2\pi} + q^1 < \frac{3}{2}, \quad -\frac{3}{2} < \frac{\theta}{2\pi} + q^2 < \frac{3}{2}. \tag{6.48}$$

This is the grade restriction rule. As long as $\theta \not\equiv \pi \pmod{2\pi\mathbf{Z}}$, this defines a set of nine weights in a square of size 3 on the diagonal. This set does not change as long as θ moves in a window, i.e. and open interval of length 2π sandwitched between singular ponts. See Fig. 10 for example.

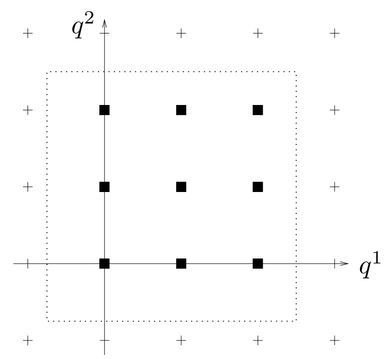


Figure 10: The grade restriction rule for the window $-3\pi < \theta < -\pi$.

We decided to look at this example because this is one of the first examples in a work by Donnovan-Segal [41] which studies aspects of (classical) grade restriction rule in a class of non-Abelian linear sigma models. There it is found that as the relevant "window category" one can take the one generated by Kapranov's exceptional collection of $G(2,4)$, which are the vector bundles associated with the following representations of $U(2)$:

$$\mathbf{C}, \quad \mathbf{C}^2, \quad \mathrm{Sym}^2\mathbf{C}^2, \quad \det, \quad \mathbf{C}^2 \otimes \det, \quad \det^{\otimes 2}, \tag{6.49}$$

where $\mathbf{C}$ is the trivial representation. We see that the weights of these representations fits precisely to the one in Fig 10.

7 Low Energy Behaviour

In this section, we check the partition function against the expected low energy physics of the theory. In the Calabi-Yau case, we fulfill the promise to show that, deep inside a phase, the partition function takes the same value for branes that descend to the same brane in the low energy theory. This in particular shows that the analytic property of the partition function is consistent with the rule of D-brane transport along a path in the parameter space. In the non Calabi-Yau case, we look at the behaviour of the partition function in the large r limit. Some consistency check can be made, and moreover, the study leads us to find the rule of D-brane map under the bulk renormalization group flow. The key is to look at the partition function of a particular class of branes, called "empty branes".

For concreteness, we consider in detail the particular $U(1)$ theory introduced in Section 6.2. Let us write down the formula for the partirion function for a brane $\mathfrak{B}$ in this theory,

$$Z_{D^2}(\mathfrak{B}) = (r\Lambda)^{\widehat{c}/2} \int_{\gamma} \mathrm{d}\sigma' \, \Gamma\left(-d\,\mathrm{i}\sigma' + 1 - \frac{d\epsilon}{2}\right) \Gamma\left(\mathrm{i}\sigma' + \frac{\epsilon}{2}\right)^N \mathrm{e}^{\mathrm{i}t_R(\sigma')} f_{\mathfrak{B}}(\sigma'), \tag{7.1}$$

where $f_{\mathfrak{B}}(\sigma') = \mathrm{tr}_M(\,\mathrm{e}^{\pi \mathrm{i}\mathbf{r}_*}\,\mathrm{e}^{2\pi\rho(\sigma')})$ and

$$\begin{aligned} \widehat{c} &= N - 2 - (N-d)\epsilon, & (7.2)\\ t_R &= t - (N-d)\log(r\Lambda). & (7.3) \end{aligned}$$

The integrand has poles at

$$\sigma' = \begin{cases} \mathrm{i}\left(n_x + \frac{\epsilon}{2}\right) & n_x = 0,1,2,\ldots \quad \text{(order } N\text{)}, \\ \mathrm{i}\left(-\frac{n_p+1}{d} + \frac{\epsilon}{2}\right) & n_p = 0,1,2,\ldots \quad \text{(simple)}. \end{cases} \tag{7.4}$$

7.1 Tachyon Condensation

Let us first describe how the branes in the linear sigma model reduce to branes in the classical low energy theory, deep in either the geometric phase $\zeta_c \gg 0$ or the Landau-Ginzburg orbifold phase $\zeta_c \ll 0$. To emphasize that the analysis is purely classical, we denoted the FI parameter by ζ_c. An important rôle is played again by the D-term equation,

$$\sum_{i=1}^{N} |x_i|^2 - d|p|^2 = \zeta_c. \tag{7.5}$$

We shall only give an outline since all the detail can be found in [11]. Only the Calabi-Yau case was discussed in [11], but the classical discussion applies equally well without such a restriction.

The descent of branes can be decomposed into two steps: (i) impose the D-term equation (7.5) strictly but keep all the chiral multiplets, and (ii) integrate out the heavy chiral multiplets. Step (i) suffices for the present purposes. Step (ii) will be described in the next section.

The main ingredient in the descent is the brane-antibrane annihilation by tachyon condensation. Recall that the matrix factorization Q enters into the boundary potential $\{Q, Q^\dagger\}$. I.e., it plays the rôle of a profile of the open string tachyon. If the D-term equation (7.5) is strictly imposed, it is possible that $\{Q, Q^\dagger\}$ is everywhere positive definite. In such a case the brane can be regarded as *empty* in the low energy limit by the complete brane-antibrane annihilation. Since the space of solutions to (7.5) depends very much on the sign of ζ_c, which branes are empty and which branes are not depends also on the sign of ζ_c.

Let us introduce two basic examples:

$$\mathfrak{B}_1: \qquad \mathbf{C}(0,0) \underset{p}{\overset{f}{\rightleftarrows}} \mathbf{C}(1-d\epsilon, d) \tag{7.6}$$

$$\mathfrak{B}_2: \qquad \mathbf{C}(0,0) \underset{pf'}{\overset{x}{\rightleftarrows}} E \underset{pf'}{\overset{x}{\rightleftarrows}} \wedge^2 E \underset{pf'}{\overset{x}{\rightleftarrows}} \cdots \underset{pf'}{\overset{x}{\rightleftarrows}} \wedge^N E \tag{7.7}$$

where $E = \mathbf{C}(1-\epsilon, 1)^{\oplus N}$. Here we used the notation of [11] except that the component $\mathcal{W}(q)_j$ of R-charge j and gauge charge q is here denoted by $\mathbf{C}(j,q)$. Let us explain what the data $(M_i, Q_i, \rho_i, \mathbf{r}_{*i})$ is for $\mathfrak{B}_i$, $i = 1, 2$. The vector space M_i is the direct sum of the spaces appearing, i.e., $M_1 = \mathbf{C}(0,0) \oplus \mathbf{C}(1-d\epsilon, d)$ and $M_2 = \wedge E$, $\mathbf{r}_{*i}$ and ρ_i are specified by the numbers (j,q) of each component $\mathbf{C}(j,q)$, and the matrix factorization is given by

$$Q_1 = \begin{pmatrix} 0 & p \\ f(x) & 0 \end{pmatrix}, \tag{7.8}$$

$$Q_2 = \sum_{i=1}^{N} \left(x_i \overline{\eta}_i + \frac{1}{d}\, p\, \partial_i f(x) \eta_i \right), \tag{7.9}$$

where η_i and $\overline{\eta}_i$ are generators of the Clifford algebra, $\{\eta_i, \overline{\eta}_j\} = \delta_{i,j}$, $\{\eta_i, \eta_j\} = \{\overline{\eta}_i, \overline{\eta}_j\} = 0$, that is used to construct $\wedge E$. We may also consider the shifts, $\mathfrak{B}_1(j,q)$ and $\mathfrak{B}_2(j,q)$, where $\mathfrak{B} \mapsto \mathfrak{B}(j,q)$ for $(j,q) \in \mathbf{Z}^{\oplus 2}$ is the uniform shift of the R-charges by j and the gauge charges by q. The boundary potentials are

$$\{Q_1, Q_1^\dagger\} = \left(|p|^2 + |f(x)|^2\right) \mathrm{id}_{M_1}, \tag{7.10}$$

$$\{Q_2, Q_2^\dagger\} \;=\; \sum_{i=1}^{N}\left(|x_i|^2 + \frac{1}{N^2}|p\partial_i f(x)|^2\right)\mathrm{id}_{M_2}. \tag{7.11}$$

In the $\zeta_c \gg 0$ phase, $\mathfrak{B}_2$ and all of its shifts are empty at low energies. This is because $\sum_i |x_i|^2 \geq \zeta_c$ by the D-term equation (7.5) and hence the boundary potential $\{Q_2, Q_2^\dagger\}$ is positive definite everywhere with a strictly positive lower bound. Likewise, in the $\zeta_c \ll 0$ phase, $\mathfrak{B}_1$ and all of its shifts are empty at low energies since $\{Q_1, Q_1^\dagger\}$ is positive definite everywhere on the D-term locus (7.5) where $|p|^2 \geq |\zeta_c/N|$. On the other hand, they are non-empty in the opposite phases, since the boundary potentials fail to be positive definite at some locus: $\{Q_1, Q_1^\dagger\}$ vanishes at $p = f(x) = 0$ which is allowed in the $\zeta_c \gg 0$ phase, while $\{Q_2, Q_2^\dagger\}$ vanishes at $x = 0$ (assuming $d > 1$) which is allowed in the $\zeta_c \ll 0$ phase. After the step (ii), see [11] or the next section, we find that $\mathfrak{B}_1$ in the $\zeta_c \gg 0$ phase is (a shift of) the structure sheaf $\mathcal{O}_{X_f}$, that is, the single D-brane wrapped on the entire target space X_f and supporting the trivial line bundle. When f is a Fermat polynomial, $\mathfrak{B}_2$ in the $\zeta_c \to -\infty$ limit is one of the $\mathbf{L} = \mathbf{0}$ Recknagel-Schomerus branes [42].

Brane-antibrane annihilation implies that the descent map of branes in the linear sigma model to branes in the low energy theory is not one to one but many to one, as is always the case in renormalization group flow. In fact it is huge to one since any number of copies of empty branes should be regarded as "nothing" in the low energy theory. It would be convenient if we have a subset, or a slice, in the set of branes in the linear sigma model such that the map is one to one when restricted to that subset. In fact, such subsets exist! In the $\zeta_c \gg 0$ phase, let us consider a set of branes whose Chan-Paton charges are within a zone $\mathbf{w}$ of length N, i.e., a set of N consecutive integers, say $\mathbf{w} = \{1, \ldots, N\}$ or $\mathbf{w} = \{17, \ldots, 16{+}N\}$. Then, the map of branes in that subset to branes in the low energy theory at $\zeta_c \gg 0$ is one to one. Likewise, in the $\zeta_c \ll 0$ phase, let us consider a set of branes whose Chan-Paton charges are within a zone $\mathbf{w}$ of length d, say $\mathbf{w} = \{0, \ldots, d{-}1\}$. Then, the map of branes in that subset to branes in the low energy theory at $\zeta_c \ll 0$ is one to one. We shall call this the *classical grade restriction rule*.

We put the adjective "classical" in order to distinguish it from the (quantum) grade restriction rule which we have discussed in the previous section, for branes of the theory sitting at or going through a window between different phases, in the Calabi-Yau case. However, these are certainly related. Note that (6.16) defines a zone of length N, and we shall call it the zone of the window. Carrying over the terminology, a brane in the subset determined by a zone $\mathbf{w}$ is said to be *grade restricted* with respect to $\mathbf{w}$.

The classical grade restriction rule means that any brane can be replaced by a unique grade restricted brane by a brane-antibrane creation and annihilation process. We can show this by employing the empty branes introduced above, i.e, $\mathfrak{B}_2$ and its shifts in the $\zeta_c \gg 0$ phase and $\mathfrak{B}_1$ and its shifts in the $\zeta_c \ll 0$ phase. It goes as follows. Let us take any brane $\mathfrak{B} = (M, Q, \rho, \mathbf{r}_*)$. If $\mathfrak{B}$ has a Chan-Paton charge outside the zone $\mathbf{w}$, then, we can bind an empty brane to $\mathfrak{B}$ at the vector of that charge so that the resulting brane has charges closer to $\mathbf{w}$. We repeat this process. Note that $\mathfrak{B}_2$ has the smallest charge 0 and the largest charge N, while $\mathfrak{B}_1$ has the smallest charge 0 and the largest charge d. This guarantees that this binding process can eventually put all the charges inside the zone $\mathbf{w}$ of length N for $\zeta_c \gg 0$ and d for $\zeta_c \ll 0$.

To summarize the discussion, let us introduce some notations. We denote the set of all linear sigma model branes by $\mathfrak{D}$, the set of branes in the classical low energy theory in the phase $\zeta_c \gg 0$ (*resp.* $\zeta_c \ll 0$) by D_+ (*resp.* D_-) and the set of grade restricted branes with respect to a zone $\mathbf{w}$ by $\mathcal{T}_{\mathbf{w}}$. Then, we have maps of branes:

$$\begin{tikzcd} \mathcal{T}_{\mathbf{w}_-} \subset \mathfrak{D} \arrow[dr, "\cong"'] \arrow[d, phantom] & \mathfrak{D} \arrow[d, "\pi_-"] \\ & D_- \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_{\mathbf{w}_+} \subset \mathfrak{D} \arrow[dr, "\cong"'] & \mathfrak{D} \arrow[d, "\pi_+"] \\ & D_+ \end{tikzcd}$$

The vertical arrow $\pi_\pm : \mathfrak{D} \to D_\pm$ is the huge-to-one descent map to the low energy theory. The restriction to the subset $\mathcal{T}_{\mathbf{w}_\pm} \subset \mathfrak{D}$ associated to a zone $\mathbf{w}_\pm$ is one to one, if the length of $\mathbf{w}_-$ is d and the length of $\mathbf{w}_+$ is N. The above diagrams of "sets" and "maps" may be regarded as diagrams of categories and functors. In that case "the one to one map" should be regarded as an equivalence of categories. There is a recent development concerning such equivalences of categories, motivated by the classical grade restriction rule [43–45, 41].

Finally let us compute the factor $f_{\mathfrak{B}}(\sigma')$ in the integrand of (7.1) for the branes $\mathfrak{B}_1$ and $\mathfrak{B}_2$:

$$\begin{aligned} f_{\mathfrak{B}_1}(\sigma') &= 1 - \mathrm{e}^{-\pi \mathrm{i} d\epsilon}\, \mathrm{e}^{2\pi d\sigma'}, && (7.12)\\ f_{\mathfrak{B}_2}(\sigma') &= 1 - N\, \mathrm{e}^{-\pi \mathrm{i}\epsilon}\, \mathrm{e}^{2\pi\sigma'} + \binom{N}{2} \mathrm{e}^{-2\pi \mathrm{i}\epsilon}\, \mathrm{e}^{4\pi\sigma'} - \cdots + (-1)^N \mathrm{e}^{-N\pi \mathrm{i}\epsilon}\, \mathrm{e}^{2N\pi\sigma'} \\ &= (1 - \mathrm{e}^{-\pi \mathrm{i}\epsilon}\, \mathrm{e}^{2\pi\sigma'})^N. && (7.13) \end{aligned}$$

Notice that $f_{\mathfrak{B}_2}(\sigma')$ cancels the poles of $\Gamma(\mathrm{i}\sigma' + \frac{\epsilon}{2})^N$ on the positive imaginary axis but cannot cancel all the poles of $\Gamma(-d\mathrm{i}\sigma' + 1 - \frac{d\epsilon}{2})$ on the negative imaginary axis. On the other hand, $f_{\mathfrak{B}_1}(\sigma')$ cancels the poles of $\Gamma(-d\mathrm{i}\sigma' + 1 - \frac{d\epsilon}{2})$ on the negative imaginary axis but cannot cancel the higher order poles of $\Gamma(\mathrm{i}\sigma' + \frac{\epsilon}{2})^N$ on the positive imaginary axis. This is a reflection of the fact that $\mathfrak{B}_2$ is empty in the $\zeta_c \gg 0$ phase but not in the $\zeta_c \ll 0$

phase, while $\mathfrak{B}_1$ is empty in the $\zeta_c \ll 0$ phase but not in the $\zeta_c \gg 0$ phase. The real significance of this observation in the quantum theory will be discussed below.

We now look at the partition function and compare it with the expectation of the low energy behaviour of the theory and of the branes, including the above descent map of branes. We separate the discussion into the three cases, $d = N$, $d < N$ and $d > N$.

7.2 $\underline{d = N}$: Family of conformal field theories

In the Calabi-Yau case, $d = N$, the family of theories parametrized by $t \in \mathbf{C}/2\pi\mathrm{i}\mathbf{Z}$ is expected to flow to a family of superconformal field theories with $c/3 = N - 2$. Note that the last number is equal to the exponent $\widehat{c}$ in (7.2) as already remaked (5.81). In the two extreme regimes, $\zeta \gg 0$ and $\zeta \ll 0$, the degrees of freedom other than those in the classical low energy theory are infinitely heavy. Therefore, the classical analysis of the previous subsection is expected to hold, with $\zeta \sim \zeta_c$.

As an examination, let us look at the partition function for the branes $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Recall that $f_{\mathfrak{B}_2}(\sigma')$ cancels the poles of the gamma function factor on the positive imaginary axis while $f_{\mathfrak{B}_1}(\sigma')$ cancels the poles on the negative imaginary axis. In view of the contour choice in Fig. 2, we see that they indeed have vanishing partition function in the phase where they are said to be empty,

$$\begin{aligned} Z_{D^2}(\mathfrak{B}_2) = 0 \quad \text{for } \zeta_{\mathit{eff}} > 0, \\ Z_{D^2}(\mathfrak{B}_1) = 0 \quad \text{for } \zeta_{\mathit{eff}} < 0. \end{aligned} \tag{7.14}$$

The same holds for $\mathfrak{B}_2(j, q)$ and $\mathfrak{B}_1(j, q)$ as the shift $\mathfrak{B} \mapsto \mathfrak{B}(j, q)$ changes the brane factor $f_{\mathfrak{B}}(\sigma')$ simply by multiplication of the entire function $(-1)^j\, \mathrm{e}^{2\pi q\sigma'}$ which does not affect the cancellation of poles. Recall also that the brane factors for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ fail to cancel the poles on the opposite sides of the imaginary axis. Thus, the partition functions do not have to vanish in the opposite phases. In the next section, we will compute them and see that they are equal to the (expected) parition function of the low energy images, i.e. the structure sheaf $\mathcal{O}_{X_f}$ of X_f for $\mathfrak{B}_1$ and the Recknagel-Schomerus brane of the Landau-Ginzburg orbifold for $\mathfrak{B}_2$.

The vanishing (7.14) means that, in each phase, branes related by binding the empty branes have exactly the same partition function. In particular, a given brane and its grade restricted replacement have the same partition function. Therefore, the partition function takes the same value on the branes which descend to the same brane in the classical low energy theory.

Now let us come back to the problem in Section 6.2.1 concerning analytic continuation of the partition function $Z_{D^2}(\mathfrak{B})$ along a path from one phase to another, say from $\zeta \gg 0$ to $\zeta \ll 0$. There was a problem if the brane $\mathfrak{B}$ is not grade restricted with respect to the window through which the path goes. We now know what to do: while in the $\zeta \gg 0$ phase, we replace $\mathfrak{B}$ by a grade restricted brane $\mathfrak{B}'$ by binding the empty branes $\mathfrak{B}_2(j, q)$. We have just learned that $Z_{D^2}(\mathfrak{B})$ is exactly equal to $Z_{D^2}(\mathfrak{B}')$ in the $\zeta \gg 0$ phase. Now that $\mathfrak{B}'$ is grade restricted with respect to the window, the partition function can be analytically continued to $\zeta \ll 0$ through that window. We therefore conclude that the partition function of the brane $\mathfrak{B}$ at $\zeta \gg 0$ is analytically continued along the path to the partiction function of the brane $\mathfrak{B}'$ at $\zeta \ll 0$.

This matches the rule of D-brane transport [11]. In the Calabi-Yau case $d = N$, the lengths of the zones for the classical grade restriction are the same between the two phases. Hence we can take a common grade restricted subset $\mathcal{T}_{\mathbf{w}}$ to make a bridge between low energy branes in one phase and the low energy branes in the other.

$$\begin{tikzcd}
 & \mathfrak{D} \arrow[ddl, bend right, "\pi_-"'] \arrow[ddr, bend left, "\pi_+"] & \\
 & \mathcal{T}_{\mathbf{w}} \arrow[u, phantom, "\cup"] \arrow[dl, "\cong"] \arrow[dr, "\cong"'] & \\
D_- & & D_+
\end{tikzcd}$$

If we take $\mathbf{w}$ to be the zone of a window, this gives the rule of D-brane transport through that window. Once again, we have seen that the analytic continuation of the partition function matches with this rule.

We may also consider a closed loop in the parameter space, starting from one phase, going to the other phase through a window, and then coming back through a different window. If we analytically continue the partition function of a brane $\mathfrak{B}$ along such a path, it comes back as the partition function of another brane $\mathfrak{B}''$.

$$\begin{tikzcd}
 & \mathfrak{D} \arrow[ddl, bend right, "\pi_-"'] \arrow[ddr, bend left, "\pi_+"] & & \\
 & \mathcal{T}_{\mathbf{w}'} \neq \mathcal{T}_{\mathbf{w}} \arrow[u, phantom, "\cup"] \arrow[dr, "\cong"] & \\
D_- \arrow[ur, "\cong"] & & D_+ \arrow[ul, "\cong"']
\end{tikzcd}$$

The tranform $\mathfrak{B} \mapsto \mathfrak{B}''$ is what is known as the D-brane monodromy. As in the above discussion, this is done by the brane replacement via binding empty branes at appropriate phases. If the loop goes around one singular point, it is to bind a brane which becomes massless at the singular point, in accord with the picture found by Strominger [47].

At the level of categories, the D-brane transport along a path from one phase to the other gives an equivalence of the categories, $D_+ \xrightarrow{\cong} D_-$. The equivalences for various windows are the same as the equivalences first found by Orlov [46]. The D-brane monodromy for a closed loop gives an autoequivalence of the category, say, $D_+ \xrightarrow{\cong} D_+$. Construction of such autoequivalences had been given in [48, 49] and is called Seidel-Thomas twist.

7.3 $\underline{d < N}$: Flow from the non-linear sigma model

When $d < N$, the FI parameter is larger at higher energies and the theory describes the asymptotically free non-linear sigma model on the Fano manifold X_f, with $c/3 = N - 2$ in the ultra-violet limit. At low energies, the theory reduces to the Landau-Ginzburg orbifold $W = f(X_1, \ldots, X_N)/\mathbf{Z}_d$ or one of the $(N-d)$ massive vacua. The Landau-Ginzburg orbifold is expected to flow to a superconformal field theory with

$$\frac{c}{3} = N\left(1 - \frac{2}{d}\right). \tag{7.15}$$

The massive vacua are at

$$\boldsymbol{\sigma}_k = -\mathrm{i}\widetilde{\Lambda}\exp\left(\mathrm{i}\frac{\theta + \pi d + 2\pi k}{N-d}\right), \quad k \in \mathbf{Z}/(N-d)\mathbf{Z}, \tag{7.16}$$

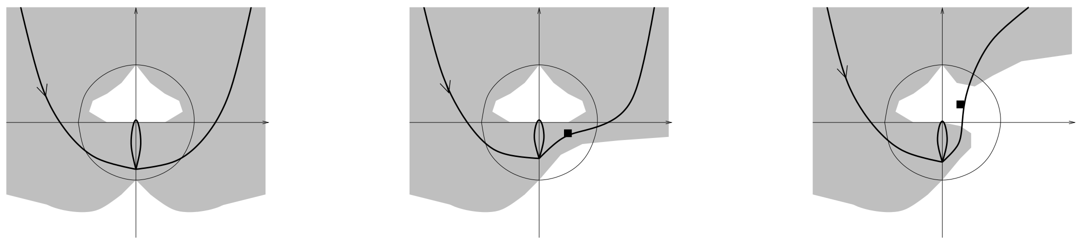
($\widetilde{\Lambda}^{N-d} := \Lambda^{N-d} d^d\, \mathrm{e}^{-\zeta}$, see Section 6.2) with the value $2\pi\widetilde{W}_{\mathit{eff}} = (N-d)\boldsymbol{\sigma}_k$ for the twisted superpotential.

We would like to ask which branes correspond to the superconformal field theory and which branes correspond to the massive vacua at low energies. Can we see that by looking at the behaviour of the partition function in the large size limit $r \to \infty$? Taking the lesson from Section 5.5, we may try to see if it has a power or exponential behaviour. We suppose that the poles on the negative imaginary axis are relevant for the Landau-Ginzburg orbifold. For $r\Lambda \gg 1$, the pole $\sigma' = \mathrm{i}(-1/d + \epsilon/2)$ closest to the origin yields the dominant contribution, which is of the order of

$$(r\Lambda)^{\widehat{c}/2}\, \mathrm{e}^{\mathrm{i}t_R \mathrm{i}(-1/d+\epsilon/2)} \;\sim\; (r\Lambda)^{N(1-2/d)/2}. \tag{7.17}$$

This is indeed the expected power behaviour for the conformal field theory of central charge (7.15). Therefore, if the partition function is dominated by the pole contribution (7.17), we may say that the brane corresponds to a brane in the superconformal field theory, or more precisely, has such a component.

For which values of q does the integral have the residue (7.17) as the dominant contribution? We recall that the contour is decided to to be as in Fig. 6-Left. We deform it so



$|\theta + 2\pi q| < \pi d$ $\qquad$ $\pi d < |\theta + 2\pi q| < \pi\frac{N+d}{2}$ $\qquad$ $\pi\frac{N+d}{2} < |\theta + 2\pi q| < \pi N$

Figure 11: Deformed contours

that it picks this and some other poles on the negative imaginary axis as in Fig. 11. (The meaning of the dots will be explained later.) The part along the negative imaginary axis will have (7.17). The question is what the other part of γ gives. If $|\theta + 2\pi q| < \frac{\pi}{2}(N+d)$, we can choose γ so that it goes through the region in which the integrand is exponentially small, $\mathrm{e}^{-C' r|\sigma'|}$, as $r \to \infty$ where C' is positive and with a strictly positive lower bound along the way. Therefore the integral from that part vanishes in the $r \to \infty$ limit, and is dominated by (7.17). If $|\theta + 2\pi q| > \frac{\pi}{2}(N+d)$, on the other hand, it is unavoidable that γ goes through a region where the integrand is exponentially large. Therefore, the integral on the other part is generically exponentially *growing* as $r \to \infty$.

For more detailed evaluation, let us see if the integrand has a critical point. We assume $\theta \not\equiv \pi d, \pi N$ so that the massive vacua (7.16) are not on the imaginary axis. For large values of σ', we may omit the power factor and only look at the exponent, which is $2\pi \mathrm{i} r \widetilde{W}_{\mathit{eff},q}(\sigma)$, now with the q dependence. The equation $\partial_\sigma \widetilde{W}_{\mathit{eff},q}(\sigma) = 0$ reads $|\sigma| = \widetilde{\Lambda}$ and

$$(N-d)\mathrm{Arg}(\mathrm{i}\sigma) = \theta + 2\pi q - \mathrm{sgn}(\sigma_1)\pi d. \tag{7.18}$$

(This is equivalent to the vanishing of the coefficient of $|\sigma_1'|$ in (6.17). See (6.8).) When $|\theta + 2\pi q| < \pi d$ and $|\theta + 2\pi q| > \pi N$, there is no solution. When $\pi d < |\theta + 2\pi q| < \pi N$, there is a unique solution which is equal to $\boldsymbol{\sigma}_k$ of (7.16) with

$$k = \begin{cases} q - d & (\pi d < \theta + 2\pi q < \pi N) \\ q & (-\pi N < \theta + 2\pi q < -\pi d). \end{cases} \tag{7.19}$$

For these values of q, a part of the integral can be evaluated by the saddle point approximation at the critical point and has the exponential behaviour as $r \to \infty$. The dots in Fig. 11-Middle and -Right are the critical points. Indeed the contour γ comes close to this point. If $|\theta + 2\pi q| < \pi d$, except the sum of poles on the negative imaginary axis, the integral vanishes more rapidly as $r \to \infty$ than any of these exponentials.

Let us see the behaviour of the partition function at large r for the branes $\mathfrak{B}_1$ and $\mathfrak{B}_2$

and their shifts. Looking at the contour γ, we immediately see that it vanishes for $\mathfrak{B}_2$ and all of its shifts,

$$Z_{D^2}(\mathfrak{B}_2(j,q)) = 0. \tag{7.20}$$

This is exact vanishing, for any value of r. For $\mathfrak{B}_1(j,q)$, let us consider the deformed contour as in Fig. 11. We know that the poles on the negative imaginary axis is cancelled by the brane factor $f_{\mathfrak{B}_1}(\sigma')$, and hence the contribution comes entirely from the other part. Recall that $\mathfrak{B}_1(j,q)$ has two components, $\mathbf{C}(j,q)$ and $\mathbf{C}(j+1-d\epsilon, q+d)$. By (7.19), the contribution of the former (*resp.* latter) component has the exponential behaviour $\sim\ \mathrm{e}^{-(N-k)\mathrm{i}r\boldsymbol{\sigma}_q}$ for $-\pi N < \theta + 2\pi q < -\pi d$ (*resp.* $\pi d < \theta + 2\pi(q+d) < \pi N$). Therefore, for such a q,

$$Z_{D^2}(\mathfrak{B}_1(j,q)) \ \sim\ \exp\Big(-(N-d)\mathrm{i}r\boldsymbol{\sigma}_q\Big), \qquad r\to\infty. \tag{7.21}$$

It is non-zero and therefore the brane $\mathfrak{B}_1(j,q)$ cannot be empty in the full quantum theory, even though it is so when reduced to the Landau-Ginzburg orbifold.

The above observations are enough to conclude the following, assuming $\theta \not\equiv \pi d, \pi N$ (mod $2\pi\mathbf{Z}$). It is enough to consider branes which are grade restricted with respect to a zone of length N. A natural zone is

$$\mathbf{w}_{+,\theta} = \left\{ q\in\mathbf{Z} \ \Big|\ -\frac{N}{2} \ <\ \frac{\theta}{2\pi}+q \ <\ \frac{N}{2} \right\}. \tag{7.22}$$

For $|\theta + \pi d + 2\pi q| < \pi(N-d)$, the brane $\mathfrak{B}_1(j,q)$ descends to a brane of the massive vacuum at $\boldsymbol{\sigma}_q$. On the other hand, branes which are grade restricted with respect to the zone

$$\mathbf{w}_{-,\theta} = \left\{ q\in\mathbf{Z} \ \Big|\ -\frac{d}{2} \ <\ \frac{\theta}{2\pi}+q \ <\ \frac{d}{2} \right\} \tag{7.23}$$

descend purely to the superconformal field theory. A picture of the descent is shown in Fig. 12 where the branes are plotted on the $\widetilde{W}_{\!\mathit{eff}}$-plane, for the value $\theta = -\pi d + \delta$ with a small positive δ. The square dots are the values of the massive vacua (7.16), and the origin is the value for the Landau-Ginzburg orbifold. We plot the large volume image $\mathcal{O}(q) = \mathcal{O}_{X_f}(q)$ of the brane $\mathfrak{B}_1(0,q)$ in the place of the critical value of $\boldsymbol{\sigma}_q$ The maximum and the minimum values of q are $q_{\max} := [\frac{N-d-1}{2}]$ and $q_{\min} := -[\frac{N-d}{2}]$.

At the special values $\theta \equiv \pi d, \pi N$, one or two of the critical points (7.16) are on the imaginary axis. When a critical point crosses the negative imaginary axis as we vary θ, the zone $\mathbf{w}_{-,\theta}$ changes. For example, if we move $\theta = -\pi d + \delta$ from a positive δ to a negative δ, the zone changes from $\mathbf{w}_- = \{0,1,\ldots,d-1\}$ to $\mathbf{w}'_- = \{1,\ldots,d\}$. If a brane $\mathfrak{B}'$ is grade restricted with respect to the latter it is not grade rerstricted with respect to the former, at the components of charge d. That can be cancelled by binding the branes

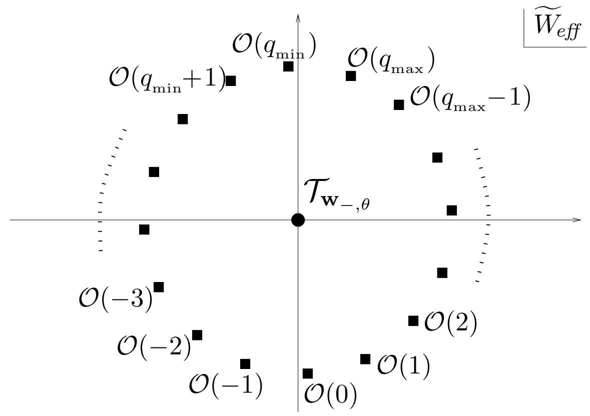


Figure 12: Low energy images of the branes

$\mathfrak{B}_1(j, 0)$ there, and we obtain a brane $\mathfrak{B}$ which is grade restricted with respect to $\mathbf{w}_-$. But $\mathfrak{B}$ and $\mathfrak{B}'$ are not the same brane. They differ by the attached branes $\mathfrak{B}_1(j, 0)$ which are not empty at the massive vacuum $\boldsymbol{\sigma}_0$. This is the "brane creation" in the sense of [13]. When the critical point crosses the positive imaginary axis, there is again the change of zones $\mathbf{w}_{+,\theta}$. The change is accompanied with a brane replacement using $\mathfrak{B}_2(j, q_*)$ for a particular q_*. Since $\mathfrak{B}_2(j, q_*)$ are genuinely empty, it is simply a change of linear sigma model representatives of the same brane.

To summarize, let us redraw the diagram of the sets and maps of the branes.

$$\begin{tikzcd}
 & \mathfrak{D} \arrow[ddl, bend right, "\pi_-"'] \arrow[ddr, bend left, "\pi_+"] \arrow[d, phantom, "\cup"] & \\
 & \mathcal{T}_{\mathbf{w}_-} \subset \mathcal{T}_{\mathbf{w}_+} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
D_- & & D_+
\end{tikzcd}$$

We emphasize that $D_\pm$ are the set of branes in the *classical* low energy theory. D_+ is the set of branes in the non-linear sigma model on the Fano-manifold X_f and D_- is the set of branes in the Landau-Ginzburg orbifold. For each θ, we have a pair of grade restricted subsets, $\mathcal{T}_{\mathbf{w}_-} \subset \mathcal{T}_{\mathbf{w}_+}$, of $\mathfrak{D}$. This gives rize to an embedding $D_- \subset D_+$, and the complement is given by the collection of $(N - d)$ branes, $\mathcal{O}(q_{\min}), \ldots, \mathcal{O}(q_{\max})$, which descend to the $(N - d)$ massive vacua. As we vary θ, one or both of the pair $\mathcal{T}_{\mathbf{w}_-} \subset \mathcal{T}_{\mathbf{w}_+}$ can jump. When $\mathcal{T}_{\mathbf{w}_-}$ jumps, the embedding $D_- \subset D_+$ will also jump.

7.4 $\underline{d > N}$: Flow from the Landau-Ginzburg orbifold

When $d < N$, the FI parameter is smaller at higher energies and the theory describes a relevant deformation of the superconformal field theory with $c/3 = N(1-2/d)$ associated to the Landau-Ginzburg orbifold $W = f(X_1, \ldots, X_N)/\mathbf{Z}_d$. At low energies, the theory reduces to the non-linear sigma model on the manifold of general type X_f or one of $(d-N)$ massive vacua. The nonilinear sigma model is free in the infra-red limit with central charge

$$\frac{c}{3} = N - 2. \tag{7.24}$$

The massive vacua are at $\boldsymbol{\sigma}_k$ in (7.16) which may be rewritten as

$$\boldsymbol{\sigma}_k = \mathrm{i}\widetilde{\Lambda}\exp\left(\mathrm{i}\frac{\theta + \pi N + 2\pi k}{N-d}\right), \qquad k \in \mathbf{Z}/(d-N)\mathbf{Z}, \tag{7.25}$$

with the value $2\pi\widetilde{W}_{\mathit{eff}} = (N-d)\boldsymbol{\sigma}_k$ for the twisted superpotential.

The analysis of the contour and the integral goes in the same way as in the $d < N$ case, and the description can be brief. Roughly speaking, we only need to exchange the rôles of d and N and flip the sign of the FI parameter and σ_2'. The contour γ can be taken as in Fig. 6-Right, coming in from and going out to th eregion where $\mathrm{Im}(\sigma')$ is negative infinity. The sum of residues at the poles on the positive imaginary axis is dominated by the one at $\sigma' = \mathrm{i}\epsilon/2$ at $r \to \infty$ which behaves as

$$(r\Lambda)^{\widehat{c}/2}\,\mathrm{e}^{\mathrm{i}t_R\mathrm{i}\epsilon/2} \;\sim\; (r\Lambda)^{(N-2)/2}. \tag{7.26}$$

This is the expected behaviour for the conformal field theory of central charge (7.24). The integral for a fixed charge q is dominated by (7.26) when $|\theta + 2\pi q| < \frac{\pi}{2}(N+d)$. Assumimg $\theta \not\equiv \pi d, \pi N$, the integrand has a unique critical point at (7.25) with

$$k = \begin{cases} q - N & (\pi N < \theta + 2\pi q < \pi d) \\ q & (-\pi d < \theta + 2\pi q < -\pi N). \end{cases} \tag{7.27}$$

For these q's, a part of the integral can be evaluated by th saddle point approximation and has the exponential behaviour as $r \to \infty$. For other values of q's, the integrand has no critical points. In particular, when $|\theta + 2\pi q| < \pi N$, the integral minus the sum of residues at the poles on the positive imaginary exis decays more rapidly than any of the above exponentials as $r \to \infty$. The branes $\mathfrak{B}_1$ and $\mathfrak{B}_2$ and their shifts have the following partition functions,

$$Z_{D^2}(\mathfrak{B}_2(j,q)) \;=\; 0, \qquad \forall r, \tag{7.28}$$

for any (j,q) and

$$Z_{D^2}(\mathfrak{B}_1(j,q)) \;\sim\; \exp\Big(-(N-d)\mathrm{i}r\boldsymbol{\sigma}_q\Big), \qquad r\to\infty. \tag{7.29}$$

for any j and for $|\theta+\pi N+2\pi q|<\pi(d-N)$.

The conclusion for $\theta\not\equiv \pi d, \pi N$ (mod $2\pi\mathbf{Z}$) is as follows: It is enough to consider branes which are grade restricted with respect to a zone of length d and a natural choice is $\mathbf{w}_{-,\theta}$ as in (7.23). For $|\theta+\pi N+2\pi q|<\pi(d-N)$, the brane $\mathfrak{B}_2(j,q)$ descends to a brane of the massive vacuum at $\boldsymbol{\sigma}_q$. On the other hand, branes which are grade restricted with respect to the zone $\mathbf{w}_{+,\theta}$ as in (7.22) descend purely to the non-linear sigma model. A picture of

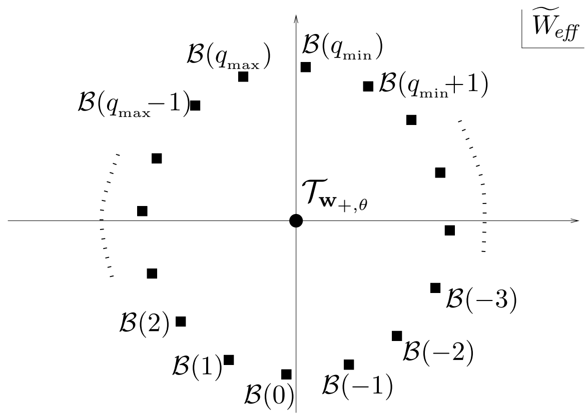


Figure 13: Low energy images of the branes

the descent is shown in Fig. 13 where the branes are plotted on the $\widetilde{W}_{\!eff}$-plane, for a small positive θ. The square dots are the values of the massive vacua (7.25), and the origin is the value for the non-linear sigma model. We plot the $\mathbf{L}=0^N$ Recknagel-Schomerus brane $\mathcal{B}(q)=\mathcal{B}_{0^N,q,0}$ which is the Landau-Ginzburg orbifold image of the brane $\mathfrak{B}_2(-N,q-N)$. The maximum and the minimum values of q are $q_{\max} := [\frac{d-N-1}{2}]$ and $q_{\min} := -[\frac{d-N}{2}]$.

When θ varies across the special values $\theta\equiv\pi d$ and πN, the critical points crosses the imaginary axis, and the zone change occurs. When a critical point crosses the positive imaginary axis, a non-linear sigma model brane creates branes at the massive vacua.

To summarize, let us draw the diagram of sets and maps of the branes.

$$\begin{tikzcd}
 & \mathfrak{D} \arrow[dddl, bend right, "\pi_-"'] \arrow[dddr, bend left, "\pi_+"] & \\
 & \cup & \\
\mathcal{T}_{\mathbf{w}_-} \arrow[d, "\cong"] & \supset & \mathcal{T}_{\mathbf{w}_+} \arrow[d, "\cong"] \\
D_- & & D_+
\end{tikzcd}$$

For each θ, we have a pair of grade restricted subsets, $\mathcal{T}_{\mathbf{w}_-} \supset \mathcal{T}_{\mathbf{w}_+}$, of $\mathfrak{D}$. This gives rize to an embedding $D_- \supset D_+$, and the complement is given by the collection of $(d-N)$ branes, $\mathcal{B}(q_{\min}), \ldots, \mathcal{B}(q_{\max})$, which descend to the $(d-N)$ massive vacua. As we vary θ, one or both of the pair $\mathcal{T}_{\mathbf{w}_-} \supset \mathcal{T}_{\mathbf{w}_+}$ can jump. When $\mathcal{T}_{\mathbf{w}_+}$ jumps, the embedding $D_- \supset D_+$ will also jump.

8 Expressions In Phases

In this section, we compute the partition function for the theory deep inside various phases. In particular, we find expressions at the Landau-Ginzburg orbifold points and in the geometric phases. The expression at a Landau-Ginzburg orbifold point agrees with the result of the purely Landau-Ginzburg orbifold found in Section 5.7 which in turn agrees with the formula for the central charge. The expression at the large volume limit mathces with the expected formula for the central charge, except that the class $\sqrt{\widehat{A}}$ should be replaced by the Gamma-class, a correction well-known among mathematicians.

8.1 Landau-Ginzburg Orbifold Phase

Let us first look at the Landau-Ginzburg orbifold phase. We start with the $U(1)$ theories introduced in Section 6.2 as a warm up, and then consider more general theories.

8.1.1 The $U(1)$ Theories

The Landau-Ginzburg orbifold appears in the regime $\zeta \ll 0$ if $d = N$, as a part of the theory in the long distance regime $r \gg \Lambda$ if $d < N$, and as the theory in the short distance regime $r \ll \Lambda$ if $d > N$. In either case, we look at the parameter region with $\zeta_R \ll 0$.

Before looking at the partition function, we describe the descent rule of branes [11]. The Landau-Ginzburg orbifold is obtained by freezing the field p at some value, say 1, which breaks the gauge group $G = U(1)$ to $G_L = \mathbf{Z}_d$. Therefore, it is natural to go to the $\epsilon = \frac{2}{d}$ frame where the R-charge of p vanishes. In this frame, the element $J \in G$ defined in (3.57) becomes

$$J = \mathrm{e}^{\pi \mathrm{i}\epsilon} \xrightarrow{\epsilon\to\frac{2}{d}} \mathrm{e}^{\frac{2\pi\mathrm{i}}{d}} =: J_L \tag{8.1}$$

The brane $\mathfrak{B} = (M, Q, \rho, \mathbf{r}_*)$ descends to the brane $\mathfrak{B}_{\mathrm{LG}} = (M_L, Q_L, \rho_L, \mathbf{r}_{*,L})$ where

$$M_L = M,$$

$$
\begin{aligned}
Q_L(x) &= Q(1,x), \\
\rho_L(\omega) &= \rho(\omega), \quad \omega^d = 1, \\
\mathbf{r}_{*,L} &= \mathbf{r}_*\Big|_{\epsilon=\frac{2}{d}}.
\end{aligned} \tag{8.2}
$$

We recall that (3.58) is satisfied, so that $\mathrm{e}^{\pi \mathrm{i}\mathbf{r}_*}\rho(J) = \mathrm{e}^{\pi\mathrm{i}\mathbf{r}_{*,L}}\rho_L(J_L)$ is the $\mathbf{Z}_2$ grading, or equivalently, $\mathrm{e}^{\pi\mathrm{i}r_j}\,\mathrm{e}^{q_j\pi\mathrm{i}\epsilon} = \mathrm{e}^{\pi\mathrm{i}r_{j,L}}\,\mathrm{e}^{q_j 2\pi\mathrm{i}/d} = (-1)^{r_j^o}$.

The formula for the partition function in the $\epsilon = \frac{2}{d}$ frame is

$$
Z_{D^2}(\mathfrak{B}) = (r\Lambda)^{\frac{\widehat{c}_{\rm LG}}{2}} \int_\gamma \mathrm{d}\sigma'\, \Gamma(-d\,\mathrm{i}\sigma')\, \Gamma\left(\mathrm{i}\sigma' + \tfrac{1}{d}\right)^N \mathrm{e}^{\mathrm{i}t_R\sigma'} f_{\mathfrak{B}}(\sigma'), \tag{8.3}
$$

with $\widehat{c}_{\rm LG} = N(1-\frac{2}{d})$. Note that the contour γ should be poked at $\sigma' = 0$ so that it goes above 0. From a glance at the contours (Figs. 2, 6, 11), and from the discussion in the previous section, we see that we only need to take the poles on the negative imaginary axis, $\sigma' = -\mathrm{i}n/d$ for $n = 0, 1, 2, \ldots$. Using (6.2), we obtain

$$
Z^{\rm LG}_{D^2}(\mathfrak{B}) = \frac{2\pi}{d}(r\Lambda)^{\frac{\widehat{c}_{\rm LG}}{2}} \sum_{n=0}^\infty \frac{(-1)^n}{n!}\Gamma\left(\tfrac{n+1}{d}\right)^N \mathrm{e}^{t_R n/d} f_{\mathfrak{B}}\left(-\mathrm{i}\tfrac{n}{d}\right). \tag{8.4}
$$

The brane factor can be written as

$$
\begin{aligned}
f_{\mathfrak{B}}\left(-\mathrm{i}\tfrac{n}{d}\right) &= \sum_j \mathrm{e}^{\pi\mathrm{i}r_j}\big|_{\epsilon=\frac{2}{d}}\, \mathrm{e}^{2\pi q_j(-\mathrm{i}\frac{n}{d})} \\
&= \sum_j (-1)^{r_j^o}\,\mathrm{e}^{-2\pi\mathrm{i}q_j\left(\frac{1+n}{d}\right)} = \mathrm{Str}_M\rho(J_L^{-1-n}).
\end{aligned} \tag{8.5}
$$

Thus, we obtain

$$
Z^{\rm LG}_{D^2}(\mathfrak{B}) = \frac{2\pi}{d}(r\Lambda)^{\frac{\widehat{c}_{\rm LG}}{2}} \sum_{n=0}^\infty \frac{(-1)^n}{n!}\Gamma\left(\tfrac{n+1}{d}\right)^N \mathrm{e}^{t_R n/d}\, \mathrm{Str}_M\rho(J_L^{-1-n}). \tag{8.6}
$$

This is the $\mathrm{e}^{t_R/d}$ expansion of the full partition function for $d \geq N$ and of a part of it for $d < N$. In the limit $\zeta_R \to -\infty$, that is, $\zeta \to -\infty$, the infra-red and the ultra-violet limits respectively for $d = N$, $d < N$ and $d > N$, only the leading term remains,

$$
Z^{\rm LG}_{D^2}(\mathfrak{B}) \;\longrightarrow\; \frac{2\pi}{d}\Gamma\left(\tfrac{1}{d}\right)^N (r\Lambda)^{\frac{\widehat{c}_{\rm LG}}{2}}\, \mathrm{Str}_M\rho(J_L^{-1}). \tag{8.7}
$$

Up to the numerical factor, this agrees with the formula (5.82) for the brane (8.2) in the Landau-Ginzburg orbifold, which in turn is the same as the formula of [6] for the central charge of the same brane.

To get back a general ϵ, say for comparison with the other phase, we use (5.78) and find that the result must be multiplied by $\exp\left(-t\left(\frac{\epsilon}{2} - \frac{1}{d}\right)\right)$.

8.1.2 More General Theories

Let us consider a theory with an Abelian and connected gauge group, $G = T$, with charge integrality. We assume the situation as discussed in [11],[1] where the fields are grouped into two, $Y_1, \ldots, Y_k$ and $X_1, \ldots, X_l$, such that $Q_{Y_1}, \ldots, Q_{Y_k}$ span $i\mathfrak{t}^*$ and that Q_{X_j} are non-positive spans of Q_{Y_i}'s,

$$Q_{X_j} = -\sum_{i=1}^{k} a_j^{\,i} Q_{Y_i}; \qquad a_j^{\,i} \geq 0 \quad \forall (i,j). \tag{8.8}$$

If ζ is a positive linear span of Q_{Y_i}'s, the D-term equation $\sum_i Q_{Y_i}|y_i|^2 = \sum_{i,j} a_j^{\,i} Q_{Y_i}|x_j|^2 + \zeta$ has a solution with y_i's all non-zero, for any value of x_j's. Therefore the gauge group is broken to a finite subgroup G_L, consisting of elements that fix all y_i's, and the classical low energy theory is the G_L-orbifold of the Landau-Ginzburg model with the superpotential

$$W_L(X_1, \ldots, X_l) = W(1, \ldots, 1, X_1, \ldots, X_l), \tag{8.9}$$

where $W(Y,X)$ is the original superpotential. By the charge integrality, we have the R^o frame in which all the R-charges of the bulk fields, $R^o_{Y_i}, R^o_{X_j}$, are 0 or 2. Since Q_{Y_i}'s span $i\mathfrak{t}^*$, there is a unique gauge shift $\mathit{\Delta}$ that annihilates the R-charges of Y_i's,

$$R_{Y_i,L} = R^o_{Y_i} + Q_{Y_i}(\mathit{\Delta}) = 0. \tag{8.10}$$

The new R-charges for X_j's

$$R_{X_j,L} = R^o_{X_j} + Q_{X_j}(\mathit{\Delta}) = R^o_{X_j} + \sum_{i=1}^{k} a_j^{\,i} R^o_{Y_i}, \tag{8.11}$$

are the R-charges of the low energy Landau-Ginzburg orbifold. The element $\mathrm{e}^{\pi \mathrm{i} \mathit{\Delta}} \in G$ acts trivially on Y_i's and acts on X_j by the phase $\mathrm{e}^{\pi \mathrm{i} R_{X_j,L}}$. That is, it is the element $J_L \in G_L$ of the Landau-Ginzburg orbfiold,

$$J_L = \mathrm{e}^{\pi \mathrm{i} \mathit{\Delta}}. \tag{8.12}$$

The brane descent is as in (8.2): $\mathfrak{B} = (M, Q, \rho, \mathbf{r}_*) \mapsto \mathfrak{B}_{\mathrm{LG}} = (M_L, Q_L, \rho_L, \mathbf{r}_{*,L})$, where

$$\begin{aligned} M_L &= M, \\ Q_L(x_1, \ldots, x_l) &= Q(1, \ldots, 1, x_1, \ldots, x_l), \\ \rho_L &= \rho|_{G_L}, \\ \mathbf{r}_{*,L} &= \mathbf{r}^o_* - \rho(\mathit{\Delta}). \end{aligned} \tag{8.13}$$

[1] Although this include a wide class of examples, this is not the most general situation. Some examples in [15] are not of this type.

The partition function in the R_L-frame is

$$Z_{D^2}(\mathfrak{B}) = (r\Lambda)^{\widehat{c}_{\mathrm{LG}}/2}\int_\gamma \mathrm{d}^k\sigma \prod_{i=1}^k \Gamma\left(\mathrm{i}Q_{Y_i}(\sigma)\right)\prod_{j=1}^l \Gamma\left(-\mathrm{i}\sum_i a_j^{\,i} Q_{Y_i}(\sigma)+\tfrac{R_{X_j,L}}{2}\right) \times \mathrm{e}^{\mathrm{i}t(\sigma)}\,\mathrm{tr}_M\left(\mathrm{e}^{\pi\mathrm{i}(\mathbf{r}_*^\circ-\varDelta)}\,\mathrm{e}^{2\pi\rho(\sigma)}\right), \tag{8.14}$$

where the contour γ should be poked near $(Q_{Y_i}(\sigma)=0)$'s to avoid the poles that came down in the R_L-frame limit. If the theory satisfies the Calabi-Yau condition, the contour γ in this phase can be taken as in (6.28). For example, we can take (the poked version of)

$$Q_{Y_i}(\upsilon) = (Q_{Y_i}(\tau))^2, \qquad i=1,\ldots,k. \tag{8.15}$$

The growth condition is satisfied since ζ is a positive span of Q_{Y_i}'s. The wedge condition to avoid poles, which is trivially satisfied for Y_i's, is also satisifed for X_j's,

$$Q_{X_j}(\upsilon) = -\sum_{i=1}^k a_j^{\,i} Q_{Y_i}(\upsilon) = -\sum_{i=1}^k a_j^{\,i}(Q_{Y_i}(\tau))^2 \le 0, \tag{8.16}$$

where (8.8) is used. If the theory is not Calabi-Yau, as in the $U(1)$ theory, the above contour may still be admissible, or appears as a part of the admissible contour in the regime where ζ_R is deep inside the positive span of Q_{Y_i}'s. In either case, we decide to take (the poked version of) (8.15) as the contour. In the Calabi-Yau case and some other cases, it is the full partition function but in some other cases it is only a part of it.

Taking the poles at $Q_{Y_i}(\sigma)=\mathrm{i}n_i$, $n_i=0,1,2,\ldots$, for $i=1,\ldots,k$, we obtain

$$Z^{\mathrm{LG}}_{D^2}(\mathfrak{B}) = \frac{(2\pi)^k}{\det Q_Y}(r\Lambda)^{\widehat{c}_{\mathrm{LG}}/2}\sum_n \frac{(-1)^{n_1+\cdots+n_k}}{n_1!\cdots n_k!}\prod_{j=1}^l \Gamma\left(a_j(n)+\tfrac{R_{X_j,L}}{2}\right) \times \mathrm{e}^{-t(Q_Y^{-1}(n))}\,\mathrm{Str}_M\,\rho\Big(J_L^{-1}\,\mathrm{e}^{2\pi\mathrm{i}Q_Y^{-1}(n)}\Big) \tag{8.17}$$

In the limit $(tQ_Y^{-1})^i\to\infty$, only the $n=0$ term remains,

$$Z^{\mathrm{LG}}_{D^2}(\mathfrak{B}) \longrightarrow \frac{(2\pi)^k}{\det Q_Y}\prod_{j=1}^l \Gamma\left(\tfrac{R_{X_j,L}}{2}\right)(r\Lambda)^{\widehat{c}_{\mathrm{LG}}/2}\ \mathrm{Str}_M\rho(J_L^{-1}). \tag{8.18}$$

Up to the numerical factor, this agrees with the formula (5.82) for the brane (8.13) in the Landau-Ginzburg orbifold.

8.2 Geometric Phase

We next consider the geometric phase.

8.2.1 The Gamma Classes

Before starting, we describe some characteristic classes which will enter into the formulae. Let us introduce some functions of one variable x with Taylor series at $x = 0$ starting with 1:

$$\widehat{\mathrm{A}}(x) = \frac{x/2}{\sinh(x/2)}, \tag{8.19}$$

$$\mathrm{td}(x) = \frac{x}{1 - \mathrm{e}^{-x}}, \tag{8.20}$$

$$\widehat{\Gamma}(x) = \Gamma\left(1 - \frac{x}{2\pi \mathrm{i}}\right), \quad \widehat{\Gamma}^*(x) = \Gamma\left(1 + \frac{x}{2\pi \mathrm{i}}\right). \tag{8.21}$$

They define characteristic classes $\widehat{\mathrm{A}}_X$, td_X, $\widehat{\Gamma}_X$ and $\widehat{\Gamma}^*_X$ of the tangent bundle of a complex manifold X via the total Chern class $c(X)$ [51], in such a way as

$$c(X) = \frac{\prod_i (1 + x_i)}{\prod_j (1 + y_j)} \implies \mathrm{td}_X = \frac{\prod_i \mathrm{td}(x_i)}{\prod_j \mathrm{td}(y_j)}. \tag{8.22}$$

These are called the *A-roof class*, *Todd class*, and *Gamma classes*. To be more precise, the A-roof class can be defined for any real manifold and can be expressed in terms of the Pontrjagin classes. Here we are considering the specialization to complex manifolds, assuming the usual relation between the Pontrjagin classes of the real tangent bundle and the Chern classes of the complex tangent bundle. Explicit expressions in terms of the Chern classes are well known for $\widehat{A}$ and td. We write down first few terms for the Gamma class:

$$\begin{aligned} \widehat{\Gamma} \;=\;& 1 - \mathrm{i}\frac{\gamma}{2\pi}c_1 + \frac{1}{24}c_2 + \left(-\frac{1}{48} - \frac{1}{2}\left(\frac{\gamma}{2\pi}\right)^2\right) c_1^2 + \mathrm{i}\frac{\zeta(3)}{(2\pi)^3}c_3 \\ & -\mathrm{i}\left(\frac{\gamma}{24 \cdot 2\pi} + \frac{\zeta(3)}{(2\pi)^3}\right) c_2 c_1 + \mathrm{i}\left(\frac{\gamma}{48 \cdot 2\pi} + \frac{1}{6}\left(\frac{\gamma}{2\pi}\right)^3 + \frac{\zeta(3)}{3(2\pi)^3}\right) c_1^3 + \cdots, \end{aligned} \tag{8.23}$$

where γ is Euler's consant. There are some relations among the above functions, $\widehat{\mathrm{A}}(x) = \mathrm{e}^{-x/2}\mathrm{td}(x) = \widehat{\Gamma}(x)\widehat{\Gamma}^*(x)$, which are copied to the relations among the associated classes,

$$\widehat{\mathrm{A}}_X \;=\; \mathrm{e}^{-c_1(X)/2}\mathrm{td}_X \;=\; \widehat{\Gamma}_X \widehat{\Gamma}^*_X. \tag{8.24}$$

Let us also recall that the A-roof and Todd classes appears in some index formula. If X is an even dimensional smooth manifold with a spin structure, and E is a smooth vector bundle on X, we can consider the Dirac operator acting on the spinors with values in E. Then, the index of the Diract operator is given by the Atiyah-Singer formula:

$$\mathrm{ind}\, \not{D}_E \;=\; \int_X \widehat{\mathrm{A}}_X \,\mathrm{ch}(E). \tag{8.25}$$

If X is a complex manifold and $\mathcal{E}$ is a holomorphic vector bundle on X, we can consider the Dolbeault operator acting on anti-holomorphic differential forms with values in $\mathcal{E}$. Then, the Euler characteristic of the Dolbeault complex is given by the Riemann-Roch formula

$$\chi(\mathcal{E},\overline{\partial}) \;=\; \int_X \mathrm{td}_X\,\mathrm{ch}(\mathcal{E}). \tag{8.26}$$

8.2.2 The $U(1)$ Theories

As a warm up, we start with the $U(1)$ theories. The geometric phase is in the regime $\zeta \gg 0$ if $d = N$, in the short distance regime $r \ll \Lambda$ if $d < N$, and in a part of the long distance regime $r \gg \Lambda$ if $d > N$, In either case, we look at the parameter region with $\zeta_R \gg 0$.

Before starting, let us describe how branes in the linear sigma model descend to branes in the non-linear sigma model [11]. What we have after imposing the Higgs mechanism (step (i) in the language of Section 7.1) is the non-linear sigma model on the total space of the line bundle $\mathcal{O}(-d)$ over $\mathbb{CP}^{N-1}$, with the superpotential $W = pf(x)$. This W is a Bott-Morse function and the critical set is the locus $p = f(x) = 0$, that is, the hypersurface X_f. We obtain the non-linear sigma model on X_f by integrating out the massive modes, p and $f(x)$. (This is the step (ii).) The brane descent for integrating out a pair of massive variables is known as Knörrer periodicity [52] and we only have to apply it in the current situation. How to do it is described in [11] and we simply record the procedure.

Branes in the non-linear sigma model with the target X_f are represented by complexes of holomorphic vector bundles on X_f, possibly of infinite lengths but with truncation to finite lengths complexes of coherent sheaves. A brane is therefore specified by a pair $(\mathcal{E}, d)$: $\mathcal{E}$ is a $\mathbf{Z}$-graded vector bundle on X_f which is of finite rank in each degree. d is a local endomorphism of $\mathcal{E}$ (holomorphic bundle map of $\mathcal{E}$) of degree 1 such that $d^2 = 0$ and that $\{d, d^\dagger\}$ has a finite rank kernel for some choice of fibre metric on $\mathcal{E}$.

We shall simply write M for $(M, \rho, \mathbf{r}_*)$ so that the information of gauge group and R-symmetry group action on the Cahn-Paton vector space is included into the notation M. The brane $\mathfrak{B} = (M, Q)$ descends to the brane $\mathfrak{B}_{\mathrm{LV}} = (M_L, Q_L)$ where

$$M_L \;=\; \bigoplus_{i=0}^{\infty} M(2i, di), \tag{8.27}$$

$$Q_L \;=\; Q(p_L, x), \tag{8.28}$$

where p_L is the shift of charges by $(2, d)$. Here we regard $\mathbf{C}(j, q)$ as the line bundle $\mathcal{O}_{X_f}(q)$

at degree j. The Chern character of this brane is given by

$$\begin{aligned} \mathrm{ch}(\mathfrak{B}_{\mathrm{LV}}) &= \sum_{i=0}^{\infty}\sum_{j}(-1)^{r_j^o+2i}\,\mathrm{e}^{(q_j+di)H} \\ &= \frac{1}{1-\exp(dH)}\,f_{\mathfrak{B}}\left(\tfrac{1}{2\pi}H\right). \end{aligned} \tag{8.29}$$

It is also noticed that the Knörrer procedure involves the shift of the Chan-Paton charge, which can be absorbed into the shift of the theta angle or a B-field:

$$2\pi B = (\theta + \pi d)H \tag{8.30}$$

In this paper, we normalize the B-field as $[B] \in H^2(X_f, \mathbf{Z})$ on the closed string sector so that the instanton factor for the degree β maps is

$$\exp\left(-\int_{\beta}(\omega - 2\pi\mathrm{i}B)\right), \tag{8.31}$$

where ω is the Kähler form.

Now we look at the partition function. The formula in the R^o frame is

$$Z_{D^2}(\mathfrak{B}) = (r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}}\int_{\gamma}\mathrm{d}\sigma'\,\Gamma(-d\,\mathrm{i}\sigma'+1)\,\Gamma\left(\mathrm{i}\sigma'\right)^N\,\mathrm{e}^{\mathrm{i}t_R\sigma'}f_{\mathfrak{B}}(\sigma'), \tag{8.32}$$

with $\widehat{c}_{\mathrm{LV}} = N-2$. Note that the contour γ should be poked at $\sigma' = 0$ so that it goes below 0. Looking at the contours (Figs. 2, 6, 11), and from the discussion in the previous section, we see that we only need to take the poles on the positive imaginary axis, $\sigma' = \mathrm{i}n$ for $n = 0, 1, 2, \dots$. At each n, we shift the integration variable as $\sigma' = \mathrm{i}n + \frac{z}{2\pi}$. This yields

$$Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = (r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}}\sum_{n=1}^{\infty}\oint_0\frac{\mathrm{d}z}{2\pi}\,\Gamma\left(dn+\tfrac{dz}{2\pi\mathrm{i}}+1\right)\Gamma\left(-n-\tfrac{z}{2\pi\mathrm{i}}\right)^N\mathrm{e}^{-t_Rn+\frac{\mathrm{i}}{2\pi}t_Rz}f_{\mathfrak{B}}\left(\mathrm{i}n+\tfrac{z}{2\pi}\right). \tag{8.33}$$

Note that $f_{\mathfrak{B}}(\mathrm{i}n+\frac{z}{2\pi}) = f_{\mathfrak{B}}(\frac{z}{2\pi})$ since $\mathrm{e}^{2\pi q_j(\mathrm{i}n)} = 1$. We also use the relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{8.34}$$

to rewrite a part of the gamma function factors. This yields

$$Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = -C(r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}}\sum_{n=0}^{\infty}\oint_0\frac{\mathrm{d}z}{2\pi\mathrm{i}}\left(\frac{(-1)^n}{2\sinh\left(\frac{z}{2}\right)}\right)^N\frac{\Gamma\left(1+\frac{dz}{2\pi\mathrm{i}}+dn\right)}{\Gamma\left(1+\frac{z}{2\pi\mathrm{i}}+n\right)^N}\,\mathrm{e}^{-nt_R+\frac{\mathrm{i}}{2\pi}t_Rz}f_{\mathfrak{B}}\left(\tfrac{z}{2\pi}\right), \tag{8.35}$$

with $C = -\mathrm{i}(-2\pi\mathrm{i})^N$. We further rewrite it as follows,

$$
\begin{aligned}
Z_{D^2}^{\mathrm{LV}}(\mathfrak{B}) \;=\; C(r\Lambda)^{\frac{\hat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} \oint_0 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z^N} \cdot dz \cdot \frac{z^{N-1}(1-\mathrm{e}^{-dz})}{d(1-\mathrm{e}^{-z})^N} \frac{\Gamma\left(1+\frac{dz}{2\pi\mathrm{i}}+dn\right)}{\Gamma\left(1+\frac{z}{2\pi\mathrm{i}}+n\right)^N} \\
\times \exp\left(-nt'_R + \frac{\mathrm{i}}{2\pi}t'_R z\right) \frac{f_{\mathfrak{B}}\left(\frac{z}{2\pi}\right)}{1-\mathrm{e}^{dz}},
\end{aligned}
\tag{8.36}
$$

where

$$
t'_R = t_R - d\pi\mathrm{i} + (N-d)\pi\mathrm{i}. \tag{8.37}
$$

This is in order to express each term as an integral over X_f, using

$$
\int_{X_f} g(H) = \int_{\mathbb{CP}^{N-1}} d\, H g(H) = \oint_0 \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{1}{z^N} \cdot dz \cdot g(z), \tag{8.38}
$$

which holds for a power series $g(z)$ in z where H is the hyperplane class on $\mathbb{CP}^{N-1}$ or its restriction on X_f.

At this point, let us write down the expressions of some characteristic classes of X_f. By the exact sequences

$$
\begin{aligned}
&0 \to \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus N} \longrightarrow T_{\mathbb{CP}^{N-1}} \to 0, \\
&0 \to T_{X_f} \longrightarrow T_{\mathbb{CP}^{N-1}}|_{X_f} \longrightarrow N_{X_f/\mathbb{CP}^{N-1}} \to 0,
\end{aligned}
$$

we have

$$
c(X_f) = \frac{(1+H)^N}{(1+dH)}, \tag{8.39}
$$

which implies $c_1(X_f) = (N-d)H$ and

$$
\mathrm{td}_{X_f} = \frac{H^{N-1}(1-\mathrm{e}^{-dH})}{d(1-\mathrm{e}^{-H})^N}. \tag{8.40}
$$

Let us also introduce a cohomology class

$$
\begin{aligned}
\widehat{\Gamma}_{X_f}(n) \;&:=\; \widehat{\mathrm{A}}_{X_f} \cdot \frac{\Gamma\left(1+d\left(\frac{H}{2\pi\mathrm{i}}+n\right)\right)}{\Gamma\left(1+\frac{H}{2\pi\mathrm{i}}+n\right)^N} \\
&=\; \mathrm{e}^{-\frac{N-d}{2}H} \frac{H^{N-1}(1-\mathrm{e}^{-dH})}{d(1-\mathrm{e}^{-H})^N} \cdot \frac{\Gamma\left(1+d\left(\frac{H}{2\pi\mathrm{i}}+n\right)\right)}{\Gamma\left(1+\frac{H}{2\pi\mathrm{i}}+n\right)^N}.
\end{aligned}
\tag{8.41}
$$

At $n=0$, it reduces to the Gamma class,

$$
\widehat{\Gamma}_{X_f}(0) \;=\; \widehat{\mathrm{A}}_{X_f} \cdot \frac{\Gamma(1+\frac{d}{2\pi\mathrm{i}}H)}{\Gamma(1+\frac{1}{2\pi\mathrm{i}}H)^N} \;=\; \widehat{\mathrm{A}}_{X_f} \cdot \frac{1}{\widehat{\Gamma}^*_{X_f}} \;=\; \widehat{\Gamma}_{X_f}, \tag{8.42}
$$

where we used (8.24).

Using (8.38) and looking at (8.29) and (8.41), we can write (8.36) as

$$Z_{D^2}^{\mathrm{LV}}(\mathfrak{B}) = C(r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} \int_{X_f} \mathrm{e}^{\frac{N-d}{2}H} \widehat{\Gamma}_{X_f}(n) \exp\left(-nt'_R + \frac{\mathrm{i}}{2\pi} t'_R H\right) \mathrm{ch}(\mathfrak{B}_{\mathrm{LV}}) \tag{8.43}$$

Let us denote the renormalized Kähler form as $\omega_R = \zeta_R H$. In view of (8.30), we have $t'_R H = \omega_R - 2\pi\mathrm{i}B + \pi\mathrm{i}(N-d)H$. Then, we may also write the result as

$$Z_{D^2}^{\mathrm{LV}}(\mathfrak{B}) = C(r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} \mathrm{e}^{-nt'_R} \int_{X_f} \widehat{\Gamma}_{X_f}(n) \exp\left(B + \frac{\mathrm{i}}{2\pi}\omega_R\right) \mathrm{ch}(\mathfrak{B}_{\mathrm{LV}}) \tag{8.44}$$

This is the e^{-t_R} expansion of the full partition function for $d \leq N$ and of a part of it for $d > N$. In the limit $\zeta_R \to +\infty$, that is, $\zeta \to +\infty$, the ultra-violet and the infra-red limits respectively for $d = N$, $d < N$ and $d > N$, only the leading term remains,

$$Z_{D^2}^{\mathrm{LV}}(\mathfrak{B}) \;\longrightarrow\; C(r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}} \int_{X_f} \widehat{\Gamma}_{X_f} \exp\left(B + \frac{\mathrm{i}}{2\pi}\omega_R\right) \mathrm{ch}(\mathfrak{B}_{\mathrm{LV}}). \tag{8.45}$$

This is agrees with the expected formula for the central charge of the brane $\mathfrak{B}_{\mathrm{LV}}$, except that we have the Gama class in the place of $\sqrt{\widehat{A}_{X_f}}$. That the Gamma class rather than $\sqrt{\widehat{A}_{X_f}}$ should enter into the asymptotic formula for the central charge had been well-known to mathematicians. In fact, formula of the type (8.43), (8.44) were first presented by Hosono [7] for the quintic, $N = d = 5$, and that was a part of the motivation to define the Gamma class [8–10].

Gravitational Descendants and Loop Operators

In one of such development [8, 10], Iritani studied the D-brane central charge from the view point of Gromov-Witten theory, i.e., topological A-model, on Fano manifolds. If we compare our results with his formula, it looks like that the central charge can be expressed in terms of the genus zero topological string three point amplitudes as

$$Z_{D^2_{(-)}}(\mathfrak{B}) \;=\; \sum_{n=0}^{\infty} (-1)^n (r\Lambda)^n \langle \tau_n(F(\mathfrak{B})) PP \rangle_0, \tag{8.46}$$

where $\langle \cdots \rangle_0$ stands for the genus zero topological string amplitude with sum over all worldsheet instantons, $F(\mathfrak{B})$ is a certain cohomology class of X_f constrcuted out of $\mathrm{ch}(\mathfrak{B})$, $\widehat{\Gamma}_{X_f}$ and $(r\Lambda)^{c_1(X_f)}$. $\tau_n F(\mathfrak{B})$ is the n-th gravitational descendant of $F(\mathfrak{B})$ and P is the puncture operator. The series of the form $w(\ell) = \sum_n (-1)^n \ell^n \tau_n$ is known as the "loop operator" in the study of 2d quantum gravity which creates a hole on the worldsheet. It is interesting to observe that our formula came from the hemisphere, i.e. a genus zero Riemann surface with a big hole, and that the right hand side of (8.46) is also associated to a sphere amplitude with one hole. It would be interesting to understand the meaning of this observation.

8.2.3 More General Theories

Let us move on to a more general linear sigma model with a gauge group G and the matter fields grouped into two, an E-valued field X and an F^*-valued field P, for some representations E and F of G of dimensions d_E and d_F. We assume the superpotential of the form

$$W = \langle P, f(X)\rangle \tag{8.47}$$

where $f : E \to F$ is a G-equivariant polynomial map, and $\langle -, -\rangle$ is the pairing between F^* and F. We assign the R-charge 0 to X and 2 to P in the R^o-frame. We assume that there is a phase in which the D-term equation requires X to have non-zero values which break the gauge group G completely. We also assume that f is generic enough so that the D- and F-term equations force $P = 0$ and that the vacuum manifold is a smooth submanifold X_f, defined by $f = 0$, of a smooth compact symplectic quotient $\mathbb{P}$ of E by G. We may also regard $\mathbb{P}$ as the geometric invariant theory quotient

$$\mathbb{P} = E /\!/ G_{\mathbf{C}} \tag{8.48}$$

with respect to the stability condition defined by the FI parameter in the phase. We write the weights of E and F with respect to a maximal torus T by Q_i's and d_β's,

$$E|_T = \bigoplus_i \mathbf{C}(Q_i), \qquad F|_T = \bigoplus_\beta \mathbf{C}(d_\beta). \tag{8.49}$$

Let us write down some characteristic classes of X_f. By the exact sequences,

$$\begin{aligned} &0 \to \mathcal{O}(\mathfrak{g}_{\mathbf{C}}) \longrightarrow \mathcal{O}(E) \longrightarrow T_{\mathbb{P}} \to 0, \\ &0 \to T_{X_f} \longrightarrow T_{\mathbb{P}}|_{X_f} \longrightarrow N_{X_f/\mathbb{P}} \to 0, \end{aligned} \tag{8.50}$$

we have

$$c(X_f) = \frac{c(\mathbb{P})}{c(N_{X_f/\mathbb{P}})} = \frac{\prod_i (1 + Q_i(H))}{\prod_{\alpha>0}(1-\alpha(H)^2)\prod_\beta(1+d_\beta(H))}, \tag{8.51}$$

which implies $c_1(X_f) = \sum_i Q_i(H) - \sum_\beta d_\beta(H)$ and

$$\mathrm{td}(X_f) = \frac{\prod_i Q_i(H)\prod_{\alpha>0}\left(2\sinh(\frac{\alpha(H)}{2})\right)^2\prod_\beta(1-\,\mathrm{e}^{-d_\beta(H)})}{\prod_{\alpha>0}\alpha(H)^2\prod_\beta d_\beta(H)\prod_i(1-\,\mathrm{e}^{-Q_i(H)})} \tag{8.52}$$

For a coroot $n \in \mathrm{Q}^\vee \subset \mathrm{i}\mathfrak{t}$, we put

$$\widehat{\Gamma}_{X_f}(n) := \widehat{\mathrm{A}}_{X_f} \cdot \frac{\prod_\beta \Gamma\left(1 + d_\beta\left(\frac{H}{2\pi \mathrm{i}} + n\right)\right)}{\prod_i \Gamma\left(1 + Q_i\left(\frac{H}{2\pi \mathrm{i}} + n\right)\right)} \prod_{\alpha > 0} \Gamma\left(1 + \alpha\left(\frac{H}{2\pi \mathrm{i}} + n\right)\right) \Gamma\left(1 - \alpha\left(\frac{H}{2\pi \mathrm{i}} + n\right)\right). \tag{8.53}$$

It reduces to the Gamma class $\widehat{\Gamma}_{X_f}$ at $n = 0$.

The rule of brane descent is just as in the $U(1)$ case. We shall denote the R-charge shift by j by $M \mapsto M[j]$ — if the gauge group were Abelian we could use the notation $M \mapsto M(j, 0)$, but that would not be appropriate for non-Abelian gauge group. The brane $\mathfrak{B} = (M, Q)$ descends to the brane $\mathfrak{B}_{\mathrm{LV}} = (M_L, Q_L)$ in the non-linear sigma model on X_f where

$$M_L = M \otimes \mathrm{Sym}\, F[2], \tag{8.54}$$

$$Q_L = Q(p_L, x) \tag{8.55}$$

where p_L is the co-evaluation combined with the degree 2 shift: The component $p_L(v)$ for $v \in F$ is the multiplication by v and the shift of the R-charge by 2. The Chern character of the image brane is

$$\mathrm{ch}(\mathfrak{B}_{\mathrm{LV}}) = \frac{1}{\prod_\beta \left(1 - \exp\left(d_\beta(H)\right)\right)} f_{\mathfrak{B}}\left(\tfrac{1}{2\pi} H\right) \tag{8.56}$$

The theta angle shift is

$$2\pi B = \left(\theta + \pi \sum_\beta d_\beta\right)(H). \tag{8.57}$$

Now let us compute the partition function, which is given in the R^o-frame by

$$\begin{aligned} Z_{D^2}(\mathfrak{B}) \;=\; & (r\Lambda)^{\widehat{c}_{\mathrm{LV}}/2} \int_\gamma \mathrm{d}^{l_G}\sigma' \prod_{\alpha > 0} \alpha(\sigma') \sinh(\pi\alpha(\sigma')) \\ & \times \prod_\beta \Gamma(-\mathrm{i} d_\beta(\sigma') + 1) \prod_i \Gamma(\mathrm{i} Q_i(\sigma'))\, \mathrm{e}^{\mathrm{i} t_R(\sigma')} f_{\mathfrak{B}}(\sigma'). \end{aligned} \tag{8.58}$$

with $\widehat{c} = d_E - d_F - d_G$. The contour γ should be poked near $(Q_i(\sigma') = 0)$'s to avoid poles that came down in the R^o limit. Alternatively, we can uniformly shift γ by $-\mathrm{i}\boldsymbol{\epsilon}$ for some small $\boldsymbol{\epsilon} \in \mathrm{i}\mathfrak{t}$. As in the $U(1)$ case, we would like to deform, or close, the contour γ so that we have a sum over residues. We may try to do it for one coordinate after another, but that is not practical for high rank cases. Fortunately, a machinery is developed for the situation like this. It is called the *multivariable Jordan lemma* [53, 54].

Let $C \subset \mathrm{i}\mathfrak{t}$ be a cone with l_G faces with $-\boldsymbol{\epsilon}$ as its vertex, and suppose $\mathcal{C} = \{\mathrm{Im}(\sigma') \in C\}$ is deep inside the admissible region, i.e., the integrand decays exponentially fast at infinity of $\mathcal{C}$. We name the faces of $\mathcal{C}$ by $\{\mathcal{C}_a\}_{a=1}^{l_G}$. We assume that the charges $\{Q_i\}$ are decomposed into l_G groups, $\{Q_i\}_{i\in I_a}$, for $a=1,\dots,l_G$, so that the following condition is satisfied. Let us define holomorphic functions $f_1,\dots,f_{l_G}$ of σ' by $\frac{1}{f_a(\sigma')}:=\prod_{i\in I_a}\Gamma(\mathrm{i}Q_i(\sigma'))$. Then the condition is that the divisor $D_a:=(f_a=0)$ do not meet the face $\mathcal{C}_a$, for each a, and that the intersection of $D_1,\dots,D_{l_G}$ is a discrete point set $\mathrm{i}S\subset\mathfrak{t}$ on the imaginary plane. Under this situation, the multivariable Jordan lemma says that the integral is the sum of residues at $\sigma'=\mathrm{i}n$ for $n\in C\cap S$,

$$\begin{aligned}Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) &= (r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}}\sum_{n\in C\cap S}\oint_{\gamma_{\mathbf{G}}}\frac{\mathrm{d}^{l_G}z}{(2\pi)^{l_G}}\prod_{\alpha>0}\alpha\left(\mathrm{i}n+\tfrac{z}{2\pi}\right)\sinh\left(\pi\alpha\left(\mathrm{i}n+\tfrac{z}{2\pi}\right)\right)\\ &\times\prod_{\beta}\Gamma\left(-\mathrm{i}d_\beta\left(in+\tfrac{z}{2\pi}\right)+1\right)\prod_i\Gamma\left(\mathrm{i}Q_i\left(\mathrm{i}n+\tfrac{z}{2\pi}\right)\right)\exp\left(\mathrm{i}t_R\left(\mathrm{i}n+\tfrac{z}{2\pi}\right)\right)f_{\mathfrak{B}}\left(\mathrm{i}n+\tfrac{z}{2\pi}\right).\end{aligned}$$

The cycle $\gamma_{\mathbf{G}}$, called the Grothendieck cycle, is a small cycle of $z\in\mathfrak{t}_{\mathbf{C}}$ defined by the equation $|f_a(\mathrm{i}n+\frac{z}{2\pi})|=\varepsilon_a$ for all a, for some $0<\varepsilon_a\ll 1$. One important property is that the integral does not depend on the choice of ε_a's.

At this point, we assume that the set $C\cap S$ is a subset of the coroot lattice $\mathrm{Q}^\vee$, and denote it by $\mathrm{Q}^\vee_+$. Then, we have $f_{\mathfrak{B}}\left(\mathrm{i}n+\frac{z}{2\pi}\right)=f_{\mathfrak{B}}\left(\frac{z}{2\pi}\right)$. After some computation, we find

$$\begin{aligned}Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) &= C(r\Lambda)^{\frac{\widehat{c}_{\mathrm{LV}}}{2}}\sum_{n\in\mathrm{Q}^\vee_+}\oint_{\gamma_{\mathbf{G}}}\frac{\mathrm{d}^{l_G}z}{(2\pi\mathrm{i})^{l_G}}\frac{\prod_{\alpha>0}\alpha(z)^2\prod_\beta d_\beta(z)}{\prod_i Q_i(z)}\widehat{\Gamma}_{X_f}(n,z)\\ &\qquad\times\ \mathrm{e}^{\frac{1}{2}(\sum Q_i-\sum d_\beta)(z)}\,\mathrm{e}^{-t'_R(n)+\frac{\mathrm{i}}{2\pi}t'_R(z)}\frac{f_{\mathfrak{B}}\left(\frac{z}{2\pi}\right)}{\prod_\beta(1-\mathrm{e}^{d_\beta(z)})},\end{aligned}\tag{8.59}$$

in which $\widehat{\Gamma}_{X_f}(n,z)=\widehat{\Gamma}_{X_f}(n)|_{H\to z}$, and

$$t'_R=t_R-\pi\mathrm{i}\left(\sum_i Q_i-2\sum_\beta d_\beta\right).\tag{8.60}$$

C is a constant $(-1)^{d_F}(-2\pi\mathrm{i})^{d_E}(2\pi)^{-|\Delta_+|}\mathrm{i}^{d_G}$ ($|\Delta_+|$ is the number of positive roots).

We would now like to convert each term into an integral over X_f using an identity like (8.38). A generalization of (8.38) to a possibly non-Abelian quotient exists and is known as the Jeffrey-Kirwan localization formula [55]:

$$\int_{X_f}g(H)=\int_{\mathbb{P}}\prod_\beta d_\beta(H)g(H)=\oint_{\gamma_{\mathbf{JK}}}\frac{\mathrm{d}^{l_G}z}{(2\pi\mathrm{i})^{l_G}}\frac{\prod_{\alpha>0}\alpha(z)^2}{\prod_i Q_i(z)}\cdot\prod_\beta d_\beta(z)\cdot g(z),\tag{8.61}$$

where $\gamma_{\mathbf{JK}}$ is a middle dimensional homology class of the complement of $\prod_i Q_i(z)=0$, called the JK cycle. The question is whether the integration over $\gamma_{\mathbf{G}}$ and the one over

$\gamma_{\mathbf{JK}}$ are the same. That is indeed the case in two examples which we will present below, but we do not have a proof at the moment. We simply assume this and proceed. Then we immediately see that the partition function can be written as

$$Z^{\rm LV}_{D^2}(\mathfrak{B}) = C(r\Lambda)^{\frac{\widehat{c}_{\rm LV}}{2}} \sum_{n\in {\rm Q}^\vee_+} \int_{X_f} {\rm e}^{\frac{1}{2}c_1(X_f)}\widehat{\Gamma}_{X_f}(n)\exp\left(-t'_R(n)+\frac{\rm i}{2\pi}t'_R(H)\right){\rm ch}(\mathfrak{B}_{\rm LV}). \quad (8.62)$$

In view of (8.57), we have $t'_R(H) = \omega_R - 2\pi{\rm i}B + \pi{\rm i}c_1(X_f)$. Using this, we may also rewrite the result as

$$Z^{\rm LV}_{D^2}(\mathfrak{B}) = C(r\Lambda)^{\frac{\widehat{c}_{\rm LV}}{2}} \sum_{n\in {\rm Q}^\vee_+} {\rm e}^{-t'_R(n)} \int_{X_f} \widehat{\Gamma}_{X_f}(n)\exp\left(B+\frac{\rm i}{2\pi}\omega_R\right){\rm ch}(\mathfrak{B}_{\rm LV}). \quad (8.63)$$

In the large volume limit in this phase, only the $n = 0$ term remains,

$$Z^{\rm LV}_{D^2}(\mathfrak{B}) \;\longrightarrow\; C(r\Lambda)^{\frac{\widehat{c}_{\rm LV}}{2}} \int_{X_f} \widehat{\Gamma}_{X_f}\exp\left(B+\frac{\rm i}{2\pi}\omega_R\right){\rm ch}(\mathfrak{B}_{\rm LV}). \quad (8.64)$$

Again, this matches with the expected formula for the central charge of the brane $\mathfrak{B}_{\rm LV}$.

Let us present two examples. These are simple enough so that one by one contour deformation can be done by hand. It is instructive to do so and check that the multivariable Jordan lemma gives the correct answer.

The first is the two parameter model considered in Section 6.3. We are in Phase I. $\{Q_i\}$ is the set of charges $\{(0,1),(1,0),(1,-2)\}$ for $X_1,\dots,X_6$. As the shift, we can take $-\boldsymbol{\epsilon} = (-\epsilon_1,-\epsilon_2)$ with $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_1 > 2\epsilon_2$. The last condition comes from the requirement that $R_{X_6} > 0$ before taking the R^o-frame limit. As the cone C, we can take $C_{\rm I} - \boldsymbol{\epsilon}$, where $C_{\rm I}$ is the image cone of the map $\boldsymbol{\tau} \mapsto \boldsymbol{\upsilon}(\boldsymbol{\tau})$ given in (6.24). It is the first quadrant shifted by $-\boldsymbol{\epsilon}$. See Fig. 14. Let us regard the horizontal and vertical faces by the first and the second respectively. Let us group the charges of X_i's so that $\{(0,1)\}$ and $\{(1,0),(1,-2)\}$ are the first and the second groups respectively. Then, the grouping satisfies the condition for the multivariable Jordan lemma. And $C\cap S$ is the first quadrant of the integral lattice $\mathbf{Z}^{\oplus 2}$, that is, it is a subset of the coroot lattice ${\rm Q}^\vee = \mathbf{Z}^{\oplus 2}$. The Grothendieck cycle is therefore

$$|z_2^2| = \varepsilon_1, \qquad |z_1^3(z_1-2z_2)| = \varepsilon_2. \quad (8.65)$$

On the other hand, the JK cycle is

$$|z_1| = \widetilde{\varepsilon}_1, \qquad |z_2| = \widetilde{\varepsilon}_2, \qquad \widetilde{\varepsilon}_1 \ll \widetilde{\varepsilon}_2. \quad (8.66)$$

See for example, [56]. If we choose $\varepsilon_1 \gg \varepsilon_2$, then, the two cycles are homotopic to each other.

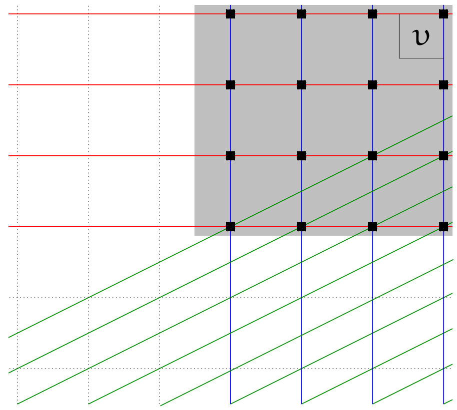


Figure 14: The cone and the poles: The cone C is the shaded region. The poles for the charges $(0,1)$, $(1,0)$ and $(1,-2)$ are shown as the red, blue and green lines respectively. The red itself forms a group while the blue and the green form the other group. The intersection of the two groups are shown as the black dots. Note that the intersection only between the blue and the green are not taken.

The second example is the Rødland model in the Grassmannian phase. The set $\{Q_i\}$ is $\{(1,0),(0,1)\}$. We can take $-\boldsymbol{\epsilon} = (-\epsilon_1, -\epsilon_2)$ with arbitrary positive ϵ_1 and ϵ_2 as the shift. The cone C is the first quadrant shifted by $-\boldsymbol{\epsilon}$. There is a unique grouping and the assumption of the lemma is trivially satisfied. $C \cap S$ is again the first quadrant of the integral lattice and hence is a subset of the coroot lattice. The Grothendieck cycle and the JK cycle are the same, $|z_1| = |z_2| = \varepsilon$.

9 Factorization Of Two-Sphere Partition Function

We have collected a lot of evidence that the parition function of the hemisphere is equal to the central charge of the brane placed at the boundary. They agree whenever both can be computed, and the expressions in some limits also match. This motivates us to conjecture that this is the case in general:

$$Z_{D^2_{(+)}}(\mathfrak{B}) = {}_{\mathrm{RR}}\langle 0|\mathfrak{B}\rangle_{\mathrm{RR}}, \qquad Z_{D^2_{(-)}}(\mathfrak{B}) = {}_{\mathrm{RR}}\langle \mathfrak{B}|0\rangle_{\mathrm{RR}}. \tag{9.1}$$

On the other hand, there is a conjecture [5] that the partition function on the whole sphere is equal to the 00 component of the tt* metric:

$$Z_{S^2} = {}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}}. \tag{9.2}$$

If we admit these, there is a certain relation between the partition functions on the whole sphere and the hemisphere. For any basis $\{|a\rangle\}_{a=1}^{\mu}$ of the space of supersymmetric ground states, we have

$$
{}_{\rm RR}\langle 0|0\rangle_{\rm RR}=\sum_{a,b=1}^{\mu}{}_{\rm RR}\langle 0|a\rangle g^{ab}\langle b|0\rangle_{\rm RR}, \tag{9.3}
$$

where (g^{ab}) is inverse to the matrix $(\langle a|b\rangle)$. Suppose there are μ D-branes $\{\mathfrak{B}_i\}_{i=1}^{\mu}$ whose boundary states have components which span the space of supersymmetric ground states. That is, the square matrix $({}_{\rm RR}\langle\mathfrak{B}_i|a\rangle)$ is invertible. Then, we may use the ground state components of $|\mathfrak{B}_i\rangle_{\rm RR}$'s as a new basis and obtain the formula like (9.3). In the place of g^{ab} we have the inverse to

$$
{}_{\rm RR}\langle\mathfrak{B}_i|P_G|\mathfrak{B}_j\rangle_{\rm RR}, \tag{9.4}
$$

where P_G is the orthogonal projection to the space of supersymmetric ground states. The matrix element (9.4) can be represented by the partition function on the infinitely long cylinder in which the fields including fermions are all periodic along the circle direction. In fact, by the supersymmetry, the length and the thickness of the cylinder does not matter. So, it is just a cylinder partition function of any size. Viewed from the open string channel, (9.4) is the open string Witten index,

$$
\chi(\mathfrak{B}_i,\mathfrak{B}_j)\;:=\;{\rm Tr}_{\mathcal{H}_{\mathfrak{B}_i,\mathfrak{B}_j}}(-1)^F\,{\rm e}^{-\beta H}, \tag{9.5}
$$

where $\mathcal{H}_{\mathfrak{B}_i,\mathfrak{B}_j}$ is the space of states of the open string with the boundary conditions $\mathfrak{B}_i$ and $\mathfrak{B}_j$ on the left and the right ends of the string. H and F are the Hamiltonian and a fermion number operator. Given (9.1) and (9.2), we must have

$$
Z_{S^2}=\sum_{i,j}Z_{D^2_{(+)}}(\mathfrak{B}_i)\,\chi^{ij}\,Z_{D^2_{(-)}}(\mathfrak{B}_j), \tag{9.6}
$$

where χ^{ij} is the inverse to $\chi(\mathfrak{B}_i,\mathfrak{B}_j)$. In this section, we shall examine whether this factorization equation holds.

9.1 The Sphere

First let us write down the formula for the two-sphere partition function. The result of [3, 4] is essentially as follows:

$$
\begin{aligned}
Z_{S^2}\;&=\;(r\Lambda)^{\widehat{c}}\sum_{m\in{\rm Q}^\vee}\int_{{\rm i}\mathfrak{t}}{\rm d}^{l_G}\sigma'\,\exp\Big(2\,{\rm i}\,\zeta_R(\sigma')+{\rm i}\,(\theta+2\pi\rho)(m)\Big) \\
&\qquad\times\prod_{\alpha>0}\left(\frac{\alpha(m)^2}{4}+\alpha(\sigma')^2\right)\prod_i\frac{\Gamma\left({\rm i}Q_i(\sigma')-\frac{Q_i(m)}{2}+\frac{R_i}{2}\right)}{\Gamma\left(1-{\rm i}Q_i(\sigma')-\frac{Q_i(m)}{2}-\frac{R_i}{2}\right)}.
\end{aligned} \tag{9.7}
$$

We say "essentially" because we have done one modification: a shift of the theta angle,

$$\theta \;\longrightarrow\; \theta + \pi \sum_{\alpha>0} \pm\alpha \;\equiv\; \theta + 2\pi\rho \quad \mod 2\pi\mathrm{P}. \tag{9.8}$$

Note that the choice of sign assignment $\pm\alpha$ does not matter since a root is always a weight $\alpha \in \mathrm{P}$ (so that it takes integer values on coroots $m \in \mathrm{Q}^\vee$). ρ is half the sum of positive roots, $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$, which may fail to land on the weight lattice P depending on the group G. For example, for a $U(k)$ gauge theory this matters if and only if k is even. As we will see, this is needed for the factorization. Necessity of the same modification is also noticed in [35] from a different point of view. The factor $(r\Lambda)^{\widehat{c}}$ is not in [3, 4] but is noticed by the authors of these papers, [28, 29].

9.2 The Annulus

Next, we dicuss the open string Witten index $\chi(\mathfrak{B}_i, \mathfrak{B}_j)$, or equivalently, the cylinder, or annulus, partition function. At this moment, we do not have a complete results concerning the computation, but let us make some preliminary remarks.

We may try to apply the localization, sending the gauge coupling to zero and the Kähler metric of the matter to infinity. However, that is plagued by the presence of bosonic as well as fermionic zero modes. It is similar to the situation of the elliptic genus [57, 56] but it is worse than that. In the case of elliptic genus, we have, by definition, the twist by R-symmetry, which usualy separates the singular loci for "positively charged" and the "negatively charged" matter fields. That separation made it possible to justify a certain manipulation of the path integral. For the case of open string Witten index, we do not have that, so that singular loci may collide and cannot be separated. That makes the justification of computation based on the free approximation difficult. But we may hope that there is a way to justify it some way, and try to see if we obtain a reasonable answer.

The annulus partition function of each multiplet in the free approximation is straightforward. We choose the real boundary condition (3.54) for the vector multiplet, so that we may need to consider only grade restricted branes. For the matter sector, the computation is almost done in [11]. It immediately gives the result in the operator formalism but the mode expansion presented there can also be used for the path integral. The result is

$$Z_{\mathrm{chiral}} = \frac{1}{\prod_i 2\sinh\left(\frac{Q_i(u)}{2}\right)}. \tag{9.9}$$

where u parametrizes the bosonic zero mode of the vector multiplet,

$$u\ =\ \beta\sigma_1 - \mathrm{i}a\ \in \mathfrak{t}_{\mathbf{C}}/2\pi\mathrm{i}\mathrm{Q}^{\vee}. \tag{9.10}$$

In the last expression, β is the circumference of the annulus, σ_1 is the scalar zero mode and $a \in \mathrm{i}\mathfrak{t}/2\pi\mathrm{Q}^{\vee}$ parametrizes the gauge holonomy along the circle. The one for the vector multiplet can also be computed. The W-boson pair with the roots $\pm\alpha$ yields

$$Z_{\mathrm{vector},\alpha} = \left(2\sinh\left(\tfrac{\alpha(u)}{2}\right)\right)^2. \tag{9.11}$$

The path integral is presented as the integration over the whole moduli space

$$(\mathfrak{t}_{\mathbf{C}}/2\pi\mathrm{i}\mathrm{Q}^{\vee})/W_G \tag{9.12}$$

of the vector multiplet bosonic zero modes. A proper treatment of the bosonic zero modes from the matter and the fermionic zero modes from the vector may results in an expression of the integrand as a total derivative, which by Stokes theorem leads to the integration over a lower dimensional subspace, just as in [57, 56]. This and some evidences which we will describe below motivates us to make the following conjecture: The annulus partition function is given by a contour integral

$$\chi(\mathfrak{B}_1,\mathfrak{B}_2) = \frac{1}{|W_G|}\int\limits_{\Gamma}\frac{\mathrm{d}^{l_G}u}{(2\pi\mathrm{i})^{l_G}}\frac{\prod_{\alpha>0}\left(2\sinh\left(\frac{\alpha(u)}{2}\right)\right)^2}{\prod_i 2\sinh\left(\frac{Q_i(u)}{2}\right)} f_{\mathfrak{B}_1}\left(-\tfrac{u}{2\pi}\right) f_{\mathfrak{B}_2}\left(\tfrac{u}{2\pi}\right) \tag{9.13}$$

where $f_{\mathfrak{B}}(\frac{u}{2\pi})$ is the brane factor in the R^o-frame,

$$f_{\mathfrak{B}}\left(\tfrac{u}{2\pi}\right) = \mathrm{Str}_M\rho\left(\mathrm{e}^u\right), \tag{9.14}$$

and $\Gamma \subset \mathfrak{t}_{\mathbf{C}}/2\pi\mathrm{i}\mathrm{Q}^{\vee}$ is *some* middle dimensional cycle which represents a homology class of the complement of the divisor $\prod_i \sinh(Q_i(u)/2) = 0$.

Let us comment on some anomaly, which was already noticed in [11]. The integrand of (9.13) is not always single valued on the moduli space (9.12). If one shifts u by $2\pi\mathrm{i}n$ with $n \in \mathrm{Q}^{\vee}$, then, the integrand changes by a sign, $(-1)^{\sum_i Q_i(n)}$. The integrand is single valued if and only if the sum of weights is even,

$$\sum_i Q_i \in 2\mathrm{P}. \tag{9.15}$$

If the theory has a usual geometric phase, with a target Kähler manifold X, this is equivalent to the condition that $c_1(X)$ is even, in other words, X admits a spin structure.

Let us describe some evidences for the conjecture. We first consider the $U(1)$ theories. We assume $N-d$ is even. The formula is written as

$$I_\Gamma = \int_\Gamma \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{f_{\mathfrak{B}_1}(-\frac{u}{2\pi}) f_{\mathfrak{B}_2}(\frac{u}{2\pi})}{\left(\mathrm{e}^{\frac{u}{2}} - \mathrm{e}^{-\frac{u}{2}}\right)^N \left(\mathrm{e}^{-d\frac{u}{2}} - \mathrm{e}^{d\frac{u}{2}}\right)}, \tag{9.16}$$

where Γ is some cycle in $\mathbf{C}/2\pi\mathrm{i}\mathbf{Z}$ minus the pole location which is $\{\,\mathrm{e}^{2\pi\mathrm{i}n/d}\}_{n=0}^{d-1}$. It can be rewitten as

$$I_\Gamma = \int_\Gamma \frac{\mathrm{d}u}{2\pi\mathrm{i}} \frac{1}{u^N} \cdot du \cdot \frac{u^{N-1}\left(\mathrm{e}^{\frac{du}{2}} - \mathrm{e}^{-\frac{du}{2}}\right)}{d\left(\mathrm{e}^{\frac{u}{2}} - \mathrm{e}^{-\frac{u}{2}}\right)^N} \frac{f_{\mathfrak{B}_1}(-\frac{u}{2\pi})}{(1-\mathrm{e}^{-du})} \frac{f_{\mathfrak{B}_2}(\frac{u}{2\pi})}{(1-\mathrm{e}^{du})} \tag{9.17}$$

Suppose the cycle is the small contour γ_0 around $u=0$. Then, we can use the identity (8.38) to write it as an integration over X_f and in fact it is nothing but

$$I_{\gamma_0} = \int_{X_f} \widehat{\mathrm{A}}_{X_f}\, \mathrm{ch}(\mathfrak{B}_{1\,\mathrm{LV}})^\vee\, \mathrm{ch}(\mathfrak{B}_{2\,\mathrm{LV}}). \tag{9.18}$$

(For a $2i$ form ω we define $\omega^\vee := (-1)^i\omega$.) This is indeed an expected answer in the geometric phase. For the Witten index, we may employ the zero mode approximation. In the zero mode sector, open string states are spinors valued in $Hom(E_1, E_2)$, where E_i is the vector bundle for $\mathfrak{B}_{i\,\mathrm{LV}}$ and a linear combination of the supercharges is essentially the Dirac operator. Therefore, the Witten index is the Dirac index gievn by the Atiyah-Singer formula (8.25), which is (9.18) in the present context. To be more precise, the geometric phase can represent the full theory only for $d \le N$. So, we obtain the expected correct answer if we choose $\Gamma = \gamma_0$ in the case $d \le N$.

This can be generalized to any theory with the usual geometric phase. In the set up of Section 8.2.3, assuming $\sum_i Q_i - \sum_\beta d_\beta$ is even, if we take the JK cycle near $u=0$, $\Gamma = \gamma_{\mathbf{JK}}$, then the same computation yields the Dirac-type index,

$$I_{\gamma_{\mathbf{JK}}} = \int_{X_f} \widehat{\mathrm{A}}_{X_f}\, \mathrm{ch}(\mathfrak{B}_{1\,\mathrm{LV}})^\vee\, \mathrm{ch}(\mathfrak{B}_{2\,\mathrm{LV}}). \tag{9.19}$$

Let us come back to the $U(1)$ theory. Recall that there are also $(d-1)$ poles at $u = \mathrm{e}^{2\pi\mathrm{i}n/d}$. If we start from $\Gamma = \gamma_0$ and deform it, provided the behaviour $\mathrm{Re}(u) \to \pm\infty$ is good enough, we can arrive at the $(d-1)$ small cycles around these poles, with the clockwise orientation. Since each is a simple pole, it is easy to evaluate the residues. The result is

$$\sum_{n=1}^{d-1} I_{-\gamma_n} = \frac{1}{d} \sum_{n=1}^{d-1} \mathrm{e}^{\frac{\pi\mathrm{i}(d-N)n}{d}} \frac{f_{\mathfrak{B}_1}\left(-\frac{\mathrm{i}n}{d}\right) f_{\mathfrak{B}_2}\left(\frac{\mathrm{i}n}{d}\right)}{\left(1-\mathrm{e}^{-\frac{2\pi\mathrm{i}n}{d}}\right)^N} \tag{9.20}$$

When $d=N$ this is precisely the open string Witten index in the Landau-Ginzburg orbifold [6]. When $d<N$ it does not agree with that. Indeed, we do not expect an agreement since the Landau-Ginzburg orbifold is only a part of the whole theory. However, some of the branes descends purely to the Landau-Ginzburg orbifold. It would be interesting to see if the above gives the correct answer for a pair of such branes. When $d>N$ it *is* the whole theory, but the starting choice $\Gamma=\gamma_0$ would not be the right choice in general since it gives the formula in the non-linear sigma model, which is only a part of the theory.

We would like to make a final comment on the behaviour at $\mathrm{Re}(u)\to\pm\infty$ in this $U(1)$ theory. The charge $(q^{(1)},q^{(2)})$ term of the integrand behaves as

$$\text{integrand}_{q^{(1)},q^{(2)}}\longrightarrow\exp\left(-q^{(1)}u+q^{(2)}u-\frac{N+d}{2}|u|\right)\quad\text{as }\mathrm{Re}(u)\to\pm\infty.\tag{9.21}$$

A good behaviour is guaranteed only if

$$\left|\,q^{(1)}-q^{(2)}\,\right|\;<\;\frac{N+d}{2}\tag{9.22}$$

for any pair of Chan-Paton charges of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. In the Calabi-Yau case, $d=N$, if the two branes are grade restricted with respect to a common window, $-\frac{N}{2}<\frac{\theta}{2\pi}+q^{(a)}_{j_a}<\frac{N}{2}$, $a=1,2$, then the above condition is indeed satisfied.

9.3 Factorization

Let us now come back to the question of factorization. Since we do not yet know the general formula for the annulus, we cannot make the most general check at this moment. However, we do know the formula for the theory with a usual geometric phase — it is given by the Dirac index (9.18) and (9.19). So, we shall test the factorization in such theories.

Let us first examine the $U(1)$ theory introduced in Section 6.2. We shall only consider the case $d\leq N$ where the large volume expression (8.43) is an expansion of the full partition function. In this case, the formula (9.8) is

$$Z_{S^2}\;=\;\sum_{m\in\mathbf{Z}}\int_{\mathbf{R}-\mathrm{i}0}\mathrm{d}\sigma'\,\mathrm{e}^{2\mathrm{i}\zeta_R\sigma'+\mathrm{i}\theta m}\frac{\Gamma\left(1-\mathrm{i}d\sigma'+\frac{dm}{2}\right)}{\Gamma\left(\mathrm{i}d\sigma'+\frac{dm}{2}\right)}\frac{\Gamma\left(\mathrm{i}\sigma'-\frac{m}{2}\right)^N}{\Gamma\left(1-\mathrm{i}\sigma'-\frac{m}{2}\right)^N}\tag{9.23}$$

We look at the geometric regime $\zeta_R\gg0$ in which the integrand decays exponentially fast in the positive imaginary direction. due to the factor $\mathrm{e}^{2\in\zeta_R\sigma'}$. Then, we can bend both ends of the contour upwards and we only have to take the poles on the upper half plane. The poles are at

$$\mathrm{i}\sigma'-\frac{m}{2}=-l\,;\qquad l\geq0,\;\;l\geq m.\tag{9.24}$$

They come from the factor $\Gamma(\mathrm{i}\sigma' - \frac{m}{2})^N$. The condition $l \geq m$ is to omit the poles which are cancelled by the zeroes from the gamma functions in the denominator. The other gamma function in the numerator may have poles on the upper half plane but they are all cancelled from the other gamma function in the denominator. With the reparametrization $l = n$, $m = n - \overline{n}$, the condition $l \geq 0, m$ becomes $n, \overline{n} \geq 0$. If we shift the integration variable as $\sigma' = \mathrm{i}\left(l - \frac{m}{2}\right) + \frac{z}{2\pi}$ at each pole, we have

$$Z_{S^2} = \sum_{n,\overline{n}\geq 0} \oint_0 \frac{\mathrm{d}z}{2\pi}\, \mathrm{e}^{-t_R n - \bar{t}_R \overline{n} + \mathrm{i}(t_R + \bar{t}_R)\frac{z}{2\pi}} \frac{\Gamma\left(1 + dn + \frac{dz}{2\pi\mathrm{i}}\right)}{\Gamma\left(-d\overline{n} - \frac{dz}{2\pi\mathrm{i}}\right)} \frac{\Gamma\left(-n - \frac{z}{2\pi\mathrm{i}}\right)^N}{\Gamma\left(1 + \overline{n} + \frac{z}{2\pi\mathrm{i}}\right)^N}. \tag{9.25}$$

On the other hand, we use the large volume formula (8.43) for the hemisphere partition function,

$$\begin{aligned} Z_{D^2_{(+)}}(\mathfrak{B}_i) &= \sum_{n=0}^{\infty} \int_{X_f} \mathrm{e}^{\frac{N-d}{2}H} \widehat{\Gamma}_{X_f}(n) \exp\left(-nt'_R + \frac{\mathrm{i}}{2\pi} t'_R H\right) \mathrm{ch}(\mathfrak{B}_{i\mathrm{LV}}), & (9.26)\\ Z_{D^2_{(-)}}(\mathfrak{B}_j) &= \sum_{\overline{n}=0}^{\infty} \int_{X_f} \mathrm{e}^{-\frac{N-d}{2}H} \widehat{\Gamma}_{X_f}(\overline{n}) \exp\left(-\overline{n}\overline{t}'_R + \frac{\mathrm{i}}{2\pi} \overline{t}'_R H\right) \mathrm{ch}(\mathfrak{B}_{j\mathrm{LV}})^\vee. & (9.27) \end{aligned}$$

$\overline{t}'_R$ is the complex conjugate of t'_R. The latter expression (9.27) is obtained from the former by using (5.73) and the sign change of H. Note that $\widehat{A}(-x) = \widehat{A}(x)$. We ignore overall nemerical factors. To evaluate the right hand siade of (9.6), we employ the identity

$$\sum_{i,j} \int_{X_f} \omega\, \mathrm{ch}(\mathfrak{B}_{i\mathrm{LV}})\, \chi^{ij} \int_{X_f} \eta\, \mathrm{ch}(\mathfrak{B}_{j\mathrm{LV}})^\vee = \int_{X_f} \omega \frac{1}{\widehat{\mathrm{A}}_{X_f}} \eta \tag{9.28}$$

Then the right hand side is

$$\mathrm{RHS} = \sum_{n,\overline{n}\geq 0} \int_{X_f} \frac{\widehat{\Gamma}_{X_f}(n)\, \widehat{\Gamma}_{X_f}(\overline{n})}{\widehat{\mathrm{A}}_{X_f}} \exp\left(-nt'_R - \overline{n}\overline{t}'_R + \frac{\mathrm{i}}{2\pi}(t'_R + \overline{t}'_R)H\right) \tag{9.29}$$

Recalling the definition (8.41) and (8.37), after some computation using the gamma function identity (8.34), we find

$$\mathrm{RHS} = \sum_{n,\overline{n}\geq 0} \oint_0 \frac{\mathrm{d}z}{2\pi} (-1)^{(N-d)\overline{n}}\, \mathrm{e}^{-t_R n - \bar{t}_R \overline{n} + \mathrm{i}(t_R + \bar{t}_R)\frac{z}{2\pi}} \frac{\Gamma\left(1 + dn + \frac{dz}{2\pi\mathrm{i}}\right)}{\Gamma\left(-d\overline{n} - \frac{dz}{2\pi\mathrm{i}}\right)} \frac{\Gamma\left(-n - \frac{z}{2\pi\mathrm{i}}\right)^N}{\Gamma\left(1 + \overline{n} + \frac{z}{2\pi\mathrm{i}}\right)^N}. \tag{9.30}$$

This agrees with the expression (9.25) when $(N - d)$ is even, i.e., when X_f is a spin manifold, which is the case where the Dirac index makes sense.

We next consider the more general theory with a geometric phase from Section 8.2.3. We take over the assumptions made in that section (which are confirmed in the examples).

The two sphere partition function is

$$Z_{S^2} = \sum_{m\in Q^\vee} \int_{\mathrm{i}\mathfrak{t}-\mathrm{i}0} \mathrm{d}^{l_G}\sigma' \exp\Big(2\,\mathrm{i}\,\zeta_R(\sigma') + \mathrm{i}\,(\theta+2\pi\rho)(m)\Big) \tag{9.31}$$
$$\times \prod_{\alpha>0}\left(\frac{\alpha(m)^2}{4}+\alpha(\sigma')^2\right)\prod_\beta \frac{\Gamma\left(1+d_\beta(-\mathrm{i}\sigma'+\frac{m}{2})\right)}{\Gamma\left(d_\beta(\mathrm{i}\sigma'+\frac{m}{2})\right)}\prod_i \frac{\Gamma\left(Q_i(\mathrm{i}\sigma'-\frac{m}{2})\right)}{\Gamma\left(1+Q_i(-\mathrm{i}\sigma'-\frac{m}{2})\right)}.$$

We deform the contour in the direction of the cone $\mathcal{C}$. By the multi-dimensional Jordan lemma, we only have to take the poles at $\mathrm{i}\sigma' - \frac{m}{2} = -l$ with $l \in Q_+^\vee$, but we also need to omit the poles that are cancelled by the zeroes from the gamma function on the denominator. We assume that it can be done by requiring $l-m \in Q_+^\vee$. We also assume that the gamma function factors from the P-fields do not have poles that contribute to this integral. We do not have a proof of these claims, although these indeed hold in the examples. To summarize, we take poles at

$$\mathrm{i}\sigma' - \frac{m}{2} = -l\,; \quad ,\ l\in Q_+^\vee,\ \ l-m\in Q_+^\vee. \tag{9.32}$$

With the same reparametrization of l and m and the shift of integration variables as in the $U(1)$ theory, we find that the S^2 partition function can be written as

$$Z_{S^2} = (-1)^{|\Delta_+|}\sum_{n,\overline{n}\in Q_+^\vee}\oint_{\gamma_{\mathbf{G}}} \frac{\mathrm{d}^{l_G}z}{(2\pi)^{l_G}}\, \mathrm{e}^{-(t_R-2\pi\mathrm{i}\rho)(n)-(\overline{t}_R+2\pi\mathrm{i}\rho)(\overline{n})+\frac{\mathrm{i}}{2\pi}(t_R+\overline{t}_R)(z)} \tag{9.33}$$
$$\times\prod_{\alpha>0}\alpha\left(n+\tfrac{z}{2\pi\mathrm{i}}\right)\alpha\left(\overline{n}+\tfrac{z}{2\pi\mathrm{i}}\right)\prod_\beta\frac{\Gamma\left(1+d_\beta(n+\frac{z}{2\pi\mathrm{i}})\right)}{\Gamma\left(d_\beta(-\overline{n}-\frac{z}{2\pi\mathrm{i}})\right)}\prod_i\frac{\Gamma\left(Q_i(-n-\frac{z}{2\pi\mathrm{i}})\right)}{\Gamma\left(1+Q_i(\overline{n}+\frac{z}{2\pi\mathrm{i}})\right)}.$$

On the other hand, we use the expression (8.62) for the hemisphere partition function in the geometric phase. Using the identity (9.28), we see that the right hand side of (9.6) can be written in the same way as (9.29) where the sum is over $n,\overline{n}\in Q_+^\vee$ and the exponent is $-t'_R(n)-\overline{t}'_R(\overline{n})+\frac{\mathrm{i}}{2\pi}(t'_R+\overline{t}'_R)(H)$. Applying the Jeffrey-Kirwan fomula (8.61) and after some computation using the identity (8.34), we find

$$\mathrm{RHS} = \mathrm{const}\sum_{n,\overline{n}\in Q_+^\vee}\oint_{\gamma_{\mathbf{JK}}}\frac{\mathrm{d}^{l_G}z}{(2\pi)^{l_G}}\,\mathrm{e}^{-t_R(n)-\overline{t}_R(\overline{n})+\frac{\mathrm{i}}{2\pi}(t_R+\overline{t}_R)(z)}(-1)^{2\rho(n+\overline{n})+(\sum_i Q_i-\sum_\beta d_\beta)(\overline{n})} \tag{9.34}$$
$$\times\prod_{\alpha>0}\alpha\left(n+\tfrac{z}{2\pi\mathrm{i}}\right)\alpha\left(\overline{n}+\tfrac{z}{2\pi\mathrm{i}}\right)\prod_\beta\frac{\Gamma(1+d_\beta\left(n+\frac{z}{2\pi\mathrm{i}}\right))}{\Gamma\left(d_\beta(-\overline{n}-\frac{z}{2\pi\mathrm{i}})\right)}\prod_i\frac{\Gamma(Q_i\left(-n-\frac{z}{2\pi\mathrm{i}}\right))}{\Gamma\left(1+Q_i(\overline{n}+\frac{z}{2\pi\mathrm{i}})\right)}.$$

The sign $(-1)^{2\rho(n+\overline{n})}$ comes out during the process of the following type,

$$\sin\left(\pi\alpha\left(n+\tfrac{z}{2\pi\mathrm{i}}\right)\right) = (-1)^{\alpha(n)}\sin\left(\pi\alpha\left(\tfrac{z}{2\pi\mathrm{i}}\right)\right).$$

We see that it agrees with (9.33) up to constant, provided X_f is a spin manifold, $c_1(X_f) \equiv 0 \bmod 2$, so that $\sum_i Q_i - \sum_\beta d_\beta$ takes even numbers on the coroot lattice. And we see why the shift (9.8) is needed in order for the factorization to work out.

10 Mirror Symmetry

In this final section, we use one more property of the gamma function. That is, the Euler integral of the second kind,

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}\, t^{z-1}\,\mathrm{d}t, \qquad \mathrm{Re}(z) > 0. \tag{10.1}$$

which is usually used as the *definition* of the gamma function.

For convenience, let us write once again the formula for the hemisphere partition function,

$$\begin{aligned} Z_{D^2}(\mathfrak{B}) \;&=\; (r\Lambda)^{\widehat{c}/2} \int_\gamma \mathrm{d}^{l_G}\sigma' \prod_{\alpha>0} \alpha(\sigma') \sinh(\pi\alpha(\sigma')) \prod_i \Gamma\left(\mathrm{i}Q_i(\sigma') + \frac{R_i}{2}\right) \\ &\qquad \times \exp\left(\mathrm{i}t_{\mathrm{R}}(\sigma')\right) \sum_j \mathrm{e}^{\pi \mathrm{i} r_j}\, \mathrm{e}^{2\pi q_j(\sigma')}. \end{aligned} \tag{10.2}$$

Let us apply (10.1) to the gamma function factor in (10.2). Using the variable $\mathrm{e}^{-y_i'}$ instead of t, we have

$$\Gamma\left(\mathrm{i}Q_i(\sigma') + \frac{R_i}{2}\right) \;=\; \int_{-\infty}^\infty \mathrm{d}y_i' \,\exp\left(-y_i'\left(\mathrm{i}Q_i(\sigma') + \frac{R_i}{2}\right) - \mathrm{e}^{-y_i}\right) \tag{10.3}$$

which is valid when $\mathrm{Im}(Q_i(\sigma')) < \frac{R_i}{2}$. Using this, we can write (10.2) as

$$Z_{D^2}(\mathfrak{B}) = (r\Lambda)^{\widehat{c}/2} \sum_{\varepsilon, j} \left(\prod_{\alpha>0} \frac{\varepsilon_\alpha}{2}\right) \mathrm{e}^{\pi \mathrm{i} r_j} \int\limits_{\gamma \times \mathbf{R}^{d_V}} \mathrm{d}^{l_G}\sigma'\, \mathrm{d}^{d_V} y' \prod_{\alpha>0} \alpha(\sigma') \cdot \delta' \cdot \mathrm{e}^{F_{\varepsilon, q_j}(\sigma', y')} \tag{10.4}$$

where the sum is over j and the choice of $\varepsilon_\alpha = \pm 1$ for each $\alpha > 0$, $\delta' := \prod_i \mathrm{e}^{-y_i' R_i/2}$ and

$$F_{\varepsilon, q_j} := \mathrm{i}t_R(\sigma') - \mathrm{i}\sum_i y_i' Q_i(\sigma') - \sum_i \mathrm{e}^{-y_i'} + \sum_{\alpha>0} \varepsilon_\alpha \pi \alpha(\sigma') + 2\pi q_j(\sigma'). \tag{10.5}$$

Recalling $t_R = t - \sum_i Q_i \log(r\Lambda)$ and shifting the variables as $y_i' = y_i - \log(r\Lambda)$, we find that the partition function can be rewritten as

$$\begin{aligned} Z_{D^2}(\mathfrak{B}) \;&=\; \frac{(r\Lambda)^{\frac{d_V + l_G}{2}}}{\Lambda^{\frac{d_G + l_G}{2}}} \sum_{\varepsilon, j} \left(\prod_{\alpha>0} \frac{\varepsilon_\alpha}{2}\right) \mathrm{e}^{\pi \mathrm{i} r_j} \\ &\quad \times \int\limits_{\widetilde{\gamma} \times \mathbf{R}^{d_V}} \mathrm{d}^{l_G}\sigma\, \mathrm{d}^{d_V} y \prod_{\alpha>0} \alpha(\sigma) \cdot \delta \cdot \exp\left(-2\pi r \mathrm{i} \widetilde{W}_{\varepsilon, q_j}(\sigma, y)\right) \end{aligned} \tag{10.6}$$

where

$$\delta = \prod_i \exp\left(-\frac{R_i}{2} y_i\right) \tag{10.7}$$

and

$$2\pi \widetilde{W}_{\varepsilon,q_j} = \left(\sum_i Q_i y_i - t_{\varepsilon,q_j}\right)(\sigma) + (-\mathrm{i}\Lambda) \sum_i \exp\Big(-y_i\Big), \tag{10.8}$$

in which $t_{\varepsilon,q_j} = \zeta - \mathrm{i}\theta_{\varepsilon,q_j}$ with

$$\theta_{\varepsilon,q_j} = \theta + 2\pi q_j + \sum_{\alpha>0} \varepsilon_\alpha \pi\alpha \tag{10.9}$$

This is valid when the contour $\widetilde{\gamma}$ lies in the region with $\mathrm{Im}(\sigma) < \frac{R_i}{2r}$. For convergence of the integral, we may need to consider only the grade restricted branes.

This is the same as, or more precisely, similar to the expression for the D-brane central charge found in [12] during the derivation of mirror symmetry. $2\pi\widetilde{W}_{\varepsilon,q}$ is essentially the mirror superpotential found in [12]. The factor δ is the factor found in [12] following [39], also denoted by δ, which is required if there is a tree level superpotential in the original side. The factor $\prod_{\alpha>0}\alpha(\sigma)$ is also in [12]. We say "essentially", because they are not the same, even modulo $2\pi\mathrm{i}\mathrm{P}(\sigma)$, because of the shift $\sum_{\alpha>0} \pm\pi\alpha(\sigma)$ of the theta angle. This is a simple mistake in [12]. More importantly, even the integral part from $2\pi\mathrm{i}\mathrm{P}$ matters. Our formula shows precisely how to fix this integral part and then how to sum over the integrals with appropriate signs/phases, depending on the choice of D-brane.

Our formula may be used as a string point to find explicit correspondence between B-branes in the linear sigma model and A-branes in the mirror theory, at least at the level of Ramond-Ramond charge. We leave this problem for future works.

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Appendix

A Conventions

A.1 Spinors On A Two-Manifold

A two-dimensionsal oriented Riemannian manifold (Σ, g) has a natural complex structure. The holomorphic and antiholomorphic cotangent bundles are isomorphic as unitary bundles to the anti-holomorphic and holomorphic tangent bundles, $K_\Sigma \cong \overline{T}_\Sigma$, $\overline{K}_\Sigma \cong T_\Sigma$. A spin structure defines square roots of these bundles, $S_- = \sqrt{K_\Sigma} \cong \sqrt{\overline{T}_\Sigma}$ and $S_+ = \sqrt{\overline{K}_\Sigma} \cong \sqrt{T_\Sigma}$. We assume them be dual to each other. The total spin bundle is the direct sum, $S = S_- \oplus S_+$. A local complex coordinate z of Σ yields a local frame $(\sqrt{\mathrm{d}z}, \sqrt{\mathrm{d}\overline{z}})$ of S, with respect to which the gamma matrices are expressed as

$$\gamma^z \doteq \begin{pmatrix} 0 & (2g^{z\overline{z}})^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \quad \gamma^{\overline{z}} \doteq \begin{pmatrix} 0 & 0 \\ (2g^{z\overline{z}})^{\frac{1}{2}} & 0 \end{pmatrix}. \tag{A.1}$$

The chirality operator γ_3 is defined to have the expression

$$\gamma_3 \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

that is, $\gamma_3 = +1$ on S_- and -1 on S_+. (We hope that this is not too confusing.) The natural projections $P_\mp : S \to S_\mp$ have expressions

$$P_- = \frac{1+\gamma_3}{2} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_+ = \frac{1-\gamma_3}{2} \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.3}$$

Spinors are expressed as

$$\epsilon = \epsilon_-^{\{z\}} \sqrt{\mathrm{d}z} + \epsilon_+^{\{z\}} \sqrt{\mathrm{d}\overline{z}}. \tag{A.4}$$

We shall often suppress the superscript "$\{z\}$" when it is obvious. The pairing between S_- and S_+ is extended to an antisymmetric bilinear form on S,

$$\langle \epsilon, \eta \rangle = (2g_{z\overline{z}})^{-\frac{1}{2}} (\epsilon_+ \eta_- - \epsilon_- \eta_+). \tag{A.5}$$

If the spinors are anticommuting, then it is *symmetric*, $\langle \epsilon, \eta \rangle = \langle \eta, \epsilon \rangle$. It obeys other relations including Fierz identities,

$$\begin{aligned}
&\langle \epsilon, \gamma^\mu \eta \rangle = -\langle \gamma^\mu \epsilon, \eta \rangle, \qquad \langle \epsilon, \gamma_3 \eta \rangle = -\langle \gamma_3 \epsilon, \eta \rangle, \\
&\epsilon \langle \eta, \lambda \rangle + \eta \langle \lambda, \epsilon \rangle + \lambda \langle \epsilon, \eta \rangle = 0, \\
&\gamma^\mu \epsilon \langle \eta, \gamma_\mu \lambda \rangle + \gamma_3 \epsilon \langle \eta, \gamma_3 \lambda \rangle + \epsilon \langle \eta, \lambda \rangle + 2\lambda \langle \epsilon, \eta \rangle = 0, \\
&\epsilon \langle \eta, \lambda \rangle - \gamma_3 \epsilon \langle \gamma_3 \eta, \lambda \rangle + 2 \langle P_- \epsilon, \eta \rangle P_- \lambda + 2 \langle P_+ \epsilon, \eta \rangle P_+ \lambda = 0.
\end{aligned} \tag{A.6}$$

We may also write spinors as

$$\epsilon = \epsilon^+_{\{z\}} \sqrt{\frac{\partial}{\partial \overline{z}}} + \epsilon^-_{\{z\}} \sqrt{\frac{\partial}{\partial z}}, \tag{A.7}$$

where

$$\epsilon^{\{z\}}_{\pm} = \pm (2g_{z\overline{z}})^{\frac{1}{2}} \epsilon^{\mp}_{\{z\}}, \tag{A.8}$$

so that the bilinear form has the expression

$$\langle \epsilon, \eta \rangle = (2g_{z\overline{z}})^{\frac{1}{2}} (-\epsilon^- \eta^+ + \epsilon^+ \eta^-) = \epsilon^- \eta_- + \epsilon^+ \eta_+. \tag{A.9}$$

These mean $\langle \sqrt{\frac{\partial}{\partial z}}, \sqrt{\mathrm{d}z} \rangle = \langle \sqrt{\frac{\partial}{\partial \overline{z}}}, \sqrt{\mathrm{d}\overline{z}} \rangle = 1$, $\langle \sqrt{\mathrm{d}\overline{z}}, \sqrt{\mathrm{d}z} \rangle = (2g_{z\overline{z}})^{-\frac{1}{2}}$, $\langle \sqrt{\frac{\partial}{\partial \overline{z}}}, \sqrt{\frac{\partial}{\partial z}} \rangle = (2g_{z\overline{z}})^{\frac{1}{2}}$, and

$$\sqrt{\mathrm{d}z} = -(2g_{z\overline{z}})^{-\frac{1}{2}} \sqrt{\frac{\partial}{\partial \overline{z}}}, \qquad \sqrt{\mathrm{d}\overline{z}} = (2g_{z\overline{z}})^{-\frac{1}{2}} \sqrt{\frac{\partial}{\partial z}}. \tag{A.10}$$

On the flat space with metric $\mathrm{d}^2 s = |\mathrm{d}z|^2$ we have $2g_{z\overline{z}} = 1$. The above spinor convention matches with the dimensionally reduced and Wick rotated version of the standard one in four dimensions [14].

A conformal Killing spinor is a section ϵ of S obeying

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon' \tag{A.11}$$

for some other section ϵ'. Obviously, $\epsilon' = \frac{1}{2} \not{\nabla} \epsilon$, and the condition is equivalent to

$$\partial_z \epsilon^+ = 0, \qquad \partial_{\overline{z}} \epsilon^- = 0. \tag{A.12}$$

That is, the S_- and S_+ components of ϵ are antiholomorphic and holomorphic sections of $\sqrt{\overline{T}_\Sigma}$ and $\sqrt{T_\Sigma}$ respectively. If Σ is closed, such a spinor exists only when Σ is a sphere or a torus.

When the manifold Σ has a boundary $\partial\Sigma$, a spin structure includes, as a part of the information, an identification

$$\varsigma : S_\mp|_{\partial\Sigma} \longrightarrow S_\pm|_{\partial\Sigma}, \qquad \varsigma^2 = \mathrm{id}, \tag{A.13}$$

whose second tensor power equals a canonical isomorphism between $K_\Sigma|_\Sigma$ and $\overline{K}_\Sigma|_\Sigma$. As the canonical isomorphism, we may take the one that sends $\mathrm{d}\zeta$ to $-\mathrm{d}\overline{\zeta}$ for a complex coordinate ζ near the boundary that maps the chart of Σ to the upper half plane. A conformal Killing spinor is assumed to be anti-invariant under ς at the boundary. Then it defines a conformal Killing spinor of the double, $\Sigma \sharp \overline{\Sigma}$, which exists only when the latter is a sphere or a torus. That is, a conformal Killing spinor exists only when Σ is a hemisphere or an annulus.

A.2 Sphere And Hemisphere

A two-sphere is $\mathbb{CP}^1$ as a complex manifold and is covered by two charts. One with coordinate z and the other with w which are related by $zw = 1$. The round sphere metric of radius r is

$$\mathrm{d}s^2 = \frac{4r^2|\mathrm{d}z|^2}{(1+|z|^2)^2} \quad \text{or} \quad g_{z\overline{z}} = \frac{2r^2}{(1+|z|^2)^2}, \tag{A.14}$$

with the Christoffel symbols given by $\Gamma^z_{zz} = -\frac{2\overline{z}}{1+|z|^2}$, $\Gamma^{\overline{z}}_{\overline{z}\overline{z}} = -\frac{2z}{1+|z|^2}$. The expressions in terms of the w coordinate are the same. It is useful to note

$$(2g_{z\overline{z}})^{\frac{1}{2}} = \frac{2r}{1+|z|^2} \xrightarrow{|z|\to 1} r. \tag{A.15}$$

There is a unique spin structure on $\mathbb{CP}^1$. $\sqrt{T_{\mathbb{CP}^1}}$ as a holomorphic bundle is isomorphic to $\mathcal{O}(1)$ and has two holomorphic sections. Thus, there are four conformal Killing spinors,

$$\mathbf{s}_{-\frac{1}{2}} = \sqrt{\frac{\partial}{\partial z}}, \quad \mathbf{s}_{\frac{1}{2}} = z\sqrt{\frac{\partial}{\partial z}}, \quad \widetilde{\mathbf{s}}_{-\frac{1}{2}} = \sqrt{\frac{\partial}{\partial \overline{z}}}, \quad \widetilde{\mathbf{s}}_{\frac{1}{2}} = \overline{z}\sqrt{\frac{\partial}{\partial \overline{z}}}. \tag{A.16}$$

It is useful to note that

$$\not{\nabla}\, \mathbf{s}_{\pm\frac{1}{2}} = \mp\frac{1}{r}\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}, \qquad \not{\nabla}\, \widetilde{\mathbf{s}}_{\pm\frac{1}{2}} = \mp\frac{1}{r}\mathbf{s}_{\mp\frac{1}{2}}. \tag{A.17}$$

Let us consider the southern hemisphere $D^2_0 = \{|z| \leq 1\}$. There are two spin structures, $(+)_0$ and $(-)_0$, given by

$$\varsigma_{(\pm)_0} : \sqrt{\frac{\mathrm{d}z}{z}} \longleftrightarrow \pm\sqrt{\frac{\mathrm{d}\overline{z}}{\overline{z}}}, \quad \sqrt{z\frac{\partial}{\partial z}} \longleftrightarrow \mp\sqrt{\overline{z}\frac{\partial}{\partial \overline{z}}}, \qquad \text{at } |z| = 1. \tag{A.18}$$

There are two conformal Killing spinors for each,

$$(+)_0 : \qquad \mathbf{s}_{(+)+} = \mathbf{s}_{\frac{1}{2}} + \widetilde{\mathbf{s}}_{-\frac{1}{2}}, \quad \mathbf{s}_{(+)-} = \mathbf{s}_{-\frac{1}{2}} + \widetilde{\mathbf{s}}_{\frac{1}{2}}, \tag{A.19}$$

$$(-)_0 : \qquad \mathbf{s}_{(-)+} = \mathbf{s}_{\frac{1}{2}} - \widetilde{\mathbf{s}}_{-\frac{1}{2}}, \quad \mathbf{s}_{(-)-} = \mathbf{s}_{-\frac{1}{2}} - \widetilde{\mathbf{s}}_{\frac{1}{2}} \tag{A.20}$$

For the outward unit normal vector at the boundary

$$\widehat{n} = \frac{1}{r}\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right), \tag{A.21}$$

$\gamma^{\widehat{n}} = g_{\mu\nu}\widehat{n}^\mu\gamma^\nu$ acts on the above conformal Killing spinors as

$$\gamma^{\widehat{n}}\, \mathbf{s}_{(\pm)\nu} = \mp\, \mathbf{s}_{(\pm)\nu} \qquad \text{at } |z| = 1. \tag{A.22}$$

Finally, let us consider the northern hemisphere $D^2_\infty = \{|w| \leq 1\}$. We define two spin structures $(\pm)_\infty$ in the same way as (A.18) but with the replacement $z, \overline{z} \to w, \overline{w}$. Conformal Killing spinors are $\mathbf{s}_{(-)\pm}$ for $(+)_\infty$ and $\mathbf{s}_{(+)\pm}$ for $(-)_\infty$.

B Graded Chan-Paton Factor

Chan-Paton factors which appear in this paper takes the following form

$$\mathrm{tr}_M\left[P\exp\left(\oint_{S^1}(\psi^a T_a+V)\,\mathrm{d}\tau\right)\right]\quad\text{or}\quad \mathrm{Str}_M\left[P\exp\left(\oint_{S^1}(\psi^a T_a+V)\,\mathrm{d}\tau\right)\right],\tag{B.1}$$

where M is a $\mathbf{Z}_2$-graded vector space, $\tau\equiv\tau+\beta$ is a periodic coordinate of a circle S^1, T_a and V are functions on S^1 with values in $\mathrm{End}^{od}(M)$ and $\mathrm{End}^{ev}(M)$ respectively, ψ^a are fermionic fields (i.e. anticommuting functions) on S^1. tr_M is the usual trace over M and Str_M is the supertrace defined by $\mathrm{Str}_M(U)=\mathrm{tr}_{M^{ev}}(U)-\mathrm{tr}_{M^{od}}(U)$. We take the usual trace when the fermions are anit-periodic $\psi^a(\tau+\beta)=-\psi^a(\tau)$ and the supertrance when they are periodic $\psi^a(\tau+\beta)=\psi^a(\tau)$. In this appendix, we give a definition to the expression like (B.1), and explain why we take the trace or the supertrace depending on the periodicity of $\psi^a(\tau)$.[1]

We start with defining

$$U(\tau_f,\tau_i)=P\exp\left(\int_{\tau_i}^{\tau_f}(\psi^a T_a+V)\,\mathrm{d}\tau\right),\tag{B.2}$$

for an interval $[\tau_i,\tau_f]$. First, we formally apply the usual rule of path ordered exponential. If we set $V=0$ for simplicity just for now, the n-th order term is of the form

$$\int_{\tau_i}^{\tau_f}\mathrm{d}\tau_n\cdots\int_{\tau_i}^{\tau_3}\mathrm{d}\tau_2\int_{\tau_i}^{\tau_2}\mathrm{d}\tau_1\,(\psi^{a_n}T_{a_n})(\tau_n)\cdots(\psi^{a_2}T_{a_2})(\tau_2)(\psi^{a_1}T_{a_1})(\tau_1)\tag{B.3}$$

We *now define* this expression by

$$\begin{aligned}:=\int_{\tau_i}^{\tau_f}\mathrm{d}\tau_n\cdots\int_{\tau_i}^{\tau_3}\mathrm{d}\tau_2\int_{\tau_i}^{\tau_2}\mathrm{d}\tau_1\,(-1)^{1+2+\cdots+(n-1)}\,\psi^{a_n}(\tau_n)\cdots\psi^{a_2}(\tau_2)\psi^{a_1}(\tau_1)&\\ \times T_{a_n}(\tau_n)\cdots T_{a_2}(\tau_2)T_{a_1}(\tau_1).&\end{aligned}\tag{B.4}$$

The last line is the usual matrix multiplcation of $T_{a_j}(\tau_j)$'s. We can recover $V\neq 0$ by inserting $U_0(\tau_{j+1},\tau_j):=P\exp\left(\int_{\tau_j}^{\tau_{j+1}}V(\tau)\mathrm{d}\tau\right)$ between $T_{a_{j+1}}(\tau_{j+1})$ and $T_{a_j}(\tau_j)$, as well as $U_0(\tau_f,\tau_n)$ to the left of $T_{a_n}(\tau_n)$ and $U_0(\tau_1,\tau_i)$ to the right of $T_{a_1}(\tau_1)$. By the sign $(-1)^{1+\cdots+(n-1)}$, we may treat $T_a(\tau)$'s as fermionic quantities inside the formal expressions like (B.2) and (B.3). But in the actual definition (B.4), they are genuine ("bosonic")

[1]**Warning:** We are not reviewing the well understood rule of quantum mechanics that the trace and the supertrace correspond respectively to path integrals over fermions with the antiperiodic and periodic boundary conditions along a (time) circle. The present problem can be related to that, as we will mention below, but only in a special case.

functions with values in the space $\mathrm{End}^{od}(M)$ of usual matrices. Let us express $U = U(\tau_f, \tau_i)$ with respect to a basis of M, where the first entries are even and the last entries are odd,

$$U \doteq \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{B.5}$$

In view of the above definition of U, we see that A and D have even powers of $\psi^a(\tau)$'s and hence are bosonic while B and C have odd powers of $\psi^a(\tau)$'s and hence are fermionic.

We next consider the case where τ is a coordinate of a circle with periodocity $\tau \equiv \tau + \beta$. In the usual case, say the case $T^a = 0$, we can simply take the trace of $U(\tau_0 + \beta, \tau_0)$ to define an invariant. This does not depend on the choice of the initial time τ_0, because

$$\begin{aligned} \mathrm{tr}\, U(\tau_0+\beta,\tau_0) &= \mathrm{tr}\,[U(\tau_0+\beta,\tau_1)U(\tau_1,\tau_0)] = \mathrm{tr}\,[U(\tau_1,\tau_0)U(\tau_0+\beta,\tau_1)] \\ &= \mathrm{tr}\,[U(\tau_1+\beta,\tau_0+\beta)U(\tau_0+\beta,\tau_1)] = \mathrm{tr}\, U(\tau_1+\beta,\tau_1). \end{aligned} \tag{B.6}$$

In this proof, we used the following properties

$$\begin{aligned} \text{composition rule} \quad & U(\tau_2,\tau_1) = U(\tau_2,\tau_*)U(\tau_*,\tau_1), \\ \text{cyclicity of the trace} \quad & \mathrm{tr}\,[U_1U_2] = \mathrm{tr}\,[U_2U_1], \\ \text{periodicity} \quad & U(\tau_2+\beta,\tau_1+\beta) = U(\tau_2,\tau_1). \end{aligned}$$

In the graded case, the composition rule holds for (B.2). However, the cyclicity of the trace or supertrace may fail since some of the matrix entries are fermionic. To examine how it may fail or hold, let us write

$$U_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \qquad i = 1, 2, \tag{B.7}$$

with respect to the basis where the first entries are even and last entries are odd. We have

$$\begin{aligned} \mathrm{tr}\,[U_1U_2] &= \mathrm{tr}\,[A_1A_2 + B_1C_2 + C_1B_2 + D_1D_2], \\ \mathrm{Str}\,[U_1U_2] &= \mathrm{tr}\,[A_1A_2 + B_1C_2 - C_1B_2 - D_1D_2], \end{aligned}$$

When A, D are bosonic and B, C are fermionic as in (B.5), then we see that the supertrace has the right cyclicity

$$\mathrm{Str}\,[U_1U_2] = \mathrm{Str}\,[U_2U_1] \tag{B.8}$$

but the usual trace violates it in the middle two terms. However, we can say

$$\mathrm{tr}\,[U_1U_2] = \mathrm{tr}\,[U_2U_1]\big|_{\substack{B_2\to -B_2, \\ C_2\to -C_2}} \tag{B.9}$$

For U in (B.5), the sign flip of the B and C components can be realized by $\tau \to \tau + \beta$ provided $\psi^a(\tau)$ are antiperiodic. This proves that the following is independent of the choice of the initial time τ_0:

$$\begin{array}{rl} \mathrm{tr}_M\, U(\tau_0+\beta,\tau_0) & \text{if } \psi^a(\tau) \text{ are antiperiodic,} \\ \mathrm{Str}_M\, U(\tau_0+\beta,\tau_0) & \text{if } \psi^a(\tau) \text{ are periodic.} \end{array}$$

In the special case where the rank of M is a power of 2, there is a very familar way to understand the above construction. Let us consider the simplest case where $\mathrm{rank}(M) = 2$, $\mathrm{rank}(M^{ev}) = \mathrm{ranl}(M^{od}) = 1$. Let us write

$$T_a = \begin{pmatrix} 0 & f_a \\ g_a & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} V_0 & 0 \\ 0 & V_0 \end{pmatrix}. \tag{B.10}$$

(We take this special form for V for simplicity.) Using

$$\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \overline{\eta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{B.11}$$

the matrix $\psi^a T_a + V$ may be written as $\psi^a(f_a\eta + g_a\overline{\eta}) + V_0 =: -H$. We may regard H as a time dependent Hamiltonian of a quantum mechanical system whose space of states is M. In the path-integral formulation, such a system can be realized by a pair of anticommuting variables $\eta(t)$, $\overline{\eta}(t)$, with the Lagrangian $L = \mathrm{i}\overline{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\eta - H$. The matrix (B.2), which can be regarded as the evolution in the imaginary time, $\tau = \mathrm{i}t$, is represented by the path-integral with an appropriate boundary condition $\mathbf{B}_{\tau_i}^{\tau_f}$

$$U(\tau_f,\tau_i) = \int_{\mathbf{B}_{\tau_i}^{\tau_f}} \mathcal{D}\overline{\eta}\mathcal{D}\eta \exp\left(\int_{\tau_i}^{\tau_f}\left(-\overline{\eta}\frac{\mathrm{d}}{\mathrm{d}\tau}\eta + \psi^a(f_a\eta + g_a\overline{\eta}) + V_0\right)\mathrm{d}\tau\right). \tag{B.12}$$

Let us now discuss the case where τ is a periodic coordinate, $\tau \equiv \tau + \beta$. If $\psi^a(\tau)$ is anti-periodic (*resp.* periodic), we need $\eta(\tau)$ and $\overline{\eta}(\tau)$ to be also anti-periodic (*resp.* periodic), in order for the Lagrangian to be periodic. By the standard quantization rule, we have

$$\mathrm{tr}_M\, U(\beta,0) = \int_{\mathbf{A}} \mathcal{D}\overline{\eta}\mathcal{D}\eta \exp\left(\int_{\tau_i}^{\tau_f}\left(-\overline{\eta}\frac{\mathrm{d}}{\mathrm{d}\tau}\eta + \psi^a(f_a\eta + g_a\overline{\eta}) + V_0\right)\mathrm{d}\tau\right), \tag{B.13}$$

$$\mathrm{Str}_M\, U(\beta,0) = \int_{\mathbf{P}} \mathcal{D}\overline{\eta}\mathcal{D}\eta \exp\left(\int_{\tau_i}^{\tau_f}\left(-\overline{\eta}\frac{\mathrm{d}}{\mathrm{d}\tau}\eta + \psi^a(f_a\eta + g_a\overline{\eta}) + V_0\right)\mathrm{d}\tau\right), \tag{B.14}$$

where $\mathbf{A}$ and $\mathbf{P}$ stand for the anti-periodic and the periodic boundary conditions for both $\eta(\tau), \overline{\eta}(\tau)$ and $\psi^a(\tau)$. In this presentation, we explicitly see that $T_a = f_a\eta + g_a\overline{\eta}$ is a fermionic opeartor which is anti-periodic (*resp.* periodic) in the former (*resp.* latter) case.

C Explict Expressions For Supersymmetry Transformations

For the study of supersymmetry of the boundary conditions, we explicitly write down the $\mathrm{A}_{(\pm)}$-type supersymmetry transformation of the chiral multiplet and vector multiplet fields. Spinors are written in components (3.43) with respect to the natural frames near the boundary ∂D^2. We also use the variational parameter $\varepsilon(\tau)$ and $\overline{\varepsilon}(\tau)$ defined in (3.29). Expressions are simplified a little by partially using $\psi'_- = |z|^{\mp\frac{1}{2}}\psi_-$, $\psi'_+ = |z|^{\pm\frac{1}{2}}\psi_+$, $\overline{\psi}'_- = |z|^{\pm\frac{1}{2}}\overline{\psi}_-$, $\overline{\psi}'_+ = |z|^{\mp\frac{1}{2}}\overline{\psi}_+$, $\lambda'_- = |z|^{\pm\frac{1}{2}}\lambda_-$, $\lambda'_+ = |z|^{\mp\frac{1}{2}}\lambda_+$, $\overline{\lambda}'_- = |z|^{\mp\frac{1}{2}}\overline{\lambda}_-$, $\overline{\lambda}'_+ = |z|^{\pm\frac{1}{2}}\overline{\lambda}_+$. (Here and elsewhere the multiple signs $\pm$ or $\mp$ are always correlated with the spin structure $(\pm)$ or equivalently the type $\mathrm{A}_{(\pm)}$ of supersymmetry.) We also use

$$|x_{1,2}| = \frac{2|z|}{1+|z|^2} \overset{\partial D^2}{=} 1, \qquad x_3 = \frac{|z|^2-1}{1+|z|^2} \overset{\partial D^2}{=} 0.$$

The transformation of the chiral multiplet fields is

$$\begin{aligned}
&\delta\phi = \varepsilon(\psi'_- + \psi'_+), \qquad \delta\overline{\phi} = -\overline{\varepsilon}(\overline{\psi}'_- + \overline{\psi}'_+), \\
&\delta(\psi'_- + \psi'_+) = 2\overline{\varepsilon}\left[D_\tau\phi + \left(\pm\left(\frac{\mathrm{i}}{2r}R - \sigma_1\right) + \mathrm{i}x_3\sigma_2\right)\phi\right], \\
&\delta(\overline{\psi}'_- + \overline{\psi}'_+) = 2\varepsilon\left[-D_\tau\overline{\phi} + \overline{\phi}\left(\pm\left(\frac{\mathrm{i}}{2r}R - \sigma_1\right) + \mathrm{i}x_3\sigma_2\right)\right], \\
&\delta(\psi'_- - \psi'_+) = 2\overline{\varepsilon}\left[\mathrm{i}D_\rho\phi - \left(x_3\left(\frac{\mathrm{i}}{2r}R - \sigma_1\right) \pm \mathrm{i}\sigma_2\right)\phi\right] \mp 2\mathrm{i}\varepsilon|x_{1,2}|f, \\
&\delta(\overline{\psi}'_- - \overline{\psi}'_+) = 2\varepsilon\left[-\mathrm{i}D_\rho\overline{\phi} + \overline{\phi}\left(x_3\left(\frac{\mathrm{i}}{2r}R - \sigma_1\right) \pm \mathrm{i}\sigma_2\right)\right] \mp 2\mathrm{i}\overline{\varepsilon}|x_{1,2}|\overline{f}, \\
&\delta f = \overline{\varepsilon}\Bigg[\pm\frac{2}{r|x_{1,2}|}\left(|z|^{\pm\frac{1}{2}}zD_z\psi_+ + |z|^{\mp\frac{1}{2}}\overline{z}D_{\overline{z}}\psi_-\right) \\
&\qquad - |z|^{\pm\frac{1}{2}}\left(\frac{R}{2r} + \mathrm{i}\overline{\sigma}\right)\psi_- + |z|^{\mp\frac{1}{2}}\left(\frac{R}{2r} + \mathrm{i}\sigma\right)\psi_+ - \left(|z|^{\pm\frac{1}{2}}\overline{\lambda}_- + |z|^{\mp\frac{1}{2}}\overline{\lambda}_+\right)\phi\Bigg], \\
&\delta\overline{f} = \varepsilon\Bigg[\pm\frac{2}{r|x_{1,2}|}\left(|z|^{\mp\frac{1}{2}}zD_z\overline{\psi}_+ + |z|^{\pm\frac{1}{2}}\overline{z}D_{\overline{z}}\overline{\psi}_-\right) \\
&\qquad + |z|^{\mp\frac{1}{2}}\overline{\psi}_-\left(\frac{R}{2r} + \mathrm{i}\sigma\right) - |z|^{\pm\frac{1}{2}}\overline{\psi}_+\left(\frac{R}{2r} + \mathrm{i}\overline{\sigma}\right) - \overline{\phi}\left(|z|^{\mp\frac{1}{2}}\lambda_- + |z|^{\pm\frac{1}{2}}\lambda_+\right)\Bigg].
\end{aligned} \tag{C.1}$$

The transformation of the vector multiplet fields is

$$\begin{aligned}
&\delta v_\tau = \mp\frac{|x_{1,2}|}{2}\varepsilon\left(|z|^{\pm\frac{1}{2}}\overline{\lambda}_- - |z|^{\mp\frac{1}{2}}\overline{\lambda}_+\right) \mp \frac{|x_{1,2}|}{2}\overline{\varepsilon}\left(|z|^{\mp\frac{1}{2}}\lambda_- - |z|^{\pm\frac{1}{2}}\lambda_+\right), \\
&\delta v_\rho = \pm\mathrm{i}\frac{|x_{1,2}|}{2}\varepsilon\left(|z|^{\pm\frac{1}{2}}\overline{\lambda}_- + |z|^{\mp\frac{1}{2}}\overline{\lambda}_+\right) \pm \mathrm{i}\frac{|x_{1,2}|}{2}\overline{\varepsilon}\left(|z|^{\mp\frac{1}{2}}\lambda_- + |z|^{\pm\frac{1}{2}}\lambda_+\right), \\
&\delta\sigma_1 = -\frac{\mathrm{i}}{2}\varepsilon(\overline{\lambda}'_- - \overline{\lambda}'_+) - \frac{\mathrm{i}}{2}\overline{\varepsilon}\left(\lambda'_- - \lambda'_+\right),
\end{aligned} \tag{C.2}$$

$$
\begin{aligned}
\delta\sigma_2 &= \frac{1}{2}\varepsilon(\overline{\lambda}'_- + \overline{\lambda}'_+) - \frac{1}{2}\overline{\varepsilon}\left(\lambda'_- + \lambda'_+\right),\\
\delta(\lambda'_- - \lambda'_+) &= 2\varepsilon\left[\mathrm{i}D_\tau\sigma_1 - \mathrm{i}D_\rho\sigma_2 \pm \left(D_E + \frac{\sigma_1}{r}\right) + \mathrm{i}x_3\left(\frac{v_{12}}{\sqrt{g}} + \frac{\sigma_2}{r} + \frac{1}{2}[\sigma,\overline{\sigma}]\right)\right],\\
\delta(\overline{\lambda}'_- - \overline{\lambda}'_+) &= 2\overline{\varepsilon}\left[\mathrm{i}D_\tau\sigma_1 + \mathrm{i}D_\rho\sigma_2 \mp \left(D_E + \frac{\sigma_1}{r}\right) - \mathrm{i}x_3\left(\frac{v_{12}}{\sqrt{g}} + \frac{\sigma_2}{r} - \frac{1}{2}[\sigma,\overline{\sigma}]\right)\right],\\
\delta(\lambda'_- + \lambda'_+) &= 2\varepsilon\left[-D_\rho\sigma_1 - D_\tau\sigma_2 + x_3\left(D_E + \frac{\sigma_1}{r}\right) \pm \mathrm{i}\left(\frac{v_{12}}{\sqrt{g}} + \frac{\sigma_2}{r} + \frac{1}{2}[\sigma,\overline{\sigma}]\right)\right],\\
\delta(\overline{\lambda}'_- + \overline{\lambda}'_+) &= 2\overline{\varepsilon}\left[-D_\rho\sigma_1 + D_\tau\sigma_2 + x_3\left(D_E + \frac{\sigma_1}{r}\right) \pm \mathrm{i}\left(\frac{v_{12}}{\sqrt{g}} + \frac{\sigma_2}{r} - \frac{1}{2}[\sigma,\overline{\sigma}]\right)\right],\\
\delta D_E &= \frac{\mathrm{i}}{r|x_{1,2}|}\left\{\mp\overline{\varepsilon}\left(|z|^{\pm\frac{1}{2}}zD_z\lambda_+ + |z|^{\mp\frac{1}{2}}\overline{z}D_{\overline{z}}\lambda_-\right) \pm \varepsilon\left(|z|^{\mp\frac{1}{2}}zD_z\overline{\lambda}_+ + |z|^{\pm\frac{1}{2}}\overline{z}D_{\overline{z}}\overline{\lambda}_-\right)\right\}\\
&\quad + \frac{\mathrm{i}}{2r}\left\{\varepsilon(\overline{\lambda}'_- - \overline{\lambda}'_+) + \overline{\varepsilon}(\lambda'_- - \lambda'_+)\right\}\\
&\quad + \frac{1}{2}\left[\sigma_1, \varepsilon(\overline{\lambda}'_- - \overline{\lambda}'_+) - \overline{\varepsilon}(\lambda'_- - \lambda'_+)\right] + \frac{\mathrm{i}}{2}\left[\sigma_2, \varepsilon(\overline{\lambda}'_- + \overline{\lambda}'_+) + \overline{\varepsilon}(\lambda'_- + \lambda'_+)\right],\\
\delta\frac{v_{12}}{\sqrt{g}} &= \frac{1}{r|x_{1,2}|}\left\{\pm\overline{\varepsilon}\left(|z|^{\pm\frac{1}{2}}zD_z\lambda_+ - |z|^{\mp\frac{1}{2}}\overline{z}D_{\overline{z}}\lambda_-\right) \pm \varepsilon\left(|z|^{\mp\frac{1}{2}}zD_z\overline{\lambda}_+ - |z|^{\pm\frac{1}{2}}\overline{z}D_{\overline{z}}\overline{\lambda}_-\right)\right\}\\
&\quad + \frac{1}{2r}\left\{\varepsilon(-\overline{\lambda}'_- - \overline{\lambda}'_+) + \overline{\varepsilon}(\lambda'_- + \lambda'_+)\right\}.
\end{aligned}
$$

If we set $|z| = 1$, the above transformation rule simplifies. The rule (C.1) for the chiral multiplet just becomes (3.47). For the vector multiplet, we write $\sigma^a = \sigma_a$ for $a = 1, 2$ and introduce

$$
\begin{aligned}
&\lambda^1 = \frac{\mathrm{i}}{2}(\lambda_- - \lambda_+) - \frac{\mathrm{i}}{2}(\overline{\lambda}_- - \overline{\lambda}_+), \quad \lambda^2 = \frac{1}{2}(\lambda_- + \lambda_+) + \frac{1}{2}(\overline{\lambda}_- + \overline{\lambda}_+),\\
&\widetilde{\lambda}^1 = -\frac{\mathrm{i}}{2}(\lambda_- + \lambda_+) + \frac{\mathrm{i}}{2}(\overline{\lambda}_- + \overline{\lambda}_+), \quad \widetilde{\lambda}^2 = -\frac{1}{2}(\lambda_- - \lambda_+) - \frac{1}{2}(\overline{\lambda}_- - \overline{\lambda}_+),\\
&D^1_0 = \mp\left(\frac{v_{12}}{\sqrt{g}} + \frac{\sigma_2}{r}\right), \quad D^2_0 = \pm\left(D_E + \frac{\sigma_1}{r}\right),
\end{aligned} \tag{C.3}
$$

and $N^a = D_\rho\sigma^a + \mathrm{i}D^a_0$. We also use ε_1 and ε_2 given by $\varepsilon = \mathrm{i}\varepsilon_1 - \varepsilon_2$ and $\overline{\varepsilon} = -\mathrm{i}\varepsilon_1 - \varepsilon_2$. Then a part of (C.2) at $|z| = 1$ is written as (3.48).

References

[1] E. Witten, "Constraints on Supersymmetry Breaking," Nucl. Phys. B **202** (1982) 253.

[2] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824 [hep-th]].

[3] F. Benini and S. Cremonesi, "Partition functions of N=(2,2) gauge theories on S^2 and vortices," arXiv:1206.2356 [hep-th].

[4] N. Doroud, J. Gomis, B. Le Floch and S. Lee, "Exact Results in D=2 Supersymmetric Gauge Theories," JHEP **1305** (2013) 093 [arXiv:1206.2606 [hep-th]].

[5] H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison and M. Romo, "Two-Sphere Partition Functions and Gromov-Witten Invariants," arXiv:1208.6244 [hep-th].

[6] J. Walcher, "Stability of Landau-Ginzburg branes," J. Math. Phys. **46** (2005) 082305 [hep-th/0412274].

[7] S. Hosono, "Central charges, symplectic forms, and hypergeometric series in local mirro r symmetry," hep-th/0404043.

[8] H. Iritani, "An integral structure in quantum cohomology and mirror symmetry for toric o rbifolds," Adv. Math. **222** (2009), no.3, 1016-1079 arXiv:0903.1463 [math.AG].

[9] L. Katzarkov, M. Kontsevich and T. Pantev, "Hodge theoretic aspects of mirror symmetry," arXiv:0806.0107 [math.AG].

[10] H. Iritani, "Quantum Cohomology and Periods," arXiv:1101.4512 [math.AG].

[11] M. Herbst, K. Hori and D. Page, "Phases Of N=2 Theories In 1+1 Dimensions With Boundary," arXiv:0803.2045 [hep-th].

[12] K. Hori and C. Vafa, "Mirror symmetry," hep-th/0002222.

[13] K. Hori, A. Iqbal and C. Vafa, "D-branes and mirror symmetry," hep-th/0005247.

[14] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, (Princeton Univ. Press, 1992).

[15] E. Witten, "Phases of N=2 theories in two-dimensions," Nucl. Phys. B **403** (1993) 159 [hep-th/9301042].

[16] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *Mirror Symmetry*, (AMS/Clay Math. Inst., 2003).

[17] W. Lerche, C. Vafa and N. P. Warner, "Chiral Rings in N=2 Superconformal Theories," Nucl. Phys. B **324** (1989) 427.

[18] H. Ooguri, Y. Oz and Z. Yin, "D-branes on Calabi-Yau spaces and their mirrors," Nucl. Phys. B **477** (1996) 407 [hep-th/9606112].

[19] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory," Nucl. Phys. B **241** (1984) 333.

[20] S. Cecotti and C. Vafa, "Topological antitopological fusion," Nucl. Phys. B **367** (1991) 359.

[21] J. Gomis and S. Lee, "Exact Kahler Potential from Gauge Theory and Mirror Symmetry," JHEP **1304** (2013) 019 [arXiv:1210.6022 [hep-th]].

[22] N. P. Warner, "Supersymmetry in boundary integrable models," Nucl. Phys. B **450** (1995) 663 [hep-th/9506064].

[23] A. Kapustin and Y. Li, "D branes in Landau-Ginzburg models and algebraic geometry," JHEP **0312** (2003) 005 [hep-th/0210296].

[24] I. Brunner, M. Herbst, W. Lerche and B. Scheuner, "Landau-Ginzburg realization of open string TFT," JHEP **0611** (2006) 043 [hep-th/0305133].

[25] K. Hori and J. Walcher, "D-branes from matrix factorizations," Comptes Rendus Physique **5** (2004) 1061 [hep-th/0409204].

[26] L. Pontrjagin, *Topological Groups*, (Princeton Univ. Press, 1939).

[27] M. E. Rose, *Elementary Theory of Angular Momentum*, (John Wiley & Sons, 1957).

[28] S. Lee, talks at *Geometry and Physics of the Gauged Linear Sigma Model*, Univ. Michigan, March 4-8, 2013 and at *Strings 2013*, Seoul, June 24-28, 2013.

[29] Private communication with J. Gomis, March 1-3, 2013.

[30] A. M. Polyakov, "Quantum Geometry of Bosonic Strings," Phys. Lett. B **103** (1981) 207.

[31] E. J. Martinec, "Algebraic Geometry and Effective Lagrangians," Phys. Lett. B **217** (1989) 431.

[32] C. Vafa and N. P. Warner, "Catastrophes and the Classification of Conformal Theories," Phys. Lett. B **218** (1989) 51.

[33] E. Silverstein and E. Witten, "Global U(1) R symmetry and conformal invariance of (0,2) models," Phys. Lett. B **328** (1994) 307 [hep-th/9403054].

[34] K. Hori and D. Tong, "Aspects of Non-Abelian Gauge Dynamics in Two-Dimensional N=(2,2) Theories," JHEP **0705** (2007) 079 [hep-th/0609032].

[35] K. Hori, C.Y. Park and Y. Tachikawa, "2d SCFT from M2-branes" to appear.

[36] K. Hori, "Duality In Two-Dimensional (2,2) Supersymmetric Non-Abelian Gauge Theories," arXiv:1104.2853 [hep-th].

[37] E. Witten, "The Verlinde algebra and the cohomology of the Grassmannian," In *Cambridge 1993, Geometry, topology, and physics* 357-422 [hep-th/9312104].

[38] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, "A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," Nucl. Phys. B **359** (1991) 21.

[39] D. R. Morrison and M. R. Plesser, "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties," Nucl. Phys. B **440** (1995) 279 [hep-th/9412236].

[40] E.A. Rødland, "The Pfaffian Calabi-Yau, its mirror, and their link to the Grassmannian $G(2,7)$," Composito Math. bf 122 (2000) 135-149; arXiv:math/9801092.

[41] W. Donovan and E. Segal, "Window shifts, flop equivalences and Grassmannian twists," arXiv:1206.0219 [math.AG].

[42] A. Recknagel and V. Schomerus, "D-branes in Gepner models," Nucl. Phys. B **531** (1998) 185 [hep-th/9712186].

[43] E. Segal, "Equivalences between GIT quotients of Landau-Ginzburg B-models," Commun. Math. Phys.**304** (2011) 411-432.

[44] D. Halpern-Leistner, "The derived category of a GIT quotient," arXiv:1203.0276 [math.AG].

[45] M. Ballard, D. Favero and L. Katzarkov, "Variation of geometric invariant theory quotients and derived categories," arXiv:1203.6643 [math.AG]

[46] D. Orlov, "Derived categories of coherent sheaves and triangulated categories of singularities," Progress. Math. **270** (2009) 503-531; [arXiv:math/0503632].

[47] A. Strominger, "Massless black holes and conifolds in string theory," Nucl. Phys. B **451**, 96 (1995) [arXiv:hep-th/9504090].

[48] P. Seidel and R. Thomas, "Braid group actions on derived categories of coherent sheaves," arXiv:math/0001043; Duke Math. Jour. **108** (2001) 37–108.

[49] R. P. Horja, "Hypergeometric functions and Mirror Symmetry in Toric Varieties," [arXiv:math.AG/9912109].

[50] D. Halpern-Leistner and I. Shipman, "Autoequivalences of derived categories via geometric invariant theory," arXiv:1303.5531 [math.AG].

[51] F. Hirzebruch, *Topological methods in algebraic geometry*, (Springer 1966).

[52] H. Knörrer, "Cohen-Macaulay modules on hypersurface singularities. I" Invent. Math. **88** (1987) 153-164.

[53] Zhdanov, O. N., and A. K. Tsikh, "Studying the multiple Mellin-Barnes integrals by means of multidimensional re sidues," Siberian Mathematical Journal **39**, 2 (1998): 245-260.

[54] M. Passare, A. K. Tsikh and A. A. Cheshel, "Multiple Mellin-Barnes integrals as periods of Calabi-Yau manifolds with se veral moduli," Theor. Math. Phys. **109**, 1544 (1997) [Teor. Mat. Fiz. **109N3**, 381 (1996)] [hep-th/9609215].

[55] L.C. Jeffrey and F.C. Kirwan, "Localization for nonabelian group actions," Topology **34** (1995) 291-327; [arXiv:alg-geom/9307001].

[56] F. Benini, R. Eager, K. Hori and Y. Tachikawa, "Elliptic genera of 2d $\mathcal{N} = 2$ gauge theories," to appear.

[57] F. Benini, R. Eager, K. Hori and Y. Tachikawa, "Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups," arXiv:1305.0533 [hep-th].