Brane transport in anomalous $(2,2)$ models and localization

Joel Clingempeel[1†] Bruno Le Floch[2‡] and Mauricio Romo[3*]

† Department of Mathematics, Rutgers University, Hill Center - Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

‡ Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544, USA

‡ Philippe Meyer Institute, Physics Department, École Normale Supérieure, PSL Research University, 24 rue Lhomond, F-75231 Paris Cedex 05, France

* School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA

* Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China

arXiv:1811.12385v1 [hep-th] 29 Nov 2018

Abstract

We study B-branes in two-dimensional $\mathcal{N} = (2,2)$ anomalous models, and their behaviour as we vary bulk parameters in the quantum Kähler moduli space. We focus on the case of $(2,2)$ theories defined by abelian gauged linear sigma models (GLSM). We use the hemisphere partition function as a guide to find how B-branes split in the IR into components supported on Higgs, mixed and Coulomb branches: this generalizes the band restriction rule of Herbst–Hori–Page to anomalous models.

As a central example, we work out in detail the case of GLSMs for Hirzebruch-Jung resolutions of cyclic surface singularities. In these non-compact models we explain how to compute and regularize the hemisphere partition function for a brane with compact support, and check that its Higgs branch component explicitly matches with the geometric central charge of an object in the derived category.

[1] jac634@math.rutgers.edu
[2] lefloch@lpt.ens.fr
[3] mromoj@ias.edu

Contents

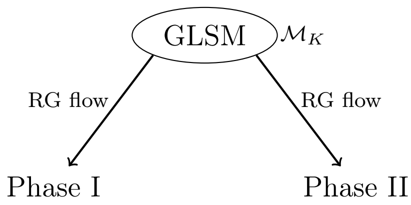
1 Introduction

The study of the dynamics of gauged linear sigma models (GLSMs) [1] has been a continuous source of new results in physics and mathematics. GLSMs are two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories that can describe the world-sheet theories of strings propagating on certain space-times. In this context, boundary conditions of GLSMs can describe D-branes.

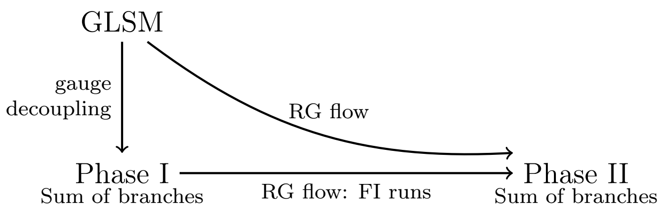
Classically, GLSMs have left and right $U(1)$ R-symmetries, or equivalently vector $U(1)_V$ and axial $U(1)_A$ R-symmetries, and $U(1)_V$ can only be broken by the superpotential term. For appropriate superpotentials W that are quasi-homogeneous under $U(1)_V$, non-renormalization theorems ensures that the quantum theory has this symmetry too. Coefficients of F-terms (superpotentials) are protected by supersymmetry, which makes them invariant under the RG flow. On the other hand, $U(1)_A$ may fail to be a symmetry of the quantum theory due to an anomaly, in which case we refer to the theory as an anomalous GLSM. Explicitly, if the matter content of the GLSM transforms in a representation $\rho\colon G \to GL(V)$ of the gauge group G, the anomaly is proportional to the weight b of the character $\det(\rho)\colon G \to \mathbb{C}^*$. The theory is thus non-anomalous when ρ factors through $SL(V)$. The same weight b controls the renormalization of twisted F-terms in the action: the FI-theta coefficient $t = \zeta_{(\mathrm{FI})} - i\theta$ receives a 1-loop correction proportional to $b \log \mu$, where μ is the energy scale. This renormalization is important to understand the IR dynamics of anomalous GLSMs.

In a GLSM, we can consider boundary conditions that preserve a particular subset of the supersymmetries. We will consider those that preserve the B-type supersymmetry algebra $\mathbf{2}_B$ generated by supercharges that have charge $+1$ under $U(1)_A$. Boundary conditions invariant under this subalgebra of the $(2, 2)$ supersymmetry algebra are termed B-branes. While B-branes have been studied in detail in superconformal field theories, the same definition applies to any $\mathcal{N} = (2, 2)$ theory. In the context of GLSMs [2], B-branes are known to form a category that can be defined and studied mathematically [3, 4, 5]. One of the main insights of [2] was to define transport of B-branes as one varies the twisted chiral parameters t. These parameters belong to the so called quantum Kähler moduli space $\mathcal{M}_K$, and in the abelian non-anomalous cases studied in [2] this fact is exploited to find equivalences between categories of B-branes, which may have different descriptions on the various Kähler cones in $\mathcal{M}_K$. B-brane transport functors found in this way have been extended to nonabelian non-anomalous GLSMs [6, 7] by the third-named author, and an equivalent approach to these functors has been considered recently in the mathematical literature [8, 9, 10], for the same classes of models.

The anomalous case, however, has been explored much less using this approach. One of the main differences is that one cannot be oblivious to the renormalization of the FI-theta parameters. In the non-anomalous case, at fixed W, each point in $\mathcal{M}_K$ corresponds to a $\mathcal{N} = (2, 2)$ SCFT defined by RG-flow of the GLSM. The SCFTs typically have concrete descriptions in different Kähler cones inside $\mathcal{M}_K$, schematically as follows.



For an anomalous GLSM there are two interesting limits: the RG flow decouples gauge degrees of freedom and moves the FI-theta parameter deep in specific Kähler cones, while another limit is to decouple gauge degrees of freedom while keeping the FI-theta parameter $t(\mu)$ fixed at some fixed finite energy scale μ. The second limit (which we call gauge-decoupling limit) is more general than the first one (IR limit under RG flow), as it allows the FI-theta parameter to explore arbitrary Kähler cones. Deep in a Kähler cone, an anomalous GLSM can have disjoint branches, for example a Higgs branch and some massive vacua. Its IR limit is described by a direct sum of one possibly trivial SCFT for each of these branches. Schematically, the situation can be as follows.



Each point in the classical space $\mathcal{M}_K^{\text{bare}}$ of bare FI-theta parameters flows in general to one of these direct-sum phases in the gauge-decoupling or IR limits, so that a B-brane can split into components on the Higgs branch and other branches. In terms of B-brane categories of Higgs branches in different Kähler cones, this means that one should at best expect an embedding of one into the other instead of an equivalence. We make this more precise for abelian GLSM dynamics in section 6.

We use as our central example Hirzebruch-Jung resolutions of singularities of the form $\mathbb{C}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts diagonally with weights $(1, p)$. The singularity is Gorenstein only if $p = -1$, and otherwise the minimal resolution is always non-crepant. This is reflected in the fact that the GLSMs we use to study these singularities and their (partial) resolutions are anomalous for $p \neq -1$. In these models, one should ask how B-branes in the $\mathbb{Z}_n$ orbifold phase are transported to B-branes in the various partial resolutions, and what is the map between them. For resolution of quotient singularities this has been studied previously in the math literature. For instance for quotients of $\mathbb{C}^2$ by finite subgroups of $GL(2, \mathbb{C})$ [11][1]. In physics,

[1]Derived categories on quotients of $\mathbb{C}^3$ by finite subgroups of $GL(3, \mathbb{C})$ and the relation with their resolutions have been studied in [12] and for certain cyclic quotients of projective varieties in [13].

the K-theory, i.e., the lattice of charges of B-branes, and the chiral rings have been studied using a GLSM approach for the different resolutions of $\mathbb{C}^2/\mathbb{Z}_n$ singularities [14, 15, 16] and for other nonsupersymmetric orbifolds [17, 18, 19, 20, 21].

The mathematical references [11, 12, 13] deals with the map between categories while references [14, 15, 16, 17, 18, 19, 20, 21] mainly focus on the projection to K-theory. In the former case the picture of transport of B-branes along the moduli space is lost and in the latter case, the analysis is limited only to K-theory. We would like to unify these two approaches using the GLSM and recent results on supersymmetric localization as a guide. B-branes on anomalous models have been studied in the context of mirror symmetry of massive theories [22] and the interpretation in terms of flows in the moduli space in [23] for Fano and general type hypersurfaces in $\mathbb{P}^N$.[2] We want to unify [14, 15, 16] with the categorical approach of [3, 4, 5, 2]. The GLSM approach and the localization formula of [23] provide us with the perfect setup for this purpose. One can say that we are making modest steps into extending the work of [2] to anomalous models.

We use the definition of B-branes on the GLSM that is given by two pieces of data, the algebraic one $\mathcal{B}$ and the contour L (see subsection 4.1). By a careful study of the admissible contour L behaviour, as we move in $\mathcal{M}_K^{\mathrm{bare}}$ and the energy scale, we are able to tell how the B-branes $(\mathcal{B}, L)$ are mapped into the different mixed phases and hence, derive their splitting into Coulomb and Higgs components.

The hemisphere partition function computed in [23] has been conjectured to reproduce the geometric central charge [27, 28] with all its instanton corrections, for geometric phases corresponding to compact Calabi-Yau (CY) varieties. The conjecture can also be extended to nongeometric phases and there are some checks and evidence for it in [23, 29, 30]. For anomalous models, when the Higgs branches corresponds to a compact non-CY variety, some generalization of such conjecture have been proposed [30]. In this work, we study the anomalous case, but when the Higgs branches are non-compact toric varieties (not necessarily CY). We then face two problems: first, the fact that we are working with anomalous models and second, the necessity of develop some regularization scheme to get sensible results in the non-compact case. We apply our scheme to the Hirzebruch-Jung resolutions, and use it to compute the hemisphere partition function of B-branes corresponding to sheaves with compact support on the Higgs branch. We check to leading order, i.e., in the zero instanton sector, with the geometric central charge. This central charge is a map from (compactly supported) K-theory to $\mathbb{C}$, whose computation requires the machinery of differential topology on toric varieties. We find perfect agreement.

The paper is organized as follows. In section 2 we review the basics of abelian anomalous GLSM and we perform a very careful analysis on how the mixed phases arise in the different Kähler cones. In section 3 we review the necessary background on Hirzebruch-Jung resolutions and their corresponding GLSMs. We apply results of the previous section to these models in order to have a complete picture of their phase structure. In section 4 we analyze the image of the GLSM B-branes $(\mathcal{B}, L)$ into the different phases, focusing on their projection on the Higgs branch and working mostly in Hirzebruch-Jung models. We also review the hemisphere partition function (HPF) of GLSMs, that can be seen as the central charge of $(\mathcal{B}, L)$, and compute it for several classes of branes in Hirzebruch-Jung models, when some restrictions apply. In section 5 we review the necessary machinery of K-theory and cohomology of toric

[2]In mathematics similar cases have been analyzed in [24, 25, 26], but mostly limited to Fano varieties or general type hypersurfaces in $\mathbb{P}^N$.

varieties to define the geometrical central charge of B-branes on the different Higgs branches and compare it to the HPF computed in section 4. Finally, in section 6 we turn to the question on how to map (transport) B-branes between the different phases. This is done by studying the contours of $(\mathcal{B}, L)$ as we vary the Kähler parameters and we derive this way a version of the grade restriction rule [2] for Hirzebruch-Jung models. Our derived rule is analogous to the ones in [23] and [5].

2 Branches of abelian gauged linear sigma models

In this section we review the general properties of gauged linear sigma models (GLSM). We define a GLSM by specifying the following data.

- **Gauge group**: a compact Lie group G.
- **Chiral matter fields**: a faithful unitary representation $\rho_m\colon G \to U(V)$ of G on some complex vector space V.
- **Superpotential**: a holomorphic, G-invariant polynomial $W\colon V \to \mathbb{C}$, namely $W \in \mathrm{Sym}(V^*)^G$.
- **FI-theta parameters**, or **stringy Kähler moduli**: a set of complex parameters t such that $\exp(t) \in \mathrm{Hom}(\pi_1(G), \mathbb{C}^*)^{\pi_0(G)}$ i.e., $\exp(t)$ is a group homomorphism from $\pi_1(G)$ to $\mathbb{C}^*$ that is invariant under the adjoint action of G.
- **R-symmetry**: a vector $U(1)_V$ and axial $U(1)_A$ R-symmetries that commute with the action of G on V. To preserve the $U(1)_V$ symmetry the superpotential must have weight 2 under it. As we explain below, $U(1)_A$ is anomalous in general.
- **Twisted masses**: an element of the Cartan algebra of the flavour symmetry group F. This group is the quotient by G of the normalizer of $G \times U(1)_V \times U(1)_A$ in $U(V)$.

In this paper we only consider **abelian GLSMs**, namely an abelian gauge group $G \simeq U(1)^r \times \Gamma$ with Γ a finite abelian group. Since we want non-compact Higgs branches, we also restrict to cases with **zero superpotential** and **zero twisted masses**. There are no discrete θ angles because $\pi_1(G) \simeq \mathbb{Z}^r$ has no torsion. Choosing a basis of $\mathfrak{g} = \mathrm{Lie}(G)$ we can write coordinates of t as $t_\alpha = \zeta_\alpha - i\theta_\alpha \in \mathbb{C}/2\pi i\mathbb{Z}$ for $1 \leq \alpha \leq r$. The action of $U(1)^r$ on V is characterized by a **charge matrix** with integer entries Q^j_α, where $1 \leq j \leq \dim V$ is a flavour index and $1 \leq \alpha \leq r$ is a gauge index.

We often take Γ trivial. Otherwise the charges of each chiral multiplet under Γ must also be specified, and the theory is an orbifold by Γ of the theory with gauge group $U(1)^r$.

2.1 Classical phases [review]

Vacua are thus common solutions of the **mass**, **D-term**, and **F-term** equations

$$\sum_{\alpha=1}^{r} \sigma^\alpha Q^i_\alpha X_i = 0 \quad \forall i, \qquad \sum_{i=1}^{\dim V} Q^i_\alpha |X_i|^2 = \zeta_\alpha \quad \forall \alpha, \qquad \frac{\partial W}{\partial X_i} = 0 \quad \forall i, \tag{2.1}$$

modulo gauge transformations, namely G acting on the X_i. Here, σ^α are vector multiplet scalars and X_i are chiral multiplet scalars and we sometimes denote the chiral multiplet itself in the same way.

Let us introduce some notation. We denote the set of non-negative linear combinations of the Q^i for i in some subset $I \subset [\![1, \dim V]\!]$ by

$$\mathrm{Cone}_I = \left\{ \sum_{i\in I} \lambda_i Q^i \;\middle|\; \lambda_i \in \mathbb{R}_{\geq 0} \quad \forall i \in I \right\}. \tag{2.2}$$

Each set I of $r-1$ linearly independent charge vectors Q^i defines a codimension 1 **wall** (phase boundary) Cone_I in FI parameter space. The complement of the union of all walls is typically disconnected, and each connected component is called a (classical) **phase** of the GLSM.

The D-term equation expresses ζ as a non-negative linear combination of charge vectors Q^i. When ζ does not belong to a wall, the charge vectors with non-zero coefficient necessarily span $\mathbb{Z}^r$, hence the mass equations for the corresponding $X_i \neq 0$ impose linear constraints on σ that set all $\sigma^\alpha = 0$. In addition, the non-zero X_i Higgs the gauge group down to a (possibly trivial) discrete subgroup because they are not fixed by any infinitesimal gauge transformation. This set of vacua is called the **Higgs branch** ($\sigma = 0$, $X \neq 0$). Within a phase, the possible sets of non-zero X_i do not change, and only the magnitudes of various $|X_i|^2$ are affected by the precise values of ζ_α.

We focus on the case of a zero superpotential: $W = 0$.

Then the Higgs branch is a GIT (geometric invariant theory) quotient. As a complex manifold or orbifold it is a complex quotient $(V \setminus \Delta)/G_{\mathbb{C}}$ where $G_{\mathbb{C}} = (\mathbb{C}^*)^r \times \Gamma$. The **deleted set** Δ is a union of complex subspaces of V that are intersections of hyperplanes $\{X_i = 0\}$. This set and its complement are:

$$\Delta = \bigcap_{I|\zeta\in\mathrm{Cone}_I} \bigcup_{i\in I} \{X \mid X_i = 0\} = \bigcup_{I|\zeta\notin\mathrm{Cone}_I} \{X \mid \forall i \in \complement I, X_i = 0\}, \tag{2.3}$$

$$V \setminus \Delta = \bigcup_{I|\zeta\in\mathrm{Cone}_I} \{X \mid \forall i \in I, X_i \neq 0\}. \tag{2.4}$$

Since every Cone_I that contains ζ is a union of phases and phase boundaries, the deleted set, hence the complex manifold or orbifold, only depends on the phase in which ζ is. (The Kähler structure of the Higgs branch depends on ζ even within a phase.) When ζ crosses a phase boundary, the Higgs branch typically undergoes a change of toplogy called **flop**. The Higgs branch may even be empty in some phases.

For ζ on a wall there are solutions of (2.1) where only $r-1$ chiral multiplet scalars X_i are non-zero. The mass equation then allows σ to take a non-zero value transverse to the wall. Such a branch of vacua with $\sigma \neq 0$ and $X \neq 0$ is called a **mixed branch**. It opens up at walls in FI parameter space, and at intersections of walls there are further mixed branches in which σ can vary in a higher-dimensional subspace of $\mathbb{R}^r$, culminating in a **Coulomb branch** ($X = 0$, σ arbitrary) at $\zeta = 0$. Therefore classically one expects the theory to be singular whenever ζ belongs to any wall.

2.2 Quantum effects [review]

The classical phases get corrected in several ways by quantum effects.

As a warm-up to understand one of the energy scales involved, we consider Higgs branches. Classically, for ζ not in a wall, solutions of the D-term equations (Higgs branch vacua) are such that the charge vectors Q^i of non-zero chirals X_i span $\mathbb{C}^r$, and the mass equation sets $\sigma = 0$. The quantum version is that these chirals get vevs (vacuum expectation values) $\langle X_i \rangle$ which make all σ massive, and fluctuations of X transverse to the Higgs branch are also massive. We now show that both masses are of order $e|\zeta|^{1/2}$ for ζ deep in a phase, where e is the gauge coupling, of mass dimension 1. It is useful to display the classical potential

$$U = \sum_{i=1}^{\dim V} |Q^i \cdot \sigma|^2 |X_i|^2 + \frac{e^2}{2} \sum_{\alpha=1}^{r} \biggl(\zeta_\alpha - \sum_{i=1}^{\dim V} Q^i_\alpha |X_i|^2 \biggr)^2 + \sum_{i=1}^{\dim V} \biggl| \frac{\partial W}{\partial X_i} \biggr|^2 \tag{2.5}$$

in which $Q^i \cdot \sigma = \sum_\alpha Q^i_\alpha \sigma^\alpha$ and we have already integrated out the vector multiplet's auxiliary field D. The vector multiplet scalar with a canonical kinetic term is σ^α/e, to which the first term in U schematically gives a mass $2eQ\langle X \rangle \sim e|\zeta|^{1/2}$. More precisely, the mass-squared of the scalars σ^α/e is the positive-definite matrix

$$(m^2_{\sigma,\text{eff}})_{\alpha\beta} = 4e^2 \sum_{i=1}^{\dim V} Q^i_\alpha Q^i_\beta |\langle X_i \rangle|^2. \tag{2.6}$$

For ζ deep in a phase, Higgs branch vacua are such that $|\langle X_i \rangle| \gtrsim |\zeta|^{1/2}$ for a set of indices i such that the corresponding Q^i span $\mathbb{C}^r$. Eigenvalues of the mass-squared matrix of the canonically normalized σ/e are thus all of order $e^2|\zeta|$. A similar calculation shows that non-gauge transverse fluctuations of chiral multiplets around the Higgs branch have mass-squared of order $e^2|\zeta|$ too. We conclude that at energies well below $e|\zeta|^{1/2}$ the theory is well-described by a non-linear sigma model with target the Higgs branch. A convenient way to ensure this regime is to take the limit $e \to \infty$ with all other parameters fixed, at some fixed energy scale.

To keep the formula simple we took all gauge couplings to be equal to some e. Upon a $GL(r,\mathbb{Z})$ change of basis on $\mathfrak{g}$, which is useful in concrete models, e^2 is replaced by a quadratic form on $\mathfrak{g}^* \simeq \mathbb{R}^r$, dual to the quadratic form $1/e^2$ on $\mathfrak{g}$ that appears in gauge kinetic terms. The second term in U becomes schematically $\frac{1}{2}\sum_{\alpha,\beta}(e^2)^{\alpha\beta}(\zeta_\alpha - \cdots)(\zeta_\beta - \cdots)$. As explained just above we eventually only care about the limit $e^2 \to \infty$, unaffected by such changes of basis.

The first quantum effect is that **the FI parameter is renormalized**:

$$\zeta(\mu) = \zeta_{\text{UV}} + Q^{\text{tot}} \log\Bigl(\frac{\mu}{M_{\text{UV}}}\Bigr) \tag{2.7}$$

where M_{UV} is a UV mass scale and $Q^{\text{tot}} = \sum_i Q^i$ is the $U(1)_A$ (axial R-symmetry) anomaly. Models with $Q^{\text{tot}} = 0$ are called Calabi-Yau models because their Higgs branch is a Calabi-Yau orbifold. At energies far below $e|\zeta|^{1/2}$ the gauge theory flows to a nonlinear sigma model (NLSM) with target space the Higgs branch. Further RG flow is expected to change the Kähler metric (given by the GIT construction) to one that gives a conformal NLSM.

In non-Calabi-Yau models, flowing to the IR shifts the FI parameter in the direction $-Q^{\text{tot}} \neq 0$. The deep IR limit can thus only explore some of the phases, specifically the phases whose closure contains the vector $-Q^{\text{tot}}$, interpreted as a point in FI parameter space. For example, if $-Q^{\text{tot}}$ is not parallel to any wall, then it belongs to one specific phase, and the deep IR limit is described by that phase of the GLSM regardless of ζ_{UV}. Nevertheless, for every phase we can arrange parameters so that the phase gives a good description of the physics at some intermediate energy scale μ: tune $\zeta_{\rm UV}$ so that the renormalized $\zeta(\mu)$ lies deep in the given phase, then take e sufficiently large to ensure $e|\zeta(\mu)|^{1/2} \gg \mu$.

The counterpart to the fact that FI parameter varies under scale transformations as (2.7) is that the theta angle varies by $\alpha Q^{\rm tot}$ under $e^{i\alpha} \in U(1)_A$ due to that symmetry's anomaly. These symmetries act on twisted chiral parameters such as twisted masses (vector multiplet scalars) by scaling and phase rotations. As a result, the anomalous transformation of $t = \zeta - i\theta$ can be repackaged into a dependence of all observables on a **complexified energy scale** μ. Both ζ and θ have one unphysical component that can be traded for this complexified energy scale μ, but we find it more convenient to take the point of view of fixing μ and keeping the $Q^{\rm tot}$ component of t.

A second quantum effect makes some classical walls (**Calabi-Yau walls**) into **complex codimension** 1 singular loci in the FI-theta parameter space while others correspond to **no wall** quantum mechanically. A classical wall is a real codimension 1 cone in the FI parameter space, where a mixed (or Coulomb) branch opens up. Let u_0 be some nonzero vector orthogonal to that wall. In this branch, $\sigma = \sigma_0 u_0$ can be an arbitrary multiple of u_0. There may then be vacua whose wave-function explores large values of σ_0. Such large values give mass to all chirals for which $Q^i \cdot u_0 \neq 0$, and integrating these out gives an effective action for σ_0 that is given by the twisted superpotential

$$\widetilde{W}_{\rm eff} = -(t(\mu)\cdot u_0)\sigma_0 - \sum_{i|Q^i\cdot u_0\neq 0} (Q^i\cdot u_0)\sigma_0 \left(\log\left(\frac{(Q^i\cdot u_0)\sigma_0}{\mu}\right) - 1\right), \tag{2.8}$$

which is actually μ-independent due to (2.7). Vacua are critical points of this twisted superpotential, namely solutions of

$$t(\mu)\cdot u_0 = -(Q^{\rm tot}\cdot u_0)\log\left(\frac{\sigma_0}{\mu}\right) - \sum_{i|Q^i\cdot u_0\neq 0} (Q^i\cdot u_0)\log(Q^i\cdot u_0). \tag{2.9}$$

The ambiguity of log by $2\pi i$ shifts has no effect since t is $2\pi i$ periodic and all $Q^i \in \mathbb{Z}$. Then we have two very different cases: the classical walls parallel to $Q^{\rm tot}$ correspond to walls in the quantum theory (up to some shift), while others are not quantum walls.

- Calabi-Yau walls are those for which $Q^{\rm tot}\cdot u_0 = 0$, that is, $Q^{\rm tot}$ is parallel to the wall. Then there is a whole mixed (or Coulomb) branch of vacua at a specific locus $t\cdot u_0 = -\sum_i (Q^i\cdot u_0)\log(Q^i\cdot u_0)$ in the FI-theta parameter space. Quantum effects thus shift the wall away from $\zeta\cdot u_0 = 0$ and give it complex (rather than real) codimension 1.[3]

  More precisely, our analysis is valid infinitely deep in the wall. In Calabi-Yau models, we typically have a collection of loci that asymptote to this wall and at which the theory is singular due to a non-compact Coulomb branch opening up. In some non-Calabi-Yau models, the Coulomb branch that opens up has a finite size controlled by the FI parameters along the wall. Then, the theory has no singular locus near the classical wall. We plan to explore this subtle issue in the future.

[3] A more pedestrian point of view on the lack of singularity for $\theta\cdot u_0 \neq 0 \bmod 2\pi$ is that the Lagrangian includes terms $\frac{1}{2e^2}E^2 - \frac{i}{2\pi}(\theta\cdot u_0)E$ where E is the electric field $\partial_1 A_2 - \partial_2 A_1$ in the direction u_0. Completing the square and taking into account quantization of E gives an energy contribution proportional to $\min(\theta\cdot u_0 - 2\pi\mathbb{Z})^2$, which prevents the singularity except at $\theta\cdot u_0 = 0 \bmod 2\pi$.

- If $Q^{\mathrm{tot}} \cdot u_0 \neq 0$, there are $|Q^{\mathrm{tot}} \cdot u_0|$ solutions at

$$\sigma_0 \simeq \mu \exp \frac{-t(\mu) \cdot u_0 + 2\pi i k}{Q^{\mathrm{tot}} \cdot u_0}, \qquad k = 0, \dots, |Q^{\mathrm{tot}} \cdot u_0| - 1. \tag{2.10}$$

The approximation used to derive these is good at energies well below $|\sigma|$, namely provided $(t(\mu) \cdot u_0)/(Q^{\mathrm{tot}} \cdot u_0) \ll 0$. (The phase to which the model flows obeys this for any UV FI parameter.) In other words these solutions should be ignored in the other phase $(t(\mu) \cdot u_0)/(Q^{\mathrm{tot}} \cdot u_0) \gg 0$ as they merge into the $\sigma_0 = 0$ Higgs branch. Note that there is no wall between the two phases $t(\mu)/Q^{\mathrm{tot}} \ll 0$ and $t(\mu)/Q^{\mathrm{tot}} \gg 0$, even though the low-energy descriptions are quite different.

In $U(1)$ GLSMs the solutions (2.10) are isolated **quantum Coulomb vacua**. Excitations around these vacua are all massive: indeed, chiral multiplets have mass $|Q^i\sigma| \gg \mu$, while fluctuations of σ/e have a mass $e^2/|\sigma|$, which can be taken much larger than μ by choosing a sufficiently large e. This is a further condition on e besides the condition $e|\zeta|^{1/2}$ that was needed for the Higgs branch NLSM to give a good approximation.

2.3 Interlude: nonlinear twisted superpotential

Before studying in detail the possibility of mixed branches, let us consider a slight generalization of usual GLSMs. Usually, the twisted superpotential of a GLSM is taken to be linear: $\widetilde{W} = -t \cdot \sigma$ with t the FI and σ the twisted chiral field strength of the vector multiplet. We now consider a gauge theory with a more complicated unspecified $\widetilde{W}(\sigma)$. We allow the charges Q^i not to span $\mathbb{R}^r$, namely the gauge group action not to be faithful. In other words, the charge lattice $\mathbb{Z}^r = \mathrm{Hom}(U(1)^r, U(1))$ of the gauge group contains all integer linear combinations of the Q^i, but may contain more elements.

As we explain shortly, vacua in which σ gives a mass to none of the chirals (analogous to Higgs branches) are solutions of

$$(Q^i \cdot \sigma) X_i = 0, \tag{2.11}$$

$$\zeta_{\mathrm{eff}} \coloneqq -\operatorname{Re}\biggl(\frac{\partial \widetilde{W}}{\partial \sigma}\biggr) = \sum_i \bigl(Q^i |X_i|^2\bigr), \tag{2.12}$$

$$\theta_{\mathrm{eff}} \coloneqq \operatorname{Im}\biggl(\frac{\partial \widetilde{W}}{\partial \sigma}\biggr) \in \operatorname{Span}_{\mathbb{R}}\bigl(\{Q^i\}\bigr) + 2\pi \mathbb{Z}^r, \tag{2.13}$$

modulo gauge transformations. While the equations are real, this is a complex orbifold thanks to the fact that the $d = \dim(\operatorname{Span}\{Q^i\})$ "missing" constraints on $\operatorname{Im}(\partial\widetilde{W}/\partial\sigma)$ are accounted for by the $U(1)^d$ (times discrete factor) gauge transformations that act non-trivially on the chirals. These equations reduce to well-known ones when $d = 0$ or r. When there are no (charged) chiral multiplet they state that $\partial\widetilde{W}/\partial\sigma$ vanishes modulo 2π. Instead, when charges span $\mathbb{R}^r$, they reduce to the mass equation $(Q^i \cdot \sigma) X_i = 0$ and to D-term equations $\mu(X) = \zeta_{\mathrm{eff}}$, modulo gauge transformations, where $\mu = \sum_i \bigl(Q^i |X_i|^2\bigr)$ is the moment map of the gauge group.

Equations (2.11)–(2.12) come from the same classical potential as (2.5), with $\zeta \to \zeta_{\mathrm{eff}}$. The third equation is found by considering the action for the gauge field components along directions $u \in \mathfrak{g}$ such that all $Q^i \cdot u = 0$. These components only appear in the gauge kinetic term and in the twisted superpotential term:

$$L = \frac{1}{2e^2} E^2 - \frac{i}{2\pi} \bigl(\theta_{\mathrm{eff}} \cdot u\bigr)\, E, \tag{2.14}$$

where E is the electric field $\partial_1 A_2 - \partial_2 A_1$ in the direction u. This is well-known to have vacua at $\theta_{\text{eff}} = 0 \bmod 2\pi$.

2.4 Mixed branches

We now go back to a standard GLSM and generalize our earlier discussion from $U(1)$ to $U(1)^r$ models to find mixed branches. We learn that mixed branches are essentially products of Coulomb and Higgs branches. Furthermore some phases subdivide beyond the classical analysis.

In one phase of a $U(1)$ GLSM in which $Q^{\text{tot}} \neq 0$, we found quantum Coulomb branch vacua (2.10). In $U(1)^r$ models, Coulomb branch vacua ($\sigma \neq 0$, $X = 0$) are found as follows. Assume σ has a generic large vev so that all chirals are massive. Integrate out all the massive chirals to get an effective twisted superpotential $\widetilde{W}_{\text{eff}}(\sigma)$. Find classical solutions for σ (critical points of $\widetilde{W}_{\text{eff}}$). Check whether chiral multiplets in these solutions are indeed all massive, or not: if yes we found a Coulomb branch vacuum.

We follow a similar procedure to find all branches. In each branch we expect some set of chiral multiplets to be made massive by σ, and some set not to be. Let us search for vacua in which a set $I \subset [\![1, \dim V]\!]$ of flavours have $Q^i \cdot \sigma = 0$ for all $i \in I$, that is, $\sigma \in \mathfrak{q}_{\mathbb{C}}^{\perp} \subset \mathfrak{g}_{\mathbb{C}}$ where

$$\mathfrak{q} \coloneqq \operatorname{Span}_{\mathbb{R}}(\{Q^i \mid i \in I\}) \subset \mathfrak{g}^*. \tag{2.15}$$

Of course, any chiral multiplet with charge $Q^i \in \mathfrak{q}$ is given no mass by σ, so we restrict our attention without loss of generality to cases where I contains all such flavours.

Integrate out all the chirals X_i for $i \notin I$ since we expect them to be massive. The effective twisted superpotential is

$$\widetilde{W}_{\text{eff}} = -t(\mu) \cdot \sigma - \sum_{i \notin I} (Q^i \cdot \sigma) \left(\log\left(\frac{Q^i \cdot \sigma}{\mu}\right) - 1 \right), \tag{2.16}$$

and we search for solutions of (2.11), (2.12), (2.13) for the resulting gauge theory. We are only interested in solutions for which $\sigma \in \mathfrak{q}_{\mathbb{C}}^{\perp}$ so that the remaining chirals X_i for $i \in I$ are not given a mass, and for which all $X_i = 0$ for $i \notin I$ since they should be massive.

Focus now on components of (2.12) and (2.13) along $\mathfrak{q}_{\mathbb{C}}$. They give

$$\frac{\partial \widetilde{W}}{\partial \sigma} = -t - \sum_{i \notin I} Q^i \log\left(\frac{Q^i \cdot \sigma}{\mu}\right) \in \mathfrak{q}_{\mathbb{C}} + 2\pi i \mathbb{Z}^r. \tag{2.17}$$

Once these equations are solved for $\sigma \in \mathfrak{q}_{\mathbb{C}}^{\perp}$, one must check that masses are large ($|Q^i \cdot \sigma| \gg \mu$) for $i \notin I$. These are precisely the condition for quantum Coulomb branch vacua of a sub-theory, with smaller gauge Lie algebra $\mathfrak{q}^{\perp} \subset \mathfrak{g}$ and one chiral multiplet of charge ($Q^i \bmod \mathfrak{q}$) for each $i \notin I$ (one can naturally include gauge-neutral chirals for $i \in I$). Indeed, the Coulomb branch equation for this sub-theory is

$$(t \bmod \mathfrak{q}_{\mathbb{C}}) + \sum_{i \notin I} (Q^i \bmod \mathfrak{q}) \log\left(\frac{(Q^i \bmod \mathfrak{q}) \cdot \sigma}{\mu}\right) = (0 \bmod \mathfrak{q}) \tag{2.18}$$

and the condition that X_i, $i \notin I$ be massive reads $\left|(Q^i \bmod \mathfrak{q}) \cdot \sigma\right| = |Q^i \cdot \sigma| \gg \mu$. Solutions are typicaly isolated. In addition, scaling $t \to \infty$ in a fixed direction, σ can have multiple scales: for a given solution the various $Q^i \cdot \sigma$ may behave as exponentials $\exp(t \cdot \lambda^i)$ with different parameters λ^i.

Then, for each solution $\sigma \in \mathfrak{q}^\perp$, we solve the D-term equations $\sum_{i\in I} Q^i |X_i|^2 = \zeta_{\text{eff}}$ with

$$\zeta_{\text{eff}} = -\operatorname{Re}\biggl(\frac{\partial \widetilde{W}}{\partial \sigma}\biggr) = \zeta + \sum_{i\notin I} Q^i \log\biggl|\frac{Q^i \cdot \sigma}{\mu}\biggr| \in \mathfrak{q}, \tag{2.19}$$

modulo gauge transformations. Infinitesimal gauge transformations along $\mathfrak{q}^\perp$ act trivially, so this is the Higgs branch of a "quotient" theory with gauge Lie algebra $\mathfrak{q}^* = \mathfrak{g}/\mathfrak{q}^\perp$ and one chiral multiplet of charge Q^i for each $i \in I$. The Higgs branch is empty whenever ζ_{eff} cannot be written as a positive linear combination of the Q^i for $i \in I$.

The branch we found is a Coulomb branch if all X vanish, a mixed branch if both chiral and vector multiplet scalars have non-zero vev, and a Higgs branch if σ vanishes. Importantly, the dimension of the branch is

$$|I| - \dim \mathfrak{q} = \bigl\{ i \bigm| Q^i \in \mathfrak{q} \bigr\} - \dim \mathfrak{q}. \tag{2.20}$$

If the Q^i that lie in $\mathfrak{q}$ are linearly independent, this implies that the branch is an isolated vacuum. In all models we study later in the paper, only the Higgs branch has positive dimension. The remaining branches are isolated vacua, and it is then irrelevant to work out whether they are mixed or Coulomb branches.

There is an interesting phase structure upon varying t. The set of solutions to the Coulomb branch equation (2.18) can change when the solution σ is such that one mass $|Q^i \cdot \sigma|$ becomes of order μ for $i \notin I$. Then one chiral becomes massless and the vacua are described by some larger choice of I and $\mathfrak{q}$. On the other hand, the Higgs branch of the quotient theory has change of geometry when ζ_{eff} given in (2.19) changes sign. The location of this phase transition depends on all components of the FI parameter (but not the theta angle), and we find in examples that the transition takes place along a wall subdividing classical phases. See subsection 3.4 for examples.

3 Hirzebruch-Jung model

Our main example in this paper is an abelian GLSM considered in [14], whose classical Higgs branch in different phases is a certain orbifold of $\mathbb{C}^2$ and its (partial) resolutions. More precisely we consider the orbifold $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ in which the generator $\omega = \exp(2\pi i/n)$ of $\mathbb{Z}_n$ acts on $\mathbb{C}^2$ by $(z_1, z_2) \mapsto (\omega z_1, \omega^p z_2)$ for some $0 < p < n$ such that $\gcd(p, n) = 1$.[4] In addition to the classical Higgs branch the model also has isolated quantum Coulomb/mixed branch vacua in many phases.

[4]The action $(z_1, z_2) \mapsto (\omega^j z_1, \omega^k z_2)$ of $\mathbb{Z}_n$ on $\mathbb{C}^2$ is faithful if $\gcd(j, k, n) = 1$, so one may naively expect the existence of more general quotients $\mathbb{C}^2/\mathbb{Z}_{n(j,k)}$. If $\gcd(j, n) = 1 = \gcd(k, n)$ then $\mathbb{C}^2/\mathbb{Z}_{n(j,k)}$ is $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ where $pj = k \mod n$. Otherwise $\mathbb{C}^2/\mathbb{Z}_{n(j,k)}$ is actually a quotient of $(\mathbb{C}/\mathbb{Z}_{\gcd(k,n)}) \times (\mathbb{C}/\mathbb{Z}_{\gcd(j,n)})$ so we first rescale $z_1 \mapsto z_1^{\gcd(k,n)}$ and $z_2 \mapsto z_2^{\gcd(j,n)}$ (this does not affect the complex structure) before applying this argument: we obtain a quotient $\mathbb{C}^2/\mathbb{Z}_{n'(p)}$ with $n' = n/\bigl(\gcd(j, n) \gcd(k, n)\bigr)$ and $p\frac{j}{\gcd(j,n)} = \frac{k}{\gcd(k,n)} \mod n'$.

3.1 Notations

Before describing the GLSM let us introduce some notations. Like every rational number in $(1, +\infty)$ the fraction n/p has a unique continued fraction expansion

$$\frac{n}{p} = [a_1, \dots, a_r] \coloneqq a_1 - \frac{1}{a_2 - \frac{1}{\dots - 1/a_r}} \tag{3.1}$$

in terms of integers $a_\alpha \geq 2$. This also defines r.

3.1.1 Determinants

Then we consider the generalized Cartan matrix

$$(C_{\alpha\beta})_{1\leq\alpha,\beta\leq r} \coloneqq \begin{pmatrix} a_1 & -1 & & 0 \\ -1 & a_2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & a_r \end{pmatrix} \tag{3.2}$$

and its (diagonal) minors

$$\begin{aligned} d_{ij} \coloneqq -d_{ji} &\coloneqq \det\bigl(C_{\alpha\beta}\bigr)_{i<\alpha,\beta<j} && \text{for } 0 \leq i < j \leq r+1, \\ d_{ii} &\coloneqq 0 && \text{for } 0 \leq i \leq r+1. \end{aligned} \tag{3.3}$$

Here strict inequalities imply that the submatrix has size $j - i - 1$, so $d_{i(i+1)} = 1$ is the determinant of a 0×0 matrix, and we extended the notation to $i \geq j$ by antisymmetry for later convenience. These determinants obey two recursion relations

$$\begin{aligned} d_{i(j-1)} + d_{i(j+1)} &= a_j d_{ij} && \text{for } 0 \leq i \leq r+1 \text{ and } 1 \leq j \leq r, \\ d_{(i-1)j} + d_{(i+1)j} &= a_i d_{ij} && \text{for } 1 \leq i \leq r \text{ and } 0 \leq j \leq r+1, \end{aligned} \tag{3.4}$$

and they can be related to partial continued fractions through

$$\begin{aligned} [a_i, a_{i+1}, \dots, a_{j-1}] &= \frac{d_{(i-1)j}}{d_{ij}} && \text{for } 1 \leq i < j \leq r+1, \\ [a_j, a_{j-1}, \dots, a_{i+1}] &= \frac{d_{i(j+1)}}{d_{ij}} && \text{for } 0 \leq i < j \leq r. \end{aligned} \tag{3.5}$$

Note that since the continued fractions are all in $(1, +\infty)$ we learn that $d_{ij} < d_{(i-1)j}$ and $d_{ij} < d_{i(j+1)}$ for $i < j$ and these inequalities extend to all i, j by antisymmetry.

We then define $p_j = d_{j(r+1)}$ and $q_j = d_{0j}$ for $0 \leq j \leq r+1$, which can alternatively be defined as in [14] by $p_{r+1} = q_0 = 0$ and $p_r = q_1 = 1$ and

$$[a_j, a_{j+1}, \dots, a_r] = \frac{p_{j-1}}{p_j} \qquad \text{and} \qquad [a_j, a_{j-1}, \dots, a_1] = \frac{q_{j+1}}{q_j}. \tag{3.6}$$

They obey $0 = p_{r+1} < p_r < \dots < p_1 = p < p_0 = n$ and $0 = q_0 < q_1 < \dots < q_{r+1} = n$. The recursion relations (3.4) read $p_{j-1} + p_{j+1} = a_j p_j$ and $q_{j-1} + q_{j+1} = a_j q_j$ for $1 \leq j \leq r$, from which we deduce by induction that

$$p_i q_j - p_j q_i = n d_{ij} \qquad \text{for } 0 \leq i, j \leq r+1 \tag{3.7}$$

because both quantities obey the same recursion relations (3.4) and agree for $0 \le i,j \le 1$. In fact this explicit formula for d_{ij} implies (and is a special case of)

$$d_{ij}d_{kl} - d_{ik}d_{jl} + d_{il}d_{jk} = 0 \qquad \text{for } 0 \le i,j,k,l \le r+1. \tag{3.8}$$

This reproduces the recursion relations thanks to $d_{(i-1)(i+1)} = a_i$. Taking $l = k \pm 1$ so that $d_{kl} = \pm 1$ we find $\gcd(d_{ik}, d_{jk}) | d_{ij}$. Combining with permutations of i, j, k we deduce

$$\gcd(d_{ij}, d_{ik}) = \gcd(d_{ij}, d_{jk}) = \gcd(d_{ik}, d_{jk}). \tag{3.9}$$

Another consequence of the recursion relation is that $j \mapsto p_j$ is convex since its discrete Laplacian $p_{j-1} - 2p_j + p_{j+1} = (a_j - 2)p_j$ is non-negative (all $a_j \ge 2$), and likewise $j \mapsto q_j$ is convex. Their sum is convex and $p_0 + q_0 = p_{r+1} + q_{r+1} = n$ so $p_j + q_j \le n$ for all $0 \le j \le r+1$. It is easy to check that equality only happens for $j = 0$ or $j = r+1$ or when $p = n-1$ (all $a_j = 2$). Given the definitions of p_j and q_j the inequality reads $d_{j(r+1)} + d_{0j} \le d_{0(r+1)}$. It generalizes to $d_{ij} + d_{jk} \le d_{ik}$ for $i < j < k$ with equality if and only if $a_\alpha = 2$ for all $i < \alpha < k$.

Integer solutions $(x_0, \dots, x_{r+1}) \in \mathbb{Z}^{r+2}$ of the recursion relation $x_{i-1} - a_i x_i + x_{i+1} = 0$ for all $1 \le i \le r$ appear in a few places in our work. The lattice of solutions is a rank 2 sublattice of $\mathbb{Z}^{r+2}$, and any pair of solutions $(d_{ij})_{0 \le i \le r+1}$ and $(d_{ik})_{0 \le i \le r+1}$ spans an index $|d_{jk}|$ sublattice inside it.

3.1.2 GLSM in one basis

The GLSM has gauge group $G = U(1)^r$ and $r+2$ chiral multiplets $X_0, \dots, X_{r+1}$ (for convenience we label flavours starting at 0). There are two convenient choices of bases for the Lie algebra, leading to two different charge matrices that are of course related to each other by a change of basis.

The factor $U(1)_\alpha$, namely the α-th factor in $U(1)^r$, acts with charges $(1, -a_\alpha, 1)$ on $(X_{\alpha-1}, X_\alpha, X_{\alpha+1})$ and does not act on other X_β. In other words the charge matrix is

$$\left(Q_\alpha{}^i\right)_{1 \le \alpha \le r, 0 \le i \le r+1} = \begin{pmatrix} 1 & -a_1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -a_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -a_r & 1 \end{pmatrix}, \tag{3.10}$$

and in particular charges of $X_1, \dots, X_r$ are minus the generalized Cartan matrix. The action of G is faithful: if an element $(g_1, \dots, g_r) \in U(1)^r$ acts trivially then $g_1 = 1$ (because of the action on X_0), then $g_2 = 1$ (because of the action on X_1) and so on, so all $g_\alpha = 1$. We denote components of ζ in this basis by ζ_α.

3.1.3 GLSM in the second basis

We change basis by multiplying the charge matrix by nC^{-1}, whose components are integers $(nC^{-1})^{\alpha\beta} = p_{\max(\alpha,\beta)} q_{\min(\alpha,\beta)}$. This yields

$$\left(\sum_{\beta=1}^{r} n(C^{-1})^{\alpha\beta} Q_\beta{}^i\right)_{1 \le \alpha \le r, 0 \le i \le r+1} = \begin{pmatrix} p_1 & -n & 0 & \cdots & 0 & q_1 \\ p_2 & 0 & -n & \ddots & \vdots & q_2 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ p_r & 0 & \cdots & 0 & -n & q_r \end{pmatrix}. \tag{3.11}$$

Again, each factor acts on three chiral multiplets, but now these are X_0, X_α and X_{r+1}. In this basis the gauge group takes the form $U(1)^r/(\mathbb{Z}_n)^{r-1}$. Indeed, all elements $(g^1,\dots,g^r)\in(\mathbb{Z}_n)^r\subset U(1)^r$ such that $\prod_\alpha(g^\alpha)^{p_\alpha}=\prod_\alpha(g^\alpha)^{q_\alpha}=1$ act trivially, and in fact these two conditions are equivalent thanks to $p_\alpha=p_1q_\alpha \mod n$, see (3.7).

In terms of the components ζ_α in the first basis, components of ζ in this basis are $\zeta'_\alpha=\sum_{\beta=1}^r\bigl(n(C^{-1})^{\alpha\beta}\zeta_\beta\bigr)$ for $1\leq\alpha\leq r$.

Consider the phase where all of these sums are negative. The classical vacuum equations imply $X_\alpha\neq 0$ for $1\leq\alpha\leq r$, and these fields are fixed up to a phase in terms of ζ, X_0 and X_{r+1}. The phase is absorbed by a gauge transformation, and the gauge group is Higgsed down to the discrete subgroup $(\mathbb{Z}_n)^r/(\mathbb{Z}_n)^{r-1}\subset U(1)^r/(\mathbb{Z}_n)^{r-1}$ that leaves $X_1,\dots,X_r$ invariant. Altogether the Higgs branch is spanned by X_0 and X_{r+1}, modulo the remaining $\mathbb{Z}_n$ gauge transformations, which multiply X_{r+1} and X_0 by powers of (ω,ω^p) since $p_\alpha=pq_\alpha \mod n$. As a complex orbifold, the Higgs branch is thus $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ in this phase. This phase is called the **orbifold phase**. We discuss other phases in detail in subsection 3.2.

The case $a_1=\cdots=a_r=2$ is interesting because charge vectors sum to zero precisely in that case. The generalized Cartan matrix is the Cartan matrix of the Lie algebra A_r, and we compute $d_{ij}=j-i$. In particular, $n=r+1$ and $p=r$, namely $\mathbb{Z}_n$ acts by multiplication by (ω,ω^{n-1}), hence acts on $\mathbb{C}^2$ as a subgroup of $SU(2)$. The Higgs branch is in this case a Calabi-Yau manifold or orbifold depending on the phase. The model then flows to a superconformal field theory.

3.2 Higgs branch geometry

We now turn to describing the geometry of the Higgs branch in each phase. The Hirzebruch-Jung models also admit Coulomb and mixed branch vacua (see subsection 3.4). By a slight abuse of notations we denote the vev of the bottom component of a chiral multiplet by the same letter: the coordinate ring of the UV target space is thus $\mathcal{R}=\mathbb{C}[X_0,X_1,\dots,X_r,X_{r+1}]$.

3.2.1 In one phase

The IR Higgs branch X in a given phase admits a $(\mathbb{C}^*)^2$ action obtained as the $(\mathbb{C}^*)^{r+2}$ symmetry rotating individual X_i, quotiented by the $(\mathbb{C}^*)^r$ gauge symmetry. Orbits of the $(\mathbb{C}^*)^2$ action are parametrized by the values of $|P|^2$ and $|Q|^2$ (or any other pair of chirals). Orbits are typically $(\mathbb{C}^*)^2$, but they reduce to $\mathbb{C}^*$ in each locus $\{X_i=0\}$ and to a point at pairwise intersections thereof.

To describe the allowed values $(|P|^2,|Q|^2)$, consider the D-term equations for the GLSM written in the second basis (3.11): under the α-th $U(1)$ gauge factor the fields P, X_α, Q have charges p_α, $-n$, q_α and other multiplets are neutral. We denote by $\zeta'_\alpha=\sum_{\beta=1}^r\bigl(n(C^{-1})^{\alpha\beta}\zeta_\beta\bigr)$ the FI parameters in this basis (in terms of those in the first basis). The D-term equations are then

$$p_\alpha|P|^2+q_\alpha|Q|^2=\zeta'_\alpha+n|X_\alpha|^2,\quad \text{for } 1\leq\alpha\leq r. \tag{3.12}$$

Up to the toric action, this fixes all X_α in terms of $(|P|^2,|Q|^2)$, provided that the linear inequalities $p_\alpha|P|^2+q_\alpha|Q|^2\geq\zeta'_\alpha$ are obeyed. The toric diagram of X depicted in Figure 1 thus consists of the subset S of the upper quadrant that lies above all lines

$$p_\alpha|P|^2+q_\alpha|Q|^2=\zeta'_\alpha\,. \tag{3.13}$$

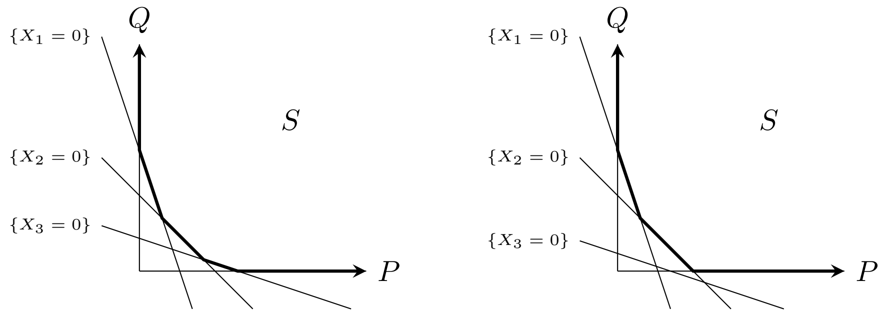


Figure 1: Toric diagrams of two resolutions of $\mathbb{C}^2/\mathbb{Z}_{4(-1)}$. These are Higgs branches in two different phases of the same GLSM of rank $r = 3$. On the left, all exceptional divisors are blown up, while on the right the third exceptional divisor is blown down.

Extending notations to include $\zeta'_0 = \zeta'_{r+1} = 0$ and $X_0 = P$ and $X_{r+1} = Q$, the $\alpha = 0$ and $\alpha = r + 1$ lines are the two axes $\{|P|^2 = 0\}$ and $\{|Q|^2 = 0\}$.

In a given phase, the boundary of S consists of a collection of line segments along some of the lines (3.13). Each such line segment corresponds to an exceptional divisor of the toric geometry, that can be blown down by varying from $\zeta'_\alpha \gg 0$ to $\zeta'_\alpha \ll 0$. The 2^r phases of the GLSM are characterized by the set $A \subset [\![1, r]\!]$ of divisors that are blown up, namely such that $\{X_\alpha = 0\}$ is neither empty nor a point. Since q_α/p_α increases with α we learn that edges of the set S are segments of lines corresponding to $\{X_i = 0\}$ for $i \in \{0\} \cup A \cup \{r+1\}$, in increasing order. Later, we need the deleted set Δ. It is the union of the hyperplanes $\{X_\alpha = 0\}$ for each $\alpha \in [\![1, r]\!] \setminus A$, and of the intersections $\{X_\alpha = X_\beta = 0\}$ for $\alpha, \beta \in \{0\} \cup A \cup \{r + 1\}$ that are not consecutive elements of this set (namely such that there exists $\gamma \in A$ with $\alpha < \gamma < \beta$).

Note that X typically has orbifold singularities (see subsection 3.3 for details). For instance in the phase $A = \emptyset$ the non-zero vevs of $X_1, \ldots, X_r$ only break the gauge group down to $\mathbb{Z}_n$, which acts on P and Q with charges $p_1 = p$ and $q_1 = 1$ (one could equally well choose $p_r = 1$ and q_r because $p_1 q_r = p_r q_1 = 1 \mod n$). Thus, in that phase, $X = \mathbb{C}^2/\mathbb{Z}_{n(p)}$.

3.2.2 Phase boundaries

Phase boundaries occur when one of the lines (3.13) touches S at a single point, namely when three of these lines intersect at a point that is above any other line (3.13). The lines for $\alpha = i, j, k$ (with $0 \le i < j < k \le r + 1$) have such a common intersection when

$$\det \begin{pmatrix} p_i & q_i & \zeta'_i \\ p_j & q_j & \zeta'_j \\ p_k & q_k & \zeta'_k \end{pmatrix} = 0, \quad \text{and, for any } i < \alpha < k, \quad \det \begin{pmatrix} p_i & q_i & \zeta'_i \\ p_\alpha & q_\alpha & \zeta'_\alpha \\ p_k & q_k & \zeta'_k \end{pmatrix} \ge 0 \tag{3.14}$$

where we recall $\zeta'_0 = \zeta'_{r+1}$. The same linear condition on FI parameters can also be derived in the first basis as follows by solving D-term equations together with $X_i = X_j = X_k = 0$. The D-term equations for $i < \alpha < j$ together with $X_i = X_j = 0$ give a unique solution for $|X_{i+1}|^2, \ldots, |X_{j-1}|^2$ in terms of $\zeta_{i+1}, \ldots, \zeta_{j-1}$. Likewise $X_j = X_k = 0$ and the D-term equations for $i < \alpha < j$ give $|X_\alpha|^2$ for $j < \alpha < k$ in terms of ζ_α for the same range of α. The j-th D-term equation $|X_{j-1}|^2 + |X_{j+1}|^2 = \zeta_j$ then provides a linear constraint on ζ_α for $i < \alpha < k$. This linear equation must be combined in general with an analogue of the inequalities in (3.14).

The more conceptual point of view is to consider a $U(1) \subset U(1)^r$ that acts trivially on all chiral multiplets except X_i, X_j, X_k, as we do in subsubsection 3.3.2. Its FI parameter ζ_{loc} is given in (3.20) as a linear combination of ζ_α for $i < \alpha < k$. The corresponding D-term equation writes ζ_{loc} as a linear combination of $|X_i|^2$, $|X_j|^2$, $|X_k|^2$, so the point $X_i = X_j = X_k = 0$ characterizing wall-crossing occurs at $\zeta_{\text{loc}} = 0$. In the phase with the exceptional divisor E_j blown up, its volume is controlled by $|\zeta_{\text{loc}}|$, and is independent of ζ_i and ζ_k. In particular in the fully resolved phase we have

$$\text{Vol}(E_j) \sim \zeta_j. \tag{3.15}$$

Besides the (classical) position of walls, it is interesting to determine which ones are Calabi-Yau walls, because these are true codimension 1 singularities around which it makes sense to study monodromies in the Kähler moduli space. The wall at which X_i, X_j, X_k can vanish at the same point is a cone of the charges $Q^{\neq i,j,k}$. It is Calabi-Yau if $Q^{\text{tot}} = \sum_\ell Q^\ell$ can be written as a linear combination of these. We prove now that the condition is that $a_{i+1} = \cdots = a_{k+1} = 2$.

Consider first the case $i = 0$, $k = r + 1$, namely a wall-crossing from the orbifold phase. In basis II, Q^{tot} has components $p_\alpha + q_\alpha - n$ for $1 \leq \alpha \leq r$ and we want to write it as a linear combination of the charges $Q^{\neq i,j,k}$, which in basis II are $-n$ times each basis vector except the j-th one. This exactly requires the j-th component $p_j + q_j - n = 0$. We proved in subsubsection 3.1.1 that this only happens in the Calabi-Yau case (all $a_\alpha = 2$). Of course, since bases are equivalent, we could have obtained the same conclusion in basis I, with more work.

Now consider the general case and work in basis I of the GLSM. Start from Q^{tot}. By subtracting a linear combination of the $X_{<i}$ we can cancel components $1, \ldots, i$ of Q^{tot}. Likewise subtracting a linear combination of the $X_{>k}$ cancels components $k, \ldots, r$. In this way the problem reduces to the case $i = 0$, $k = r + 1$ which we have analysed, so we learn that the wall is CalabiâĂŞYau if and only if $a_{i+1} = \cdots = a_{k-1} = 2$.

An alternative point of view is to look at the local $U(1) \times \mathbb{Z}_m$ model of subsubsection 3.3.2 that describes the wall-crossing. The wall is a true singularity if and only that local model has a true singularity, namely is Calabi-Yau. We compute that the sum of charges is $d_{ik} - d_{ij} - d_{jk}$ up to a scaling. We proved in subsubsection 3.1.1 that this only happens when $a_{i+1} = \cdots = a_{k-1} = 2$.

3.3 Local models

Let $E_i = \{X_i = 0\}$ for $0 \leq i \leq r + 1$. The sets E_0 and E_{r+1} are always non-compact, while the E_α for $1 \leq \alpha \leq r$ are exceptional divisors or are empty depending on whether $\alpha \in A$ or not. We now describe the geometry near intersections $E_i \cap E_j$ for $0 \leq i < j \leq r + 1$ in phases where they exist, then near E_j for $0 \leq j \leq r + 1$.

3.3.1 Local model near an intersection

An intersection $E_i \cap E_j$ is non-empty only in phases such that i and j are elements of $\{0\} \cup A \cup \{r + 1\}$ (to have $E_i \neq \emptyset$ and $E_j \neq \emptyset$) and such that no other element $\alpha \in A$ is between i and j. The second condition ensures E_i and E_j intersect, as is manifest in the $(|P|^2, |Q|^2)$ plane. Near the intersection point $E_i \cap E_j$, all chiral multiplets other than X_i and X_j have a vev. By the Higgs mechanism this vev breaks the gauge symmetry down to the subgroup of elements of $U(1)^r$ that fix all X_l other than X_i and X_j.

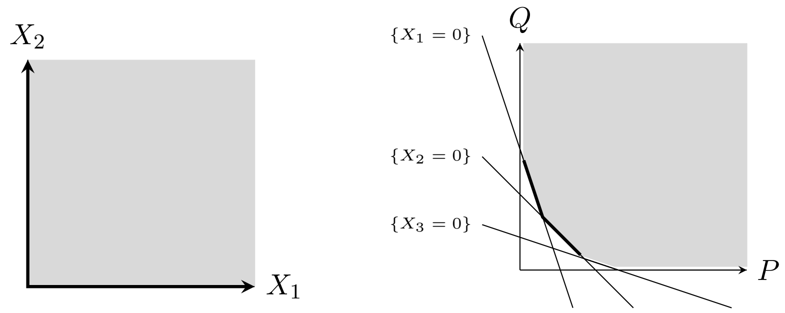


Figure 2: Toric diagrams of the Higgs branch of: (a) a local model of an intersection $E_i \cap E_j$, and (b) the full GLSM. The region shaded in gray is the image of (a) inside (b) under the embedding of complex manifolds/orbifolds. The toric diagram does not depic the orbifold group $\mathbb{Z}_{d_{ij}}$ since it acts purely on phases of chiral multiplets.

Let us work in basis I of the GLSM, and let $g = (g_1, \dots, g_r) \in U(1)^r$ be such an element. Fixing X_0 requires $g_1 = 1$, in which case fixing X_1 requires $g_2 = 1$ and so on, so $g_1, \dots, g_i = 1$. Likewise $g_j = \dots = g_r = 1$. Next, we consider in turn the constraints coming from the fact that g fixes X_α for $\alpha = j-1, j-2, \dots, i+1$. Each step α gives one component $g_{\alpha-1} = (g_\alpha)^{a_\alpha}(g_{\alpha+1})^{-1}$, and an explicit expression is

$$g_\alpha = (g_{j-1})^{d_{\alpha j}} \qquad \text{for } i \le \alpha \le j. \tag{3.16}$$

This relies on the initial cases $d_{jj} = 0$ and $d_{(j-1)j} = 1$ and the recursion relation (3.4) $d_{(\alpha+1)j} + d_{(\alpha-1)j} = a_\alpha d_{\alpha j}$, applied for $i < \alpha < j$. Then, $g_i = 1$ forces g_{j-1} to be a d_{ij}-th root of unity. Altogether we are left with a gauge group $\mathbb{Z}_{d_{ij}}$. The two remaining chirals X_i and X_j have charges $d_{(i+1)j}$ and 1 as summarized in the following table

	X_i	X_j
$\mathbb{Z}_{d_{ij}}$	$d_{(i+1)j}$	1

or in another basis

	X_i	X_j
$\mathbb{Z}_{d_{ij}}$	1	$d_{i(j-1)}$

(3.17)

where we used $d_{i(j-1)}d_{(i+1)j} - d_{ij}d_{(i+1)(j-1)} = d_{i(i+1)}d_{(j-1)j} = 1$ to invert $d_{(i+1)j}$ modulo d_{ij}. We conclude that near $E_i \cap E_j$ the Higgs branch is close to the orbifold $\mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{(i+1)j})}$.

This reproduces our earlier conclusions for the orbifold phase $A = \emptyset$, which is the only phase in which $E_0 \cap E_{r+1}$ is non-empty. We had found that the Higgs branch is $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ in that phase and indeed $d_{0(r+1)} = n$ and $d_{1(r+1)} = p_1 = p$. In fact all cases reduce to this one by noting that vevs of $X_0, \dots, X_{i-1}$ and $X_{j+1}, \dots, X_{r+1}$ break completely the gauge factors with indices $1 \le \alpha \le i$ and $j \le \alpha \le r$ (see above), thus reducing the problem to the orbifold phase of a $U(1)^{j-i-1}$ Hirzebruch-Jung model with $j-i+1$ chirals. From this point of view, the orbifold singularities for $i+2 \le j$ are due to the presence of blown-down exceptional divisors E_α for $i < \alpha < j$. In contrast, the residual gauge group trivial for $i+1 = j$ because $d_{ij} = d_{i(i+1)} = 1$, so intersections $E_i \cap E_{i+1}$ are smooth.

Recall that as a complex orbifold the Higgs branch of an abelian GLSM is a quotient by the complexified gauge group $G_\mathbb{C}$ of $V \setminus \Delta$, the space of chiral multiplets minus some deleted set consisting of coordinate subspaces. This construction enables us to embed the Higgs branch of the local model (3.17), as a complex orbifold, into that of the full model in the given phase. A point $(X_i, X_j) \in \mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{(i+1)j})}$ is mapped to $(1, \dots, 1, X_i, 1, \dots, 1, X_j, 1, \dots, 1) \in V = \mathbb{C}^{r+2}$, up to $(\mathbb{C}^*)^r$ gauge transformations, where the non-trivial entries are in positions $0 \leq i < j \leq r+1$. Our earlier analysis of what subgroup of $U(1)^r$ leaves the non-zero vevs of X_l for $l \neq i, j$ extends trivially to the complexified setting and shows that the map is well-defined. Its image consists of all $(X_0, \dots, X_{r+1}) \in V$ that can be gauge-fixed to have $X_l = 1$ for $l \neq i, j$, namely to the set $V \setminus \bigcup_{l \neq i,j} E_l$. The toric diagram is depicted in Figure 2.

3.3.2 Local model near an exceptional divisor

We repeat the same analysis for an exceptional divisor E_j for $1 \leq j \leq r$ in some phase. Let E_i and E_k be non-empty and intersect E_j at one point each, namely the indices should obey $i < j < k$ and be successive elements in the set $\{0\} \cup A \cup \{r+1\}$. From the $(|P|^2, |Q|^2)$ toric diagrams we know that E_j is topologically a two-sphere and it has a $U(1)$ isometry with two fixed points: $E_i \cap E_j$ and $E_j \cap E_k$. In E_j, hence on a neighborhood thereof, all chiral multiplets except X_i, X_j, X_k get a vev. The Higgs mechanism breaks the gauge group to the subgroup of elements $g = (g_1, \dots, g_r) \in U(1)^r$ that leave all the vevs invariant.

Again we work in basis I and find that fixing $X_0, \dots, X_{i-1}$ and $X_{k+1}, \dots, X_{r+1}$ forces $g_1 = \cdots = g_i = 1$ and $g_k = \cdots = g_r = 1$. Next, the calculations near (3.16) give $g_\alpha = (g_{i+1})^{d_{i\alpha}}$ for $i \leq \alpha \leq j$, and $g_\alpha = (g_{k-1})^{d_{\alpha k}}$ for $j \leq \alpha \leq k$. The compatibility of these two expressions of g_j means that the gauge group is parametrized by solutions of $g_{i+1}^{d_{ij}} = g_{k-1}^{d_{jk}}$, namely

$$g_{i+1} = h^{d_{jk}/m} \omega^u \text{ and } g_{k-1} = h^{d_{ij}/m} \omega^v \quad \text{for} \quad (h, \omega) \in U(1) \times \mathbb{Z}_m, \tag{3.18}$$

where $m \in \mathbb{Z}_{\geq 1}$ and $u, v \in \mathbb{Z}$ are chosen to obey

$$m = \gcd(d_{ij}, d_{jk}) = u d_{ij} - v d_{jk}. \tag{3.19}$$

Different choices of (u, v) amount to different choices of basis for $\mathbb{Z}_m$. The GLSM can be expressed in various choices of basis, related by automorphisms of $U(1) \times \mathbb{Z}_m$. Besides conjugation that changes signs of all charges, one can add to the $\mathbb{Z}_m$ charges any multiple of the $U(1)$ charges, and multiply the $\mathbb{Z}_m$ charges by any invertible element of $\mathbb{Z}_m$.

From how the $U(1)$ factor of the gauge group of the local model embeds into the $U(1)^r$ gauge group of the full GLSM we work out the FI-theta parameter of the local model,

$$t_{\text{loc}} = \frac{1}{m} \sum_{\alpha=i+1}^{k-1} d_{i \min(\alpha,j)} d_{\max(\alpha,j) k} \, t_\alpha. \tag{3.20}$$

The chiral multiplet X_i transforms by $g_{i-1} g_i^{-a_i} g_{i+1} = h^{d_{jk}/m} \omega^u$ and X_k by $g_{k-1} g_k^{-a_k} g_{k+1} = h^{d_{ij}/m} \omega^v$ while X_j transforms by $g_{j-1} g_j^{-a_j} g_{j+1} = h^{\ell/m} \omega^s$ with

$$\begin{aligned} \ell &= d_{i(j-1)} d_{jk} - a_j d_{ij} d_{jk} + d_{ij} d_{(j+1)k} = -d_{i(j+1)} d_{jk} + d_{ij} d_{(j+1)k} = -d_{ik} \\ s &\equiv -d_{i(j+1)} u + d_{(j+1)k} v \equiv d_{i(j-1)} u - d_{(j-1)k} v \mod m \end{aligned} \tag{3.21}$$

where we used the recursion relation and (3.8) and $d_{j(j+1)} = 1$ to simplify ℓ and to give two equally complicated expressions for s. Charges are summarized in the following table, with m, u, s, v given above:

	X_i	X_j	X_k
$U(1)$	d_{jk}/m	$-d_{ik}/m$	d_{ij}/m
$\mathbb{Z}_m$	u	s	v

(3.22)

It will be useful later that $X_i^{d_{i(j+1)}} X_j X_k^{-d_{(j+1)k}}$ and $X_i^{d_{ij}} X_k^{-d_{jk}}$ and $X_i^{-d_{i(j-1)}} X_j X_k^{d_{(j-1)k}}$ are gauge neutral.

In this analysis we first reduced the model to a $U(1)^{k-i-1}$ Hirzebruch-Jung model with $k-i+1$ chirals. Up to relabeling this is the case $i=0$ and $k=r+1$, namely where a single exceptional divisor, E_j, is blown up. In that case, $m=\gcd(p_j,q_j)$ and the local model has $U(1)$ charges $(p_j,-n,q_j)/m$ and $\mathbb{Z}_m$ charges given above. It is instructive to reproduce some of these results in basis II. Elements $g=(g^1,\dots,g^r)\in U(1)^r/\mathbb{Z}_n^{r-1}$ that only acts on X_0, X_j, X_{r+1} are those for which all $g^\alpha\in\mathbb{Z}_n$ except $g^j\in U(1)$. The residual gauge group is thus $(U(1)_j\times\mathbb{Z}_n^{r-1})/\mathbb{Z}_n^{r-1}$ and the question is how the $\mathbb{Z}_n^{r-1}$ quotient is taken. The $U(1)_j$ factor acts with charges $(p_j,-n,q_j)$, hence its $\mathbb{Z}_m$ subgroup acts trivially. By a volume argument the residual gauge group must be $U(1)_j/\mathbb{Z}_m$ times a discrete abelian group of order m.

Another consistency check is to determine which $(h,\omega)\in U(1)\times\mathbb{Z}_m$ fix X_i. Write $\omega=\exp(2\pi ia/m)$ and $h=\exp(-2\pi ib/d_{jk})$ with $a\in\mathbb{Z}$ by construction. The condition is that $au\equiv b \bmod m$. It is solved exactly by the d_{jk} powers of $(h,\omega)=\big(\exp(-2\pi i/m),\exp(2\pi iu/d_{jk})\big)$. The residual gauge group is $\mathbb{Z}_{d_{jk}}$, and it is easy to check that it acts on X_j and X_k with charges $d_{(j+1)k}$ and 1. As expected from subsubsection 3.3.1 we find the orbifold $\mathbb{C}^2/\mathbb{Z}_{d_{jk}(d_{(j+1)k})}$. Near the other pole $E_i\cap E_j$ we similarly find $\mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{i(j-1)})}\simeq\mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{(i+1)j})}$. The local model embeds as a complex orbifold into the full Higgs branch, and its image is the region in which no chiral vanishes except X_i, X_j, X_k.

Does the local model approximate well the metric on the divisor E_j in the full Higgs branch? Not always. Integrating out chirals that are nonzero near E_j and removing gauge fields that their vev breaks is an approximation that is valid provided the chiral multiplets that we keep have vevs that are much less than those that we integrate out. However, the vev of X_i near the intersection $E_j\cap E_k$ may be bigger than some other chiral multiplets X_α for $j<\alpha<k$, especially when FI parameters are taken close to a wall that corresponds to blowing up E_α (see right side of Figure 1 for instance). It may be interesting to make quantitative comparisons between the $U(1)$-invariant metrics on E_j for different models. When discussing the metric we will assume that the regime of FI parameters is such that all chirals other than X_i, X_j, X_k have large vevs in the neighborhood of E_j that we are considering.

Very close to $E_i\cap E_j$ we know from subsubsection 3.3.1 that the metric is that of $\mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{i(j-1)})}$, parametrized by X_i and X_j. The submanifold $E_j=\{X_j=0\}$ has the same deficit angle at $E_i\cap E_j$ as $\mathbb{C}/\mathbb{Z}_{d_{ij}}$. The exceptional divisor is thus a topological two-sphere with $U(1)$ isometry and two conical singularities. As a complex manifold/orbifold it is simply $\mathbb{P}^1$, with projective coordinates $(X_i^{d_{ij}}:X_k^{d_{jk}})$, for the same reason that $\mathbb{C}/\mathbb{Z}_n\simeq\mathbb{C}$ under the map $X\to X^n$. As a Kähler manifold the divisor $\{X_j=0\}$ of the local model (3.22) could be called $\mathbb{WCP}^1_{b,a}$ with $b=d_{jk}$ and $a=d_{ij}$. When $m=\gcd(d_{ij},d_{jk})=1$ the local model is a $U(1)$ GLSM, and the Kähler quotient construction of its Higgs branch coincides with a standard construction of weighted projective spaces (in any dimensions). When $m>1$ the same $U(1)$ construction would simply construct $\mathbb{WCP}^1_{b/m,a/m}$ and one needs a further $\mathbb{Z}_m$ orbifold to obtain the correct conical singularities $\mathbb{C}/\mathbb{Z}_a$ and $\mathbb{C}/\mathbb{Z}_b$.

3.3.3 Line bundles on $\mathbb{WCP}^1$

In the resolved phase of the local model (3.22), the Higgs branch is the total space of the normal line bundle of the exceptional divisor E_j. (Away from E_j the metric receives strong corrections.) Let us determine what line bundle it is and discuss more general line bundles, as this is essential for our study of B-branes on Hirzebruch-Jung models in subsection 4.5. For brevity we denote $a = d_{ij}$ and $b = d_{jk}$ so that the exceptional divisor is $\mathbb{WCP}^1_{b,a}$ with conical singularities $\mathbb{C}/\mathbb{Z}_a$ and $\mathbb{C}/\mathbb{Z}_b$. Let $m = \gcd(a, b)$ and $\omega_p = \exp(2\pi i/p)$ for all $p > 0$.

Geometric point of view. We discussed above how $\mathbb{WCP}^1_{b,a}$ is constructed by gluing the cones $\mathbb{C}/\mathbb{Z}_a$ with coordinate x and $\mathbb{C}/\mathbb{Z}_b$ with coordinate y. We choose these coordinates to be single-valued before quotienting so the well-defined expressions on the orbifold are x^a and y^b. The change of coordinates between the two is then $x^a = y^{-b}$.

A line bundle on $\mathbb{WCP}^1_{b,a}$ is built by gluing an orbifold (or equivariant) line bundle on $\mathbb{C}/\mathbb{Z}_a$ and one on $\mathbb{C}/\mathbb{Z}_b$ through a transition map. The orbifold line bundles are characterized by their charge under the orbifold group, which in view of later identifications we denote respectively by $-\bar{\gamma} \in \mathbb{Z}_a$ under $\mathbb{Z}_a$ and $\bar{\delta} \in \mathbb{Z}_b$ under $\mathbb{Z}_b$. Sections of the $\mathbb{C}/\mathbb{Z}_a$ bundle are $f_{\mathrm{N}} \colon \mathbb{C}^* \to \mathbb{C}$ such that $f_{\mathrm{N}}(x) = \omega_a^{-\bar{\gamma}} f_{\mathrm{N}}(\omega_a x)$. The transition map must map that to a section $f_{\mathrm{S}} \colon \mathbb{C}^* \to \mathbb{C}$ such that $f_{\mathrm{S}}(y) = \omega_b^{\bar{\delta}} f_{\mathrm{S}}(\omega_b y)$ of the other orbifold bundle, by a relation of the form $f_{\mathrm{N}}(x) = (\cdots) f_{\mathrm{S}}(y)$ for $x^a = y^{-b}$. Since neither x nor y determines the other uniquely in general, and since the orbifold line bundles may have non-zero charges, the coefficient $(\cdots)$ defining the transition map typically depends on both x and y, subject to the relation $x^a = y^{-b}$. The transition map is thus

$$f_{\mathrm{N}}(x) = x^{\gamma} y^{\delta} f_{\mathrm{S}}(y). \tag{3.23}$$

The transition map should reproduce the orbifold group actions on f_{N} and f_{S} when one keeps y or x fixed, respectively. This implies $-\bar{\gamma} = -\gamma \bmod a$ and $\bar{\delta} = \delta \bmod b$. Altogether the line bundle is characterized by

$$(\gamma, \delta) \in \mathbb{Z}^2/\big((a, -b)\mathbb{Z}\big). \tag{3.24}$$

This group is isomorphic to $\mathbb{Z} \times \mathbb{Z}_m$ with $m = \gcd(a, b)$. In particular, on weighted projective spaces $\mathbb{WCP}^1_{b,a}$ with $\gcd(a, b) = 1$, all line bundles are tensor powers of one line bundle that we call $\mathcal{O}(1)$. In terms of the $U(1)$ GLSM discussed next, that line bundle is parametrized by a scalar of $U(1)$ charge 1.

GLSM point of view. The $\mathbb{WCP}^1_{b,a}$ exceptional divisor is the Higgs branch of the following $U(1) \times \mathbb{Z}_m$ GLSM, obtained from (3.22) by dropping the chiral multiplet X_j:

	X_i	X_k
$U(1)$	b/m	a/m
$\mathbb{Z}_m$	u	v

(3.25)

where $u, v \in \mathbb{Z}$ obey $(a/m)u - (b/m)v = 1$, which implies for instance that a/m and v are coprime. As a complex orbifold, $\mathbb{WCP}^1_{b,a}$ is parametrized by homogeneous coordinates $(x_i : x_k) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ with the identification $(x_i : x_k) \sim (h^{b/m}\omega^u x_i : h^{a/m}\omega^v x_k)$ for all $(h, \omega) \in \mathbb{C}^* \times \mathbb{Z}_m$, the complexified gauge group. The coordinates x and y of the gluing description are obtained from $(x_i : x_k)$ by gauge-fixing $x_k = 1$ or $x_i = 1$ so $(x_i : x_k) \sim (x : 1) \sim (1 : y)$.

While not strictly necessary it is instructive to check that x and y are subject to $\mathbb{Z}_a$ and $\mathbb{Z}_b$ orbifold identifications. Let us gauge-fix $x_k = 1$. The elements (h, ω) that leave x_k fixed are those such that $h^{a/m}\omega^v = 1$. For each $\omega \in \mathbb{Z}_m$ there are a/m possible h, so in total there are $m(a/m)=a$ solutions. On the other hand, $\ell\mapsto(\omega_a^{-\ell v},\omega_m^\ell)$ defines an injective group morphism from $\mathbb{Z}_a$ to the space of solutions: its kernel consists of ℓ such that $\ell=0 \bmod m$ and $\ell v=0 \bmod a$, hence $\ell=m\ell'$ and $\ell' v=0 \bmod a/m$, hence (because a/m and v are coprime) $\ell'=0 \bmod a/m$ and finally $\ell=0 \bmod a$. The residual gauge group is thus $\mathbb{Z}_a$ consisting of all $(\omega_a^{-\ell v},\omega_m^\ell)\in U(1)\times\mathbb{Z}_m$. The group $\mathbb{Z}_a$ acts by $(x:1)\mapsto(\omega_a^{-\ell vb/m+\ell ua/m}x:1)=(\omega_a^\ell x:1)$, namely a standard orbifold $\mathbb{C}/\mathbb{Z}_a$. The situation is the same for the other pole.

The fiber of a line bundle is parametrized by a scalar with some charges (α,β) under $U(1)\times\mathbb{Z}_m$. A section of that line bundle is then (a meromorphic function) $f\colon\mathbb{C}^2\backslash\{(0,0)\}\to\mathbb{C}$ such that

$$h^\alpha\omega^\beta f\big(h^{b/m}\omega^u x_i : h^{a/m}\omega^v x_k\big)=f(x_i:x_k),\quad \text{for all } (h,\omega)\in\mathbb{C}^*\times\mathbb{Z}_m. \tag{3.26}$$

In the $\mathbb{C}/\mathbb{Z}_a$ patch, (3.26) becomes

$$\omega_a^{\ell(-v\alpha+\beta a/m)} f\big(\omega_a^\ell x_i:1\big)=f(x_i:1), \tag{3.27}$$

which describes an equivariant line bundle with charge $-v\alpha+\beta a/m \bmod a$ on $\mathbb{C}/\mathbb{Z}_a$. The combination $-v\alpha+\beta a/m$ can be found more directly: the charge vector (α,β) is an integer linear combination of those of x_i and x_k,

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=-\gamma\begin{pmatrix}b/m\\ u\end{pmatrix}+\delta\begin{pmatrix}a/m\\ v\end{pmatrix},\ \text{where } \gamma=v\alpha-\beta a/m,\ \text{and } \delta=u\alpha-\beta b/m \tag{3.28}$$

are defined up to shifting (γ,δ) by multiples of (a,b).

A gauge transformation that maps $(x_i:x_k)\mapsto(\lambda x_i:\mu x_k)$ acts on the section as $f\mapsto\lambda^{-\gamma}\mu^\delta f$. We deduce that the equivariant line bundles from which our line bundle is built have charges $-\gamma \bmod a$ and $\delta \bmod b$, respectively. We also deduce the transition map by converting (3.26) to γ and δ and imposing $x_k=1$ and $h^{b/m}\omega^u x_i=1$:

$$f(x:1)=x^\gamma y^\delta f(1:y). \tag{3.29}$$

This is exactly (3.23) since $f_{\mathrm{N}}(x)=f(x:1)$ and $f_{\mathrm{S}}(y)=f(1:y)$.

The normal bundle. The normal bundle of E_j in the Higgs branch of the Hirzebruch-Jung model is the same line bundle as in the local model (3.22). It is thus parametrized by a scalar X_j with charges $(\alpha,\beta)=(-d_{ik}/m,s)$, with $s=d_{i(j-1)}u-d_{(j-1)k}v$, under the $U(1)\times\mathbb{Z}_m$ gauge group. As observed below (3.22) these are the same charges as $X_i^{-d_{i(j+1)}}X_k^{d_{(j+1)k}}$ and also the same charges as $X_i^{d_{i(j-1)}}X_k^{-d_{(j-1)k}}$. In the notations above, $(\gamma,\delta)=(d_{i(j+1)},d_{(j+1)k})$.

3.4 Coulomb and mixed branches

The Hirzebruch-Jung models we consider also admit Coulomb and mixed branch vacua. As explained in subsection 2.4, these are found by searching, for each subspace of the chirals, some vacua in which these chirals (and no others) are not given a mass by σ. The dimension of the branch is (2.20), which vanishes unless the set of charges of these chiral multiplets obey linear relations. In our models, any $r-1$ of the charge vectors Q^i are linearly independent, so the only branch of positive dimension is the one in which all chiral multiplets can get vevs, namely the Higgs branch. Other branches only consist of isolated vacua, and the distinction between Coulomb and mixed vacua is unimportant.[5]

To keep notations short, let $\hat{\sigma} = \sigma/\mu$.

Coulomb branch vacua for example are found by extremising the effective twisted superpotential

$$W_{\text{eff}} = -t \cdot \hat{\sigma} - \sum_i (Q^i \cdot \hat{\sigma})\big(\log(Q^i \cdot \hat{\sigma}) - 1\big) \tag{3.30}$$

where t is only defined up to multiples of $2\pi i$. Exponentiating $\frac{\partial W_{\text{eff}}}{\partial \hat{\sigma}^\alpha} \in 2\pi i \mathbb{Z}$ gives the relations

$$\prod_i (Q^i \cdot \hat{\sigma})^{Q^i_\alpha} = e^{-t_\alpha}. \tag{3.31}$$

We recall that $t = \zeta - i\theta$ is renormalized as (2.7), consistent with the power of μ on the left-hand side.

Let us consider solutions of these equations for some instructive examples.

3.4.1 One parameter model

For the $r = 1$ Hirzebruch-Jung models, $n/p = a_1$ namely $p = 1$ and $a_1 = n$. We get one equation $\hat{\sigma}(-a_1\hat{\sigma})^{-a_1}\hat{\sigma} = e^{-t}$ namely

$$\hat{\sigma}^{a_1 - 2} = e^t (-a_1)^{-a_1}. \tag{3.32}$$

This has $a_1 - 2$ solutions, which have large $|\hat{\sigma}|$ (namely the approximation makes sense) in the phase $t \ll 0$. The picture that emerges is that the one-parameter model has two phases

- $\zeta \ll 0$ with Higgs branch $\mathbb{C}^2/\mathbb{Z}_{n(1)}$ and no Coulomb branch;
- $\zeta \gg 0$ with Higgs branch the total space of $\mathcal{O}(-n) \to \mathbb{CP}^1$ and $n - 2$ Coulomb branch vacua.

In the Coulomb branch case, both phases are pure-Higgs of course. At $t = a_1 \log(-a_1) \bmod 2\pi i$ a non-compact Coulomb branch opens up, since $\hat{\sigma}$ is arbitrary in (3.32), and the theory is singular.

3.4.2 Two-parameter models with $p = 2$

Next we consider $\mathbb{C}^2/\mathbb{Z}_{n(2)}$ (for $n = 2k - 1$), namely $r = 2$, $a_1 = k$ and $a_2 = 2$. We recall the charge matrix for convenience:

	X_0	X_1	X_2	X_3
$U(1)_1$	1	$-k$	1	0
$U(1)_2$	0	1	-2	1

(3.33)

Critical points of the effective twisted superpotential, namely solutions of

$$\begin{aligned} \hat{\sigma}_1(\hat{\sigma}_1 - 2\hat{\sigma}_2) &= e^{-t_1}(-k\hat{\sigma}_1 + \hat{\sigma}_2)^k \\ (-k\hat{\sigma}_1 + \hat{\sigma}_2)\hat{\sigma}_2 &= e^{-t_2}(\hat{\sigma}_1 - 2\hat{\sigma}_2)^2 \end{aligned} \tag{3.34}$$

[5] In future work we plan to work on models in which mixed branches have positive dimension.

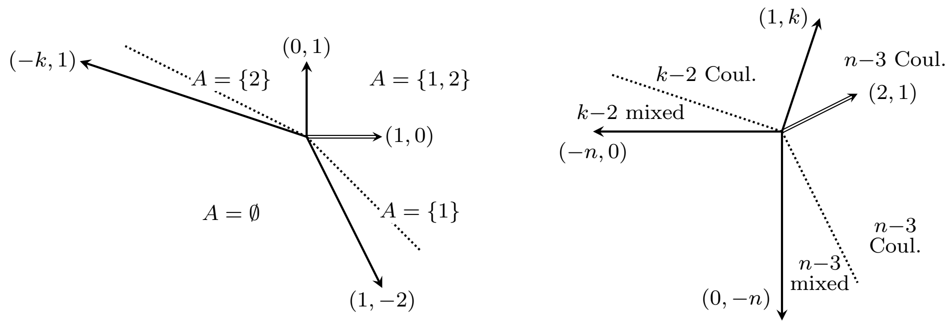


Figure 3: Phase structure for $\mathbb{C}^2/\mathbb{Z}_{n(2)}$ with $n=2k-1$ and $k>2$. The FI parameter runs towards the IR in the direction given by the double arrow, which in this model is parallel to a wall. Dotted lines denote transitions between sub-phases in which isolated massive vacua are Coulomb branch vacua or mixed branch vacua. To avoid clutter, we only indicate the sets A in one diagram and the numbers of mixed and Coulomb branch vacua in the other diagram (note that $n-3=2(k-2)$). Left: basis I. Right: basis II.

can be found explicitly by solving the second equation then the first:

$$\begin{aligned} \hat\sigma_1 &= v_\pm \hat\sigma_2, \qquad \hat\sigma_2^{k-2} = \frac{e^{t_1} v_\pm (v_\pm - 2)}{(1-kv_\pm)^k}, \quad \text{with} \\ v_\pm &= 2 - ke^{t_2}/2 \pm \sqrt{(1-2k)e^{t_2} + k^2 e^{2t_2}/4}. \end{aligned} \tag{3.35}$$

Altogether we get $2(k-2)=n-3$ solutions. In the Calabi-Yau case $\mathbb{C}^2/\mathbb{Z}_{3(2)}$, namely $k=2$, there are no Coulomb branch vacua for generic t_1, t_2. The theory has a codimension 1 singular locus where a non-compact one-dimensional Coulomb branch ($\hat\sigma_1 = v_\pm\hat\sigma_2$) opens up. This singular locus asymptotes to (shifted) classical walls. We henceforth consider the non-Calabi-Yau case $k>2$.

Besides being solutions of (3.35), Coulomb branch vacua must also be such that $|\hat\sigma_1|$, $|\hat\sigma_2|$, $|\hat\sigma_1 - 2\hat\sigma_2|$, $|-k\hat\sigma_1+\hat\sigma_2|$ are all large so as to make all chiral multiplets massive. Depending on the phase, only some of these $n-3$ values of $\hat\sigma$ are genuine Coulomb branch vacua. We now consider in turn each of the $2^r=4$ phases we found when analysing the Higgs branch. These are recapitulated in Figure 3. Recall that they are classified by the set of exceptional divisors that are blown up.

- **No divisor blown up** ($A=\emptyset$): $\zeta_1', \zeta_2' \ll 0$, that is, $2\zeta_1+\zeta_2 \ll 0$ and $\zeta_1 + k\zeta_2 \ll 0$. The equations (3.34) characterizing Coulomb branch vacua can be usefully combined into

$$\begin{aligned} (-k\hat\sigma_1+\hat\sigma_2)^{2k-1} &= e^{2t_1+t_2}\hat\sigma_1^2\hat\sigma_2, \\ (\hat\sigma_1 - 2\hat\sigma_2)^{2k-1} &= e^{t_1+kt_2}\hat\sigma_1\hat\sigma_2^k, \end{aligned} \tag{3.36}$$

  and in addition $|\hat\sigma_1|$, $|\hat\sigma_2|$, $|\hat\sigma_1-2\hat\sigma_2|$, $|-k\hat\sigma_1+\hat\sigma_2|$ must all be large. Dividing the equations by $(|\hat\sigma_1|+|\hat\sigma_2|)^{2k-1}$ and using that the exponentials are small in this phase, we learn that both $-k\hat\sigma_1+\hat\sigma_2$ and $\hat\sigma_1-2\hat\sigma_2$ must be parametrically smaller than $|\hat\sigma_1|+|\hat\sigma_2|$. This is impossible since $k\neq 1/2$. There are no Coulomb branch vacua in this phase.

- **First divisor blown up** ($A=\{1\}$): $2\zeta_1+\zeta_2 \gg 0$ and $\zeta_2 \ll 0$. For this phase we work out the $t_2\to-\infty$ asymptotics of (3.35) to be

$$\hat\sigma_1 \sim \hat\sigma_2 \sim (-k\hat\sigma_1+\hat\sigma_2) \sim e^{(2t_1+t_2)/(2k-4)}, \qquad (\hat\sigma_1 - 2\hat\sigma_2) \sim e^{(2t_1+(k-1)t_2)/(2k-4)}. \tag{3.37}$$

While the first three combinations are large throughout the phase, the last one is large only in a sub-phase $2\zeta_1 + (k-1)\zeta_2 \gg 0$. In that sub-phase we get $n-3 = 2k-4$ Coulomb branch vacua.

In the other sub-phase, the mass $|\hat\sigma_1 - 2\hat\sigma_2|$ of the chiral multiplet X_2 becomes small, so that a better approximation is to only integrate out X_0, X_1, X_3 and get an effective twisted superpotential for a vector multiplet scalar constrained to have $\hat\sigma_1 - 2\hat\sigma_2 = 0$. The asymptotics (3.37) remain correct for $\hat\sigma_1$, $\hat\sigma_2$, $(-k\hat\sigma_1 + \hat\sigma_2)$. The non-trivial D-term equation reads, up to unimportant constant shifts,

$$\begin{pmatrix}1\\-2\end{pmatrix}|X_2|^2 = \begin{pmatrix}\zeta_1\\\zeta_2\end{pmatrix} + \left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-k\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)\frac{2\zeta_1+\zeta_2}{2k-4} = -\frac{2\zeta_1+(k-1)\zeta_2}{2k-4}\begin{pmatrix}1\\-2\end{pmatrix}. \tag{3.38}$$

There are solutions, hence mixed branch vacua, when $2\zeta_1+(k-1)\zeta_2 \ll 0$. The number of solutions is $2k-4$ because that is the number of possible overall phases for $\hat\sigma$, just like in the other sub-phase. Upon crossing the wall, the $2k-4$ vacua remain well-separated in the direction transverse to $\hat\sigma_1 - 2\hat\sigma_2 \simeq 0$, hence each vacuum gets deformed continuously to a vacuum on the other side. There is no phase transition: the isolated massive vacua simply correspond to different combinations (Coulomb versus mixed) of the UV fields.

- **Second divisor blown up** ($A = \{2\}$): $\zeta_1 + k\zeta_2 \gg 0$ and $\zeta_1 \ll 0$. In this phase we rewrite (3.34) as

$$\begin{aligned}\hat\sigma_1(\hat\sigma_1 - 2\hat\sigma_2) &= e^{-t_1}(-k\hat\sigma_1+\hat\sigma_2)^k,\\ (\hat\sigma_1-2\hat\sigma_2)^{2k-1} &= e^{t_1+kt_2}\hat\sigma_1\hat\sigma_2^k.\end{aligned} \tag{3.39}$$

with $|\hat\sigma_1|$, $|\hat\sigma_2|$, $|\hat\sigma_1 - 2\hat\sigma_2|$, $|-k\hat\sigma_1+\hat\sigma_2|$ all big. Since $\zeta_1 \ll 0$, the first equation requires $(-k\hat\sigma_1+\hat\sigma_2)$ to be parametrically smaller than $|\hat\sigma_1|+|\hat\sigma_2|$. Plugging $\hat\sigma_2 \simeq k\hat\sigma_1$ in the second equation gives

$$\hat\sigma_1^{k-2} \simeq e^{t_1+kt_2}k^k(1-2k)^{1-2k}, \tag{3.40}$$

which has $k-2$ solutions. From the first equation we then deduce

$$(-k\hat\sigma_1+\hat\sigma_2) \simeq e^{(t_1+2t_2)/(k-2)}k^{2/(k-2)}(1-2k)^{-3/(k-2)} \tag{3.41}$$

which is large or not depending on the sign of $\zeta_1 + 2\zeta_2$. The line $\zeta_1+2\zeta_2 = 0$ splits the phase into two parts. In the sub-phase $\zeta_1+2\zeta_2 \gg 0$ we have $k-2$ Coulomb branch vacua, and in the other sub-phase, none.

The next step is to look for mixed branch vacua for which the mass $|-k\hat\sigma_1+\hat\sigma_2|$ of X_1 is small. The asymptotics (3.40) are unchanged and the D-term equation reads essentially

$$\begin{pmatrix}-k\\1\end{pmatrix}|X_1|^2 = \begin{pmatrix}\zeta_1\\\zeta_2\end{pmatrix} + \left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)\frac{\zeta_1+k\zeta_2}{k-2} = -\frac{\zeta_1+2\zeta_2}{k-2}\begin{pmatrix}-k\\1\end{pmatrix}. \tag{3.42}$$

There are solutions, hence mixed branch vacua, when $\zeta_1+2\zeta_2 \ll 0$. Each of the $k-2$ Coulomb branch vacua of one subphase gets deformed continuously to a mixed branch vacuum in the other subphase.

- **Both divisors blown up** ($A = \{1,2\}$): $\zeta_1 \gg 0$, $\zeta_2 \gg 0$. The $t_2 \to +\infty$ asymptotics of (3.35) are different for the v_+ and the v_- solutions. For v_+,

$$\hat\sigma_2 \sim \hat\sigma_1 \sim (\hat\sigma_1 - 2\hat\sigma_2) \sim e^{(t_1+kt_2)/(k-2)}, \qquad (-k\hat\sigma_1+\hat\sigma_2) \sim e^{(t_1+2t_2)/(k-2)} \tag{3.43}$$

and for v_-,

$$\hat{\sigma}_1 \sim (\hat{\sigma}_1 - 2\hat{\sigma}_2) \sim (-k\hat{\sigma}_1 + \hat{\sigma}_2) \sim e^{t_1/(k-2)}, \qquad \hat{\sigma}_2 \sim e^{t_1/(k-2)} e^{-t_2}. \tag{3.44}$$

All of these $n - 3 = 2k - 4$ solutions are genuine Coulomb branch vacua.

3.4.3 General rank

In general models it is difficult to determine Coulomb branch and mixed branch vacua explicitly, but we can count them. The idea is to turn on generic twisted masses so that the Higgs branch reduces to isolated massive vacua too. Since one can smoothly vary between any two phases, by turning on a theta angle to avoid singularities in FI-theta parameter space, the number of vacua in all phases must be the same. In particular, in the orbifold phase there are n Higgs branch vacua due to twisted sectors, and no Coulomb/mixed branch vacua.

The question then boils down to finding the number of vacua on the Higgs branch when generic twisted masses are turned on. We reiterate that the Higgs branch only changes at classical phase boundaries, shifted according to the discussion below (2.9).

The effect of twisted masses m_i is to change the mass equation from $(Q^i \cdot \sigma)X_i = 0$ to $(Q^i \cdot \sigma + m_i)X_i = 0$ for all i. For generic m_i, at most r of the masses $Q^i \cdot \sigma + m_i$ can vanish at the same time, so at most r of the X_i may be non-zero. In other words, at least two of the X_i must vanish. The Higgs branch thus reduces to the intersections $E_i \cap E_j = \{X_i = X_j = 0\}$. At each such intersection there are $d_{ij} \geq 1$ vacua, due to the twisted sectors for the $\mathbb{Z}_{d_{ij}}$ orbifold group. Altogether there are

$$\begin{aligned} d_{0\alpha_1} + d_{\alpha_1\alpha_2} + \cdots + d_{\alpha_{\ell-1}\alpha_\ell} + d_{\alpha_\ell(r+1)} &\quad \text{Higgs branch vacua,} \\ n - \big(d_{0\alpha_1} + d_{\alpha_1\alpha_2} + \cdots + d_{\alpha_{\ell-1}\alpha_\ell} + d_{\alpha_\ell(r+1)}\big) &\quad \text{Coulomb/mixed vacua,} \end{aligned} \tag{3.45}$$

where we denoted $A = \{\alpha_1, \ldots, \alpha_\ell\}$ the set of blown up exceptional divisors. When we turn off twisted masses the isolated Coulomb/mixed branch vacua are unaffected while Higgs branch vacua spread onto the whole Higgs branch.

In subsection 6.2 we explain how the number of Coulomb/mixed branch vacua jumps when crossing a wall, by restricting to a local model of the wall with gauge group $U(1) \times \Gamma$ for Γ a discrete group. In Hirzebruch-Jung models, crossing a wall means blowing up an exceptional divisor E_j. The local model (3.22) for this transition has gauge group $U(1) \times \mathbb{Z}_m$ with $m = \gcd(d_{ij}, d_{jk})$ where E_i and E_k are the two divisor intersecting E_j. Then the number of Coulomb branch vacua should increase by m times[6] the sum of $U(1)$ charges, so

$$m\left(\frac{d_{ik}}{m} - \frac{d_{ij}}{m} - \frac{d_{jk}}{m}\right) \geq 0. \tag{3.46}$$

This is consistent with (3.45). The jump vanishes if and only if the local model is Calabi-Yau, namely $a_{i+1} = \cdots = a_{k-1} = 2$.

4 Hemisphere partition function and B-branes

B-branes are a special class of boundary states in $\mathcal{N} = (2,2)$ 2d SCFTs that preserve a particular subalgebra of the superconformal algebra. In this work we mainly deal with theories that

[6] The gauge group is abelian so $\mathbb{Z}_m$ acts trivially on the non-zero vevs of σ in Coulomb branch vacua. Then $\mathbb{Z}_m$ converts each vacuum of the $U(1)$ theory to m due to twisted sectors.

do not flow to an SCFT in the IR. More precisely we work with $\mathcal{N} = (2,2)$ supersymmetric UV descriptions with an anomalous axial R-symmetry. B-branes can still be defined in these theories as boundary states that preserve a subalgebra $\mathbf{2}_B \subset (2,2)$ [22, 31, 32].

In this section we focus on the Higgs branch. Deep in a phase different branches of the theory are well-separated, hence a B-brane decomposes into one part for each branch in a way that we explore in section 6. For some class of B-branes, called grade restricted, the Higgs branch part has a simple construction.

We review how to compute the central charge of a B-brane, defined directly in the UV via supersymmetric localization [23, 33, 34]. This provides a powerful tool to analyze the behaviour of B-branes along the RG flow. The localization formula is singular in our non-compact setting and we determine how to regularize it using R-charges. Among several classes of branes in abelian GLSM we find that compactly-supported branes have finite central charge. We also analyse B-branes on Higgs branches of the Hirzebruch-Jung models and compute their (zero-instanton) central charges, which we compare with a K-theory calculation in section 5.

4.1 Field theory description of B-branes in GLSMs [review]

Consider a GLSM on a half space $\mathcal{H} = \mathbb{R} \times \mathbb{R}_{\leq 0}$. We denote left and right supercharges by $(\mathbf{Q}_\pm, \overline{\mathbf{Q}}_\pm)$. Then the $\mathbf{2}_B \subset (2,2)$ supersymmetry we want to preserve is the subalgebra generated by $\mathbf{Q}_+ + \mathbf{Q}_-$ and its conjugate $\overline{\mathbf{Q}}_+ + \overline{\mathbf{Q}}_-$.

The boundary term

$$\operatorname{Tr}_{R_G}\left(\operatorname{Pexp}\int_{\partial\mathcal{H}} A\right), \qquad A = (\operatorname{Re}(\sigma) + iv_\tau)\mathrm{d}\tau \tag{4.1}$$

is invariant under $\mathbf{2}_B$ supersymmetry and gauge-invariant for any representation R_G of G. When the superpotential W is nonzero we also need to add an extra term to the action in order to preserve $\mathbf{2}_B$. This term is also a holonomy,

$$\operatorname{Tr}_{R_G}\left(\operatorname{Pexp}\int_{\partial\mathcal{H}} \mathcal{A}(\mathbf{T})\right), \qquad \mathcal{A}(\mathbf{T}) \coloneqq \frac{1}{2}\psi^i\partial_i\mathbf{T} - \frac{1}{2}\bar{\psi}^{\bar{i}}\bar{\partial}_{\bar{i}}\mathbf{T}^\dagger - \frac{1}{2}\{\mathbf{T}, \mathbf{T}^\dagger\}, \tag{4.2}$$

where $\mathbf{T}$ is a matrix factorization of W (the tachyon profile) [2, 35]. The boundary contribution is thus the holonomy of $A + \mathcal{A}(\mathbf{T})$.

Altogether, specifying the B-brane requires the following **algebraic data** $\mathcal{B} = (M, \rho, \mathbf{r}_*, \mathbf{T})$.

- A $\mathbb{Z}_2$-graded, finite dimensional free $\operatorname{Sym}(V^*)$ module (**Chan-Paton vector space**) denoted by $M = M_0 \oplus M_1$. For the cases when $W \neq 0$, we will need $\operatorname{rank}(M_0) = \operatorname{rank}(M_1)$.
- Two representations, $\rho\colon G \to GL(M)$ and $\mathbf{r}_*\colon \mathfrak{u}(1)_V \to \mathfrak{gl}(M)$.
- A **matrix factorization** $\mathbf{T} \in \operatorname{End}_{\operatorname{Sym}(V^*)}(M)$ of the superpotential $W \in \operatorname{Sym}(V^*)$, i.e., a $\mathbb{Z}_2$ odd endomorphism such that $\mathbf{T}^2 = iW \cdot \mathrm{id}_M$ and such that the group actions ρ and $\mathbf{r}_*$ are compatible with the action of G and $U(1)_V$ on the chiral matter $X \in V$: for all $\lambda \in U(1)_V$ and $g \in G$,

$$\begin{aligned} \lambda^{\mathbf{r}_*}\mathbf{T}(\lambda^R X)\lambda^{-\mathbf{r}_*} &= \lambda\mathbf{T}(X), \\ \rho(g)^{-1}\mathbf{T}(\rho_{\mathrm{m}}(g)\cdot X)\rho(g) &= \mathbf{T}(X). \end{aligned} \tag{4.3}$$

However, there is still a piece of data that we need to fix to fully define a B-brane on a GLSM: a profile for the vector multiplet scalar σ. This data consists of a gauge-invariant middle-dimensional subvariety of $\mathfrak{g}_{\mathbb{C}}$, or equivalently its intersection $L \subset \mathfrak{t}_{\mathbb{C}}$ with the Cartan algebra, which we refer to as the contour. An **admissible contour** is a gauge invariant, middle dimensional L that is a continuous deformation of the real contour $L_{\mathbb{R}} := \{\operatorname{Im}\sigma = 0\}$ such that the boundary effective twisted superpotential

$$\widetilde{W}_{\mathrm{eff},\rho}(\sigma) := \biggl(\sum_{\alpha>0} \pm i\pi\,\alpha\cdot\sigma\biggr) - \biggl(\sum_j (Q^j\cdot\sigma)\Bigl(\log\Bigl(\frac{iQ^j\cdot\sigma}{\Lambda}\Bigr) - 1\Bigr)\biggr) - t\cdot\sigma + 2\pi i\,\rho\cdot\sigma \quad (4.4)$$

approaches $+\infty$ in all asymptotic directions of L. Signs in the sum over positive roots α of G depend on the Weyl chamber in which $\operatorname{Re}\sigma$ lies; this sum is absent in abelian GLSMs. The full B-brane is then given by $(\mathcal{B}, L)$.

The brane's central charge is given by [23][7]

$$Z_{D^2}(\mathcal{B}) = C(\mathfrak{r}\Lambda)^{\hat{c}/2}\int_\gamma \mathrm{d}^{l_G}\hat{\sigma}\biggl(\prod_{\alpha>0}(\alpha\cdot\hat{\sigma})\sinh(\pi\alpha\cdot\hat{\sigma})\biggr)\prod_j \Gamma\Bigl(iQ^j\cdot\hat{\sigma} + \frac{R_j}{2}\Bigr)e^{it\cdot\hat{\sigma}} f_{\mathcal{B}}(\hat{\sigma}). \quad (4.5)$$

Here $\mathfrak{r}$ is the radius of the disk D^2 and Λ the UV energy scale. So we identify $\mu = \mathfrak{r}^{-1}$ and then

$$t_\alpha(\mu) = \zeta_\alpha - i\theta_\alpha - (\sum_j Q^j_\alpha)\log\mathfrak{r}\Lambda \quad (4.6)$$

Note that the only dependence of the partition function on the choice of brane is through the **brane factor**

$$f_{\mathcal{B}}(\hat{\sigma}) := \operatorname{Tr}_M\Bigl(e^{i\pi\mathbf{r}_*}e^{2\pi\rho(\hat{\sigma})}\Bigr), \quad (4.7)$$

which itself does not depend on the matrix factorization $\mathbf{T}$.

In the following we focus on abelian GLSMs with zero superpotentials.[8] The partition function is then

$$Z_{D^2}(\mathcal{B}) = C(\mathfrak{r}\Lambda)^{\hat{c}/2}\int_L \mathrm{d}^r\hat{\sigma}\prod_j \Gamma\Bigl(iQ^j\cdot\hat{\sigma} + \frac{R_j}{2}\Bigr)e^{it\cdot\hat{\sigma}} f_{\mathcal{B}}(\hat{\sigma}). \quad (4.8)$$

To be precise, the localization formula was only derived when R-charges of all chiral multiplets obey $0 < R_j < 2$. This ensures that none of the poles of one-loop determinants $\Gamma(iQ^j\cdot\hat{\sigma}+R_j/2)$ lie on the contour $L_{\mathbb{R}}$ that L is a deformation of. The contour $L_{\mathbb{R}}$ must then be deformed, without crossing any poles, into a contour L that ensures convergence at large $|\hat{\sigma}|$.

Calabi-Yau abelian GLSM. Consider first the Calabi-Yau case. For each phase, the contour integral can be closed and expressed as a sum of residues. Each Gamma function has poles along an infinite family of parallel hyperplanes $Q^j\cdot\hat{\sigma} = i(R_j/2 + k)$ for $k \geq 0$. For generic R_j, hyperplanes have at most r-fold intersections, which are solutions $\hat{\sigma}_{J,k}$ of $Q^j\cdot\hat{\sigma} = i(R_j/2 + k_j)$ for $j \in J$, where J is a set of r flavours and $k_j \geq 0$ are integers that physically count world-sheet instantons (vortices).

[7] We give the result for a non-abelian gauge group.

[8] Then $\mathbf{T}$ squares to zero, namely $M_0 \xrightarrow{\mathbf{T}} M_1 \xrightarrow{\mathbf{T}} M_0$ is a $\mathbb{Z}_2$-graded complex.

When closing the contours one picks up a residue from some of these families of points. The choiceof what J to pick up depends on the direction in which contours are closed, which depends on the asymptotics of the integrand and in particular on the phase in which the FI parameter $\zeta = \operatorname{Re} t$ lies. A convenient shortcut is to use that the sum of residues should converge: the factor $e^{it\cdot\hat{\sigma}}$ must involve positive powers of all $e^{-\zeta k_j}$. This occurs precisely when ζ is a positive linear combination of the Q^j, namely when $\zeta \in \mathrm{Cone}_J$:

$$Z_{D^2,\text{residue}}(\mathcal{B}) = \frac{(\mathfrak{r}\Lambda)^{\hat{c}/2}}{(-2\pi i)^{\dim V - r}} \sum_{J|\zeta\in\mathrm{Cone}_J} \sum_{k:J\to\mathbb{Z}_{\geq 0}} \pm \operatorname*{res}_{i\hat{\sigma}=i\hat{\sigma}_{J,k}} \left(\prod_j \Gamma\left(iQ^j\cdot\hat{\sigma} + \frac{R_j}{2}\right) e^{it\cdot\hat{\sigma}} f_{\mathcal{B}}(\hat{\sigma}) \right), \tag{4.9}$$

where we fixed the normalization constant C for later convenience. The shortcut does not fix the sign with which residues should be summed.[9] A more precise analysis gives that the sign $\pm$ in this formula is $\operatorname{sign}\big(\det(Q^j)_{j\in J}\big)$. Incidentally, the sign and the poles that contribute coincide with those selected by the Jeffrey-Kirwan (JK) prescription with JK parameter ζ.

We include the subscript "residue" to conveniently refer to the same equation in cases where this is not the complete hemisphere partition function. Here, $Z_{D^2}(\mathcal{B}) = Z_{D^2,\text{residue}}(\mathcal{B})$.

Non-Calabi-Yau abelian GLSMs. We turn to theories with anomalous $U(1)_A$. In section 6 we discuss the shape of the contour L and the behaviour of the integrand at large $|\hat{\sigma}|$, which depends on the brane factor. In phases with only a Higgs branch, namely such that $Q^{\text{tot}} = \sum_j Q^j$ belongs to the closure of the phase, the contour can be closed and $Z_{D^2}(\mathcal{B})$ gives the sum of residues (4.9).

In other phases, the series appearing in (4.9) are asymptotic series. Finitely deep in the phase, the contour integral is evaluated by deforming L to pass finitely many poles. This expresses $Z_{D^2}(\mathcal{B})$ as a sum of finitely many residues in the asymptotic series, plus a remaining contour integral. Provided the brane is "grade restricted" (a bound on the charge vectors of the representation ρ hence on the degrees of $f_{\mathcal{B}}$ as a multivariate polynomial in the $\exp 2\pi\hat{\sigma}^\alpha$), the remaining contour integral is computed using a saddle-point approximation that identifies it with a contribution due to mixed or Coulomb branches. For such grade restricted branes, the Higgs branch contribution is exclusively due to the asymptotic series of residues.

We focus on resolving the singularity at $\hat{\sigma} = 0$ that occurs in our models because of vanishing R-charges. We denote by $Z^{\text{0-instanton}}_{D^2,\text{residue}}(\mathcal{B})$ this zero-instanton term, $k = 0$ in (4.9).

4.2 B-brane category of Higgs branches in abelian GLSMs [review]

Branes preserving B-type supersymmetry naturally form a category whose morphisms are given by quantizing string states between them. For a sigma model to a target X this category is equivalent [36] to the derived category $D^b(X)$ whose objects are given by bounded chain complexes of vector bundles[10] and whose morphisms are given by chain complexes modulo chain homotopy with quasi-isomorphisms formally inverted.

For targets X that can be realized as the Higgs branch of a GLSM in some phase, it is useful to start with the brane category of the UV GLSM, denoted by $D^b(V, G)$. This is

[9] To cancel some factors of i we take the residue of the integrand as a function of $i\hat{\sigma}$ rather than $\hat{\sigma}$. For instance $\operatorname{res}_{\hat{\sigma}=0}\Gamma(i\hat{\sigma}) = -i$ while $\operatorname{res}_{i\hat{\sigma}=0}\Gamma(i\hat{\sigma}) = 1$.

[10] A useful fact is that it is equivalent to work with complexes of vector bundles when coherent sheaves admits locally free resolutions, which is the case in all the cases considered in this work.

a G-equivariant[11] version of $D^b(V)$; for instance for $G = U(1)^r$, objects in this category are equivalent to complexes of Wilson lines (equivariant line bundles) $\mathcal{W}(q)$ for $q \in \mathrm{Hom}(G, U(1))$. Any vector bundle on V is of course trivial, but G may act nontrivially. Then one can flow from the UV GLSM down to the IR, or as discussed below (2.7) to some intermediate energy scale well-described by the phase of interest. Each brane in $D^b(V, G)$ flows to a brane in this phase, which typically has contributions from all branches of the theory. In particular it has a Higgs branch part which lies in $D^b(X)$. Besides this functor $F_{\text{flow,Higgs}} \colon D^b(V, G) \to D^b(X)$, the restriction and projection from V to $X = (V \setminus \Delta)/G$ provide $F_{\text{geom}} \colon D^b(V, G) \to D^b(X)$.

Unless stated otherwise, in this section (section 4) we work with F_{geom}. In section 6 we learn that these two functors are equal in phases with only a Higgs branch (such as all phases in Calabi-Yau models), and that they otherwise agree for B-branes that are grade restricted, namely that are built from Wilson lines with charges in some range. In other words, for these GLSM branes, naive geometric considerations give the correct Higgs branch image. We also explain in section 6 how every GLSM brane is equivalent to one that is suitably grade restricted. Together this allows to determine the Higgs branch image of every GLSM brane.

The derived category of a toric variety or orbifold is generated by the line bundles [37, 2]

$$\mathcal{O}(q) \coloneqq F_{\text{geom}}\big(\mathcal{W}(q)\big). \tag{4.10}$$

We also introduce the notation $\mathcal{F}(q) \coloneqq \mathcal{F} \otimes \mathcal{O}(q)$ for any sheaf $\mathcal{F}$. From the GIT viewpoint, the deleted set in a given phase is $\Delta = \bigcup_J \Delta_J$ for a collection of linear subspaces $\Delta_J = \{X_i = 0\ \forall i \in J\}$. Each structure sheaf $\mathcal{O}_{\Delta_J} \in D^b(V, G)$ is mapped by F_{geom} to a trivial brane in $D^b(X)$. This sheaf admits a Koszul resolution as a complex of equivariant line bundles, mapped by F_{geom} to a complex of line bundles $\mathcal{O}(q)$. The resulting complex is trivial. This explains why the derived category $D^b(X)$ is generated by the $\mathcal{O}(q)$ subject to one relation for each deleted set Δ_J. One interpretation is that we start in the UV with a "free" category consisting of all possible bound states of Wilson lines and then in different phases we impose different sets of relations to construct the B-brane categories $D^b(X)$; these relations in turn encode the geometry of X.

Let us describe the Koszul resolution of the structure sheaf $\mathcal{O}_{\{X_j=0\}}$ of a hyperplane. For explicit calculations it is helpful to treat coherent sheaves as modules. Then we wish to resolve the $R = \mathbb{C}[X_1, \dots, X_N]$-module $R/(X_j)$. The resolution is a two-term complex of line bundles with multiplication by X_j as its sole morphism: indeed, the sequence

$$0 \to R \xrightarrow{X_j} R \twoheadrightarrow R/(X_j) \to 0, \quad \text{that is,} \quad 0 \to \mathcal{O} \xrightarrow{X_j} \mathcal{O} \twoheadrightarrow \mathcal{O}_{\{X_j=0\}} \to 0, \tag{4.11}$$

is exact. Note that it is important to keep up with the gauge charges at each stage of the resolution: different gauge charges lead to different bundles on the quotient X. To resolve the module $R/(X_j)$ the two R above should be taken to have gauge charges $-Q^j$ and 0, respectively, where Q^j is the charge vector of X_j.

The Koszul resolution of $\mathcal{O}_{\Delta_J}$ is then obtained by using that $\mathcal{O}_{\Delta_J} = \bigotimes_{j \in J} \mathcal{O}_{\{X_j=0\}}$. Denoting m the number of elements of J we get the resolution

$$0 \to \mathcal{O} \to \mathcal{O}^{\oplus m} \to \cdots \to \mathcal{O}^{\oplus \binom{m}{k}} \to \cdots \to \mathcal{O}^{\oplus m} \to \mathcal{O} \to 0 \tag{4.12}$$

where the $2^m = \sum_{k=0}^m \binom{m}{k}$ copies of $\mathcal{O}$ in the resolution are labeled by subsets $I \subset J$, and the non-zero maps are as follows: from the copy of $\mathcal{O}$ labelled $I \sqcup \{j\}$ to that labelled I the map

[11] Our notations leave implicit the action of G on the vector space V in which chiral multiplets take values.

is multiplication by $\pm X_j$. Signs are chosen to make (4.12) exact. The resolution of $\mathcal{O}_{\Delta_J}(q)$ is then this complex in which the copy of $\mathcal{O}$ labelled I has gauge charge $q - \sum_{j\in I} Q^j$ and R-charge $-\sum_{j\in I} R_j/2$. We deduce that $\mathcal{O}_{\Delta_J}(q)$ has brane factor

$$f_J(\hat{\sigma}) = e^{2\pi q\cdot\hat{\sigma}} \prod_{j\in J} (1 - e^{-2\pi Q^j\cdot\hat{\sigma}+i\pi R_j}). \tag{4.13}$$

Interestingly, this brane factor has zeros at every pole of one-loop determinants of X_j for $j \in J$. We will return to this.

In phases with only a Higgs branch there is no grade restriction so we learn that the Wilson line analogue $0 \to \mathcal{W} \to \mathcal{W}^{\oplus m} \to \cdots \to \mathcal{W}^{\oplus m} \to \mathcal{W} \to 0$ of (4.12) flows to an empty brane. In other phases, it turns out that this complex of Wilson lines is grade-restricted for a suitable range of q. For such q, the brane flows to some brane whose Higgs branch part is empty but whose mixed and Coulomb parts are typically non-trivial.

4.3 Regularization for one-parameter models

Before coming back to general considerations for multiparameter models in the next subsection, let us do some calculations in one-parameter ($r = 1$) Hirzebruch-Jung models. These have gauge group $U(1)$ and three chiral multiplets X_0, X_1, X_2 of charges 1, $-n$, 1. The orbifold phase $\zeta \ll 0$ has Higgs branch $\mathbb{C}^2/\mathbb{Z}_{n(1)}$. The resolved phase $\zeta \gg 0$ has Higgs branch the total space of $\mathcal{O}(-n) \to \mathbb{P}^1$, and has $n-2$ massive vacua. As explained in section 6, there is a grade restriction rule in the resolved phase: only branes built from Wilson lines $\mathcal{W}(q)$ within the window $|\theta/(2\pi)+q| < n/2$ are such that the sum of residues gives the correct Higgs branch contribution to the hemisphere partition function. Nevertheless, we work out in the next subsections that the quantity relevant to the Higgs branch geometry is $Z_{D^2,\text{residue}}(\mathcal{B})$, whose 0-instanton part we focus on now.

The localization formula for the hemisphere partition function (4.8) is only valid when R-charges are all in the interval $(0, 2)$. We thus turn on positive R-charges R_0, R_1, R_2 for X_0, X_1, X_2. The localization result

$$Z_{D^2}(\mathcal{B}) = \frac{(\mathfrak{r}\Lambda)^{\hat{c}/2}}{(-2\pi i)^2} \int_L \frac{\mathrm{d}\hat{\sigma}}{2\pi} \Gamma(i\hat{\sigma} + R_0/2)\Gamma(-in\hat{\sigma} + R_1/2)\Gamma(i\hat{\sigma} + R_2/2)e^{it\hat{\sigma}} f_{\mathcal{B}}(\hat{\sigma}) \tag{4.14}$$

is then well-defined, and poles due to positively and negatively charged chiral multiplets are on different sides of the contour:

$$i\hat{\sigma} = -R_j/2 - k < 0 \qquad \text{for } j \in \{0, 2\} \text{ and } k \geq 0, \tag{4.15}$$

$$i\hat{\sigma} = \frac{1}{n}\big(R_1/2 + k\big) > 0 \quad \text{for } k \geq 0. \tag{4.16}$$

By mixing R-symmetry with the gauge symmetry, we take $R_1 = 0$ and deform L slightly to keep all poles of each Gamma function on the same side of L as for positive R-charges. Contrarily to compact models, the remaining R-charges R_0 and R_2 are needed for regularization and the localization result depends non-trivially on them: there are divergences as $R_0, R_2 \to 0$.

With $R_1 = 0$, consider the limit $R_0 \to 0$; then the contour gets pinched between a pair of poles at negative and zero $i\hat{\sigma}$. (A slightly more complicated pinching occurs as both $R_0, R_2 \to 0$.) On very general grounds such a pinching makes the integral blow up like the inverse distance between the poles. For any function f that is holomorphic in x in a neighborhood of the contour,

$$\int_{\mathbb{R}} \frac{f(x)\,dx}{(x-i\varepsilon)(x+i\varepsilon)} = 2\pi i \frac{f(i\varepsilon)}{2i\varepsilon} + \int_{\rightarrow\frown\rightarrow} \frac{f(x)\,dx}{(x-i\varepsilon)(x+i\varepsilon)} = \frac{2\pi i f(0)}{2i\varepsilon} + O(1) \tag{4.17}$$

as $\varepsilon \to 0$, where the contour is a slight deformation of $\mathbb{R}$ that goes above both poles. We will not need the ε^0 term but a quick calculation shows that it is given by a principal value prescription $\lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} (f(x) - f(0))dx/x^2$. Observe also that the singular term can be computed directly as $f(0) = \operatorname{res}_{x\to 0}\bigl(x \lim_{\varepsilon\to 0}(\text{integrand})\bigr)$, without explicitly decomposing the integrand into two singular factors $(x \pm i\varepsilon)^2$ and a holomorphic function f.

Coming back to our $U(1)$ model with charges 1, $-n$, 1, we discuss five instructive cases in the resolved phase:

- the image $\mathcal{O}(q)$ of a Wilson line,
- the structure sheaf $\mathcal{O}_{E_1}$ of the exceptional divisor,
- other branes with compact support E_1,
- noncompact branes with the same brane factor but different regularized partition functions,
- noncompact branes that only respect part of the Higgs branch isometry but can still be regularized.

As our first example of brane we specialize (4.14) to a Wilson line of charge q, whose brane factor is $e^{2\pi q\hat{\sigma}}$. Like in (4.17) we can shift the contour through the pole at $i\hat{\sigma} = 0$ and the remaining integral is smooth as $R_j \to 0$, so the only singular contribution as $R_0, R_2 \to 0$ is the residue

$$Z^{\text{0-instanton}}_{D^2,\text{residue}}(\mathcal{O}(q)) \simeq \frac{(\mathfrak{r}\Lambda)^{\hat{c}/2}}{n} \frac{\Gamma(R_0/2)}{-2\pi i} \frac{\Gamma(R_2/2)}{-2\pi i} + O(1) = -\frac{(\mathfrak{r}\Lambda)^{\hat{c}/2}}{n\pi^2 R_0 R_2} + O\biggl(\frac{1}{R_0}\biggr) + O\biggl(\frac{1}{R_2}\biggr). \tag{4.18}$$

This quadratic divergence is not an artifact of how we regularized. Up to a factor it is the $U(1)^2$ equivariant volume of an orbifold of $\mathbb{C}^2$, the $\mathbb{Z}_n$ orbifold group being responsible for the $1/n$ factor. More precisely, the divergent terms are ($1/n$ times) the partition function of a pair of free chiral multiplets of R-charges R_0 and R_2, which parametrize the two non-compact directions in the support of the brane. None of the divergent terms depends on the charge q.

Next, consider the structure sheaf of the exceptional divisor. The structure sheaf has Koszul resolution $\mathcal{O}(n) \xrightarrow{X_1} \mathcal{O}$ hence brane factor $f_1(\hat{\sigma}) = 1 - e^{2\pi n\hat{\sigma}}$, that is, a difference of Wilson line brane factors with $q = 0$ and $q = n$. Taking the difference cancels all $R_j \to 0$ divergences of (4.18). As we already discussed in (4.25),

$$\Gamma(-in\hat{\sigma}) f_1(\hat{\sigma}) = \frac{-2\pi i e^{\pi n\hat{\sigma}}}{\Gamma(1 + in\hat{\sigma})} \tag{4.19}$$

has no pole. Then the localization formula does not exhibit contour pinching since all poles on one side have been cancelled. We compute, through a contour integral or directly through (4.9),

$$\begin{aligned} Z^{\text{0-instanton}}_{D^2,\text{residue}}\Bigl(\mathcal{W}(n) \xrightarrow{X_1} \mathcal{W}(0)\Bigr) &= \frac{(\mathfrak{r}\Lambda)^{\hat{c}/2}}{(-2\pi i)^2} \int_0 \frac{\mathrm{d}\hat{\sigma}}{2\pi} \frac{-2\pi i e^{\pi n\hat{\sigma}}}{\Gamma(1+in\hat{\sigma})} \Gamma(i\hat{\sigma})^2 e^{ti\hat{\sigma}} \\ &= (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{t - i\pi n + (n-2)\gamma}{-2\pi i}, \end{aligned} \tag{4.20}$$

where $\int_0$ denotes an integral around the pole at $i\hat{\sigma} = 0$. The key aspect of this brane that leads to a finite partition function is that poles of the one-loop determinant of X_1 are cancelled by the brane factor. If we had given a positive R-charge R_1 to $X = X_1$ too, it would appear in the brane factor in exactly the correct way to cancel the pole of the one-loop determinant, namely through $-in\hat{\sigma} \to -in\hat{\sigma} + R_1/2$. Note that this brane is not grade-restricted.

Any brane that is supported on the exceptional divisor E_1 has a brane factor that is a multiple of f_1, before introducing R-charges R_j. For branes that do not respect the flavour symmetries (isometries of the Higgs branch) there is no preferred way to include R-charges in their brane factors. However, it is natural to impose that the brane factor is still a multiple of f_1. The brane factor then cancels poles of $\Gamma(-in\hat{\sigma})$, which avoids contour pinching. The regularized partition function is then finite as $R_j \to 0$, and unambiguous since adding any $O(R_j)$ terms (times f_1) to the brane factor simply shifts the regularized partition function by $O(R_j)$. For instance the twist $\mathcal{O}_E(q)$ with resolution $\mathcal{O}(q+n) \xrightarrow{X_1} \mathcal{O}(q)$ has brane factor $e^{2\pi q\hat{\sigma}} f_1(\hat{\sigma})$ which gives

$$Z^{\text{0-instanton}}_{D^2,\text{residue}}\Big(\mathcal{W}(q+n) \xrightarrow{X_1} \mathcal{W}(q)\Big) = (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{t - i\pi(n+2q) + (n-2)\gamma}{-2\pi i}, \tag{4.21}$$

found by shifting $t \to t - 2\pi i q$. Another example is the brane with resolution $\mathcal{O}(kn) \xrightarrow{X_1^k} \mathcal{O}(0)$, which results in $(\mathfrak{r}\Lambda)^{\hat{c}/2}\, k\big(t - i\pi kn + (n-2)\gamma\big)/(-2\pi i)$. Again this is finite, consistent with the fact that this brane's support is compact.

We now illustrate that non-compact branes with the same brane factor at $R_j = 0$ can have different, geometrically meaningful, regularized partition functions. Consider a brane $\mathcal{B}_{k_0,k_1,k_2}$ with resolution $\mathcal{O}(0) \xrightarrow{X_0^{k_0} X_1^{k_1} X_2^{k_2}} \mathcal{O}(k_0 - nk_1 + k_2)$ for some $k_i \geq 0$, where the arrow denotes multiplication by the monomial $X_0^{k_0} X_1^{k_1} X_2^{k_2}$. This brane is supported on the base $\mathbb{P}^1$ and two noncompact fibers: $\{X_0 = 0\} \cup \{X_1 = 0\} \cup \{X_2 = 0\}$. The brane factor, including R-charges, is then

$$f_{\mathcal{B}_{k_0,k_1,k_2}}(\hat{\sigma}) = 1 - e^{2\pi(k_0 - nk_1 + k_2)\hat{\sigma} - i\pi(k_0 R_0 + k_2 R_2)} \tag{4.22}$$

and its $R_j \to 0$ limit only depends on $k_0 - nk_1 + k_2$. For instance when $k_0 + k_2 = nk_1$ the brane factor is zero, as for an empty brane. The partition function computed using (4.18) is in general divergent:

$$\begin{aligned} Z^{\text{0-instanton}}_{D^2,\text{residue}}(\mathcal{B}_{k,l}) &= (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{1 - e^{-i\pi(k_0 R_0 + k_2 R_2)}}{(-2\pi i)^2} \left(\frac{1}{n}\Gamma\left(\frac{R_0}{2}\right)\Gamma\left(\frac{R_2}{2}\right) + O(1)\right) \\ &= (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{1}{n(2\pi i)} \left(k_0 \Gamma\left(\frac{R_2}{2}\right) + k_2 \Gamma\left(\frac{R_0}{2}\right)\right) + \dots \end{aligned} \tag{4.23}$$

Ignoring the factor $2\pi i$, the two terms have a geometric interpretation as contributions of the non-compact supports $\{X_0 = 0\}$ and $\{X_2 = 0\}$ of the brane, which have multiplicity k_0 and k_2 respectively.

A last instructive case is a brane $\mathcal{O}(0) \xrightarrow{G_k(X_0,X_2)} \mathcal{O}(k)$ for G_k a homogenous polynomial of degree k. Roots of G_k define points with homogenous coordinates $(X_0 : X_2)$ on $\mathbb{P}^1$. In the resolved phase the brane is supported on the corresponding fibers of the total space of $\mathcal{O}(-n) \to \mathbb{P}^1$. Unless G_k is a monomial, it does not transform in a definite way under the $U(1)^2$ isometries acting on X_0 and X_2 that we used to introduce R-charges R_0 and R_2. Thus one cannot deform the brane factor to introduce R_0 and R_2. However, G_k has definite charge under the diagonal subgroup $U(1) \subset U(1)^2$, so that it makes sense to turn on $R_0 = R_2 > 0$. Such an R-charge is enough to avoid contour pinching and regularize the partition function.

To summarize, the hemisphere partition function can be regularized for branes with support on E_1, and for non-compact branes that preserve an isometry of the Higgs branch. Contour pinching as $R_j \to 0$ reflects the existence of non-compact branes for which the correct partition function is infinite.

4.4 Regularization for compact branes in abelian GLSMs

We are interested in models with non-compact Higgs branch. In Calabi-Yau models the superconformal algebra of the IR limit contains a $U(1)$ R-symmetry that factorizes between left-moving and right-moving parts. Such a factorized symmetry must act trivially on non-compact directions.[12] This IR R-symmetry is typically visible in the UV (it could also be emergent), in which case the relevant localization calculation is the one involving that R-symmetry. As we just argued it must assign R-charge 0 to gauge-invariant polynomials in chiral multiplets that span the non-compact directions. The natural generalization to non-compact models that are not Calabi-Yau is to apply the localization result in which all non-compact directions have R-charge 0.

At face value the localization result is singular whenever any R-charge vanishes, because the contour passes through a pole at $\hat{\sigma} = 0$ (other poles are not problematic). The obvious regularization is to turn on a small positive R-charge R_j for each chiral multiplet X_j, that is, mix the R-symmetry with gauge and flavour symmetries. Geometrically, the mixing with flavour symmetries amounts to working equivariantly with respect to isometries of the Higgs branch. In principle such a regularization is only adapted for branes that preserve an isometry of the Higgs branch, but we find in examples that the regularization can be extended to some other branes with compact support.

In compact models it is typically possible to mix the R-symmetry with ε times a gauge symmetry so as to shift all R-charges into $(0, 2)$. This mixing with gauge charges amounts to a shift of the integration variable $\hat{\sigma}$. The localization result thus only depends on ε through an overall factor $e^{t\varepsilon/2}$, independent of the brane, and which disappears as $\varepsilon \to 0$. Since the resulting integral is regular, any $O(\varepsilon)$ correction to the brane factor drops out as $\varepsilon \to 0$. Therefore the result is finite for any brane, and is insensitive to the precise regularisation. As an example, consider the quintic hypersurface GLSM, a $U(1)$ model with chiral multiplets P and X_i, $1 \leq i \leq 5$ of charges -5 and 1 and with a superpotential $W = PG_5(X)$ for G_5 a generic degree 5 homogenous polynomial; instead of the usual R-charges 2 and 0 for P and X one uses $2 - 5\varepsilon$ and ε for $\varepsilon \in (0, 2/5)$. An alternative point of view, rather than shifting $\hat{\sigma}$, is that the contour L is not $\mathbb{R}$ but a shift (more generally a deformation) thereof such that all poles of $\Gamma(iQ^j\hat{\sigma})$ for $Q^j > 0$ are on one side of L, and those with $Q^j < 0$ on the other side.

In non-compact models, such as the quintic GLSM above without its superpotential, the non-compact directions are spanned by some gauge invariants with R-charge 0. Mixing R-symmetry with gauge symmetry does not affect their R-charge, thus it cannot make all chiral multiplets have positive R-charge.

Let

$$E_J = \{X_j = 0 \;\forall j \in J\} \tag{4.24}$$

[12] More precisely, in a nonlinear sigma model, the $U(1)$ isometry given by a Killing vector ξ_I (namely such that $\nabla_{(I}\xi_{J)} = 0$) factorizes if $\nabla_{[I}\xi_{J]} = 0$ too. A $U(1)$ rotation of a cylinder factorizes, but not a $U(1)$ rotation of a cone or plane.

denote subsets of the Higgs branch (in Hirzebruch-Jung models these include the exceptional divisors). For branes supported in a union of sets E_J that is compact we argue that the regularized partition function has an unambiguous finite $R_j \to 0$ limit, which we compute for Hirzebruch-Jung models to match it with a geometric calculation in section 5. In contrast, branes that are non-compact only have a meaningful regularized partition function if they respect enough isometries of the Higgs branch. The result typically diverges as $R_j \to 0$ and different branes that have the same brane factor at $R_j = 0$ may give different regularized partition functions.

4.4.1 Empty branes

For now let us just discuss GLSM branes whose geometric image (image under F_{geom}) in the Higgs branch category $D^b(X)$ is empty. Besides showing in a simple case that noncompactness of the Higgs branch is not an issue, the main purpose is to clarify the relation between the residue contribution and Higgs branch contribution to the partition function on the one hand, and the functors F_{geom} and $F_{\mathrm{flow,Higgs}}$ on the other hand.

Consider the brane $\mathcal{O}_{\Delta_K}$ for $\Delta_K = \{X_j = 0 \ \forall j \in K\}$ an irreducible component of the deleted set $\Delta \subset V$. Given the second description of Δ in (2.3), the possible K are characterized by the fact that $\zeta \notin \mathrm{Cone}_{(\complement K)}$, namely the FI parameter cannot be written as a linear combination of $\{Q^j \mid j \notin K\}$. Recall now that the Higgs branch hemisphere partition function (4.9) picks up residues labeled by a set of r flavours J such that $\zeta \in \mathrm{Cone}_J$. Together this implies that $J \not\subset \complement K$ namely $J \cap K \neq \emptyset$. Each pole that contributes obeys $Q^j \cdot \hat\sigma \in i(R_j/2 + \mathbb{Z}_{\geq 0})$ for all $j \in J$, hence for at least one $j \in K$. However, the brane factor (4.13) has zeros whenever $Q^j \cdot \hat\sigma \in i(R_j/2 + \mathbb{Z}_{\geq 0})$ for any $j \in K$. To reiterate, the brane factor cancels all poles of one-loop determinants of the chirals X_j, $j \in K$: Euler's formula $\Gamma(x)\sin\pi x = \pi/\Gamma(1-x)$ yields

$$\big(1 - e^{-2\pi Q^j\cdot\hat\sigma + i\pi R_j}\big)\Gamma\left(iQ^j \cdot \hat\sigma + \frac{R_j}{2}\right) = -2\pi i\, e^{-\pi Q^j\cdot\hat\sigma + i\pi R_j/2} \Big/ \Gamma\left(1 - iQ^j \cdot \hat\sigma - \frac{R_j}{2}\right) \tag{4.25}$$

which has no pole. All residue contributions in that phase are thus eliminated, namely $Z_{D^2,\mathrm{residue}} = 0$ for that brane.

Given a Higgs branch brane $\mathcal{B} \in D^b(X)$ we would now like to compute its hemisphere partition function, defined as the Higgs branch part of $Z_{D^2}(\mathcal{B}_{\mathrm{GLSM}})$ for $\mathcal{B} = F_{\mathrm{flow,Higgs}}(\mathcal{B}_{\mathrm{GLSM}})$. First, realize the brane geometrically as $\mathcal{B} = F_{\mathrm{geom}}(\mathcal{B}_1)$. Then the key observation is that one can bind $\mathcal{B}_1 \in D^b(V,G)$ with a collection of branes $\mathcal{O}_{\Delta_K}(q)$, for Δ_K part of the deleted set, until getting a brane $\mathcal{B}_2 \in D^b(V,G)$ that is grade restricted. In the language of section 6 it is enough to restrict to the big window. We then have that $\mathcal{B} = F_{\mathrm{geom}}(\mathcal{B}_1) = F_{\mathrm{geom}}(\mathcal{B}_2) = F_{\mathrm{flow,Higgs}}(\mathcal{B}_2)$ so we want to get the Higgs branch part of $Z_{D^2}(\mathcal{B}_2)$. As we already outlined (see details in section 6) that Higgs branch part is simply $Z_{D^2,\mathrm{residue}}(\mathcal{B}_2)$ because $\mathcal{B}_2$ is grade restricted. This, in turn is equal to $Z_{D^2,\mathrm{residue}}(\mathcal{B}_1)$ by the calculation we just made. We learn that for the purposes of computing the Higgs branch hemisphere partition function of $\mathcal{B}$ it is enough to compute $Z_{D^2,\mathrm{residue}}$ of any complex of Wilson lines that reduces geometrically to $\mathcal{B}$.

One should however be careful about the physical meaning of these calculations: the RG flow of complexes of Wilson lines only gives the geometric answer for grade-restricted branes. Even for such branes, if the phase has mixed/Coulomb branches the GLSM brane can flow to a non-trivial image on these branches. The same GLSM brane typically also has a non-empty image when considered in another phase: a pole with $J \subset {}^{\complement}K$ contributes in any phase such that $\zeta \in \mathrm{Cone}_J$.

In our calculations it was crucial that R-charges appear in the brane factor (4.13) in the same way as in chiral multiplet one-loop determinants, so that the cancellation (4.25) took place. We deduced the brane factor from the Koszul complex (4.12) whose morphisms are multiplications by chiral multiplets with well-defined R-charges. An arbitrary complex may in general fail to have well-defined R-charges; our regularization of the hemisphere partition function can then fail to be defined.

4.4.2 Compact branes

Next we discuss more generally the case of branes supported on distinguished subsets $E_K = \{X_j = 0 \mid j \in K\}$ of the Higgs branch. Depending on K this may be empty, compact, or noncompact.

The structure sheaf of E_K has a Koszul resolution (4.12) with brane factor (4.13).[13] As explained in (4.25) this brane factor cancels all poles of the one-loop determinant of X_j for $j \in K$: explicitly their product gives a factor $-2\pi i\, e^{\cdots}/\Gamma(1-\cdots)$ with no pole. Recall now that the residue part of the hemisphere partition function (4.9) is a sum, over sets J of r flavours such that $\zeta \in \mathrm{Cone}_J$, of residues at common poles of the chiral multiplets X_j, $j \in J$. Any such residue with $J \cap K \neq \emptyset$ vanishes due to the brane factor. The sum is thus restricted to $J \subset {}^{\complement}K$. Altogether,

$$Z_{D^2,\mathrm{residue}} \simeq (\mathfrak{r}\Lambda)^{\hat{c}/2} \sum_{\substack{J \subset {}^{\complement}K \mid \zeta \in \mathrm{Cone}_J \\ k: J \to \mathbb{Z}_{\geq 0}}} \pm \operatorname*{res}_{i\hat{\sigma} = i\hat{\sigma}_{J,k}} \left(\frac{e^{it\cdot\hat{\sigma}} e^{\sum_{j\in K}(-\pi Q^j \cdot \hat{\sigma} + i\pi R_j/2)} \prod_{j\notin K} \Gamma\big(iQ^j \cdot \hat{\sigma} + R_j/2\big)}{(-2\pi i)^{\#({}^{\complement}K \setminus J)} \prod_{j\in K} \Gamma\big(1 - iQ^j \cdot \hat{\sigma} - R_j/2\big)} \right), \tag{4.26}$$

where the sign $\pm$ is $\mathrm{sign}(\det(Q^\ell)_{\ell\in J})$.

The set E_K is defined as solutions to D-term equations with the further constraint $X_j = 0$ for $j \in K$, so

$$\sum_{i\notin K} Q^i |X_i|^2 = \zeta, \tag{4.27}$$

modulo gauge transformations. This has solutions ($E_K \neq \emptyset$) if and only if $\zeta \in \mathrm{Cone}_{({}^{\complement}K)}$. Under this condition, let us prove that E_K is compact if and only if there exists $\hat{s}$ such that $Q^i \cdot \hat{s} > 0$ for all $i \notin K$. If there exists such a $\hat{s}$ then the norm of points in E_K is bounded: $\|X\|^2 \leq (\zeta \cdot \hat{s}) / \min_{i\notin K}(Q^i \cdot \hat{s})$. Conversely, if there exists no such $\hat{s}$ then the polygonal cone $\mathrm{Cone}_{({}^{\complement}K)}$ with finitely many edges does not lie in any half-space, which implies that the cone contains a line through 0. In turn this implies that there exists a vanishing linear combination $\sum_{i\notin K} \lambda_i Q^i = 0$ with positive coefficients $\lambda_i > 0$. From any $X \in E_K$ we can then build arbitrarily large solutions by shifting each $|X_i|^2$ by the same multiple of λ_i, thus E_K is noncompact. We discuss each of these cases in turn: E_K empty, E_K compact, and E_K noncompact.

We have already treated near (4.25) the case where E_K is empty, namely where the subspace $\Delta_K = \{X_j = 0 \mid j \in K\}$ of V belongs to the deleted set Δ. This means $\zeta \notin \mathrm{Cone}_{({}^{\complement}K)}$. As we just discussed, any J that appears in the sum (4.26) obeys $J \subset {}^{\complement}K$ and $\zeta \in \mathrm{Cone}_J$,

[13] In case E_K is contained in an orbifold singularity, its structure sheaf is actually a fractional brane rather than a usual D-brane wrapping E_K.

hence $\zeta \in \mathrm{Cone}_J \subset \mathrm{Cone}_{(\complement K)}$, which is a contradiction. Thus, the hemisphere partition function has no residue contribution in that case: provided it is grade restricted, the brane is a **Higgs-empty** brane in the given phase.

Consider next the case of a compact E_K, such that there exists $\hat{s}$ with $Q^i \cdot \hat{s} > 0$ for all $i \notin K$. In the contour formula for the hemisphere partition function, shift the integration variable $\hat{\sigma}$ to $\hat{\sigma} - i\varepsilon\hat{s}$ for some small $\varepsilon > 0$. This shifts the argument of all numerator Gamma functions (those with $j \notin K$) by a positive amount, just like R-charges, thus none of their poles intersect the contour as all $R_j \to 0$. We have no control on the signs of $Q^i \cdot \hat{s}$ for $i \in K$, which are shifts of arguments of Gamma functions in the denominator, but these factors do not contribute any pole. Altogether the contour integral remains finite as $R_j \to 0$. Just as in one parameter examples, the brane factor of any brane supported on E_K should be a multiple of the brane factor of the structure sheaf of E_K. That brane factor cancels poles from all chiral multiplets with $j \in K$, hence the regularized partition function remains finite as $R_j \to 0$ too. This should generalize readily to branes supported on the union of all compact E_K: their brane factor is a sum of brane factors supported on each E_K. These compact branes exhibit no contour pinching.

Finally, for a non-compact E_K we expect the regularized partition function of its structure sheaf to have singular contributions at $R_j \to 0$. We worked them out in one-parameter examples in the previous subsection. It would be very interesting to relate these singular contribution to an equivariant integral on the support of the brane, but we postpone such an investigation to later work.

4.5 B-brane category of Hirzebruch-Jung models

We apply here the general considerations of the previous subsections to Hirzebruch-Jung models of arbitrary rank. We describe the derived category $D^b(X)$ of coherent sheaves on the Higgs branch X in terms of generators and relations. We determine the pull-back of each generator to local models of the orbifold points and of exceptional divisors. Finally, in subsubsection 4.5.3 and subsubsection 4.5.4 we calculate the regularized hemisphere partition function for some compact branes that we compare with geometry in section 5.

4.5.1 Generators and relations

Fix a phase specified by the collection A of blown up divisors. Recall that $D^b(X)$ is generated by the line bundles $\mathcal{O}(b_1, \dots, b_r)$ on X, defined to be the images (under F_{geom}) of the Wilson line branes $\mathcal{W}(b_1, \dots, b_r)$ with charges b under the $U(1)^r$ gauge group of the GLSM. The tensor product $\mathcal{O}(b_1, \dots, b_r) \otimes \mathcal{O}(c_1, \dots, c_r) = \mathcal{O}(b_1 + c_1, \dots, b_r + c_r)$ means we could restrict our attention to branes with a single non-zero $b_i = 1$, but it will be clearer to keep all b_i.

Sections of $\mathcal{O}(0, \dots, 0)$ are just G-invariant functions on $V \setminus \Delta$, hence are functions on the Higgs branch $(V \setminus \Delta)/G$, so this is the structure sheaf of the Higgs branch, $\mathcal{O} = \mathcal{O}(0, \dots, 0)$. Multiplication by X_j maps from $\mathcal{O}$ to $\mathcal{O}(\dots, 0, 1, -a_j, 1, 0, \dots)$, so the latter sheaf is $\mathcal{O}$ twisted by the divisor $E_j = \{X_j = 0\}$. Explicitly,

$$\begin{aligned} \mathcal{O}(E_0) &= \mathcal{O}(1, 0, \dots), \qquad \mathcal{O}(E_1) = \mathcal{O}(-a_1, 1, 0, \dots), \\ \mathcal{O}(E_\alpha) &= \mathcal{O}(\dots, 0, 1, -a_\alpha, 1, 0, \dots) \text{ for } 1 < \alpha < r, \\ \mathcal{O}(E_r) &= \mathcal{O}(\dots, 0, 1, -a_r), \qquad \mathcal{O}(E_{r+1}) = \mathcal{O}(\dots, 0, 1), \end{aligned} \tag{4.28}$$

Since the Cartan matrix of charges has determinant n rather than 1, tensor products of the line bundles $\mathcal{O}(E_\alpha)$ for $1 \leq \alpha \leq r$ do not give all $\mathcal{O}(b_1, \dots, b_r)$. On the other hand the line bundles $\mathcal{O}(E_j)$ for $0 \leq j \leq r+1$ do.

Any B-brane of the GLSM whose support is in the deleted set Δ gives a trivial brane in the Higgs branch theory (we study Coulomb/mixed parts of the brane in section 6). Therefore, the line bundles $\mathcal{O}(b_1, \dots, b_r)$ are subject to one relation for each irreducible component of the deleted set Δ. This gives two types of relations.

- For each divisor that is not blown up (each $\alpha \in [\![1, r]\!] \setminus A$), Δ contains the hyperplane $\{X_\alpha = 0\}$. Its structure sheaf has Koszul resolution (4.11) $\mathcal{W} \xrightarrow{X_\alpha} \mathcal{W}$ by line bundles, thus the complex $\mathcal{O}(-E_\alpha) \xrightarrow{X_\alpha} \mathcal{O}$ is trivial in $D^b(X)$.
- For each pair of non-consecutive blown-up divisor (each $\alpha, \beta \in A$ such that no $\gamma \in A$ obeys $\alpha < \gamma < \beta$), Δ contains the intersection $\{X_\alpha = X_\beta = 0\}$. Its structure sheaf has Koszul resolution (4.12), hence the following complex on X is trivial:

$$\mathcal{O}(-E_\alpha - E_\beta) \xrightarrow[(X_\alpha, X_\beta)]{} \mathcal{O}(-E_\beta) \oplus \mathcal{O}(-E_\alpha) \xrightarrow[(X_\beta, -X_\alpha)]{} \mathcal{O}. \tag{4.29}$$

4.5.2 Pull-backs

Our goal now is to clarify what the generators $\mathcal{O}(b_1, \dots, b_r)$ are by determining their pullbacks to Higgs branches of local models discussed in subsection 3.3. Recall that these local models were found by determining that the $U(1)^r$ gauge group is Higgsed down to some subgroup H when some chiral multiplets have a non-zero vev. The Wilson line $\mathcal{W}(b_1, \dots, b_r)$ can be realized by the insertion of a 1d Fermi multiplet with charges $b\colon U(1)^r \to U(1)$. After Higgsing its charge under H is deduced from $H \subset U(1)^r \xrightarrow{b} U(1)$. The resulting Wilson line in the local model has a clear geometric meaning.

Consider first the local model (3.17) for an intersection point $E_{\{i,j\}} = E_i \cap E_j$ of two divisors, with $0 \leq i < j \leq r+1$. The residual gauge group $H = \mathbb{Z}_{d_{ij}}$ embeds into $U(1)^r$ (in basis I) as

$$\mathbb{Z}_{d_{ij}} \ni 1 \mapsto (1, \dots, 1, \omega^{d_{(i+1)j}}, \omega^{d_{(i+2)j}}, \dots, \omega^{d_{(j-1)j}}, 1, \dots, 1) \tag{4.30}$$

where $\omega = \exp(2\pi i / d_{ij})$ and the entries in positions $\alpha \in [\![i, j]\!]$ are $\omega^{d_{\alpha j}}$. Note that $d_{(j-1)j} = 1$. The Higgs branch image of a Wilson line with charges $(b_1, \dots, b_r)$ in basis I therefore has the following equivariant line bundle as its pullback to the neighborhood of $E_i \cap E_j$:

$$\mathcal{W}(q) \text{ on } \mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{(i+1)j})} \text{ with } \mathbb{Z}_{d_{ij}}\text{-charge } q = \sum_{\alpha=i+1}^{j-1} d_{\alpha j} b_\alpha. \tag{4.31}$$

While the expression is asymmetric between i and j one can change basis in $\mathbb{Z}_{d_{ij}}$ by multiplying all charges by $d_{i(j-1)}$. Using $d_{i(j-1)} d_{\alpha j} = d_{ij} d_{\alpha(j-1)} + d_{i\alpha} d_{(j-1)j} \equiv d_{i\alpha} \bmod d_{ij}$, we find

$$\mathcal{W}(q) \text{ on } \mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{i(j-1)})} \text{ with } \mathbb{Z}_{d_{ij}}\text{-charge } q = \sum_{\alpha=i+1}^{j-1} d_{i\alpha} b_\alpha. \tag{4.32}$$

In both bases, the charge q only involves charges b_α for indices α such that E_α is not blown up.

Consider next the local model (3.22) of an exceptional divisor E_j, in a phase where that divisor intersects E_i and E_k for $0 \leq i < j < k \leq r+1$. The gauge group is $H = U(1) \times \mathbb{Z}_m$ with $m = \gcd(d_{ij}, d_{jk})$, and $(h, \omega) \in U(1) \times \mathbb{Z}_m$ is mapped to $g \in U(1)^r$ with the following coordinates in basis I: $g_\alpha = 1$ for $\alpha \leq i$ or $\alpha \geq k$, while

$$\begin{aligned} g_\alpha &= \left(h^{d_{jk}/m} \omega^u\right)^{d_{i\alpha}} \text{ for } i \leq \alpha \leq j, \\ g_\alpha &= \left(h^{d_{ij}/m} \omega^v\right)^{d_{\alpha k}} \text{ for } j \leq \alpha \leq k. \end{aligned} \tag{4.33}$$

Note that $d_{ii} = d_{kk} = 0$ hence these formulas are compatible. A Wilson line with charges $(b_1, \ldots, b_r)$ in basis I thus maps to a Wilson line with charges

$$\begin{aligned} \frac{d_{jk}}{m} \sum_{\alpha=i+1}^{j-1} d_{i\alpha} b_\alpha + \frac{d_{ij} d_{jk} b_j}{m} + \frac{d_{ij}}{m} \sum_{\alpha=j+1}^{k-1} d_{\alpha k} b_\alpha \text{ under } U(1), \\ u \sum_{\alpha=i+1}^{j-1} d_{i\alpha} b_\alpha + v \sum_{\alpha=j+1}^{k-1} d_{\alpha k} b_\alpha \text{ under } \mathbb{Z}_m. \end{aligned} \tag{4.34}$$

For a Wilson line with a single non-zero b_β all of these charges vanish except one ($j = \beta$) if E_β is blown up, and two otherwise. In the first case, the Higgs branch image of the Wilson line is a non-trivial line bundle on the weighted projective space E_β but has trivial pullback near each orbifold point or any other exceptional divisor. In the second case (E_β not blown-up) the Higgs branch image has a non-trivial pullback, with charge $d_{\beta j} b_\beta$, near the orbifold point $E_i \cap E_j$ with $i < \beta < j$ and non-trivial pullbacks on E_i and E_j.

The case of the fully resolved phase is instructive: then all d_{ij} and m appearing above are equal to 1 and the Higgs branch image of the Wilson line $\mathcal{W}(b_1, \ldots, b_r)$ is a line bundle (on the Hirzebruch-Jung resolution) whose pull-back to each $\mathbb{P}^1$ exceptional divisor E_j is $\mathcal{O}(b_j)$. This is consistent with the fact that the gauge group of the local model near E_j is in that case the j-th $U(1)$ factor in $U(1)^r$ (in basis I).

4.5.3 Central charge: intersection

We explain in subsubsection 4.4.1 why the residue part $Z_{D^2,\text{residue}}$ of the hemisphere partition function correctly captures the Higgs branch contribution of a GLSM brane that flows to a given Higgs branch brane, regardless of grade restriction. Let us apply this to (fractional) D0 branes at the intersection $E_i \cap E_j$, which is a $\mathbb{Z}_{d_{ij}}$-orbifold point. The brane factor is

$$f(\hat{\sigma}) = \left(1 - e^{2\pi i (iQ^i \cdot \hat{\sigma} + R_i/2)}\right)\left(1 - e^{2\pi i (iQ^j \cdot \hat{\sigma} + R_j/2)}\right) e^{2\pi \rho \cdot \hat{\sigma}} \tag{4.35}$$

where the twist by a Wilson line $\mathcal{W}(\rho)$ affects the $\mathbb{Z}_{d_{ij}}$ charge of the fractional brane. Incidentally, this brane can often not be made grade restricted for any choice of ρ.

The brane's support is compact, thus as explained in subsection 4.4 the brane factor cancels enough poles to avoid contour pinching. Specifically, the brane factor cancels poles from one-loop determinants of X_i and X_j. Notice that there are only r chiral multiplets other than X_i and X_j, which is exactly the rank of the gauge group, so the r-fold integral picks up exactly one family of residues, labeled by $J = \complement\{i, j\}$ in the notations of (4.26). We find

$$Z^{\text{0-instanton}}_{D^2,\text{residue}} = \pm(\mathfrak{r}\Lambda)^{\hat{c}/2} \operatorname*{res}_{i\hat{\sigma}=i\hat{\sigma}_{\{i,j\}}} \left(\frac{e^{(it+2\pi\rho)\cdot\hat{\sigma}}}{(-2\pi i)^2} \prod_{\ell=i,j} \frac{-2\pi i\, e^{-\pi Q^\ell \cdot \hat{\sigma} + i\pi R_\ell/2}}{\Gamma\left(1 - iQ^\ell \cdot \hat{\sigma} - R_\ell/2\right)} \prod_{\ell \neq i,j} \Gamma\left(iQ^\ell \cdot \hat{\sigma} + R_\ell/2\right) \right), \tag{4.36}$$

where the sign is $\operatorname{sign}(\det(Q^\ell)_{\ell\neq i,j})$ and $i\hat{\sigma}_{\{i,j\}}$ is the solution of $iQ^\ell\cdot\hat{\sigma}+R_\ell/2=0$ for all $\ell\neq i,j$. This solution is linear in the R-charges, and we do not need its explicit expression (4.40) now.

Computing the residue gives a factor $1/\det(Q^\ell)_{\ell\neq i,j}$, which combines with the sign to give an absolute value. The matrix has a block form, so

$$\det(Q^\ell)_{\ell\neq i,j} = \det\begin{pmatrix} U & N_1 & 0 \\ 0 & -C_{(ij)} & 0 \\ 0 & N_2 & L \end{pmatrix} = \det(-C_{(ij)}) = (-1)^{j-i} d_{ij}, \tag{4.37}$$

where U and L are upper and lower triangular matrices with 1 on the diagonal, N_1 and N_2 are matrices with a single non-zero entry equal to 1 in the corner closest to the diagonal of the main matrix, and $C_{(ij)}$ consists of rows and columns from $(i+1)$ to $(j-1)$ included of the generalized Cartan matrix. We deduce

$$Z^{\text{0-instanton}}_{D^2,\text{residue}} = (\mathfrak{r}\Lambda)^{\hat{c}/2} \lim_{R\to 0} \frac{1}{d_{ij}} \left(\frac{e^{(it+2\pi\rho)\cdot\hat{\sigma}}}{(-2\pi i)^2} \prod_{\ell=i,j} \frac{-2\pi i\, e^{-\pi Q^\ell\cdot\hat{\sigma}+i\pi R_\ell/2}}{\Gamma\bigl(1-iQ^\ell\cdot\hat{\sigma}-R_\ell/2\bigr)} \right)_{i\hat{\sigma}=i\hat{\sigma}_{\{i,j\}}} = \frac{(\mathfrak{r}\Lambda)^{\hat{c}/2}}{d_{ij}}. \tag{4.38}$$

Recall that the intersection $E_i\cap E_j$ is a fixed point of the orbifold group $\mathbb{Z}_{d_{ij}}$. This central charge does not depend on the $\mathbb{Z}_{d_{ij}}$-charge of the fractional brane (no ρ dependence). A collection of d_{ij} fractional branes with all possible $\mathbb{Z}_{d_{ij}}$ charges gives a D0 brane that can move away from the orbifold point, which is consistent with the fact that this collection has central charge 1 independent of which orbifold point we start from. For $d_{ij}=1$ (so $j=i+1$) we are simply discussing a usual D0 brane. We chose the normalization of the hemisphere partition function to make this case very simple.

4.5.4 Central charge: exceptional divisor

We now turn to the structure sheaf of an exceptional divisor E_j, $1\leq j\leq r$, in a phase in which it is blown up. As usual we denote by $i<j<k$ the neighboring exceptional divisors. Again, the brane's support E_j is compact so the brane factor

$$f(\hat{\sigma}) = \bigl(1-e^{2\pi i(iQ^j\cdot\hat{\sigma}+R_j/2)}\bigr)e^{2\pi\rho\cdot\hat{\sigma}} \tag{4.39}$$

cancels enough poles to avoid contour pinching.

Specializing (4.26) to the present case, the residue part of the regularized hemisphere partition function is a sum over sets J of r flavours with $j\notin J$ and $\zeta\in\mathrm{Cone}_J$. The D-term equation gives a criterion: $\zeta\in\mathrm{Cone}_J$ if and only if $E_{(\complement J)}=\{X_i=0\mid i\notin J\}$ is a non-empty subset of the Higgs branch. Given that $j\in\complement J$ and $\complement J$ has $r+2-r=2$ elements we find $\complement J$ can be $\{i,j\}$ or $\{j,k\}$.

We first compute the term with $J=\complement\{i,j\}$. The leading term of (4.26) is given by taking all vorticities (denoted $k_j\geq 0$ there) to be zero. One of the residues is then taken at the unique solution of $Q^\ell\cdot\hat{\sigma}=iR_\ell/2$ for all $\ell\neq i,j$, which we denote $\hat{\sigma}_{\{i,j\}}$. Explicitly,

$$i\hat{\sigma}^\alpha_{\{i,j\}} = \begin{cases} -\sum_{\ell=0}^{\alpha-1} d_{\ell\alpha}\frac{R_\ell}{2} & \text{for } 1\leq\alpha\leq i, \\ -\frac{1}{d_{ij}}\sum_{\ell=0}^{r+1} d_{i\min(\alpha,\ell)}d_{j\max(\alpha,\ell)}\frac{R_\ell}{2} & \text{for } i\leq\alpha\leq j, \\ -\sum_{\ell=\alpha+1}^{r+1} d_{\alpha\ell}\frac{R_\ell}{2} & \text{for } j\leq\alpha\leq r. \end{cases} \tag{4.40}$$

The two formulas for $\hat{\sigma}^i_{\{i,j\}}$ agree, as do the two formulas for $\hat{\sigma}^j_{\{i,j\}}$. Using the recursion relation $d_{\ell(i-1)} - a_i d_{\ell i} + d_{\ell(i+1)} = 0$ and $d_{ij} d_{\ell(i+1)} - d_{i\ell} d_{j(i+1)} = d_{i(i+1)} d_{\ell j} = d_{\ell j}$, we work out

$$iQ^i \cdot \hat{\sigma}_{\{i,j\}} + \frac{R_i}{2} = \sum_{\ell=0}^{r+1} \frac{d_{\ell j}}{d_{ij}} \frac{R_\ell}{2} \tag{4.41}$$

and likewise

$$iQ^j \cdot \hat{\sigma}_{\{i,j\}} + \frac{R_j}{2} = \sum_{\ell=0}^{r+1} \frac{d_{i\ell}}{d_{ij}} \frac{R_\ell}{2}. \tag{4.42}$$

The residue that appears in the hemisphere partition function is then

$$\begin{aligned} &\operatorname{sign}\bigl(\det(Q^\ell)_{\ell\neq i,j}\bigr) \operatorname*{res}_{i\hat{\sigma}=i\hat{\sigma}_{\{i,j\}}} \left(\frac{e^{it\cdot\hat{\sigma}-\pi Q^j\cdot\hat{\sigma}+i\pi R_j/2} \prod_{\ell\neq j} \Gamma\bigl(iQ^\ell\cdot\hat{\sigma}+R_\ell/2\bigr)}{-2\pi i\,\Gamma\bigl(1-iQ^j\cdot\hat{\sigma}-R_j/2\bigr)} \right) \\ &= \frac{1}{d_{ij}} \frac{e^{it\cdot\hat{\sigma}_{\{i,j\}}+i\pi\sum_{\ell=0}^{r+1}(d_{i\ell}/d_{ij})(R_\ell/2)} \Gamma\bigl(\sum_{\ell=0}^{r+1}\frac{d_{\ell j}}{d_{ij}}\frac{R_\ell}{2}\bigr)}{-2\pi i\,\Gamma\bigl(1-\sum_{\ell=0}^{r+1}\frac{d_{i\ell}}{d_{ij}}\frac{R_\ell}{2}\bigr)} \\ &= \frac{i}{2\pi}\left(\frac{2}{\sum_{\ell=0}^{r+1} d_{\ell j}R_\ell} + \left(\frac{2it\cdot\hat{\sigma}_{\{i,j\}}}{\sum_{\ell=0}^{r+1} d_{\ell j}R_\ell} + \frac{(i\pi-\gamma)\sum_{\ell=0}^{r+1} d_{i\ell}R_\ell}{d_{ij}\sum_{\ell=0}^{r+1} d_{\ell j}R_\ell} - \frac{\gamma}{d_{ij}} \right) + O(R) \right) \end{aligned} \tag{4.43}$$

where the factor $1/d_{ij}$ comes from the determinant of the matrix of charges Q^ℓ, $\ell \neq i,j$ when computing the residue, $t\cdot\hat{\sigma}_{\{i,j\}}$ can be computed from (4.40), and we used $\Gamma(x) = \frac{1}{x} - \gamma + O(x)$ and $\Gamma(1-x) = 1 + \gamma x + O(x^2)$.

The same steps give the residue corresponding to $J = {}^\complement\{j,k\}$. In fact, most intermediate calculations can be skipped: for example $iQ^k \cdot \hat{\sigma}_{\{j,k\}} + R_k/2$ is immediately obtained from $iQ^j \cdot \hat{\sigma}_{\{i,j\}} + R_j/2$ by replacing $(i,j) \to (j,k)$ in (4.42). The residue that appears in the hemisphere partition function is then

$$\begin{aligned} &\operatorname{sign}\bigl(\det(Q^\ell)_{\ell\neq j,k}\bigr) \operatorname*{res}_{i\hat{\sigma}=i\hat{\sigma}_{\{j,k\}}} \left(\frac{e^{it\cdot\hat{\sigma}-\pi Q^j\cdot\hat{\sigma}+i\pi R_j/2} \prod_{\ell\neq j} \Gamma\bigl(iQ^\ell\cdot\hat{\sigma}+R_\ell/2\bigr)}{-2\pi i\,\Gamma\bigl(1-iQ^j\cdot\hat{\sigma}-R_j/2\bigr)} \right) \\ &= \frac{i}{2\pi}\left(\frac{2}{\sum_{\ell=0}^{r+1} d_{j\ell}R_\ell} + \left(\frac{2it\cdot\hat{\sigma}_{\{j,k\}}}{\sum_{\ell=0}^{r+1} d_{j\ell}R_\ell} + \frac{(i\pi-\gamma)\sum_{\ell=0}^{r+1} d_{\ell k}R_\ell}{d_{jk}\sum_{\ell=0}^{r+1} d_{j\ell}R_\ell} - \frac{\gamma}{d_{jk}} \right) + O(R) \right). \end{aligned} \tag{4.44}$$

Summing the two residues, the $O(1/R)$ divergence cancels as expected. Using

$$i\hat{\sigma}^\alpha_{\{i,j\}} - i\hat{\sigma}^\alpha_{\{j,k\}} = \begin{cases} 0 & \text{for } 1 \leq \alpha \leq i, \\ (d_{i\alpha}/d_{ij}) \sum_{\ell=0}^{r+1} d_{\ell j}R_\ell/2 & \text{for } i \leq \alpha \leq j, \\ (d_{\alpha k}/d_{jk}) \sum_{\ell=0}^{r+1} d_{\ell j}R_\ell/2 & \text{for } j \leq \alpha \leq k, \\ 0 & \text{for } k \leq \alpha \leq r, \end{cases} \tag{4.45}$$

and relations between the $d_{\beta\gamma}$, we finally get the finite $R \to 0$ limit

$$Z^{\text{0-instanton}}_{D^2,\text{residue}} = (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{i}{2\pi} \left(\left(\sum_{\alpha=i+1}^{j-1} \frac{d_{i\alpha}}{d_{ij}} t_\alpha \right) + t_j + \left(\sum_{\alpha=j+1}^{k-1} \frac{d_{\alpha k}}{d_{jk}} t_\alpha \right) - i\pi \frac{d_{ik}}{d_{ij}d_{jk}} + \frac{d_{ik}-d_{jk}-d_{ij}}{d_{ij}d_{jk}} \gamma \right) \tag{4.46}$$

where we ignored an overall constant factor of $C(\mathfrak{r}\Lambda)^{\hat{c}/2}$ and powers of 2π, and the t_j term could be included in either of the two sums by extending the bounds to $\alpha = j$.

Whenever $i = j-1$ and $k = j+1$, in particular in the fully-resolved phase, this formula reduces to

$$Z^{\text{0-instanton}}_{D^2,\text{residue}} = (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{i}{2\pi}\Big(t_j - i\pi a_j + (a_j-2)\gamma\Big), \tag{4.47}$$

which coincides with the result (4.20) for the one-parameter model. More generally, the central charge coincides with the central charge one can compute from the local model (3.22):

$$Z^{\text{0-instanton}}_{D^2,\text{residue}} = (\mathfrak{r}\Lambda)^{\hat{c}/2} \frac{i}{2\pi}\frac{1}{m}\frac{t_{\text{loc}} - i\pi d_{ik}/m + (d_{ik}/m - d_{jk}/m - d_{ij}/m)\gamma}{(d_{ij}/m)(d_{jk}/m)} \tag{4.48}$$

where t_{loc} is given in (3.20), $m = \gcd(d_{ij}, d_{jk})$, and the $1/m$ factor is due to the orbifold.

5 K-theoretic aspects

In this section we study the central charge of B-branes on abelian GLSMs, when projected to their image in the Higgs branch. We saw in subsection 3.2 that the geometry of the Higgs branch corresponds to a toric variety with at most abelian quotient singularities. Denote it by X_ζ, where ζ is the real part of the FI parameter. The derived category $D(X_\zeta)$ of such spaces is a well known mathematical object, as we previously reviewed in subsection 4.2.

In models with compact target, the central charge Z of a B-brane $\mathcal{V} \in D(X_\zeta)$ is a map

$$Z\colon D(X_\zeta) \longrightarrow \mathbb{C} \tag{5.1}$$

that is holomorphic in ζ and multivalued in the Kähler moduli space. Moreover, Z factors through $K_0(X_\zeta)$, the Grothendieck group of $D(X_\zeta)$ spanned by holomorphic vector bundles:

$$Z\colon K_0(X_\zeta) \longrightarrow \mathbb{C}. \tag{5.2}$$

Specifically, the central charge is given by [38]

$$Z(\mathcal{V}) = \int_{X_\zeta} e^{\tau} \hbar^{-\frac{i}{2\pi}c_1(X_\zeta)} \widehat{\Gamma}(TX_\zeta)\,\mathrm{ch}(\mathcal{V}) + \ldots \tag{5.3}$$

Here, τ denotes the complexified Kähler class of X_ζ, so $\tau \in H^2(X_\zeta, \mathbb{R}/\mathbb{Z}) + i\mathcal{K}_{X_\zeta}$, with $\mathcal{K}_{X_\zeta}$ the Kähler cone of X_ζ, and the "$+\ldots$" denote instanton corrections. The meaning of the parameter $\hbar$ can be traced to a $\mathbb{C}^*$-equivariant cohomology on the worldsheet $\mathbb{P}^1$ (see e.g., [39, section 10.2.3]). Finally, $\widehat{\Gamma}(TX_\zeta)$ denote the gamma class of the tangent bundle. It is defined as

$$\widehat{\Gamma}(TX_\zeta) \coloneqq \prod_{l=1}^{\dim X_\zeta} \Gamma\left(1 - \frac{\lambda_l}{2\pi i}\right) \tag{5.4}$$

where λ_l are the Chern roots of TX_ζ. The real part of Z is related to the RR-charge of the B-brane $\mathcal{V}$ [40, 41]. The phase of the central charge plays an important role on the stability of B-branes [42, 43]. An exact expression for Z (including all instanton corrections) in geometric phases of local and compact Calabi–Yau manifolds was proposed by Hosono in [27] and used to define an integral structure on the A-model compatible with mirror symmetry [38, 44].

In the following we review the necessary mathematical framework to write (5.3) for branes with compact support on local toric geometries and compare it with localization calculations for the projection of GLSM branes $(\mathcal{B}, L)$ into the Higgs branch. All our analysis will concern only the leading term of Z and we ignore the instanton corrections, leaving them for future work.

5.1 K-theory and characteristic classes of toric varieties [review]

We need to define the K-group $K_0(X_\zeta)$, the cohomology $H^*(X)$ and the Chern character $\mathrm{ch}\colon K_0(X_\zeta)\otimes\mathbb{Q}\to H^{\mathrm{even}}(X_\zeta)$ when X_ζ is toric.[14] Since we are dealing with X_ζ toric and noncompact, we must make a distinction between objects with compact support and the ones with non-compact support. We will see that we can make sense of the central charge geometrically for B-branes with compact support. For this purpose we have to define in addition the compact K-group $K_0^c(X_\zeta)$ and cohomology $H_c^*(X_\zeta)$ which are modules of their noncompact counterparts, and the compact Chern character $\mathrm{ch}^c\colon K_0^c(X_\zeta)\to H_c^{\mathrm{even}}(X_\zeta)$.

Our starting point is the data of a fan Σ on a lattice N of rank d. A fan consists of a consistent collection of rational polyhedral cones in N. For a review and conventions used here, the reader can consult [45]. Denote $\Sigma(1)=\{v_1,\dots,v_n\}$ the rays of Σ, let $I=\{1,\dots,n\}$ be the set of these indices, and $\mathbb{P}_\Sigma$ the toric variety associated to Σ.

Then the untwisted sector of the cohomology of $\mathbb{P}_\Sigma$ has a well known description,

$$H_0^*(\mathbb{P}_\Sigma)=\frac{\mathbb{C}[D_1,\dots,D_n]}{\left\{\sum_i m(v_i)D_i | m\in N^*\right\},\mathcal{I}_{\mathrm{SR}}},\tag{5.5}$$

where the Stanley-Reisner (SR) ideal $\mathcal{I}_{\mathrm{SR}}$ is spanned by products $\prod_{i\in J}D_i$ for every $J\subset I$ that does not span a cone in Σ. The classes D_i have cohomological degree 2 and correspond to toric divisors of $\mathbb{P}_\Sigma$. Here we denote the cohomology by H_0^* to remind that the full cohomology may include twisted sectors.[15] Besides this twisted sector, we have one twisted sector γ for each $\gamma=\sum_i\gamma_i v_i\in N$ with $\gamma_i\in[0,1)$,[16] where the origin $\gamma=0$ corresponds to the untwisted sector. The twisted sector cohomology is as follows:

$$H_\gamma^*(\mathbb{P}_\Sigma)=\frac{\mathbb{C}[\overline{D}_i]_{i\in S_\gamma(1)}}{\left\{\sum_i m(v_i)\overline{D}_i \mid m\in\mathrm{Ann}(v_i\in\sigma(\gamma))\right\},\mathcal{I}_{\mathrm{SR}}^\gamma},\qquad S_\gamma:=\mathrm{Star}(\sigma(\gamma))-\sigma(\gamma).\tag{5.6}$$

The SR ideal $\mathcal{I}_{\mathrm{SR}}^\gamma$ in the γ twisted sector is spanned by $\prod_{i\in J}\overline{D}_i$ for J not a cone in $\mathrm{Star}(\sigma(\gamma))$.

The compact cohomology is given by the free $H^*(\mathbb{P}_\Sigma)$-module, generated by the symbols F_J's, quotient by two types of relations, H_1 and H_2:

$$H_{c,0}^*(\mathbb{P}_\Sigma)=\bigoplus_{\sigma_J^\circ\subseteq\Sigma^\circ}\frac{\mathbb{C}[D_1,\dots,D_n]F_J}{\langle H_1,H_2\rangle}.\tag{5.7}$$

[14] In most of this section we will be working with K-theory with $\mathbb{Q}$ or $\mathbb{C}$ coefficients, hence ch is an isomorphism. Of course, when discussing questions such as integrality structure of the central charge map, one needs to consider $\mathbb{Z}$-valued K-theory

[15] For example in partial resolutions of singularities, they usually appear.

[16] If $\sigma(\gamma)\in\Sigma$ is the minimal cone in Σ that contains γ, the twisted sector is the toric substack described by the quotient fan $\Sigma/\sigma(\gamma)$, its rays are labelled by the rays in $\mathrm{Star}(\sigma(\gamma))-\sigma(\gamma)$. Recall the definition $\mathrm{Star}(\sigma)=\{\sigma'\in\Sigma|\sigma\subseteq\sigma'\}$. The set of all γ's is usually denoted $\mathrm{Box}(\Sigma)$.

The relations are given by

$$\begin{aligned} H_1 &= \{D_i F_J = F_{J\cup\{i\}},\ \text{for } i \notin J, J\cup\{i\} \in \Sigma\}, \\ H_2 &= \{D_i F_J = 0,\ \text{for } i \notin J, J\cup\{i\} \notin \Sigma\}. \end{aligned} \tag{5.8}$$

The twisted sectors $H^*_{c,\gamma}(\mathbb{P}_\Sigma)$ have a similar description as for the untwisted case, just replacing the commutative ring by $\mathbb{C}[\overline{D}_i]_{i\in S_\gamma(1)}$ and the fan by the quotient fan $\Sigma_\gamma := \Sigma/\sigma(\gamma)$. More precisely, recall that the cones on the quotient fan are given by S_γ, hence

$$H^*_{c,\gamma}(\mathbb{P}_\Sigma) = \bigoplus_{\sigma_J^\circ \subseteq \Sigma_\gamma^\circ} \frac{\mathbb{C}[\overline{D}_i]_{i\in S_\gamma}\overline{F}_J}{\langle H_1^\gamma, H_2^\gamma\rangle} \tag{5.9}$$

where

$$\begin{aligned} H_1^\gamma &= \{\overline{D}_i \overline{F}_J = \overline{F}_{J\cup\{i\}},\ \text{for } i \notin J, J\cup\{i\} \in \Sigma_\gamma\}, \\ H_2^\gamma &= \{\overline{D}_i \overline{F}_J = 0,\ \text{for } i \notin J, J\cup\{i\} \notin \Sigma_\gamma\}. \end{aligned} \tag{5.10}$$

The description of the K-theory (with complex coefficients) is given by the following ring [46, 47, 48]:

$$K_0(\mathbb{P}_\Sigma) = \frac{\mathbb{C}[R_i^\pm]_{i\in I}}{\left\{\prod_{i\in I} R_i^{m(v_i)} - 1 \;\middle|\; m \in N^*\right\}, \mathcal{I}_K}, \qquad \mathcal{I}_K = \left\langle \prod_{i\in J} (1-R_i) | J \notin \Sigma \right\rangle. \tag{5.11}$$

The compact version $K_0^c(\mathbb{P}_\Sigma)$ is a free $K_0(\mathbb{P}_\Sigma)$-module generated by the symbols G_J with $\sigma_J^\circ \subseteq \Sigma^\circ$ and relations

$$\begin{aligned} &\{(1-R_i^{-1})G_J = G_{J\cup\{i\}},\ \text{for }\ i \notin J, J\cup\{i\} \in \Sigma\} \\ &\{D_i F_J = 0, \ \text{ for }\ i \notin J, J\cup\{i\} \notin \Sigma\}. \end{aligned} \tag{5.12}$$

Finally we have the Chern character maps, which give isomorphisms (here we are always considering K-theory with complex coefficients):

$$\mathrm{ch}\colon K_0(\mathbb{P}_\Sigma) \to H^*(\mathbb{P}_\Sigma), \qquad \mathrm{ch}^c\colon K_0^c(\mathbb{P}_\Sigma) \to H_c^*(\mathbb{P}_\Sigma). \tag{5.13}$$

Explicitly, the usual Chern character is the following (where ch_γ denotes the projection to the sector γ):

$$\begin{aligned} &\mathrm{ch}_\gamma(R_i) = 1, && i \notin \mathrm{Star}(\sigma(\gamma)), \\ &\mathrm{ch}_\gamma(R_i) = e^{\overline{D}_i}, && i \in S_\gamma, \\ &\mathrm{ch}_\gamma(R_i) = e^{2\pi i\gamma_i} \prod_{j\notin\sigma(\gamma)} \mathrm{ch}(R_j)^{m_i(v_j)}, && i \in \sigma(\gamma). \end{aligned} \tag{5.14}$$

Here all the inclusivity conditions are understood as a condition for the rays of $\mathrm{Star}(\sigma(\gamma))$, S_γ and $\sigma(\gamma)$. The map ch^c_γ is

$$\begin{aligned} &\mathrm{ch}^c_\gamma\Big(\prod_i R_i^{k_i} G_I\Big) \\ &= \begin{cases} 0 & \text{for } I \not\subseteq \mathrm{Star}(\sigma(\gamma)), \\ \Big(\prod_i \mathrm{ch}_\gamma(R_i)^{k_i} \prod_{i\in I, i\notin\sigma(\gamma)} \frac{1-e^{-\overline{D}_i}}{\overline{D}_i}\Big) \prod_{i\in I\cap\sigma(\gamma)} (1-\mathrm{ch}_\gamma(R_i)^{-1})\overline{F}_I, & \text{for } I \subseteq \mathrm{Star}(\sigma(\gamma)), \end{cases} \end{aligned} \tag{5.15}$$

where for $I \subseteq \mathrm{Star}(\sigma(\gamma))$, the subindex of $\overline{F}_I$ denotes the projection to S_γ. The second factor is defined by Taylor expanding $(1-e^{-x})/x = 1 - x/2 + \ldots$ then replacing x by $\overline{D}_i$.

One last thing we will need is the integration map

$$\int : H^*_c(\mathbb{P}_\Sigma) \longrightarrow \mathbb{C} \tag{5.16}$$

it is given, in the different sectors, by

$$\int \overline{F}_I = \frac{1}{|\operatorname{Vol}(I)|}, \qquad \text{for } |I| = \operatorname{rank}(N_\gamma) \tag{5.17}$$

and zero otherwise. Here $\operatorname{Vol}(I)$ is the index of the lattice spanned by I inside N_γ, and N_γ is the quotient lattice $N/\operatorname{Span}(\sigma(\gamma))$.

5.2 Central charges of B-branes in Hirzebruch-Jung models

In order to apply the machinery reviewed in the previous subsection, we first need to connect the fan description with the GLSM data. This a well known construction. Part of the data of the abelian GLSM is given by a $r \times d$ matrix of charges Q^j_α. This matrix gives relations between d vectors on the lattice N, of rank $d-r$. This lattice can be constructed as a quotient

$$N = \mathbb{Z}^d \;/\; \operatorname{Span}_{\mathbb{Z}}\big(\{Q_\alpha, 1 \le \alpha \le r\}\big). \tag{5.18}$$

Basis vectors in $\mathbb{Z}^d$ project to vectors $S = \{v_1, \ldots, v_d\} \subset N$ whose relations are given by the charge matrix. The other important piece of the data is the D-terms. For a choice of ζ in the interior of a Kähler cone, the D-terms have no solution when certain subsets of the chirals vanish simultaneously. Denote the collection of these subsets as

$$\mathcal{P}_\zeta = \{\Delta_{J_\zeta}\}_{J_\zeta \subseteq S}. \tag{5.19}$$

Here, the sets Δ_j are of the form $\{X_{i_1}, \ldots, X_{i_k}\}$ and are labeled by some subsets of indices J_ζ of S, which depend on ζ. Take all the subsets of rank one, i.e., all Δ_{J_ζ} such that $|J_\zeta| = 1$ and call their union $\mathcal{I}_\zeta$. Then we associate the vectors in $S - \mathcal{I}_\zeta$ to the rays of a fan Σ_ζ. The cones of Σ_ζ are chosen in a unique way that is consistent with the SR ideal given by $\mathcal{P}_\zeta$, i.e., the rays that does not span a cone in Σ_ζ are given by the sets J_ζ.

Next we need to actually determine the lattice N. In practice, given the charge matrix we determine S as a subset of $\mathbb{Z}^d$, but we still need to determine which are the generators of N. Consider then the dual lattice $N^* = \operatorname{Hom}(N, \mathbb{Z})$ and the map

$$f\colon \begin{cases} N^* \to \operatorname{Hom}((\mathbb{C}^*)^d, \mathbb{C}^*) \\ m \;\mapsto \big(\lambda \mapsto \prod_{i=1}^d \lambda_i^{m(v)}\big) \end{cases}. \tag{5.20}$$

Then N^* is given by the points in $\mathbb{R}^d$ such that $\ker(f)$ reproduces the gauge group $G = U(1)^r$. Another equivalent way to determine N from Q^j_α is to look at the dual of the lattice of gauge invariant Laurent monomials. Consider the gauge invariant monomials $\prod_i X_i^{l_i}$, $Q^j_\alpha l_j = 0$ for all α, with $l_j \in \mathbb{Z}$. Then, all these monomials can be written in terms of d basis monomials $M_1, \ldots, M_d$. Assign charges $n_1, \ldots, n_d \in \mathbb{Q}$ to them and demand that all the gauge invariant monomials have integer charges. This defines a lattice inside $\mathbb{Q}^d$ which we identify with N.

5.2.1 Geometric central charge in Hirzebruch-Jung models

If $\mathcal{F}$ is a compact sheaf in a noncompact variety X, denote $\mathcal{F}$ its K-theory class in K_0^c as well. Then, we define its central charge by

$$Z(\mathcal{F}) = \int e^{\tau} \hbar^{-\frac{i}{2\pi}c_1(X)} \widehat{\Gamma}(TX)\,\mathrm{ch}^c(\mathcal{F}) \tag{5.21}$$

where τ is the complexified Kähler class. The gamma class $\widehat{\Gamma}$, $c_1(X)$ and τ are elements of $H^*(X, \mathbb{C})$, hence the act over $\mathrm{ch}^c(\mathcal{F}) \in H_c^*(X)$ giving another element of $H_c^*(X)$ where the integral map previously defined, makes sense.

We used two GLSMs to study the quantum geometry of the Hirzebruch-Jung resolutions. The two are simply related by a change of basis on the $U(1)^r$ weights of the chiral fields. Since this change of variables was characterized by a transformation matrix of determinant not 1, the gauge groups of the corresponding models differ by a nontrivial finite quotient. This is just and artifact of the presentation and since the two charge matrices $Q_\alpha^{\ j}$ and $n(C^{-1})_\alpha^{\ \beta} Q_\beta^{\ j}$ are related by an invertible linear transformation, it does not matter which one we use to determine the set $S \subset N$. Then in some basis, S is given by:

$$S = \{v_j\}_{j=0}^{r+1} \qquad v_j := (p_j, q_j) \tag{5.22}$$

and the lattices N and N^* are straightforward to determine:

$$N^* = \mathrm{Span}_{\mathbb{Z}} \left\{ \begin{pmatrix} 1/n \\ -p/n \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad N = \mathrm{Span}_{\mathbb{Z}}\big\{(n, 0), (p, 1)\big\}. \tag{5.23}$$

We consider the fully resolved phase,[17] call its fan Σ_{geom}. From the previous discussion we are instructed to look at the deleted sets imposed by the D-term equations. These are given by the sets of the form $\{X_i = X_j = 0\}$ with $j \neq i \pm 1$. Hence the 2-dimensional cones of Σ_{geom} are given by $\{j, j+1\}$ for $j = 0, \ldots, r$ and $\Sigma_{\mathrm{geom}}(1) = S$. Is easy to see that there are no twisted sectors. Then, the cohomology ring is given by

$$H_0^*(\mathbb{P}_{\Sigma_{\mathrm{geom}}}) = \frac{\mathbb{C}[D_0, \ldots, D_{r+1}]}{\left\{ \sum_{i=0}^{r+1} q_i D_i = \sum_{i=0}^{r+1} p_i D_i = 0 \right\}, \mathcal{I}_{\mathcal{SR}}} \tag{5.24}$$

where $\mathcal{I}_{SR} = \langle D_i D_j | j \neq i \pm 1 \rangle$.

It is very convenient to note that the linear relations between the generators of $H_0^*(\mathbb{P}_{\Sigma_{\mathrm{geom}}})$ imply

$$\sum_j d_{ij} D_j = 0 \qquad \text{for all } \ i = 0, \ldots, r+1. \tag{5.25}$$

Of course, only two are linearly independent (over $\mathbb{C}$), but this fact is very convenient for computations. Simple manipulations of the ring relations leads us to conclude

$$D_i D_j = 0 \ \text{ for all } \ i, j \in \{0, \ldots, r+1\}, \tag{5.26}$$

as expected since the top cohomology should vanish for noncompact surfaces: $H^4(\mathbb{P}_{\Sigma_{\mathrm{geom}}}) = 0$. The linear relations can be solved by writing

$$D_i = \sum_{\alpha=1}^{r} Q_\alpha^i \eta^\alpha, \qquad \eta^\alpha := \sum_{\beta=1}^{r} -(C^{-1})^{\alpha\beta} D_\beta. \tag{5.27}$$

[17] In arXiv v2 we will extend these computations to partially resolved phases and match with subsection 4.5.

We thus write the Kähler class as

$$\tau = \sum_{\alpha=1}^{r} \tau_\alpha \eta^\alpha. \tag{5.28}$$

The Chern class of the tangent bundle $T\mathbb{P}_{\Sigma_{\mathrm{geom}}}$ can be determined from the Euler sequence:

$$0 \to \mathcal{O}^r \to \bigoplus_{j=0}^{r+1} \mathcal{O}(D_j) \to T\mathbb{P}_{\Sigma_{\mathrm{geom}}} \to 0 \tag{5.29}$$

therefore determining its gamma class is also straightforward

$$\mathrm{ch}(T\mathbb{P}_{\Sigma_{\mathrm{geom}}}) = 2 + \sum_{\alpha=1}^{r}(2-a_\alpha)\eta^\alpha, \qquad \widehat{\Gamma}(T\mathbb{P}_{\Sigma_{\mathrm{geom}}}) = 1 - i\frac{\gamma}{2\pi}\sum_{\alpha=1}^{r}(2-a_\alpha)\eta^\alpha \tag{5.30}$$

where γ is the Euler-Mascheroni constant.

The compact K-theory group $K_0^c(\mathbb{P}_{\Sigma_{\mathrm{geom}}})$ and the compact cohomology $H_c^*(\mathbb{P}_{\Sigma_{\mathrm{geom}}})$ are spanned by the symbols G_I and F_I respectively where I can be either $\{\alpha\}$ or $\{j, j+1\}$ with $\alpha = 1, \dots, r$ and $j = 0, \dots, r$. We identify the compact K-theory class of (branes wrapping) exceptional divisors as G_α and it is a straightforward computation, using the previously defined formulas that

$$\mathrm{ch}^c(G_\alpha) = \left(1 - \frac{1}{2}D_\alpha\right)F_\alpha. \tag{5.31}$$

Acting with the relations (5.25) on F_α we can show

$$\begin{aligned} &F_{\alpha,\alpha+1} = F_{\{0,1\}} \quad \text{for} \quad \alpha = 1, \dots, r, \\ &D_\alpha F_\beta = (\delta_{\alpha,\beta-1} - a_\alpha \delta\alpha, \beta + \delta_{\alpha,\beta+1})F_{\{0,1\}} = -C_\alpha{}^\beta F_{\{0,1\}}, \end{aligned} \tag{5.32}$$

and therefore $\eta_\alpha F_\beta = \delta_{\alpha,\beta}F_{\{0,1\}}$. Putting everything together,

$$\begin{aligned} &e^\tau \hbar^{-\frac{i}{2\pi}c_1(\mathbb{P}_{\Sigma_{\mathrm{geom}}})}\widehat{\Gamma}(T\mathbb{P}_{\Sigma_{\mathrm{geom}}})\,\mathrm{ch}^c(G_\alpha) \\ &= F_\beta + \left(-i\frac{\gamma}{2\pi}(2-a_\beta) + \tau^\beta - \frac{i}{2\pi}(2-a_\beta)\log\hbar + \frac{a_\beta}{2}\right)F_{\{0,1\}}. \end{aligned} \tag{5.33}$$

The lattice spanned by $\{v_0, v_1\}$ is precisely N, hence $|\,\mathrm{Vol}_{\{0,1\}}\,| = 1$, therefore we finally conclude that the central charge of the brane wrapping the exceptional divisor E_β, corresponding to the sheaf $\mathcal{O}_{E_\beta}$, is given by

$$\begin{aligned} Z(\mathcal{O}_{E_\beta}) = Z(G_\beta) &= \int e^\tau \widehat{\Gamma}(T\mathbb{P}_{\Sigma_{\mathrm{geom}}})\,\mathrm{ch}^c(G_\beta) \\ &= -i\frac{\gamma}{2\pi}(2-a_\beta) + \tau_\beta - \frac{i}{2\pi}(2-a_\beta)\log\hbar + \frac{a_\beta}{2}. \end{aligned} \tag{5.34}$$

By comparing with (4.47), up to an overall factor of $(\mathfrak{r}\Lambda)^{\frac{\hat{c}}{2}}$, which can be absorbed in a rescaling of the classes η^β , we can see that the match is perfect, by identifying

$$t_\beta(\mathfrak{r}^{-1}) = -2\pi i\tau_\beta - (2-a_\beta)\log\hbar \quad \text{and} \quad \hbar = \mathfrak{r}\Lambda. \tag{5.35}$$

6 Grade restriction rule in non-supersymmetric models

Our discussion of B-branes and their central charges in section 4 (localization approach) and section 5 (geometric approach) only determined the Higgs branch contributions for a restricted class of branes, and it focussed on one phase at a time. In this section we explain the grade restriction rule that characterises that restricted class, and explain how the Higgs and mixed/Coulomb branch parts of a GLSM brane are distinguished in general. This partly relies on transporting a B-brane from one phase of a GLSM to another by continuously varying the FI-theta parameter t at some fixed energy scale. We thus explore brane transport in this section too.

B-brane transport is defined in [2, section 3.5] by enforcing that the full bulk plus boundary action is only deformed by D-terms of the algebra $\mathbf{2}_B$, namely bulk D-terms and twisted F-terms, and boundary D-terms. This defines a functor between the categories of B-branes (modulo $\mathbf{2}_B$ D-terms) at different values of t, and this functor only depends on the homotopy class of the path in FI-theta parameter space. In addition, since the hemisphere partition function is independent of bulk D-terms and is holomorphic in twisted F-terms, B-brane transport amounts to analytically continuing the hemisphere partition function.

We first review RG flow and transport of B-branes in $U(1)$ GLSMs through the lens of the hemisphere partition function, stressing how different the Calabi-Yau and non-Calabi-Yau cases are. Then we determine how transporting branes through a wall in general abelian models reduces to that of a gauge group $U(1)$ times a discrete factor whose effect we also explain. For Hirzebruch-Jung models we work out the images of B-branes in each phase of the GLSM by transporting them from the orbifold phase. We characterize which branes are transported to pure-Higgs branes in the fully resolved phase: this reproduces the set of special representations of the orbifold group $\mathbb{Z}_n \subset GL(2,\mathbb{C})$ in the geometric McKay correspondence. We end by examining brane transport between all phases of a rank 2 example, including mixed branch vacua.

6.1 $U(1)$ models [careful review]

As a practice, and a building block for the higher rank models, we give a streamlined discussion of $U(1)$ GLSMs with no superpotential, following [2, 23]. Consider a $U(1)$ GLSM with chiral multiplets of charges Q^j for $1 \leq j \leq \dim V$. Let $N_\pm \geq 0$ be the total positive/negative charges, so $N_+ - N_- = \sum_j Q^j$ and $N_+ + N_- = \sum_j |Q^j|$. Let $t_{\mathrm{sh}} = t + \sum_j Q^j \log Q^j \bmod 2\pi i)$.

For $N_+ = N_-$ this model is Calabi-Yau. In FI-theta parameter space there is a singularity at $t_{\mathrm{sh}} = 0 \bmod 2\pi i$. There are two phases $\zeta \gg 0$ and $\zeta \ll 0$, well-described by non-linear sigma models (NLSMs) on two different classical Higgs branches, which are line bundles on different weighted projective spaces.

Otherwise, up to charge conjugation we can assume $N_+ > N_-$. The FI-theta parameter is shifted upon scale and axial R-symmetry transformations. We fix these by considering the theory at some fixed complexified energy scale μ. The physics at that scale is well-described for $\zeta \gg 0$ by an NLSM on the classical Higgs branch, and for $\zeta \ll 0$ by the direct sum of such an NLSM and of Q^{tot} massive vacua. These vacua are located on the classical Coulomb branch at

$$\sigma \sim \mu \exp\left(\frac{-t_{\mathrm{sh}} + 2\pi i k}{N_+ - N_-}\right), \qquad k \in \mathbb{Z}_{N_+ - N_-}. \tag{6.1}$$

A large part of this subsection is devoted to how GLSM branes split into Higgs and Coulomb parts. Then we move on to brane transport.

6.1.1 B-branes and their central charge

Admissible contour. Recall from subsection 4.1 that B-branes must come with the data of a Lagrangian contour for $\sigma \in \mathfrak{t}_{\mathbb{C}}$ which is a deformation of $\mathfrak{t}_{\mathbb{R}}$ and that is admissible in the sense that the contour integral (4.5) converges. It is typically enough to work with contours that can be written as graphs $\sigma = \tau + iv(\tau)$ of some function $v\colon \mathfrak{t}_{\mathbb{R}} \to \mathfrak{t}_{\mathbb{R}}$. For our rank-one case the contour is simply a deformation of $\mathbb{R} \subset \mathbb{C}$. We work out the integrand's behaviour for $\hat{\sigma}$ large away from the imaginary axis:

$$\begin{aligned} \log|\text{integrand}| &= -A_q(\hat{\sigma}) + O(\log\hat{\sigma}) \\ A_q(\hat{\sigma} = \hat{\tau} + i\hat{v}) &= \big(\zeta_{\text{sh}} + (N_+ - N_-)\big(\log|\hat{\tau} + i\hat{v}| - 1\big)\big)\hat{v} \\ &\quad + \left(\frac{\pi}{2}(N_+ + N_-) + (N_+ - N_-)\arctan\frac{\hat{v}}{|\hat{\tau}|} - (\operatorname{sign}\hat{\tau})(\theta + 2\pi q)\right)|\hat{\tau}| \end{aligned} \tag{6.2}$$

where we have split $\hat{\sigma} = \hat{\tau} + \hat{v}$. The integral converges provided $A_q(\hat{\sigma}) \to +\infty$ fast enough at infinity in the contour. For cases other than $\zeta_{\text{sh}} = 0 = N_+ - N_-$, we can take the contour[18] $\hat{v}\colon \hat{\tau} \mapsto \pm\hat{\tau}^2$ with the positive sign for $N_+ > N_-$, or for $N_+ = N_-$ and $\zeta_{\text{sh}} > 0$, and the negative sign for $N_+ < N_-$, or for $N_+ = N_-$ and $\zeta_{\text{sh}} < 0$. Since $A_q \gtrsim |\hat{v}|$ along this contour, it is admissible regardless of q, hence it is also admissible for arbitrary B-branes of the GLSM.

The Calabi-Yau case $N_+ = N_-$, with in addition $\zeta_{\text{sh}} = 0$, deserves attention. Then $A_q(\hat{\tau} + i\hat{v}) = \big(\pi N_+ - (\operatorname{sign}\hat{\tau})(\theta + 2\pi q)\big)|\hat{\tau}|$. For Wilson lines $\mathcal{W}(q)$ in the window

$$-\frac{N_+}{2} < \frac{\theta}{2\pi} + q < \frac{N_+}{2} \tag{6.3}$$

even the contour $\mathbb{R}$ is admissible (it is then also admissible away from $\zeta_{\text{sh}} = 0$). Outside this window, A_q becomes arbitrarily negative for $\hat{\tau} \to +\infty$ or $-\infty$ hence no deformation of $\mathbb{R}$ is admissible. This **grade restriction rule** allows $N_+ = N_-$ Wilson line branes for $\zeta_{\text{sh}} = 0$ and $\theta \neq N_+\pi \bmod 2\pi$. As we vary θ, the window jumps for $\theta = N_+\pi \bmod 2\pi$, where the theory is singular.

Pure-Higgs phases. Consider the phase $\zeta \gg 0$ in the case $N_+ \geq N_-$ (the phase $\zeta \ll 0$ for $N_+ \leq N_-$ is similar). This phase is pure-Higgs in the sense that it has no mixed/Coulomb branch. Its Higgs branch is a quotient $X = (V \setminus \Delta)/\mathbb{C}^*$ where V is spanned by all chiral multiplets while the deleted set Δ is the subspace where all positively charged chiral multiplets vanish.

A brane of the GLSM is described by a complex of Wilson lines and by an admissible contour L. Its image in the Higgs branch NLSM is obtained by sending $e \to \infty$ at some fixed complexified energy scale μ. This turns D into a Lagrange multiplier and decouples the gauge degrees of freedom, thus quotienting V to X. The complex is sent to a complex of line bundles on X obtained by restriction and pushforward from V to X. In the language of subsection 4.2, $F_{\text{flow,Higgs}} = F_{\text{geom}}$. We state the argument more precisely in subsection 6.2.

Given the shape of the integration contour L, namely $\hat{\sigma} = \hat{\tau} + i\hat{\tau}^2$, the integral picks up poles on the positive imaginary axis, at $\hat{\sigma} = i(R_j/2 + k)/Q^j$ for $Q^j > 0$ and $k \in \mathbb{Z}_{\geq 0}$. These

[18] The choice of superlinear function $\hat{\tau} \mapsto \hat{\tau}^2$ is arbitrary.

poles are simple for generic R_j and we get

$$\begin{aligned}
Z_{D^2}(\mathcal{B}) &= C(\mathfrak{r}\Lambda)^{\hat{c}/2} \int_L \mathrm{d}\hat{\sigma} \prod_j \Gamma\Big(iQ^j\hat{\sigma} + \frac{R_j}{2}\Big) e^{it\hat{\sigma}} f_{\mathcal{B}}(\hat{\sigma}) \\
&= C(\mathfrak{r}\Lambda)^{\hat{c}/2} \sum_{j|Q^j>0} \sum_{k=0}^{\infty} \frac{2\pi(-1)^k}{k!\,Q^j} e^{-t(R_j/2+k)/Q^j} f_{\mathcal{B}}\Big(\frac{i(R_j/2+k)}{Q^j}\Big) \prod_{i\neq j} \Gamma\Big(\frac{R_i}{2} - \frac{Q^i}{Q^j}\Big(\frac{R_j}{2}+k\Big)\Big).
\end{aligned} \tag{6.4}$$

For non-generic R_j there can be higher-order poles, in which case the explicit expression is more difficult to obtain. However, the asymptotics are the same. Using Stirling's formula and $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$ we show that for a fixed j, and fixed ($k \bmod Q^j$), the summand behaves at large k according to

$$\begin{aligned}
\log|\text{summand}| &= -\frac{\zeta}{Q^j}k - \sum_i \frac{Q^i k}{Q^j} \log\left|\frac{Q^i k}{e\,Q^j}\right| + O(\log k) \\
&= -\frac{N_+ - N_-}{Q^j}\big(k\log k - k - k\log|Q^j|\big) - \frac{\zeta + \sum_i Q^i \log|Q^i|}{Q^j} k + O(\log k).
\end{aligned} \tag{6.5}$$

For $N_+ > N_-$ the sum converges regardless of t, as is consistent with the fact that there is no preferred value of t in non-Calabi-Yau models. For $N_+ = N_-$ the sum converges in the phase $\zeta_{\text{sh}} = \zeta + \sum_i Q^i \log|Q^i| > 0$. The similar sum for $N_+ < N_-$ defines an asymptotic series in (fractional) powers of e^{-t}, which never converges. We wrote the asymptotics in a form valid for summing residues with $Q^j < 0$ instead: in that case the series converges for $N_+ < N_-$, or $N_+ = N_-$ and $\zeta_{\text{sh}} < 0$, while it is an asymptotic series if $N_+ > N_-$.

Phases with massive vacua. We turn to the phase $\zeta \ll 0$ in models with $N_+ > N_-$ (the phase $\zeta \gg 0$ with $N_+ < N_-$ is similar). We argued that for the pure-Higgs phase $\zeta \gg 0$, branes flow to their geometric image. In the phase $\zeta \ll 0$, there is both a Higgs branch and $N_+ - N_-$ massive vacua, so we need to determine the image of the brane on both branches.

Let us try to expand at large $(-\zeta)$ the hemisphere partition function with a Wilson line $\mathcal{W}(q)$. Our experience with the $\zeta \gg 0$ phase suggests one part of the hemisphere partition function should be a sum over poles along $-i\mathbb{R}_{>0}$. Below (6.5) we work out that such a sum is an asymptotic series. For any finite ζ we should expect that a good approximation is given by a finite number of terms in the series: terms shrink until the term with

$$\frac{k}{|Q^j|} \simeq \lambda, \qquad \lambda := \exp\Big(-\frac{\zeta_{\text{sh}}}{N_+ - N_-}\Big), \tag{6.6}$$

then terms grow again. This leads us to rewriting the contour integral over L into a sum of residues at $\hat{\sigma} = i(R_j/2+k)/Q^j$ for $Q^j < 0$ and $k \in \mathbb{Z}_{\geq 0}$ bounded above by $\operatorname{Im}(-\hat{\sigma}) < \lambda$, plus a contour integral over some contour L_λ that crosses the imaginary axis at $\hat{\sigma} = -i\lambda$.

Our key task is to evaluate the contour integral over L_λ. We do this by a saddle-point approximation, which is only good if we can deform the contour to pass close to a steepest descent contour, with some remaining piece in regions where the potential A_q is sufficiently large. This last constraint depends drastically on q.

Small window: no saddle-point. The simplest case is $\mathcal{W}(q)$ in the **small window**

$$\left|\frac{\theta}{2\pi}+q\right| < \frac{1}{2}\min(N_-,N_+). \tag{6.7}$$

Again we take $N_+ > N_-$ and $\zeta \ll 0$. Then the coefficient of $|\hat{\tau}|$ in (6.2) is bounded below by some $c > 0$, so

$$A_q(\hat{\tau}-i\lambda) \geq A_q(-i\lambda)+c|\hat{\tau}| = (N_+-N_-)\lambda + c|\hat{\tau}|. \tag{6.8}$$

We can thus choose $L_\lambda = \mathbb{R}-i\lambda$. Its contribution will then scale like $\exp(-(N_+-N_-)\lambda)$ and we deduce an asymptotic expansion of the form

$$Z_{D^2}\big(\mathcal{W}(q)\big) = \left(\sum_{j|Q^j<0}\ \sum_{k=0}^{-R_j/2-Q^j\lambda}(\cdots)\lambda^{-\frac{N_+-N_-}{|Q^j|}k}\right) + O\Big(e^{-(N_+-N_-)\lambda}\Big). \tag{6.9}$$

We will soon check that the residual term is more exponentially suppressed than any of the N_+-N_- possible Coulomb branch contributions. The asymptotic expansion we found thus shows that a Wilson line brane in the small window (6.7) flows purely to a Higgs branch brane.

Big window and saddle-point. In this paragraph we find the image in the Higgs and Coulomb branches of any complex of Wilson lines $\mathcal{W}(q)$ whose charges fit in the **big window**

$$\left|\frac{\theta}{2\pi}+q\right| < \frac{1}{2}\max(N_-,N_+). \tag{6.10}$$

We call such branes **grade-restricted**.

We continue with $N_+ > N_-$ and $\zeta \ll 0$ and assume $\theta \neq N_\pm\pi \bmod 2\pi$ to avoid singularities. The contour integral is now computed using a saddle-point analysis. Away from the imaginary axis, the integrand obeys

$$\log\big(\text{integrand}(\hat{\sigma})\big) = \begin{cases}(N_+-N_-)i\hat{\sigma}\big(\log(\hat{\sigma})-\ell_+-1\big)+O(\log|\hat{\sigma}|), & \operatorname{Re}\hat{\sigma}>0\\ (N_+-N_-)i\hat{\sigma}\big(\log(-\hat{\sigma})-\ell_--1\big)+O(\log|\hat{\sigma}|), & \operatorname{Re}\hat{\sigma}<0\end{cases} \tag{6.11}$$

where

$$\ell_\pm = -\frac{\zeta_{\text{sh}}}{N_+-N_-} + i\frac{\theta+2\pi q\mp\frac{\pi}{2}(N_++N_-)}{N_+-N_-} \tag{6.12}$$

and we also note that $\operatorname{Re}\ell_+ = \operatorname{Re}\ell_- = \log\lambda$. These formulae generalize (6.2) by including the phase of the integrand. Note that the two asymptotic expansions do not have the same $\operatorname{Re}\hat{\sigma}\to 0$ limits: at $\hat{\sigma}=i\lambda$ with $\lambda>0$ there is a jump by $-2\pi i N_+\lambda$; at $\hat{\sigma}=-i\lambda$ with $\lambda>0$ there is a jump by $2\pi i N_-\lambda$.

There is one saddle-point at $\log(\hat{\sigma})=\ell_+$ if $|\operatorname{Im}\ell_+|<\pi/2$ (to ensure $\operatorname{Re}\hat{\sigma}>0$), and one at $\log(-\hat{\sigma})=\ell_-$ if $|\operatorname{Im}\ell_-|<\pi/2$ (to ensure $\operatorname{Re}\hat{\sigma}<0$), and otherwise none. At a fixed θ, the collection of these saddle-points for $q\in\mathbb{Z}$ coincides with the set of N_+-N_- Coulomb branch vacua, which are solutions of

$$(i\hat{\sigma})^{N_+-N_-} = \exp(-t_{\text{sh}}), \qquad t_{\text{sh}} = t+\sum_i Q_i\log Q_i \bmod 2\pi i. \tag{6.13}$$

When the hemisphere partition function of a brane has a contribution from one of these saddle-points, the brane has a non-trivial part along the corresponding massive vacuum.

Let us focus on the first case $|\mathrm{Im}\,\ell_+| < \pi/2$, namely $N_-\pi < \theta + 2\pi q < N_+\pi$. The steepest descent contour L_{steep} passing through the saddle $\hat\sigma = \exp(\ell_+)$ is (one branch of) the locus where the integrand has the same phase and smaller norm than its value at the saddle. In terms of the logarithm,

$$\hat\sigma\bigl(\log(\hat\sigma) - \ell_+ - 1\bigr) + \exp(\ell_+) \in i\mathbb{R}_{>0}. \tag{6.14}$$

Simple numerics indicate[19] that this contour, together with its continuation to the half-space $\mathrm{Re}\,\hat\sigma < 0$, is homotopic to a parabola intersecting the imaginary axis at some point $-i\lambda\, c(\mathrm{Im}\,\ell_+)$ where $c(\varphi)$ is some function of the argument $\varphi \in (-\pi/2, \pi/2)$ that interpolates between $c(-\pi/2) = 1$ and $c(\pi/2) = 0$.

The original contour integral over the parabola L thus decomposes into a sum of $O(\lambda)$ residues at poles along $-i\mathbb{R}_{>0}$, plus an integral over L_{steep}. The latter scales like

$$\left|\int_{L_{\text{steep}}} \mathrm{d}\hat\sigma\,(\text{integrand})\right| \sim \exp\biggl((N_+ - N_-)\cos\biggl(\frac{N_+\pi - (\theta + 2\pi q)}{N_+ - N_-}\biggr)\lambda\biggr), \tag{6.15}$$

where the cosine takes discrete values in $(-1, 1)$ that increase with $\theta + 2\pi q \in (N_-\pi, N_+\pi)$. All of these exponential behaviours are larger than the residual term we found for Wilson lines in the small window in (6.9).

Altogether, we find the following cases (assuming $\theta \neq N_\pm\pi \bmod 2\pi$):

- If $|\theta + 2\pi q| < N_-\pi$ (small window), the contour integral has no saddle. It is given by the asymptotic series of residues, so that the image of the Wilson line is the line bundle on the Higgs branch expected from geometry, with no Coulomb branch part.
- If $N_-\pi < |\theta + 2\pi q| < N_+\pi$, the contour integral has one saddle-point $\hat\sigma = \hat\sigma_q$, in the half-space where $\operatorname{sign}\mathrm{Re}\,\hat\sigma = \operatorname{sign}(\theta + 2\pi q)$. It is equal to the residue sum plus the integral along a steepest descent contour. The image of the Wilson line has two parts: the expected line bundle on the Higgs branch, and a brane on the Coulomb branch vacuum corresponding to the given saddle-point.
- If $N_+\pi < |\theta + 2\pi q|$, there is no saddle-point, and in addition the contour cannot be chosen to keep the potential $A_q(\hat\sigma)$ positive along the contour. The integral cannot be computed directly using saddle-point approximation. In fact, using other techniques we show that the naive Higgs branch contribution obtained by picking up the asymptotic series of residues in the same way as before is incorrect. This is due to an unavoidably large contribution from the remaining contour integral.

Empty brane. Consider the case $N_+ \geq N_-$ and the phase $\zeta \gg 0$. As explained in subsection 4.2, the Koszul complex

$$\mathcal{K}^+ := \bigotimes_{j|Q^j>0} \Bigl(\mathcal{W}(-Q^j) \xrightarrow{X_j} \mathcal{W}(0)\Bigr) \tag{6.16}$$

is a resolution of the structure sheaf of the deleted set Δ, hence that complex has a trivial geometric image on the Higgs branch. Since the phase has no massive vacuum, the GLSM brane $\mathcal{K}^+$ is simply empty in that phase.

[19] It would be pleasant to obtain an analytic proof.

If $N_+ > N_-$ (rather than $N_+ = N_-$), then this (empty) B-brane can be transported to the other phase, where it is also empty because adding twisted F-terms does not spoil emptyness. We can check this picture using the hemisphere partition function. The integrand in (4.8) has poles along the positive and negative imaginary axes. The integral picks up poles on the positive imaginary axis as in (6.4). But the brane factor

$$f_{\mathcal{K}^+}(\hat{\sigma}) = \prod_{j|Q^j>0} \left(1 - e^{-2\pi Q^j \hat{\sigma}}\right) \tag{6.17}$$

is such as to precisely cancel these poles, so $Z_{D^2}(\mathcal{K}^+) = 0$ exactly. Brane transport analytically continues this vanishing partition function, which thus vanishes in all phases, consistent with the claim that this GLSM brane is **empty in all phases**.

The situation is different for Calabi-Yau models ($N_+ = N_-$): again $Z_{D^2}(\mathcal{K}^+) = 0$, but we cannot conclude that the same complex of Wilson lines (6.16) is empty in all phases. Indeed, transporting the brane or its twists to another phase is not directly possible because none fit in the window (6.3) that is required to transport the brane from one phase to the other. The brane $\mathcal{K}^+$ that is empty in one phase and its analogue $\mathcal{K}^-$ with $Q^j < 0$ that is empty in the other phase can be quite different in general.

6.1.2 RG flow image of all branes.

So far, we found the image of any GLSM brane $\mathcal{B}$ under RG flow to a pure-Higgs phase. It is

$$F_{\text{flow}}(\mathcal{B}) = F_{\text{geom}}(\mathcal{B}) \in D^b(X) \tag{6.18}$$

where the "geometric" functor $F_{\text{geom}}\colon D^b(V, U(1)) \to D^b(X)$ projects complexes of $U(1)$-equivariant vector bundles from V to the Higgs branch $X = (V \setminus \Delta)/\mathbb{C}^*$. In phases with both a Higgs branch and massive vacua, we found the image of grade-restricted branes, namely those that fit in the big window $|\theta + 2\pi q| < \max(N_+, N_-)\pi$. We now generalize to all branes.

For definiteness take $N_+ > N_-$ and $\zeta \ll 0$.

The twist $\mathcal{K}^+ \otimes \mathcal{W}(q)$ of the Koszul brane (6.16) is empty in the pure-Higgs phase $\zeta \gg 0$ hence in all phases. It is a complex with minimum and maximum charges $q_{\min} = q - N_+$ and $q_{\max} = q$. We deduce that the brane $\mathcal{W}(q)$ is equivalent in all phases to a complex with charges among $q - N_+, \dots, q - 1$, and likewise to a complex with charges among $q + 1, \dots, q + N_+$.

Consider now an arbitrary complex $\mathcal{B}_1$ of Wilson lines. Repeatedly replace every Wilson line $\mathcal{W}(q)$ with a charge outside the big window $|\theta + 2\pi q| < N_+$ by an equivalent complex of Wilson lines with charges closer to the big window. This eventually ends and yields a complex $\mathcal{B}_2$ of Wilson lines with charges in the big window.

Coulomb branch. The brane factors of $\mathcal{B}_1$ and $\mathcal{B}_2$ differ by a multiple of the brane factor (6.17) of $\mathcal{K}^+$. The decomposition

$$f_{\mathcal{B}_1}(\hat{\sigma}) = f_{\mathcal{B}_2}(\hat{\sigma}) + P(e^{2\pi\hat{\sigma}}) f_{\mathcal{K}^+}(\hat{\sigma}), \tag{6.19}$$

where P is some Laurent polynomial in $e^{2\pi\hat{\sigma}}$, is essentially given by polynomial Euclidean division. The remainder $f_{\mathcal{B}_2}$ is unique and does not depend on details of the binding with Higgs-empty branes.

From its coefficients

$$f_{\mathcal{B}_2}(\hat{\sigma}) = \sum_{q=q_{\min}}^{q_{\max}} a_q e^{2\pi q\hat{\sigma}} \tag{6.20}$$

where $q_{\min}$ and $q_{\max}$ are the bounds of the big window, we can deduce the Coulomb branch part of the image of $\mathcal{B}_1$ or equivalently $\mathcal{B}_2$: for each q such that $N_-\pi < |\theta + 2\pi q| < N_+\pi$, there are a_q D0 branes, or $-a_q$ $\overline{\text{D0}}$ if $a_q < 0$, on the massive vacuum at

$$i\hat{\sigma} = \exp\biggl(\frac{-\zeta_{\text{sh}} + i(\theta + 2\pi q - \operatorname{sign}(\theta + 2\pi q)\pi N_-)}{N_+ - N_-}\biggr). \tag{6.21}$$

Indeed, each Wilson line $\mathcal{W}(q)$ in the big window goes to an empty brane for $|\theta + 2\pi q| < N_-\pi$ and otherwise goes to a single D0 brane on one Coulomb branch vacuum.

Higgs branch. The Higgs branch image $F_{\text{flow,Higgs}}(\mathcal{B}_1)$ under RG flow cannot be read off from the brane factor only. Since F_{geom} and $F_{\text{flow,Higgs}}$ agree on grade-restricted branes we have

$$F_{\text{flow,Higgs}}(\mathcal{B}_1) = F_{\text{flow,Higgs}}(\mathcal{B}_2) = F_{\text{geom}}(\mathcal{B}_2), \tag{6.22}$$

which differs in general from $F_{\text{geom}}(\mathcal{B}_1)$. The two functors only agree in general for grade-restricted branes.

Let us say some words about the brane $\mathcal{K}^-(q) \coloneqq \mathcal{W}(q) \otimes \bigotimes_{j|Q^j<0}\bigl(\mathcal{W}(-Q^j) \xrightarrow{X_j} \mathcal{W}(0)\bigr)$, in which we selected the other sign of charges compared to the brane $\mathcal{K}^+$ (6.16) that is empty in all phases. By construction the brane $\mathcal{K}^-(q)$ is mapped to an empty brane by F_{geom}. In addition, $\mathcal{K}^-(q)$ is grade-restricted if $q + \theta/(2\pi) \in (N_- - N_+/2, N_+/2)$, in which case it has no Higgs part under RG flow. This pure-Coulomb brane typically goes to several Coulomb branch vacua according to the discussion near (6.21), but one can take linear combinations of the brane factors of $\mathcal{K}^-(q)$ such that a single coefficient outside the small window vanishes. The corresponding direct sums of $\mathcal{K}^-(q)$ branes are complexes that flow purely to a **D0 brane on one massive vacuum**. Finally, despite having an empty image under F_{geom}, branes $\mathcal{K}^-(q)$ that are not grade-restricted typically acquire a non-empty Higgs part under RG flow, due to the round-about construction (6.22).

In conclusion, it should be stressed again that the effect of massive vacua on the RG flow of GLSM branes goes beyond simply adding a Coulomb branch part to the resulting brane: even the Higgs branch part is different from the natural geometric one. Furthermore, the Higgs branch image depends on the theta angle because θ affects windows.

6.1.3 Brane transport

We now turn to brane transport between phases.

Calabi-Yau case We recall that for a complex $\mathcal{B}$ of Wilson lines $\mathcal{W}(q)$ in the window (6.3), namely $\mathbf{w} \coloneqq \{q \mid -N_+\pi < \theta + 2\pi q < N_+\pi\}$, the contour $\mathbb{R}$ is admissible for all ζ. Such a grade-restricted B-brane can thus be transported trivially from $\zeta \gg 0$ to $\zeta \ll 0$ by varying ζ without changing charges or morphisms in the complex.

Consider next a B-brane $\mathcal{B}_+ \in D^b(X_+)$ on the Higgs branch in the phase $\zeta \gg 0$ (hence the subscript $+$). It can be written as the image of some B-brane $\mathcal{B}_1 \in D^b(V, U(1))$ of the GLSM. Recall that the Koszul brane $\mathcal{K}^+$ (6.16) that is empty in the phase $\zeta \gg 0$ has charges from

$-N_+$ to 0, so $\mathcal{B}_1$ is equivalent to another complex $\mathcal{B}_2$ with charges $q \in \mathbf{w}$. That brane can be transported to the phase $\zeta \ll 0$ and projected to the Higgs branch category $D^b(X_-)$ there. Brane transport from $\zeta \gg 0$ to $\zeta \ll 0$ only depends on θ via the window $\mathbf{w}$ of charges that are allowed at $\zeta_{\mathrm{sh}} = 0$. This is consistent with the fact that brane transport only depends on the path in FI-theta parameter space up to homotopy.

Formally, for each interval $\mathbf{w}$ of N_+ consecutive integers one defines the window category $\mathcal{T}_{\mathbf{w}} \subset D^b(V, U(1))$ as consisting of complexes of Wilson lines with charges $q \in \mathbf{w}$. Then one considers the functors $F_\pm \colon D^b(V, U(1)) \to D^b(X_\pm)$ that project to the Higgs branch in each phase. Their restriction to any window category $\mathcal{T}_{\mathbf{w}}$ can be shown to be an equivalence of categories. Brane transport $D^b(X_+) \to D^b(X_-)$ is then defined by composing these equivalences. Pictorially,

$$\begin{tikzcd}[column sep=small]
 & D^b(V,U(1)) \arrow[ddl, "F_-"'] \arrow[ddr, "F_+"] & \\
 & \mathcal{T}_{\mathbf{w}} \arrow[u, phantom, "\cup"] \arrow[dl, "\cong"] \arrow[dr, "\cong"'] & \\
D^b(X_-) & & D^b(X_+)
\end{tikzcd} \tag{6.23}$$

Non-Calabi-Yau case: starting from pure-Higgs phase For definiteness, $N_+ > N_-$. We use subscripts $+$ and $-$ for the phases $\zeta \gg 0$ and $\zeta \ll 0$, respectively.

Start from a brane $\mathcal{B}_+ \in D^b(X_+)$ on the Higgs branch X_+ of the pure-Higgs phase $\zeta \gg 0$. Lift to a GLSM brane $\mathcal{B}_1 \in D^b(V, U(1))$ in the sense that $F_{\mathrm{geom},+}(\mathcal{B}_1) = \mathcal{B}_+$. Since there is no constraint on charges for an admissible contours to exist, the brane can be transported to the other phase. However, finding the image in the other phase is delicate.

Using that the Koszul brane $\mathcal{K}^+$ (6.16) is empty in both phases and has charges from $-N_+$ to 0, the brane $\mathcal{B}_1$ can be replaced by a grade-restricted brane $\mathcal{B}_2$ whose images in both phases are the same as those of $\mathcal{B}_1$. Thanks to grade restriction, the Higgs branch part of the image in the phase $\zeta \ll 0$ is then $F_{\mathrm{geom},-}(\mathcal{B}_2)$. The Coulomb branch (massive vacuum) part $F_{\mathrm{flow,Coulomb}}(\mathcal{B}_2)$ is deduced from the brane factor of $\mathcal{B}_2$.

The procedure simplifies if one starts from images $\mathcal{O}(q) = F_{\mathrm{geom},+}(\mathcal{W}(q))$ of Wilson lines in the pure-Higgs phase, with $|\theta + 2\pi q| < N_+\pi$. These generate $D^b(X_+)$. The Wilson lines in the small window $|\theta + 2\pi q| < N_-\pi$ map to generators of $D^b(X_-)$. The $N_+ - N_-$ Wilson lines with $N_-\pi < |\theta + 2\pi q| < N_+\pi$ map to a combination of one massive vacuum and some component along $D^b(X_-)$. Informally, one can say that these $N_+ - N_-$ Wilson lines have "gone away" to the Coulomb branch, but to be more precise what goes to the Coulomb branch is a complex of these Wilson lines with some in the small window.

Non-Calabi-Yau case: going towards pure-Higgs phase For definiteness, $N_+ > N_-$. Start now from a brane in the phase $\zeta \ll 0$. This requires giving both a Higgs branch part $\mathcal{B}_- \in D^b(X_-)$ and a Coulomb branch part $\mathcal{C}_-$. The non-trivial step now is to find a complex of Wilson lines that flows to $\mathcal{B}_-$ and $\mathcal{C}_-$.

What is often easily available is a complex $\mathcal{B}_1 \in D^b(V, G)$ such that $F_{\mathrm{geom},-}(\mathcal{B}_1) = \mathcal{B}_-$. Then one uses the Koszul brane $\mathcal{K}^-$ that has an empty image under $F_{\mathrm{geom},-}$ to construct a complex $\mathcal{B}_2 \in D^b(V, G)$ built from Wilson lines with $|\theta + 2\pi q| < N_-\pi$, and such that $F_{\mathrm{geom},-}(\mathcal{B}_2) = \mathcal{B}_-$. Separately, one combines grade-restricted branes that flow to branes on single Coulomb branch massive vacua into a brane $\mathcal{B}_3 \in D^b(V, G)$ that flows to $\mathcal{C}_-$. The sought-after lift is then $\mathcal{B}_2 \oplus \mathcal{B}_3$, which is then projected down to $F_{\text{geom},+}(\mathcal{B}_2 \oplus \mathcal{B}_3) \in D^b(X_+)$ in the pure-Higgs phase.

The relevant diagram is as follows, where $\mathcal{T}_{\mathbf{w}_\pm}$ denote window categories for the big and small windows. The B-brane category in the $\zeta \ll 0$ phase in fact has a semi-orthogonal decomposition $\langle C, D^b(X_-)\rangle$ into the category C of Coulomb branch branes (itself further decomposed into individual massive vacua) and the derived category of the Higgs branch X_-. It would be interesting to clarify the physical meaning of this semi-orthogonal decomposition, as it appears to allow strings stretching in one direction between the Coulomb branch vacua and the Higgs branch.

$$\begin{tikzcd}
 & & D^b(V,U(1)) \arrow[ddll, "F_{\text{flow}}"'] \arrow[ddddrr, "F_+"] & & \\
 & & \cup & & \\
\langle C, D^b(X_-)\rangle & & \mathcal{T}_{\mathbf{w}_+} \arrow[ll, "\cong"'] \arrow[ddrr, "\cong"'] & & \\
\cup & & \cup & & \\
D^b(X_-) & & \mathcal{T}_{\mathbf{w}_-} \arrow[ll, "\cong"'] & & D^b(X_+)
\end{tikzcd} \tag{6.24}$$

6.2 Wall-crossing and Higgsing

Given a B-brane in a $U(1)^r$ GLSM with no superpotential, we can ask for its image in some phase upon RG flow. More precisely, as we explained, we take the large gauge coupling limit $e \to \infty$ while keeping the complexified energy scale μ and FI-theta parameters fixed, so that the phase gives a good description of the physics at scale μ.

Deep in a pure-Higgs phase, at every point on the Higgs branch all continuous gauge symmetries are Higgsed and the vector multiplets have mass $\gtrsim e$. The massive vector multiplets can thus be integrated out. Their effect is thus to impose D-term equations and quotient out by gauge symmetry, so that the theory is well-described by an NLSM on the Higgs branch $X_\zeta = V//_\zeta G$. A B-brane $\mathcal{B} \in D^b(V,G)$ of the GLSM, namely a complex of equivariant vector bundles on V, is mapped by these steps to the brane obtained by restricting the vector bundles then pushing forward to the quotient $X_\zeta \simeq (V \setminus \Delta)/G_{\mathbb{C}}$. In other words,

$$F_{\text{flow}} = F_{\text{geom}} \colon D^b(V,G) \to D^b(X_\zeta) \tag{6.25}$$

in pure-Higgs phases, for instance in all phases of a Calabi-Yau model.

To integrate out the gauge degrees of freedom it was essential to have no mixed or Coulomb branches. As we saw in $U(1)$ models, the existence of such branches affects even the Higgs branch image of GLSM branes. For other phases, the general strategy that we apply to Hirzebruch-Jung models in subsection 6.3, and extend in upcoming work, is to transport B-branes starting from a pure-Higgs phase.

In this subsection we focus on crossing a single wall, deep in that wall, far from other walls. First we give a physical explanation for band restriction rules, then we justify it using the hemisphere partition function.

6.2.1 Higgsing argument

The wall has codimension 1 in $\mathfrak{g}^*$. Let $\mathfrak{h} \in \mathfrak{g}$ be the one-dimensional subspace orthogonal to the hyperplane containing the wall, and let I be the set of flavours i such that $Q^i \in \mathfrak{h}^\perp$. Since the wall is spanned by some charge vectors, $\mathfrak{h}$ contains a non-zero vector with rational coordinates so it generates a compact subgroup isomorphic to $U(1)$ inside $G = U(1)^r$. We let

$u \in \mathfrak{h}$ be the generator normalized such that $\exp(2\pi i \alpha u) = 1 \in G \Leftrightarrow \alpha \in \mathbb{Z}$ and defined up to a sign.

The D-term equation expresses ζ as a positive linear combination of charge vectors. As ζ touches the wall, it belongs to the cone of Q^i, $i \in I$, which means that the Higgs branch has a locus $P \subset X_\zeta$ at which only X^i, $i \in I$, are non-zero. In other words, P consists of common zeros of all chiral multiplets charged under $\mathfrak{h}$. Configurations in P do not break gauge symmetries completely, but rather to $\mathfrak{h}$. Besides the $U(1)$ factor generated by $\mathfrak{h}$, the unbroken gauge group H_p at $p \in P$ can have a discrete factor.

Let us assume for simplicity that $P = \{p\}$ is a single point, as this is the situation for Hirzebruch-Jung models. D-term equations fix $(X_i)_{i\in I}$ completely at p, hence they also fix $(X_i)_{i\in I}$ as functions of $(X_j)_{j\notin I}$ near p. Their generic vev in $\mathfrak{h}^\perp$ gives masses to all vector multiplets except those along $\mathfrak{h}$. We integrate them out, as well as fluctuations for the $(X_i)_{i\in I}$, in a neighborhood of p. This gives a **local model**, with

$$\begin{aligned}&\text{gauge group } H_p \simeq U(1) \times \Gamma, \qquad \Gamma \text{ discrete,}\\ &\text{FI-theta parameter } t \cdot u,\\ &\text{chiral multiplets } (X_j)_{j\notin I} \text{ of charges } Q^j \cdot u.\end{aligned} \tag{6.26}$$

Of course the local model is only a good description near p, but this is precisely the neighborhood whose topology changes upon crossing the wall. We thus expect wall-crossing to only affect branes in the region near p, which is well-described by the local model. The two phases $\zeta \cdot u \gtrless 0$ correspond to the two sides of the wall in the full model.

Assume first that the local model is Calabi-Yau, namely that $\sum_{j\notin I} Q^j \cdot u = 0$, itself equivalent to $Q^{\text{tot}} \cdot u = 0$, namely the wall is Calabi-Yau. In each phase, the local model only has a Higgs branch. A Wilson line with some charge under $U(1) \subset H_p$ can be transported between the phases $\zeta \cdot u \gg 0$ and $\zeta \cdot u \ll 0$ provided its $U(1)$ charge is grade-restricted. Translating back to the full GLSM, **band-restricted** branes are $\mathcal{W}(q)$ such that [2]

$$\bigl|(\theta \cdot u) + 2\pi(q \cdot u)\bigr| < \pi N_{u,+} = \pi N_{u,-}, \qquad N_{u,\pm} = \sum_j (Q^j \cdot u)^\pm \tag{6.27}$$

where $(x)^\pm := (|x| \pm x)/2$ and we extended the sum to all j since $Q^j \cdot u = 0$ for $j \in I$. Complexes of band-restricted Wilson lines are transported unchanged through the wall.

If the local model is not Calabi-Yau, we choose the sign of u such that $Q^{\text{tot}} \cdot u > 0$. The local model then has a pure-Higgs phase at $\zeta \cdot u \gg 0$ and a phase with a Higgs branch and some massive vacua at $\zeta \cdot u \ll 0$. These massive vacua lie at $Q^{\text{tot}} \cdot u$ values on the classical Coulomb branch, on which the discrete abelian group Γ acts trivially, so twisted sectors increase the number of massive vacua to $|\Gamma|\, Q^{\text{tot}} \cdot u$. Brane transport from $\zeta \cdot u \gg 0$ to $\zeta \cdot u \ll 0$ now involves two nested windows, which translate to bands in the full GLSM:

$$\begin{aligned}&\textbf{small band} \qquad \bigl|(\theta \cdot u) + 2\pi(q \cdot u)\bigr| < \pi \min(N_{u,+}, N_{u,-}),\\ &\textbf{big band} \qquad \bigl|(\theta \cdot u) + 2\pi(q \cdot u)\bigr| < \pi \max(N_{u,+}, N_{u,-}),\end{aligned} \tag{6.28}$$

where $N_{u,\pm} = \sum_j (Q^j \cdot u)^\pm$ are defined as before. We call **band-restricted** a brane whose charges are in the big band. A Wilson line in the small band gets transported to the Higgs branch, while a Wilson line in the big band but not the small one is transported to a combination of the Higgs branch and one massive vacuum.

The charge of the Wilson line under $\Gamma \subset H \subset G$ only plays a role in determining which massive vacuum appears. We defer to later work a careful analysis of the contribution from massive vacua. If the locus P with unbroken gauge symmetry has non-zero dimension, the analogue of these massive vacua is a collection of mixed branches on which $\sigma \in \mathfrak{h}$ and the chiral multiplets $(X_i)_{i\in I}$ both get a vev. A natural conjecture is that the band restriction rule still applies: a Wilson line in the small band is transported to a brane supported purely on the Higgs branch, while a Wilson line in the big band maps to a combination of a Higgs branch brane and a brane supported on a specific mixed branch.

We caution that the approximations we made are only valid for crossing a single wall in an asymptotic regime. If we wish to cross multiple walls, each wall crossing will be in a different asymptotic regime and thus involve a different $U(1)$ subgroup and hence lead to a different band restriction rule. Crossing several walls in a row gives successive band restriction rules.

6.2.2 Hemisphere partition function

In subsection 4.1 we discussed the hemisphere partition function $Z_{D^2}(\mathcal{B})$ with a B-brane. In Calabi-Yau models it can be converted by closing contours to an infinite sum of residues (4.9).

For $U(1)$ models that are not Calabi-Yau, say with $Q^{\text{tot}} > 0$, we reviewed how brane transport can be understood by comparing two expansions of $Z_{D^2}(\mathcal{B})$. The first, corresponding to the phase $\zeta \gg 0$, is to write $Z_{D^2}(\mathcal{B})$ as a sum of residues (4.9) at poles of chiral multiplet one-loop determinants with $Q^j > 0$. The second expansion corresponds to the phase $\zeta \ll 0$, and it can only be carried out straightforwardly for branes in the big window. It consists of decomposing $Z_{D^2}(\mathcal{B})$ into a Higgs branch contribution from residues with $Q^j < 0$, and a massive vacuum contribution from a steepest descent contour. For branes outside the big window, the sum of residues (4.9) does not correctly give the Higgs branch contribution.

We generalize these ideas now to $U(1)^r$ non-Calabi-Yau models. In generic phases we expect one Higgs branch and a collection of mixed and Coulomb branches. A mixed branch is characterized by the space $\mathfrak{q}^\perp \subset \mathfrak{g}$ in which $\hat{\sigma}$ varies, and correspondingly by the set $I = \{i \mid Q^i \in \mathfrak{q}\}$ of flavours that are not given a mass by the vev of $\hat{\sigma}$. The mixed branch is roughly a product of some Higgs branch for $(X_i)_{i\in I}$ and some Coulomb branch vacua with $\hat{\sigma} \in \mathfrak{q}^\perp$. The mixed branch is found by integrating out all chiral multiplets $(X_j)_{j\notin I}$ to get the effective twisted superpotential for $\hat{\sigma} \in \mathfrak{q}^\perp$, then by solving a D-term equation for the remaining $(X_i)_{i\in I}$. How is $\hat{\sigma}$ reduced to $\mathfrak{q}^\perp$? Physically, this reduction is due to the non-zero vevs of $(X_i)_{i\in I}$. Computationally, the reduction is done by closing some of the integrals to pick up residues at common poles of $\dim \mathfrak{q}$ chiral multiplet one-loop determinants of $(X_i)_{i\in I}$. Indeed, the leading pole[20] of such a one-loop determinant $\Gamma(iQ^i \cdot \hat{\sigma} + R_i/2)$ is at $Q^i \cdot \hat{\sigma} = iR_i/2$, which is close to the constraint imposed by the non-zero vev of X_i. Intersections of $\dim \mathfrak{q}$ hyperplanes impose $\hat{\sigma}$ approximately in $\mathfrak{q}^\perp$, as we want. Our $U(1)$ experience then suggests to compute the integral by a saddle-point approximation at large generic $|\hat{\sigma}|$ within $\hat{\sigma} \in \mathfrak{q}^\perp$. In summary,

- pick up residues at common poles of $\dim \mathfrak{q}$ chiral multiplet one-loop determinants;
- find saddles of the integral over $\hat{\sigma} \in \mathfrak{q}^\perp$.

This process of decomposing $Z_{D^2}(\mathcal{B})$ into contributions of various branches is quite difficult to carry out from first principles. Instead, we relate decompositions upon wall-crossing. For

[20] Other poles are related to vortex configurations and are subleading in the series over poles.

simplicity we focus on the Higgs branch.

In pure-Higgs phases, the partition function is given by the convergent series of residues (4.9), which gives the Higgs branch (and only) contribution. Starting from such a phase, we cross the walls from $\zeta\cdot u \gg 0$ to $\zeta\cdot u \ll 0$, where the sign of u is chosen as above such that $Q^{\text{tot}}\cdot u\geq 0$. At each wall, provided the brane "goes through the wall", namely fits in the big band (6.28), we find that the sum of residues (4.9) splits into the analogous sum for the other phase, and a mixed branch contribution (with $\mathfrak{q}^{\perp}=\mathfrak{h}$ in the notations above). If the brane does not fit in the big band, the sum of residues does not give the correct Higgs branch contribution. The picture that emerges is that the sum of residues only correctly gives the Higgs branch contribution, in some given phase, for branes that go through all walls between a pure-Higgs phase and that phase. It is not clear whether for every phase there should exist a collection of branes that go through all walls and generates the Higgs branch B-brane category. For (the resolved phase of) Hirzebruch-Jung models, there is.

Let us thus start with the sum of residues (4.9), ranging over collections J of r flavours such that $\zeta\in\text{Cone}_J$. Some collections appear on both sides of the wall and are uninteresting for us: these are analogous to parts of the Higgs branch whose topology doesn't change under wall-crossing. Collections that are only allowed on one side of the wall must take the form $J=\{j\}\cup J'$, where charge vectors $(Q^i)_{i\in J'}$ lie in the wall, hence $J'\subset I$. The sum of residue then takes the form

$$Z_{D^2,\text{residue}}(\mathcal{B}) \simeq \sum_{J'\subset I,\, k:J'\to\mathbb{Z}_{\geq 0}} \left(\sum_{j\notin I|\zeta\in\text{Cone}_{J'\cup\{j\}}} \sum_{k_j\geq 0} \pm \operatorname*{res}_{i\hat{\sigma}=i\hat{\sigma}_{J'\cup\{j\},k}} (\cdots) \right) + \dots, \tag{6.29}$$

where the trailing dots denote other collections J, and we recall that $\hat{\sigma}_{J,k}$ is the common solution of $iQ^i\cdot\hat{\sigma}+R_i/2=-k_i$ for $i\in J$.

The sums over j and k_j recombine into a one-dimensional contour integral which can be found in several ways. The simplest way is to start from the original contour integral and select the residue at $iQ^i\cdot\hat{\sigma}+R_i/2=-k_i$ for all $i\in J'$. These conditions on $Q^i\cdot\sigma$ force $\hat{\sigma}$ to belong to $\mathfrak{h}$ up to some imaginary offset $r_{J',k}>0$, so $\hat{\sigma}=ir_{J',k}+su$ with an integral over s. We get schematically

$$Z_{D^2,\text{residue}}(\mathcal{B}) \simeq \sum_{J'\subset I,\, k:J'\to\mathbb{Z}_{\geq 0}} \int_{\hat{\sigma}=ir_{J',k}+su} \mathrm{d}s \Big(\pm \operatorname*{res}_{i\hat{\sigma}=ir_{J',k}+su} (\cdots) \Big) + \dots, \tag{6.30}$$

Each Gamma function $\Gamma\big(iQ^i\cdot\hat{\sigma}+R_i/2\big)$ in the integrand with $i\notin J'$ becomes $\Gamma\big(i(Q^i\cdot u)s+\text{real}\big)$, namely one of the one-loop determinants of the local model. Other parts simplify similarly and the resulting one-dimensional contour integral is itself a hemisphere partition function: that of the local model (6.26) with various R-charge assignments. Picking up residues on one side or the other of the contour gives (6.29) on either side of the wall. This was expected since performing all r contour integrals should be the same as preforming $r-1$ and then the last one.

As we reviewed extensively in subsection 6.1, if the local model is not Calabi-Yau, the sums of residues on both sides of the wall differ by a contour integral that should be evaluated by a saddle-point calculation. This reproduces the band-restricted rule in a more rigorous way than we did previously.

6.3 GLSM branes in phases of Hirzebruch-Jung models

We determine here the Higgs branch image of Wilson lines in various phases of Hirzebruch-Jung models. For Calabi-Yau cases, this image is given in each phase by a geometric functor (6.25), so we only consider non-Calabi-Yau models. The strategy is then to transport branes from the orbifold phase to the phase of interest, taking into account the band restriction rules (6.28). For general rank we focus on reaching the fully resolved phase, then we explore all phases of a rank 2 example.

6.3.1 Contour and empty branes

In $U(1)$ non-Calabi-Yau models there was a universal admissible contour for all branes. Let us show in our Hirzebruch-Jung model that the contour[21]

$$\Big\{\big(\hat\tau_1 - i(\hat\tau_1)^2, \dots, \hat\tau_r - i(\hat\tau_r)^2\big) \;\Big|\; \hat\tau \in \mathbb{R}^r\Big\} \subset \mathfrak{g}_{\mathbb{C}} \quad \text{in basis II} \tag{6.31}$$

is a deformation of $\mathbb{R}^r$ that ensures convergence of the hemisphere partition function for Wilson lines $\mathcal{W}(q)$ with arbitrary ζ, θ and q.

The condition to be a deformation of $\mathbb{R}^r$ is that no pole of one-loop determinants are encountered when deforming from $\mathbb{R}^r$ to the contour. To be precise, we must turn on small positive R-charges to avoid the contour pinching discussed starting in subsection 4.3. The poles are at $Q^j \cdot \hat\sigma \in iR_j/2 + i\mathbb{Z}_{\geq 0} \subset \mathbb{R}_{>0}$, so the condition to avoid crossing any pole is that none of the $Q^j \cdot \hat\sigma$ should touch the line $i\mathbb{R}_{>0}$ anywhere on the contour. We compute $Q^j \cdot \hat\sigma$ separately for $j = 0$ (and $j = r+1$ by replacing $p_\alpha \to q_\alpha$)

$$\operatorname{Im}(Q^0 \cdot \hat\sigma) = \operatorname{Im}\bigg(\sum_{\alpha=1}^{r} p_\alpha(\hat\tau_\alpha - i\hat\tau_\alpha^2)\bigg) = -\sum_{\alpha=1}^{r} p_\alpha \hat\tau_\alpha^2 \leq 0 \tag{6.32}$$

and for $1 \leq j \leq r$ where we also see that $Q^j \cdot \hat\sigma = -n\hat\tau_j + in\hat\tau_j^2 \notin i\mathbb{R}_{>0}$.

Next, we study the integrand of the hemisphere partition function far along the contour. For this we write $\hat\tau_\alpha = \lambda \hat n_\alpha$ for real $\lambda \gg 0$ and the direction $\hat n$ normalized in an arbitrary way, say, $\sum_\alpha \hat n_\alpha^2 = 1$. The arguments of Gamma functions are quadratic in λ so we need the asymptotics

$$\log|\Gamma(\alpha\lambda^2 + i\beta\lambda + \gamma)| = 2\alpha\lambda^2 \log(\lambda) + O(\lambda^2) \tag{6.33}$$

for $(\alpha, \beta) \in \mathbb{R}^2$, $\gamma \in \mathbb{C}$, except obviously for $\alpha = \beta = 0$ and $\gamma \in \mathbb{Z}_{\leq 0}$. We deduce

$$\log|\text{integrand}| = \sum_{j=0}^{r+1} 2(Q^j \cdot \hat n^2)\lambda^2 \log(\lambda) + O(\lambda^2) \leq -2\lambda^2 \log(\lambda) + O(\lambda^2) \tag{6.34}$$

where we denoted abusively $\hat n^2$ the vector whose components in basis II are $\hat n_\alpha^2$. To show the inequality, note that components of $Q^{\text{tot}} = \sum_j Q^j$ in basis II are $p_\alpha + q_\alpha - n \leq -1$ (except in the Calabi-Yau case), and we normalized $\hat n$. We thus find that the integrand is exponentially suppressed at infinity along the contour. The FI-theta and Wilson line contributions are subleading (of order λ^2) hence the same contour works for all t and q, hence also arbitrary complexes of Wilson lines.

[21] We find it useful to switch back and forth between basis I and basis II of the GLSM throughout this discussion, hence we state explicitly which basis is used.

As in $U(1)$ models, we now find empty branes as Koszul resolutions in a pure-Higgs phase.

Pure-Higgs phases are those whose closure contains Q^{tot}. For our Hirzebruch-Jung model, components $p_\alpha + q_\alpha - n$ of Q^{tot} in basis II are negative, hence Q^{tot} lies in the interior of the orbifold phase $\mathbb{R}^r_{<0}$ (see subsection 3.2). The orbifold phase is thus the only pure-Higgs phase. It has Higgs branch $\mathbb{C}^2/\mathbb{Z}_{n(p)}$. The gauge group is broken to the $\mathbb{Z}_n$ subgroup that leaves $X_1, \dots, X_r$ fixed and acts on $P = X_0$ and $Q = X_{r+1}$ with charges p and 1, respectively. The n-th root of unity ω embeds in $U(1)^r$ with coordinates ω^{p_α} in basis I. The Higgs branch image of a Wilson line with charges $(b_1, \dots, b_r)$ in basis I is thus an equivariant line bundle on $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ with $\mathbb{Z}_n$ charge

$$\sum_{\alpha=1}^{r} p_\alpha b_\alpha. \tag{6.35}$$

By construction of the $\mathbb{Z}_n$ subgroup, a Wilson line with the same charges as X_α for some $1 \le \alpha \le r$ becomes an equivariant bundle with vanishing $\mathbb{Z}_n$ charge. This simply restates the fact that the Koszul branes $(1 \le \alpha \le r)$

$$\mathcal{K}_\alpha \coloneqq \Big(\mathcal{W}(\dots, 0, -1, a_\alpha, -1, 0, \dots) \xrightarrow{X_\alpha} \mathcal{W}(0)\Big) \tag{6.36}$$

are empty in the orbifold phase. The brane can then be transported to an arbitrary phase since the contour remains admissible for arbitrary ζ. The image of this same complex $\mathcal{K}_\alpha$ in each phase remains empty.[22] At the level of hemisphere partition functions we are simply stating that the analytic continuation of a function that is identically zero is zero.

Given any complex of Wilson line, our first step to find its image in some phase is to bind it with the empty branes $\mathcal{K}_\alpha$ to reduce all charges to a fundamental domain of the quotient

$$\mathbb{Z}^r \Big/ \Big(\operatorname{Span}_{\mathbb{Z}}\{Q^j \mid 1 \le j \le r+1\}\Big) \simeq \mathbb{Z}_n. \tag{6.37}$$

Our choice of fundamental domain is guided by the band-restriction rule: we wish to find some Wilson lines that go through all walls in the sense of being in the big band upon crossing each wall. This ensures that the Higgs branch image of the Wilson line matches its image under F_{geom} in each phase. Since the bands depend on θ, our choice of fundamental domain depends on θ.

6.3.2 Going through all the walls

Let us find some Wilson line branes whose images generate the B-brane category of the Higgs branch in the fully resolved phase. We transport branes from the orbifold phase to the fully resolved phase through a particular sequence of walls, which we choose to be blowing up exceptional divisors in the order $E_1, \dots, E_r$.

Consider a Wilson line with charges $(b_1, \dots, b_r)$ in basis I. The local model describing how some E_j is blown up was given in (3.22). In our present case $i = j-1$ and $k = r+1$, the local model is a $U(1)$ GLSM with chiral multiplets X_i, X_j, X_k of $U(1)$ charges $d_{jk} = p_j$, $-d_{ik} = -p_{j-1}$, and $d_{ij} = 1$. The embedding $U(1) \subset U(1)^r$ means that $\mathcal{W}(b)$ has charge

$$b_{U(1)} = \sum_{\alpha=j}^{r} p_\alpha \, b_\alpha. \tag{6.38}$$

[22]The situation is quite different in Calabi-Yau models, where the contour may stop being admissible (and has no admissible deformation) as we cross a wall because some of the Wilson lines constituting $\mathcal{K}_\alpha$ cannot be defined at the wall itself.

To write the bands (6.28) we work out $N_{u,+} = p_j + 1 \leq N_{u,-} = p_{j-1}$. The bands are:

$$\begin{aligned} &\text{small band} && \left|\sum_{\alpha=j}^{r} p_\alpha(\theta_\alpha + 2\pi b_\alpha)\right| < \pi(p_j+1), \\ &\text{big band} && \left|\sum_{\alpha=j}^{r} p_\alpha(\theta_\alpha + 2\pi b_\alpha)\right| < \pi p_{j-1}. \end{aligned} \tag{6.39}$$

We fix the theta angles to values that make small windows more convenient, namely we take θ to be the solution of

$$\sum_{\alpha=j}^{r} p_\alpha \theta_\alpha = -\pi(p_j+1) + 2\pi\varepsilon \text{ for } 1 \leq j \leq r, \tag{6.40}$$

for some unimportant $\varepsilon \in (0,1)$. The j-th small band is then

$$\sum_{\alpha=j}^{r} p_\alpha b_\alpha \in [0, p_j]. \tag{6.41}$$

Since all $p_\alpha \leq p_j$ for $\alpha \geq j$, it is clear that the $r+1$ Wilson lines (in basis I)

$$\mathcal{W}(0,\ldots,0) \text{ and } \mathcal{W}(\ldots,0,1,0,\ldots) \tag{6.42}$$

with either zero or one non-zero entries 1 go through all small bands. The converse holds: any Wilson line $\mathcal{W}(b)$ that fits in all small bands (6.41) must be one of these. If $b=0$ we are done. Otherwise, let $1 \leq j \leq r$ be the position of the last non-zero entry of b, namely $b_j \neq 0$ and $b_{j+1} = \cdots = b_r = 0$. The j-th small band restriction rule simplifies to $p_j b_j \in [\![0, p_j]\!]$ hence $b_j = 1$ (we assumed $b_j \neq 0$. If b_j is the only non-zero entry we are done. Otherwise, let $1 \leq i < j$ be the position of the last non-zero entry before b_j, namely $b_i \neq 0$ and $b_{i+1} = \cdots = b_{j-1} = 0$. The i-th small band restriction rule simplifies to $p_i b_i + p_j \in [\![0, p_i]\!]$. This has no non-zero b_i solution because $p_i > p_j$.

Since they fit in small bands, the image of these Wilson lines (6.42) in any of the phases that we visited is purely along the Higgs branch, with no massive vacuum part.

In the fully resolved phase, the image of $\mathcal{W}(0)$ is the structure sheaf, while the image of $\mathcal{W}(\ldots,0,1,0,\ldots)$ is a line bundle whose pull-back to each exceptional divisor $\mathbb{P}^1$ except one is $\mathcal{O}(0)$, and the last one $\mathcal{O}(1)$ (see subsubsection 4.5.2). In the orbifold phase, the image of $\mathcal{W}(0)$ is the trivial line bundle on $\mathbb{C}^2/\mathbb{Z}_{n(p)}$, while $\mathcal{W}(\ldots,0,1,0,\ldots)$ maps to an equivariant line bundle with charge p_α according to (6.35). Thus we see that the branes which can be passed through all the walls to the large volume phase are precisely the fractional branes corresponding to (i) the trivial representation and (ii) the "special" representations in the language of [49] (we refer to the notes [50] for further mathematical references).

6.3.3 $\mathbb{C}^2/\mathbb{Z}_{n(2)}$

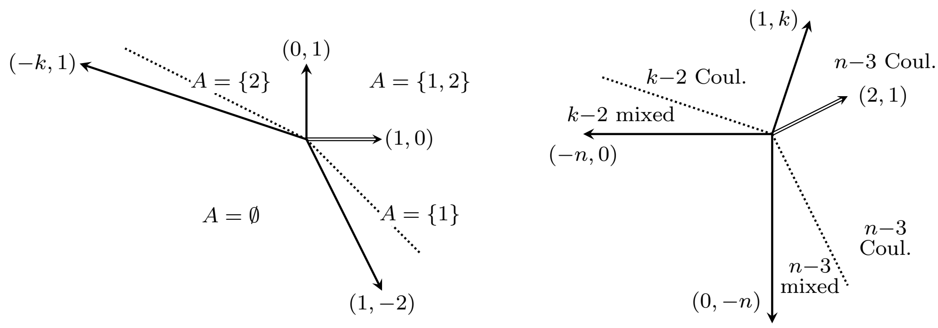
We now apply our considerations to the two-parameter model $\mathbb{C}^2/\mathbb{Z}_{n(2)}$ with $n = 2k-1$. Its charge matrices in basis I and II are respectively as follows.

	X_0	X_1	X_2	X_3
$U(1)_1$	1	$-k$	1	0
$U(1)_2$	0	1	-2	1

	X_0	X_1	X_2	X_3
$U(1)'_1$	2	$-n$	0	1
$U(1)'_2$	1	0	$-n$	k

(6.43)

We carefully analysed its phases in subsubsection 3.4.2, finding all Coulomb branch and mixed branch vacua. For convenience we reproduce the phase diagram given in Figure 3: the left diagram is in basis I and the right one in basis II.



Consider a Wilson line $\mathcal{W}(b_1,b_2)$ (in basis I) and transport it from $A=\emptyset$ to $A=\{1,2\}$ through either $A=\{1\}$ or $A=\{2\}$. Notice that we use a different value $\theta\simeq 0$ here than in (6.42).

The band restriction rules for passing from the orbifold phase $A=\emptyset$ to the phase $A=\{1\}$ where E_1 is blown up are (6.39)

$$\left|2b_1+b_2+\frac{2\theta_1+\theta_2}{2\pi}\right| < \begin{cases} 3/2 & \text{small band,} \\ n/2 & \text{big band.} \end{cases} \tag{6.44}$$

The band restriction rules for passing then to the fully resolved phase $A=\{1,2\}$ are the same for small and big bands:

$$\left|b_2+\frac{\theta_2}{2\pi}\right| < 1. \tag{6.45}$$

Taking $\theta_1,\theta_2>0$ small for definiteness, the Wilson lines that go through both small bands are

$$\mathcal{W}(0,0), \quad \mathcal{W}(0,-1), \quad \mathcal{W}(1,-1), \tag{6.46}$$

while those going through big bands are

$$\mathcal{W}(b_1,0) \text{ for } 1-\lceil k/2\rceil \le b_1 \le \lceil k/2\rceil -1 \text{ and } \mathcal{W}(b_1,-1) \text{ for } 1-\lfloor k/2\rfloor \le b_1 \le \lfloor k/2\rfloor. \tag{6.47}$$

If instead we go from $A=\emptyset$ to $A=\{2\}$ to $A=\{1,2\}$, the bands are

$$\left|b_1+kb_2+\frac{\theta_1+k\theta_2}{2\pi}\right| < \begin{cases} (k+1)/2 & \text{small band,} \\ n/2 & \text{big band,} \end{cases} \tag{6.48}$$

and

$$\left|b_1+\frac{\theta_1}{2\pi}\right| < \begin{cases} 1 & \text{small band,} \\ k/2 & \text{big band.} \end{cases} \tag{6.49}$$

For some values of θ_1 and θ_2, such as small $\theta_1,\theta_2>0$, only two Wilson lines go through both small bands: $\mathcal{W}(0,0)$ and $\mathcal{W}(-1,0)$. The number of Wilson lines that go through both small bands depends on the value of θ. This is not in contradiction with the rank of K-theory of the Higgs branch being 3 (in this case), because it is also possible to take a Wilson line, transport it through the first wall provided its charge is in the first small band, then change the charge by binding the brane with an empty brane $\mathcal{K}_\alpha$ given in (6.36), in such a way that the line goes through the second small band.

Acknowledgments

We would like to thank L. Borisov, T. Okuda and D. Pomerleano for enlightening discussions and we would like to thank especially G. Moore for suggesting this project and providing references. JC and BLF thank the IHÉS for hospitality during the school "Supersymmetric localization and exact results". MR would like to thank Rutgers University, Steklov Mathematical Institute, Higher School of Economics and Heidelberg University for hospitality while part of this work has been performed. MR gratefully acknowledges the support of the Institute for Advanced Study, DOE grant DE-SC0009988 and the Adler Family Fund.

References

[1] E. Witten, "Phases of N=2 theories in two-dimensions," *Nucl.Phys.* **B403** (1993) 159–222, `hep-th/9301042`.

[2] M. Herbst, K. Hori, and D. Page, "Phases Of N=2 Theories In 1+1 Dimensions With Boundary," `0803.2045`.

[3] E. Segal, "Equivalences between GIT quotients of Landau-Ginzburg B-models," *Commun. Math. Phys.* **304** (2011) 411–432, `0910.5534`.

[4] D. Halpern-Leistner, "The derived category of a git quotient," *Journal of the American Mathematical Society* **28** (2015), no. 3, 871–912.

[5] M. Ballard, D. Favero, and L. Katzarkov, "Variation of geometric invariant theory quotients and derived categories," *Journal für die reine und angewandte Mathematik (Crelles Journal)* (2012).

[6] R. Eager, K. Hori, J. Knapp, and M. Romo, "work in progress.".

[7] J. K. R. Eager, K. Hori and M. Romo, "Beijing Lectures on the Grade Restriction Rule," *Chinese Annals of Mathematics, Series B.* **38** (2017).

[8] N. Addington, W. Donovan, and E. Segal, "The Pfaffian-Grassmannian equivalence revisited," `1401.3661`.

[9] W. Donovan and E. Segal, "Window shifts, flop equivalences and Grassmannian twists," *Compos. Math.* **150** (2014), no. 6, 942–978, `01206.0219`.

[10] J. Vold Rennemo and E. Segal, "Hori-mological projective duality," *ArXiv e-prints* (Sept., 2016) `1609.04045`.

[11] A. Ishii, K. Ueda, *et al.*, "The special mckay correspondence and exceptional collections," *Tohoku Mathematical Journal* **67** (2015), no. 4, 585–609.

[12] Y. Kawamata, "Derived mckay correspondence for gl (3, c)," *Advances in Mathematics* **328** (2018) 1199–1216.

[13] A. Krug, D. Ploog, and P. Sosna, "Derived categories of resolutions of cyclic quotient singularities," *The Quarterly Journal of Mathematics* **69** (2017), no. 2, 509–548.

[14] E. J. Martinec and G. W. Moore, “On decay of K theory,” `hep-th/0212059`.

[15] G. W. Moore and A. Parnachev, “Localized tachyons and the quantum McKay correspondence,” *JHEP* **11** (2004) 086, `hep-th/0403016`.

[16] G. W. Moore and A. Parnachev, “Profiling the brane drain in a nonsupersymmetric orbifold,” *JHEP* **01** (2006) 024, `hep-th/0507190`.

[17] D. R. Morrison, K. Narayan, and M. R. Plesser, “Localized tachyons in C**3 / Z(N),” *JHEP* **08** (2004) 047, `hep-th/0406039`.

[18] T. Sarkar, “On localized tachyon condensation in C**2 / Z(n) and C**3 / Z(n),” *Nucl. Phys.* **B700** (2004) 490–520, `hep-th/0407070`.

[19] D. R. Morrison and K. Narayan, “On tachyons, gauged linear sigma models, and flip transitions,” *JHEP* **02** (2005) 062, `hep-th/0412337`.

[20] K. Narayan, “Closed string tachyons, flips and conifolds,” *JHEP* **03** (2006) 036, `hep-th/0510104`.

[21] K. Narayan, “On nonsupersymmetric C**4/Z(N), tachyons, terminal singularities and flips,” *JHEP* **03** (2010) 019, `0912.3374`.

[22] K. Hori, A. Iqbal, and C. Vafa, “D-branes and mirror symmetry,” `hep-th/0005247`.

[23] K. Hori and M. Romo, “Exact Results In Two-Dimensional (2,2) Supersymmetric Gauge Theories With Boundary,” `1308.2438`.

[24] P. Acosta, “Asymptotic Expansion and the LG/(Fano, General Type) Correspondence,” `1411.4162`.

[25] S. Galkin, V. Golyshev, and H. Iritani, “Gamma classes and quantum cohomology of Fano manifolds: Gamma conjectures,” *Duke Math. J.* **165** (2016), no. 11, 2005–2077, `1404.6407`.

[26] S. Galkin and H. Iritani, “Gamma conjecture via mirror symmetry,” *arXiv preprint arXiv:1508.00719* (2015).

[27] S. Hosono, “Central charges, symplectic forms, and hypergeometric series in local mirror symmetry,” `hep-th/0404043`.

[28] H. Iritani, “An integral structure in quantum cohomology and mirror symmetry for toric orbifolds,” *Adv. Math.* **222** (2009), no. 3, 1016–1079, `0903.1463`.

[29] J. Knapp, M. Romo, and E. Scheidegger, “work in progress.”.

[30] K. Hori and M. Romo, “Proceedings of the conference "Primitive forms and related subjects" at IPMU (Feb 2014).”.

[31] P. S. Aspinwall, T. Bridgeland, A. Craw, M. R. Douglas, A. Kapustin, G. W. Moore, M. Gross, G. Segal, B. SzendrÃűi, and P. M. H. Wilson, *Dirichlet branes and mirror symmetry*, vol. 4 of *Clay Mathematics Monographs.* AMS, Providence, RI, 2009.

[32] E. Witten, "Chern-Simons gauge theory as a string theory," *Prog. Math.* **133** (1995) 637–678, `hep-th/9207094`.

[33] D. Honda and T. Okuda, "Exact results for boundaries and domain walls in 2d supersymmetric theories," `1308.2217`.

[34] S. Sugishita and S. Terashima, "Exact Results in Supersymmetric Field Theories on Manifolds with Boundaries," *JHEP* **1311** (2013) 021, `1308.1973`.

[35] C. I. Lazaroiu, "On the boundary coupling of topological Landau-Ginzburg models," *JHEP* **05** (2005) 037, `hep-th/0312286`.

[36] P. S. Aspinwall and A. E. Lawrence, "Derived categories and zero-brane stability," *JHEP* **08** (2001) 004, `hep-th/0104147`.

[37] P. S. Aspinwall, "D-Branes on Toric Calabi-Yau Varieties," `0806.2612`.

[38] H. Iritania, "An integral structure in quantum cohomology and mirror symmetry for toric orbifolds," *Advances in Mathematics* **222** (2009) 1016–1079.

[39] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, vol. 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[40] M. B. Green, J. A. Harvey, and G. W. Moore, "I-brane inflow and anomalous couplings on d-branes," *Class.Quant.Grav.* **14** (1997) 47–52, `hep-th/9605033`.

[41] R. Minasian and G. W. Moore, "K theory and Ramond-Ramond charge," *JHEP* **9711** (1997) 002, `hep-th/9710230`.

[42] M. R. Douglas, B. Fiol, and C. Romelsberger, "Stability and BPS branes," *JHEP* **0509** (2005) 006, `hep-th/0002037`.

[43] T. Bridgeland, "Stability conditions on triangulated categories," *Annals of Mathematics* (2007) 317–345.

[44] L. Katzarkov, M. Kontsevich, and T. Pantev, "Hodge theoretic aspects of mirror symmetry," `0806.0107`.

[45] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, vol. 1 of *Clay mathematics monographs*. AMS, Providence, USA, 2003.

[46] L. A. Borisov and R. P. Horja, "On the k-theory of smooth toric dm stacks," *arXiv preprint math/0503277* (2005).

[47] L. A. Borisov and R. P. Horja, "Applications of homological mirror symmetry to hypergeometric systems: duality conjectures," *Advances in Mathematics* **271** (2015) 153–187.

[48] L. Borisov, L. Chen, and G. Smith, "The orbifold chow ring of toric deligne-mumford stacks," *Journal of the American Mathematical Society* **18** (2005), no. 1, 193–215.

[49] O. Riemenschneider, *Characterization and application of special reflexive modules on rational surface singularities.* Institut Mittag-Leffler, 1987.

[50] O. Riemenschneider *et al.*, "Special representations and the two-dimensional mckay correspondence," *Hokkaido mathematical journal* **32** (2003), no. 2, 317–333.