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Notes on the Hemisphere

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Abstract.

In these notes, we provide an introduction to the hemisphere partition function of 2d $(2,2)$ supersymmetric gauge theories, and discuss its relation to the "D-brane central charge" which were studied in superstring theory, in 2d supersymmetric quantum field theory, and in topological string theory. We also discuss relation to "macroscopic loop" in matrix models. They are mostly reviews of the work by the authors, but contains some new results such as the partition function for a rotated supersymmetry as well as the differential equations.

Contents

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1. Introduction

Quantum field theory provides a place where mathematicians and physicists interact with each other, and supersymmetry has been the key in the interaction for more than thirty years since the introduction of Witten index [1]. Having infinite degrees of freedom, quantum field theory so far resisted mathematical definition, and also, even the basic physical property of a given theory is usually difficult to understand. In the presence of supersymmetry, cancellation of infinities happens to a class of observables, and they are sometimes exactly computable and often provide important physical information of the theory. Attempts to define such observables have generated new areas of mathematical research, and certain relations among those observables, which hold for trivial or non-trivial physical reasons, may have dramatic mathematical consequences. For example, as a consequence of mirror symmetry, the number of rational curves in a Calabi-Yau manifold was predicted [2], and that motivated mathematicians to develop the theory of Gromov-Witten invariants and further led to surprising relationship between algebraic geometry and symplectic geometry.

More recently, in the physics side, the class of computable observables are enlarged by using *superconformal* transformations, which become symmetries of the system in special spacetime backgrounds even if the theory is not conformally invariant. Starting from [3] the partition functions of various supersymmetric gauge theories on spheres of dimensions ≤ 5 were computed and certain new information of the theories were obtained. In particular, the partition function of 2d $(2,2)$ supersymmetric gauge theories on the two sphere was computed in [4, 5]. When a certain condition called "the Calabi-Yau condition" is met, it was observed in some examples [6] and later explained in [7, 8] that the partition function determines the Kähler potential of the space of superconformal fixed points of the theory. Motivated by these works, the authors of these notes studied the partition function of 2d $(2,2)$ supersymmetric gauge theories on the hemisphere [9]. (Related works [10, 11] also appeared at the same time.) When the Calabi-Yau condition is met, it was observed in examples that the hemisphere partition function computes the central charge of the D-brane that is placed at the boundary.

The present notes consists of a review of that work [9] and some introductory materials. We also include some new results such as the partition function for a rotated supersymmetry as well as the differential equations. We elaborate on the relation to the central charge in the Calabi-Yau case as well as the discussion of non-Calabi-Yau case. In a non-Calabi-Yau case where the theory is related to non-linear sigam model with a Fano target space X, the hemisphere partition function computes the "central charge" in the Gromov-Witten theory of X. There is also a rather surprising but suggestive relation to "macroscopic loop" in matrix model. We shall discuss these matters in some detail as they were only briefly mentioned in [9].

Notational Remarks

In these notes, we take the following convention. For a compact Lie group G, we write G_0, T and Z_G for the identity component, a maximal torus and the center, respectively. We write the Weyl group of G and G_0 by W and W_0. We write $\mathfrak{g} \supset \mathfrak{t} \supset \mathfrak{z}$ for the Lie algebras of G (or G_0) $\supset T \supset Z_G$ and regard them "pure imaginary". "Reals" are $i\mathfrak{g} \supset i\mathfrak{t} \supset i\mathfrak{z}$ in the complexfied Lie algebras $\mathfrak{g}_{\mathbb{C}} \supset \mathfrak{t}_{\mathbb{C}} \supset \mathfrak{z}_{\mathbb{C}}$. The weight lattice of T is denoted by $\mathrm{P} \subset i\mathfrak{t}^*$. For a $\mathfrak{g}_{\mathbb{C}}$ valued quantity X, we write $X = \mathrm{Re}(X) + i\mathrm{Im}(X)$ where both $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ are $i\mathfrak{g}$ valued, and write $\overline{X} = \mathrm{Re}(X) - i\mathrm{Im}(X)$.

2. 2d $(2, 2)$ Supersymmetric QFTs

Let us first describe the basics of quantum field theories (QFTs) in two dimensions with $(2, 2)$ supersymmetry, with emphasis on the mathematical aspects.

2.1. 2d $(2, 2)$ supersymmetry

When formulated on the Minkowski spacetime, a 2d $(2, 2)$ supersymmetric QFT has the following symmetry operators: the time translation H (Hamiltonian), the space translation P (momentum), the Lorentz transformation M, the supercharges $Q_+, \overline{Q}_+, Q_-, \overline{Q}_-$, and possibly the vector R-charge F_V and/or the axial R-charge F_A. They act on the Hilbert space of states $\mathcal{H}$ which is $\mathbb{Z}_2$-graded. The supercharges $Q_\pm$ and $\overline{Q}_\pm$ are odd and are the adjoint of each other, while the other operators are even and self-adjoint. With respect to the Lorentz group, H and P form a vector, $i[M, H \pm P] = \mp 2(H \pm P)$, the supercharges are spinors, $i[M, Q_\pm] = \mp Q_\pm$, $i[M, \overline{Q}_\pm] = \mp \overline{Q}_\pm$, and the R-charges are scalars, $[M, F_V] = [M, F_A] = 0$. The supercharges obey[1]

$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{2.1}$$

$$\text{all other anticommutators} = 0. \tag{2.2}$$

R-charges are phase rotations of the supercharges

$$[F_V, Q_\pm] = -Q_\pm, \quad [F_V, \overline{Q}_\pm] = \overline{Q}_\pm, \tag{2.3}$$

$$[F_A, Q_\pm] = \mp Q_\pm, \quad [F_A, \overline{Q}_\pm] = \pm \overline{Q}_\pm. \tag{2.4}$$

2.2. A and B

Let us put

$$Q_A := \overline{Q}_+ + Q_-, \qquad Q_B := \overline{Q}_+ + \overline{Q}_-. \tag{2.5}$$

Then, $(Q, F) = (Q_A, F_A)$ or (Q_B, F_V) obey

$$Q^2 = 0, \qquad [F, Q] = Q. \tag{2.6}$$

This means that the space of states forms a complex with differential Q and grading F.[2] The same applies also for the space of local operators. In particular, cohomology classes of local operators form a ring called the

[1] There is a possible modification to (2.2) by central terms which we do not consider in these notes.

[2] In this subsection, we make statements assuming that F_A or F_V is present, but that is not necessary. We can use the $\mathbb{Z}_2$-grading instead.

chiral ring, which we denote by $\mathcal{R}_A$ for (Q_A, F_A) and $\mathcal{R}_B$ for (Q_B, F_V). It is a graded commutative algebra.

When formulated on a half of the Minkowski space, say, where the space coordinate is bounded as $x \leq 0$ and the time t is unbounded, a boundary condition on the fields must be specified at $x = 0$. There are essentially two types of boundary conditions that preserve maximal number of supercharges:

$$\begin{aligned} &\textit{A-type}: \qquad Q_A \text{ and } Q_A^\dagger \text{ conserved.}\\ &\textit{B-type}: \qquad Q_B \text{ and } Q_B^\dagger \text{ conserved.} \end{aligned}$$

Boundary conditions of such types are called *A-branes* and *B-branes* respectively. The pair $(Q, F) = (Q_A, F_A)$ (*resp.* (Q_B, F_V)) acts on local operators inserted on the boundary with an A-type (*resp.* B-type) boundary condition, obeying the same relation as (2.6). The cohomology classes form an algebra, which is non-commutative in general. Two differnt boundary conditions of the same type can be placed on the boundary with a local operator inserted inbetween. The pair (Q, F) acts also on such local operators obeying (2.6),[3] and we may consider the cohomology classes. Then, we have a *category*, which is denoted by $\mathcal{C}_A$ for A-branes and $\mathcal{C}_B$ for B-branes. Objects are boundary conditions and morphisms between boundary conditions are Q-cohomology classes of local operators inserted between them, with the composition represented by the product of operators.

We may also consider combinations of supercharges which are rotated by "the other" R-charge: the vector rotation for the A-type, $\mathrm{e}^{i\alpha F_V} Q_A \,\mathrm{e}^{-i\alpha F_V} = \mathrm{e}^{i\alpha}\overline{Q}_+ + \mathrm{e}^{-i\alpha} Q_-$, and the axial rotation for the B-type, $\mathrm{e}^{i\beta F_A} Q_B \,\mathrm{e}^{-i\beta F_A} = \mathrm{e}^{i\beta}\overline{Q}_+ + \mathrm{e}^{-i\beta}\overline{Q}_-$. D-branes preserving these supercharges and their conjugates are called $A_{\mathrm{e}^{2i\alpha}}$*-branes* and $B_{\mathrm{e}^{2i\beta}}$*-branes* respectively.

2.3. RG flow

In a general QFT, the behaviour of observables depends very much on the energy scale, or inversely, the distance scale. For example, the correlation function of two operators inserted as two points of distance r is in general a complicated function of r. The behaviour at small r i.e. short distance, or equivalently, high energy or ultra-violet (UV), is in general very much different from the behaviour at large r i.e. long distance, or equivalently, low energy or infra-red (IR). If the length scale is increased (i.e. the energy scale is lowered), the behaviour of the theory

[3] There is a potential anomaly to the relations [12].

changes — it may be identified as the behaviour of a different theory before the change of the scale. This change of the theory under the change of the scale is called the *renormalization group flow*, or *RG flow* in short. A theory is scale invariant if it is invariant under the RG flow. The two point correlation function of an operator $\mathcal{O}$ in such a theory depends on the distance as its power,

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{\mathrm{dist}(x,y)^{2\Delta_{\mathcal{O}}}}. \tag{2.7}$$

The number $\Delta_{\mathcal{O}}$ is called the *dimension* of the operator.

In two-dimensions, scale invariance of a QFT is proven to be equivalent to conformal invariance. For a general QFT, we have conformally invariant field theories (CFTs) in the UV and IR limits. An invariant of a CFT is its central charge c, and it is known that it descreases under the RG flow, $c_{\mathrm{UV}} \geq c_{\mathrm{IR}}$.

The same applies of course to QFTs with supersymmetry. In 2d $(2,2)$ supersymmetric QFTs, there are a class of observables which are invariant under the RG flow, even if the theory is not scale invariant. The chiral ring and the category of branes, which are introduced above, are examples of such observables which are "protected" from renormalization. In a $(2,2)$ superconformal field theory (SCFT), it is convenient to use $\widehat{c} = c/3$ for the central charge.

2.4. Deformations

A QFT can be deformed by adding a local operator $\mathcal{O}$ to its Lagrangian density. If the theory is scale invariant, the deformation is called *irrelevant*, *marginal* and *relevant* if the dimension of the operator minus the dimension of the spacetime, which is $\Delta_{\mathcal{O}} - 2$ in a 2d theory, is positive, zero and negative, respectively. Under the RG flow, an irrelevant deformation decays and a relevant deformation grows. A marginal deformation is called *exactly marginal* if it remains invariant under the RG flow, while it is *marginally irrelevant* (*resp. marginally relevant*) if it decays (*resp.* grows) under the RG. The moduli space of conformal field theories is coordinatized by exactly marginal deformation parameters. It has a natural metric G called the Zamolodchikov metric [13]: Let v_1 and v_2 be tangent vectors at one theory that correspond to exactly marginal operators $\mathcal{O}_1$ and $\mathcal{O}_2$. Then, their inner product $G(v_1, v_2)$ is provided (for 2d) by

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = \frac{G(v_1,v_2)}{\mathrm{dist}(x,y)^4}. \tag{2.8}$$

In a 2d $(2,2)$ supersymmetric QFT, deformation operators that preserve the supersymmetry are of the following three types,

$$\Delta_D\mathcal{L} = Q_+Q_-\overline{Q}_-\overline{Q}_+\mathcal{K}, \tag{2.9}$$

$$\Delta_A\mathcal{L} = Q_+\overline{Q}_-\mathcal{O}_A \text{ and its adjoint,} \tag{2.10}$$

$$\Delta_B\mathcal{L} = Q_+Q_-\mathcal{O}_B \text{ and its adjoint,} \tag{2.11}$$

for scalar operators $\mathcal{K}$, $\mathcal{O}_A$ and $\mathcal{O}_B$, where $\mathcal{K}$ is arbitrary, $\mathcal{O}_A$ is *A-chiral*, $\overline{Q}_+\mathcal{O}_A = Q_-\mathcal{O}_A = 0$, and $\mathcal{O}_B$ is *B-chiral*, $\overline{Q}_\pm\mathcal{O}_B = 0$. We shall call them *D-term*, *A-term*, and *B-term*, respectively.[4] We see from (2.1)–(2.2) that a D-term is an A-term and a B-term at the same time, up to total derivatives. It turns out that A-term deformations modulo D-term deformations are in one to one correspondence with elements of the chiral ring $\mathcal{R}_A$. Similarly, B-term deformations modulo D-term deformations are in one to one correspondence with elements of the chiral ring $\mathcal{R}_B$.

It follows from the algebra (2.1)–(2.2) that D-terms and A-terms are Q_B-exact while D-terms and B-terms are Q_A-exact, up to total derivatives. In particular, the ring $\mathcal{R}_B$ and the category $\mathcal{C}_B$ are invariant under D-term and A-term deformations, while $\mathcal{R}_A$ and the category $\mathcal{C}_A$ are invariant under D-term and B-term deformations.

In a $(2,2)$ SCFT, the D-term deformations are irrelevant, since each supercharge carry dimension $\frac{1}{2}$. Only a part of A-term and B-term deformations are marginal or relevant.

The spaces of parameters of the theory corresponding to A-term deformations and B-term deformations are denoted by $\mathfrak{M}_A$ and $\mathfrak{M}_B$. They have complex structures: tangent vectors of $\mathfrak{M}_A$ (*resp.* $\mathfrak{M}_B$) of type $(1,0)$ correspond to operators of the form $Q_+\overline{Q}_-\mathcal{O}_A$ (*resp.* $Q_+Q_-\mathcal{O}_B$) and can naturally be identified as elements of the chiral ring $\mathcal{R}_A$ (*resp.* $\mathcal{R}_B$). For a $(2,2)$ SCFT, the subspaces of exactly marginal parameters, $\mathfrak{M}_A^0 \subset \mathfrak{M}_A$ and $\mathfrak{M}_B^0 \subset \mathfrak{M}_B$, are complex submanifolds. They are also submanifolds of the moduli space of conformal field theories. The Zamolodchikov metric induced on $\mathfrak{M}_A^0$ and $\mathfrak{M}_B^0$ are known to be Kähler [14].

2.5. Topological Twists

The theory can be formulated not just on the (half of) Minkowski space. For example, we can consider the cylinder or the strip, again with the Minkowski metric, which yield closed or open string states. We

[4] The standard terminology is: *twisted F-term* instead of A-term, and *F-term* intead of B-term.

may also formulate the theory on these manifolds with Euclidean metric, by Wick rotation of the time line. Furthermore, we may formulate the system on a two-dimensional manifold with an arbitrary metric and spin structure, via the standard covariantization. Does the supersymmetry survive? We can certainly extend the definition of supersymmetry transformation of fields $\mathcal{O}$

$$\delta\mathcal{O} = i\epsilon_+ Q_- \mathcal{O} - i\epsilon_- Q_+ \mathcal{O} - i\overline{\epsilon}_+ \overline{Q}_- \mathcal{O} + i\overline{\epsilon}_- \overline{Q}_+ \mathcal{O}, \tag{2.12}$$

by covariantization of the expressions $Q_\pm \mathcal{O}$ and $\overline{Q}_\pm \mathcal{O}$, and by taking the variational parameters $\epsilon_\pm$ and $\overline{\epsilon}_\pm$ to be sections of the spin bundles $S_\pm$. However, invariance of the covariantized action requires the variational parameters to be covariantly constant, which is impossible on a curved manifold. There are several ways to restore a part of the supersymmetry. One is the *topological twisting* which we now describe. In Section 4, we will consider an alternative way, which yields the main character of the present notes.

Let us assume that $F = F_V$ or F_A is conserved and has *charge integrality*, that is, it generates a $U(1)$ symmetry group under which the non-spinorial and spinorial fields have even and odd charges respectively. The topological twisting is to replace a field of R-charge q with values in a vector bundle E by a field with values in $E \otimes T_\Sigma^{\otimes q/2}$, when we consider the theory on an oriented Riemannian manifold Σ. Here, T_Σ is the holomorphic tangent bundle equipped with the Levi-Civita connection. Note that, due to the constraint on the parity of the R-charge, $E \otimes T_\Sigma^{\otimes q/2}$ makes sense without choice of spin structure of Σ. The same change occurs also for the variational parameters $\epsilon_\pm$ and $\overline{\epsilon}_\pm$, and some of them become scalars. We can take such a parameter to be constant, and the corresponding supercharge is conserved. It is called the *A-twist* for $F = F_V$ and *B-twist* for $F = F_A$. In the A-twisted (*resp.* B-twisted) theory, the supercharges $\overline{Q}_+$ and Q_- and hence their sum $Q = Q_A$ (*resp.* $\overline{Q}_\pm$ and their sum $Q = Q_B$) are conserved. Furthermore, the correlation functions of Q-closed operators depend only on the Q-cohomology classes of the operators and are invariant under deformation of the Riemannian metric on Σ. In particular, they depend only on the topology of Σ — we obtain a topological field theory.

We can further consider *topological string theory* by coupling the twisted theory to a certain theory of 2d gravity called *topological gravity*. A g-loop amplitude is obtained by integration over the moduli space of curves of genus g.

2.6. Examples

Non-linear sigma model

Let X be a compact Kähler manifold. Then, there is a 2d $(2,2)$ supersymmetric QFT called the *non-linear sigma model* with target X. As a part of the data, we also choose a class $[B] \in \mathrm{H}^2(X, \mathbb{R}/2\pi\mathbb{Z})$ called the *B-field*. The model classically has both vector and axial $U(1)$ R-symmetries with charge integrality, but the axial R-symmetry is anomalous if the first Chern class $c_1(X)$ is non-zero: the axial rotation shifts $[B]$ by $c_1(X)$. The model is classically scale invariant, but the target metric changes under the RG flow. The Kähler class runs according to the Ricci flow: $[\omega] \to [\omega'] = [\omega] + c_1(X)\log(\mu'/\mu)$ for the change $\mu \to \mu'$ of the energy scale.

The sigma model coupling is proportional to the curvature of X which, roughly, is inversely proportional to the size of X or the Kähler class $[\omega]$. Therefore, the energy dependence of the sigma model coupling is controlled by the signs of the eigenvalues of $c_1(X) \simeq$ Ricci curvature. When $c_1(X) > 0$, i.e., when X is a Fano manifold, the Kähler class is larger at higher energies and diverges in the limit $\mu \to +\infty$. That is, the sigma model is free in the ultra-violet limit (asymptotically free and UV complete). At lower energies, the Kähler class is smaller and the coupling is stronger. Finding the infra-red behaviour is a non-trivial problem. When $c_1(X) = 0$, i.e., when X is a Calabi-Yau manifold, the Kähler class does not run under the RG. The theory flows in the infra-red limit to an SCFT with central charge $\widehat{c} = \dim_{\mathbb{C}} X$. When $c_1(X)$ has a negative component, the corresponding component of the Kähler class becomes smaller at higher energies. That is, the coupling partly diverges at some high energy (Landau pole) and the sigma model is not UV complete. In particular, this is the case when $c_1(X) < 0$, i.e., when X is of general type. In that case, the sigma model is free in the infra-red limit.

The chiral ring etc of the model are

$$
\begin{aligned}
\mathcal{R}_A &= \mathrm{QH}^*(X) \quad \text{quantum cohomology ring,}\\
\mathcal{R}_B &= \mathrm{H}^*(X, \wedge^* T_X) \quad \text{cohomology ring of polyvector fields,}\\
\mathcal{C}_A &= \mathrm{Fuk}(X) \quad \text{Fukaya category,}\\
\mathcal{C}_B &= \mathrm{D}^b_{Coh}(X) \quad \text{derived category of sheaves with coherent}\\
&\qquad \text{cohomologies,}\\
\mathfrak{M}^{0,c}_A &= \text{the space of complexified Kähler class}\\
&\qquad [\omega - iB] \in \mathrm{H}^2(X, \mathbb{R}/2\pi i\mathbb{Z}),\\
\mathfrak{M}^{0,c}_B &= \text{the moduli space of complex structures of } X.
\end{aligned}
$$

$\mathfrak{M}^{0,c}_{A/B}$ is a submanifold of $\mathfrak{M}_{A/B}$ that corresponds to the marginal deformations of the classical system. When $c_1(X) = 0$, the space $\mathfrak{M}^{0,c}_A \times \mathfrak{M}^{0,c}_B$ is identified as an open subset of the moduli space of the IR SCFTs. When $c_1(X) \neq 0$, there is an RG low on $\mathfrak{M}^{0,c}_A$ in the direction of $c_1(X)$, and the shift in the direction of $ic_1(X)$ is absorbed by the axial rotaion.

As the model has vector $U(1)$ R-symmetry with charge integrality, A-twist is always possible. The corresponding topological string theory is known as the *Gromov-Witten theory* in mathematics. B-twist is possible if and only if X is a Calabi-Yau manifold.

Landau-Ginzburg model

Let $W(x)$ be a polynomial of N variables $x = (x_1, \ldots, x_N)$ with complex coefficients, having only isolated critical points. Then, there is a 2d $(2,2)$ supersymmetric QFT called the *Landau-Ginzburg model* with superpotential $W(x)$. It always have an axial $U(1)$ R-symmetry with charge integrality. A vector R-symmetry exists if and only if $W(x)$ is quasi-homogeneous, that is, with a change of coordinates if necessary, there are some numbers $R = (R_1, \ldots, R_N)$ such that $W(\lambda^R x) = \lambda^2 W(x)$, where $\lambda^R x = (\lambda^{R_1} x_1, \ldots, \lambda^{R_N} x_N)$. In that case, the theory flows in the infra-red limit to an SCFT with $\widehat{c} = \mathrm{tr}(1 - R)$. The chiral ring etc of the model are

$$
\begin{aligned}
\mathcal{R}_A &= ?,\\
\mathcal{R}_B &= \mathrm{Jac}(W) \quad \text{Jocobi ring } \mathbb{C}[x_1, \ldots, x_N]/(\partial_1 W, \ldots, \partial_N W),\\
\mathcal{C}_A &= \mathrm{Fuk}(W) \quad \text{Fukaya category controled by } W,\\
\mathcal{C}_B &= \mathrm{MF}(W) \quad \text{category of matrix factorizations of } W,\\
\mathfrak{M}_A &= ?,\\
\mathfrak{M}_B &= \text{the moduli space of versal deformations of } W.
\end{aligned}
$$

The authors do not know what are $\mathcal{R}_A$ and $\mathfrak{M}_A$ at this moment.

As the model has axial $U(1)$ R-symmetry with charge integrality, B-twist is always possible. The corresponding topological string theory at the tree level (genus zero) is closely related to K. Saito's theory of primitive forms [15–17]. When W is a Morse function, A. Givental proposed a recipe to construct the higher genus amplitudes [18], and C. Teleman proved that they satisfy a mathematical axiom of topological string theory [19].

When W is quasi-homogeneous, there is also a vector R-symmetry. However, it does not possess the charge integrality and the A-twist is not possible. That problem may be cured by orbifolding. Gauge the system by a finite group $\Gamma \subset GL(N, \mathbb{C})$ of symmetries of $W(x)$ that include $\mathrm{e}^{\pi i R}$ as its element. Then, the charge integrality holds for gauge invariant fields, and the A-twist becomes possible. The corresponding topological string theory is developed in [20] and is called the *FJRW theory*.

2.7. Mirror Symmetry

The 2d $(2,2)$supersymmetry algebra has an automorphism: $Q_- \leftrightarrow \overline{Q}_-$, $F_V \leftrightarrow F_A$, and the other generators kept intact. A pair of 2d $(2,2)$ supersymmetric QFTs are said to be *mirror* to each other when there is an isomorphism between them under which the supersymmetry generators undergo the above automorphism. There are immediate consequences of the mirror symmetry: the ring $\mathcal{R}_A$ of one theory is isomorphic to the ring $\mathcal{R}_B$ of the mirror, the category $\mathcal{C}_A$ of one theory is equivalent to the category $\mathcal{C}_B$ of the mirror, the parameter space $\mathfrak{M}_A$ of one theory is isomorphic to the parameter space $\mathfrak{M}_B$ of the mirror, and the topological A-model (the A-twisted model or the corresponding topological string theory) of one theory is isomorphic to the topological B-model of the other.

The most famous example of mirror symmetry is the one for the sigma models with Calabi-Yau targets, say X and Y. As a part of the above consequences, we have the relation between the Hodge numbers, $h^{p,q}(X) = h^{n-p,q}(Y)$, where $n = \dim X = \dim Y$. Other well known examples are mirror symmetry between the sigma model with a non-Calabi-Yau target and the Landau-Ginzburg model, and the one between Landau-Ginzburg orbifolds. Some of the consequences in these examples are mathematically proven.

3. Gauged Linear Sigma Models

The main characters of the present notes are a class of 2d $(2,2)$ supersymmetric gauge theories called gauged linear sigma models (GLSMs). In this section, we provide an introduction to GLSMs.

3.1. The Bulk Theory

A 2d $(2,2)$ gauge theory is specified by a choice of

- *gauge group* G**:** a compact Lie group,
- *matter representation* V**:** a finite dimensional complex representation of G,
- *superpotential* $W(\phi)$**:** a G invariant polynomial function of $\phi \in V$, and
- *twisted superpotential* $\widetilde{W}(\sigma)$**:** a G invariant polynomial function of $\sigma \in \mathfrak{g}_{\mathbb{C}}$.

As a minor part of the data, we also choose a G-invariant norm $X \in i\mathfrak{g} \mapsto \frac{1}{e^2}(X)^2 \in \mathbb{R}_{\geq 0}$ on $i\mathfrak{g}$ (e is called the *gauge coupling constant*), and a G-invariant hermitian inner product on V. The latter defines a G-invariant symplectic structure on V, and we denote by $\mu : V \to i\mathfrak{g}^*$ the moment map that vanishes at the origin.

A vector $U(1)$ R-symmetry exists when there is a linear map $R : V \to V$ commuting with the G-action such that

$$W(\lambda^R \phi) = \lambda^2 W(\phi). \tag{3.1}$$

The charge integrality holds when $\mathrm{e}^{\pi i R} : V \to V$ is the same as the action of a gauge group element, say $J \in G$. An axial $U(1)$ R-symmetry with charge integrality exists at the classical level when $\widetilde{W}(\sigma)$ is linear, and it remains to be a symmetry of the quantum system under *Calabi-Yau condition*: $G \subset SL(V)$. In the present notes, we assume all of the above but the Calabi-Yau condition. We write the linear twisted superpotential as

$$\widetilde{W}(\sigma) = -t(\sigma), \tag{3.2}$$

for an adjoint invariant linear form

$$t = \zeta - i\theta \in \mathfrak{g}_{\mathbb{C}}^{*G}, \tag{3.3}$$

where ζ and θ, both in $i\mathfrak{g}^{*G}$, are called the Fayet-Illiopoulos (FI) parameter and the theta parameter respectively. Note that ζ and θ can also be regarded as elements of $i\mathfrak{t}^{*\mathrm{W}}$ or $i\mathfrak{z}^*$ thanks to the natural isomorphisms

$\mathfrak{g}^{*G} \cong \mathfrak{t}^{*\mathrm{W}} \cong \mathfrak{z}^*$. To be precise, the theta parameter is subject to a discrete identification,

$$\theta \equiv \theta + 2\pi n, \tag{3.4}$$

for an image n of a character $G \to U(1)$ under the differential map $\mathrm{Hom}(G, U(1)) \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{u}(1)) = i\mathfrak{g}^*$. Therefore, the space of theta parameter (or theta *angle*) is the compact torus $i\mathfrak{g}^{*G}/2\pi\Lambda_G$, where $\Lambda_G := \mathrm{Image}(\mathrm{Hom}(G, U(1)) \to i\mathfrak{g}^{*G})$. When G is connected, Λ_G is isomorphic to $\mathrm{Hom}(G, U(1))$ and is equal to the lattice P^{W} of Weyl invariant weights of T embedded in $i\mathfrak{g}^{*G}$ via $i\mathfrak{t}^{*\mathrm{W}} \cong i\mathfrak{g}^{*G}$.

The theory consists of two sets of fields called a *matter multiplet* and a *gauge multiplet*, that include scalar fields, ϕ and σ, with values in V and $\mathfrak{g}_{\mathbb{C}}$ respectively. The classical potential for the scalar fields is

$$U(\sigma, \phi) = \frac{1}{8e^2}[\sigma, \overline{\sigma}]^2 + \frac{1}{2}|\sigma\phi|^2 + \frac{1}{2}|\overline{\sigma}\phi|^2 + \frac{e^2}{2}\left(\mu(\phi) - \zeta\right)^2 + |\mathrm{d}W(\phi)|^2. \tag{3.5}$$

Note that each term is non-negative. The space of zero points of U, called *classical vacua*, provides us with a first hint to understand the low energy behaviour of the theory. The vacuum equation $U = 0$ reads

$$[\sigma, \overline{\sigma}] = 0, \quad \sigma\phi = \overline{\sigma}\phi = 0, \quad \mu(\phi) = \zeta, \quad \mathrm{d}W(\phi) = 0. \tag{3.6}$$

The last two equations require ϕ to be in $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$ and the first two equations require σ to be in the Cartan subalgebra of the stabilizer subgroup at ϕ. The space of the FI parameter ζ is separated into chambers, called *phases*, according to the topology of the G-space $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$. Inside a phase, typically, the stabilizer subgroup is finite at each point of $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$, so that σ is forced to vanish — the space of classical vacua is the quotient $X_\zeta = (\mathrm{Crit}(W) \cap \mu^{-1}(\zeta))/G$, called the *Higgs branch*. If that is the case, the theory reduces at low energies to the Landau-Ginzburg model $(\mu^{-1}(\zeta)/G, W_\zeta)$ where W_ζ is the function on $\mu^{-1}(\zeta)/G$ induced from W. If, in addition, W_ζ is Bott-Morse, the theory reduces further to the non-linear sigma model on $\mathrm{Crit}(W_\zeta)$, which is nothing but the Higgs branch X_ζ. Such a phase is called a *geometric phase*. On a wall between chambers (*phase boundary*), there are continuous stabilizer subgroups at some loci of $\mathrm{Crit}(W) \cap \mu^{-1}(\zeta)$. There develops a component of the space of classical vacua, called the *Coulomb branch*, in which σ can take any value in the Cartan subalgebra of the stabilizer subgroup. Emergence of this non-compact space may be regarded as a singularity.

Quantum effects will yield significant modification of this picture. In particular, classical Coulomb branch may be lifted, or quantum Coulomb vacua may emerge even in the absence of classical one. To see this, we explore the region in the field space where σ takes large generic values in a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. Then, the second and the third terms of (3.5) provide masses to many of the components of ϕ, typically all (which we assume for now). Integrating out the massive modes, we obtain the effective theory consisting of the gauge multiplet of the maximal torus T only, with the effective twisted superpotential

$$\widetilde{W}_{\rm eff}(\sigma) = -t(\sigma) + 2\pi i\rho(\sigma) - \sum_i Q_i(\sigma)\left(\log(Q_i(\sigma)/\Lambda) - 1\right). \tag{3.7}$$

Here, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is half the sum of positive roots of $\mathfrak{g}$, Q_i's are the weights of V, and Λ is a scale parameter which is needed for renormalization. The gauge coupling constant of the effective theory is a complicated function $e_{\rm eff}(\sigma)$ but it approaches the given value e at $|\sigma/\Lambda| \gg 1$. The effective potential is

$$U_{\rm eff}(\sigma) = \min_{n\in \mathrm{P}} \frac{e^2_{\rm eff}(\sigma)}{2}\left|{\rm d}\widetilde{W}_{\rm eff}(\sigma) + 2\pi i n\right|^2. \tag{3.8}$$

Note that the choice of branch of the logarithms in (3.7) has no physical effect — a different choice would shift $\widetilde{W}_{\rm eff}(\sigma)$ by an element of $2\pi i\mathrm{P}(\sigma)$, but that does not affect (3.8). A point σ_* is a true Coulomb vacuum in the quantum theory when $U_{\rm eff}(\sigma_*) = 0$, that is, ${\rm d}\widetilde{W}_{\rm eff}(\sigma_*) \in 2\pi i\mathrm{P}$.[1] In particular, a classical Coulomb branch may not survive in the quantum theory, or true Coulomb vacua might appear even in the absence of classical Coulomb branch. For completeness, one should also explore the region in the field space where σ takes large generic values in a Cartan subalgebra for a subgroup $H \subset G$ and large values of H-neutral components of ϕ, and examine whether there are true mixed Coulomb-Higgs vacua.

The character of the theory depends very much on whether the infinitesimal version of the Calabi-Yau condition, $\mathfrak{g} \subset \mathfrak{sl}(V)$, is satisfied or not. That is, whether b_1 defined by

$$b_1(X) := {\rm tr}_V(X) \qquad X \in \mathfrak{g} \tag{3.10}$$

[1]The vacuum value of the twisted superpotential is

$$\widetilde{W}_{\rm eff}(\sigma_*) = \sum_i Q_i(\sigma_*). \tag{3.9}$$

This is the value of $\widetilde{W}_{\rm eff}(\sigma)$ on the branch of logarithms on which it is genuinely critical at σ_*, ${\rm d}\widetilde{W}_{\rm eff}(\sigma_*) = 0$. Note that it is not affected by the $2\pi i\mathrm{P}(\sigma)$ ambiguity of $\widetilde{W}_{\rm eff}(\sigma)$.

is zero or not. Note that b_1 may be regarded as an element of $\mathrm{P}^W \cong \mathrm{P}_{Z_G}$. As an element of P^W, it can also be written as $b_1 = \sum_i Q_i$.

Calabi-Yau case

Suppose it is satisfied, $b_1 = 0$. In this case, the FI parameter is invariant under the renormalization group and the axial $U(1)$ R-symmetry exists in the quantum theory. Accordingly, $\widetilde{W}_{\text{eff}}$ in (3.7) is independent of the parameter Λ. In particular, the vacuum equation $\mathrm{d}\widetilde{W}_{\text{eff}}(\sigma) \in 2\pi i\mathrm{P}$ is invariant under the scaling, $\sigma \to \lambda\sigma$ for $\lambda \in \mathbb{C}^\times$. This means that if σ is a Coulomb vacuum, then, any of its scaling is also. In particular, the space of such vacua, the Coulomb branch, must be non-compact. Also, presence of Coulomb branch imposes a non-trivial constraint on the FI-theta parameter t. In fact, the equation $\mathrm{d}\widetilde{W}_{\text{eff}}(\sigma) \in 2\pi i\mathrm{P}$ produces a parametric representation of t in terms of ratio of σ coordinates, defining a complex hypersurface in the space of t. Let $\Delta \subset \mathfrak{g}_{\mathbb{C}}^{*G}/2\pi i\Lambda_G$ be the discriminant locus on which there is a Coulomb branch and/or mixed branches. It is a union of hyeprsurfaces. When projected to the the ζ space, the discriminant locus Δ projects to an amoeba, in the sense of [21], which asymptotes to the phase boundary. The space of regular values of t is thus

$$\mathfrak{M}_t = \mathfrak{g}_{\mathbb{C}}^{*G}/2\pi i\Lambda_G - \Delta. \tag{3.11}$$

Since $\Delta \subset \mathfrak{g}_{\mathbb{C}}^{*G}/2\pi i\Lambda_G$ has complex codimension one, one can go from one phase to another without meeting it. In particular, there is no sharp transition between different phases.

The theory flows in the infra-red limit to an SCFT with $\widehat{c} = \mathrm{tr}_V(1 - R) - \dim G$, and the FI-theta parameter (*resp.* parameters of W) are exactly marginal A-term (*resp.* B-term) deformation parameters of the SCFT. That is, they parameterize submanifolds of the moduli space of SCFTs

$$\mathfrak{M}_t \subset \mathfrak{M}_A^0, \qquad \mathfrak{M}_W \subset \mathfrak{M}_B^0. \tag{3.12}$$

Quite often, the inclusion $\subset$ is equality $=$.

In a geometric phase, $\mathfrak{M}_t$ and $\mathfrak{M}_W$ are respectively (parts of) the moduli space of complexified Kähler class and the moduli space of complex structures of the Higgs branch, respectively.

Non Calabi-Yau case

Suppose the condition is violated, $b_1 \neq 0$. In this case, the FI parameter runs under the renormalization group — for a change of energy scale $\mu \to \mu'$ it changes as

$$\zeta \to \zeta' = \zeta + \log(\mu'/\mu) b_1, \tag{3.13}$$

and the classical axial $U(1)$ R-symmetry is anomalous — the axial rotation $\mathrm{e}^{i\beta} \in U(1)_A$ shifts the theta angle as

$$\theta \to \theta + 2\beta b_1. \tag{3.14}$$

Accordingly, $\widetilde{W}_{\mathrm{eff}}$ in (3.7) depends non-trivially on Λ. The parameter t in $\widetilde{W}_{\mathrm{eff}}$ is the FI-theta parameter at the scale Λ. Since the vacuum equation $\mathrm{d}\widetilde{W}_{\mathrm{eff}}(\sigma) \in 2\pi i \mathrm{P}$ has no scaling invariance, the space of Coulomb vacua does not have to be non-compact. Quite often, there are isolated Coulomb vacua. Such a vacuum cannot be found by the classical analysis of $U(\sigma, \phi)$ but should be taken into account as a sound vacuum of the quantum theory. Of course, there can be vacua at special values of σ, such as $\sigma = 0$, which can be found by the classical analysis.

The theory flows in the infra-red limit to one of the isolated Coulomb vacua, which is typically a massive vacuum, or to the Higgs branch theory $(\mu^{-1}(\zeta_{\mathrm{IR}})/G, W_{\zeta_{\mathrm{IR}}})$ at $\sigma = 0$ where ζ_{IR} is the IR value of the FI parameter, or to a mixture of these two types. Some of the Higgs branch theory can be a non-trivial SCFT.

One should be careful for the use of the term "phase" for two reasons; one is that the FI parameter runs under the renormalization group and another is that there are other vacua at different regions of the field space, such as Coulomb vacua. When we say "phase", we mean the theory at certain range of energy scales where ζ is in a certain chamber *and* in the region of the field space where the gauge symmetry is broken to a finite group and the classical analysis is valid. When we want to make it clear, we shall sometimes use the term "regime" instead. In a geometric regime, the flow of ζ corresponds to the flow of the Kähler class and the axial shift of θ corresponds to that of the B-field.

Example $\mathbf{T}^{\boldsymbol{U(1)}}_{\boldsymbol{N,d}}$

Let us consider a model $\mathrm{T}^{U(1)}_{N,d}$ labelled by two positive integers N and d, with the following data

$$\begin{aligned}
G &= U(1), \\
V &= \mathbb{C}(-d) \oplus \mathbb{C}(1)^{\oplus N} \ni (p, x_1, \ldots, x_N), \\
W &= p f(x_1, \ldots, x_N), \\
\widetilde{W} &= -t\sigma.
\end{aligned}$$

$\mathbb{C}(i)$ is the representation of $U(1)$ of weight i, $f(x_1,\ldots,x_N)$ is a degree d polynomial which is generic in the sense that $\partial f/\partial x_i = 0$ for all i implies $x_1 = \cdots = x_N = 0$, and $t = \zeta - i\theta \in \mathbb{C}/2\pi i\mathbb{Z}$.

The R-charge is unique up to gauge, $R = (2 - d\epsilon, \epsilon, \ldots, \epsilon)$, and satisfies the charge integrality condition with $J = \mathrm{e}^{\pi i\epsilon}$. The space of FI-parameter $i\mathfrak{z}^* \cong \mathbb{R}$ is separated into two phases — the geometric phase $\zeta > 0$ and the Landau-Ginzburg phase $\zeta < 0$. In $\zeta > 0$, the classical vacuum equation $U = 0$ forces x to have a non-zero value which breaks the gauge group completely. The theory reduces to the sigma model whose target space is the degree d hypersurface $X_f \subset \mathbb{CP}^{N-1}$ defined by $f = 0$. The Kahler and the B-field classes are approximately given by

$$(3.15)\quad [\omega] \simeq \zeta H \in \mathrm{H}^2(X_f, \mathbb{R}), \quad [B] \simeq [(\theta + \pi d)H] \in \mathrm{H}^2(X_f, \mathbb{R}/2\pi\mathbb{Z}),$$

where H is the hyperplane class of $\mathbb{P}^{N-1}$ restricted on X_f. More precisely, the correction is exponentially small in the $\zeta \to +\infty$ limit, $[\omega - iB] = [(t - d\pi i)H] + O(\mathrm{e}^{-t})$. In $\zeta < 0$, the classical vacuum equation $U = 0$ forces p to have a non-zero value which breaks the gauge group to the subgroup $\mathbb{Z}_d \subset U(1)$ consisting of d-th roots of unity. The theory reduces to the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_d, W = f)$ at low energies. The FI parameter runs as $\zeta' = \zeta + (N - d)\log(\mu'/\mu)$, except when the Calabi-Yau condition $d = N$ is satisfied. The effective twisted superpotential is

$$(3.16)\quad \widetilde{W}_{\mathrm{eff}}(\sigma) = -t\sigma + d\sigma(\log(-d\sigma/\Lambda) - 1) - N\sigma(\log(\sigma/\Lambda) - 1),$$

and the equation for the Coulomb vacuum is $\partial_\sigma \widetilde{W}_{\mathrm{eff}} \equiv 0 \bmod 2\pi i\mathbb{Z}$, or

$$(3.17)\qquad (\sigma/\Lambda)^{N-d} = (-d)^d\,\mathrm{e}^{-t}.$$

When $d = N$, we have a family of superconformal field theories with $\widehat{c} = N - 2$ parametrized by t as well as the parameters for f. Since the equation (3.17) has solutions (i.e. arbitrary $\sigma \neq 0$) only for $\mathrm{e}^t = (-N)^N$, the discriminant locus Δ is one point at $t \equiv N\log N + N\pi i$,

$$(3.18)\qquad \mathfrak{M}_t = \mathbb{C}/2\pi i\mathbb{Z} - \{[N\log N + N\pi i]\}.$$

We see that the non-linear sigma model on the Calabi-Yau manifold X_f is continuously connected to the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_N, f)$. This is the basic example of *CY/LG correspondence*. In the present model the inclusions in (3.12) are both equalities, $\mathfrak{M}_t = \mathfrak{M}_A^0$ and $\mathfrak{M}_W = \mathfrak{M}_B^0$.

When $d < N$, the FI parameter ζ runs from positive to negative. The high energy theory is the non-linear sigma model on the Fano manifold X_f whose size decreases as the energy scale is lowered. At lower energies, the sigma model description is no longer valid. There are $(N-d)$ Coulomb vacua with mass gap at $\sigma^{N-d} = \Lambda^{N-d}(-d)^d\,\mathrm{e}^{-t}$, as well as one Higgs branch theory at $\sigma = 0$ (for $d > 1$) which is the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$. When $3 \leq d < N$, the Higgs branch theory further flows in the infra-red limit to a superconformal field theory with $\widehat{c} = N(1-2/d)$. When $d = 2$, the Higgs branch theory has two (*resp.* one) supersymmetric ground states with a mass gap for even (*resp.* odd) N.

When $d > N$, the FI parameter ζ runs from negative to positive. The theory can be regarded as the superconformal field theory with $\widehat{c} = N(1-2/d)$ corrsponding to the Landau-Ginzburg orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$ which is perturbed by a relevant operator of dimension $2N/d$. There are $(d-N)$ Coulomb vacua with mass gap at $\sigma^{d-N} = \Lambda^{d-N}(-d)^{-d}\,\mathrm{e}^{t}$ as well as one Higgs branch theory at $\sigma = 0$ which is the non-linear sigma model on the hypersurface X_f of general type.

3.2. Boundary Conditions

Our main interests are B-branes in GLSM and their low energy behaviour. The classical data for a B-brane is

- *Chan-Paton vector space* $M = M^{\mathrm{ev}} \oplus M^{\mathrm{od}}$**:** a $\mathbb{Z}_2$-graded representation of G,
- *matrix factorization* $Q(\phi)$**:** a G-equivariant polynomial function of $\phi \in V$ with values in $\mathrm{End}^{\mathrm{od}}(M)$ satisfying $Q(\phi)^2 = W(\phi)\mathrm{id}_M$, and
- $\gamma \subset \mathfrak{t}_{\mathbb{C}}$**:** a Weyl invariant Lagrangian submanifold which is a deformation of the real locus $i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$

G-equivariance reads $g^{-1}Q(g\phi)g = Q(\phi)$. We assume that the vector $U(1)$ R-symmetry extends to the boundary: M is also a representation of $U(1)_V$, commuting with G, such that $Q(\phi)$ has R-charge 1. That is, there is a linear map $\mathbf{r} : M \to M$ commuting with the G-action such that

$$\lambda^{\mathbf{r}} Q(\lambda^R \phi)\lambda^{-\mathbf{r}} = \lambda Q(\phi). \tag{3.19}$$

We further assume that the charge integrality is maintained: $\mathrm{e}^{\pi i \mathbf{r}} J$ agrees with the $\mathbb{Z}_2$ grading on M, which is $+1$ (*resp.* -1) on even (*resp.* odd) elements of M. When the Lie algebra $\mathfrak{g}$ has a non-zero center $\mathfrak{z}$, there is an unphysical ambiguity in the R-charge: for each $\varepsilon \in i\mathfrak{z}$, we may do

$$R \to R + \varepsilon, \qquad J \to J\,\mathrm{e}^{\pi i \varepsilon}, \qquad \mathbf{r} \to \mathbf{r} - \varepsilon. \tag{3.20}$$

Given this data, we have a boundary condition on the fields as well as interaction terms at the boundary [22, 9]. Neumann boundary condition is imposed on the scalar ϕ in the matter multiplet while the scalar σ in the gauge multiplet is required to have boundary values in the adjoint orbit of γ. Also, we have boundary interaction determined by the data (M, Q), called the *Chan-Paton factor*: For $\Sigma = (-\infty, 0] \times \mathbb{R}$ with coordinate (x, t), it is $\mathrm{Pexp}\left(-i \int_{\partial\Sigma} \mathcal{A}_t \mathrm{d}t\right) : M \to M$ where[2]

$$\mathcal{A}_t = v_t - \mathrm{Re}(\sigma) - \frac{1}{2\sqrt{2\pi}} \psi^i \partial_i Q(\phi) + \frac{1}{2\sqrt{2\pi}} \overline{\psi}^{\bar{\imath}} \partial_{\bar{\imath}} Q(\phi)^\dagger + \frac{1}{4\pi} \{Q(\phi), Q(\phi)^\dagger\}. \tag{3.21}$$

Here, $v_t \mathrm{d}t$ is the gauge potential on the boundary, while ψ^i and $\overline{\psi}^{\bar{\imath}}$ are the boundary values of the fermionic components of the matter multiplet. Of course, $v_t - \mathrm{Re}(\sigma)$ in (3.21) should be understood as its representation on M.

In the presence of boundary, the theta parameter is no longer periodic: the shift $\theta \to \theta + 2\pi n$ for $n \in \Lambda_G$ generates a boundary term $-\oint_{\partial\Sigma} n(v)$ in the action. This, however, can be compensated by a change of M: since $n \in \Lambda_G$ is the infinitesimal version of a character $f_n \in \mathrm{Hom}(G, U(1))$, the shift is compensated if the Chan-Paton representation M, $m \mapsto gm$, is replaced by a new representation $M(f_n^{-1})$, $m \mapsto f_n(g)^{-1} gm$. In other words,

$$\theta \to \theta + 2\pi n \quad \text{is equivalent to} \quad M \to M(f_n). \tag{3.22}$$

We may also consider $\mathrm{B}_{\mathrm{e}^{2i\beta}}$-branes in the model. Since $\mathrm{B}_{\mathrm{e}^{2i\beta}}$-type supersymmetry is obtained from the B-type supersymmetry by the classical axial rotation $\mathrm{e}^{i\beta F_A}$, we may obtain a $\mathrm{B}_{\mathrm{e}^{2i\beta}}$-brane from the B-brane (M, Q, γ) by performing a change of variable $X \to \mathrm{e}^{i\beta F_A} X$. In particular, σ in (3.21) is replaced by $\mathrm{e}^{2i\beta}\sigma$, and the submanifold $\gamma \subset \mathfrak{t}_{\mathbb{C}}$ is rotated as

$$\gamma \to \mathrm{e}^{-2i\beta}\gamma. \tag{3.23}$$

There is a possible mathematical application. The general principle of supersymmetry dictates that the category of B-branes is invariant under A-term deformations as well as under the renormalization group.

[2]Somewhat unusual appearance of powers of $\sqrt{2\pi}$ in (3.21) is due to our convention that the entire bulk action is divided by 2π. See (A.10).

When the infinitesimal Calabi-Yau condition is satisfied $\mathfrak{g} \subset \mathfrak{sl}(V)$, the GLSM defines a family of SCFTs over the parameter space $\mathfrak{M}_t \times \mathfrak{M}_W$. Since the FI-theta parameter is an A-term parameter, the category of B-branes in the SCFT is locally constant on $\mathfrak{M}_t$. In particular, if we draw a path in $\mathfrak{M}_t$, there must be an equivalence of categories at two different points on the path. If the path connects two different phases, there must be an equivalence of the categories of B-branes in the two phases, and the equivalence should not change under deformation of the path. If we draw a loop in $\mathfrak{M}_t$ around the discriminant locus Δ, we must have an autoequivalence of the category at any point of the loop. When the infinitesimal Calabi-Yau condition is violated $\mathfrak{g} \not\subset \mathfrak{sl}(V)$, the GLSM defines an RG flow from a UV theory to an IR theory. From the RG-invariance, the category of B-branes in the UV theory must be equivalent to the category of B-branes in the IR theory.

A natural problem is to learn about these equivalences, both in the Calabi-Yau and non Calabi-Yau cases, by studying B-branes in GLSM. The hemisphere partition function will play an important rôle in this problem, as will be discussed in later sections.

At this stage, let us mention that there is a category $\mathfrak{D}_{\rm LSM} = \mathrm{MF}_G(V, W)$ whose objects are B-brane data (M, Q). A morphism from (M_1, Q_1) to (M_2, Q_2) is an $\mathbb{C}[\phi]$-module map $a : M_1 \otimes \mathbb{C}[\phi] \to M_2 \otimes \mathbb{C}[\phi]$ compatible with the $U(1)_V \times G$ action obeying $Q_2 a = a Q_1$. This may be regarded as the category of branes of the theory where the gauge coupling constant is turned off, $e \searrow 0$. In a regime where the gauge symmetry is broken to a finite subgroup by the non-zero values of the matter fields and the theory reduces to a Higgs branch theory $(\mu^{-1}(\zeta)/G, W_\zeta)$, the rôle of the gauge multiplet is expected to be unimportant, and the data (M, Q) alone, without reference to $\gamma \subset \mathfrak{t}_{\mathbb{C}}$, should be enough to determine a D-brane. In such a regime, there is a functor $\pi_\zeta : \mathfrak{D}_{\rm LSM} \to \mathrm{MF}(\mu^{-1}(\zeta)/G, W_\zeta)$ that represents the reduction.[3] Since $e \searrow 0$ is a singular limit, unlike in the usual RG flow, this is far from being an equivalence.

Example $\mathbf{T}^{U(1)}_{N,d}$

The reduction of a B-brane data $\mathfrak{B} = (M, Q)$ in the geometric and LG regimes of the theory $\mathrm{T}^{U(1)}_{N,d}$ are described as follows. Suppose the

[3] In a geometric regime where $W_\zeta : \mu^{-1}(\zeta)/G \to \mathbb{C}$ is a Bott-Morse function with a smooth critical locus $X_\zeta = \mathrm{Crit}(W_\zeta)$, the target category $\mathrm{MF}(\mu^{-1}(\zeta)/G, W_\zeta)$ is expected to be equivalent to the derived category $D^b_{\rm Coh}(X_\zeta)$. This is a global version of Knörrer periodicity [23]. Proofs in various set ups have been given in [24–26].

Chan-Paton vector space is

$$M = \bigoplus_j \mathbb{C}(r_j, q_j) \tag{3.24}$$

where $\mathbb{C}(r_i, q_i)$ stands for a component of R-charge r_i and $U(1)$ gauge charge q_i. In the geometric regime, it is convenient to set $\epsilon = 0$ so that x_i's have R-charge zero, and we write r_j^0 for r_j at $\epsilon = 0$. Then, the geometric image of $\mathfrak{B}$ is the complex of vector bundles on X_f,

$$\pi_+(\mathfrak{B}) = (\mathcal{E}, d) : \mathcal{E} = \bigoplus_{n=0}^{\infty} \bigoplus_j \mathcal{O}_{X_f}(q_j + dn)[r_j^0 + 2n], \quad d = Q(p_+, x), \tag{3.25}$$

where p_+ stands for the shift operator $\otimes\mathcal{O}_{X_f}(d)[2]$. To elaborate on "$d = Q(p_+, x)$", if $Q(p, x)$ has a term $p^m x^\alpha$ in the component that sends $\mathbb{C}(r_{j_2}^0, q_{j_2})$ to $\mathbb{C}(r_{j_1}^0, q_{j_1})$, which is possible only if $-q_{j_1} + q_{j_2} - dm + |\alpha| = 0$ and $r_{j_1}^0 - r_{j_2}^0 + 2m = 1$, the corresponding term in d is multiplication by x^α in the components that send $\mathcal{O}_{X_f}(q_{j_2} + dn)[r_{j_2}^0 + 2n]$ to $\mathcal{O}_{X_f}(q_{j_1} + d(n + m))[r_{j_2}^0 + 2(n + m)]$ for $n = 0, 1, \ldots$. This $(\mathcal{E}, d)$ is a complex of vector bundles on X_f. It is infinitely long to the right but is exact beyond a certain degree, and hence defines a bounded complex of coherent sheaves on X_f. We should note that the B-field class is no longer periodic in the presence of boundary, just like the theta parameter. The *absolute* B-field class is approximately

$$[B] \simeq (\theta + \pi d)H \in \mathrm{H}^2(X_f, \mathbb{R}). \tag{3.26}$$

In the LG regime, it is convenient to set $\epsilon = 2/d$ so that p has R-charge zero, and we write r_j^{LG} for r_j at $\epsilon = 2/d$. Then, the LG image of $\mathfrak{B}$ is simply

$$\pi_-(\mathfrak{B}) = (M_-, Q_-) : M_- = \bigoplus_j \mathbb{C}(r_j^{\mathrm{LG}}, q_j) \text{ as } U(1)_V \times \mathbb{Z}_d\text{-module}, \quad Q_-(x) = Q(1, x). \tag{3.27}$$

This is indeed a brane data in the LG orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$, that is, the GLSM with gauge group $\mathbb{Z}_d$, matter representation V and superpotential $f(x)$.

Let us present two examples of B-brane data, $\mathfrak{B}_1 = (M_1, Q_1)$ and $\mathfrak{B}_2 = (M_2, Q_2)$, with

$$M_1 = \mathbb{C}(0,0) \oplus \mathbb{C}(1 - d\epsilon, d), \tag{3.28}$$

$$Q_1 = \begin{pmatrix} 0 & p \\ f(x) & 0 \end{pmatrix}, \tag{3.29}$$

and

$$M_2 = \bigoplus_{j=0}^{N} \mathbb{C}(j - j\epsilon, j)^{\oplus \binom{N}{j}}, \tag{3.30}$$

$$Q_2 = \sum_{i=1}^{N} \left(x_i \overline{\eta}_i + \frac{1}{d}\, p \frac{\partial f(x)}{\partial x_i} \eta_i \right). \tag{3.31}$$

In the second data, M_2 is a representation of the Clifford algebra $\{\eta_i, \overline{\eta}_j\} = \delta_{i,j}$, $\{\eta_i, \eta_j\} = \{\overline{\eta}_i, \overline{\eta}_j\} = 0$, in such a way that $\overline{\eta}_i$'s raise the (R, gauge) charge by $(1-\epsilon, 1)$ while η_i's lower it by the same amount. We may also consider $\mathfrak{B}_1(i,q)$ and $\mathfrak{B}_2(i,q)$ where the (R, gauge) charge of each component of M_1 and M_2 is shifted by (i,q). For example, $\mathfrak{B}_1(i,q)$ has $M = \mathbb{C}(i,q) \oplus \mathbb{C}(1 - d\epsilon + i, d + q)$ with everything else unchanged.

In fact, these data play crucial rôles in the study of low energy behaviour of D-branes in the geometric regime or in the LG regime. For $Q = Q_1$ and Q_2, let us compute $\{Q, Q^\dagger\}$ that enters into (3.21) as the boundary potential $\frac{1}{4\pi}\{Q, Q^\dagger\}$:

$$\{Q_1, Q_1^\dagger\} = \left(|p|^2 + |f(x)|^2\right) \mathrm{id}_{M_1}, \tag{3.32}$$

$$\{Q_2, Q_2^\dagger\} = \sum_{i=1}^{N} \left(|x_i|^2 + \frac{1}{d^2} \left| p \frac{\partial f(x)}{\partial x_i} \right|^2 \right) \mathrm{id}_{M_2}. \tag{3.33}$$

In the geometric regime where $x \neq 0$, $\{Q_2, Q_2^\dagger\}$ is positive definite which means that $\mathfrak{B}_2$ and any of its shifts are empty at low energies. In the Landau-Ginzburg regime where $p \neq 0$, $\{Q_1, Q_1^\dagger\}$ is positive definite which means that $\mathfrak{B}_1$ and its shifts are empty at low energies. We can also see the emptiness by looking at the reductions, (3.25) and (3.27), $\pi_+(\mathfrak{B}_2(i,q)) \cong 0$ and $\pi_-(\mathfrak{B}_1(i,q)) \cong 0$.

A side remark: In the geometric regime, $\mathfrak{B}_1$ and its shifts can be non-trivial, as $\{Q_1, Q_1^\dagger\}$ vanishes at $p = f(x) = 0$. Indeed, $\mathfrak{B}_1(i,q)$ descends under the reduction (3.25) to the line bundle $\mathcal{O}(q)[j]$ over the hypersurface X_f. In the LG regimes, $\mathfrak{B}_1$ and its shifts can be non-trivial, as $\{Q_2, Q_2^\dagger\}$ vanishes at $x = 0$. Indeed, $\mathfrak{B}_2(i,q)$ descends under the reduction (3.27) to a non-trivial B-brane in the LG orbifold, which further flows to a famous B-brane called "Recknagel-Schomerus brane" in Gepner model (when f is Fermat polynomial) [27].

Presence of empty branes makes it manifest that different D-branes in GLSM can lead to the same D-brane at low energies. In the geometric regime, the branes $\mathfrak{B}_2(i,q)$ can be added to any D-brane $\mathfrak{B}$ without changing its low energy behaviour. This holds even when we bind $\mathfrak{B}_2(i,q)$'s and $\mathfrak{B}$ with a non-trivial map between them (cone construction). The resulting brane data can be simplified by cancelling a pair of components. In this way, $\mathfrak{B}$ can be replaced by another brane which bahaves in the same way at low energies, but with a different set of Chan-Paton gauge charges. If you wish, repeating this process if necessary, you can increase the minimum gauge charge or lower the maximal gauge charge. Since the set of gauge charges of $\mathfrak{B}_2(j,q)$ is $\{q, q+1, \ldots, q+N\}$, the given brane $\mathfrak{B}$ can be replaced by another brane $\mathfrak{B}'$ whose gauge charges are in a set of N consecutive integers, say $\{0, 1, \ldots, N-1\}$. The same holds in the LG regime, where empty branes are $\mathfrak{B}_1(i,q)$ whose set of gauge charges is $\{q, q+d\}$. A given brane can be replaced by another brane whose gauge charges are in a set of d consecutive integers, say $\{1, 2, \ldots, d\}$, without changing the low energy behaviour.

This result has a categorical interpretation. The functors $\pi_+ : \mathfrak{D}_{\mathrm{LSM}} \to D^b_{\mathrm{Coh}}(X_f)$ and $\pi_- : \mathfrak{D}_{\mathrm{LSM}} \to \mathrm{MF}_{\mathbb{Z}_d}(f)$ that represent the reduction in the geometric and LG regimes have huge kernels — many many different objects are sent to the same object. But there are nice *slices*. For a subset $I \subset \mathbb{Z}$, we write $\mathcal{T}_I \subset \mathfrak{D}_{\mathrm{LSM}}$ for the full subcategory consisting of data whose gauge charges are in the set I. Then, for sets $I_+ \subset \mathbb{Z}$ and $I_- \subset \mathbb{Z}$ of N and d consecutive integers respectively, the functors

$$\mathcal{T}_{I_+} \hookrightarrow \mathfrak{D}_{\mathrm{LSM}} \xrightarrow{\pi_+} D^b_{\mathrm{Coh}}(X_f), \tag{3.34}$$

$$\mathcal{T}_{I_-} \hookrightarrow \mathfrak{D}_{\mathrm{LSM}} \xrightarrow{\pi_-} \mathrm{MF}_{\mathbb{Z}_d}(f), \tag{3.35}$$

are equivalences of categories. A proof can be found in [28] with a combination of [24–26]. This was extended to a more general situation in [29, 30].

4. The Hemisphere Partition Function

In this section, we present the partition function of gauged linear sigma model on the hemisphere, and describe some of its properties, such as the behaviour at large values of the radius r of the hemisphere, expressions in the geometric and the Landau-Ginzburg regimes, and the differential equations with respect to the radius r as well as the FI-thtea parameters.

4.1. Supersymmetry on the sphere and the hemisphere

We first present the supersymmetry on the sphere and the hemisphere. The key is to employ the *superconformal* transformations. A superconformal transformation has conformal Killing spinors as the variational parameters: holomorphic sections ϵ_+ and $\overline{\epsilon}_+$ of $S_+ \cong \sqrt{T_\Sigma}$ and anti-holomorphic sections ϵ_- and $\overline{\epsilon}_-$ of $S_- \cong \sqrt{\overline{T}_\Sigma}$. It reduces to the supersymmetry transformation (2.12) for constant variational parameters on a flat Euclidean or Minkowski space. The transformation is specified by a choice of vector and axial R-transformations of the fields. The commutator of the superconformal transformations $\delta_1^{\rm sc}$ and $\delta_2^{\rm sc}$ with different variational parameters is the sum of conformal and R transformations

$$[\delta_1^{\rm sc}, \delta_2^{\rm sc}] = \delta_{X_{12}}^{\rm conformal} + \delta_{\Theta_{V,12}}^{\rm vector} + \delta_{\Theta_{A,12}}^{\rm axial}. \tag{4.1}$$

Let us first consider the sphere. It is covered by two charts, the z-plane and the w-plane, which are related by $zw = 1$. We give it an $O(3)$ symmetric round metric of radius r, which is ${\rm d}s^2 = 4r^2|{\rm d}z|^2/(1+|z|^2)^2$ on the z-plane. There are four conformal Killing spinors

$$\mathbf{s}_{-\frac{1}{2}} = \sqrt{\frac{\partial}{\partial z}}, \quad \mathbf{s}_{\frac{1}{2}} = z\sqrt{\frac{\partial}{\partial z}} \text{ and } \widetilde{\mathbf{s}}_{-\frac{1}{2}} = \sqrt{\frac{\partial}{\partial \overline{z}}}, \quad \widetilde{\mathbf{s}}_{\frac{1}{2}} = \overline{z}\sqrt{\frac{\partial}{\partial \overline{z}}}. \tag{4.2}$$

Thus, we may consider the superconformal transformations $\delta^{\rm sc}(\epsilon_+, \epsilon_-, \overline{\epsilon}_+, \overline{\epsilon}_-)$ where the variational parameters are chosen from the four. However, we can accept only the combinations whose commutators are symmetries of the system. In particular, since we consider theories which are not necessarily conformally invariant, like GLSM, the vector field X_{12} that appear in (4.1) must be an $\mathfrak{o}(3)$ isometry. There are essentially two types of such combinations — A-type and B-type:

A-type supercharges are the four combinations

$$\begin{aligned} Q_{(+)}^{A+} &= \delta^{\rm sc}(0, 0, \mathbf{s}_{\frac{1}{2}}, \widetilde{\mathbf{s}}_{-\frac{1}{2}}), \quad & Q_{(+)}^{A-} &= \delta^{\rm sc}(\mathbf{s}_{-\frac{1}{2}}, \widetilde{\mathbf{s}}_{\frac{1}{2}}, 0, 0), \\ Q_{(-)}^{A+} &= \delta^{\rm sc}(0, 0, \mathbf{s}_{-\frac{1}{2}}, -\widetilde{\mathbf{s}}_{\frac{1}{2}}), \quad & Q_{(-)}^{A-} &= \delta^{\rm sc}(\mathbf{s}_{\frac{1}{2}}, -\widetilde{\mathbf{s}}_{-\frac{1}{2}}, 0, 0). \end{aligned} \tag{4.3}$$

The $\mathfrak{o}(3)$ rotations and the vector R-transformation appear as their anti-commutators.

B-type supercharges are the four combinations

$$
\begin{aligned}
&Q^{B+}_{(+)} = \delta^{\mathrm{sc}}(\mathbf{s}_{\frac{1}{2}}, 0, 0, \widetilde{\mathbf{s}}_{-\frac{1}{2}}), \qquad Q^{B-}_{(+)} = \delta^{\mathrm{sc}}(0, \widetilde{\mathbf{s}}_{\frac{1}{2}}, \mathbf{s}_{-\frac{1}{2}}, 0),\\
&Q^{B+}_{(-)} = \delta^{\mathrm{sc}}(\mathbf{s}_{-\frac{1}{2}}, 0, 0, -\widetilde{\mathbf{s}}_{\frac{1}{2}}), \quad Q^{B-}_{(-)} = \delta^{\mathrm{sc}}(0, -\widetilde{\mathbf{s}}_{-\frac{1}{2}}, \mathbf{s}_{\frac{1}{2}}, 0).
\end{aligned} \tag{4.4}
$$

The $\mathfrak{o}(3)$ rotations and the axial R-transformation appear as their anti-commutators.

To be precise, there are variants obtained by the other R-rotations. $\mathrm{A}^{\mathrm{e}^{2i\beta}}$-type is obtained from A-type by the axial R-rotation: $\epsilon_\pm \to \mathrm{e}^{\pm i\beta}\epsilon_\pm$ and $\overline{\epsilon}_\pm \to \mathrm{e}^{\mp i\beta}\overline{\epsilon}_\pm$; $\mathrm{B}^{\mathrm{e}^{2i\alpha}}$-type is obtained from B-type by the vector R-rotation: $\epsilon_\pm \to \mathrm{e}^{-i\alpha}\epsilon_\pm$ and $\overline{\epsilon}_\pm \to \mathrm{e}^{i\alpha}\overline{\epsilon}_\pm$. Together with the $\mathfrak{o}(3)$ rotations and the R-charge (F_V for $\mathrm{A}^{\mathrm{e}^{2i\beta}}$-type and F_A for $\mathrm{B}^{\mathrm{e}^{2i\alpha}}$-type), they form a Lie super-algebra isomorphic to $\mathfrak{osp}(2|2)$.

Next, let us consider the hemisphere, which is realized as a half of the sphere, say, the region $|z| \le 1$ ("southern" hemisphere) in the z-plane. We again give it the round metric of radius r. There are infinitely many conformal Killing spinors, $\mathbf{s}_r = z^{r+\frac{1}{2}}\sqrt{\frac{\partial}{\partial z}}$ and $\widetilde{\mathbf{s}}_r = \overline{z}^{r+\frac{1}{2}}\sqrt{\frac{\partial}{\partial \overline{z}}}$ with $r \in \mathbb{Z}_{\ge 0} - \frac{1}{2}$, but the boundary condition at $|z| = 1$ can admit only the pairs $(\mathbf{s}_{\frac{1}{2}}, \widetilde{\mathbf{s}}_{-\frac{1}{2}})$ and $(\mathbf{s}_{-\frac{1}{2}}, \widetilde{\mathbf{s}}_{\frac{1}{2}})$ as the variational parameters. Note also that the hemisphere only has $O(2)$ isometry, and hence we can accept only the combinations whose anti-commutators (4.1) have the $\mathfrak{o}(2)$ isometry as X_{12}. These constraints leave us with only four possibilities: $\mathrm{A}_{(+)}$, $\mathrm{A}_{(-)}$, $\mathrm{B}_{(+)}$ and $\mathrm{B}_{(-)}$ where

$\mathbf{A}_{(\pm)}$**-type supercharges** are the two combinations

$$
Q^{A+}_{(\pm)} = \delta^{\mathrm{sc}}(0, 0, \mathbf{s}_{\pm\frac{1}{2}}, \pm\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}) \ \text{ and } \ Q^{A-}_{(\pm)} = \delta^{\mathrm{sc}}(\mathbf{s}_{\mp\frac{1}{2}}, \pm\widetilde{\mathbf{s}}_{\pm\frac{1}{2}}, 0, 0). \tag{4.5}
$$

Their anticommutator is the sum of the $\mathfrak{o}(2)$ rotation and the vector R-rotation.

$\mathbf{B}_{(\pm)}$**-type supercharges** are the two combinations

$$
Q^{B+}_{(\pm)} = \delta^{\mathrm{sc}}(\mathbf{s}_{\pm\frac{1}{2}}, 0, 0, \pm\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}) \ \text{ and } \ Q^{B-}_{(\pm)} = \delta^{\mathrm{sc}}(0, \pm\widetilde{\mathbf{s}}_{\pm\frac{1}{2}}, \mathbf{s}_{\mp\frac{1}{2}}, 0). \tag{4.6}
$$

Their anticommutator is the sum of the $\mathfrak{o}(2)$ rotation and the axial R-rotation.

Again, there are variants, $\mathrm{A}^{\mathrm{e}^{2i\beta}}_{(\pm)}$ and $\mathrm{B}^{\mathrm{e}^{2i\alpha}}_{(\pm)}$, by applying the other R-rotation. The commutator algebra for $(Q, \overline{Q}, F) = (\mathrm{e}^{-\frac{\pi i}{4}}Q^{A\pm}_{(\pm)}, \mathrm{e}^{-\frac{\pi i}{4}}Q^{A\mp}_{(\pm)}, \pm F_V)$ or $(\mathrm{e}^{-\frac{\pi i}{4}}Q^{B\pm}_{(\pm)}, \mathrm{e}^{-\frac{\pi i}{4}}Q^{B\mp}_{(\pm)}, \pm F_A)$ and the $\mathfrak{o}(2)$ rotation generator L

is

$$\begin{aligned} Q^2 = \overline{Q}^2 = 0, \quad \{Q, \overline{Q}\} = -2L + F, \\ [L, Q] = \tfrac{1}{2}Q, \quad [L, \overline{Q}] = -\tfrac{1}{2}\overline{Q}, \\ [F, Q] = Q, \quad [F, \overline{Q}] = -\overline{Q}. \end{aligned} \tag{4.7}$$

If we zoom in to a point on the boundary $|z| = 1$ and take the flat space limit $r \to \infty$, we see that the supercharges become

$$Q^{A+}_{(\pm)} \propto \mathrm{e}^{i\beta}\overline{Q}_+ \pm \mathrm{e}^{-i\beta}\overline{Q}_-, \quad Q^{A-}_{(\pm)} \propto \mathrm{e}^{-i\beta}Q_+ \pm \mathrm{e}^{i\beta}Q_-, \tag{4.8}$$

$$Q^{B+}_{(\pm)} \propto \mathrm{e}^{i\alpha}\overline{Q}_+ \mp \mathrm{e}^{-i\alpha}Q_-, \quad Q^{B-}_{(\pm)} \propto \mathrm{e}^{-i\alpha}Q_+ \mp \mathrm{e}^{i\alpha}\overline{Q}_-. \tag{4.9}$$

Therefore, the boundary condition at $|z| = 1$ must be a $\mathrm{B}_{\pm\,\mathrm{e}^{2i\beta}}$-brane (*resp.* $\mathrm{A}_{\mp\,\mathrm{e}^{2i\alpha}}$-brane) for the $\mathrm{A}^{\mathrm{e}^{2i\beta}}_{(\pm)}$-type (*resp.* $\mathrm{B}^{\mathrm{e}^{2i\alpha}}_{(\pm)}$-type) supersymmetry.

Supergravity approach

A general approach for supersymmetry on a curved space is to couple the system to the supergravity and to choose a supergravity background that is invariant under a part of the supersymmetry. Topological twisting, introduced in Section 2.5, is an example of such a procedure — turn on the $U(1)$ connection for an R-symmetry so that some of the variational parameters can be constant scalars. The supersymmetry on the (hemi)sphere discussed above is another example where a different component of the supergravity multiplet is turned on. Certain linear combinations, ϵ and $\overline{\epsilon}$, of the variational parameters satisfy the following equations for some $\omega \in \mathbb{R}/2\pi\mathbb{Z}$ [1]

$$\nabla_\mu \epsilon = \frac{i}{2r}\gamma_\mu\, \mathrm{e}^{i\omega\gamma_3}\epsilon, \qquad \nabla_\mu \overline{\epsilon} = \frac{i}{2r}\gamma_\mu\, \mathrm{e}^{-i\omega\gamma_3}\overline{\epsilon}, \tag{4.10}$$

where γ_μ are the gamma matrices, γ_3 is the chirality operator, $\gamma_3 = \pm 1$ on $S_\mp$. This is indeed the condition for supersymmetry in a certain supergravity background [31].

4.2. Formulation

We consider the GLSM on the sphere and the hemisphere preserving A-type supersymmetry. In what follows, we describe the hemisphere with $\mathrm{A}_{(\pm)}$-type supersymmetry and B-branes at the boundary. The

[1] $\epsilon = \epsilon_- + \epsilon_+$, $\overline{\epsilon} = \overline{\epsilon}_- + \overline{\epsilon}_+$, $\omega = 2\beta - \frac{\pi}{2}$ for $\mathrm{A}^{\mathrm{e}^{2i\beta}}$; and $\epsilon = \epsilon_- + \overline{\epsilon}_+$, $\overline{\epsilon} = \overline{\epsilon}_- + \epsilon_+$, $\omega = 2\alpha - \frac{\pi}{2}$ for $\mathrm{B}^{\mathrm{e}^{2i\alpha}}$.

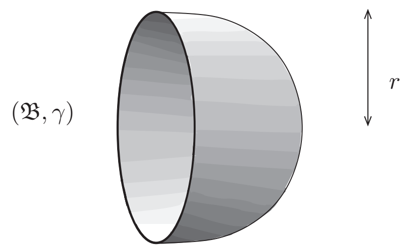

Fig. 1. The hemisphere

case of the sphere can be obtained from that easily. The supersymmetry is specified for a choice of vector R-transformation of the matter field ϕ, and for this we use the R-symmetry $R : V \to V$ of the system. We shall write $Q^- = Q^{A-}_{(\pm)}$ and $Q^+ = Q^{A+}_{(\pm)}$, to simplify the notations.

The action consists of four scalar terms

$$S = S_{\mathrm{g}} + S_{\mathrm{m}} + S_W + S_t, \tag{4.11}$$

and a matrix term $\int_{\partial D^2} \mathcal{A}$ that enters into the Chan-Paton factor

$$\mathrm{tr}_M \mathrm{Pexp}\left(-\int_{\partial D^2} \mathcal{A}\right). \tag{4.12}$$

These are obtained by modifying the covariantized version of the Lagrangian by terms that depend on the radius r as well as the bulk and the boundary R-charges R and $\mathbf{r}$. See Appendix A.4 for the detail. S_{g} and S_{m} are respectively the kinetic terms for the gauge multiplet and the matter multiplet. They are individually supersymmetric and Q-exact,

$$S_{\mathrm{g}} = Q^- Q^+(\cdots), \qquad S_{\mathrm{m}} = Q^- Q^+(\cdots). \tag{4.13}$$

S_W is the superpotential term. It is not supersymetric by itself, but the combination $\mathrm{e}^{-S_W} \mathrm{tr}_M \mathrm{Pexp}\left(-\int_{\partial D^2} \mathcal{A}\right)$ is. If we deform the superpotential $W \to W + \Delta W$ and the matrix factorization $Q \to Q + \Delta Q$ while maintaining the condition of supersymmetry, $\Delta Q Q + Q \Delta Q = \Delta W \mathrm{id}_M$, then, the combination changes by a Q-exact term

$$\Delta\left[\mathrm{e}^{-S_W} \mathrm{tr}_M \mathrm{Pexp}\left(-\int_{\partial D^2} \mathcal{A}\right)\right] = (Q^+ + Q^-)(\cdots). \tag{4.14}$$

Finally, S_t depends on the FI-theta parameter and is supersymmetric by itself. It is the sum of terms which are holomorphic and anti-holomorphic in t, $S_t = S_t^{\rm hol} + S_t^{\rm antihol}$. For the $\mathrm{A}_{(+)}$-type (*resp.* $\mathrm{A}_{(-)}$-type) supersymmetry, the latter (*resp.* former) is Q-exact,

$$\begin{aligned} \mathrm{A}_{(+)} :\ S_t^{\rm antihol} &= Q^- Q^+(\cdots), \\ \mathrm{A}_{(-)} :\ S_t^{\rm hol} &= Q^- Q^+(\cdots). \end{aligned} \tag{4.15}$$

The partition function is defined to be the path-integral over the fields on the hemisphere

$$Z_{D^2} = \int \mathcal{D}\,[\text{fields}]\ \exp\left(-S\right) \mathrm{tr}_M \mathrm{Pexp}\left(-\int_{\partial D^2} \mathcal{A}\right). \tag{4.16}$$

Due to the exactness (4.14), the partition function is invariant under the deformation of the superpotential W and the matrix factorization Q that maintain the condition $Q^2 = W \mathrm{id}_M$. Also, for the system preseving the $\mathrm{A}_{(+)}$-type (*resp.* $\mathrm{A}_{(-)}$-type) supersymmetry, due to the exactness (4.15), the partition function is annihilated by the antiholomorphic (*resp.* holomorphic) derivative with respect to the FI-theta parameter t, that is, it depends holomorphically (it resp. antiholomorphically) on t.

The exactness of the kinetic terms (4.13) is relevant for the computation. This means that the result does not change when these terms are scaled up. In particular, the path integral localizes on the supersymmetric locus where the bosonic part of the kinetic terms vanish — the one-loop approximation around such locus yields the exact answer.

4.3. The result

Let us write down the result. Let

$$V = \bigoplus_i \mathbb{C}(R_i, Q_i) \tag{4.17}$$

be the weight decomposition of the matter representation V of $U(1)_V \times G$. We assume that the R-charges are chosen in the range

$$0 < R_i < 2. \tag{4.18}$$

The partition function of the system preserving the $\mathrm{A}_{(+)}$-type supersymmetry on the hemisphere with the B-brane (M, Q, γ) at the boundary is

$$\begin{aligned} Z_{D^2}^{\mathrm{A}_{(+)}}(M, Q, \gamma) = C(r\Lambda)^{\widehat{c}/2} &\int_{\gamma'} \mathrm{d}^\ell \sigma' \\ &\times \prod_{\alpha>0} \alpha(\sigma') \sinh\left(\pi\alpha(\sigma')\right) \prod_i \Gamma\left(iQ_i(\sigma') + \frac{R_i}{2}\right) \\ &\times \exp\Big(it_R(\sigma')\Big)\, f_M(\sigma'), \end{aligned} \tag{4.19}$$

with

$$f_M(\sigma') := \mathrm{tr}_M \exp\Big(\pi i \mathbf{r} + 2\pi\sigma'\Big). \tag{4.20}$$

For the system preserving the $\mathrm{A}_{(-)}$-type supersymmetry on the hemisphere with the B_{-1}-brane $(M, Q, -\gamma)$ at the boundary (see (3.23) for the replacement $\gamma \to -\gamma$), it is its complex conjugate,

$$Z_{D^2}^{\mathrm{A}_{(-)}}(M, Q, -\gamma) = \left(Z_{D^2}^{\mathrm{A}_{(+)}}(M, Q, \gamma)\right)^*. \tag{4.21}$$

In the above expressions, r is the radius of the (hemi)sphere, Λ is the energy scale that is needed to renormalize the theory, and $\widehat{c} = \mathrm{tr}_V(1 - R) - \dim G$. The integration variable σ' takes values in $\mathfrak{t}_{\mathbb{C}}$ and comes from the Cartan zero mode of the field σ times the radius r. The contour $\gamma' \subset \mathfrak{t}_{\mathbb{C}}$ in (4.19) is γ times r, and $\mathrm{d}^\ell\sigma'$ is the flat holomorphic volume form

$$\mathrm{d}^\ell\sigma' = \mathrm{d}\sigma_1' \wedge \cdots \wedge \mathrm{d}\sigma_\ell'. \tag{4.22}$$

The product $\prod_{\alpha>0}$ is over positive roots of the gauge group. t_R is the renormalized FI-theta parameter defined by

$$t_R = t - b_1 \log(r\Lambda). \tag{4.23}$$

Information of the D-brane enters into $f_M(\sigma')$ which we call the brane factor. Note that it depends only on M as the representation of $U(1)_V \times G$. The detail of the matrix factorization $Q(\phi)$ does not matter.

The above result is first derived for the case where γ is the real locus $i\mathfrak{t}$ and then for a deformed γ using holomorphy. The integrand has poles at the hyperplanes

$$iQ_i(\sigma') + \frac{R_i}{2} = 0, -1, -2, \ldots \tag{4.24}$$

which misses the real locus $i\mathfrak{t}$ under the condition (4.18). We propose that γ is acceptable only when (i) it can be continuously deformed to the real locus $i\mathfrak{t}$ without meeting the pole hyperplanes (4.24). Since the integral (4.19) is over a non-compact contour, whether it is convergent is a non-trivial question. The asymptotic behaviour of the integrand can be found easily using Stirling's formula for the Gamma function. For the irreducible decomposition $M = \oplus_j U_j$ with respect to the identity component G_0 of G, the brane factor $f_M(\sigma')$ decomposes into the sum $\sum_j \mathrm{tr}_{U_j}\, \mathrm{e}^{\pi i \mathbf{r} + 2\pi\sigma'}$. Stirling's formula says that the integrand for the term corresponding to the representation U_j of highest weight λ_j behaves as

$$\text{a power of } \sigma \times \exp\left(-ir\widetilde{W}_{\mathrm{eff},\lambda_j}(\sigma)\right), \tag{4.25}$$

with[2]

(4.26)

$$\widetilde{W}_{\mathrm{eff},\lambda_j}(\sigma) = -t(\sigma) + 2\pi i(\lambda_j + \rho)(\sigma) - \sum_i Q_i(\sigma)\left(\log\left(\frac{Q_i(\sigma)}{-i\Lambda}\right) - 1\right).$$

We see that the integrand may grow exponentially depending on the direction of $\mathfrak{t}_{\mathbb{C}}$. This leads us to another condition — (ii) the integrand must decay at infinity of γ. We shall call γ *admissible* for (M, Q) when both of the conditions (i) and (ii) are satisfied. We may also say that γ is admissible with respect to a representation U of G_0 when the decay condition holds if the brane factor is replaced by $\mathrm{tr}_U\, \mathrm{e}^{2\pi\sigma'}$.

We propose that *the classical data* (M, Q, γ) *defines a B-brane in the quantum theory when* γ *is admissible for* (M, Q).

The case of Landau-Ginzburg model

Let us also present the partition functions for the Landau-Ginzburg model of N variables $x_1, \ldots, x_N$ with superpotential $W(x_1, \ldots, x_N)$, on the hemisphere with $\mathrm{B}_{(\pm)}$-type supersymmetry. For the hemisphere with $\mathrm{B}_{(\pm)}$-type supersymmetry, we need to put an $\mathrm{A}_{\mp 1}$-type boundary condition at $|z| = 1$ which is specified by a Lagrangian submanifold $L_\pm \subset \mathbb{C}^N$ such that $\mp\mathrm{Im}(W)$ are bounded below on $L_\pm$. The partition function is

$$Z_{D^2}^{\mathrm{B}_{(+)}}(L_+) = (r\Lambda)^{\frac{N}{2}} \int_{L_+} \exp\Bigl(-irW(x)\Bigr)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_N, \tag{4.27}$$

$$Z_{D^2}^{\mathrm{B}_{(-)}}(L_-) = (r\Lambda)^{\frac{N}{2}} \int_{L_-} \exp\Bigl(-ir\overline{W(x)}\Bigr)\mathrm{d}\overline{x}_1 \wedge \cdots \wedge \mathrm{d}\overline{x}_N. \tag{4.28}$$

These are absolutely convergent or convergent oscillatory integrals.

If $W(x)$ is quasihomogeneous, $W(\lambda^R x) = \lambda^2 W(x)$ for some linear map $x \mapsto Rx$, and if the class of L_+ is invariant under $x \mapsto \lambda^R x$, we see from a change of variables that the partition function (4.27) depends on the radius r simply as an overall power

$$Z_{D^2}^{\mathrm{B}_{(+)}}(L_+) = (r\Lambda)^{\widehat{c}/2} \cdot Z_{D^2}^{\mathrm{B}_{(+)}}(L_+)|_{r\Lambda=1}, \tag{4.29}$$

[2]Each logarithm is required to have imaginary parts in the interval $(-\pi, \pi)$. Note that $\widetilde{W}_{\mathrm{eff},\lambda}(\sigma)$ looks similar to the effective twisted superpotential $\widetilde{W}_{\mathrm{eff}}(\sigma)$ on the Coulomb branch (3.7). There is a differnce though — While $\widetilde{W}_{\mathrm{eff}}(\sigma)$ is defined only modulo shifts by $2\pi i\mathrm{P}(\sigma)$, $\widetilde{W}_{\mathrm{eff},\lambda}(\sigma)$ is defined absolutely by the specific choice of branch of the logarithms. Nevertheless, there is a reason for the similarity, as will be explained momentarily.

where $\widehat{c} = \mathrm{tr}(1-R)$ is one third of the central charge of the SCFT to which the LG model flows in the IR limit. The other function (4.28) has the same dependence. This power behaviour is expected to be a characteristic of the partition function of a superconformal field theory.

On the other hand, if $W(x)$ is a Morse function and if L_+ passes through one of the critical points, the partition function (4.27) in the large radius limit is dominated by the Gaussian integral near the critical point x_*,

$$Z_{D^2}^{\mathrm{B}_{(+)}}(L_+) \sim C_* \exp\Big(-irW(x_*)\Big), \qquad r \to \infty. \tag{4.30}$$

The other function (4.28) has a similar behaviour. This exponential behaviour is expected to be a characteristic of the partition function of a theory with massive vacua.

Rotated supersymmetry

We may also consider the partition functions on the hemisphere with the rotated supersymmetry, $\mathrm{A}_{(\pm)}^{\mathrm{e}^{2i\beta}}$ or $\mathrm{B}_{(\pm)}^{\mathrm{e}^{2i\alpha}}$.

The effect of rotation is easiest to state in the LG model with B-type supersymmetry. It is simply to make the replacement

$$W(x) \to \mathrm{e}^{2i\alpha}W(x), \qquad \overline{W(x)} \to \mathrm{e}^{-2i\alpha}\overline{W(x)} \tag{4.31}$$

in the result, such as (4.27)–(4.28) and (4.30). Note that the brane $L_\pm$ must be an $\mathrm{A}_{\mp\,\mathrm{e}^{2i\alpha}}$-brane on which $\mp\mathrm{Im}(\mathrm{e}^{2i\alpha}W)$ is bounded below, so that the result remains to be an absolutely convergent or convergent oscillatory integral.

For GLSM with A-type supersymmetry, we may employ the classical axial R-symmetry $\mathrm{e}^{i\beta F_A}$. Then, the effect is simply the change of variables $X \to \mathrm{e}^{i\beta F_A}X$ and the rotation of the contour $\gamma \to \mathrm{e}^{-2i\beta}\gamma$, which does not seem to change anything: we may simply denote $r\,\mathrm{e}^{2i\beta}\sigma$ again by σ'. However, this change of variables is possibly anomalous, and will shift the theta angle as $\theta \to \theta + 2\beta\mathrm{tr}_V$. The precise effect is executed by the replacement

$$\Lambda \to \mathrm{e}^{2i\beta}\Lambda, \quad \sigma \to \mathrm{e}^{2i\beta}\sigma, \quad \gamma \to \mathrm{e}^{-2i\beta}\gamma, \tag{4.32}$$

with σ' and γ' unchanged, in all of the formula above for GLSM within this subsection. This in particular changes the asymptotic behaviour of the integrand (4.25) to

$$\text{a power of } \sigma \times \exp\left(-ir\,\mathrm{e}^{2i\beta}\widetilde{W}_{\mathrm{eff},\lambda_j}(\sigma)\right), \tag{4.33}$$

with the expression (4.26) for $\widetilde{W}_{\mathrm{eff},\lambda_j}(\sigma)$ unchanged.

Remarks

(i) Let us write down the expression for the partition functions on the sphere. Let $\mathrm{Q}^\vee \subset i\mathfrak{t}$ be the cocharacter lattice, that is, the set of elements that takes integer values on the weight lattice $\mathrm{P} \subset i\mathfrak{t}^*$. Then, the partition function of the GLSM on the sphere with the A-type supersymmetry is

$$\begin{aligned} Z^{\mathrm{A}}_{S^2} = C|r\Lambda|^{\widehat{c}} \sum_{m\in \mathrm{Q}^\vee} \int_{i\mathfrak{t}} \mathrm{d}^\ell \sigma' \\ \times \prod_{\alpha>0}\left(\frac{\alpha(m)^2}{4}+\alpha(\sigma')^2\right)\prod_i \frac{\Gamma\left(iQ_i(\sigma')-\frac{Q_i(m)}{2}+\frac{R_i}{2}\right)}{\Gamma\left(1-iQ_i(\sigma')-\frac{Q_i(m)}{2}-\frac{R_i}{2}\right)} \\ \times \exp\Big(2i\zeta_R(\sigma')+i(\theta_R+2\pi\rho)(m)\Big). \end{aligned} \tag{4.34}$$

Here $t_R = \zeta_R - i\theta_R$. The integrand grows as a power in magnitude but it rapidly oscillates along the contour $i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$ of integration. Thus, it is convergent though not absolutely. The partition function of the LG model on the sphere with B-type supersymmetry is

$$Z^{\mathrm{B}}_{S^2} = |r\Lambda|^N \int_{\mathbb{C}^N} \exp\left(-ir(W(x)+\overline{W(x)})\right) \mathrm{d}^{2N}x. \tag{4.35}$$

This is also a convergent oscillatory integral.

(ii) The similarity between $\widetilde{W}_{\mathrm{eff},\lambda_j}(\sigma)$ in (4.26) and $\widetilde{W}_{\mathrm{eff}}(\sigma)$ in (3.7) is not a coincidence. If $\widetilde{W}_{\mathrm{eff}}(\sigma)$ has a non-degenerate critical point σ_*, there is a Coulomb vacuum with a mass gap. If γ passes through σ_*, the hemisphere partition function should behave in the large radius limit $r \to \infty$ as (4.30), that is, $\sim \exp\left(-ir\widetilde{W}_{\mathrm{eff}}(\sigma_*)\right)$. Note that the critical value is $\widetilde{W}_{\mathrm{eff}}(\sigma_*) = b_1(\sigma_*)$ and is not affected by the $2\pi i \mathrm{P}(\sigma)$ ambiguity of $\widetilde{W}_{\mathrm{eff}}(\sigma)$ — see the discussion around (3.9). This is indeed the behaviour of the integral (4.19) provided $\widetilde{W}_{\mathrm{eff},\lambda_j}(\sigma) \equiv \widetilde{W}_{\mathrm{eff}}(\sigma) \bmod 2\pi i\mathrm{P}(\sigma)$. This comparison yields the precise relationship between the two Λ's, the one in (3.7) for $\widetilde{W}_{\mathrm{eff}}(\sigma)$ and the one for the hemisphere partition function which appears in (4.26):

$$\Lambda|_{(3.7)} = -i\Lambda|_{(4.26)}. \tag{4.36}$$

(iii) Recall that there is an unphysical ambiguity (3.20) in the R-charge when $\mathfrak{g}$ has a non-zero center $\mathfrak{z}$. Let us see the effect of this change, $R \to R+\varepsilon$ and $\mathbf{r} \to \mathbf{r}-\varepsilon$, on the hemisphere partition function (4.19).

This affects the exponent $\widehat{c}$ of the prefactor as $\widehat{c} \to \widehat{c} - b_1(\varepsilon)$ as well as the Gamma function factors and the brane factor. The effect of these functions is absorbed by the change of variables $\sigma' \to \sigma' + \frac{i}{2}\varepsilon$. The net effect is an overall factor $\mathrm{e}^{-t(\varepsilon)/2}$ and the shift of the contour $\gamma' \to \gamma' - \frac{i}{2}\varepsilon$. As long as the bound (4.18) is not violated by the shift $R \to R + \varepsilon$, we can bring the contour back to γ' without hitting the poles. Thus, the effect is simply

$$Z_{D^2}^{\mathrm{A}(+)} \longrightarrow \mathrm{e}^{-\frac{1}{2}t(\varepsilon)} Z_{D^2}^{\mathrm{A}(+)}. \tag{4.37}$$

The effect on the other functions are $Z_{D^2}^{\mathrm{A}(-)} \to \mathrm{e}^{-\frac{1}{2}\overline{t}(\varepsilon)} Z_{D^2}^{\mathrm{A}(-)}$ and $Z_{S^2}^{\mathrm{A}} \to \mathrm{e}^{-\frac{1}{2}t(\varepsilon) - \frac{1}{2}\overline{t}(\varepsilon)} Z_{S^2}^{\mathrm{A}}$.

(iv) The expression (4.19) can be simplified a little. Let $M = \oplus_j U_j(r_j)$ be the irreducible decomposition with respect to the identity component G_0 of G. If U is the irreducible representation of G_0 with highest weight λ, we have the Weyl character formula

$$\mathrm{tr}_U\, \mathrm{e}^{2\pi\sigma'} = \frac{\sum_{w \in W_0} (-1)^{\ell(w)}\, \mathrm{e}^{2\pi w(\lambda+\rho)(\sigma')}}{\prod_{\alpha>0}\left(\mathrm{e}^{\pi\alpha(\sigma')} - \mathrm{e}^{-\pi\alpha(\sigma')}\right)}. \tag{4.38}$$

Notice that the denominator is nothing but the factor $\prod_{\alpha>0}\sinh(\pi\alpha(\sigma'))$ in (4.19) up to a factor of 2. Note also that $\prod_{\alpha>0}\alpha(\sigma')$ is Weyl odd. Using these as well as the Weyl invariance of γ, we see that the partition function can be written as

(4.39)
$$\begin{aligned} Z_{D^2}^{\mathrm{A}(+)}(M, Q, \gamma) &= C\frac{|W_0|}{2^{|\Delta^+|}}(r\Lambda)^{\widehat{c}/2}\sum_j \mathrm{e}^{\pi i r_j}\int_{\gamma'} \mathrm{d}^\ell\sigma' \\ &\times \prod_{\alpha>0}\alpha(\sigma')\prod_i \Gamma\left(iQ_i(\sigma') + \frac{R_i}{2}\right)\exp\Big(it_R(\sigma') + 2\pi(\lambda_j+\rho)(\sigma')\Big), \end{aligned}$$

where $|\Delta^+|$ is the number of positive roots. In fact, this has been used in finding the behaviour (4.25).

(v) The gamma function has an integral expression

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\mathrm{d}t, \tag{4.40}$$

when the real part of z is positive. The same holds for other range of z provided the contour of integration is chosen appropriately. Applying this for each gamma function in (4.39), we find

$$Z_{D^2}^{\mathrm{A}_{(+)}}(M,Q,\gamma) = C\frac{|W_0|}{2^{|\Delta^+|}\Lambda^{\ell+|\Delta^+|}}(r\Lambda)^{\frac{\ell+d_V}{2}}\sum_j \mathrm{e}^{\pi i r_j} \times \int_\Gamma \mathrm{d}^\ell\sigma\,\mathrm{d}^{d_V}y\prod_{\alpha>0}\alpha(\sigma)\prod_i \mathrm{e}^{-\frac{R_i}{2}y_i}\exp\Big(-ir\widetilde{W}(\sigma,y)\Big), \tag{4.41}$$

with $d_V = \dim_{\mathbb{C}} V$ and

$$\widetilde{W}(\sigma,y) = \left(\sum_i Q_i y_i - t + 2\pi i(\lambda_j+\rho)\right)(\sigma) - i\Lambda\sum_i \mathrm{e}^{-y_i}, \tag{4.42}$$

for a contour $\Gamma \subset \mathfrak{t}_{\mathbb{C}} \times \mathbb{C}^{d_V}$ that projects onto $\gamma \subset \mathfrak{t}_{\mathbb{C}}$. Compared to (4.27), the formula (4.41) looks to be the hemisphere partition function $Z_{D^2}^{\mathrm{B}}$ for an A-brane in the LG model with superpotential (4.42). In fact, (4.41) is nothing but the formula for the central charge of the B-brane in the mirror theory [32], except that the dependence on the brane is made more precise and that the mirror superpotential modulo $2\pi i\mathrm{P}(\sigma)$ is corrected by a shift by $2\pi i\rho(\sigma)$.

4.4. The example $\mathrm{T}_{N,d}^{U(1)}$

Let us look at the hemisphere partition function $Z_{D^2} = Z_{D^2}^{\mathrm{A}_{(+)}}$ in the example $\mathrm{T}_{N,d}^{U(1)}$. We assign R-charge $2-d\epsilon$ on p and ϵ on x_i's with $0<\epsilon<2/d$, so that the bound (4.18) is satisfied. Then, we have

$$Z_{D^2}(M,Q,\gamma) = (r\Lambda)^{\widehat{c}/2}\int_\gamma \mathrm{d}\sigma'\,\Gamma\left(-di\sigma'+1-\tfrac{d\epsilon}{2}\right)\Gamma\left(i\sigma'+\tfrac{\epsilon}{2}\right)^N \times \mathrm{e}^{it_R\sigma'}f_M(\sigma') \tag{4.43}$$

where $\widehat{c} = N-2-(N-d)\epsilon$, $t_R = t-(N-d)\log(r\Lambda)$, and

$$f_M(\sigma') = \mathrm{tr}_M\,\mathrm{e}^{\pi i\mathbf{r}+2\pi\sigma'} = \sum_j \mathrm{e}^{\pi i r_j+2\pi q_j\sigma'}, \tag{4.44}$$

for the $U(1)_V \times U(1)$ weight decomposition

$$M = \bigoplus_j \mathbb{C}(r_j,q_j). \tag{4.45}$$

The Gamma function factor of the integrand has order N poles at $\sigma' = i\left(n_x+\frac{\epsilon}{2}\right)$ with $n_x = 0,1,2,\ldots$ and simple poles at $\sigma' = i\left(-\frac{n_p+1}{d}+\frac{\epsilon}{2}\right)$ with $n_p = 0,1,2,\ldots$. We shall call them *x-poles* and *p-poles* respectively.

The x-poles are on the positive imaginary axis and the p-poles are on the negative imaginary axis. By Stirling's formula, the j-th term of the integrand behaves as $\mathrm{e}^{-A_{q_j}(\sigma')}$ up to a power factor, where

$$(4.46)\quad \begin{aligned} A_q(\sigma') = \mathrm{Im}(\sigma')\left(\zeta - d\log d + (N-d)\left(\log\left|\frac{\sigma'}{r\Lambda}\right| - 1\right)\right) \\ + |\mathrm{Re}(\sigma')|\left(\frac{N+d}{2}\pi + (N-d)\arctan\left[\frac{\mathrm{Im}(\sigma')}{|\mathrm{Re}(\sigma')|}\right]\right) \\ - \mathrm{Re}(\sigma')(\theta + 2\pi q). \end{aligned}$$

The contour γ is admissible with repect to the charge q when A_q goes to positive infinity at the ends of γ. We see that the behaviour of A_q depends very much on whether $(N-d)$ is zero, positive or negative. Below, we shall discuss these cases separately.

Important rôles will be played by the brane data $\mathfrak{B}_1$ and $\mathfrak{B}_2$ introduced earlier. They have the following brane factors

$$(4.47)\qquad f_{M_1}(\sigma') = 1 - \mathrm{e}^{-\pi i d\epsilon}\,\mathrm{e}^{2\pi d\sigma'},$$

$$(4.48)\qquad f_{M_2}(\sigma') = \sum_{j=0}^{N}\binom{N}{j}\mathrm{e}^{\pi i j(1-\epsilon)}\,\mathrm{e}^{2\pi j\sigma'} = (1 - \mathrm{e}^{-\pi i\epsilon}\,\mathrm{e}^{2\pi\sigma'})^N.$$

We see that $f_{M_1}(\sigma')$ has simple zero at $\sigma' = i\left(\frac{n}{d} + \frac{\epsilon}{2}\right)$, while $f_{M_2}(\sigma')$ has N-th order zero at $\sigma' = i\left(n + \frac{\epsilon}{2}\right)$, both for $n \in \mathbb{Z}$. In particular, the integrand for $\mathfrak{B}_1$ has vanishing residue at the p-poles but has non-zero residues at the x-poles, while the integrand for $\mathfrak{B}_2$ has vanishing residue at the x-poles but has non-zero residues at some of the p-poles (four out of the five series). Such a property does not change under the shift $\mathfrak{B} \mapsto \mathfrak{B}(i,q)$ which does $f_M(\sigma') \mapsto f_M(\sigma')(-1)^i\,\mathrm{e}^{2\pi q\sigma'}$.

$\underline{d = N}$ Family of conformal field theories

Recall that the model with $d = N$ defines a family of SCFTs parameterized by $t = \zeta - i\theta$, with $\widehat{c} = N - 2$. In the presence of boundary, the 2π periodicity of θ is lost, and we should consider the unwrapped moduli space $\widetilde{\mathfrak{M}}_t = \mathbb{C} - \widetilde{\Delta}$, where $\widetilde{\Delta}$ is the set of singular points, $\zeta = N\log N$ and $\theta \in N\pi + 2\pi\mathbb{Z}$. $\zeta \gg 0$ is the geometric phase and $\zeta \ll 0$ is the Landau-Ginzburg orbifold phase. By the general principle of supersymmetry, we expect to have an equivalence $D^b_{\mathrm{Coh}}(X_f) \cong \mathrm{MF}_{\mathbb{Z}_N}(f)$ for each homotopy class of paths in $\widetilde{\mathfrak{M}}_t$ between the two phases, as well as an autoequivalence for each loop around a singular point.

The dependence of Z_{D^2} on the radius r is simply the overall power

$$(4.49)\qquad Z_{D^2}(\mathfrak{B},\gamma) = (r\Lambda)^{\frac{N-2}{2}} \cdot Z_{D^2}(\mathfrak{B},\gamma)|_{r\Lambda=1}.$$

This is indeed the characteristic behaviour of the partition function for a superconformal field theory with $\widehat{c} = N - 2$. The growth function A_q also simplifies for $d = N$ to

$$(4.50)\quad A_q(\sigma') = (\zeta - N\log N)\mathrm{Im}(\sigma') + N\pi|\mathrm{Re}(\sigma')| - (\theta + 2\pi q)\mathrm{Re}(\sigma').$$

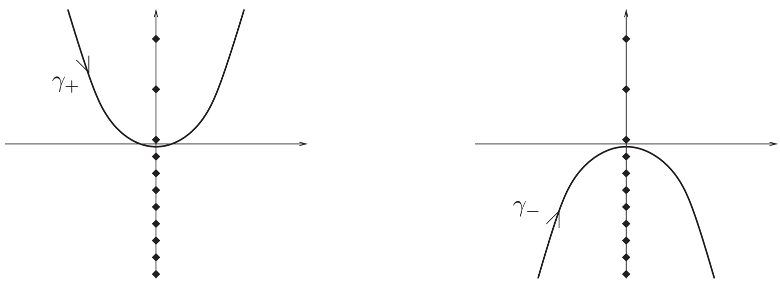

Fig. 2. Contours γ_+ and γ_-

In the geometric phase $\zeta \gg 0$, the function A_q for any q blows up linearly and the integrand decays expotentially in the direction where $\mathrm{Im}(\sigma')/|\mathrm{Re}(\sigma')|$ is large positive. Therefore, if we take $\gamma = \gamma_+$ as in Fig 2, it is admissible with respect to any q and hence for any (M, Q). The contour γ_+ can be deformed further up toward the positive imaginary direction, so that the integral becomes the sum of residues at the x-poles. This shows $Z_{D^2}(\mathfrak{B}_2(i,q), \gamma_+) = 0$ and $Z_{D^2}(\mathfrak{B}_1(i,q), \gamma_+) \neq 0$. This is consistent with the fact in the geometric regime that $\mathfrak{B}_2(i,q)$ is empty but $\mathfrak{B}_1(i,q)$ is not. In particular, the hemisphere partition function does not change under the brane replacement $\mathfrak{B} \rightsquigarrow \mathfrak{B}'$ introduced in Section 3.2, $Z_{D^2}(\mathfrak{B}, \gamma_+) = Z_{D^2}(\mathfrak{B}', \gamma_+)$, since $\mathfrak{B}'$ is obtained from $\mathfrak{B}$ by binding $\mathfrak{B}_2$ and its shifts and by cancelling identical pairs.

In the Landau-Ginzburg phase $\zeta \ll 0$, the function A_q for any q blows up linearly and the integrand decays expotentially in the direction where $\mathrm{Im}(\sigma')/|\mathrm{Re}(\sigma')|$ is large negative. Therefore, if we take $\gamma = \gamma_-$ as in Fig 2, it is admissible with respect to any q and hence for any (M, Q). The contour γ_- can be deformed further down toward the negative imaginary direction, so that the integral becomes the sum of residues at the p-poles. This shows $Z_{D^2}(\mathfrak{B}_1(i,q), \gamma_-) = 0$ and $Z_{D^2}(\mathfrak{B}_2(i,q), \gamma_-) \neq 0$. This is consistent with the fact in the Landau-Ginzburg regime that $\mathfrak{B}_1(i,q)$ is empty but $\mathfrak{B}_2(i,q)$ is not. In particular, the hemisphere partition function does not change under the brane replacement, $Z_{D^2}(\mathfrak{B}, \gamma_-) = Z_{D^2}(\mathfrak{B}', \gamma_-)$, since $\mathfrak{B}'$ is obtained from $\mathfrak{B}$ by binding $\mathfrak{B}_1$ and its shifts and by cancelling identical pairs.

On the line $\zeta = N \log N$, the dependence on $\mathrm{Im}(\sigma')$ disappears from (4.50),

$$A_q(\sigma') = N\pi|\mathrm{Re}(\sigma')| - (\theta + 2\pi q)\mathrm{Re}(\sigma'). \tag{4.51}$$

Note that $\theta+2\pi q$ never coinsides with $\pm N\pi$ unless t is one of the singular points. If $|\theta+2\pi q| > N\pi$, the function A_q goes down to negative infinity for either $\mathrm{Re}(\sigma') \to +\infty$ or $\mathrm{Re}(\sigma') \to -\infty$; No γ is admissible with respect to q. On the other hand, if $|\theta + 2\pi q| < N\pi$, it goes up to positive infinity for both $\mathrm{Re}(\sigma') \to \pm\infty$; Admissibles are the real locus $\gamma = \mathbb{R}$ as well as any of its deformation as long as it remains to extend to $\mathrm{Re}(\sigma') \to \pm\infty$. To summarize, at $\zeta = N \log N$, there is a unique homotopy class of admissible contours if and only if

$$-\frac{N}{2} < q + \frac{\theta}{2\pi} < \frac{N}{2}. \tag{4.52}$$

This yields the *grade restriction rule* concerning D-brane transport along paths in $\widetilde{\mathfrak{M}}_t$ from one phase to another. Such a path must go through a window on the interface $\zeta = N \log N$, $-N\pi + 2\pi n < \theta < -N\pi + 2(n+1)\pi$ for some $n \in \mathbb{Z}$. For each window $\mathbf{w}$, the bound (4.52) defines a set $[\mathbf{w}]$ of N consecutive integers. For example, if $\mathbf{w}$ is $(-N-2)\pi < \theta < -N\pi$ at $\zeta = N \log N$, the set is $[\mathbf{w}] = \{1, 2, \ldots, N\}$. If $\mathbf{w}'$ is the next one on the right, $-N\pi < \theta < (-N+2)\pi$, the set is $[\mathbf{w}'] = \{0, 1, 2, \ldots, N-1\}$. We shall say that a brane data $\mathfrak{B} = (M, Q)$ is *grade restricted with respect to* $\mathbf{w}$ when all the gauge charges of M belong to the set $[\mathbf{w}]$.

Suppose a brane data $\mathfrak{B}$ is grade restricted with respect to a window $\mathbf{w}$. Then, one can find a family of contours γ along a path through $\mathbf{w}$, interpolating γ_+ at $\zeta \gg 0$ and γ_- at $\zeta \ll 0$, so that it is admissible for $\mathfrak{B}$ all the way. This defines a family of quantum B-branes along the path. This is the rule of D-brane transport in the grade restricted case. Since the integral (4.43) is absolutely convergent all the way, $Z(\mathfrak{B}, \gamma_+)$ at $\zeta \gg 0$ and $Z(\mathfrak{B}, \gamma_-)$ at $\zeta \ll 0$ are related by the analytic continuation along the path.

Suppose, on the other hand, $\mathfrak{B}$ is not grade restricted with respect to $\mathbf{w}$. Then, a family of admissible contours does not exist along any path through $\mathbf{w}$. The brane $(\mathfrak{B}, \gamma_+)$ does make sense in the geometric phase $\zeta \gg 0$, but it cannot go to the LG phase $\zeta \ll 0$ through the window $\mathbf{w}$. In such a situation, we employ the idea of *brane replacement* introduced in Section 3.2. While in the geometric phase $\zeta \gg 0$, we can replace $\mathfrak{B}$ by another data $\mathfrak{B}'$ whose gauge charges belong to a set of N consecutive integers without changing the low energy behaviour

— without changing the hemisphere partition function $Z_{D^2}(\mathfrak{B},\gamma_+) = Z_{D^2}(\mathfrak{B}',\gamma_+)$ in particular. If we choose the set to be $[\mathbf{w}]$, then, $\mathfrak{B}'$ is grade restricted with respect to $\mathbf{w}$. Then, we can find a family of contours γ along a path through $\mathbf{w}$, starting from γ_+ in $\zeta \gg 0$ and ending with γ_- in $\zeta \ll 0$, which is admissible for $\mathfrak{B}'$ all the way. This defines a family of quantum B-branes along the path, starting with $(\mathfrak{B},\gamma_+) \cong (\mathfrak{B}',\gamma_+)$ in the geometric phase and ending with $(\mathfrak{B}',\gamma_-)$ in the LG phase. This is the rule of D-brane tranport. In particular, $Z_{D^2}(\mathfrak{B},\gamma_+)$ in $\zeta \gg 0$ analytically continues along the path to $Z_{D^2}(\mathfrak{B}',\gamma_-)$ in $\zeta \ll 0$. The same works for the transport backward. If we start from a brane $(\mathfrak{B},\gamma_-)$ in the LG phase $\zeta \ll 0$, we first find a grade restricted representative $(\mathfrak{B}'',\gamma_-)$ before the transport, and then go through the window $\mathbf{w}$. The analytic continuation of $Z_{D^2}(\mathfrak{B},\gamma_-)$ in $\zeta \ll 0$ ends with $Z_{D^2}(\mathfrak{B}'',\gamma_+)$ in $\zeta \gg 0$.

This rule allows us to find the monodromy. Let us consider a loop around one of the singular points, say, $t_* = N\log N + N\pi i$, with the base point in the geometric phase $\zeta \gg 0$. If the loop goes counter-clockwise, it may be regarded as a concatenation of a path from $\zeta \gg 0$ to $\zeta \ll 0$ through the window $\mathbf{w}$ on the left of t_* and a path from $\zeta \ll 0$ to $\zeta \gg 0$ through the window $\mathbf{w}'$ on the right of t_*. Let us start from a brane $(\mathfrak{B},\gamma_+)$ at $\zeta \gg 0$. While in the geometric phase, we replace $\mathfrak{B}$ by $\mathfrak{B}'$ which is grade restricted with respect to $\mathbf{w}$ and then move $(\mathfrak{B}',\gamma)$ through $\mathbf{w}$ to $\zeta \ll 0$. While in the LG phase $\zeta \ll 0$, we replace $\mathfrak{B}'$ by $\mathfrak{B}''$ which is grade restricted with respect to $\mathbf{w}'$, and then move $(\mathfrak{B}'',\gamma)$ through $\mathbf{w}'$ back to $\zeta \gg 0$. Thus, we end up with $(\mathfrak{B}'',\gamma_+)$.

By now it should be clear how to describe the (auto)equivalences of categories. Recall the definition of $\mathcal{T}_I \subset \mathfrak{D}_{\mathrm{LSM}}$ and the equivalences (3.34) and (3.35) in the two phases. The transport along a path through a window $\mathbf{w}$ results in the equivalence

$$D^b_{\mathrm{Coh}}(X_f) \xrightarrow{\pi_+^{-1}} \mathcal{T}_{[\mathbf{w}]} \xrightarrow{\pi_-} \mathrm{MF}_{\mathbb{Z}_N}(f). \tag{4.53}$$

The monodromy along the loop in the previous paragraph is

$$D^b_{\mathrm{Coh}}(X_f) \xrightarrow{\pi_+^{-1}} \mathcal{T}_{[\mathbf{w}]} \xrightarrow{\pi_-} \mathrm{MF}_{\mathbb{Z}_N}(f) \xrightarrow{\pi_-^{-1}} \mathcal{T}_{[\mathbf{w}']} \xrightarrow{\pi_+} D^b_{\mathrm{Coh}}(X_f). \tag{4.54}$$

$\underline{d < N}$ Flow from sigma model

The model with $d < N$ describes an RG flow from the sigma mode with target X_f to the LG orbifold $W = f(x)/\mathbb{Z}_d$ or one of the $(N-d)$ massive vacua at

$$\sigma_k = -i\widetilde{\Lambda}\exp\left(i\,\frac{\theta+\pi d+2\pi k}{N-d}\right), \qquad k \in \mathbb{Z}/(N-d)\mathbb{Z}, \tag{4.55}$$

where $\widetilde{\Lambda} = \Lambda d^{\frac{d}{N-d}}\,\mathrm{e}^{-\frac{\zeta}{N-d}}$. Here Λ is the scale parameter for the hemisphere partition function which we assume to be real positive. See (3.17) and (4.36). The value of the twisted superpotential at these vacua are $\widetilde{W}_{\text{eff}}(\sigma_k) = (N-d)\sigma_k$. Note that the UV limit has $\widehat{c}_{\text{LV}} = N-2$ while the LG orbifold flows further to an SCFT with $\widehat{c}_{\text{LG}} = N(1-2/d)$. We expect that $D^b_{\text{Coh}}(X_f)$ is equivalent to a category that includes $\text{MF}_{\mathbb{Z}_d}(f)$ and $(N-d)$ exceptional objects.

Let us study the behaviour of the growth function (4.46). It can be written as

$$A_q(\sigma') = \zeta_{\text{eff}}\text{Im}(\sigma') + N_{\text{eff}}\pi|\text{Re}(\sigma')| - (\theta+2\pi q)\text{Re}(\sigma'), \tag{4.56}$$

where

$$\begin{aligned} \zeta_{\text{eff}} &= (N-d)\left(\log\left|\frac{\sigma'}{r\widetilde{\Lambda}}\right| - 1\right), \\ N_{\text{eff}} &= \frac{N+d}{2} + \frac{N-d}{\pi}\arctan\left[\frac{\text{Im}(\sigma')}{|\text{Re}(\sigma')|}\right]. \end{aligned} \tag{4.57}$$

ζ_{eff} is positive outside the circle $|\sigma'| = r\widetilde{\Lambda}\,\mathrm{e}$ and negative inside. N_{eff} is bounded as $d < N_{\text{eff}} < N$ — the lower (*resp.* upper) bound is approached in the direction of the negative (*resp.* positive) imaginary axis.

The fact that ζ_{eff} is large positive for large enough $|\sigma'|$ and that N_{eff} is bounded as $d < N_{\text{eff}} < N$ means that γ_+ in Fig. 2 is admissible with respect to any q. Thus for any brane data $\mathfrak{B} = (M, Q)$, γ_+ is admissible and $(\mathfrak{B}, \gamma_+)$ defines a D-brane in the quantum theory. If we wish, we can deform γ_+ further up in the positive imaginary direction, and the integral (4.19) is written as the sum of residues at the x-poles. This shows

$$Z_{D^2}(\mathfrak{B}_2(i,q), \gamma_+) = 0. \tag{4.58}$$

This vanishing allows us to employ the brane replacement using $\mathfrak{B}_2$ and its shifts, so that the brane data $\mathfrak{B}$ can be taken from $\mathcal{T}_{I_+}$ for a set I_+ of N consecutive integers. In what follows, we study the behaviour of $Z_{D^2}(\mathfrak{B}, \gamma_+)$ in the small radius limit $r\Lambda \searrow 0$ (UV limit) as well as in the large radius limit $r\Lambda \nearrow \infty$ (IR limit).

When $r\Lambda \ll 1$, ζ_{eff} quickly becomes large positive as σ' goes away from the origin, and the above deformation of contour is the best choice, resulting in the sum of residues at the x-poles. Since t_R is large positive for $r\Lambda \ll 1$, the dominant is the residue at the first pole, $\sigma' = i\epsilon/2$, whose depenence on the radius is simply

$$Z_{D^2} \sim (r\Lambda)^{\widehat{c}/2}\,\mathrm{e}^{it_R\cdot i\epsilon/2} \sim (r\Lambda)^{\frac{N-2}{2}}. \tag{4.59}$$

This is indeed the characteristic behaviour for an SCFT with $\widehat{c}_{\rm LV} = N-2$. We will find the precise expression of the residue in Section 4.5 which will show that it is non-zero whenever the geometric image of the brane data $\mathfrak{B}$ is non-trivial.

When $r\Lambda \gg 1$, $\zeta_{\rm eff}$ is negative over an extended region $|\sigma'| < r\widetilde{\Lambda}\,{\rm e}$ inside which A_q is positive (*resp.* negative) in a neighborhood of the negative (*resp.* positive) imaginary axis. Therefore, we may deform the contour γ_+ to $\widetilde{\gamma}_+$ as in Fig. 3. The new contour can be separated into two

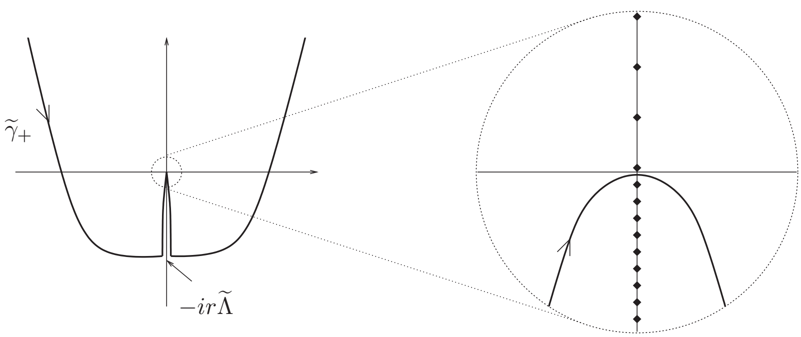

Fig. 3. Deformed contour $\widetilde{\gamma}_+$

parts, $\widetilde{\gamma}_+ = \widetilde{\gamma}_{\rm cent} + \widetilde{\gamma}_{\rm rest}$, where $\widetilde{\gamma}_{\rm cent}$ is the central part that encircles a large number of p-poles. Integration over $\widetilde{\gamma}_{\rm cent}$ yields the sum of residues at the p-poles. Since t_R is large negative for $r\Lambda \gg 1$, the dominant is the residue at the first pole, $\sigma' = i(-1/d + \epsilon/2)$, whose dependence on the radius is simply

$$Z_{D^2}|_{\rm central} \sim (r\Lambda)^{\widehat{c}/2}\,{\rm e}^{it_R\cdot i(-1/d+\epsilon/2)} \sim (r\Lambda)^{\frac{N(1-2/d)}{2}}. \tag{4.60}$$

This is the characteristic behaviour for a brane in an SCFT with $\widehat{c}_{\rm LG} = N(1-2/d)$. To see what the rest $\widetilde{\gamma}_{\rm rest}$ gives, let us examine whether the integrand has critical points. When $|\sigma'| \sim r\widetilde{\Lambda} \gg 1$ and ${\rm Re}(\sigma') \neq 0$, we can use the asymptotic behaviour (4.25) and look for the critical points of $\widetilde{W}_{{\rm eff},q}(\sigma)$. That is to find solutions to $\partial_\sigma \widetilde{W}_{{\rm eff},q} = 0$ literally, not modulo $2\pi i\mathbb{Z}$. It turns out that this equation is equivalent to

$$|\sigma'| = r\widetilde{\Lambda}, \qquad (\theta + 2\pi q){\rm sgn}({\rm Re}(\sigma')) = N_{\rm eff}\pi. \tag{4.61}$$

We see that there is a solution only when $d\pi < |\theta + 2\pi q| < N\pi$. In such a case, the solution is unique and is nothing but one of the Coulomb vacua in (4.55) — it is $\sigma_{k(q)}$ with $k(q) = q$ (*resp.* $q - d$) when $\theta + 2\pi q$ is in $(-N\pi, -d\pi)$ (*resp.* $(d\pi, N\pi)$). The contour $\widetilde{\gamma}_{\rm rest}$ can be chosen to go through it, and the saddle point approximation to the integral gives

$$\text{const} \times \exp\left(-ir\widetilde{W}_{\text{eff},q}(\sigma_{k(q)})\right). \tag{4.62}$$

This is the characteristic behaviour for a brane supported at the massive vacuum $\sigma_{k(q)}$. When $|\theta + 2\pi q| < d\pi$, there is no critical point. In this case, we can choose $\widetilde{\gamma}_{\rm rest}$ along which the integrand is very small entirely, and can show that the integral vanishes exponentially as $r\Lambda \to \infty$, much faster than any of $\mathrm{e}^{-ir\widetilde{W}_{\rm eff}(\sigma_k)}$'s. When $|\theta+2\pi q| > N$, there is no solution again. In this case, one cannot avoid $\widetilde{\gamma}_{\rm rest}$ to go through a region where the integrand is large, and it is hard to estimate the integral. However, we may avoid the case $|\theta+2\pi q| > N$ to begin with, via brane replacement using $\mathfrak{B}_2$ and its shifts. The special case $|\theta+2\pi q| = d\pi$ (*resp.* $N\pi$) arizes when $\theta \equiv d\pi$ (*resp.* $N\pi$) modulo $2\pi\mathbb{Z}$. Then, one of the Coulomb vacua σ_k is on the negative (*resp.* positive) imaginary axis where Stirling's formula breaks down. In order to avoid possible complications, we shall assume $\theta \not\equiv d\pi, N\pi$ modulo $2\pi\mathbb{Z}$.

Under this assumption, $[\theta; d]$ and $[\theta; N]$, with

$$[\theta; m] := \left\{ q \in \mathbb{Z} \,\middle|\, -\frac{m}{2} < q + \frac{\theta}{2\pi} < \frac{m}{2} \right\}, \tag{4.63}$$

are sets of d and N consecutive integers. The above analysis leads us to the following claim concerning the low energy behaviour of a brane $(\mathfrak{B}, \gamma_+)$. If $\mathfrak{B}$ belongs to $\mathcal{T}_{[\theta;d]}$, the brane flows purely to a brane in the LG orbifold. If $\mathfrak{B}$ belongs to $\mathcal{T}_{[\theta;N]}$ but not to $\mathcal{T}_{[\theta;d]}$, there is at least one gauge charge q such that $d\pi < |\theta + 2\pi q| < N\pi$. Let q_* be the one that maximizes $\mathrm{Im}\widetilde{W}_{\text{eff},q}(\sigma_{k(q)})$ among such q's. Then, the brane at low energies has a component supported at the Coulomb vacuum $\sigma_{k(q_*)}$.

Let us examine the case of $\mathfrak{B} = \mathfrak{B}_1(i, q)$ which reduces to the shifted line bundle $\mathcal{O}(q)[i]$ in the high energy sigma model. For any q, it does not belong to $\mathcal{T}_{[\theta;d]}$ since the set of gauge charges, which is $\{q, q + d\}$, cannot fit into $[\theta; d]$. It belongs to $\mathcal{T}_{[\theta;N]}$ when q is one of the $(N - d)$ integers satisfying $-N\pi < \theta + 2\pi q < (N - 2d)\pi$. In this case, q_* is q if $\theta + 2\pi q$ is in $(-N\pi, -d\pi)$ and $q + d$ if $\theta + 2\pi q$ is in $(-d\pi, (N - 2d)\pi)$. In either case, $k(q_*)$ is q and hence the brane has a component supported at σ_q. In fact, one can show that it is supported purely at σ_q. Therefore, for q in this range, the low energy limit of the line bundle $\mathcal{O}(q)$ on X_f is a brane supported purely at the Coulomb vacuum σ_q. When q is outside this range, we need to use its replacement in $\mathcal{T}_{[\theta;N]}$, obtained by binding $\mathfrak{B}_2$ and its shifts. Since $\mathfrak{B}_2$ and its shifts have non-zero residues at the p-poles, the low energy brane may have an SCFT component as well as components supported at Coulomb vacua. For example, let us consider $\mathfrak{B}_1$ in the theory with $\theta = -N\pi + \delta$ for a small negative δ, for which $[\theta; N] = \{1, 2, \ldots, N\}$. Obviously $\mathfrak{B}_1$ is not in $\mathcal{T}_{[\theta;N]}$ since its set of gauge charges is $\{0, d\}$. Its representative in $\mathcal{T}_{[\theta;N]}$ can be obtained by binding $\mathfrak{B}_1$ with $\mathfrak{B}_2(-1, 0)$,[3] which gives

(4.64)

$$\mathbb{C}(1,d) \underset{f'}{\overset{px}{\rightleftarrows}} \mathbb{C}(0,1)^N \underset{pf'}{\overset{x}{\rightleftarrows}} \mathbb{C}(1,2)^{\binom{N}{2}} \underset{pf'}{\overset{x}{\rightleftarrows}} \cdots \underset{pf'}{\overset{x}{\rightleftarrows}} \mathbb{C}(N-1,N)$$

where $x = \sum_i x_i \overline{\eta}_i$ and $f' = \sum_i \partial_i f \eta_i / d$. Note that the last component $\mathbb{C}(N-1, N)$ yields a brane supported at σ_0 which has the highest value of $\mathrm{Im}\widetilde{W}_{\mathrm{eff}}(\sigma_k)$. This shows that the low energy limit of the structure sheaf $\mathcal{O}$ of X_f includes the Recknagel-Shomerus brane $\pi_-(\mathfrak{B}_2(-1,0))$ of the SCFT and is also supported at some of the Coulomb vacua including σ_0.

What happens when θ crosses the special values, $d\pi$ and $N\pi$ modulo $2\pi\mathbb{Z}$? As remarked above, as θ crosses $d\pi$ (*resp.* $N\pi$) modulo $2\pi\mathbb{Z}$, one of the Coulomb vacua crosses the negative (*resp.* positive) imaginary axis on which there are p-poles (*resp.* x-poles). In fact, this is where there is a zero mode for p (*resp.* x) localized near the boundary and something special can happen. The example in the last paragraph can be used to illustrate what happens when θ crosses $N\pi$ (mod 2π), say, from $\theta > -N\pi$ to $\theta < -N\pi$, under which the vacuum σ_0 crosses the positive imaginary axis. Under this, $[\theta; N]$ changes from $\{0, 1, \ldots, N-1\}$ to $\{1, 2, \ldots, N\}$, and $\mathfrak{B}_1$ moves from inside $\mathcal{T}_{[\theta;N]}$ to the outside. Before the crossing, the brane $(\mathfrak{B}_1, \gamma_+)$ is supported purely at the vacuum σ_0 at low energies, but after the crossing, it flows to an SCFT brane as well as branes supported at some of the Coulomb vacua including σ_0. To see what happens when θ crosses $d\pi$ (mod 2π), let us look at a brane whose low energy limit is supported purely in the SCFT. To be specific, we consider a move from $\theta < -d\pi$ to $\theta > d\pi$, under which $[\theta; d]$ changes from $\{1, 2, \ldots, d\}$ to $\{0, 1, \ldots, d-1\}$. We are looking at a brane $(\mathfrak{B}, \gamma_+)$ with $\mathfrak{B} \in \mathcal{T}_{\{1,\ldots,d\}}$. Note that at each component of charge d, $\mathfrak{B}$ is of the form

(4.65)
$$\mathbb{C}(i,d) \underset{b}{\overset{p\,a}{\rightleftarrows}} \mathfrak{B}_{\mathrm{rest}},$$

[3]We take the limit $\epsilon \searrow 0$ for a while, to simplify the expressions.

for some $a : \mathbb{C}(i-2,0) \to \mathfrak{B}_{\mathrm{rest}}$ such that $b \cdot a$ is $\mathbb{C}(i-2,0) \xrightarrow{f(x)\times} \mathbb{C}(i,d)$. This can be expressed as

$$\begin{tikzcd} \mathbb{C}(i-2,0) \arrow[r, shift left, "a"] \arrow[d, "-\mathrm{id}"'] & \mathfrak{B}_{\mathrm{rest}} \arrow[l, shift left, "p\,b"] \arrow[d, "b"] \\ \mathbb{C}(i-1,0) \arrow[r, shift left, "f"] & \mathbb{C}(i,d) \arrow[l, shift left, "p"] \end{tikzcd} \tag{4.66}$$

We see that $\mathbb{C}(i,d)$ can be replaced by $\mathbb{C}(i-2,0)$ by forming a bound state with $\mathfrak{B}_1(i-1,0)$. Applying this for each component of charge d, $\mathfrak{B} \in \mathcal{T}_{\{1,\ldots,d\}}$ can be expressed as a bound state of $\mathfrak{B}' \in \mathcal{T}_{\{0,\ldots,d-1\}}$ and a number of copies of $\mathfrak{B}_1(j,0)$'s. Note that the LG images of $\mathfrak{B}$ and $\mathfrak{B}'$ are equivalent, $\pi_-(\mathfrak{B}) \cong \pi_-(\mathfrak{B}')$, and also that $\mathfrak{B}_1(j,0)$'s are supported purely at the Coulomb vacuum σ_0 that crosses the negative imaginary axis under the move. To summarize, as θ crosses $-d\pi$, a brane supported purely in the SCFT at low energies acquires components supported at the Coulomb vacuum that crossed the negative imaginary axis while the SCFT component remains the same. These effects of the move can be regarded as the "brane creation" as discussed in [33]. See Fig. 4.

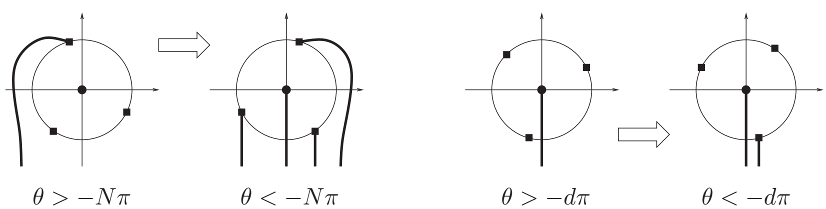

Fig. 4. Brane creations

Under the expected equivalence between $D^b_{\mathrm{Coh}}(X_f)$ and a category including $\mathrm{MF}_{\mathbb{Z}_d}(f)$ and $(N-d)$ exceptional objects, $\mathrm{MF}_{\mathbb{Z}_d}(f)$ is included in $D^b_{\mathrm{Coh}}(X_f)$ by

$$\mathrm{MF}_{\mathbb{Z}_d}(f) \cong \mathcal{T}_{[\theta;d]} \hookrightarrow \mathcal{T}_{[\theta;N]} \cong D^b_{\mathrm{Coh}}(X_f). \tag{4.67}$$

As θ crosses $d\pi$ (*resp.* $N\pi$) mod 2π, the category $\mathcal{T}_{[\theta;d]}$ (*resp.* $\mathcal{T}_{[\theta;N]}$) moves inside $\mathfrak{D}_{\mathrm{LSM}}$, and accordingly, the inclusion (4.67) changes. This change yields the brane creation discussed above.

$\underline{d > N}$ Flow from Landau-Ginzburg model

The model with $d > N$ describes a deformation of an SCFT corresponding to the LG orbifold $W = f(x)/\mathbb{Z}_d$, which flows to the sigma model with target X_f or one of the $(d-N)$ massive vacua at

$$\sigma_k = i\widetilde{\Lambda}\exp\left(i\,\frac{\theta+\pi N+2\pi k}{N-d}\right), \qquad k \in \mathbb{Z}/(d-N)\mathbb{Z}, \tag{4.68}$$

where $\widetilde{\Lambda} = \Lambda d^{\frac{d}{N-d}}\,\mathrm{e}^{-\frac{\zeta}{N-d}}$, with $\widetilde{W}_{\mathrm{eff}}(\sigma_k) = (N-d)\sigma_k$. Note that the UV SCFT has $\widehat{c}_{\mathrm{LG}} = N(1-2/d)$ while the IR sigma model has $\widehat{c}_{\mathrm{LV}} = N-2$. We expect that $\mathrm{MF}_{\mathbb{Z}_d}(f)$ is equivalent to a category that includes $D^b_{\mathrm{Coh}}(X_f)$ and $(d-N)$ exceptional objects. The analysis of the model goes similarly to the case $d < N$ and hence we shall only highlight the main points.

For any brane data $\mathfrak{B} = (M, Q)$, γ_- as in Fig. 2 is admissible and $(\mathfrak{B}, \gamma_-)$ defines a D-brane in the quantum theory. The hemisphere partition function $Z_{D^2}(\mathfrak{B}, \gamma_-)$ may be written as the sum of residues at the p-poles. In particular, we have

$$Z_{D^2}(\mathfrak{B}_1(i,q), \gamma_-) = 0. \tag{4.69}$$

This vanishing allows us to assume $\mathfrak{B} \in \mathcal{T}_{I_-}$ for a set I_- of d consecutive integers.

When $r\Lambda \ll 1$, t_R is large negative and the series of the p-pole residues is dominated by the first term. Its radius dependence is simply

$$Z_{D^2} \sim (r\Lambda)^{\widehat{c}/2}\,\mathrm{e}^{it_R\cdot i(-1/d+\epsilon/2)} \sim (r\Lambda)^{\frac{N(1-2/d)}{2}}, \tag{4.70}$$

which is indeed the characteristic behaviour for an SCFT with $\widehat{c}_{\mathrm{LG}} = N(1-2/d)$. This leading term is non-vanishing whenever the LG image of $\mathfrak{B}$ is non-trivial.

When $r\Lambda \gg 1$, the integrand is very small (*resp.* very large) in a neighborhood of the positive (*resp.* negative) imaginary axis within $|\sigma'| < r\widetilde{\Lambda}\,\mathrm{e}$. Therefore, we may deform the contour γ_- to $\widetilde{\gamma}_-$ as in Fig. 5. The new contour can be separated into two parts, $\widetilde{\gamma}_- = \widetilde{\gamma}_{\mathrm{cent}} + \widetilde{\gamma}_{\mathrm{rest}}$, where $\widetilde{\gamma}_{\mathrm{cent}}$ is the central part that encircles a large number of x-poles. Integration over $\widetilde{\gamma}_{\mathrm{cent}}$ yields the sum of residues at the x-poles. Since t_R is large positive for $r\Lambda \gg 1$, the dominant is the residue at the first pole, $\sigma' = i\epsilon/2$, whose dependence on the radius is simply

$$Z_{D^2}|_{\mathrm{central}} \sim (r\Lambda)^{\widehat{c}/2}\,\mathrm{e}^{it_R\cdot i\epsilon/2} \sim (r\Lambda)^{\frac{N-2}{2}}. \tag{4.71}$$

This is the characteristic behaviour for a brane in an SCFT with $\widehat{c}_{\mathrm{LV}} = N-2$. The bahaviour of the contribution from $\widetilde{\gamma}_{\mathrm{rest}}$ depends on the range

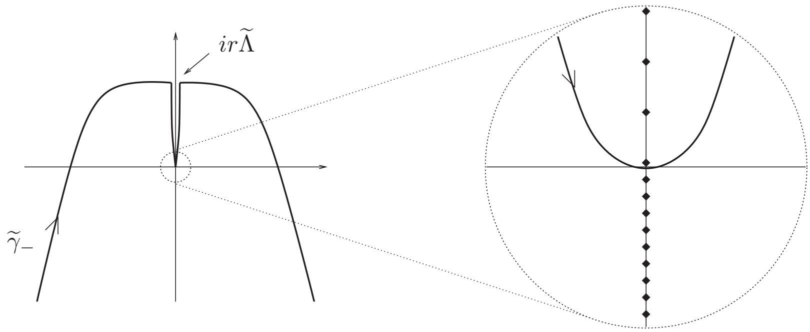

Fig. 5. Deformed contour $\widetilde{\gamma}_-$

of the gauge charge q. We assume that $\theta \not\equiv d\pi, N\pi$ modulo $2\pi\mathbb{Z}$ so that the Coulomb vacua σ_k are all away from the imaginary axis. When $N\pi < |\theta + 2\pi q| < d\pi$, the integrand has a unique critical point at the Coulomb vacuum $\sigma_{k(q)}$ with $k(q) = q$ (*resp.* $q - N$) when $\theta + 2\pi q$ is in $(-d\pi, -N\pi)$ (*resp.* $(N\pi, d\pi)$), and the integral behaves as $\sim \mathrm{e}^{-ir\widetilde{W}_{\mathrm{eff},q}(\sigma_{k(q)})}$ as $r \to \infty$. When $|\theta + 2\pi q| < N\pi$, the integral on $\widetilde{\gamma}_{\mathrm{rest}}$ is much smaller than any of $\mathrm{e}^{-ir\widetilde{W}_{\mathrm{eff}}(\sigma_k)}$'s. The case $|\theta + 2\pi q| > N\pi$ where it is hard to estimate the integral can be avoided by a choice of $\mathfrak{B}$. From this analysis, we may make the following claim concerning the low energy behaviour of the brane $(\mathfrak{B}, \gamma_+)$: If $\mathfrak{B}$ belongs to $\mathcal{T}_{[\theta;N]}$, the brane flows purely to a brane in the sigma model on X_f. If $\mathfrak{B}$ belongs to $\mathcal{T}_{[\theta;d]}$ but not to $\mathcal{T}_{[\theta;N]}$, some of the gauge charges q are in $N\pi < |\theta + 2\pi q| < d\pi$. Let q_* be the one that maximizes $\mathrm{Im}\widetilde{W}_{\mathrm{eff},q}(\sigma_{k(q)})$ among such q's. Then, the brane at low energies has a component supported at the Coulomb vacuum $\sigma_{k(q_*)}$.

Let us examine the case of $\mathfrak{B} = \mathfrak{B}_2(i, q)$ which reduces to the Recknagel-Schomerus brane in the UV SCFT. For any q, it does not belong to $\mathcal{T}_{[\theta;N]}$ since the set of gauge charges $\{q, q+1, \ldots, q+N\}$ cannot fit into $[\theta; N]$. When $-d\pi < \theta + 2\pi q < (d - 2N)\pi$, it belongs to $\mathcal{T}_{[\theta;d]}$, and the brane $(\mathfrak{B}_2(i, q), \gamma_-)$ is supported purely at the vacuum σ_q at low energies. When q is outside this range, we need to use a replacement of in $\mathcal{T}_{[\theta;d]}$ to find the low energy behaviour of $(\mathfrak{B}_2(i, q), \gamma_-)$.

When θ crosses the special values, $d\pi$ and $N\pi$ modulo $2\pi\mathbb{Z}$, one (or two) of the Coulomb vacua crosses the imaginary axis. This is where there is a matter zero mode localized near the boundary, and something special can happen. When θ crosses $d\pi$ (mod 2π), a brane supported at the vacuum that crosses the negative imaginary axis will acuire a component supported at the sigma model on X_f. When θ croses $N\pi$

(mod 2π), a brane that flows to the sigma model brane will acquire a component supported at the vacuum that crosses the positive imaginary axis.

Under the expected equivalence between $\mathrm{MF}_{\mathbb{Z}_d}(f)$ and a category including $D^b_{\mathrm{Coh}}(X_f)$ and $(d-N)$ exceptional objects, $D^b_{\mathrm{Coh}}(X_f)$ is included in $\mathrm{MF}_{\mathbb{Z}_d}(f)$ by

$$\mathrm{MF}_{\mathbb{Z}_d}(f) \cong \mathcal{T}_{[\theta;d]} \hookleftarrow \mathcal{T}_{[\theta;N]} \cong D^b_{\mathrm{Coh}}(X_f). \tag{4.72}$$

As θ crosses $d\pi$ (*resp.* $N\pi$) mod 2π, the category $\mathcal{T}_{[\theta;d]}$ (*resp.* $\mathcal{T}_{[\theta;N]}$) moves inside $\mathfrak{D}_{\mathrm{LSM}}$, and accordingly, the inclusion (4.72) changes. This change yields the creation of new components as discussed above.

4.5. Geometric and LG expressions

In a regime where the gauge symmetry is broken to a finite subgroup, the GLSM reduces to the Higgs branch theory $(\mu^{-1}(\zeta)/G, W_\zeta)$. From the general result (4.19), we may attempt to extract the hemisphere partition function of such a Higgs branch theory. The result for a brane data $\mathfrak{B}$ should be expressed in terms of the image $\pi_\zeta(\mathfrak{B})$ in the reduced theory.

In this subsection, we present the partition function of the theory in the geometrc and the LG regimes of the model $\mathrm{T}^{U(1)}_{N,d}$. As is clear from the previous subsection, the one in the geometric regime is the sum of residues at the x-poles, while the one in the LG regime is the sum of residues at the p-poles. They may or may not capture the partition function of the full theory. In the Calabi-Yau case $d = N$, the geometric and the LG expressions are the limiting behaviour of the full partition function in the two phases. In the non-Calabi-Yau case $d \neq N$, one is the full partition function in the short distance limit $r\Lambda \to 0$, while the other is a part of the full partition function in the long distance limit $r\Lambda \to \infty$.

To simplify some of the expressions, in what follows, we shall take the limit $\epsilon \searrow 0$ in which we write $f^0_M(\sigma')$ for the brane factor. If we wish, we can always bring back the ganeral ϵ using (4.37).

Geometric regime

We first consider the hemisphere partition function in the geometric regime. The geometric image of a brane data $\mathfrak{B} = (M, Q)$ is $\pi_+(\mathfrak{B})$ given in (3.25), with $[B] \simeq (\theta + d\pi)H$. In particular, its Chern character is

$$\mathrm{ch}(\pi_+(\mathfrak{B})) = \sum_{n=0}^{\infty} \sum_j (-1)^{r^0_j}\, \mathrm{e}^{(q_j+dn)H} = \frac{f^0_M(\frac{1}{2\pi}H)}{1-\mathrm{e}^{dH}}. \tag{4.73}$$

Since the x-poles are at $\sigma' = in$ with $n = 0, 1, 2, \ldots$, the partition function in the geometric regime is

$$(4.74)\quad Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = (r\Lambda)^{\frac{\hat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} \oint_0 \frac{\mathrm{d}z}{2\pi}\, \Gamma\left(dn + \tfrac{dz}{2\pi i} + 1\right) \Gamma\left(-n - \tfrac{z}{2\pi i}\right)^N \times \mathrm{e}^{-t_R n + \frac{i}{2\pi} t_R z} f^0_M\left(in + \tfrac{z}{2\pi}\right).$$

This is a power series in e^{-t_R} which is a small parameter when $t_R \gg 1$. Note that $f^0_M(in + \frac{z}{2\pi}) = f^0_M(\frac{z}{2\pi})$ since $\mathrm{e}^{2\pi q_j(in)} = 1$. Using this and $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$, it can be written as

(4.75)

$$\begin{aligned} Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) &= -C(r\Lambda)^{\frac{\hat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} \oint_0 \frac{\mathrm{d}z}{2\pi i} \left(\frac{(-1)^n}{2\sinh\left(\frac{z}{2}\right)}\right)^N \frac{\Gamma\left(1 + \frac{dz}{2\pi i} + dn\right)}{\Gamma\left(1 + \frac{z}{2\pi i} + n\right)^N} \\ &\qquad \times \mathrm{e}^{-nt_R + \frac{i}{2\pi} t_R z} f^0_M\left(\tfrac{z}{2\pi}\right) \\ &= C(r\Lambda)^{\frac{\hat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} \oint_0 \frac{\mathrm{d}z}{2\pi i} \frac{1}{z^N} \cdot d \cdot z \cdot \frac{z^{N-1} 2\sinh\left(\frac{dz}{2}\right)}{d\left(2\sinh\left(\frac{z}{2}\right)\right)^N} \frac{\Gamma\left(1 + \frac{dz}{2\pi i} + dn\right)}{\Gamma\left(1 + \frac{z}{2\pi i} + n\right)^N} \\ &\qquad \times (-1)^{Nn}\, \mathrm{e}^{-nt_R}\, \mathrm{e}^{\frac{i}{2\pi}(t_R - d\pi i) z} \cdot \frac{f^0_M\left(\frac{z}{2\pi}\right)}{1 - \mathrm{e}^{dz}}, \end{aligned}$$

with $C = -i(-2\pi i)^N$. Using the identity

$$(4.76)\qquad \oint_0 \frac{\mathrm{d}z}{2\pi i} \frac{1}{z^N} \cdot d \cdot z \cdot g(z) = \int_{\mathbb{P}^{N-1}} d \cdot H \cdot g(H) = \int_{X_f} g(H),$$

for a power series $g(z)$ in z, and the expression (4.73) for the Chern character of $\pi_+(\mathfrak{B})$, we find

$$(4.77)\qquad Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = C(r\Lambda)^{\frac{\hat{c}_{\mathrm{LV}}}{2}} \sum_{n=0}^{\infty} (-1)^{Nn}\, \mathrm{e}^{-nt_R} \int_{X_f} \widehat{\Gamma}_{X_f}(n) \times \exp\left(\frac{i}{2\pi}(t_R - d\pi i) H\right) \mathrm{ch}(\pi_+(\mathfrak{B})),$$

with

$$(4.78)\qquad \widehat{\Gamma}_{X_f}(n) := \frac{H^{N-1} 2\sinh\left(\frac{dH}{2}\right)}{d\left(2\sinh\left(\frac{H}{2}\right)\right)^N} \cdot \frac{\Gamma\left(1 + d\left(\frac{H}{2\pi i} + n\right)\right)}{\Gamma\left(1 + \frac{H}{2\pi i} + n\right)^N}.$$

Note that

$$(4.79)\qquad \widehat{\Gamma}_{X_f}(0) = \widehat{\mathrm{A}}_{X_f} \cdot \frac{\Gamma(1 + \frac{d}{2\pi i} H)}{\Gamma(1 + \frac{1}{2\pi i} H)^N} = \widehat{\mathrm{A}}_{X_f} \cdot \frac{1}{\widehat{\Gamma}^*_{X_f}} = \widehat{\Gamma}_{X_f},$$

where $\widehat{\mathrm{A}}_{X_f}$, $\widehat{\Gamma}_{X_f}$ and $\widehat{\Gamma}^*_{X_f}$ are the characteristic classes of the holomorphic tangent bundle of X_f obtained by inserting the roots of the total Chern class

$$c(X_f) = \frac{(1+H)^N}{(1+dH)} \tag{4.80}$$

into the functions $\widehat{\mathrm{A}}(x) = \frac{x/2}{\sinh(x/2)}$, $\widehat{\Gamma}(x) = \Gamma\left(1 - \frac{x}{2\pi i}\right)$ and $\widehat{\Gamma}^*(x) = \Gamma\left(1 + \frac{x}{2\pi i}\right)$. We said earlier that the complexified Kähler class is related to the FI-theta parameter by $[\omega - iB] = (t - d\pi i)H + O(\mathrm{e}^{-t})$, but this should be understood to be the relation between the renormalized parameters, that is, $[\omega_R - iB]$ versus t_R, where

$$\omega_R := \omega - \underbrace{c_1(X_f)}_{(N-d)H} \log(r\Lambda). \tag{4.81}$$

We see that the asymptotic behaviour at $\zeta_R \to \infty$ is

(4.82)

$$Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = C(r\Lambda)^{\frac{\hat{c}_{\mathrm{LV}}}{2}}\left[\int_{X_f} \widehat{\Gamma}_{X_f} \exp\left(\frac{1}{2\pi}(B + i\omega_R)\right) \mathrm{ch}(\pi_+(\mathfrak{B})) + O(\mathrm{e}^{-\omega_R + iB})\right],$$

where $O(\mathrm{e}^{-\omega_R+iB})$ stands for a correction of the form $\sum_{n=1}^{\infty} c_n\, \mathrm{e}^{-n\langle \omega_R - iB, l\rangle}$ where l is a second homology class of X_f such that $\langle H, l\rangle = 1$.

As seen in the previous subsection, in the case $d < N$ (*resp.* $d = N$), this is the expansion of the full partition function in the ultra-voilet limit $r\Lambda \to 0$ (*resp.* the large target volume limit $\zeta \to +\infty$), while in the case $d > N$, this is the expansion of a part of the full partition function in the infra-red limit $r\Lambda \to \infty$. Note the relation $\mathrm{e}^{-t_R} = \mathrm{e}^{-t}(r\Lambda)^{N-d}$. This difference can also be seen from the growth of the power series (4.77). The n-th term is roughly $\frac{\Gamma(1+dn)}{\Gamma(1+n)^N}\,\mathrm{e}^{-nt_R}$ times a constant. By Stirling's formula, it behaves for $n \gg 1$ as

(4.83)

$$\frac{\Gamma(1+dn)}{\Gamma(1+n)^N}\,\mathrm{e}^{-nt_R} \sim \mathrm{e}^{n(-t_R+N-d+d\log d)} n^{(d-N)n - \frac{\hat{c}_{\mathrm{LV}}+1}{2}}\left(1 + O\left(\tfrac{1}{n}\right)\right).$$

We see that the series is abolutely convergent for any e^{-t_R} if $d < N$ and for $|\,\mathrm{e}^{-t}| < N^{-N}$ if $d = N$, while it is divergent if $d > N$. In the case $d \leq N$, the series can be summed up to an analytic function of e^{-t_R} in an appropriate domain and may be used as a definition of the full partition function there. In the case $d > N$, on the other hand, one needs to resort to some summation technique to define an analytic function, but that usually involves choices. This reflects the fact that the sigma model is ultra-violet complete for $d \leq N$ but not for $d > N$. In the latter case, to say something about short distance $r\Lambda \ll 1$, the theory needs to be embedded into a ultra-violet complete theory, but things can depend on the choice of the completion. Of course, the GLSM provides one choice of UV completion, and the full partition function is the fully analytic completion of the divergent series (4.77).

Landau-Ginzburg regime

We next consider the hemisphere partition function in the LG regime. The LG image of a brane data $\mathfrak{B} = (M, Q)$ is $\pi_-(\mathfrak{B}) = (M_-, Q_-)$ given in (3.27). From the charge integrality of $\mathfrak{B}$ and $\pi_-(\mathfrak{B})$, we see that $(-1)^{\mathbf{r}^0}$ plays the rôle of the $\mathbb{Z}_2$-grading operator of M_-.

Since the p-poles are simple poles at $\sigma' = -i(n+1)/d$ with $n = 0, 1, 2, \ldots$, the partition function in the LG regime is

$$(4.84) \quad Z^{\mathrm{LG}}_{D^2}(\mathfrak{B}) = \frac{2\pi}{d}(r\Lambda)^{\frac{N-2}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\Gamma\left(\tfrac{n+1}{d}\right)^N \mathrm{e}^{t_R \frac{n+1}{d}} f^0_M\left(-i\tfrac{n+1}{d}\right).$$

Note that $(r\Lambda)^{\frac{N-2}{2}} \mathrm{e}^{t_R/d} = \mathrm{e}^{t/d}(r\Lambda)^{\frac{N(1-2/d)}{2}} = \mathrm{e}^{t/d}(r\Lambda)^{\frac{\hat{c}_{\mathrm{LG}}}{2}}$ and

$$(4.85) \qquad f^0_M\left(-i\tfrac{n+1}{d}\right) = \mathrm{tr}_M\left(\mathrm{e}^{\pi i \mathbf{r}^0}\,\mathrm{e}^{-2\pi i\frac{n+1}{d}}\right) = \mathrm{Str}_{M_-}\left(\omega_1^{-n-1}\right),$$

where ω_1 is the generator $\mathrm{e}^{2\pi i/d}$ of $\mathbb{Z}_d$ and $\mathrm{Str}_{M_-}(?)$ is the supertrace $\mathrm{tr}_{M_-}((-1)^{\mathbf{r}^0}?)$. Thus, we find

$$\begin{aligned}(4.86) \quad & Z^{\mathrm{LG}}_{D^2}(\mathfrak{B}) \\ &= \frac{2\pi}{d}\,\mathrm{e}^{t/d}(r\Lambda)^{\frac{\hat{c}_{\mathrm{LG}}}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\mathrm{e}^{nt_R/d}\Gamma\left(\tfrac{n+1}{d}\right)^N \mathrm{Str}_{M_-}\left(\omega_1^{-n-1}\right) \\ &= \frac{2\pi}{d}\,\mathrm{e}^{t/d}\Gamma\left(\tfrac{1}{d}\right)^N \cdot \underbrace{(r\Lambda)^{\frac{\hat{c}_{\mathrm{LG}}}{2}}\mathrm{Str}_{M_-}\left(\omega_1^{-1}\right)} + \mathrm{e}^{t/d}O(\mathrm{e}^{t_R/d}).\end{aligned}$$

The underbraced factor of the leading term in the limit $t_R \to -\infty$ is indeed the hemisphere partition function for the brane $\pi_-(\mathfrak{B})$ of the LG orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$, obtained by applying the result (4.19) to the theory with gauge group $\mathbb{Z}_d$, matter representation V and superpotential $f(x)$.

In the case $d > N$ (*resp.* $d = N$), this is the expansion of the full partition function in the ultra-voilet limit $r\Lambda \to 0$ (*resp.* the LG limit $\zeta \to -\infty$), while in the case $d < N$, this is the expansion of a part of the full partition function in the infra-red limit $r\Lambda \to \infty$. Note the relation $\mathrm{e}^{t_R/d} = \mathrm{e}^{t/d}(r\Lambda)^{1-N/d}$. This difference can also be seen from the growth of the power series (4.86). The n-th term behaves for $n \gg 1$ as

(4.87)
$$\frac{\mathrm{e}^{nt_R/d}}{n!}\Gamma\left(\tfrac{n+1}{d}\right)^N \sim \mathrm{e}^{n(t_R-N+d-N\log d)/d} n^{(N-d)n/d-\frac{\widehat{c}_{\mathrm{LG}}+1}{2}}\left(1+O\left(\tfrac{1}{n}\right)\right).$$

We see that the series is abolutely convergent for any $\mathrm{e}^{t_R/d}$ if $d > N$ and for $|\,\mathrm{e}^{t/N}| < N$ if $d = N$, while it is divergent if $d < N$. In the case $d \geq N$, the series can be summed up to an analytic function of $\mathrm{e}^{t_R/d}$ in an appropriate domain and may be used as a definition of the full partition function there. In the case $d < N$, on the other hand, one needs to resort to some summation technique to define an analytic function, but that usually involves choices. This reflects the fact that the theory under question is a deformation of the LG orbifold by a relevant, exactly marginal and irrelevant operator respectively for the case $d > N$, $d = N$ and $d < N$. In the latter case, to say something about short distances $r\Lambda \ll 1$, the theory needs to be embedded into a ultra-violet complete theory, but things can depend on the choice of the completion. Of course, the GLSM provides one choice of UV completion, and the full partition function is the fully analytic completion of the divergent series (4.86).

4.6. Differential Equations

As a function of the radius r and the FI-theta parameters t, the hemisphere partition function of the GLSM satisfies certain differential equations. We shall write down such equations for $Z_{D^2} = Z_{D^2}^{\mathrm{A}(+)}$.

Renormalization group eqaution

Let us first write down the renormalization group equation that shows how the partition function Z_{D^2} changes as the distance scale is varied. That is, we look at the responce to the differentiation with respect to the radius r of the hemisphere. The radius enters into the prefactor $(r\Lambda)^{\widehat{c}/2}$ and possibly also in the factor $\mathrm{e}^{it_R(\sigma')}$ of the integrand as $t_R(\sigma') = t(\sigma) - b_1(\sigma')\log(r\Lambda)$, where we recall $b_1 = \sum_i Q_i$ from (3.10). Noting $i\sigma'_a\,\mathrm{e}^{it_R(\sigma')} = \frac{\partial}{\partial t^a}\,\mathrm{e}^{it_R(\sigma')}$, we find

$$r\frac{\partial}{\partial r}Z_{D^2} = \left(\frac{\widehat{c}}{2} - b_1\left(\frac{\partial}{\partial t}\right)\right)Z_{D^2}, \tag{4.88}$$

where $b_1\left(\frac{\partial}{\partial t}\right) := \sum_a b_1^a \frac{\partial}{\partial t^a}$. The equation is invariant under the unphysical shift of R-charge, which does $\widehat{c} \to \widehat{c} - b_1(\varepsilon)$ and $Z_{D^2} \to \mathrm{e}^{-t(\varepsilon)/2}Z_{D^2}$.

For example, in the model $\mathrm{T}^{U(1)}_{N,d}$ (with $\epsilon \searrow 0$), the equation is

$$r\frac{\partial}{\partial r}Z_{D^2} = \left(\frac{N-2}{2} - (N-d)\frac{\partial}{\partial t}\right) Z_{D^2}. \tag{4.89}$$

Picard-Fuchs equations

The partition function Z_{D^2} satisfies another set of differential equations associated to relations of local operators. We may consider inserting an operator $\mathcal{O}$, say, at the center $z = 0$ of the hemisphere. The

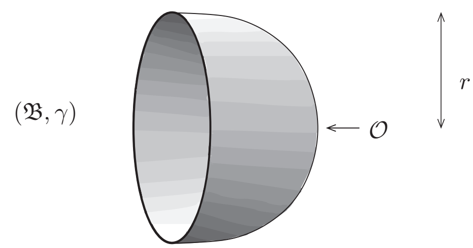

Fig. 6. The hemisphere with an insertion of an operator

supersymmetric localization still work if $\mathcal{O}$ is invariant under the $\mathrm{A}_{(+)}$-type supersymmetry, $Q^{A+}_{(+)}\mathcal{O} = Q^{A-}_{(+)}\mathcal{O} = 0$. For example, an adjoint invariant polynomial $\Phi(\sigma)$ of the scalar σ in the gauge multiplet, inserted at $z = 0$, satisfies this condition. With insertion of such an operator, the computation of the partition function goes through as before and the result is simply (4.19) with an insertion of $\Phi(\sigma'/r)$ in the integrand. If the operator is $Q^{A+}_{(+)}$ or $Q^{A-}_{(+)}$ exact, then the path-integral should vanish. In particular, if there is an operator relation at $z = 0$, $F(t; \Phi_1(\sigma), \Phi_2(\sigma), \ldots) \equiv 0$ modulo $Q^{A+}_{(+)}$ or $Q^{A-}_{(+)}$ exact operators, the partition function must vanish. Note that inserting a central component of σ' in the integrand is implemented by differentiation with respect to it. Therefore, for each relation $F(t; \sigma_{z_1}, \sigma_{z_2}, \ldots) \equiv 0$ among the central components $\sigma_{z_1}, \sigma_{z_2}, \ldots$, we have a differential equation

$$F\left(t; \frac{1}{ir}\frac{\partial}{\partial t^{z_1}}, \frac{1}{ir}\frac{\partial}{\partial t^{z_2}}, \ldots\right) Z_{D^2} = 0. \tag{4.90}$$

Note that $Q^{A+}_{(+)}$ and $Q^{A-}_{(+)}$ at $z = 0$ approach the supercharges $\overline{Q}_+$ and Q_- in the limit $r \to \infty$ where we get back the standard flat background. This implies that the operator relation $F(t; \Phi_1(\sigma), \Phi_2(\sigma), \ldots) \equiv 0$ approaches in this limit to a relation of the chiral ring $\mathcal{R}_A$. In other words, it should be a deformation of a chiral ring relation by terms that vanish as $r \to \infty$.

Let us illustrate this in the model $\mathrm{T}_{N,d}^{U(1)}$. The equation can be found by using the Gamma function identity

$$z\Gamma(z) = \Gamma(z+1), \tag{4.91}$$

and the property that the brane factor vanishes at $\sigma' = 0$,

$$f_M^0(0) = \mathrm{tr}_M(-1)^{\mathbf{r}_0} = \dim M^{\mathrm{ev}} - \dim M^{\mathrm{od}} = 0, \tag{4.92}$$

which follows from the matrix factorization equation $Q^2 = W\mathrm{id}_M$. Indeed, repeatedly using (4.91), we find

$$\begin{aligned}(i\sigma')^{N-1} \cdot \Gamma(-id\sigma'+1)\Gamma(i\sigma')^N &= -d\Gamma(-id\sigma')\Gamma(i\sigma'+1)^N \\ &\xrightarrow{\sigma' \to \sigma'+i} -d\Gamma(-id\sigma'+d)\Gamma(i\sigma')^N \\ &= -d(-id\sigma'+d-1)\cdots(-id\sigma'+1)\cdot\Gamma(-id\sigma'+1)\Gamma(i\sigma')^N.\end{aligned} \tag{4.93}$$

Because of $f_M^0(0) = 0$, the function $(i\sigma')^{N-1}\Gamma(-id\sigma'+1)\Gamma(i\sigma')^N\,\mathrm{e}^{it_R\sigma} f_M^0(\sigma')$ has no pole between γ' and $\gamma'+i$. Note also that $f_M^0(\sigma'+i) = f_M^0(\sigma')$. Therefore, for an admissible $(\mathfrak{B}, \gamma)$ we have

$$\begin{aligned}0 &= \left[\int_{\gamma'} - \int_{\gamma'+i}\right] \mathrm{d}\sigma' (i\sigma')^{N-1}\cdot\Gamma(-id\sigma'+1)\Gamma(i\sigma')^N\,\mathrm{e}^{it_R\sigma} f_M^0(\sigma') \\ &\overset{(4.93)}{=} \int_{\gamma'} \mathrm{d}\sigma' \mathcal{D}(\sigma')\Gamma(-id\sigma'+1)\Gamma(i\sigma')^N\,\mathrm{e}^{it_R\sigma} f_M^0(\sigma')\end{aligned} \tag{4.94}$$

with

$$\mathcal{D}(\sigma') = (i\sigma')^{N-1} + d\,\mathrm{e}^{-t_R}(-id\sigma'+d-1)\cdots(-id\sigma'+1). \tag{4.95}$$

The admissibility is used to ensure that the contour $\gamma' - (\gamma'+i)$ can be closed at infinity. This shows that the partition function is annihilated by $\mathcal{D}(-i\frac{\partial}{\partial t})$, that is, it satisfies the differential equation

$$\left[\left(\frac{\partial}{\partial t}\right)^{N-1} - (-d)^d\,\mathrm{e}^{-t_R}\prod_{n=1}^{d-1}\left(\frac{\partial}{\partial t} - \frac{n}{d}\right)\right] Z_{D^2} = 0. \tag{4.96}$$

For $N = d = 5$, this is the famous Picard-Fuchs equation for the periods of the mirror quintic [2]. The corresponding operator relation must be $\mathcal{D}(r\sigma) = 0$. Using $\mathrm{e}^{-t_R} = \mathrm{e}^{-t}(r\Lambda)^{N-d}$, it reads

$$\sigma^{N-1} - (-d)^d\,\mathrm{e}^{-t}(-i\Lambda)^{N-d}\prod_{n=1}^{d-1}\left(\sigma + i\frac{n}{dr}\right) = 0. \tag{4.97}$$

In the limit $r \to \infty$, this becomes

$$\sigma^{N-1} - (-d)^d \, \mathrm{e}^{-t}(-i\Lambda)^{N-d}\sigma^{d-1} = 0. \tag{4.98}$$

This is indeed the chiral ring relation of the model that follows partially from $\partial_\sigma \widetilde{W}_{\text{eff}}(\sigma) \equiv 0 \bmod 2\pi i\mathbb{Z}$ for (3.16) with (4.36). See [34]. The relation can also be derived using the mirror description [32]. The relation in the sigma model for the case $d < N$ is proved mathematically in [35].

We may try to find the equation in other models, say, in a model with $U(1)$ gauge group. Let us put

$$\mathcal{D}_\pm(\sigma') := \prod_{\pm Q_i > 0} \prod_{n=0}^{|Q_i|-1} \left(iQ_i\sigma' + \frac{R_i}{2} + n \right). \tag{4.99}$$

Using (4.91), we find

$$\begin{aligned} \mathcal{D}_\pm(\sigma') \prod_i \Gamma\left(iQ_i\sigma' + \frac{R_i}{2} \right) \\ = \prod_{\pm Q_i<0} \Gamma\left(iQ_i\sigma' + \frac{R_i}{2} \right) \prod_{\pm Q_i>0} \Gamma\left(iQ_i(\sigma' \mp i) + \frac{R_i}{2} \right), \end{aligned} \tag{4.100}$$

which means that

$$\mathcal{D}_+(\sigma') \prod_i \Gamma\left(iQ_i\sigma' + \frac{R_i}{2} \right) \overset{\sigma' \to \sigma' + i}{\longrightarrow} \mathcal{D}_-(\sigma') \prod_i \Gamma\left(iQ_i\sigma' + \frac{R_i}{2} \right). \tag{4.101}$$

Note that the left hand side of (4.101) has no pole between γ' and $\gamma' + i$ as long as the bound $0 < R_i < 2$ is satisfied. Therefore, for an admissible $(\mathfrak{B}, \gamma)$ we have

$$\begin{aligned} 0 &= \left[\int_{\gamma'} - \int_{\gamma'+i} \right] \mathrm{d}\sigma' \mathcal{D}_+(\sigma') \prod_i \Gamma\left(iQ_i\sigma' + \frac{R_i}{2} \right) \mathrm{e}^{it_R\sigma'} f_M(\sigma') \\ &= \int_{\gamma'} \mathrm{d}\sigma' \left(\mathcal{D}_+(\sigma') - \mathrm{e}^{-t_R}\mathcal{D}_-(\sigma') \right) \\ &\qquad \times \prod_i \Gamma\left(iQ_i\sigma' + \frac{R_i}{2} \right) \mathrm{e}^{it_R\sigma'} f_M(\sigma'), \end{aligned} \tag{4.102}$$

which yields the differential equation

(4.103)
$$\left[\prod_{Q_i>0}\prod_{n=0}^{Q_i-1}\left(Q_i\frac{\partial}{\partial t}+\frac{R_i}{2}+n\right)-\mathrm{e}^{-t_R}\prod_{Q_i<0}\prod_{n=0}^{|Q_i|-1}\left(Q_i\frac{\partial}{\partial t}+\frac{R_i}{2}+n\right)\right]Z_{D^2} = 0.$$

This holds in any model, with or without superpotential W. In the model $\mathrm{T}^{U(1)}_{N,d}$, this equation is one order higher than (4.96). In fact, it is nothing but $\frac{\partial}{\partial t}$(4.96). To get (4.96), we needed to use the property (4.92) of the brane factor. Similarly, if the model has a non-zero superpotential W, the brane factor $f_M(\sigma')$ has an extra property by which we may be able to find a lower order equation.

Let us show this in the model $\mathrm{T}^{U(1)}_{\vec{w},d}$ labelled by $\vec{w}=(w_1,\ldots,w_N)\in(\mathbb{Z}_{>0})^N$ and $d\in\mathbb{Z}_{>0}$. The model is similar to $\mathrm{T}^{U(1)}_{N,d}$, but is different just in that $\mathbb{C}(1)^{\oplus N}$ is replaced by $\oplus_{i=1}^N\mathbb{C}(w_i)$ and $f(x)$ is quasihomogeneous, $f(c^{w_1}x_1,\ldots,c^{w_N}x_N) = c^d f(x_1,\ldots,x_N)$. The R-charge is $R=(2-d\epsilon, w_1\epsilon,\ldots,w_N\epsilon)$, and we shall work again in the limit $\epsilon\searrow 0$. $\zeta\gg 0$ is the geometric phase with the target space being the hypersurface X_f in the weighted projective space $\mathbb{P}(w_1,\ldots,w_N)$, and $\zeta\ll 0$ is the phase where we have the LG orbifold $(\oplus_{i=1}^N\mathbb{C}(w_i)/\mathbb{Z}_d, f)$. We assume that X_f misses the orbifold loci. This requires that f contains a term $c_i x_i^{d_i}$ ($c_i\neq 0$) for each i with $w_i>1$, which means $d=d_iw_i$, and that w_i's are pairwise coprime. Then, for each i with $w_i>1$, for a brane data $\mathfrak{B}=(M,Q)$, if Q_i denotes Q with all x_j with $j\neq i$ set equal to zero, (M,Q_i) is a matrix factorization of $c_ipx_i^{d_i}$. By the gauge invariance, Q_i commutes with the gauge action of $g_i=\mathrm{e}^{2\pi i/w_i}$ on M and hence preserves each subspace $M_q^{(i)}\subset M$ of weight $q\in\mathbb{Z}/w_i\mathbb{Z}$. That is, the matrix factorization (M,Q_i) decomposes into the sum of $(M_q^{(i)},Q_i)$'s. Since the vanishing (4.92) holds for each component $(M_q^{(i)},Q_i)$, we find that, for any $l\in\mathbb{Z}$,

$$f_M^0(il/w_i)=\mathrm{tr}_M(-1)^{\mathbf{r}_0}g_i^l=\sum_{q\in\mathbb{Z}/w_i\mathbb{Z}}\mathrm{e}^{2\pi ilq/w_i}\mathrm{tr}_{M_q^{(i)}}(-1)^{\mathbf{r_0}}=0. \tag{4.104}$$

In the present case, $\mathcal{D}_\pm$ are

$$\mathcal{D}_+(\sigma')=\prod_{i=1}^N\prod_{n_i=0}^{w_i-1}(iw_i\sigma'+n_i),\quad \mathcal{D}_-(\sigma')=\prod_{n=0}^{d-1}(-id\sigma'+1+n). \tag{4.105}$$

Note that the factor $C(\sigma') = i(\sigma' + i)\prod_{w_i>1}\prod_{n_i=1}^{w_i-1}(iw_i(\sigma' + i) + n_i)$ of $\mathcal{D}_+(\sigma' + i)$ divides $\mathcal{D}_-(\sigma')$. Therefore, if we write $\mathcal{D}_+(\sigma' + i) = \widetilde{\mathcal{D}}_+(\sigma' + i)C(\sigma')$ and $\mathcal{D}_-(\sigma') = \widetilde{\mathcal{D}}_-(\sigma')C(\sigma')$, we find

(4.106)
$$\widetilde{\mathcal{D}}_+(\sigma')\Gamma(-id\sigma'+1)\prod_i \Gamma(iw_i\sigma') \overset{\sigma'\to\sigma'+i}{\longrightarrow} \widetilde{\mathcal{D}}_-(\sigma')\Gamma(-id\sigma'+1)\prod_i \Gamma(iw_i\sigma').$$

Between γ' and $\gamma' + i$, the left hand side of (4.106) has simple poles at $\sigma' = 0$ and $\sigma' = il_i/w_i$ for $w_i > 1$ and $l_i = 1, \ldots, w_i - 1$, but the brane factor $f_M^0(\sigma')$ vanishes at each of these points, thanks to (4.104). Therefore, for an admissible $(\mathfrak{B}, \gamma)$ we have

(4.107)
$$\begin{aligned} 0 &= \left[\int_{\gamma'} - \int_{\gamma'+i}\right] \mathrm{d}\sigma' \widetilde{\mathcal{D}}_+(\sigma')\Gamma(-id\sigma' + 1)\prod_i \Gamma(iw_i\sigma')\mathrm{e}^{it_R\sigma'} f_M^0(\sigma') \\ &= \int_{\gamma'} \mathrm{d}\sigma' \left(\widetilde{\mathcal{D}}_+(\sigma') - \mathrm{e}^{-t_R}\widetilde{\mathcal{D}}_-(\sigma')\right) \\ &\qquad\qquad \times \Gamma(-id\sigma' + 1)\prod_i \Gamma(iw_i\sigma')\mathrm{e}^{it_R\sigma'} f_M^0(\sigma'), \end{aligned}$$

which yields $\left[\widetilde{D}_+(-i\frac{\partial}{\partial t}) - \mathrm{e}^{-t_R}\widetilde{\mathcal{D}}_-(-i\frac{\partial}{\partial t})\right] Z_{D^2} = 0$, that is,

$$\left[\left(\frac{\partial}{\partial t}\right)^{N-1} - \frac{(-d)^d}{\prod_i w_i^{w_i}}\,\mathrm{e}^{-t_R} \prod_{\substack{1\le n\le d-1 \\ \frac{n}{d}\neq\frac{n_i}{w_i}}} \left(\frac{\partial}{\partial t} - \frac{n}{d}\right)\right] Z_{D^2} = 0. \tag{4.108}$$

The corresponding operator relation is

$$\sigma^{N-1} - \frac{(-d)^d}{\prod_i w_i^{w_i}}\,\mathrm{e}^{-t}(-i\Lambda)^{\sum_i w_i - d} \prod_{\substack{1\le n\le d-1 \\ \frac{n}{d}\neq\frac{n_i}{w_i}}} \left(\sigma + i\frac{n}{dr}\right) = 0. \tag{4.109}$$

It reduces in the flat space limit $r \to \infty$ to the chiral ring relation.

For example, for the cases $(\vec{w}; d) = (1, 1, 1, 1, 2; 6), (1, 1, 1, 1, 4; 8), (1, 1, 1, 2, 5; 10)$, the above equations agree with the Picard-Fuchs equations for the periods in the mirror of the Calabi-Yau three-folds X_f with $h^{1,1}(X_f) = 1$ [36–38]. The analysis can be extended to more general models. For example, if we apply it to models for Calabi-Yau hypersurfaces in toric varieties, we obtain the system of Picard-Fuchs equations as studied in, say, [39].

5. D-Brane Central Charges

In this section, we discuss what the hemisphere partition function is computing. It turns out that it is related to what is known as the central charge of the brane. The central charge (in this context) is originally defined in a particular situation in superstring theory, but is generalized to other areas of reserach. The hemisphere is related both to the original definition and to the generalizations. We will also find a surprising and suggestive relation to "macroscopic loop" in matrix models.

5.1. The Central Charge in 4d $\mathcal{N} = 2$ Compactifications

As is clear from the formulation and manifest in the resulting expression, the hemisphere partition function is invariant under continuous deformations of the brane. In this sense, it may be regarded as a "topological invariant" of the brane. In particle mechanics, the basic examples of invariant quantity of a particle are its electric and magnetic charges. In superstraing theory, D-branes are dynamical objects and are charged under Ramond-Ramond (RR) gauge potentials [40],[1] just like the electron is charged under the electro-magnetic gauge potential. Therefore, the first guess is that the hemisphere partition function is determined by the RR-charges of the brane. Indeed, we have seen in the model $\mathrm{T}_{N,d}^{U(1)}$ that the geometric expression depends on the brane $\mathcal{E} = \pi_+(\mathfrak{B})$ via its Chern character $\mathrm{ch}(\mathcal{E})$, while the LG expression depends on the brane $(M_-, Q_-) = \pi_-(\mathfrak{B})$ via $\mathrm{Str}_{M_-}(\omega)$, $\omega \in \mathbb{Z}_d \setminus \{1\}$. They are known to determine the RR charges of the branes in the sigma model [41, 42] and in the LG orbifold [43] respectively.

There are several RR potentials in general and hence each D-brane $\mathcal{B}$ has several RR charges. When we compactify Type II superstring theory on a three-dimensional Calabi-Yau manifold, we obtain an $\mathcal{N} = 2$ supersymmetric theory in the remaining four dimensions, and there is a distiguished linear combination of the RR charges, denoted by $Z(\mathcal{B})$ and called the *central charge*. The name is after the fact that it is the eigenvalue of the central element of the 4d $\mathcal{N} = 2$ supersymmetry algebra.[2] The central charge plays a crucial rôle in determining the stability of D-brane states, and motivated the mathematical study of the stability condition of the category of branes. For Type IIA (*resp.* Type IIB) string on a Calabi-Yau three-fold X, the relevant D-branes are B-branes (*resp.* A-branes) in the sigma model with target X. For

[1] See Appendix B for what "RR" means.

[2] This should not be confused with the central charge c (or $\widehat{c}$) which is the central part of the 2d (super)conformal symmetry algebra.

Type IIB, the central charge of the A-brane wrapped on a Lagrangian submanifold L is the period integral $Z(L) = \int_L \Omega$ of a holomorphic volume form Ω of X. The corresponding state is stable (or BPS) when L is a special Lagrangian submanifold of X so that $|Z(L)|$ is equal to the volume of L times $|\int_X \Omega \wedge \overline{\Omega}|^{1/2}$.

The theory $\mathrm{T}^{U(1)}_{5,5}$ yields a family of SCFTs on $\mathfrak{M}^0_A = \{\mathrm{e}^t\} = \mathbb{C} \setminus \{(-5)^5\}$ — a compactification of (3.18) at the LG point $\mathrm{e}^t = 0$ — that can be used for a 4d $\mathcal{N} = 2$ compactification, say, of Type IIA string theory. By mirror symmetry, it is equivalent to Type IIB string theory on a family of Calabi-Yau three-folds Y_t, and a detailed study of the theory is performed by Candelas, de la Ossa, Green and Parkes in [2]. In particular, they computed the period integrals of Y_t, that is, the central charges of A-branes in Y_t. Later, using mirror symmetry, the results are rephrased as the formula for the central charge for B-branes in the quintic three-fold X_f [44]. Comparing these results with the results obtained in the previous section, we see that the central charge agrees with the hemisphere partition function,

$$Z_{D^2}(\mathcal{B}) = Z(\mathcal{B}) \quad \text{in } \mathrm{T}^{U(1)}_{5,5}, \tag{5.1}$$

up to the factor $(r\Lambda)^{3/2}$ of radius dependence. Indeed, the integral formula for the periods given in [2] matches exactly our integral formula for the hemisphere (4.19), more specifically (4.43) with $d = N = 5$. For example, see Eqn (3.14) in [2],

$$\varpi_0(\psi) = \frac{1}{2\pi i} \int_C \mathrm{d}s \frac{\Gamma(-s)\Gamma(5s+1)}{\Gamma(s+1)^4}\, \mathrm{e}^{\pi i s}(5\psi)^{-5s}, \tag{5.2}$$

where $0 < \mathrm{Arg}(\psi) < 2\pi/5$ and C is a contour parallel to the imaginary axis located in $-1/5 < \mathrm{Re}(s) < 0$, as depicted in Fig. 5 of [2]. The parameter ψ is related to our t via $\mathrm{e}^t = -(5\psi)^5$. Using $1/\Gamma(s+1) = 2i\sin(\pi s)\Gamma(-s)/(-2\pi i)$, after the change of integration variables $s = -i\sigma'$, we see that this integral is nothing but the integral (4.43) with $\epsilon \searrow 0$ and $f_M(\sigma') = (\mathrm{e}^{\pi\sigma'} - \mathrm{e}^{-\pi\sigma'})^4$ for $-\pi < \theta < \pi$, up to an overall constant and $(r\Lambda)^{3/2}$, where the contour γ' is parallel to the real axis located in $-1/5 < \mathrm{Im}(\sigma') < 0$. If this $f_M(\sigma')$ comes from a brane data $\mathfrak{B}$, its Chan-Paton charges would be from $\{0, \pm 1, \pm 2\}$. Thus, this would be grade restricted with respect to the window $-\pi < \theta < \pi$, and the contour is the same as the one we proposed for the hemisphere for any value of ζ. In the geometric phase, the corresponding Chern character would be

$$\text{“ch}(\pi_+(\mathfrak{B}))\text{”} = \frac{f_M(\frac{H}{2\pi})}{1 - \mathrm{e}^{5H}} = -\frac{1}{5}H^3 = \text{a generator of } \mathrm{H}^6(X_f, \mathbb{Z}), \tag{5.3}$$

which is the Chern character of the skyscraper sheaf at a point pf X_f (a D0-brane). Another decisive support of the relation (5.1) can be found in the geometric regime. The formula for the central charge for a general B-brane in the quintic X_f is extracted in [44] from [2] via mirror symmetry, and it agrees with the large volume expansion (4.77) for the hemisphere partition function, up to $(r\Lambda)^{3/2}$.

The relation (5.1) is supported also by the differential equation. It was found in [2] that the period integrals in Y_t satisfy the Picard-Fuchs equation, which is the same as the equation (4.96) with $d = N = 5$ satisfied by the hemisphere partition function, as noted earlier. This extends also to other models. Therefore, it is very plausible that the relation between the hemisphere and the central charge holds in a general model for 4d $\mathcal{N} = 2$ compactification. In fact, the relation can be extended to a wider range of models.

5.2. The Central Charge in 2d (2, 2) Supersymmetric QFTs

From the viewpoint of the string worldsheet, the RR charges of a D-brane $\mathcal{B}$ are measured by taking the overlap of the boundary state of the D-brane ${}_{\mathrm{RR}}\langle\mathcal{B}|$ and the RR ground states. See Appendix B for the definition of boundary states. And the central charge is the overlap with a distingusihed ground state $|0\rangle_{\mathrm{RR}}$,

$$Z(\mathcal{B}) = {}_{\mathrm{RR}}\langle\mathcal{B}|0\rangle_{\mathrm{RR}}. \tag{5.4}$$

If $\mathcal{B}$ is a B-brane (*resp.* an A-brane), $|0\rangle_{\mathrm{RR}}$ is the state that corresponds to the identity operator under the A-type (*resp.* B-type) spectral flow. That is, the overlap (5.4) can be represented in path-integral as the partition function on the semi-infinite cigar, with the D-brane $\mathcal{B}$ as the boundary condition and A-twist (*resp.* B-twist) in the curved region. See Fig. 7. This formulation allows us to generalize the definition of

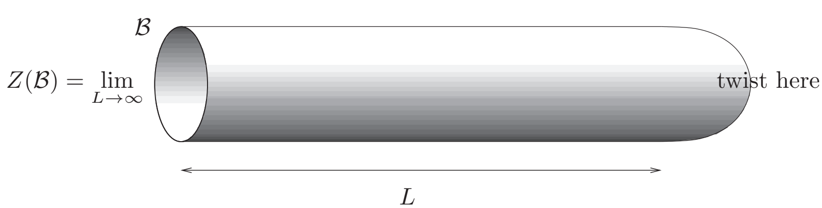

Fig. 7. The CV Central Charge

the central charge in much wider range of theories than the SCFTs for 4d $\mathcal{N} = 2$ string compactifications. The overlap (5.4) or the path-integral Fig. 7 makes sense for a B-brane (*resp.* A-brane) in a 2d $(2,2)$ supersymmetric QFT as long as the theory admits an A-twist (*resp.* a B-twist). This is so in SCFTs with an arbitrary $\widehat{c}$ and even in models which are not necessarily conformal. The only thing we need is A-twist (*resp.* B-twist). We shall call it the *CV central charge*, after the work by Cecotti and Vafa [45] who studied the inner products of RR ground states using worldsheets like Fig. 7. We may then ask whether the hemisphere partition function is equal to the CV central charge in general

$$Z_{D^2}(\mathcal{B}) \stackrel{?}{=} Z_{\mathrm{CV}}(\mathcal{B}). \tag{5.5}$$

The space of RR ground states form a vector bundle over the parameter space $\mathfrak{M} = \mathfrak{M}_A$ (*resp.* $\mathfrak{M}_B$), which we shall call the *vacuum bundle* $\mathcal{H}_{\mathrm{v}}$. To be specific, we consider states defined on the circle of circumference $\boldsymbol{\beta}$. This bundle is equipped with the hermitian inner product g induced from the one on the full space of states, and also with a holomorphic structure; the worldsheet like Fig. 7 with an insertion of a chiral operator ϕ at the tip defines a RR ground state $|\phi\rangle_{\mathrm{RR}}$, and if $\phi(t)$ is a holomorphic family of chiral operaors over $U \subset \mathfrak{M}$, then $|\phi(t)\rangle_{\mathrm{RR}}$ is a holomorphic section of $\mathcal{H}_{\mathrm{v}}$ over U. We denote the associated hermitian connection by D. For a local operator $\mathcal{O}$, we denote by $C_{\mathcal{O}}$ the operator on $\mathcal{H}_{\mathrm{v}}$ defined by multiplication by $\mathcal{O}$ followed by the projection to $\mathcal{H}_{\mathrm{v}}$. In [45], it was shown that these structures satisfies a system of equations called the tt^* equations, which amounts to the flatness of the connection ∇ of $\mathcal{H}_{\mathrm{v}}$ defined by

$$\nabla_{\phi} := D_{\phi} + \boldsymbol{\beta}\,\mathrm{e}^{iv} C_{\phi}, \qquad \nabla_{\overline{\phi}} := D_{\overline{\phi}} + \boldsymbol{\beta}\,\mathrm{e}^{-iv} C_{\overline{\phi}}, \tag{5.6}$$

for an arbitrary phase e^{iv}. Here, ϕ and $\overline{\phi}$ are A-chiral and anti-A-chiral operators (*resp.* B-chiral and anti-B-chiral operators) and may be identified respectively as the holomorphic and anti-holomorphic tangent vectors of $\mathfrak{M} = \mathfrak{M}_A$ (*resp.* $\mathfrak{M}_B$) that correspond to the deformation of the (Minkowski) action by $-Q_+\overline{Q}_-\phi$ and $\overline{Q}_+ Q_- \overline{\phi}$ (*resp.* $-Q_+Q_-\phi$ and $\overline{Q}_+\overline{Q}_-\overline{\phi}$).

Properties of $Z_{\mathrm{CV}}(\mathcal{B}) = {}_{\mathrm{RR}}\langle \mathcal{B}|0\rangle_{\mathrm{RR}}$, or more generally, of the overlap of the boundary state and all the RR ground states are studied in [47, 33]. It was found that ${}_{\mathrm{RR}}\langle \mathcal{B}|$ defines a parallel section of $\mathcal{H}_{\mathrm{v}}^*$ with respect to the connection ∇ for the phase $\mathrm{e}^{iv} = -i$.[3] In terms of local holomorphic

[3] More generally, the phase is $\mathrm{e}^{iv} = -i\,\mathrm{e}^{iu}$ if $\mathcal{B}$ is a $\mathrm{B}_{\mathrm{e}^{iu}}$-brane (*resp.* $\mathrm{A}_{\mathrm{e}^{iu}}$-brane).

coordinates t^α, this is rephrased as follows. Let ϕ_α be the chiral operator corresponding to $\frac{\partial}{\partial t^\alpha}$ and put $\Pi^{\mathcal{B}}_\alpha = {}_{\rm RR}\langle \mathcal{B}|\phi_\alpha\rangle_{\rm RR}$, which is obtained by the path-integral as in Fig. 7 with ϕ_α inserted at the tip of the cigar. Then,

$$\frac{\partial}{\partial t^\alpha}\Pi^{\mathcal{B}}_\beta = A^\gamma_{\alpha\beta}\Pi^{\mathcal{B}}_\gamma - i\boldsymbol{\beta} C^\gamma_{\alpha\beta}\Pi^{\mathcal{B}}_\gamma, \qquad \frac{\partial}{\partial \overline{t^\alpha}}\Pi^{\mathcal{B}}_\beta = i\boldsymbol{\beta} C^\gamma_{\overline{\alpha}\beta}\Pi^{\mathcal{B}}_\gamma. \tag{5.7}$$

Here $A^\gamma_{\alpha\beta}$, $C^\gamma_{\alpha\beta}$ and $C^\gamma_{\overline{\alpha}\beta}$ are the representating matrices for D_{ϕ_α}, C_{ϕ_α} and $C_{\overline{\phi}_\alpha}$ with respect to the holomorphic frame $\{|\phi_\alpha\rangle_{\rm RR}\}$. $C^\gamma_{\alpha\beta}$ is the same as the structure function $\phi_\alpha\phi_\beta = \phi_\gamma C^\gamma_{\alpha\beta}$ of the chiral ring. Using the tt^* metric $g_{\overline{\alpha}\beta} := {}_{\rm RR}\langle\phi_\alpha|\phi_\beta\rangle_{\rm RR}$, the other two elements can also be written as $A^\gamma_{\alpha\beta} = g^{\gamma\overline{\lambda}}\partial_\alpha g_{\overline{\lambda}\beta}$ and $C^\gamma_{\overline{\alpha}\beta} = (g_{\overline{\beta}\nu}C^\nu_{\alpha\lambda}g^{\lambda\overline{\gamma}})^*$. It was also found in [47, 33] that $\Pi^{\mathcal{B}}_\alpha$'s are independent of the B-chiral (*resp.* A-chiral) parameters.

When we consider a family of superconformal field theories over a subspace of $\mathfrak{M}^0_A$ (*resp.* $\mathfrak{M}^0_B$), it follows from (5.7) that the CV central charge $Z_{\rm CV}(\mathcal{B}) = \Pi^{\mathcal{B}}_0$ depends holomorphically on the parameters $(t^i)^\ell_{i=1}$ of the family and that it obeys holomorphic differential equations which take the form of deformation of the chiral ring relation among the corresponding marginal operators $\{\phi_i\}^\ell_{i=1}$.

This can be shown as folows. Recall that the vector (*resp.* axial) $U(1)$ R-symmetry with charge integrality is used for the twist. A superconformal field theory also have an axial (*resp.* vector) $U(1)$ R-symmetry, which we shall call R'-symmetry for the moment. The chiral ring $\mathcal{R}_A$ (*resp.* $\mathcal{R}_B$) is graded by the R'-charge. The identity is the unique operator of the minimum R'-charge zero, and there is a unique operator of the maximal R'-charge $2\widehat{c}$. The R'-charge of the state $|\phi_\alpha\rangle_{\rm RR}$ is equal to the R'-charge of the operator ϕ_α minus $\widehat{c}$. This yields the selection rule: $g_{\overline{\alpha}\beta} = 0$ unless ϕ_α and ϕ_β have the same R'-charge, and in particular $g_{\overline{0}\alpha} = \delta_{\alpha,0}g_{\overline{0}0}$. Therefore, $C^\gamma_{\overline{\imath}0} = (g_{\overline{0}\nu}C^\nu_{i\lambda}g^{\lambda\overline{\gamma}})^* = (g_{\overline{0}0}C^0_{i\lambda}g^{\lambda\overline{\gamma}})^*$. By the R'-grading of the chiral ring, the product of a marginal operator ϕ_i with any operator cannot include the identity component, $C^0_{i\lambda} = 0$ for any λ. This shows $C^\gamma_{\overline{\imath}0} = 0$ for any γ and hence the second equations of (5.7) read $\partial_{\overline{\imath}}\Pi^{\mathcal{B}}_0 = 0$. That is, $\Pi^{\mathcal{B}}_0$ is a holomorphic function of the marginal parameters t^i. Next, let us look at the first equations in (5.7) for the components $\Pi^{\mathcal{B}}_\nu$ where ν ranges over the basis of the subspace of the chiral ring generated by the marginal operators under question, including the identity. Let us align the components in the order of increasing R'-charge, starting from $\Pi^{\mathcal{B}}_0$, and put them in a column vector $\vec{\Pi}$. Then, the equations takes the form

$$\partial_i\vec{\Pi} = (A_i - i\boldsymbol{\beta} C_i)^T\vec{\Pi}, \tag{5.8}$$

where A_i are R'-block diagonal and non-holomorphic matrices and C_i are lower-triangular, holomorphic, and mutually commutative matrices. By definition, all the components of $\vec{\Pi}$ can be written as linear combinations of holomorphic derivatives of the first component $\Pi_0^{\mathcal{B}}$, where the coefficients are not necessarily holomorphic functions. Suppose there is a chiral ring relation $R(t; \phi_1, \ldots, \phi_\ell) = 0$ which we may assume homogeneous. It implies a matrix identity $R(t; C_1, \ldots, C_\ell) = 0$ or its transposed version. Then, using (5.8), we find an equation of the form $R\left(t; \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^\ell}\right) \vec{\Pi} = M\vec{\Pi}$, for some matrix M which is the sum of products of A_i^T's, C_i^T's and their derivatives. Expressing $\vec{\Pi}$ on the right hand side in terms of the derivatives of Π_0 and taking the first component, we obtain a differential equation

$$\left[R\left(t; \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^\ell}\right) + \delta D_R\right] \Pi_0^{\mathcal{B}} = 0, \tag{5.9}$$

for some differential operator δD_R of lower order than $R\left(t; \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^\ell}\right)$. The differentials in δD_R are holomorphic ones by construction, but the coefficients are not a priori holomorphic. However, the flatness of ∇ guarantees that they must be holomorphic.

This argument holds even when the theories are not exactly conformal — the only requirement is the existence of R'-symmetry with the properties quoted at the begining of the previous paragraph. For example, the sigma model with Calabi-Yau target X over the moduli space of complexified Kähler class (*resp.* complex structure), the GLSM obeying the Calabi-Yau condition over $\mathfrak{M}_t$ (*resp.* $\mathfrak{M}_W$), and LG model with a family of quasi-homogenious superpotentials. All these models are expected to flow in the infra-red limit to a family of superconformal field theories over the same parameter space. In the present discussion, we shall call such theories "conformal" and the parameters "marginal" by a slight abuse of language.

In the conformal case, we have seen that the CV central charge $Z_{\rm CV}(\mathcal{B})$ has all the properties that the hemisphere partition function $Z_{D^2}(\mathcal{B})$ has, concerning the dependence on the marginal parameters: no dependence on the B-chiral (*resp.* A-chiral) parameters, holomorphic dependence on the A-chiral (*resp.* B-chiral) parameters, and a differential equation for each relation among marginal A-chiral (*resp.* B-chiral) operators. This is a strong support for the relation (5.5).

Another support for (5.5) can be obtained through the sphere partition function Z_{S^2}. Right after Z_{S^2} for GLSM was computed in [4, 5], it was observed in [6] in some Calabi-Yau examples including $T_{5,5}^{U(1)}$ that it is related to the Kähler potential K for the Zamolodchikov metric on

$\mathfrak{M}_t \subset \mathfrak{M}_A^0$ via

$$Z_{S^2} = \mathrm{e}^{-K}. \tag{5.10}$$

Later, this was shown to hold in general [7, 8]. In [45], it was shown that the Kähler potential K on $\mathfrak{M}_A^0$ (*resp.* $\mathfrak{M}_B^0$) is related to the RR-ground state $|0\rangle_{\mathrm{RR}}$ obtained via A-twist (*resp.* B-twist) by

$$\mathrm{e}^{-K} = {}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}}. \tag{5.11}$$

Combining (5.10) and (5.11) we have

$$Z_{S^2} = {}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}}. \tag{5.12}$$

This was observed also in the Landau-Ginzburg model with quasihomogeneous superpotential: The inner product ${}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}}$ is determined from the tt^* equation in [45], and it was found to be given by the same integral as in (4.35) for Z_{S^2}. The relation (5.12) results in an important consequence concerning (5.5). Note that, for any pair of basis $\{|i\rangle_{\mathrm{RR}}\}$ and $\{|j'\rangle_{\mathrm{RR}}\}$ of the space $\mathcal{H}_{\mathrm{v}}^0$ of RR ground states with vector (*resp* axial) R-charge zero[4] we have

$$ {}_{\mathrm{RR}}\langle 0|0\rangle_{\mathrm{RR}} = \sum_{i,j'} {}_{\mathrm{RR}}\langle 0|j'\rangle_{\mathrm{RR}}\, g^{j'\,\bar{\imath}}\, {}_{\mathrm{RR}}\langle i|0\rangle_{\mathrm{RR}}, \tag{5.13}$$

where $g^{j'\,\bar{\imath}}$ is the inverse matrix to $g_{\bar{\imath}j'} = {}_{\mathrm{RR}}\langle i|j'\rangle_{\mathrm{RR}}$. If we can take basis of $\mathcal{H}_{\mathrm{v}}^0$ corresponding to sets of D-branes, this can be regarded as a "bilinear identity" expressing the sphere partition function as a sum of products of the hemisphere partition functions, in view of the fact (5.12) and supposing that the conjectural relation (5.5) is indeed true. Indeed, such a relation seems to hold. Let $\{\mathcal{B}_i\}$ be a set of B-branes (*resp.* A_--branes) such that the ground state projections of their RR boundary states, $\{P_{\mathrm{v}}|\mathcal{B}_i\rangle_{\mathrm{RR}}\}$, form a basis of $\mathcal{H}_{\mathrm{v}}^0$. This is the case when $\mathcal{B}_i$'s represent a basis of the Grothendieck group of the category of B-branes (*resp.* A_--branes). Let $\{\mathcal{B}'_i\}$ be a set of B_--branes (*resp.* A-branes) with the same properties. Then, we claim the factorization formula

$$Z_{S^2} \overset{!}{=} \sum_{i,j} Z_{D^2}^{(-)}(\mathcal{B}'_j)\cdot \mathrm{I}^{\mathcal{B}_i,\overline{\mathcal{B}}'_j}\cdot Z_{D^2}^{(+)}(\mathcal{B}_i), \tag{5.14}$$

[4]Note that the state $|0\rangle_{\mathrm{RR}}$ obtained by the A-twist (*resp.* B-twist) as well as the boundary state of a brane preserving the vector (*resp.* axial) R-symmetry have vanishing vector (*resp.* axial) R-charge.

where $\mathrm{I}^{\mathcal{B}_i,\overline{\mathcal{B}}'_j}$ is the inverse matrix of

$$\mathrm{I}_{\overline{\mathcal{B}}'_j,\mathcal{B}_i} := \mathrm{Tr}_{\mathcal{H}_{\overline{\mathcal{B}}'_j,\mathcal{B}_i}}(-1)^F\,\mathrm{e}^{-\beta H}, \tag{5.15}$$

which is the Witten index for the open string with the boundary conditions $\overline{\mathcal{B}}'_j$ on the left and the boundary condition $\mathcal{B}_i$ on the right. See Appendix B for the definition of the conjugation $\mathcal{B} \mapsto \overline{\mathcal{B}}$ of boundary conditions. Here $Z^{(\pm)}_{D^2}$ is the partition function on the hemisphere with $\mathrm{A}_{(\pm)}$-type (*resp.* $\mathrm{B}_{(\pm)}$-type) supersymmetry. (Recall that we had been looking at $Z_{D^2} = Z^{(+)}_{D^2}$ for a while.) Note that the conjecture (5.5) is $Z^{(+)}_{D^2}(\mathcal{B}_i) = {}_{\mathrm{RR}}\langle\mathcal{B}_i|0\rangle_{\mathrm{RR}}$ and extends to $Z^{(-)}_{D^2}(\mathcal{B}'_i) = {}_{\mathrm{RR}}\langle\mathcal{B}'_i|\overline{0}\rangle_{\mathrm{RR}}$, where ${}_{\mathrm{RR}}\langle\mathcal{B}'_i|\overline{0}\rangle_{\mathrm{RR}}$ is obtained from the path-integral on the worldsheet as Fig. 7 but with *anti*-topological twist in the curved region. By rotation, we can simply identify ${}_{\mathrm{RR}}\langle\mathcal{B}'_i|\overline{0}\rangle_{\mathrm{RR}}$ with ${}_{\mathrm{RR}}\langle 0|\mathcal{B}'_i\rangle_{\mathrm{RR}}$. Note also that the open string Witten index is given by $\mathrm{I}_{\overline{\mathcal{B}}'_j,\mathcal{B}_i} = {}_{\mathrm{RR}}\langle\mathcal{B}_i|P_{\mathrm{v}}|\mathcal{B}'_j\rangle_{\mathrm{RR}}$. Therefore, if the conjecture is true, the factorization formula (5.14) is nothing but the identity (5.13) and hence must hold. In [9], the formula was indeed shown to hold for GLSM in the geometric regime where we can use the Riemann-Roch formula for the index, $\mathrm{I}_{\overline{\mathcal{E}}_j,\mathcal{E}_i} = \int_X \mathrm{ch}(\mathcal{E}_j^*)\mathrm{ch}(\mathcal{E}_i)\widehat{\mathrm{A}}_X$. The formula also holds in the LG model due to Poincaré-Lifshetz duality: Let $B_\pm \subset \mathbb{C}^n$ be the subset defined by $\pm\mathrm{Im}W > R$ for a large positive R. Then, $\mathrm{H}^n(\mathbb{C}^n, B_\pm)$ is the group of RR-charges of $\mathrm{A}_\pm$-branes. Let us choose a basis of these groups, $\{\gamma_i\} \subset \mathrm{H}^n(\mathbb{C}^n, B_-)$ and $\{\gamma'_i\} \subset \mathrm{H}^n(\mathbb{C}^n, B_+)$. Then, the intersection matrix $\mathrm{I}_{\overline{\jmath},i} = \#(\gamma'_j \cap \gamma_i)$ has an inverse $\mathrm{I}^{i,\overline{\jmath}}$ and the folowing identity holds

$$\int_{\mathbb{C}^n} \mathrm{e}^{-ir(W+\overline{W})}\overline{\Omega}\wedge\Omega = \sum_{i,j}\int_{\gamma'_j}\mathrm{e}^{-ir\overline{W}}\overline{\Omega}\cdot \mathrm{I}^{i,\overline{\jmath}}\cdot\int_{\gamma_i}\mathrm{e}^{-irW}\Omega, \tag{5.16}$$

for a holomorphic volume form Ω such as $\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$. The intersection number $\#(\gamma'_j \cap \gamma_i)$ is identified as the Witten index for the open string between $\overline{\gamma}'_j$ and γ_i, and hence the identity (5.16) is nothing but (5.14).

The perfect similarity bewteen Z_{D^2} and Z_{CV} goes away if we consider non-conformal theoies, such as a non-Calabi-Yau GLSM or a LG model with a non-quasi-homogeneous superpotential. While they are both independent of B-chiral (*resp.* A-chiral) parameters, they differ in the dependence on the A-chiral (*resp.* B-chiral) parameters: $Z_{D^2}(\mathcal{B})$ is holomorphic but $Z_{\mathrm{CV}}(\mathcal{B})$ is not — the proof requires R'-symmetry which is absent in a non-conformal theory. However, it is true that $Z_{D^2}(\mathcal{B})$ depends on the brane $\mathcal{B}$ only through its RR charge. Therefore, we may still say $Z_{D^2}(\mathcal{B}) = {}_{\mathrm{RR}}\langle\mathcal{B}|\Omega\rangle_{\mathrm{RR}}$ for some RR ground state $|\Omega\rangle_{\mathrm{RR}}$. On the other hand, it was shown in [48] that the sphere partition function is invariant under a one parameter family of deformation of the background, and the limit of the deformation can be interpreted as the inner product of a pair of RR ground states, $Z_{S^2} = {}_{\mathrm{RR}}\langle\Omega'|\Omega''\rangle_{\mathrm{RR}}$. If the ground states $|\Omega\rangle_{\mathrm{RR}}$, $|\Omega'\rangle_{\mathrm{RR}}$, etc that appear in the above relations are related to each other, then, via the identity like (5.13), we should again have a factorization formula for the sphere partition function. Indeed, the formula (5.14) was shown in [9] to hold in GLSM with a geometric regime with Fano targets space as well, such as $\mathrm{T}^{U(1)}_{N,d}$ with $d < N$. Also, the identity (5.16) holds whether or not W is quasihomogeneous. The formula suggests a more precise version of the relations: There are families of RR ground states $|\Omega^{\mathrm{e}^{iu}}_{(\pm)}\rangle_{\mathrm{RR}} = |\Omega, \mathrm{A}^{\mathrm{e}^{iu}}_{(\pm)}\rangle_{\mathrm{RR}}$ (*resp.* $|\Omega, \mathrm{B}^{\mathrm{e}^{iu}}_{(\pm)}\rangle_{\mathrm{RR}}$) parametrized by $\mathrm{e}^{iu} \in U(1)$, such that the partition function on the sphere with the A-type (*resp.* B-type) supersymmetry is given by

$$Z_{S^2} = {}_{\mathrm{RR}}\langle\Omega^{-1}_{(+)}|\Omega^{1}_{(+)}\rangle_{\mathrm{RR}} = {}_{\mathrm{RR}}\langle\Omega^{-1}_{(-)}|\Omega^{1}_{(-)}\rangle_{\mathrm{RR}}, \tag{5.17}$$

and the partition function on the hemisphere with the $\mathrm{A}_{(\pm)}$-type (*resp.* $\mathrm{B}_{(\pm)}$-type) supersymmetry and a $\mathrm{B}_{\pm 1}$-brane (*resp.* $\mathrm{A}_{\mp 1}$-brane) $\mathcal{B}_{\pm}$ at the boundary is given by

$$Z^{(+)}_{D^2}(\mathcal{B}_+) = {}_{\mathrm{RR}}\langle\mathcal{B}_+|\Omega^{1}_{(+)}\rangle_{\mathrm{RR}} = {}_{\mathrm{RR}}\langle\Omega^{-1}_{(-)}|\mathcal{B}_+\rangle_{\mathrm{RR}}, \tag{5.18}$$

$$Z^{(-)}_{D^2}(\mathcal{B}_-) = {}_{\mathrm{RR}}\langle\mathcal{B}_-|\Omega^{1}_{(-)}\rangle_{\mathrm{RR}} = {}_{\mathrm{RR}}\langle\Omega^{-1}_{(+)}|\mathcal{B}_-\rangle_{\mathrm{RR}}. \tag{5.19}$$

For the sphere or hemisphere with $\mathrm{A}^{\mathrm{e}^{iu}}$ or $\mathrm{A}^{\mathrm{e}^{iu}}_{(\pm)}$-type (*resp.* $\mathrm{B}^{\mathrm{e}^{iu}}$ or $\mathrm{B}^{\mathrm{e}^{iu}}_{(\pm)}$-type) supersymmetry, we replace $\Omega^{\pm 1}_{(\pm)}$ by $\Omega^{\pm\mathrm{e}^{iu}}_{(\pm)}$ in the above expressions.

5.3. The Central Charge in Topological String Theory

The hemisphere partition function can also be related to a natural observable in topological field/string theory. A central element in this relation is what is known as Dubrovin's connection of a vector bundle on the space $\mathfrak{M} \times \mathbb{C}^*$, where $\mathfrak{M}$ is the space of supersymmetry preserving parameters of the theory, $\mathfrak{M}_A$ or $\mathfrak{M}_B$, and $\mathbb{C}^*$ is the space of an additional parameter. It turns out that this additional parameter is related to the radius r of the hemisphere. We will also find a relation to what is known as *macroscopic loops* in the theory of 2d quantum gravity. In this connection, the radius r is related to the length ℓ of the loop.

In the literature, several different symbols are used for Dubrovin's parameter. For convenience, we list below these symbols and the parameter z used in this section as well as the corresponding parameters in the 2d gravity and the hemisphere:

Reference	A	B	C	D	E	this section	2d gravity	hemisphere
Notation	δ	$\hbar^{-1}$	z	z^{-1}	u^{-1}	$-z^{-1}$	$-\ell$	$-r\Lambda$

The references are: (A) Kyoji Saito's original papers [15–17]; (B) Givental's earlier paper [49]; (C) Dubrovin's paper [51]; (D) Givental's later paper [52] and Iritani [53]; (E) Katzarkov et al [54].

Frobenius Manifold

Recall that a 2d $(2,2)$ supersymmetric field theory with a vector (*resp.* an axial) $U(1)$ R-symmetry with charge integrality admits an A-twist (*resp.* a B-twist). Local operators of the resulting topological field theory are elements of the chiral ring $\mathcal{R} = \mathcal{R}_A$ (*resp.* $\mathcal{R}_B$), and the sphere three point functions $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_{S^2}$ define a structure of *Frobenius algebra* on $\mathcal{R}$: it has a symmetric non-degenerate bilinear form $\eta(\mathcal{O}_1,\mathcal{O}_2) = \langle\,\mathrm{id}\;\mathcal{O}_1\mathcal{O}_2\rangle_{S^2}$ under which the product is self-adjoint, $\eta((\mathcal{O}_1\cdot\mathcal{O}_2),\mathcal{O}_3) = \eta(\mathcal{O}_1,(\mathcal{O}_2\cdot\mathcal{O}_3))$. Symmetry and self-adjointness follow from the properties of topological correlation functions, and non-degeneracy can be shown, say, by the spectral flow to RR ground states. Recall that elements of $\mathcal{R}$ are naturally identified as tangent vectors of type $(1,0)$ of the deformation space $\mathfrak{M} = \mathfrak{M}_A$ (*resp.* $\mathfrak{M}_B$), and hence we have an isomorphism $T_{\mathfrak{M}} \cong \mathcal{R}$ of holomorphic vector bundles on $\mathfrak{M}$. In particular, each fiber of $T_{\mathfrak{M}}$ is equipped with a structure of Frobenuius algebra. It varies holomorphically on $\mathfrak{M}$ and the bilinear form η can be regarded as a holomorphic metric on $\mathfrak{M}$, called the *topological metric.*

In [56], Dijkgraaf, Verline and Verlinde studied properties of the sphere three point functions of the family of topological field theories obtained by twisting a 2d $(2,2)$ superconformal field theory of central charge $c = 3\,\widehat{c}$ and its deformations. Later in [50, 51], Dubrovin extracted the finding and summarized into the structure of *Frobenius manifold* on the space $\mathfrak{M}$, which turned out to be the same as the *flat structure* found by K. Saito a decade earlier [15–17]. In the formulation of [51], it is stated as follows:

(i) The topological metric η is flat, i.e. the Levi-Civita connection ∇^η has zero curvature.

(ii) The vector field corresponding to the identity is ∇^η-parallel.

(iii) There is a holomorphic vectore field, called *Euler vector field* E, such that $L_E\eta = (2-\widehat{c})\eta$ and that $\nabla^\eta E : T_{\mathfrak{M}} \to T_{\mathfrak{M}}$ defined by $v \mapsto \nabla^\eta_v E$ is diagonalizable.

(iv) The connection $\widetilde{\nabla}$ of the vector bundle $T_{\mathfrak{M}}$ over $\mathfrak{M}\times\mathbb{C}^*_z$ defined below is flat,

$$\widetilde{\nabla}_v := \nabla^\eta_v - z^{-1}C_v, \tag{5.20}$$

$$\widetilde{\nabla}_{z\frac{\partial}{\partial z}} := z\frac{\partial}{\partial z} + z^{-1}C_E + \mu \tag{5.21}$$

where C_v is the multiplication by $v \in T_{\mathfrak{M}}$ and

$$\mu := 1 - \frac{\widehat{c}}{2} - \nabla^\eta E. \tag{5.22}$$

We emphasize that the connections ∇^η and $\widetilde{\nabla}$ of $T_{\mathfrak{M}}$ are holomorphic connections and are not the same as the C^∞ connections D and ∇ of $\mathcal{H}_{\mathrm{v}}$ introduced in the previous subsection, even though $T_{\mathfrak{M}} \cong \mathcal{R}$ and $\mathcal{H}_{\mathrm{v}}$ are isomorphic as holomorphic bundles via the spectral flow. The relation between them was discussed in [57] (see also [58]).

We can find *flat coordinates* $(\mathrm{t}^\alpha)_{\alpha=0}^{N-1}$ of $(\mathfrak{M},\eta)$, such that $\frac{\partial}{\partial \mathrm{t}^0}$ corresponds to the identity operator, the Euler vector field is of the form $E = \sum_\alpha \left(E_0^\alpha + (1-q_\alpha)\mathrm{t}^\alpha\right)\frac{\partial}{\partial \mathrm{t}^\alpha}$, and $\eta_{\alpha\beta}$ is non-zero only if $q_\alpha + q_\beta = \widehat{c}$, for some constants E_0^α and q_α. In particular $\mu^\alpha{}_\beta = \left(q_\alpha - \frac{\widehat{c}}{2}\right)\delta^\alpha_\beta$. If the theory at the origin $\mathrm{t}=0$ is a superconformal field theory, then E_0^α are zero and $2q_\alpha$ is the R'-charge, that is, the axial (*resp.* vector) R-charge, of the operator ϕ_α corresponding to $\frac{\partial}{\partial \mathrm{t}^\alpha}$. In particular, q_0 is zero and t^i with $q_i = 1$ are marginal parameters. In the flat coordinate system, a section $\xi = \xi_\alpha \mathrm{d}\mathrm{t}^\alpha$ of $T^*_{\mathfrak{M}}$ over $\mathfrak{M}\times\mathbb{C}^*$ is $\widetilde{\nabla}$-parallel when

$$\frac{\partial}{\partial \mathrm{t}^\beta}\,\xi_\alpha = -z^{-1}C^\gamma_{\beta\alpha}\xi_\gamma, \tag{5.23}$$

$$z\frac{\partial}{\partial z}\,\xi_\alpha = \left(-E^\beta\frac{\partial}{\partial \mathrm{t}^\beta} + q_\alpha - \frac{\widehat{c}}{2}\right)\xi_\alpha. \tag{5.24}$$

The identity component satisfies

$$z\frac{\partial}{\partial z}\xi_0 = \left(-E^\beta\frac{\partial}{\partial \mathrm{t}^\beta} - \frac{\widehat{c}}{2}\right)\xi_0. \tag{5.25}$$

For each chiral ring relation $R(\mathrm{t};\phi_{i_1},\ldots,\phi_{i_s}) = 0$, following the same procedure as in the derivation of (5.9), we find a differential equation of the form

$$\left[R\left(\mathrm{t}; z\frac{\partial}{\partial \mathrm{t}^{i_1}},\ldots,z\frac{\partial}{\partial \mathrm{t}^{i_s}}\right) + \delta\mathcal{D}_R\right]\xi_0 = 0, \tag{5.26}$$

where $\delta\mathcal{D}_R$ is a holomorphic differential operator of lower degree than R.

The equation (5.25) looks similar to the renormalization group equation (4.88) on the hemisphere partition function if we identify z with r^{-1} and the Euler vector field E as $-b_1\left(\frac{\partial}{\partial t}\right)$. The fact that we have an equation (5.26) for each chiral ring relation R is also similar to the fact that we have an equation (4.90) for each deformed relation among the central components of σ (see (4.96) and (4.108) for concrete examples). This suggests that the hemisphere partition function $Z_{D^2}(\mathcal{B})$ for the GLSM may be regarded as the identity component of a $\widetilde{\nabla}$-parallel section of $T^*_{\mathfrak{M}}$ over $\mathfrak{M}_t \times \mathbb{C}^*$, where the coordinate z of $\mathbb{C}^*$ is identified as a complexification of the inverse radius r^{-1}.

We next confirm this in GLSM for the example $\mathrm{T}^{U(1)}_{N,d}$ for $d \leq N$ where the full partition function can be captured in the geometric regime.

Gromov-Witten Theory

For the non-linear sigma model on a Kähler manifold X, as quoted earlier, the chiral ring $\mathcal{R}_A$ is the quantum cohomology ring $(\mathrm{H}^*(X;\mathbb{C}),\cdot)$ and the space $\mathfrak{M}_A$ may be identified as an open subset of $\mathrm{H}^*(X;\mathbb{C})$. The topological metric is simply the Poincaré pairing $\eta(\omega_1,\omega_2) = \int_X \omega_1 \wedge \omega_2$ and therefore affine coordinates of $\mathfrak{M}_A \subset \mathrm{H}^*(X;\mathbb{C})$ are flat. The sigma model is not conformal in general, but may be regarded as a "deformation" of a conformal theory with $\widehat{c} = \dim_{\mathbb{C}} X =: D$, which is the model with a fine-tuned metric when X is Calabi-Yau and the model with infinite volume when X is not Calabi-Yau. The conformal limit has an axial R-symmetry where the axial R-charge of a chiral ring element is the degree of the corresponding cohomology class.

Let $\{\phi_\alpha\}_{\alpha=0}^{N-1} \subset \mathrm{H}^*(X;\mathbb{C})$ be a basis consisting of classes of definite degrees, $\phi_\alpha \in \mathrm{H}^{2q_\alpha}(X;\mathbb{C})$, and let $(\mathrm{t}^\alpha)_{\alpha=0}^{N-1}$ be the corresponding affine coordinates. We take $\phi_0 = 1 \in \mathrm{H}^0(X;\mathbb{C})$ and $\{\phi_i\}_{i=1}^{\ell}$ to be the basis of $\mathrm{H}^{1,1}(X)$. Note that $\eta_{\alpha\beta} = \int_X \phi_\alpha \wedge \phi_\beta$ is non-zero only when $q_\alpha + q_\beta = D$. The Euler vector field is given by

$$E = \sum_{i=1}^{\ell} c_1(X)^i \frac{\partial}{\partial \mathrm{t}^i} - \sum_{\alpha=0}^{N-1} (q_\alpha - 1)\mathrm{t}^\alpha \frac{\partial}{\partial \mathrm{t}^\alpha} \tag{5.27}$$

where $\sum c_1(X)^i \phi_i = c_1(X)$ is the first Chern class of X.

$\widetilde{\nabla}$-parallel sections of $T^*_{\mathfrak{M}_A}$ over $\mathfrak{M}_A \times \mathbb{C}^*$ can be constructed using genus zero amplitudes in the associated topological string theory, that is, the Gromov-Witten theory. In this theory, each element $\mathcal{O} \in \mathrm{H}^*(X;\mathbb{C}) = \mathcal{R}_A$ defines a *primary operator* denoted by the same symbol (the one for $\mathcal{O} = \phi_0 = 1$ is called the *puncture operator* P), as well as operators $\tau_n\mathcal{O} = \psi^n\mathcal{O}$ called the *gravitational descendants*, for

$n = 1, 2, \ldots$. See for example [59–63] for the definition and properties.[5] For each $\mathcal{O} \in \mathrm{H}^*(X; \mathbb{C})$, let us put

$$w[\mathcal{O}] := \frac{\mathcal{O}}{1 + z^{-1}\psi} = \sum_{n=0}^{\infty} (-1)^n z^{-n} \tau_n \mathcal{O}, \tag{5.28}$$

and

$$\xi_\alpha[\mathcal{O}] := \left\langle P\, \phi_\alpha\, w\left[z^{-\mu} \left(z^{c_1(X)} \mathcal{O} \right) \right] \right\rangle, \tag{5.29}$$

where $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle$ is the genus zero amplitude in the background t. Then, $\xi[\mathcal{O}] = \xi_\alpha[\mathcal{O}]\mathrm{dt}^\alpha$ is a $\widetilde{\nabla}$-parallel section of $T^*_{\mathfrak{M}_A}$. The construction is due to Givental [49] and Dubrovin [51]. Givental introduced the combination $w[\mathcal{O}]$ and showed that it defines parallel sections in the $\mathfrak{M}_A$ direction, and Dubrovin put $z^{-\mu} z^{c_1(X)}$ to make it parallel also in the z-direction. In view of its importance, let us give a proof of the assertion that $\xi[\mathcal{O}]$ is $\widetilde{\nabla}$-parallel, that is, $\xi_\alpha[\mathcal{O}]$ satisfy (5.23) and (5.24). Equation (5.23) is an immediate consequence of the topological recursion relation [59],

$$\begin{aligned} \frac{\partial}{\partial \mathrm{t}_\beta} \xi_\alpha[\mathcal{O}] &= \sum_{n=0}^{\infty} (-1)^n z^{-n} \langle\, \phi_\beta\, P\, \phi_\alpha\, \tau_n(\mathcal{O}_z)\, \rangle \\ &= \sum_{n=1}^{\infty} (-1)^n z^{-n} \langle\, \tau_{n-1}(\mathcal{O}_z)\, P\, \phi_\gamma\, \rangle \langle\, \phi^\gamma \phi_\beta \phi_\alpha\, \rangle \\ &= -z^{-1} C^\gamma_{\beta\alpha} \xi_\gamma[\mathcal{O}], \end{aligned} \tag{5.30}$$

with $\mathcal{O}_z := z^{-\mu}(z^{c_1(X)}\mathcal{O})$ where we used $\langle \phi^\gamma P \phi_\beta \phi_\alpha \rangle = 0$. To see (5.24), first note that

$$\begin{aligned} z\frac{\partial}{\partial z} \xi_\alpha[\mathcal{O}] &= \sum_{n=1}^{\infty} (-1)^n (-n) z^{-n} \langle\, P\, \phi_\alpha\, \tau_n(\mathcal{O}_z)\, \rangle \\ &\quad + \sum_{n=0}^{\infty} (-1)^n z^{-n} \langle\, P\, \phi_\alpha\, \tau_n(-\mu \mathcal{O}_z + z^{-1} c_1(X) \wedge \mathcal{O}_z)\, \rangle. \end{aligned} \tag{5.31}$$

[5] $\sigma_n \mathcal{O}$ in [59, 61] is equal to $n! \times \tau_n \mathcal{O}$.

On the other hand, the divisor equation [64] for $c_1(X)$ and the selection rule (degree-matching) yield

(5.32)

$$\begin{aligned} & c_1(X)^i \frac{\partial}{\partial \mathrm{t}^i} \xi_\alpha[\mathcal{O}] = \\ & \sum_{n=0}^{\infty} (-1)^n z^{-n} \left[\left(-\frac{D}{2} + q_\alpha + n + \sum_\beta (q_\beta - 1) \mathrm{t}^\beta \frac{\partial}{\partial \mathrm{t}^\beta} \right) \langle\, P\, \phi_\alpha\, \tau_n(\mathcal{O}_z)\,\rangle \right. \\ & \qquad \left. + \langle\, P\, \phi_\alpha\, \tau_n(\mu \mathcal{O}_z)\,\rangle \right] + \sum_{n=1}^{\infty} (-1)^n z^{-n} \langle\, P\, \phi_\alpha\, \tau_{n-1}(c_1(X) \wedge \mathcal{O}_z)\,\rangle. \end{aligned}$$

Summing the above two, we find that (5.24) holds.

For a B-brane in the sigma model, $\mathcal{E} \in \mathrm{D}^b_{Coh}(X)$, following [53], we define its GW central charge as the identity component of $\xi[\mathcal{O}]$ for a particular $\mathcal{O}$ determined by the brane,

(5.33)

$$\begin{aligned} & Z_{\mathrm{GW}}(\mathcal{E}) \\ & \quad := \frac{1}{(-2\pi i)^D} \left\langle P\, P\, w \left[z^{-\mu} \left(z^{c_1(X)} (-2\pi i)^{\frac{\deg}{2}} \left(\widehat{\Gamma}_X\, \mathrm{ch}(\mathcal{E}) \right) \right) \right] \right\rangle \Big|_{\mathfrak{M}_A^{0,c} \times \mathbb{C}^*} \\ & \quad = \frac{1}{(-2\pi i)^D} \xi_0 \left[(-2\pi i)^{\frac{\deg}{2}} \left(\widehat{\Gamma}_X\, \mathrm{ch}(\mathcal{E}) \right) \right] \Big|_{\mathfrak{M}_A^{0,c} \times \mathbb{C}^*}, \end{aligned}$$

where $\mathfrak{M}_A^{0,c} \subset \mathfrak{M}_A$ is the subspace in which only the Kähler parameters $-\mathrm{t}^i$ are turned on.[6] The equation (5.25) reads

$$z \frac{\partial}{\partial z} Z_{\mathrm{GW}}(\mathcal{E}) = \left(-c_1(X)^i \frac{\partial}{\partial \mathrm{t}^i} - \frac{D}{2} \right) Z_{\mathrm{GW}}(\mathcal{E}), \tag{5.34}$$

which looks a lot closer to the renormalization group equation (4.88), epsecially to (4.89), with $z \to r^{-1}$. This means that $Z_{\mathrm{GW}}(\mathcal{E})$ is $z^{-\frac{D}{2}}$ times a function of

$$\mathrm{t}_R := \mathrm{t} - c_1(X) \log(z). \tag{5.35}$$

Using the divisor equation repeatedly, we find

$$Z_{\mathrm{GW}}(\mathcal{E}) = z^{-\frac{D}{2}} \left[\int_X \exp\left(-\frac{i}{2\pi}\, \mathrm{t}_R \right) \widehat{\Gamma}_X\, \mathrm{ch}(\mathcal{E}) + O(\mathrm{e}^{\mathrm{t}_R}) \right], \tag{5.36}$$

[6] Z_{GW} is related to the central charge Z_{Iritani} of [53] by $Z_{\mathrm{GW}}(\mathcal{E}) = (-1)^D z^{-\frac{D}{2}} Z_{\mathrm{Iritani}}(\mathcal{E}^\vee)|_{\mathfrak{M}_A^{0,c} \times \mathbb{C}^*}$.

where $O(\mathrm{e}^{\mathrm{t}_R})$ is a power series of the form $\sum c_\beta \,\mathrm{e}^{\langle \mathrm{t}_R,\beta\rangle}$ where the sum is over the effective classes of X. This matches with the large volume behaviour of the hemisphere partition function (4.82) up to the numerical constant C, under

$$z = (r\Lambda)^{-1}, \qquad \mathrm{t}_R = -\omega_R + iB. \tag{5.37}$$

Let us examine the relation more closely, for the model $\mathrm{T}^{U(1)}_{N,d}$ with $d \leq N$ where we have a large volume formula (4.77) that captures the full partition function. For convenience, we copy the formula here with $r\Lambda \to z^{-1}$

$$\begin{aligned} Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = Cz^{-\frac{D}{2}} \sum_{n=0}^{\infty} (-1)^{Nn}\,\mathrm{e}^{-nt_R} \int_{X_f} \widehat{\Gamma}_{X_f}(n) \\ \times \exp\left(\frac{i}{2\pi}(t_R - d\pi i)H\right)\mathrm{ch}(\mathcal{E}), \end{aligned} \tag{5.38}$$

where $t_R = t + (N-d)\log(z)$, $\mathcal{E} = \pi_+(\mathfrak{B})$ and

$$\begin{aligned} \widehat{\Gamma}_{X_f}(n) &= \frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^N}{\Gamma\left(1+\frac{H}{2\pi i}+n\right)^N} \frac{\Gamma\left(1+d\left(\frac{H}{2\pi i}+n\right)\right)}{\Gamma\left(1+d\frac{H}{2\pi i}\right)} \cdot \widehat{\Gamma}_{X_f}(0) \\ &= \frac{\prod_{a=1}^{dn}\left(a+\frac{dH}{2\pi i}\right)}{\prod_{b=1}^{n}\left(b+\frac{H}{2\pi i}\right)^N} \cdot \widehat{\Gamma}_{X_f}. \end{aligned} \tag{5.39}$$

The integral remains the same if we operate $\left(-\frac{2\pi i}{z}\right)^{\frac{\deg}{2}}$ to the integrand, which does $H \to -\frac{2\pi i}{z}H$ for example, and then divide the result by $\left(-\frac{2\pi i}{z}\right)^{D}$. This transforms the expression to

$$Z^{\mathrm{LV}}_{D^2}(\mathfrak{B}) = \frac{C}{(-2\pi i)^D} \int_{X_f} I_{X_f}(\widetilde{t}, z)\cdot z^{-\mu}\left(z^{c_1(X_f)}(-2\pi i)^{\frac{\deg}{2}}\left(\widehat{\Gamma}_{X_f}\mathrm{ch}(\mathcal{E})\right)\right), \tag{5.40}$$

where

$$\widetilde{t} := t - d\pi i \tag{5.41}$$

and

$$I_{X_f}(\widetilde{t}, z) := \mathrm{e}^{\widetilde{t}H/z} \sum_{n=0}^{\infty} \mathrm{e}^{-n\widetilde{t}} \cdot \frac{\prod_{a=1}^{dn}(dH - za)}{\prod_{b=1}^{n}(H - zb)^N}. \tag{5.42}$$

At this point, we quote Givental's mirror theorem [49] on the relationship between this $I_{X_f}(\widetilde{t}, z)$ and the Gromov-Witten theory of X_f. Its states that (see also [62])

$$\int_{X_f} I_{X_f}(\widetilde{t}, z)\alpha = \mathcal{Y}(\widetilde{t}, z)\,\langle\, P\, P\, w[\alpha]\,\rangle\Big|_{\mathrm{t}=-\mathcal{T}(\widetilde{t})}, \tag{5.43}$$

for any $\alpha \in \mathrm{H}^*(X_f, \mathbb{C})$, where

(5.44)

	$d \leq N$	$d = N - 1$	$d = N$
$\mathcal{Y}(\widetilde{t}, z)$	1	$\exp\left(-d!\,\mathrm{e}^{-\widetilde{t}}/z\right)$	$C^{-1} z^{\frac{D}{2}} Z(\mathrm{D0})$
$\mathcal{T}(\widetilde{t})$	$\widetilde{t}H$	$\widetilde{t}H$	$-2\pi i \dfrac{Z(\mathrm{D2})}{Z(\mathrm{D0})} H$

in which $Z(\mathrm{D0})$ and $Z(\mathrm{D2})$ are $Z_{D^2}^{\mathrm{LV}}$ for the brane data $\mathfrak{B}$ such that $\mathrm{ch}(\pi_+(\mathfrak{B}))$ is the point class [pt] and the line class l respectively. Applying this mirror theorem to (5.40), we find the relationship between the hemisphere partition function of $\mathrm{T}_{N,d}^{U(1)}$ and the GW central charge of X_f for the case $d \leq N$:

$$Z_{D^2}^{\mathrm{LV}}(\mathfrak{B})\Big|_{r\Lambda = z^{-1}} = C \cdot \mathcal{Y}(\widetilde{t}, z) \cdot Z_{\mathrm{GW}}(\pi_+(\mathfrak{B}))|_{\mathrm{t}=-\mathcal{T}(\widetilde{t})}\,. \tag{5.45}$$

That is, they are equal up to a multiplicative renormalization and a change of parameter. As a consistency test, let us compute Z_{GW} for the D0-brane in a Calabi-Yau manifld X:

$$Z_{\mathrm{GW}}(\mathrm{D0}) = z^{-\frac{D}{2}}\,\langle P\, P\; w\,[[\mathrm{pt}]]\rangle = z^{-\frac{D}{2}} \sum_{n=0}^{\infty} (-z)^{-n} \langle P\, P\, \tau_n[\mathrm{pt}]\rangle. \tag{5.46}$$

On $\mathfrak{M}_A^0 \times \mathbb{C}^*$, it follows from the selection rule that only the $n = 0$ term is non-zero, and from the puncture equation [61] that only the degree zero map without Kähler perturbation contributes, finding

$$Z_{\mathrm{GW}}(\mathrm{D0})|_{\mathfrak{M}_A^0 \times \mathbb{C}^*} = z^{-\frac{D}{2}}. \tag{5.47}$$

Therefore, for $d = N$ we indeed have $Z_{D^2}^{\mathrm{LV}}(\mathrm{D0}) = C \cdot C^{-1} z^{\frac{D}{2}} Z(\mathrm{D0}) \cdot z^{-\frac{D}{2}} = Z(\mathrm{D0})$.

Macroscopic Loop

The development of topological string in [59–61], especially the introduction of gravitational descendants, is motivated by the solution of two-dimensional quantum gravity via matrix models [65]. The partition function of the one matrix model is

$$Z(N,V)=\int_{H_N}\mathrm{d}M\exp\Bigl(-N\,\mathrm{tr}\,V(M)\Bigr) \tag{5.48}$$

where H_N is the space of $N\times N$ hermitian matrices, $\mathrm{d}M$ is the flat measure and V is a polynomial with a quadratic and higher order terms. The Feynman diagram expansion of this integral can be organized into the sum over triangulated surfaces [66]. In a certain limit where the size N is sent to infinity and the potential V is tuned at the same time (double scaling limit), the expansion can be regarded as the sum over closed oriented 2d Riemannian manifolds, including the sum over the genus, with the weight given by the partition function of a conformal field theory. That is, the theory can be regarded as the 2d quantum gravity coupled to a CFT. There is a series of double scaling limits, labelled by $m=1,2,3,\ldots$, where the CFT is the $(p,q)=(2m-1,2)$ minimal model. The $m=2$ theory, where the CFT is empty, is the pure gravity. The $m=1$ theory is identified [67] with the pure topological gravity, that is, the Gromov-Witten theory of a point. The model with ν matrix variables yields the theory of gravity coupled to the $(p,\nu+1)$ minimal CFT, and the $(1,\nu+1)$ theory is the topological string associated with the LG B-model with superpotential $W=x^{\nu+1}$, or equivalently, the FJRW theory for $(\mathbb{C}/\mathbb{Z}_{\nu+1},x^{\nu+1})$.

Insertion of $\mathrm{tr}M^j$ into the integrand of (5.48) creates a j-gonal hole in the triangulated surfaces. See Fig. 8. In the double scaling limit, certain linear combinations of $\mathrm{tr}M^j$ with finite j's become local operators $\sigma_0=:P,\sigma_1,\sigma_2,\ldots$ while a certain limit of $\mathrm{tr}M^j$ with $j\to\infty$ yields a *macroscopic loop operator* $w(\ell)$ that creates a hole of length ℓ in the 2d manifold [68]. The puncture operator and the gravitational descendants, $\tau_n=\tau_n\phi_0$ with $n=0,1,2,\ldots$, in the topological gravity are the local operators in the $m=1$ theory. Correlation functions involving macroscopic loops have been studied in [68–70] (see [71] for a review). For example, at genus zero

$$\langle P\,P\,w(\ell)\rangle=\mathrm{const}\cdot\sqrt{\ell}\,\mathrm{e}^{-\ell\langle PP\rangle}\langle PPP\rangle, \tag{5.49}$$

$$\langle P\,w(\ell_1)w(\ell_2)\rangle=\mathrm{const}\cdot\sqrt{\ell_1\ell_2}\,\mathrm{e}^{-(\ell_1+\ell_2)\langle PP\rangle}\langle PPP\rangle. \tag{5.50}$$

The macroscopic loop $w(\ell)$ for small ℓ can be expanded as a power series in ℓ with coefficients given by the local operators. In the $m=1$ theory

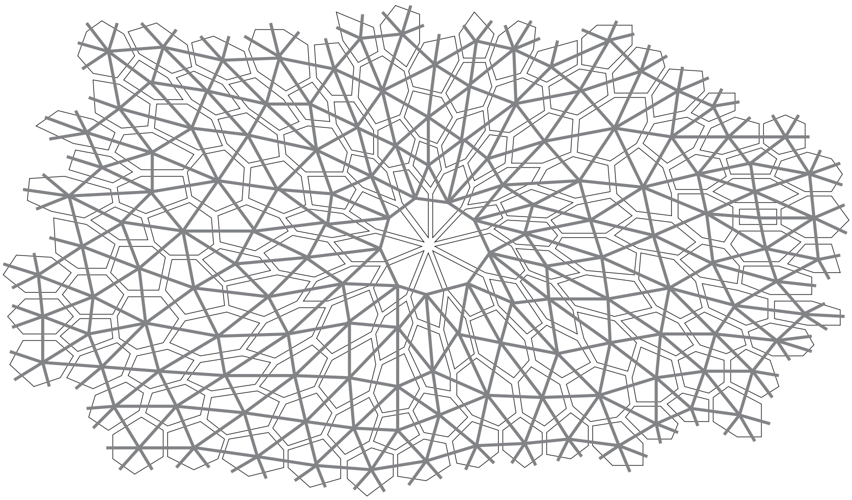

Fig. 8. Insertion of $\mathrm{tr}M^9$ creates a nine-gonal hole.

that corresponds to topological gravity, the series is

$$w(\ell) = \mathrm{const} \cdot \sqrt{\ell} \sum_{n=0}^{\infty} (-1)^n \ell^n \tau_n. \tag{5.51}$$

Indeed, under this, the results (5.49) and (5.50) can be reproduced from the constitutive relations [61], $n!m!\langle \tau_n \tau_m \rangle = \langle PP \rangle^{n+m+1}/(n+m+1)$. We would also like to quote that the Virasoro constraint in the double scaled theory [73, 74] can be written via (5.51) as a loop equation on $w(\ell)$ or its Laplace transform $\widehat{w}(\zeta)$, and that it can be stated as regularity of the energy-momentum tensor of a free chiral CFT on the ζ-space [74]. For the ν matrix model, there are ν different macroscopic loop operators $w_1(\ell), \ldots, w_\nu(\ell)$, and it was suggested [72] that the small ℓ expansion of $w_i(\ell)$ in the $(1, \nu+1)$ theory is given by (5.51) where τ_n is replaced by $\tau_n O_i$ for some primary O_i.

We see that $w[P]$ (5.28) in the Gromov-Witten theory of a point is nothing but the small ℓ expansion (5.51) of the macroscopic loop $w(\ell)$ with $\ell = z^{-1}$, up to a normalization factor including $z^{-\frac{1}{2}}$. This suggests that we may view Givental's operator $w[\mathcal{O}]$ in the Gromov-Witten theory of a general target X, or in a general topological string theory, as an expansion of the macroscopic loop operator labelled by $\mathcal{O}$ that creates a hole of length $\ell = z^{-1}$. Furthermore, if the relation of the type (5.45) holds generally, then, the partition function on the hemisphere of radius r is proportional to a genus zero correlation function involving a macroscopic loop of length r in the topological string,

$$Z_{D^2(r)}(\mathcal{B}) \sim \langle P\, P\, w_{[\mathcal{B}]}(r) \rangle. \tag{5.52}$$

This would be a strikingly simple relation. It is a very interesting problem to find a precise formulation of macroscopic loops in the topological string theory that leads to (5.28) in the small length expansion, and to see whether such a relation can be derived naturally.

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Appendix

A. Lagrangian and supersymmetry

In this appendix, we describe the supersymmetric action on the Minkowski spacetime as well as on the sphere and the hemisphere for GLSMs and LG models. For the GLSMs on the (hemi)sphere, we describe the one that preserves A (or $\mathrm{A}_{(\pm)}$) type supercharges. For the LG models on the (hemi)sphere, we describe the one that preserves B (or $\mathrm{B}_{(\pm)}$) type supercharges.

A.1. Minkowski space

We describe the supersymmetry and the Lagrangian of GLSM on the Minkowski space, with time and space coordinates x^0 and x^1 and the metric $\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2$. We often use the light-cone coordinates $x^{\pm} = x^0 \pm x^1$.

The gauge multiplet with gauge group G consists of a G connection v_μ, as well as a scalar σ, a Dirac fermion λ and a scalar D with values in $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}$ and $i\mathfrak{g}$. The matter multplet in representation V consists of a scalar ϕ, a Dirac fermion ψ and a scalar F, all with values in V. The gauge connection v_μ is "real-valued" so that the curvature is $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + i[v_\mu, v_\nu]$ and the gauge covariant derivative is $D_\mu\phi = \partial_\mu\phi + iv_\mu\phi$.

The supersymmetry transformation $\delta = i\epsilon_+ Q_- - i\epsilon_- Q_+ - i\overline{\epsilon}_+\overline{Q}_- + i\overline{\epsilon}_-\overline{Q}_+$ is

$$
\begin{aligned}
\delta v_{\pm} &= \frac{i}{2}\overline{\epsilon}_{\pm}\lambda_{\pm} + \frac{i}{2}\epsilon_{\pm}\overline{\lambda}_{\pm},\\
\delta\sigma &= -i\overline{\epsilon}_+\lambda_- - i\epsilon_-\overline{\lambda}_+,\\
\delta\lambda_{\pm} &= i\epsilon_{\pm}\left((D \pm iv_{01}) \pm \frac{1}{2}[\sigma,\overline{\sigma}]\right) + \epsilon_{\mp}(D_0 \pm D_1)\sigma_{\mp},\\
\delta D &= \left[\frac{1}{2}\epsilon_+\big((D_0 - D_1)\overline{\lambda}_+ + i[\sigma,\overline{\lambda}_-]\big)\right.\\
&\qquad \left. + \frac{1}{2}\epsilon_-\big((D_0 + D_1)\overline{\lambda}_- + i[\overline{\sigma},\overline{\lambda}_+]\big)\right] + c.c.,
\end{aligned}
\tag{A.1}
$$

and

$$\text{(A.2)}\qquad \begin{aligned} \delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+,\\ \delta\psi_\pm &= \pm i\overline{\epsilon}_\mp(D_0 \pm D_1)\phi \mp \overline{\epsilon}_\pm\sigma_\mp\phi + \epsilon_\pm F,\\ \delta F &= -i\overline{\epsilon}_+(D_0 - D_1)\psi_+ - i\overline{\epsilon}_-(D_0 + D_1)\psi_-\\ &\quad + \overline{\epsilon}_+\overline{\sigma}\psi_- + \overline{\epsilon}_-\sigma\psi_+ + i(\overline{\epsilon}_-\overline{\lambda}_+ - \overline{\epsilon}_+\overline{\lambda}_-)\phi. \end{aligned}$$

where we use the notation $\sigma_+ = \sigma$ and $\sigma_- = \overline{\sigma}$ just in here. The transformation of the complex conjugates are obtained from these by the rule $(AB)^\dagger = B^\dagger A^\dagger$. For example, $\delta\overline{\sigma} = i\overline{\lambda}_-\epsilon_+ + i\lambda_+\overline{\epsilon}_- = -i\epsilon_+\overline{\lambda}_- - i\overline{\epsilon}_-\lambda_+$. Note that σ is A-chiral and ϕ is B-chiral

$$\text{(A.3)}\qquad \overline{Q}_+\sigma = Q_-\sigma = 0, \qquad \overline{Q}_+\phi = \overline{Q}_-\phi = 0.$$

Before writing down the Lagrangian, we choose an adjoint invariant innder product on $i\mathfrak{g}$ and G-invariant hermitian inner product on V,

$$\text{(A.4)}\quad (X, Y) \in i\mathfrak{g} \times i\mathfrak{g} \longmapsto \frac{1}{e^2}XY \in \mathbb{R}, \ \ (\phi_1, \phi_2) \in V \times V \longmapsto \overline{\phi}_1\phi_2 \in \mathbb{C},$$

which are both positive definite. We also write $\frac{1}{e^2}XX = \frac{1}{e^2}X^2$, $\overline{\phi}\phi = |\phi|^2$, etc. The supersymmetric Lagrangian is

$$\text{(A.5)}\qquad \begin{aligned} \mathcal{L} &= Q_+Q_-\overline{Q}_+\overline{Q}_-\left(-\frac{1}{2e^2}|\sigma|^2 + |\phi|^2\right)\\ &\quad + \operatorname{Re} Q_+Q_-W(\phi) + \operatorname{Re} Q_+\overline{Q}_-\Big(-t(\sigma)\Big)\\ &\quad + \text{total derivative}\\ &= \mathcal{L}_{\mathrm{g}} + \mathcal{L}_{\mathrm{m}} + \mathcal{L}_W + \mathcal{L}_t, \end{aligned}$$

where

(A.6)

$$\begin{aligned} \mathcal{L}_{\mathrm{g}} = \frac{1}{2e^2}\Big[&|D_0\sigma|^2 - |D_1\sigma|^2 + i\overline{\lambda}_-(D_0 + D_1)\lambda_- + i\overline{\lambda}_+(D_0 - D_1)\lambda_+\\ &+ (v_{01})^2 + D^2 - \frac{1}{4}[\sigma, \overline{\sigma}]^2 - \lambda_+[\sigma, \overline{\lambda}_-] + [\overline{\sigma}, \lambda_-]\overline{\lambda}_+\Big], \end{aligned}$$

$$\text{(A.7)}\quad \begin{aligned} \mathcal{L}_{\mathrm{m}} &= |D_0\phi|^2 - |D_1\phi|^2 + i\overline{\psi}_-(D_0 + D_1)\psi_- + i\overline{\psi}_+(D_0 - D_1)\psi_+\\ &\quad + |F|^2 + \overline{\phi}D\phi - \frac{1}{2}|\sigma\phi|^2 - \frac{1}{2}|\overline{\sigma}\phi|^2 - \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_-\\ &\quad - i\overline{\phi}\lambda_-\psi_+ + i\overline{\phi}\lambda_+\psi_- + i\overline{\psi}_+\overline{\lambda}_-\phi - i\overline{\psi}_-\overline{\lambda}_+\phi, \end{aligned}$$

$$\mathcal{L}_W = \mathrm{Re}\left[F^i \frac{\partial W}{\partial \phi^i} - \psi_+^i \psi_-^j \frac{\partial^2 W}{\partial \phi^i \partial \phi^j}\right], \tag{A.8}$$

$$\mathcal{L}_t = -\zeta(D) + \theta(v_{01}). \tag{A.9}$$

The Lagrangian (A.5) is manifestly supersymmetric since σ is A-chiral and ϕ is B-chiral (A.3).

The fields D and F are "auxiliary fields" — they do not have the kinetic terms. They can be eliminated by the equations of motion. After doing that, we obtain the scalar potential (3.5).

In the present notes, we take the convention that the action is the integral of Lagragian density *divided by* 2π:

$$S = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}^2x\,\mathcal{L}. \tag{A.10}$$

In particular, the theta term enters into the action as $\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}^2x\,\theta(v_{01}) = \int_{\mathbb{R}^2}\theta(\frac{i}{2\pi}F_v)$ where $F_v = iv_{01}\mathrm{d}x^0\wedge\mathrm{d}x^1$ is the $\mathfrak{g}$-valued ("imaginary") curvature two-form.

A.2. Wick rotation and covariantization

The Lagrangian and supersymmetry on the Euclidean space is obtained by Wick rotation $x^0 \to -ix^2$ and $\mathcal{L} \to -\mathcal{L}_E$. The auxiliary fields are also rotated as

$$D \to iD_E, \quad D_E \in i\mathfrak{g}, \tag{A.11}$$

$$(F, \overline{F}) \to (if, i\overline{f}). \tag{A.12}$$

The fermion pairs, $(\lambda_\pm, \overline{\lambda}_\pm)$ and $(\psi_\pm, \overline{\psi}_\pm)$, are no longer related by complex conjugation.

It is starightforward to formulate the theory on a general two-manifold Σ with a Riemannian metric h and a spin structure, by the standard covariantization. Note that choice of a principal G bundle P on Σ is a part of the variables. A scalar with values in a representation U of G should now be regraded as a section of the vector bundle $P\times_G U$ on Σ. A Dirac fermion $\xi = (\xi_+, \xi_-)$ is a section of the spinor bundle $S = S_+ \oplus S_-$. The positive and negative chirality spinor bundles can also be regarded as the square roots of the (holomorphic or anti-holomorphic) tangent or cotangent bundles:

$$S_+ \cong \sqrt{\overline{K}_\Sigma} \cong \sqrt{T_\Sigma}, \qquad S_- \cong \sqrt{K_\Sigma} \cong \sqrt{\overline{T}_\Sigma}. \tag{A.13}$$

The Lagrangian and the supersymmetry transformation on a general surface (Σ, h) can then be found straightforwardly, by covariantization.

The variational parameters $\epsilon_\pm$ and $\overline{\epsilon}_\pm$ should now be regarded as sections of $S_\pm$. The variation of the Lagrangian under this is

(A.14)
$$\delta\mathcal{L} = \nabla_\mu\epsilon_+ G^\mu_- - \nabla_\mu\epsilon_- G^\mu_+ - \nabla_\mu\overline{\epsilon}_+\overline{G}^\mu_- + \nabla_\mu\overline{\epsilon}_-\overline{G}^\mu_+ + \text{total derivative},$$

where $G^\mu_\pm$ and $\overline{G}^\mu_\pm$ are the supercurrents corresponding to $Q_\pm$ and $\overline{Q}_\pm$. This shows that the condition for supersymmetry is $\nabla_\mu\epsilon_\pm = \nabla_\mu\overline{\epsilon}_\pm = 0$, which is impossible on a curved manifold.

A.3. Superconformal transformation

The superconformal transformation of the fields are obtained from the Weyl covariantization. For the fields in GLSM, this is specified by a choice vector R-charge for the B-chiral scalar ϕ. For this, we use the one $R : V \to V$ that determines the vector R-symmetry of the system. Given this choice, the superconformal transformation is

$$\begin{gathered}
\delta^{\rm sc} v_\pm = \delta v_\pm,\\
\delta^{\rm sc}\sigma = \delta\sigma, \qquad \delta^{\rm sc}\overline{\sigma} = \delta\overline{\sigma},\\
\delta^{\rm sc}\lambda = \delta\lambda + i\sigma\,\not{\nabla}\overline{\widetilde{\epsilon}}, \qquad \delta^{\rm sc}\overline{\lambda} = \delta\overline{\lambda} - i\overline{\sigma}\,\not{\nabla}\widetilde{\epsilon},\\
\delta^{\rm sc} D_E = \delta D_E - \tfrac{1}{2}\langle\not{\nabla}\widetilde{\epsilon},\overline{\lambda}\rangle - \tfrac{1}{2}\langle\not{\nabla}\overline{\widetilde{\epsilon}},\lambda\rangle,
\end{gathered} \tag{A.15}$$

and

$$\begin{gathered}
\delta^{\rm sc}\phi = \delta\phi, \qquad \delta^{\rm sc}\overline{\phi} = \delta\overline{\phi},\\
\delta^{\rm sc}\psi = \delta\psi + \tfrac{i}{2}R\phi\,\not{\nabla}\overline{\epsilon}, \qquad \delta^{\rm sc}\overline{\psi} = \delta\overline{\psi} - \tfrac{i}{2}R\overline{\phi}\,\not{\nabla}\epsilon,\\
\delta^{\rm sc} f = \delta f - \tfrac{1}{2}\langle\not{\nabla}\overline{\epsilon}, R\psi\rangle, \qquad \delta^{\rm sc}\overline{f} = \delta\overline{f} + \tfrac{1}{2}\langle R\overline{\psi},\not{\nabla}\epsilon\rangle,
\end{gathered} \tag{A.16}$$

where δ is the transformation obtained from (A.1) and (A.2) via Wick rotation and covariantization. In the above expressions, we use $\lambda = -i\lambda_- + i\overline{\lambda}_+$, $\overline{\lambda} = i\overline{\lambda}_- - i\lambda_+$, $\psi = \psi_- + \psi_+$, $\overline{\psi} = \overline{\psi}_- + \overline{\psi}_+$, $\epsilon = \epsilon_- + \epsilon_+$, $\overline{\epsilon} = \overline{\epsilon}_- + \overline{\epsilon}_+$, $\widetilde{\epsilon} = \epsilon_- + \overline{\epsilon}_+$ and $\overline{\widetilde{\epsilon}} = \overline{\epsilon}_- + \epsilon_+$; $\not{\nabla}$ is the Dirac operator; and $\langle\epsilon,\eta\rangle$ is the invariant pairing of spinors, which reduces to $\epsilon_+\eta_- - \epsilon_-\eta_+$ on the flat Minkowski space.

The commutator of the superconformal transformations $\delta^{\rm sc}_1$ and $\delta^{\rm sc}_2$ with different variational parameters is

$$[\delta^{\rm sc}_1, \delta^{\rm sc}_2] = L_{Z_{12}} + \widetilde{L}_{\widetilde{Z}_{12}} + J_{\Theta_{12}} + \widetilde{J}_{\widetilde{\Theta}_{12}}, \tag{A.17}$$

where L_Z (*resp.* $\widetilde{L}_{\widetilde{Z}}$) is the conformal transfomation by a holomorphic vector field Z (*resp.* anti-holomorphic vector field $\widetilde{Z}$) and J_Θ (*resp.* $\widetilde{J}_{\widetilde{\Theta}}$) is the right-handed R-rotation $\frac{1}{2}(F_V - F_A)$ (*resp.* left-handed R-rotation $\frac{1}{2}(F_V + F_A)$) with the variational parameter Θ (*resp.* $\widetilde{\Theta}$). $Z_{12}, \widetilde{Z}_{12}, \Theta_{12}$

and $\widetilde{\Theta}_{12}$ are the bilinears of the two superconformal variational parameters $\epsilon_{i+} = \epsilon_i(z)\sqrt{\frac{\partial}{\partial z}}$, $\overline{\epsilon}_{i+} = \overline{\epsilon}_i(z)\sqrt{\frac{\partial}{\partial z}}$, $\epsilon_{i-} = \epsilon_i(\overline{z})\sqrt{\frac{\partial}{\partial \overline{z}}}$, $\overline{\epsilon}_{i-} = \overline{\epsilon}_i(\overline{z})\sqrt{\frac{\partial}{\partial \overline{z}}}$ $(i = 1, 2)$,

$$Z_{12} = -2i\epsilon_{[1}(z)\overline{\epsilon}_{2]}(z)\frac{\partial}{\partial z}, \qquad \widetilde{Z}_{12} = 2i\epsilon_{[1}(\overline{z})\overline{\epsilon}_{2]}(\overline{z})\frac{\partial}{\partial \overline{z}},$$
$$\Theta_{12} = -2i\epsilon_{[1}(z)\overleftrightarrow{\partial}_z\overline{\epsilon}_{2]}(z), \qquad \widetilde{\Theta}_{12} = 2i\epsilon_{[1}(\overline{z})\overleftrightarrow{\partial}_{\overline{z}}\overline{\epsilon}_{2]}(\overline{z}),$$

where $A\overleftrightarrow{\partial}_x B := \frac{1}{2}A\partial_x B - \frac{1}{2}(\partial_x A)B$.

A.4. Action on the (hemi)sphere

Let $\mathcal{L}_{\mathrm{cov}}$ be the Lagrangian obtained from the one (A.5) on the Minsowski spacetime by Wick rotation and covariantization. The action $\int_\Sigma \mathcal{L}_{\mathrm{cov}}\sqrt{h}\mathrm{d}^2x$ is not invariant under any of the superconformal tranformations in general. However, on the sphere or on the hemisphere, there is a way to modify the action so that it is (mostly) invariant under a part of the superconformal transformations. The modified action is of the form

$$S_{S^2} = \int_{S^2} (\mathcal{L}_{\mathrm{cov}} + \Delta\mathcal{L})\sqrt{h}\mathrm{d}^2x, \tag{A.18}$$

$$S_{D^2} = \int_{D^2} (\mathcal{L}_{\mathrm{cov}} + \Delta\mathcal{L})\sqrt{h}\mathrm{d}^2x + \int_{\partial D^2} \mathcal{L}_{\mathrm{bdry}}\mathrm{d}\tau, \tag{A.19}$$

where $\Delta\mathcal{L}$ and $\mathcal{L}_{\mathrm{bdry}}$ are given below, h is the round (hemi)sphere metric of radius r and $\mathrm{d}\tau$ is the line element of ∂D^2. $\Delta\mathcal{L}$ is the sum of

$$\Delta\mathcal{L}_{\mathrm{g}} = \frac{1}{2e^2}\left[\frac{2}{r}\left(D_E\mathrm{Re}(\sigma) + \frac{v_{12}}{\sqrt{h}}\mathrm{Im}(\sigma)\right) + \frac{1}{r^2}\overline{\sigma}\sigma\right], \tag{A.20}$$

$$\Delta\mathcal{L}_{\mathrm{m}} = \overline{\phi}\left[\frac{2R^2 - R^2}{4r^2} - i\frac{R}{r}\mathrm{Re}(\sigma)\right]\phi, \tag{A.21}$$

and $\mathcal{L}_{\mathrm{bdry}}$ is the sum of

$$\mathcal{L}_{\mathrm{g,\,bdry}} = -\frac{1}{4e^2}\left[\partial_n(\overline{\sigma}\sigma) \pm 2i\left(D_E\mathrm{Im}(\sigma) - \frac{v_{12}}{\sqrt{h}}\mathrm{Re}(\sigma)\right)\right], \tag{A.22}$$

$$\mathcal{L}_{\mathrm{m,\,bdry}} = \pm\frac{i}{2}\langle\overline{\psi}, \psi\rangle \mp \overline{\phi}\,\mathrm{Im}(\sigma)\phi, \tag{A.23}$$

$$\mathcal{L}_{\mathrm{t,\,bdry}} = \pm\mathrm{Im}\left(\frac{1}{2\pi}t(\sigma)\right), \tag{A.24}$$

where ∂_n is the outward normal derivative. Then, S_{S^2} is invariant under A-type supercharges as defined in Section 4.1. For the hemisphere,

$S_{D^2,\mathrm{g}}$, $S_{D^2,\mathrm{m}}$, and $S_{D^2,t}$ (whose definition is hopefully obvious) are invariant under $\mathrm{A}_{(\pm)}$-type supercharges, but $S_{D^2,W}$ transforms as

$$\delta S_{D^2,\mathrm{W}} = \mp\frac{i}{2}\int_{\partial D^2}\left[\langle\overline{\epsilon},\psi^i\rangle\partial_i W + \langle\epsilon,\overline{\psi}^{\overline{\imath}}\rangle\partial_{\overline{\imath}}\overline{W}\right]\mathrm{d}\tau. \tag{A.25}$$

This failure of invariance is compensated by the transformation of the Chan-Paton factor (4.12), $\mathrm{tr}_M\mathrm{Pexp}\left(-\int_{\partial D^2}\mathcal{A}\right)$, with $\mathcal{A}=\mathcal{A}_\tau\mathrm{d}\tau$ given by

$$\mathcal{A}_\tau = iv_\tau \mp \mathrm{Re}(\sigma) - \frac{1}{2}\uppsi^i\partial_i Q + \frac{1}{2}\overline{\uppsi}^{\overline{\imath}}\partial_{\overline{\imath}}Q^\dagger + \frac{1}{2}\{Q,Q^\dagger\} \mp \frac{i}{2r}\mathbf{r}. \tag{A.26}$$

In this expression,

$$\uppsi := \frac{1}{\sqrt{r}}\left[z^{\frac{1}{2}}\psi_-^{\{z\}} \pm \overline{z}^{\frac{1}{2}}\psi_+^{\{z\}}\right],\quad \overline{\uppsi} := \frac{1}{\sqrt{r}}\left[z^{\frac{1}{2}}\overline{\psi}_-^{\{z\}} \pm \overline{z}^{\frac{1}{2}}\overline{\psi}_+^{\{z\}}\right], \tag{A.27}$$

where the superscript $\{z\}$ is put for the component with respect to the "z frames". For example $\psi_+=\psi_+^{\{z\}}\sqrt{\mathrm{d}\overline{z}}$ and $\psi_-=\psi_-^{\{z\}}\sqrt{\mathrm{d}z}$. Also, in the last term of (A.26), r in the denominator is the radius of the hemisphere while $\mathbf{r}$ in the numerator is the vector R-charge on the Chan-Paton vector space M that appears, say, in Eqn (3.19). Note that $\uppsi$ and $\overline{\uppsi}$ are fermionic and antiperiodic along ∂D^2. At first sight, it appears strange to add the terms $-\frac{1}{2}\uppsi^i\partial_i Q+\frac{1}{2}\overline{\uppsi}^{\overline{\imath}}\partial_{\overline{\imath}}Q^\dagger$ and the rest in (A.26). In fact, the Chan-Paton factor $\mathrm{tr}_M\mathrm{Pexp}\left(-\int_{\partial D^2}\mathcal{A}\right)$ is defined so that $\partial_i Q$ and $\partial_{\overline{\imath}}Q^\dagger$ are treated as fermionic, and it make a perfect sense only when $\uppsi$ and $\overline{\uppsi}$ are antiperiodic. See Appendix B of [9] for more details.

A.5. The case of Landau-Ginzburg model

The Landau-Ginzburg model associated with a polynomial $W(x_1,\ldots,x_N)$ can be regarded as the GLSM with the trivial gauge group $G=\{1\}$, the matter representation $V=\mathbb{C}^N$ and the superpotential $W(\phi)=W(x_1,\ldots,x_N)$ for $\phi^i=x_i$.[1] In particular, the Lagrangian, the supersymmetry transformation and the superconformal transformation in the theory can be obtained from the above formulae by setting (v,σ,λ,D) all zero (we shall denote this process by "$(-)|$"). That is, the Minkowski Lagrangian is $\mathcal{L}=\mathcal{L}_\mathrm{m}+\mathcal{L}_W$ given by (A.7)| and (A.8) and the supersymmetry transformation is given by (A.2)|. Wick rotation

[1] The Landau-Ginzburg orbifold associated with a pair (W,Γ) is obtained likewise by taking $G=\Gamma$ and V to be the representation corresponding to the given Γ action on the variables $x_1,\ldots,x_N$.

and covariantization are done in the standard fashion, though we follow (A.12). The superconformal transformation with a vector R-charge $R : \mathbb{C}^N \to \mathbb{C}^N$ is given by (A.16)|.

Let us describe the actions on the sphere and on the hemisphere of radius r that are invariant under the B-type and the $\mathrm{B}_{(\pm)}$-type supercharges. B-type supersymmetry does not require the theory to have a vector $U(1)$ R-symmetry, and we may pick an arbitrary R. In fact, the B-type supercharge action on (ϕ, ψ, f) with R-charge R is the same as the B-type supercharge action on $(\phi, \psi, f_!)$ with the vanishing R-charge, with

$$f_! = f + \frac{R}{2r}\phi. \tag{A.28}$$

The B-type supersymmetric Lagrangian on the sphere is $\mathcal{L} = \mathcal{L}_{\mathrm{m}} + \mathcal{L}_W$ with

$$\mathcal{L}_{\mathrm{m}} = \mathcal{L}_{\mathrm{m,\,cov}}|_{f \to f_!}, \tag{A.29}$$

$$\mathcal{L}_W = \mathcal{L}_{W,\,\mathrm{cov}}|_{f \to f_!} + \frac{i}{r}\mathrm{Re}(W). \tag{A.30}$$

When W is not quasihomogeneous, it is the simplest to choose $R = 0$, so that $f \to f_!$ is trivial. When W is quasihomogeneous, we may take R to the the one that makes $W(\lambda^R\phi) = \lambda^2 W(\phi)$. Then, we have $\mathcal{L}_W = \mathcal{L}_{W,\,\mathrm{cov}}$. On the hemisphere, the action

$$S_{D^2} = \int_{D^2} \mathcal{L}\sqrt{h}\mathrm{d}^2x + \int_{\partial D^2} \mathcal{L}_{\mathrm{bdry}}\mathrm{d}\tau, \tag{A.31}$$

in invariant under the $\mathrm{B}_{(\pm)}$-type supercharges with

$$\mathcal{L}_{\mathrm{m,\,bdry}} = -\frac{1}{2}\partial_n(\overline{\phi}\phi) \pm (\overline{f}\phi - \overline{\phi}f), \tag{A.32}$$

$$\mathcal{L}_{W,\,\mathrm{bdry}} = \mp\mathrm{Im}(W). \tag{A.33}$$

B. Boundary States Etc

In this Appendix, we describe the definition of the boundary states and related matters in order to set the convention used in Section 5.2.

Let us consider a unitary quantum field theory in 1+1 dimensions. If we quantize the system on a Minkowski cylinder, with space S^1 and time $\mathbb{R}$, we have the Hilbert space $\mathcal{H}_{S^1}$ of states on S^1 and the Hamiltonian H_c ("c" for "closed" string). We employ the standard notation where the conjugation $|a\rangle \mapsto \langle a|$ is defined by $\langle a|b\rangle = (|a\rangle, |b\rangle)$ for the hermitian inner product $(-,-)$ of $\mathcal{H}_{S^1}$. Of course, $(\langle a|b\rangle)^* = \langle b|a\rangle$. Let us consider the Euclidean path integral on the finite cylinder $[-\ell, 0] \times S^1$ where we impose the boundary condition $\mathcal{B}$ at the end $\{0\} \times S^1$ and put a state $|a\rangle$ at the beginning $\{-\ell\} \times S^1$. We define the boundary state $\langle\mathcal{B}|$ by the property that this path integral is equal to $\langle\mathcal{B}|\, \mathrm{e}^{-\ell H_c}|a\rangle$. (To be precise, $\langle\mathcal{B}|$ is not normalizable but $\langle\mathcal{B}|\, \mathrm{e}^{-\ell H_c}$ is.) By definition, $\langle a|\, \mathrm{e}^{-\ell H_c}|\mathcal{B}\rangle$ is the complex conjugate of this $\langle\mathcal{B}|\, \mathrm{e}^{-\ell H_c}|a\rangle$ and can be regarded as the Euclidean path integral on the finite cylinder $[0, \ell] \times S^1$ where we put the conjugate state $\langle a|$ at the end $\{\ell\} \times S^1$ and impose a "conjugate boundary condition" $\overline{\mathcal{B}}$ at the beggining $\{0\} \times S^1$.

For example, the path integral on the cylinder $[0, L] \times S^1$ with the boundary condition $\mathcal{B}_f$ at the end $\{L\} \times S^1$ and the boundary condition $\overline{\mathcal{B}}_i$ at the beginning $\{0\} \times S^1$ is given by $\langle\mathcal{B}_f|\, \mathrm{e}^{-LH_c}|\mathcal{B}_i\rangle$. This can also be regarded as the partition sum of the open string states on $[0, L]$ with the boundary condition $\overline{\mathcal{B}}_i$ at the left end $\{0\}$ and the boundary condition $\mathcal{B}_f$ at the right end $\{L\}$. If the circumference of the circle is $\boldsymbol{\beta}$, then, it is $\mathrm{Tr}_{\mathcal{H}_{\overline{\mathcal{B}}_i, \mathcal{B}_f}} \mathrm{e}^{-\boldsymbol{\beta} H_o}$ where $\mathcal{H}_{\overline{\mathcal{B}}_i, \mathcal{B}_f}$ is the Hilbert space of open string states and H_o is the Hamiltonian ("o" for "open" string). We obtained a relation

$$ {}_{\boldsymbol{\beta}}\langle\mathcal{B}_f|\, \mathrm{e}^{-LH_c}|\mathcal{B}_i\rangle_{\boldsymbol{\beta}} = \mathrm{Tr}_{\mathcal{H}_{\overline{\mathcal{B}}_i, \mathcal{B}_f}(L)}\, \mathrm{e}^{-\boldsymbol{\beta} H_o}, \tag{B.1} $$

where the dependence of the boundary states and the Hilbert space on the lengths $\boldsymbol{\beta}$ and L is made explicit.

More generally, let Σ be a surface with an outgoing boundary circle $\partial\Sigma \cong S^1$, having a neighborhood isomorphic to $(-\epsilon, 0] \times S^1$ for some $\epsilon > 0$. The path integral on the fields on the interior of Σ defines a state $|\Sigma\rangle \in \mathcal{H}_{S^1}$. Then, the partition function on Σ with the boundary condition $\mathcal{B}$ at $\partial\Sigma$ is given by

$$ Z(\mathcal{B}|\Sigma) = \langle\mathcal{B}|\Sigma\rangle. \tag{B.2} $$

Its complex conjugate can be regarded as the partition function on a "conjugate surface" $\overline{\Sigma}$ with the conjugate boundary condition $\overline{\mathcal{B}}$,

$$ Z(\overline{\Sigma}|\overline{\mathcal{B}}) = \langle\Sigma|\mathcal{B}\rangle. \tag{B.3} $$

The surface $\overline{\Sigma}$ has an incoming boundary circle $\partial\overline{\Sigma} \cong -S^1$, with a neighborhood isomorphic to $[0, \epsilon) \times S^1$. Note that it can be regarded as the outgoing boundary by the sign flip of the coordinates, which involves the orientation reversal of the boundary. The conjugate boundary condition $\overline{\mathcal{B}}$ at the incoming boundary is sometimes equal to the original boundary condition $\mathcal{B}$ when the boundary is regarded as outgoing. For example, for the sigma model with target X, the D-brane $\mathcal{B}$ supporting a hermitian vector bundle on X with a unitary connection (E, A) is obtained by putting the Chan-Paton factor $\mathrm{tr}_E\mathrm{Pexp}\left(-i\int_{\partial\Sigma}\phi^* A\right)$ in the path integral weight. The complex conjugate of this factor is

$$\text{(B.4)} \qquad \left[\mathrm{tr}_E\mathrm{Pexp}\left(-i\int_{\partial\Sigma}\phi^* A\right)\right]^* = \mathrm{tr}_{E^\vee}\mathrm{Pexp}\left(i\int_{-\partial\overline{\Sigma}}\phi^* A^T\right)$$
$$= \mathrm{tr}_E\mathrm{Pexp}\left(-i\int_{\partial\overline{\Sigma}}\phi^* A\right)$$

where we used the unitarity of the connection, $A^* = A^T$, in the first equality. The above equation means that the conjugate boundary condition $\overline{\mathcal{B}}$ supports the dual bundle with the dual connection $(E^\vee, -A^T)$, and that $\overline{\mathcal{B}}$ at the incoming boundary $-\partial\overline{\Sigma}$ is equal to the original $\mathcal{B}$ at the outgoing boundary $\partial\overline{\Sigma}$. In particular, we have $Z(\overline{\Sigma}|\overline{B}) = Z(\mathcal{B}|\overline{\Sigma})$, that is,

$$\text{(B.5)} \qquad (\langle\mathcal{B}|\Sigma\rangle)^* = \langle\Sigma|\mathcal{B}\rangle = \langle\mathcal{B}|\overline{\Sigma}\rangle.$$

A boundary condition is said to be *unitary* when it has this property. All the boundary conditions we consider in this paper are unitary.

When the system has spinors among the fields, we need to specify the spin structure on the domain surface. On the flat cylinder, for each of the two chiralities, there are two choices — Ramond (R) sector and Neveu-Schwarz (NS) sector in which the parallel transport along the non-trivial circle is the identity and the sign flip respectively. In total, there are four sectors on the cylinder, (R,R), (NS,NS), (R,NS) and (NS,R), or RR, NSNS, RNS and NSR for short.[1] At the boundary of a surface, we need to specify an identification of the spin bundles of the opposite chiralities (which is also a part of the spin structure). Therefore, a boundary circle has two spin structures — RR sector and NSNS sector. Accordingly, for each boundary condition $\mathcal{B}$, there are two boundary states — $|\mathcal{B}\rangle_{\mathrm{RR}}$ and $|\mathcal{B}\rangle_{\mathrm{NSNS}}$. In particular, there are two versions of the identity (B.1),

$$\text{(B.6)} \qquad {}_{\mathrm{RR}}\langle\mathcal{B}_f|\,\mathrm{e}^{-LH_c}|\mathcal{B}_i\rangle_{\mathrm{RR}} = \mathrm{Tr}_{\mathcal{H}_{\overline{\mathcal{B}}_i,\mathcal{B}_f}}(-1)^F\,\mathrm{e}^{-\beta H_o},$$

$$\text{(B.7)} \qquad {}_{\mathrm{NSNS}}\langle\mathcal{B}_f|\,\mathrm{e}^{-LH_c}|\mathcal{B}_i\rangle_{\mathrm{NSNS}} = \mathrm{Tr}_{\mathcal{H}_{\overline{\mathcal{B}}_i,\mathcal{B}_f}}\mathrm{e}^{-\beta H_o}.$$

In a supersymmetric system, when $\mathcal{B}_f$ and $\overline{\mathcal{B}}_i$ preserve a common supersymmetry on the strip $[0, L]\times\mathbb{R}$, then, the right hand side of (B.6)

[1]The "RR gauge potentials" mentioned in Section 5.1 are degrees of freedom coming from the closed string states in the RR sector.

is the open string Witten index $\mathrm{I}_{\overline{\mathcal{B}}_i,\mathcal{B}_f}$. Since the index does not depend on the lengths, it must be the same as the value in the limit $L \to \infty$. Since e^{-LH_c} approaches the projection P_{v} to the subspace of zero energy states (supersymmetric ground states) as $L \to \infty$, we find

$$\mathrm{I}_{\overline{\mathcal{B}}_i,\mathcal{B}_f} = {}_{\mathrm{RR}}\langle \mathcal{B}_f | P_{\mathrm{v}} | \mathcal{B}_i \rangle_{\mathrm{RR}}. \tag{B.8}$$

Let us now consider a $(2,2)$ supersymmetric quantum field theory. Let $\mathcal{B}$ be a B-brane boundary condition. If it is imposed at the right boundary $x = 0$ of the left-half Minkoswki spacetime $\{(x,t)|\, x \leq 0\}$, then the supercharges $\overline{Q}_+ + \overline{Q}_-$ and $Q_+ + Q_-$ are preserved. This means that the space components of the supercurrents vanishes at the boundary, $(\overline{G}^x_+ + \overline{G}^x_-)|_{x=0} = (G^x_+ + G^x_-)|_{x=0} = 0$. In order to translate this into the condition on the boundary state, we need to perform the Wick rotation $t \to -i\tau$ first, and then the 90° rotation $(x',\tau') = (-\tau, x)$ in order to trade the space and time. Note that $z = x + i\tau$ and $z' = x' + i\tau'$ are related by $z' = iz$. Since $\overline{G}_+\sqrt{\mathrm{d}\overline{z}} = \overline{G}_{+'}\sqrt{\mathrm{d}\overline{z}'}$, $\overline{G}_-\sqrt{\mathrm{d}z} = \overline{G}_{-'}\sqrt{\mathrm{d}z'}$, $G_+\sqrt{\mathrm{d}\overline{z}} = G_{+'}\sqrt{\mathrm{d}\overline{z}'}$ and $G_-\sqrt{\mathrm{d}z} = G_{-'}\sqrt{\mathrm{d}z'}$, the equations of the current become the condition on the boundary states

$$\langle \mathcal{B}|(\overline{G}^{\tau'}_{+'} + i\overline{G}^{\tau'}_{-'}) = \langle \mathcal{B}|(G^{\tau'}_{+'} + iG^{\tau'}_{-'}) = 0. \tag{B.9}$$

By conjugation, they become

$$(\overline{G}^{\tau'}_{+'} - i\overline{G}^{\tau'}_{-'})|\mathcal{B}\rangle = (G^{\tau'}_{+'} - iG^{\tau'}_{-'})|\mathcal{B}\rangle = 0. \tag{B.10}$$

These are both in RR and NSNS sectors. Undoing the 90° rotation and the Wick rotation, this means that the conjugate boundary condition $\overline{\mathcal{B}}$ at the left boundary $x = 0$ of the right-half Minkowski spacetime $\{(x,t)|\, x \geq 0\}$ preserves the supercharges $\overline{Q}_+ - \overline{Q}_-$ and $Q_+ - Q_-$. That is, $\overline{\mathcal{B}}$ is a B_{-1}-brane boundary condition. Likewise, if $\mathcal{B}$ is a $\mathrm{B}_{\mathrm{e}^{iu}}$-brane (*resp.* $\mathrm{A}_{\mathrm{e}^{iu}}$-brane), then $\overline{\mathcal{B}}$ is a $\mathrm{B}_{-\mathrm{e}^{iu}}$-brane (*resp.* $\mathrm{A}_{-\mathrm{e}^{iu}}$-brane). As noted earlier, (B.6) can be interpreted as an open string Witten index and (B.8) holds, provided $\mathcal{B}_f$ and $\overline{\mathcal{B}}_i$ preserve the same set of supercharges. This is when $\mathcal{B}_f$ and $\mathcal{B}_i$ preserve the same types of supercharges but with the opposite relative signs. For example, when $\mathcal{B}_f$ is a B-brane and $\mathcal{B}_i$ is a B_{-1}-brame.

The hemisphere partition function is the partition function on D^2 with a particular background, having an outgoing boundary $\partial D^2 = S^1$ in the NSNS sector. For example, the one preserving $\mathrm{A}_{(\pm)}$-type supersymmetry with a $\mathrm{B}_{\pm 1}$-type boundary condition $\mathcal{B}$ may be denoted by

$$Z^{\mathrm{A}_{(\pm)}}_{D^2}(\mathcal{B}) = Z(\mathcal{B}|D^2, \mathrm{A}_{(\pm)}) = {}_{\mathrm{NSNS}}\langle \mathcal{B}|D^2, \mathrm{A}_{(\pm)}\rangle_{\mathrm{NSNS}}. \tag{B.11}$$

To see what the complex conjugate is, we first note that the conjugate hemisphere is the northern hemiphere D^2_∞ (if D^2 is a southern hemisphere) obtained by $z \to \overline{z}^{-1}$. This transforms the conformal Killing spinors as $\mathbf{s}_{\pm\frac{1}{2}} \to \mp i\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}$ and $\widetilde{\mathbf{s}}_{\pm\frac{1}{2}} \to \pm i\mathbf{s}_{\mp\frac{1}{2}}$. This implies that the conjugate background is $\overline{(D^2, \mathrm{A}_{(\pm)})} = (D^2_\infty, \mathrm{A}^{-1}_{(\pm)})$ with an incoming boundary, giving $Z(\mathcal{B}|D^2, \mathrm{A}_{(\pm)})^* = Z(D^2_\infty, \mathrm{A}^{-1}_{(\pm)}|\overline{\mathcal{B}})$. Note that $\overline{\mathcal{B}}$ is an $\mathrm{B}_{\mp 1}$-brane and that is consistent with the fact that the background $D^2_\infty, \mathrm{A}^{-1}_{(\pm)}$ preserves the $\mathrm{B}_{\mp 1}$-type supersymmetry at the boundary. We can regard the northern hemisphre as the southern hemisphere by the map $z \to z^{-1}$. Under this, the conformal Killing spinors transform as $\mathbf{s}_{\pm\frac{1}{2}} \to \pm i\mathbf{s}_{\mp\frac{1}{2}}$ and $\widetilde{\mathbf{s}}_{\pm\frac{1}{2}} \to \mp i\widetilde{\mathbf{s}}_{\mp\frac{1}{2}}$. This means that $(D^2_\infty, \mathrm{A}^{-1}_{(\pm)})$ with incoming boundary can be regarded as $(D^2, \mathrm{A}^{-1}_{(\mp)})$ with outgoing boundary. If $\mathcal{B}$ is unitary, we have $Z(D^2_\infty, \mathrm{A}^{-1}_{(\pm)}|\overline{\mathcal{B}}) = Z(\mathcal{B}|D^2, \mathrm{A}^{-1}_{(\mp)})$. Combining what we have seen, we find

$$Z(\mathcal{B}|D^2, \mathrm{A}_{(\pm)})^* = Z(D^2_\infty, \mathrm{A}^{-1}_{(\pm)}|\overline{\mathcal{B}}) = Z(\mathcal{B}|D^2, \mathrm{A}^{-1}_{(\mp)}), \tag{B.12}$$

that is, ${}_{\mathrm{NSNS}}\langle D^2, \mathrm{A}_{(\pm)}|\mathcal{B}\rangle_{\mathrm{NSNS}} = {}_{\mathrm{NSNS}}\langle \mathcal{B}|D^2, \mathrm{A}^{-1}_{(\mp)}\rangle_{\mathrm{NSNS}}$. The same holds when $(\mathrm{A}_{(\pm)},\ \mathrm{A}^{-1}_{(\pm)})$ is replaced by $(\mathrm{A}^{\mathrm{e}^{iu}}_{(\pm)},\ \mathrm{A}^{-\mathrm{e}^{iu}}_{(\pm)})$ or $(\mathrm{B}^{\mathrm{e}^{iu}}_{(\pm)},\ \mathrm{B}^{-\mathrm{e}^{iu}}_{(\pm)})$. Note that the expressiond for the hemisphere partition functions, (4.21) and (4.32) for GLSMs and (4.27)–(4.28) and (4.31) for LG models, obey this rule.

C. Additional References

After the first version of the present note is submitted for refereeing process, we noticed a few relevant references on which we make some comments.

C.1. Hemisphere versus central charge

A proof of the relation (5.5), $Z_{D^2}(\mathcal{B}) = {}_{\mathrm{RR}}\langle \mathcal{B}|0\rangle_{\mathrm{RR}} =: \Pi_0^{\mathcal{B}}$ up to a constant multiple, between the hemisphere partition function and the CV central charge was proposed by Bachas and Plencner in [75] in the case where the theory is a $(2,2)$ superconformal field theory with charge integrality and the boundary condition $\mathcal{B}$ is superconformal and "stable". A superconformal boundary condition $\mathcal{B}$ is said to be stable when there is a phase $\mathrm{e}^{i\varphi}$ such that

$${}_{\mathrm{RR}}\langle \mathcal{B}|\phi_\alpha\rangle_{\mathrm{RR}} = \mathrm{e}^{i\varphi}\,{}_{\mathrm{NSNS}}\langle \mathcal{B}|\phi_\alpha\rangle_{\mathrm{NSNS}}, \tag{C.1}$$

for any chiral primary field ϕ_α where the states $|\phi_\alpha\rangle_{\mathrm{RR}}$ and $|\phi_\alpha\rangle_{\mathrm{NSNS}}$ are related by the spectral flow. Let us briefly review their logic. To be specific, we consider the hemisphere for a B-brane $\mathcal{B}$, so that the focus will be the dependence on the moduli space $\mathfrak{M}^0 = \mathfrak{M}^0_A$ of superconformal field theories parametrized by the exactly marginal A-term deformation parameters.

The argument closely follows the lines of [8] for the proof of the relation (5.10), $Z_{S^2} = \mathrm{e}^{-K}$, between the sphere partition function and the Kähler potential of the moduli space $\mathfrak{M}^0$. First, they formulated the system on a surface with boundary, coupled to a supergravity background, and wrote down the "anomaly formula" that determines the possible form of the response of the partition function to a change of the background. The formula includes the Kähler potential K and a holomorphic function $h^{\mathcal{B}}$ on the moduli space $\mathfrak{M}^0$, where the latter depends on the boundary condition $\mathcal{B}$. Integrating the formula, they found that the hemisphere partition function is given by

$$Z_{D^2}(\mathcal{B}) = \text{constant} \times \mathrm{e}^{h^{\mathcal{B}}}. \tag{C.2}$$

By conformal perturbation theory, K and $h^{\mathcal{B}}$ are related to the one point function of the A-term marginal operators $\mathcal{O}_i$ on the left half-space $\mathbb{C}_{\leq 0} = \{w \in \mathbb{C} \,|\, \mathrm{Re}(w) \leq 0\}$ as[1]

$$\frac{1}{4\pi}\partial_i(K + h^{\mathcal{B}}) = \langle \mathcal{O}_i(-1)\rangle^{\mathcal{B}}_{\mathbb{C}_{\leq 0}}. \tag{C.3}$$

The operator $\mathcal{O}_i$ is related to an A-chiral primary operator ϕ_i of conformal weight $(\frac{1}{2}, \frac{1}{2})$ by the descent relation $\mathcal{O}_i = Q_+\overline{Q}_-\phi_i$. By the supersymmetric Ward identity, this one point function is proportional to the one for ϕ_i, and by conformal Ward identity for $w \mapsto z = \frac{1+w}{1-w}$, it is proportional to the one point function on the disc $D^2 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$:

$$\begin{aligned}\langle \mathcal{O}_i(-1)\rangle^{\mathcal{B}}_{\mathbb{C}_{\leq 0}} &= -i\langle \phi_i(-1)\rangle^{\mathcal{B}}_{\mathbb{C}_{\leq 0}} = -\frac{i}{2}\langle \phi_i(0)\rangle^{\mathcal{B}}_{D^2} \\ &= -\frac{i}{2}\frac{{}_{\mathrm{NSNS}}\langle \mathcal{B}|\phi_i\rangle_{\mathrm{NSNS}}}{{}_{\mathrm{NSNS}}\langle \mathcal{B}|\,\mathbf{1}\,\rangle_{\mathrm{NSNS}}} = -\frac{i}{2}\frac{{}_{\mathrm{RR}}\langle \mathcal{B}|\phi_i\rangle_{\mathrm{RR}}}{{}_{\mathrm{RR}}\langle \mathcal{B}|\,\mathbf{1}\,\rangle_{\mathrm{RR}}} \\ &= -\frac{i}{2}\frac{\Pi^{\mathcal{B}}_i}{\Pi^{\mathcal{B}}_0}.\end{aligned} \tag{C.4}$$

In the second line, we used the stability condition (C.1). On the other hand, the first equation in (5.7) for the circumference $\boldsymbol{\beta} = 2\pi$ and for

[1] As noted in Section 5.2, we normalize the deformation as $\Delta S_E = \int_\Sigma \mathrm{d}^2x(\Delta t^i\mathcal{O}_i + \Delta\bar{t}^{\bar{\imath}}\overline{\mathcal{O}}_{\bar{\imath}})$. In this normalization, Zamolodchikov metric is $G(\partial_i, \partial_{\bar{\jmath}}) = \langle \mathcal{O}_i(1)\overline{\mathcal{O}}_{\bar{\jmath}}(0)\rangle_{\mathbb{C}} = \frac{1}{\pi^2}\partial_i\partial_{\bar{\jmath}}K$.

the operators $\phi_\alpha = \phi_i$ and $\phi_\beta = \mathbf{1}$ reads $\partial_i \Pi_0^{\mathcal{B}} = -\partial_i K \Pi_0^{\mathcal{B}} - 2\pi i \Pi_i^{\mathcal{B}}$ since $A_{i0}^0 = g^{0\bar{0}} \partial_i g_{\bar{0}0} = -\partial_i K$, where we used $g_{\bar{0}0} = \mathrm{e}^{-K}$ [45]. That is, $\partial_i(K + \log \Pi_0^{\mathcal{B}}) = -2\pi i \Pi_i^{\mathcal{B}} / \Pi_0^{\mathcal{B}}$. Thus, we find

$$\langle \mathcal{O}_i(-1) \rangle^{\mathcal{B}}_{\mathbb{C}_{\leq 0}} = \frac{1}{4\pi} \partial_i (K + \log \Pi_0^{\mathcal{B}}). \tag{C.5}$$

Comparing (C.3) and (C.5), we find that $h^{\mathcal{B}} = \log \Pi_0^{\mathcal{B}}$ up to a constant addition. Combining this with (C.2), we obtain the wanted relation, $Z_{D^2}(\mathcal{B}) = \Pi_0^{\mathcal{B}}$ up to a constant multiple.

Since the hemisphere partition function and the CV central charge are both invariant under continuous deformation of the boundary condition, the relation $Z_{D^2}(\mathcal{B}) = \Pi_0^{\mathcal{B}}$ holds for a general, not necessarily superconformal nor stable, boundary condition $\mathcal{B}$ provided there exists a superconformal and stable boundary condition in the same homotopy class.

C.2. Mathematical works related to the renormalization group flow

In [9] (see also Section 4.4), taking the example $\mathrm{T}_{N,d}^{U(1)}$ with $N \neq d$ for illustration, we examined the behaviour of the hemisphere partition function in the ultra-violet ($r \to 0$) and infra-red ($r \to \infty$) limits, and the result was used to learn which brane in the UV theory flows to which part of the IR theory. In [76–78], analogous problems were studied in the context of the Gromov-Witten theory of a Fano manifold X.

In [76] (see also [77]), Galkin, Golyshev and Iritani studied the behaviour of solutions to the quantum differential equation $\widetilde{\nabla} s = 0$ in the limit $z \to 0$ and presented two conjectures, *Gamma conjectures I* and *II*. Recall from Section 5.3 that the hemisphere partition function $Z_{D^2}^{\mathrm{LV}}$ of the sigma model is proportional to a component Z_{GW} of a solution to the quantum differential equation, when the parameter z in the latter is set to be the inverse radius of the hemisphere, $z \propto r^{-1}$. Therefore, $z \to 0$ corresponds to the infra-red limit $r \to \infty$. When translated into the sigma model language via this connection, the conjectures become assertions concerning the IR behaviour of B-branes in the sigma model.

Let h be the Fano index of X, the largest integer such that $c_1(X)/h$ is an integral class. Then, the group G_A of anomaly free axial R-symmetries is isomorphic to $\mathbb{Z}_{2h}$. It includes the fermion sign flip $(-1)^F$ which can never be spontaneously broken. We consider the sigma model with vanishing B-field, $B = 0$. We assume that the set of vacua that maximize $|\widetilde{W}|$ is a single G_A orbit with stabilizer $\{1, (-1)^F\}$ including one with $\widetilde{W} = |\widetilde{W}|$. Then, the conjectures read as follows.

Gamma conjecture I

The structure sheaf $\mathcal{O}_X$ descends at low energies to a brane supported purely at the vacuum with $\widetilde{W} = |\widetilde{W}|_{\max}$.

Gamma conjecture II

Suppose that all the vacua, say $\mathbf{v}_1, \dots \mathbf{v}_N$, have mass gaps and that $D^b_{\mathrm{Coh}}(X)$ has a full exceptional collection. Then, for each phase $\mathrm{e}^{i\phi}$ such that any non-zero $\widetilde{W}(\mathbf{v}_i) - \widetilde{W}(\mathbf{v}_j)$ is not parallel to $\mathrm{e}^{i\phi}\mathbb{R}$, there are branes $E^\phi_1, \dots, E^\phi_N$ in the sigma model that descend at low energies to branes supported purely at the vacua $\mathbf{v}_1, \dots \mathbf{v}_N$ respectively. Moreover, if $\mathrm{Im}(\mathrm{e}^{-i\phi}\widetilde{W}(\mathbf{v}_{\sigma(i)})) \leq \mathrm{Im}(\mathrm{e}^{-i\phi}\widetilde{W}(\mathbf{v}_{\sigma(j)}))$ for a permutation $\sigma \in \mathfrak{S}_N$, $E^\phi_{\sigma(1)}, \dots, E^\phi_{\sigma(N)}$ form a full exceptional collection of $D^b_{\mathrm{Coh}}(X)$.[2]

The model $\mathrm{T}^{U(1)}_{N,d}$ with $d < N$ corresponds to the Fano sigma model where the target space X is the degree d hypersurface $f = 0$ in the projective space $\mathbb{P}^{N-1}$, with Fano index $h = N - d$. The theory has $N - d$ massive vacua at $(\sigma/\Lambda)^{N-d} = (-d)^d\,\mathrm{e}^{-t}$ (see (3.17)) with $\widetilde{W} = (N-d)\sigma$, breaking the axial R-symmetry $G_A = \mathbb{Z}_{2(N-d)}$ to $\mathbb{Z}_2$, and, for $d \geq 2$, axial symmetry preserving vacua with $\widetilde{W} = 0$ of an SCFT with $\widehat{c} = N(1-2/d)$, the infra-red fixed point of the LG orbifold $(\mathbb{C}^N/\mathbb{Z}_d, f)$. In the case $d = 2$, the sector at $\widetilde{W} = 0$ has $\widehat{c} = 0$ and consists of one massive vacuum (*resp.* two massive vacua) when N is odd (*resp.* even). Note that B-field vanishes when $\theta = -\pi d$ (see (3.15)), in which case the G_A-breaking massive vacua are at $\sigma_k = \Lambda(d^d\,\mathrm{e}^{-\zeta})^{\frac{1}{(N-d)}}\,\mathrm{e}^{\frac{2\pi i k}{N-d}}$ for $k \in \mathbb{Z}/(N-d)\mathbb{Z}$ and have $\widetilde{W} = (N-d)\sigma_k$. Indeed, when Λ is real, the vacuum σ_0 has $\widetilde{W} = |\widetilde{W}|_{\max}$. In comparison with the results with the hemisphere, we should rotate the twisted superpotential as $\widetilde{W} \to -i\widetilde{W}$ due to the change $\Lambda \to -i\Lambda$ (4.36). We have seen in Section 4.4 that the line bundle $\mathcal{O}_X(q)$ for $-N\pi < \theta + 2\pi q < (N-2d)\pi$ descends to a brane supported purely at σ_q. Note that the bound on q for $\theta = -\pi d$ (for $B = 0$) reads $-(N-d)/2 < q < (N-d)/2$, and $q = 0$ satisfies it. In particular, the structure sheaf $\mathcal{O}_X$ descends to a brane at the vacuum σ_0 that has $\widetilde{W} = -i|\widetilde{W}|_{\max}$. Gamma conjecture I indeed holds. In the case $d = 1$ ($X \cong \mathbb{P}^{N-2}$) and $d = 2$ (X is a quadric in $\mathbb{P}^{N-1}$), all the vacua are massive and $D^b_{\mathrm{Coh}}(X)$ has an exceptional collection. Thus, the set-up of Gamma conjecture II is satisfied. The phase $\mathrm{e}^{i\phi}$ may be identified with the phase $\mathrm{e}^{-2i\beta}$ that determines the preserved supersymmetry. The case $d = 1$ is as discussed in [76] and the conjecture

[2] The inequality appears to be opposite to the one in [76, 77], but this is due to the duality $\mathcal{E} \to \mathcal{E}^\vee$ that appears in the footnote in page 195.

holds. For the case $d = 2$ we need to consider the vacua (vacuum) with $\widetilde{W} = 0$ that preserve(s) the axial R-symmetry. It would be interesting to explicitly contruct the branes in the sigma model that descend prely to these (this) vacua (vacuum). In the case $d \geq 3$, the theory has a non-trivial SCFT as an IR fixed point and the set-up of Gamma conjecture II is not satisfied. However, we certainly have a (non-full) exceptional collection in $D^b_{\mathrm{Coh}}(X)$ that descend to branes supported at the massive vacua, and $D^b_{\mathrm{Coh}}(X)$ admits a semi-orthogonal decomposition by these objects and the category $\mathrm{MF}_{\mathbb{Z}_d}(f)$ from the SCFT at $\widetilde{W} = 0$, where the latter sits in the "middle".

In [78], Acosta considered the Picard-Fuchs differential equation (4.108) for the model $\mathrm{T}^{U(1)}_{\vec{w},d}$ and studied the behaviour of solutions at $\mathrm{e}^{-t_R} \to 0$ and at $\mathrm{e}^{-t_R} \to \infty$. Let us restrict the description of this work to the model $\mathrm{T}^{U(1)}_{N,d}$ which is familiar to us, though extension to the model with general $\vec{w}$ is very easy. The focus of [78] is the comparison between the solutions at $\mathrm{e}^{-t_R} \sim 0$ determined by the Gromov-Witten theory of the hypersurface X_f and the solutions at $\mathrm{e}^{-t_R} \sim \infty$ determined by the FJRW theory of $(\mathbb{C}^N/\mathbb{Z}_d, f)$. When $d < N$ where the UV theory is the sigma model, the former is analytic while the latter is formal, and when $d > N$ where the UV theory is the LG orbifold, the former is formal while that latter is analytic. In [78], it was shown that a part of the the analytic solution, when continued to the opposite regime, has the formal solution as its asymptotic expansion. This is relevant to our question: which branes in the UV theory descend to branes in the non-trivial IR fixed point.

C.3. Picard-Fuchs equations as deformed chiral ring relations

In Section 4.6, we presented a picture of Picard-Fuchs equations as a consequence of operator relations at the origin of the hemisphere, which approach the chiral ring relations in the flat space limit $r \to \infty$.

Similar pictures were presented in [79] in a different but related context (two point functions on the Ω-deformed sphere) and also in [80] in a closer context (two point function on the sphere with the background of [4, 5]); Picard-Fuchs equations are regarded as relations among correlators and they are interpreted as consequences of the operator relations, the Ω-deformed chiral ring relations in [79] and the relations identical to ours in [80].

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