

TIGHTNESS AND EQUIVALENCE OF SEMIDEFINITE RELAXATIONS FOR MIMO DETECTION

RUICHEN JIANG*, YA-FENG LIU[†], CHENGLONG BAO[‡], AND BO JIANG[§]

Abstract. The multiple-input multiple-output (MIMO) detection problem, a fundamental problem in modern digital communications, is to detect a vector of transmitted symbols from the noisy outputs of a fading MIMO channel. The maximum likelihood detector can be formulated as a complex least-squares problem with discrete variables, which is NP-hard in general. Various semidefinite relaxation (SDR) methods have been proposed in the literature to solve the problem due to their polynomial-time worst-case complexity and good detection error rate performance. In this paper, we consider two popular classes of SDR-based detectors and study the conditions under which the SDRs are tight and the relationship between different SDR models. For the enhanced complex and real SDRs proposed recently by Lu *et al.*, we refine their analysis and derive the necessary and sufficient condition for the complex SDR to be tight, as well as a necessary condition for the real SDR to be tight. In contrast, we also show that another SDR proposed by Mobasher *et al.* is not tight with high probability under mild conditions. Moreover, we establish a general theorem that shows the equivalence between two subsets of positive semidefinite matrices in different dimensions by exploiting a special “separable” structure in the constraints. Our theorem recovers two existing equivalence results of SDRs defined in different settings and has the potential to find other applications due to its generality.

Key words. MIMO detection, semidefinite relaxation, tight relaxation, equivalent relaxation

AMS subject classifications. 90C22, 90C20, 90C46, 90C27

1. Introduction. Multiple-input multiple-output (MIMO) detection is a fundamental problem in modern digital communications [33, 36]. The MIMO channel can be modeled as

$$(1.1) \quad \mathbf{r} = \mathbf{H}\mathbf{x}^* + \mathbf{v},$$

where $\mathbf{r} \in \mathbb{C}^m$ is the vector of received signals, $\mathbf{H} \in \mathbb{C}^{m \times n}$ is a complex channel matrix, \mathbf{x}^* is the vector of transmitted symbols, and \mathbf{v} is the vector of additive Gaussian noises. Moreover, each entry of \mathbf{x}^* is drawn from a discrete symbol set \mathcal{S} determined by the modulation scheme.

The MIMO detection problem is to recover the transmitted symbol vector \mathbf{x}^* from the noisy channel output \mathbf{r} , with the information of the symbol set \mathcal{S} and the channel matrix \mathbf{H} . Under the assumption that each entry of \mathbf{x}^* is drawn uniformly and independently from the symbol set \mathcal{S} , it is known that the maximum likelihood detector can achieve the optimal detection error rate performance. Mathematically, it can be formulated as a discrete least-squares problem:

$$(1.2) \quad \begin{aligned} \min_{\mathbf{x} \in \mathbb{C}^n} \quad & \|\mathbf{H}\mathbf{x} - \mathbf{r}\|_2^2 \\ \text{s.t.} \quad & x_i \in \mathcal{S}, i = 1, 2, \dots, n, \end{aligned}$$

*Department of Electronic Engineering, Tsinghua University, Beijing 100084, China (rayjiang30@outlook.com).

[†]State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (yafliu@lsec.cc.ac.cn).

[‡]Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China (clbao@mail.tsinghua.edu.cn).

[§]School of Mathematical Sciences, Key Laboratory for NSLSCS of Jiangsu Province, Nanjing Normal University, Nanjing 210023, China (jiangbo@njnu.edu.cn).

where x_i denotes the i -th entry of the vector \mathbf{x} and $\|\cdot\|_2$ denotes the Euclidean norm. In this paper, unless otherwise specified, we will focus on the M -ary phase shift keying (M -PSK) modulation, whose symbol set is given by

$$(1.3) \quad \mathcal{S}_M := \{z \in \mathbb{C} : |z| = 1, \arg(z) \in \{2j\pi/M, j = 0, 1, \dots, M-1\}\},$$

where $|z|$ and $\arg(z)$ denote the modulus and argument of a complex number, respectively. As in most practical digital communication systems, throughout the paper we require $M = 2^b$ where $b \geq 1$ is an integer¹.

Many detection algorithms have been proposed to solve problem (1.2) either exactly or approximately. However, for general \mathbf{H} and \mathbf{r} , problem (1.2) has been proved to be NP-hard [32]. Hence, no polynomial-time algorithms can find the exact solution (unless $P = NP$). Sphere decoding [4], a classical combinatorial algorithm based on the branch-and-bound paradigm, offers an efficient way to solve problem (1.2) exactly when the problem size is small, but its expected complexity is still exponential [11]. On the other hand, some suboptimal algorithms such as linear detectors [23, 6] and decision-feedback detectors [35, 5] enjoy low complexity but at the expense of substantial performance loss: see [36] for an excellent review.

Over the past two decades, semidefinite relaxation (SDR) has gained increasing attention in non-convex optimization [7, 18, 34]. It is a celebrated technique to tackle quadratic optimization problems arising from various signal processing and wireless communication applications, such as beamforming design [24, 15], sensor network localization [3, 2, 27], and angular synchronization [25, 1, 37]. Such SDR-based approaches can usually offer superior performance in both theory and practice while maintaining polynomial-time worst-case complexity.

For MIMO detection problem (1.2), the first SDR detector [30, 20] was designed for the real MIMO channel and the binary symbol set $\mathcal{S} = \{+1, -1\}$. Notably, it is proved that this detector can achieve the maximal possible diversity order [12], meaning that it achieves an asymptotically optimal detection error rate when the signal-to-noise ratio (SNR) is high. It was later extended to the more general setting with a complex channel and an M -PSK symbol set in [28, 19], which we refer to as the conventional SDR or (CSDR). However, this conventional approach fails to fully utilize the structure in the symbol set \mathcal{S} . To overcome this issue, researchers have developed various improved SDRs and we consider the two most popular classes below. The first class proposed in [22] is based on an equivalent zero-one integer programming formulation of problem (1.2). Four SDR models were introduced and two of them will be discussed in details later (see (ESDR1-T) and (ESDR2-T) further ahead). The second class proposed in [17] further enhances (CSDR) by adding valid cuts, resulting in a complex SDR and a real SDR (see (ESDR-X) and (ESDR-Y) later on).

In this paper, we focus on two key problems in SDR-based MIMO detection: the tightness of SDRs and the relationship between different SDR models. Firstly, note that SDR detectors are suboptimal algorithms as they replace the original discrete optimization problem (1.2) with tractable semidefinite programs (SDPs). Hence, after solving an SDP, we need some rounding procedure to make final symbol decisions. However, under some favorable conditions on \mathbf{H} and \mathbf{v} , an SDR can be *tight*, i.e., it has an optimal rank-one solution corresponding to the true vector of transmitted symbols. Such tightness conditions are of great interest since they give theoretical guarantees on the optimality of SDR detectors. While it has been well studied for the

¹Our results in section 3 also hold for the more general case where M is a multiple of four.

simple case [9, 10, 14, 26] where $\mathbf{H} \in \mathbb{R}^{m \times n}$, $\mathbf{v} \in \mathbb{R}^m$, and $\mathcal{S} = \{+1, -1\}$, for the more general case where $\mathbf{H} \in \mathbb{C}^{m \times n}$, $\mathbf{v} \in \mathbb{C}^m$, and $\mathcal{S} = \mathcal{S}_M$ ($M \geq 4$), tightness conditions for SDR detectors have remained unknown until very recently. The authors in [17] showed that (CSDR) is not tight with probability one under some mild conditions. On the other hand, their proposed enhanced SDRs are tight² if the following condition is satisfied:

$$(1.4) \quad \lambda_{\min}(\mathbf{H}^\dagger \mathbf{H}) \sin\left(\frac{\pi}{M}\right) > \|\mathbf{H}^\dagger \mathbf{v}\|_\infty,$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a matrix, $(\cdot)^\dagger$ denotes the conjugate transpose, and $\|\cdot\|_\infty$ denotes the L_∞ -norm. To the best of our knowledge, this is the best condition that guarantees a certain SDR to be tight for problem (1.2) in the M -PSK settings.

Secondly, researchers have noticed some rather unexpected equivalence between different SDR models independently developed in the literature. The earliest one of such results is reported in [21], where three different SDRs for the high-order quadrature amplitude modulation (QAM) symbol sets are proved to be equivalent. Very recently, the authors in [16] showed that the enhanced real SDR proposed in [17] is equivalent to one SDR model in [22]. It is worth noting that while these two papers are of the same nature, the proof techniques are quite different and it is unclear how to generalize their results at present.

In this paper, we make contributions to both problems. For the tightness of SDRs, we sharpen the analysis in [17] to give the necessary and sufficient condition for the complex enhanced SDR to be tight, and a necessary condition for the real enhanced SDR to be tight. Specifically, for the case where $M \geq 4$, we show that the enhanced complex SDR (ESDR- \mathbf{X}) is tight if and only if

$$(1.5) \quad \mathbf{H}^\dagger \mathbf{H} + \text{Diag}(\text{Re}(\text{Diag}(\mathbf{x}^*)^{-1} \mathbf{H}^\dagger \mathbf{v})) - \cot\left(\frac{\pi}{M}\right) \text{Diag}(|\text{Im}(\text{Diag}(\mathbf{x}^*)^{-1} \mathbf{H}^\dagger \mathbf{v})|) \succeq 0,$$

while the enhanced real SDR (ESDR- \mathbf{Y}) is tight only if

$$(1.6) \quad \mathbf{H}^\dagger \mathbf{H} + \text{Diag}(\text{Re}(\text{Diag}(\mathbf{x}^*)^{-1} \mathbf{H}^\dagger \mathbf{v})) - \cot\left(\frac{2\pi}{M}\right) \text{Diag}(|\text{Im}(\text{Diag}(\mathbf{x}^*)^{-1} \mathbf{H}^\dagger \mathbf{v})|) \succeq 0,$$

where $\mathbf{A} \succeq 0$ means that the matrix \mathbf{A} is positive semidefinite (PSD), $\text{Diag}(\mathbf{x})$ denotes a diagonal matrix whose diagonals are the vector \mathbf{x} , and $\text{Re}(\cdot)$, $\text{Im}(\cdot)$, and $|\cdot|$ denote the entrywise real part, imaginary part, and absolute value of a number/vector/matrix, respectively. Moreover, we prove that one of the SDR models proposed in [22] is generally not tight: under some mild assumptions, its probability of being tight decays exponentially with respect to the number of transmitted symbols n .

For the relationship between different SDR models, we propose a general theorem showing the equivalence between two subsets of PSD cones. Specifically, we prove the correspondence between a subset of a high-dimensional PSD cone with a special “separable” structure and the one in a lower dimension. Our theorem covers both equivalence results in [21] and [16] as special cases, and has the potential to find other applications due to its generality.

The paper is organized as follows. We introduce the existing SDRs for (1.2) in section 2 and analyze their tightness in section 3. In section 4, we propose a general

²The definition of tightness in [17] is slightly different from ours since they also require the optimal solution of the SDR to be unique.

theorem that establishes the equivalence between two subsets of PSD cones in different dimensions, and discuss how our theorem implies previous results. [Section 5](#) provides some numerical results to validate our analysis. Finally, [section 6](#) concludes the paper.

We summarize some standard notations used in this paper. We use x_i to denote the i -th entry of a vector \mathbf{x} and $X_{i,j}$ to denote the (i,j) -th entry of a matrix \mathbf{X} . We use $|\cdot|$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ to denote the entrywise absolute value, the Euclidean norm, and the L_∞ norm of a vector, respectively. For a given number/vector/matrix, we use $(\cdot)^\dagger$ to denote the conjugate transpose, $(\cdot)^\top$ to denote the transpose, and $\text{Re}(\cdot)/\text{Im}(\cdot)$ to denote the entrywise real/imaginary part. We use $\text{Diag}(\mathbf{x})$ to denote the diagonal matrix whose diagonals are the vector \mathbf{x} , and $\text{diag}(\mathbf{X})$ to denote the vector whose entries are the diagonals of the matrix \mathbf{X} . Given an $m \times n$ matrix \mathbf{A} and the index sets $\alpha \subset \{1, 2, \dots, m\}$ and $\beta \subset \{1, 2, \dots, n\}$, we use $\mathbf{A}[\alpha, \beta]$ to denote the submatrix with entries in the rows of \mathbf{A} indexed by α and the columns indexed by β . Moreover, we denote the principal submatrix $\mathbf{A}[\alpha, \alpha]$ by $\mathbf{A}[\alpha]$ in short. For two matrices \mathbf{A} and \mathbf{B} of appropriate size, $\langle \mathbf{A}, \mathbf{B} \rangle := \text{Re}(\text{Tr}(\mathbf{A}^\dagger \mathbf{B}))$ denotes the inner product, $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product, and $\mathbf{A} \succeq \mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is PSD. For a set \mathcal{A} in a vector space, we use $\text{conv}(\mathcal{A})$ to denote its convex hull. For a random variable X and measurable sets \mathcal{B} and \mathcal{C} , $\text{Prob}(X \in \mathcal{B})$ denotes the probability of the event $\{X \in \mathcal{B}\}$, $\text{Prob}(X \in \mathcal{B} | \mathcal{C})$ denotes the conditional probability given \mathcal{C} , and $\mathbb{E}[X]$ denotes the expectation of X . Finally, the symbols \mathbf{i} , $\mathbf{1}_n$, \mathbf{I}_n , and \mathbb{S}_+^n represent the imaginary unit, the $n \times 1$ all-one vector, the $n \times n$ identity matrix, and the n -dimensional PSD cone, respectively.

2. Review of semidefinite relaxations. In this paper, we focus on the M -PSK setting with the symbol set \mathcal{S}_M given in [\(1.3\)](#). To simplify the notations, we let $\mathbf{s} \in \mathbb{C}^M$ be the vector of all symbols, where

$$s_j = e^{\mathbf{i}\theta_j} \text{ and } \theta_j = \frac{(j-1)2\pi}{M}, \quad j = 1, 2, \dots, M,$$

and further we let $\mathbf{s}_R = \text{Re}(\mathbf{s})$ and $\mathbf{s}_I = \text{Im}(\mathbf{s})$.

The objective in [\(1.2\)](#) can be written as

$$\|\mathbf{H}\mathbf{x} - \mathbf{r}\|_2^2 = \mathbf{x}^\dagger \mathbf{Q} \mathbf{x} + 2\text{Re}(\mathbf{c}^\dagger \mathbf{x}) + \mathbf{r}^\dagger \mathbf{r} = \langle \mathbf{Q}, \mathbf{x} \mathbf{x}^\dagger \rangle + 2\text{Re}(\mathbf{c}^\dagger \mathbf{x}) + \mathbf{r}^\dagger \mathbf{r},$$

where we define

$$(2.1) \quad \mathbf{Q} = \mathbf{H}^\dagger \mathbf{H} \text{ and } \mathbf{c} = -\mathbf{H}^\dagger \mathbf{r}.$$

By introducing $\mathbf{X} = \mathbf{x} \mathbf{x}^\dagger$ and discarding the constant $\mathbf{r}^\dagger \mathbf{r}$, we can reformulate [\(1.2\)](#) as

$$(2.2) \quad \begin{aligned} \min_{\mathbf{x}, \mathbf{X}} \quad & \langle \mathbf{Q}, \mathbf{X} \rangle + 2\text{Re}(\mathbf{c}^\dagger \mathbf{x}) \\ \text{s.t.} \quad & X_{i,i} = 1, \quad i = 1, 2, \dots, n, \\ & x_i \in \mathcal{S}_M, \quad i = 1, 2, \dots, n, \\ & \mathbf{X} = \mathbf{x} \mathbf{x}^\dagger, \end{aligned}$$

where the constraint $X_{i,i} = 1$ comes from $X_{i,i} = |x_i|^2 = 1$. The conventional SDR (CSDR) in [\[28, 19\]](#) simply drops the discrete symbol constraints $x_i \in \mathcal{S}_M$ and relaxes

the rank-one constraint to $\mathbf{X} \succeq \mathbf{x}\mathbf{x}^\dagger$, resulting in the following relaxation:

$$(CSDR) \quad \begin{aligned} \min_{\mathbf{x}, \mathbf{X}} \quad & \langle \mathbf{Q}, \mathbf{X} \rangle + 2\text{Re}(\mathbf{c}^\dagger \mathbf{x}) \\ \text{s.t.} \quad & X_{i,i} = 1, \quad i = 1, 2, \dots, n, \\ & \mathbf{X} \succeq \mathbf{x}\mathbf{x}^\dagger, \end{aligned}$$

where $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{X} \in \mathbb{C}^{n \times n}$. Since $\mathbf{X} \succeq \mathbf{x}\mathbf{x}^\dagger$ is equivalent to

$$\begin{bmatrix} 1 & \mathbf{x}^\dagger \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0,$$

the above (CSDR) is an SDP on the complex domain. Moreover, for the simple case where $\mathbf{H} \in \mathbb{R}^{m \times n}$, $\mathbf{v} \in \mathbb{R}^m$, and $M = 2$, a real SDR similar to (CSDR) has the form:

$$(2.3) \quad \begin{aligned} \min_{\mathbf{x}, \mathbf{X}} \quad & \langle \mathbf{Q}, \mathbf{X} \rangle + 2\mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & X_{i,i} = 1, \quad i = 1, 2, \dots, n, \\ & \mathbf{X} \succeq \mathbf{x}\mathbf{x}^\top, \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{R}^{n \times n}$, and we redefine $\mathbf{Q} = \mathbf{H}^\top \mathbf{H}$ and $\mathbf{c} = -\mathbf{H}^\top \mathbf{r}$ (cf. (2.1)). The problem (2.3) has also been extensively studied in the literature [30, 20, 9, 10, 14, 26]. It is proved in [9, 10] that (2.3) is tight if and only if

$$(2.4) \quad \mathbf{H}^\top \mathbf{H} + [\text{Diag}(\mathbf{x}^*)]^{-1} \text{Diag}(\mathbf{H}^\top \mathbf{v}) \succeq 0,$$

while (CSDR) is not tight for $M \geq 4$ with probability one under some mild conditions [17].

Recently, a class of enhanced SDRs was proposed in [17]. Instead of simply dropping the constraints $x_i \in \mathcal{S}_M$ as in (CSDR), the authors replaced the discrete symbol set \mathcal{S}_M by its convex hull to get a continuous relaxation:

$$(ESDR-\mathbf{X}) \quad \begin{aligned} \min_{\mathbf{t}, \mathbf{x}, \mathbf{X}} \quad & \langle \mathbf{Q}, \mathbf{X} \rangle + 2\text{Re}(\mathbf{c}^\dagger \mathbf{x}) \\ \text{s.t.} \quad & X_{i,i} = 1, \quad i = 1, 2, \dots, n, \\ & x_i = \sum_{j=1}^M t_{i,j} s_j, \quad \sum_{j=1}^M t_{i,j} = 1, \quad i = 1, 2, \dots, n, \\ & t_{i,j} \geq 0, \quad j = 1, 2, \dots, M, \quad i = 1, 2, \dots, n, \\ & \mathbf{X} \succeq \mathbf{x}\mathbf{x}^\dagger, \end{aligned}$$

where $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{X} \in \mathbb{C}^{n \times n}$, and $\mathbf{t} \in \mathbb{R}^{Mn}$ is the concatenation of M -dimensional vectors $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_n$ with $\mathbf{t}_i = [t_{i,1}, t_{i,2}, \dots, t_{i,M}]^\top$. The authors in [17] further proved that (ESDR- \mathbf{X}) is tight if condition (1.4) holds. We term the above SDP as ‘‘ESDR- \mathbf{X} ’’, where ‘‘E’’ stands for ‘‘enhanced’’ and ‘‘ \mathbf{X} ’’ refers to the matrix variable. The same naming convention is adopted for all the SDRs below.

We can also formulate (2.2) in the real domain and then use the same technique to get a real counterpart of (ESDR- \mathbf{X}). Let

$$(2.5) \quad \mathbf{y} = \begin{bmatrix} \text{Re}(\mathbf{x}) \\ \text{Im}(\mathbf{x}) \end{bmatrix}, \quad \hat{\mathbf{Q}} = \begin{bmatrix} \text{Re}(\mathbf{Q}) & -\text{Im}(\mathbf{Q}) \\ \text{Im}(\mathbf{Q}) & \text{Re}(\mathbf{Q}) \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{c}} = \begin{bmatrix} \text{Re}(\mathbf{c}) \\ \text{Im}(\mathbf{c}) \end{bmatrix},$$

then the real enhanced SDR (**ESDR- \mathbf{Y}**) is given by

$$(ESDR-\mathbf{Y}) \quad \begin{aligned} & \min_{\mathbf{t}, \mathbf{y}, \mathbf{Y}} \quad \langle \hat{\mathbf{Q}}, \mathbf{Y} \rangle + 2\bar{\mathbf{c}}^\top \mathbf{y} \\ & \text{s.t.} \quad \mathcal{Y}(i) = \sum_{j=1}^M t_{i,j} \mathbf{K}_j, \quad \sum_{j=1}^M t_{i,j} = 1, \quad i = 1, 2, \dots, n, \\ & \quad t_{i,j} \geq 0, \quad j = 1, 2, \dots, M, \quad i = 1, 2, \dots, n, \\ & \quad \mathbf{Y} \succeq \mathbf{y}\mathbf{y}^\top, \end{aligned}$$

where $\mathbf{t} \in \mathbb{R}^{Mn}$, $\mathbf{y} \in \mathbb{R}^{2n}$, $\mathbf{Y} \in \mathbb{R}^{2n \times 2n}$, and we define

$$\mathcal{Y}(i) := \begin{bmatrix} 1 & y_i & y_{n+i} \\ y_i & Y_{i,i} & Y_{i,n+i} \\ y_{n+i} & Y_{n+i,i} & Y_{n+i,n+i} \end{bmatrix}, \quad i = 1, 2, \dots, n.$$

In (**ESDR- \mathbf{Y}**), these 3×3 matrices are constrained in a convex hull whose extreme points are

$$(2.6) \quad \mathbf{K}_j = \begin{bmatrix} 1 \\ s_{R,j} \\ s_{I,j} \end{bmatrix} \begin{bmatrix} 1 \\ s_{R,j} \\ s_{I,j} \end{bmatrix}^\top, \quad j = 1, 2, \dots, M,$$

where $s_{R,j} = \text{Re}(s_j)$ and $s_{I,j} = \text{Im}(s_j)$. It has been shown that (**ESDR- \mathbf{Y}**) is tighter than (**ESDR- \mathbf{X}**) [17, Theorem 4.1], and hence (**ESDR- \mathbf{Y}**) is tight whenever (**ESDR- \mathbf{X}**) is tight.

Now we turn to another class of SDRs developed from a different perspective in [22], which is applicable to a general symbol set. The idea is to introduce binary variables to express $x_i \in \mathcal{S}_M$ by

$$(2.7) \quad x_i = \mathbf{t}_i^\top \mathbf{s}, \quad i = 1, 2, \dots, n,$$

where $\mathbf{t}_i = [t_{i,1}, t_{i,2}, \dots, t_{i,M}]^\top$, $\sum_{j=1}^M t_{i,j} = 1$, and $t_{i,j} \in \{0, 1\}$. The above constraints (2.7) can be rewritten in a compact form as $\mathbf{x} = \mathbf{S}\mathbf{t}$, where $\mathbf{S} = \mathbf{I}_n \otimes \mathbf{s}^\top$ and we concatenate all vectors \mathbf{t}_i to get $\mathbf{t} = [\mathbf{t}_1^\top, \dots, \mathbf{t}_n^\top]^\top \in \mathbb{R}^{Mn}$. Similarly, we can also formulate (2.7) in the real domain as $\mathbf{y} = \hat{\mathbf{S}}\mathbf{t}$, where

$$(2.8) \quad \mathbf{y} = \begin{bmatrix} \text{Re}(\mathbf{x}) \\ \text{Im}(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{S}} = \begin{bmatrix} \text{Re}(\mathbf{S}) \\ \text{Im}(\mathbf{S}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n \otimes \mathbf{s}_R^\top \\ \mathbf{I}_n \otimes \mathbf{s}_I^\top \end{bmatrix}.$$

By introducing $\mathbf{T} = \mathbf{t}\mathbf{t}^\top \in \mathbb{R}^{Mn \times Mn}$, the problem (1.2) is equivalent to

$$(2.9) \quad \begin{aligned} & \min_{\mathbf{t}, \mathbf{T}} \quad \langle \bar{\mathbf{Q}}, \mathbf{T} \rangle + 2\bar{\mathbf{c}}^\top \mathbf{t} \\ & \text{s.t.} \quad \sum_{j=1}^M t_{i,j} = 1, \quad i = 1, 2, \dots, n, \\ & \quad t_{i,j} \in \{0, 1\}, \quad j = 1, 2, \dots, M, \quad i = 1, 2, \dots, n, \\ & \quad \mathbf{T} = \mathbf{t}\mathbf{t}^\top, \end{aligned}$$

where

$$(2.10) \quad \bar{\mathbf{Q}} = \hat{\mathbf{S}}^\top \hat{\mathbf{Q}} \hat{\mathbf{S}} \quad \text{and} \quad \bar{\mathbf{c}} = \hat{\mathbf{S}}^\top \hat{\mathbf{c}}.$$

To derive an SDR for (2.9), we first allow $t_{i,j}$ to take any value between 0 and 1. For the rank-one constraint $\mathbf{T} = \mathbf{t}\mathbf{t}^\top$, the authors in [22] proposed four ways of relaxation and we will introduce two of them in the following³. We first partition \mathbf{T} as an $n \times n$ block matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} & \cdots & \mathbf{T}_{1,n} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} & \cdots & \mathbf{T}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_{n,1} & \mathbf{T}_{n,2} & \cdots & \mathbf{T}_{n,n} \end{bmatrix},$$

where $\mathbf{T}_{i,j} \in \mathbb{R}^{M \times M}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. In the first model, we relax $\mathbf{T} = \mathbf{t}\mathbf{t}^\top$ to $\mathbf{T} \succeq \mathbf{t}\mathbf{t}^\top$ and impose constraints on the diagonal elements:

$$\begin{aligned} & \min_{\mathbf{t}, \mathbf{T}} \quad \langle \bar{\mathbf{Q}}, \mathbf{T} \rangle + 2\bar{\mathbf{c}}^\top \mathbf{t} \\ \text{(ESDR1-T)} \quad & \text{s.t.} \quad t_{i,j} \geq 0, \sum_{j=1}^M t_{i,j} = 1, j = 1, 2, \dots, M, i = 1, 2, \dots, n, \\ & \text{diag}(\mathbf{T}_{i,i}) = \mathbf{t}_i, i = 1, 2, \dots, n, \\ & \mathbf{T} \succeq \mathbf{t}\mathbf{t}^\top, \end{aligned}$$

where $\mathbf{t} \in \mathbb{R}^{Mn}$ and $\mathbf{T} \in \mathbb{R}^{Mn \times Mn}$. The second model further requires $\mathbf{T}_{i,i}$ to be a diagonal matrix, leading to the following SDR:

$$\begin{aligned} & \min_{\mathbf{t}, \mathbf{T}} \quad \langle \bar{\mathbf{Q}}, \mathbf{T} \rangle + 2\bar{\mathbf{c}}^\top \mathbf{t} \\ \text{(ESDR2-T)} \quad & \text{s.t.} \quad t_{i,j} \geq 0, \sum_{j=1}^M t_{i,j} = 1, j = 1, 2, \dots, M, i = 1, 2, \dots, n, \\ & \mathbf{T}_{i,i} = \text{Diag}(\mathbf{t}_i), i = 1, 2, \dots, n, \\ & \mathbf{T} \succeq \mathbf{t}\mathbf{t}^\top. \end{aligned}$$

Since (ESDR2-T) puts more constraints on the variables \mathbf{t} and \mathbf{T} , (ESDR2-T) is tighter than (ESDR1-T). Notably, it is shown in [16] that (ESDR2-T) is equivalent to (ESDR-Y), and hence (1.4) is also a sufficient condition for (ESDR2-T) to be tight.

Table 1 summarizes all SDR models discussed in this paper, where we highlight our contributions on the tightness of different SDRs in bold.

3. Tightness of semidefinite relaxations.

3.1. Tightness of (ESDR-X). Let $\mathbf{X}^* = \mathbf{x}^*(\mathbf{x}^*)^\dagger$, and the key idea of showing the tightness of (ESDR-X) is to certify $(\mathbf{x}^*, \mathbf{X}^*)$ as the optimal solution by considering the Karush-Kuhn-Tucker (KKT) conditions of (ESDR-X). Our derivation is based on [17, Theorem 4.2] and we provide a simplified version for completeness.

THEOREM 3.1 ([17, Theorem 4.2]). *Suppose that $M \geq 4$. Then $(\mathbf{x}^*, \mathbf{X}^*)$ is the optimal solution of (ESDR-X) if and only if there exist*

$$\lambda_i \in \mathbb{R}, \mu_{i,-1} \geq 0, \text{ and } \mu_{i,1} \geq 0, i = 1, 2, \dots, n,$$

³Our formulations are slightly different from the original ones in [22] since they used the equality constraints to eliminate one variable for each t_i before relaxing the PSD constraint. However, in numerical tests we found that this variation only causes a negligible difference in the optimal solutions of the SDRs.

TABLE 1
Summary of SDR models in this paper.

SDR model	Origin	Domain	Dimension of PSD cone	Comments
CSDR	Ma <i>et al.</i> [19]	\mathbb{C}	$n + 1$	tight with probability 0 [17]
ESDR- X	CSDP2 in Lu <i>et al.</i> [17]	\mathbb{C}	$n + 1$	tight if and only if (1.5) holds
ESDR- Y	ERSDP in Lu <i>et al.</i> [17]	\mathbb{R}	$2n + 1$	tight only if (1.6) holds
ESDR1- T	Model II in Mobasher <i>et al.</i> [22]	\mathbb{R}	$Mn + 1$	tight with probability no greater than $(2/M)^n$
ESDR2- T	Model III in Mobasher <i>et al.</i> [22]	\mathbb{R}	$Mn + 1$	equivalent to ESDR- Y [16]

such that \mathbf{H} and \mathbf{v} in (1.1) satisfy

$$(x_i^*)^{-1}(\mathbf{H}^\dagger \mathbf{v})_i = \lambda_i + \frac{\mu_{i,-1}}{2} e^{-i\frac{\pi}{M}} + \frac{\mu_{i,1}}{2} e^{i\frac{\pi}{M}}, \quad i = 1, 2, \dots, n,$$

and $\mathbf{Q} + \text{Diag}(\boldsymbol{\lambda}) \succeq 0$.

The authors in [17] further derived the sufficient condition (1.4), under which the conditions in Theorem 3.1 are met by choosing $\lambda_i = -\lambda_{\min}(\mathbf{Q})$ for $i = 1, 2, \dots, n$. To strengthen their analysis, we view the conditions in Theorem 3.1 as a semidefinite feasibility problem. To be specific, if we define

$$(3.1) \quad z_i = (x_i^*)^{-1}(\mathbf{H}^\dagger \mathbf{v})_i, \quad i = 1, 2, \dots, n,$$

and

$$\mathcal{C}(\lambda) = \left\{ z \in \mathbb{C} : \exists \mu_{-1}, \mu_1 \geq 0 \text{ s.t. } z = \lambda + \frac{\mu_{-1}}{2} e^{-i\frac{\pi}{M}} + \frac{\mu_1}{2} e^{i\frac{\pi}{M}} \right\},$$

then Theorem 3.1 states that (ESDR-**X**) is tight if and only if the following problem is feasible:

$$(3.2) \quad \begin{aligned} &\text{find } \boldsymbol{\lambda} \in \mathbb{R}^n \\ &\text{s.t. } \mathbf{Q} + \text{Diag}(\boldsymbol{\lambda}) \succeq 0, \\ &\quad z_i \in \mathcal{C}(\lambda_i), \quad i = 1, 2, \dots, n. \end{aligned}$$

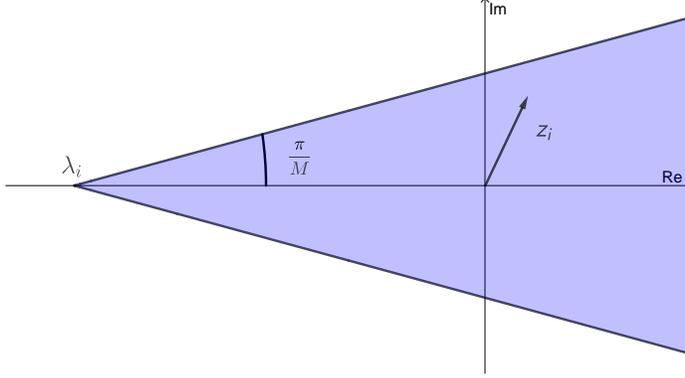
Each constraint $z_i \in \mathcal{C}(\lambda_i)$ turns out to be a simple inequality on λ_i . To see this, we plot $\mathcal{C}(\lambda_i)$ as the shaded area in Figure 1. It is clear from the figure that

$$z_i \in \mathcal{C}(\lambda_i) \Leftrightarrow |\text{Im}(z_i)| \leq (-\lambda_i + \text{Re}(z_i)) \tan\left(\frac{\pi}{M}\right),$$

which leads to

$$z_i \in \mathcal{C}(\lambda_i) \Leftrightarrow \lambda_i \leq \text{Re}(z_i) - |\text{Im}(z_i)| \cot\left(\frac{\pi}{M}\right).$$

This, together with (3.2), gives the necessary and sufficient condition for (ESDR-**X**) to be tight and we formally state it in Theorem 3.2.

FIG. 1. Illustration of $\mathcal{C}(\lambda_i)$ in the complex plane.

THEOREM 3.2. Suppose that $M \geq 4$. Then (ESDR- \mathbf{X}) is tight if and only if

$$(3.3) \quad \mathbf{Q} + \text{Diag}(\text{Re}(\mathbf{z})) - \cot\left(\frac{\pi}{M}\right) \text{Diag}(|\text{Im}(\mathbf{z})|) \succeq 0,$$

where $\mathbf{z} = [z_1, z_2, \dots, z_n]^T \in \mathbb{C}^n$.

Note that (3.3) is exactly the same as (1.5) if we recall the definitions of \mathbf{Q} in (2.1) and z_i in (3.1). Furthermore, if we set $M = 2$ and \mathbf{H} , \mathbf{v} to be real in (1.5), it becomes the same as the previous result (2.4). Hence, our result extends (2.4) to the more general case where $M \geq 4$ and \mathbf{H} , \mathbf{v} are complex. Finally, the sufficient condition (1.4) in [17] can be derived from our result. Since

$$\begin{aligned} \text{Re}(z_i) - |\text{Im}(z_i)| \cot\left(\frac{\pi}{M}\right) &= \frac{1}{\sin\left(\frac{\pi}{M}\right)} \left(\text{Re}(z_i) \sin\left(\frac{\pi}{M}\right) - |\text{Im}(z_i)| \cos\left(\frac{\pi}{M}\right) \right) \\ &\geq -\frac{1}{\sin\left(\frac{\pi}{M}\right)} |z_i| \geq -\frac{1}{\sin\left(\frac{\pi}{M}\right)} \|\mathbf{H}^\dagger \mathbf{v}\|_\infty, \end{aligned}$$

we have

$$\text{Diag}(\text{Re}(\mathbf{z})) - \cot\left(\frac{\pi}{M}\right) \text{Diag}(|\text{Im}(\mathbf{z})|) \succeq -\frac{1}{\sin\left(\frac{\pi}{M}\right)} \|\mathbf{H}^\dagger \mathbf{v}\|_\infty \mathbf{I}_n.$$

Combining this with $\mathbf{Q} \succeq \lambda_{\min}(\mathbf{Q}) \mathbf{I}_n$, we can see that (1.4) is a stronger condition on \mathbf{H} and \mathbf{v} than (1.5).

3.2. Tightness of (ESDR- \mathbf{Y}). Similar to Theorem 3.1, we have the following characterization for (ESDR- \mathbf{Y}) to be tight. Since the proof technique is essentially the same as that in [17], we put the proof in a separate technical report [13].

THEOREM 3.3. Suppose that $M \geq 4$. Let the transmitted symbol vector \mathbf{x}^* be

$$x_i^* = s_{u_i}, \quad u_i \in \{1, 2, \dots, M\}, \quad i = 1, 2, \dots, n,$$

and define

$$(3.4) \quad \begin{aligned} \hat{\mathbf{v}} &= \begin{bmatrix} \text{Re}(\mathbf{v}) \\ \text{Im}(\mathbf{v}) \end{bmatrix} \in \mathbb{R}^{2n}, \quad \hat{\mathbf{H}} = \begin{bmatrix} \text{Re}(\mathbf{H}) & -\text{Im}(\mathbf{H}) \\ \text{Im}(\mathbf{H}) & \text{Re}(\mathbf{H}) \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \\ \mathbf{y}^* &= \begin{bmatrix} \text{Re}(\mathbf{x}^*) \\ \text{Im}(\mathbf{x}^*) \end{bmatrix} \in \mathbb{R}^{2n}, \quad \mathbf{Y}^* = \mathbf{y}^* (\mathbf{y}^*)^T \in \mathbb{R}^{2n \times 2n}. \end{aligned}$$

Then $(\mathbf{y}^*, \mathbf{Y}^*)$ is the optimal solution of (ESDR- \mathbf{Y}) if and only if there exist $\boldsymbol{\lambda} \in \mathbb{R}^{2n}$, $\boldsymbol{\mu} \in \mathbb{R}^n$, and $\mathbf{g} \in \mathbb{R}^{2n}$ that satisfy

$$(3.5) \quad \hat{\mathbf{H}}^\top \hat{\mathbf{v}} = \mathbf{g} + (\boldsymbol{\Lambda} + \mathbf{M})\mathbf{y}^*,$$

$$(3.6) \quad \langle \boldsymbol{\Gamma}_i, \mathbf{K}_{u_i} \rangle \geq \langle \boldsymbol{\Gamma}_i, \mathbf{K}_j \rangle, \quad j = 1, 2, \dots, M, \quad i = 1, 2, \dots, n,$$

and

$$\hat{\mathbf{Q}} + \boldsymbol{\Lambda} + \mathbf{M} \succeq 0,$$

where \mathbf{K}_j is defined in (2.6), $\hat{\mathbf{Q}}$ is defined in (2.5), and

$$(3.7) \quad \boldsymbol{\Lambda} = \text{Diag}(\boldsymbol{\lambda}), \quad \mathbf{M} = \begin{bmatrix} \mathbf{0} & \text{Diag}(\boldsymbol{\mu}) \\ \text{Diag}(\boldsymbol{\mu}) & \mathbf{0} \end{bmatrix}, \quad \boldsymbol{\Gamma}_i = \begin{bmatrix} 0 & g_i & g_{n+i} \\ g_i & \lambda_i & \mu_i \\ g_{n+i} & \mu_i & \lambda_{n+i} \end{bmatrix}.$$

Furthermore, (3.5) and (3.6) in Theorem 3.3 can be simplified to the following inequalities on $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ (see Appendix A):

$$(3.8) \quad \begin{aligned} & \sin^2\left(\theta_{u_i} + \frac{\Delta\theta_j}{2}\right)\lambda_i + \cos^2\left(\theta_{u_i} + \frac{\Delta\theta_j}{2}\right)\lambda_{n+i} - \sin(2\theta_{u_i} + \Delta\theta_j)\mu_j \\ & \leq \text{Re}(z_i) - \cot\left(\frac{\Delta\theta_j}{2}\right)\text{Im}(z_i), \quad j \in \{1, 2, \dots, M\} \setminus \{u_i\}, \quad i = 1, 2, \dots, n. \end{aligned}$$

Here $\Delta\theta_j = \theta_j - \theta_{u_i}$, θ_{u_i} is the phase of the i -th transmitted symbol x_i^* , and z_i is defined in (3.1). Similar to (3.2), we formulate the conditions in Theorem 3.3 as a semidefinite feasibility problem as follows:

$$(3.9) \quad \begin{aligned} & \text{find } \boldsymbol{\lambda} \in \mathbb{R}^{2n} \text{ and } \boldsymbol{\mu} \in \mathbb{R}^n \\ & \text{s.t. } \hat{\mathbf{Q}} + \boldsymbol{\Lambda} + \mathbf{M} \succeq 0, \\ & \quad (3.8) \text{ is satisfied,} \end{aligned}$$

where $\boldsymbol{\Lambda}$ and \mathbf{M} are defined in (3.7). However, unlike problem (3.2) where every inequality only involves one dual variable, problem (3.9) has inequalities with three variables coupled together and it is unclear how to choose the ‘‘optimal’’ $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. In the following, we give a simple necessary condition for (ESDR- \mathbf{Y}) being tight based on (3.9).

THEOREM 3.4. *Suppose that $M \geq 4$. If (ESDR- \mathbf{Y}) is tight, then*

$$(3.10) \quad \mathbf{Q} + \text{Diag}(\text{Re}(\mathbf{z})) - \cot\left(\frac{2\pi}{M}\right)\text{Diag}(|\text{Im}(\mathbf{z})|) \succeq 0,$$

where $\mathbf{Q} = \mathbf{H}^\dagger \mathbf{H}$ and \mathbf{z} is defined in (3.1).

Before proving Theorem 3.4, we first introduce the following lemma.

LEMMA 3.5. *Suppose that \mathbf{V} is a PSD matrix in \mathbb{R}^{2n} and is partitioned as*

$$\mathbf{V} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix},$$

where $\mathbf{A} = \mathbf{A}^\top$, $\mathbf{C} = \mathbf{C}^\top$, and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times n}$. Then

$$\mathbf{U} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) + \frac{\mathbf{i}}{2}(\mathbf{B}^\top - \mathbf{B})$$

is a PSD matrix in \mathbb{C}^n .

Proof. We observe that

$$\mathbf{U} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_n & \mathbf{iI}_n \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ -\mathbf{iI}_n \end{bmatrix}.$$

The result follows immediately. \square

Proof of Theorem 3.4. If (ESDR- \mathbf{Y}) is tight, we can find $\boldsymbol{\lambda} \in \mathbb{R}^{2n}$ and $\boldsymbol{\mu} \in \mathbb{R}^n$ that satisfy the constraints in (3.9). By Lemma 3.5, the constraint $\hat{\mathbf{Q}} + \boldsymbol{\Lambda} + \mathbf{M} \succeq 0$ implies

$$(3.11) \quad \mathbf{Q} + \text{Diag}(\bar{\boldsymbol{\lambda}}) \succeq 0,$$

where $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^n$ is given by

$$\bar{\lambda}_i = \frac{1}{2}(\lambda_i + \lambda_{n+i}), \quad i = 1, 2, \dots, n.$$

Fix $i \in \{1, 2, \dots, n\}$ and let \hat{z}_i and \hat{z}_{n+i} denote $\text{Re}(z_i)$ and $\text{Im}(z_i)$, respectively. If $\hat{z}_{n+i} \geq 0$, we set $\Delta\theta_j = \frac{2\pi}{M}$ in (3.8) to get

$$\sin^2\left(\theta_{u_i} + \frac{\pi}{M}\right)\lambda_i + \cos^2\left(\theta_{u_i} + \frac{\pi}{M}\right)\lambda_{n+i} - \sin\left(2\theta_{u_i} + \frac{2\pi}{M}\right)\mu_j \leq \hat{z}_i - \cot\left(\frac{\pi}{M}\right)|\hat{z}_{n+i}|.$$

Since $M \geq 4$, we can also set $\Delta\theta_j = \frac{2\pi}{M} + \pi$ to get

$$\cos^2\left(\theta_{u_i} + \frac{\pi}{M}\right)\lambda_i + \sin^2\left(\theta_{u_i} + \frac{\pi}{M}\right)\lambda_{n+i} + \sin\left(2\theta_{u_i} + \frac{2\pi}{M}\right)\mu_j \leq \hat{z}_i + \tan\left(\frac{\pi}{M}\right)|\hat{z}_{n+i}|.$$

Adding the above two inequalities and dividing both sides by two, we have

$$(3.12) \quad \bar{\lambda}_i = \frac{1}{2}(\lambda_i + \lambda_{n+i}) \leq \hat{z}_i - \cot\left(\frac{2\pi}{M}\right)|\hat{z}_{n+i}|.$$

If $\hat{z}_{n+i} < 0$, we can also arrive at (3.12) by setting $\Delta\theta_j$ to be $-\frac{2\pi}{M}$ and $-\frac{2\pi}{M} + \pi$, respectively. Finally, Theorem 3.4 follows from (3.11) and (3.12). \square

Note that (3.10) is the same as (1.6) if we recall the definitions of \mathbf{Q} in (2.1) and z_i in (3.1). Moreover, since (ESDR- \mathbf{Y}) is tighter than (ESDR- \mathbf{X}), (ESDR- \mathbf{Y}) will also be tight if (1.5) holds. Therefore, we have both a necessary condition (1.6) and a sufficient condition (1.5) for (ESDR- \mathbf{Y}) to be tight.

3.3. Tightness of (ESDR1-T). In the same spirit, we first give a necessary and sufficient condition for (ESDR1-T) to be tight. Since the technique is essentially the same as that used in Theorem 3.3, we omit the proof details due to the space limitation.

THEOREM 3.6. *Suppose that $M \geq 4$. Let the transmitted symbol vector \mathbf{x}^* be*

$$x_i^* = s_{u_i}, \quad u_i \in \{1, 2, \dots, M\}, \quad i = 1, 2, \dots, n,$$

and define

$$\mathbf{t}_{i,u_i}^* = 1, \quad \mathbf{t}_{i,j}^* = 0, \quad j \neq u_i, \quad i = 1, 2, \dots, n,$$

$$\mathbf{T}^* = \mathbf{t}^*(\mathbf{t}^*)^\top.$$

Then $(\mathbf{t}^*, \mathbf{T}^*)$ is the optimal solution of (ESDR1-T) if and only if there exist $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\boldsymbol{\gamma} \in \mathbb{R}^{Mn}$ such that

$$(3.13) \quad \text{Diag}(1 - 2\mathbf{t}^*)\boldsymbol{\gamma} = -2\hat{\mathbf{S}}^\top \hat{\mathbf{H}}^\top \hat{\mathbf{v}} + \boldsymbol{\alpha} \otimes \mathbf{1}_M,$$

and

$$(3.14) \quad \bar{\mathbf{Q}} + \text{Diag}(\boldsymbol{\gamma}) \succeq 0.$$

where $\hat{\mathbf{S}}$ is defined in (2.8), $\hat{\mathbf{H}}$, $\hat{\mathbf{v}}$ are defined in (3.4), and $\bar{\mathbf{Q}}$ is defined in (2.10).

Now we provide a corollary that will serve as our basis for further derivation.

COROLLARY 3.7. *If (ESDR1-T) is tight, then there exist*

$$\boldsymbol{\alpha} \in \mathbb{R}^n \text{ and } \gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{R}^M$$

that satisfy

$$(3.15) \quad \gamma_{i,j} = \begin{cases} -2\text{Re}[s_j^\dagger(\mathbf{H}^\top \mathbf{v})_i] + \alpha_i, & \text{if } j \neq u_i, \\ 2\text{Re}[s_j^\dagger(\mathbf{H}^\top \mathbf{v})_i] - \alpha_i, & \text{if } j = u_i, \end{cases} \quad j = 1, 2, \dots, M, \quad i = 1, 2, \dots, n,$$

and

$$\mathbf{w}^\top \text{Diag}(\gamma_i) \mathbf{w} \geq 0, \quad i = 1, 2, \dots, n,$$

for any $\mathbf{w} \in \mathbb{R}^M$ such that $\mathbf{w}^\top \mathbf{s}_R = \mathbf{w}^\top \mathbf{s}_I = 0$.

Proof. By Theorem 3.6, if (ESDR1-T) is tight, we can find $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\boldsymbol{\gamma} \in \mathbb{R}^{Mn}$ that satisfy (3.13) and (3.14). Let $\boldsymbol{\gamma}$ be partitioned as $\boldsymbol{\gamma} = [\boldsymbol{\gamma}_1^\top, \boldsymbol{\gamma}_2^\top, \dots, \boldsymbol{\gamma}_n^\top]^\top$ where $\boldsymbol{\gamma}_j \in \mathbb{R}^M$ is the j -th block of $\boldsymbol{\gamma}$. It is straightforward to verify that (3.13) is equivalent to (3.15).

Moreover, for any $i \in \{1, 2, \dots, n\}$ and any $\mathbf{w} \in \mathbb{R}^M$ that satisfies $\mathbf{w}^\top \mathbf{s}_R = \mathbf{w}^\top \mathbf{s}_I = 0$, we set $\bar{\mathbf{w}} = [\bar{w}_1^\top, \bar{w}_2^\top, \dots, \bar{w}_n^\top]^\top \in \mathbb{R}^{Mn}$ to be

$$\bar{w}_j = \begin{cases} \mathbf{0}, & \text{if } j \neq i, \\ \mathbf{w}, & \text{if } j = i. \end{cases}$$

It is simple to check that $\hat{\mathbf{S}}\bar{\mathbf{w}} = \mathbf{0}$. Therefore, recalling that $\bar{\mathbf{Q}} = \hat{\mathbf{S}}^\top \hat{\mathbf{Q}} \hat{\mathbf{S}}$, by (3.14) we have

$$\bar{\mathbf{w}}^\top (\bar{\mathbf{Q}} + \text{Diag}(\boldsymbol{\gamma})) \bar{\mathbf{w}} = \bar{\mathbf{w}}^\top \text{Diag}(\boldsymbol{\gamma}) \bar{\mathbf{w}} = \mathbf{w}^\top \text{Diag}(\gamma_i) \mathbf{w} \geq 0.$$

The proof is complete. \square

In practice, the symbol set \mathcal{S} , such as the one in (1.3) considered in this paper, is symmetric with respect to the origin. Therefore, we can find $u'_i \in \{1, 2, \dots, M\}$ that satisfies $s_{u'_i} = -s_{u_i}$. Now let $\mathbf{w} \in \mathbb{R}^M$ be

$$w_j = \begin{cases} 0, & \text{if } j \notin \{u_i, u'_i\}, \\ 1, & \text{if } j \in \{u_i, u'_i\}, \end{cases} \quad j = 1, 2, \dots, M.$$

We have $\mathbf{w}^\top \mathbf{s}_R = s_{R,u_i} + s_{R,u'_i} = 0$ and $\mathbf{w}^\top \mathbf{s}_I = s_{I,u_i} + s_{I,u'_i} = 0$. Hence, when (ESDR1-T) is tight, Corollary 3.7 implies that

$$\mathbf{w}^\top \text{Diag}(\gamma_i) \mathbf{w} = \gamma_{u_i} + \gamma_{u'_i} = 4\text{Re}[s_{u_i}^\dagger(\mathbf{H}^\top \mathbf{v})_i] = 4\text{Re}[(x_i^*)^\dagger(\mathbf{H}^\top \mathbf{v})_i] \geq 0.$$

This immediately leads to the following upper bound on the tightness probability of **(ESDR1-T)**.

COROLLARY 3.8. *Suppose that the symbol set \mathcal{S} is symmetric with respect to the origin and $0 \notin \mathcal{S}$. We further assume that*

- (a) *The entries of \mathbf{x}^* are drawn from \mathcal{S} uniformly and independently;*
- (b) *\mathbf{x}^* , \mathbf{H} , and \mathbf{v} are mutually independent; and*
- (c) *the distribution of \mathbf{H} and \mathbf{v} are continuous.*

Then we have

$$\mathbf{Prob}(\mathbf{(ESDR1-T)} \text{ is tight}) \leq \left(\frac{1}{2}\right)^n.$$

Proof. Let $z_i = (x_i^*)^\dagger (\mathbf{H}^\dagger \mathbf{v})_i$, $i = 1, 2, \dots, n$. Since \mathbf{H} and \mathbf{v} are independent continuous random variables, the event

$$\bigcup_{i=1}^n \bigcup_{s \in \mathcal{S}} \{\operatorname{Re}(s^\dagger (\mathbf{H}^\dagger \mathbf{v})_i) = 0\}$$

happens with probability zero. Hence, because of the symmetry of \mathcal{S} , with probability one exactly half of the symbols $s \in \mathcal{S}$ satisfy $\operatorname{Re}(s^\dagger (\mathbf{H}^\dagger \mathbf{v})_i) > 0$ for each $i \in \{1, 2, \dots, n\}$ when \mathbf{H} and \mathbf{v} are given. By the assumption that x_i^* is uniformly distributed over \mathcal{S} , we obtain

$$\mathbf{Prob}(\operatorname{Re}(z_i) \geq 0 \mid \mathbf{H}, \mathbf{v}) = \frac{1}{2} \text{ almost surely.}$$

Moreover, $\{z_i\}_{i=1}^n$ are mutually independent conditioned on \mathbf{H} and \mathbf{v} . This leads to

$$\begin{aligned} \mathbf{Prob}(\operatorname{Re}(z_i) \geq 0, i = 1, 2, \dots, n) &= \mathbb{E}_{\mathbf{H}, \mathbf{v}} [\mathbf{Prob}(\operatorname{Re}(z_i) \geq 0, i = 1, 2, \dots, n \mid \mathbf{H}, \mathbf{v})] \\ &= \mathbb{E}_{\mathbf{H}, \mathbf{v}} \left[\prod_{i=1}^n \mathbf{Prob}(\operatorname{Re}(z_i) \geq 0 \mid \mathbf{H}, \mathbf{v}) \right] \\ &= \left(\frac{1}{2}\right)^n. \end{aligned}$$

Finally, **Corollary 3.8** follows from the fact that the tightness of **(ESDR1-T)** implies $\operatorname{Re}(z_i) \geq 0$, $i = 1, 2, \dots, n$. \square

It is worth noting that all the assumptions in **Corollary 3.8** are mild: they are satisfied if we use the M -PSK or QAM modulation scheme and the entries of \mathbf{H} and \mathbf{v} follow the complex Gaussian distribution.

Intuitively, we will expect that **(ESDR1-T)** is less likely to recover the transmitted symbols with an increasing symbol set size M . In the following, we present a more refined upper bound on the tightness probability specific to the M -PSK setting and the proof can be found in [13].

THEOREM 3.9. *Suppose that M -PSK is used with $M \geq 4$ and the same assumptions in **Corollary 3.8** hold. Then we have*

$$\mathbf{Prob}(\mathbf{(ESDR1-T)} \text{ is tight}) \leq \left(\frac{2}{M}\right)^n.$$

From [Corollary 3.8](#) and [Theorem 3.9](#), we can see that the tightness probability of [\(ESDR1-T\)](#) is bounded away from one regardless of the noise level, and it tends to zero exponentially fast when the number of transmitted symbols n increases. This is in sharp contrast to [\(ESDR-X\)](#) and [\(ESDR-Y\)](#), whose tightness probabilities will approach one if the noise level is sufficiently small and the number of received signals m is sufficiently large compared to n [[17](#), [Theorem 4.5](#)].

4. Equivalence between different SDRs. In this section, we focus on the relationship between different SDR models of [\(1.2\)](#). Related to the SDRs discussed so far, a recent paper [[16](#)] proved that [\(ESDR2-T\)](#) is equivalent to [\(ESDR-Y\)](#) for the MIMO detection problem with a general symbol set. An earlier paper [[21](#)] compared three different SDRs in the QAM setting and showed their equivalence. Compared with those in [section 2](#), the SDRs considered in [[21](#)] differ greatly in their motivations and structures, and the two equivalence results are proved using different techniques. In this section, we provide a more general equivalence theorem from which both results follow as special cases. This not only reveals the underlying connection between these two works, but also may potentially lead to new equivalence between SDRs.

4.1. Review of previous results. In [[16](#)], the authors established the following correspondence between a pair of feasible points of [\(ESDR2-T\)](#) and [\(ESDR-Y\)](#):

$$(4.1) \quad \mathbf{Y} = \hat{\mathbf{S}}\mathbf{T}\hat{\mathbf{S}}^\top \text{ and } \mathbf{y} = \hat{\mathbf{S}}\mathbf{t},$$

where $\hat{\mathbf{S}} \in \mathbb{R}^{2n \times Mn}$ is defined in [\(2.8\)](#). In [[21](#)], the authors considered the feasible set of a virtually-antipodal SDR (VA-SDR):

$$(VA\text{-SDR}) \quad \begin{bmatrix} 1 & \mathbf{b}^\top \\ \mathbf{b} & \mathbf{B} \end{bmatrix} \in \mathbb{S}_+^{qn+1} \\ \text{s.t. } B_{i,i} = 1, \quad i = 1, 2, \dots, qn,$$

and that of a bounded-constrained SDR (BC-SDR):

$$(BC\text{-SDR}) \quad \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \in \mathbb{S}_+^{n+1} \\ \text{s.t. } 1 \leq X_{i,i} \leq (2^q - 1)^2, \quad i = 1, 2, \dots, n,$$

where $q \geq 1$ is an integer. We refer interested readers to [[21](#)] and references therein for their derivations. The authors proved the equivalence between [\(VA-SDR\)](#) and [\(BC-SDR\)](#) by showing the following correspondence:

$$(4.2) \quad \mathbf{X} = \mathbf{W}\mathbf{B}\mathbf{W}^\top \text{ and } \mathbf{x} = \mathbf{W}\mathbf{b},$$

where

$$\mathbf{W} = [\mathbf{I}_n \quad 2\mathbf{I}_n \quad 4\mathbf{I}_n \quad \dots \quad 2^{q-1}\mathbf{I}_n] \in \mathbb{R}^{n \times qn}.$$

Note that both equivalence results in [\(4.1\)](#) and [\(4.2\)](#) fall into the following form:

$$\left\{ \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix} \in \mathcal{F}_1 \right\} = \left\{ \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{t}^\top \\ \mathbf{t} & \mathbf{T} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^\top \end{bmatrix} : \begin{bmatrix} 1 & \mathbf{t}^\top \\ \mathbf{t} & \mathbf{T} \end{bmatrix} \in \mathcal{F}_2 \right\},$$

where \mathcal{F}_1 is a subset of \mathbb{S}_+^{k+1} , \mathcal{F}_2 is a subset of \mathbb{S}_+^{d+1} , and we call $\mathbf{P} \in \mathbb{R}^{k \times d}$ as the transformation matrix. Moreover, both the transformation matrices $\hat{\mathbf{S}}$ in [\(4.1\)](#) and \mathbf{W} in [\(4.2\)](#) have a special ‘‘separable’’ property that we now define for ease of presentation.

DEFINITION 4.1. A matrix $\mathbf{P} \in \mathbb{R}^{k \times d}$ is called separable if there exist a partition of rows $\alpha_1, \alpha_2, \dots, \alpha_l$ and a partition of columns $\beta_1, \beta_2, \dots, \beta_l$ such that

$$\mathbf{P}[\alpha_i, \beta_j] = \mathbf{0}, \quad \forall i \neq j.$$

In other words, a matrix is separable if, after possibly rearranging rows and columns, it has a block diagonal structure. In particular, for the transformation matrix $\hat{\mathbf{S}}$ in (4.1), the corresponding row and column partitions are given by

$$(4.3) \quad \alpha_i = \{i, n+i\}, \quad \beta_i = \{(i-1)M+1, (i-1)M+2, \dots, iM\}, \quad i = 1, 2, \dots, n;$$

for the transformation matrix \mathbf{W} in (4.2), they are given by

$$(4.4) \quad \alpha_i = \{i\}, \quad \beta_i = \{i, i+n, i+2n, \dots, i+(q-1)n\}, \quad i = 1, 2, \dots, n.$$

4.2. A general equivalence theorem. Now we are ready to present our main equivalence result.

THEOREM 4.2. Suppose that the matrix $\mathbf{P} \in \mathbb{R}^{k \times d}$ is separable with row partition $\alpha_1, \alpha_2, \dots, \alpha_l$ and column partition $\beta_1, \beta_2, \dots, \beta_l$. Moreover, define

$$k_i := |\alpha_i|, \quad d_i := |\beta_i|, \quad \text{and } \mathbf{P}_i := \mathbf{P}[\alpha_i, \beta_i] \in \mathbb{R}^{k_i \times d_i}, \quad i = 1, 2, \dots, l,$$

where we use $|\cdot|$ to denote the cardinality of a set. Then given arbitrary constraint sets $\mathcal{A}_i \subset \mathbb{R}^{d_i \times d_i}$ for $i = 1, 2, \dots, l$, the following set

$$(4.5) \quad \begin{aligned} & \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix} \in \mathbb{S}_+^{k+1} \\ \text{s.t. } & \mathbf{Y} = \mathbf{P}\mathbf{T}\mathbf{P}^\top, \quad \mathbf{y} = \mathbf{P}\mathbf{t}, \\ & \begin{bmatrix} 1 & \mathbf{t}^\top \\ \mathbf{t} & \mathbf{T} \end{bmatrix} \in \mathbb{S}_+^{d+1}, \\ & \begin{bmatrix} 1 & \mathbf{t}[\beta_i]^\top \\ \mathbf{t}[\beta_i] & \mathbf{T}[\beta_i] \end{bmatrix} \in \mathcal{A}_i, \quad i = 1, 2, \dots, l, \end{aligned}$$

where the variables are $\mathbf{y} \in \mathbb{R}^k$, $\mathbf{Y} \in \mathbb{R}^{k \times k}$, $\mathbf{t} \in \mathbb{R}^d$, and $\mathbf{T} \in \mathbb{R}^{d \times d}$, is the same as

$$(4.6) \quad \begin{aligned} & \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix} \in \mathbb{S}_+^{k+1} \\ \text{s.t. } & \mathbf{Y}[\alpha_i] = \mathbf{P}_i \mathbf{T}^{(i)} \mathbf{P}_i^\top, \quad \mathbf{y}[\alpha_i] = \mathbf{P}_i \mathbf{t}^{(i)}, \\ & \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix} \in \mathbb{S}_+^{d_i+1}, \\ & \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix} \in \mathcal{A}_i, \quad i = 1, 2, \dots, l, \end{aligned}$$

where the variables are $\mathbf{y} \in \mathbb{R}^k$, $\mathbf{Y} \in \mathbb{R}^{k \times k}$, $\mathbf{t}^{(i)} \in \mathbb{R}^{d_i}$, and $\mathbf{T}^{(i)} \in \mathbb{R}^{d_i \times d_i}$ with $i = 1, 2, \dots, l$.

The following lemma will be useful in our proof.

LEMMA 4.3 ([8, Theorem 7.3.11]). Let $\mathbf{A} \in \mathbb{R}^{p \times n}$ and $\mathbf{B} \in \mathbb{R}^{q \times n}$ where $p \leq q$. Then $\mathbf{A}^\top \mathbf{A} = \mathbf{B}^\top \mathbf{B}$ if and only if there exists a matrix $\mathbf{U} \in \mathbb{R}^{q \times p}$ with $\mathbf{U}^\top \mathbf{U} = \mathbf{I}_p$ such that $\mathbf{B} = \mathbf{U}\mathbf{A}$.

Proof of Theorem 4.2. Without loss of generality, we assume the transformation matrix $\mathbf{P} \in \mathbb{R}^{k \times d}$ is in the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & & & \\ & \mathbf{P}_2 & & \\ & & \ddots & \\ & & & \mathbf{P}_l \end{bmatrix},$$

where $\mathbf{P}_i \in \mathbb{R}^{k_i \times d_i}$, $\sum_{i=1}^l k_i = k$, and $\sum_{i=1}^l d_i = d$.

For one direction, suppose that $(\mathbf{y}, \mathbf{Y}, \mathbf{t}, \mathbf{T})$ satisfies the constraints in (4.5). Then it is straightforward to see that (\mathbf{y}, \mathbf{Y}) also satisfies the constraints in (4.6) together with

$$\mathbf{t}^{(i)} = \mathbf{t}[\beta_i], \quad \mathbf{T}^{(i)} = \mathbf{T}[\beta_i], \quad i = 1, 2, \dots, l.$$

The other direction of the proof is more involved. Given (\mathbf{y}, \mathbf{Y}) and the variables $\{\mathbf{t}^{(i)}, \mathbf{T}^{(i)}\}_{i=1}^l$ in (4.6), our goal is to construct (\mathbf{t}, \mathbf{T}) satisfying the conditions in (4.5). To simplify the notations, we define

$$(4.7) \quad \tilde{\mathbf{Y}} := \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix}, \quad \tilde{\mathbf{T}}^{(i)} := \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{P}}_i := \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_i \end{bmatrix}.$$

Let $r = \max\{k, d\}$. Since $\tilde{\mathbf{Y}} \in \mathbb{S}_+^{k+1}$, it can be factorized as

$$(4.8) \quad \tilde{\mathbf{Y}} = \tilde{\mathbf{V}}^\top \tilde{\mathbf{V}},$$

where $\tilde{\mathbf{V}} \in \mathbb{R}^{(r+1) \times (k+1)}$. The above factorization can be done because $r \geq k$. Further, we partition $\tilde{\mathbf{V}}$ as

$$\tilde{\mathbf{V}} = [\mathbf{v} \quad \mathbf{V}_1 \quad \mathbf{V}_2 \quad \dots \quad \mathbf{V}_l],$$

where $\mathbf{v} \in \mathbb{R}^{r+1}$ and $\mathbf{V}_i \in \mathbb{R}^{(r+1) \times k_i}$ contains the columns of $\tilde{\mathbf{V}}$ indexed by α_i for $i = 1, 2, \dots, l$. Moreover, we have $\mathbf{v}^\top \mathbf{v} = \tilde{\mathbf{Y}}_{1,1} = 1$. Similarly, $\tilde{\mathbf{T}}^{(i)}$ can be factorized as

$$(4.9) \quad \tilde{\mathbf{T}}^{(i)} = (\tilde{\mathbf{Z}}^{(i)})^\top \tilde{\mathbf{Z}}^{(i)}, \quad i = 1, 2, \dots, l,$$

where $\tilde{\mathbf{Z}}^{(i)} \in \mathbb{R}^{(d_i+1) \times (d_i+1)}$ and is partitioned as

$$(4.10) \quad \tilde{\mathbf{Z}}^{(i)} = [\mathbf{z}^{(i)} \quad \mathbf{Z}^{(i)}].$$

Combining (4.9) with the equality constraints in (4.6), we get

$$\begin{bmatrix} 1 & \mathbf{y}[\alpha_i]^\top \\ \mathbf{y}[\alpha_i] & \mathbf{Y}[\alpha_i] \end{bmatrix} = \tilde{\mathbf{P}}_i \tilde{\mathbf{T}}^{(i)} \tilde{\mathbf{P}}_i^\top = (\tilde{\mathbf{Z}}^{(i)} \tilde{\mathbf{P}}_i^\top)^\top (\tilde{\mathbf{Z}}^{(i)} \tilde{\mathbf{P}}_i^\top), \quad i = 1, 2, \dots, l,$$

where $\tilde{\mathbf{Z}}^{(i)} \tilde{\mathbf{P}}_i^\top \in \mathbb{R}^{(d_i+1) \times (k_i+1)}$. On the other hand, the factorization in (4.8) implies

$$\begin{bmatrix} 1 & \mathbf{y}[\alpha_i]^\top \\ \mathbf{y}[\alpha_i] & \mathbf{Y}[\alpha_i] \end{bmatrix} = [\mathbf{v} \quad \mathbf{V}_i]^\top [\mathbf{v} \quad \mathbf{V}_i],$$

where $[\mathbf{v} \quad \mathbf{V}_i] \in \mathbb{R}^{(r+1) \times (k_i+1)}$. By Lemma 4.3, we can find $\mathbf{U}_i \in \mathbb{R}^{(r+1) \times (d_i+1)}$ with $\mathbf{U}_i^\top \mathbf{U}_i = \mathbf{I}_{d_i+1}$ such that

$$(4.11) \quad [\mathbf{v} \quad \mathbf{V}_i] = \mathbf{U}_i \tilde{\mathbf{Z}}^{(i)} \tilde{\mathbf{P}}_i^\top.$$

Substituting (4.7) and (4.10) into (4.11), we get

$$(4.12) \quad \mathbf{v} = \mathbf{U}_i \mathbf{z}^{(i)} \text{ and } \mathbf{V}_i = \mathbf{U}_i \mathbf{Z}^{(i)} \mathbf{P}_i^\top.$$

Finally, we define

$$\mathbf{R} = [\mathbf{v} \quad \mathbf{U}_1 \mathbf{Z}^{(1)} \quad \mathbf{U}_2 \mathbf{Z}^{(2)} \quad \dots \quad \mathbf{U}_l \mathbf{Z}^{(l)}] \in \mathbb{R}^{(r+1) \times (d+1)},$$

whose columns indexed by β_i are given by $\mathbf{U}_i \mathbf{Z}^{(i)}$, and construct (\mathbf{t}, \mathbf{T}) by

$$(4.13) \quad \begin{bmatrix} 1 & \mathbf{t}^\top \\ \mathbf{t} & \mathbf{T} \end{bmatrix} = \mathbf{R}^\top \mathbf{R}.$$

Next we verify that (\mathbf{t}, \mathbf{T}) in (4.13) indeed satisfies all the constraints in (4.5). The positive semidefiniteness is evident by our construction. For the equality constraints, by using (4.12) we have

$$\begin{aligned} \mathbf{R} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^\top \end{bmatrix} &= [\mathbf{v} \quad \mathbf{U}_1 \mathbf{Z}^{(1)} \mathbf{P}_1^\top \quad \mathbf{U}_2 \mathbf{Z}^{(2)} \mathbf{P}_2^\top \quad \dots \quad \mathbf{U}_l \mathbf{Z}^{(l)} \mathbf{P}_l^\top] \\ &= [\mathbf{v} \quad \mathbf{V}_1 \quad \mathbf{V}_2 \quad \dots \quad \mathbf{V}_l] \\ &= \tilde{\mathbf{V}}. \end{aligned}$$

Hence, we get

$$\begin{aligned} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{t}^\top \\ \mathbf{t} & \mathbf{T} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^\top \end{bmatrix} &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix} \mathbf{R}^\top \mathbf{R} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^\top \end{bmatrix} \\ &= \tilde{\mathbf{V}}^\top \tilde{\mathbf{V}} \\ &= \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix}, \end{aligned}$$

which is equivalent to $\mathbf{Y} = \mathbf{P} \mathbf{T} \mathbf{P}^\top$ and $\mathbf{y} = \mathbf{P} \mathbf{t}$. Lastly, note that

$$(4.14) \quad \begin{bmatrix} 1 & \mathbf{t}[\beta_i]^\top \\ \mathbf{t}[\beta_i] & \mathbf{T}[\beta_i] \end{bmatrix} = [\mathbf{v} \quad \mathbf{U}_i \mathbf{Z}^{(i)}]^\top [\mathbf{v} \quad \mathbf{U}_i \mathbf{Z}^{(i)}]$$

$$(4.15) \quad \begin{aligned} &= [\mathbf{U}_i \mathbf{z}^{(i)} \quad \mathbf{U}_i \mathbf{Z}^{(i)}]^\top [\mathbf{U}_i \mathbf{z}^{(i)} \quad \mathbf{U}_i \mathbf{Z}^{(i)}] \\ &= (\tilde{\mathbf{Z}}^{(i)})^\top \mathbf{U}_i^\top \mathbf{U}_i \tilde{\mathbf{Z}}^{(i)} \end{aligned}$$

$$(4.16) \quad \begin{aligned} &= (\tilde{\mathbf{Z}}^{(i)})^\top \tilde{\mathbf{Z}}^{(i)} \\ &= \tilde{\mathbf{T}}^{(i)} = \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix}, \end{aligned}$$

where we used (4.13) in (4.14), $\mathbf{v} = \mathbf{U}_i \mathbf{z}^{(i)}$ (cf. (4.12)) in (4.15), and $\mathbf{U}_i^\top \mathbf{U}_i = \mathbf{I}_{d_i+1}$ in (4.16). Hence, the remaining constraints in (4.5) are also satisfied because of the conditions on $\tilde{\mathbf{T}}^{(i)}$ in (4.6).

The proof of [Theorem 4.2](#) is complete. \square

Two remarks are in order. Firstly, the variables in (4.5) are in a high-dimensional PSD cone \mathbb{S}_+^{d+1} , while those in (4.6) are in the Cartesian product of smaller PSD cones $\mathbb{S}_+^{k+1} \times \mathbb{S}_+^{d_1+1} \times \mathbb{S}_+^{d_2+1} \times \dots \times \mathbb{S}_+^{d_l+1}$. When k, d_1, d_2, \dots, d_l are much smaller than d , using (4.6) instead of (4.5) can achieve dimension reduction without any additional cost. This can bring substantially higher computational efficiency for solving

the corresponding SDP in practice (see [section 5](#)). Secondly, [Theorem 4.2](#) is very general and thus could be applicable to a potentially wide range of problems. It is worth highlighting that we require no assumptions on the sets \mathcal{A}_i that constrain the submatrices, as well as the row and column partitions of the separable matrix \mathbf{P} . This enables us to accommodate both the equivalence results [\(4.1\)](#) and [\(4.2\)](#), as we will show next.

4.2.1. Equivalence between (ESDR2-T) and (ESDR-Y). As we noted before, the transformation matrix $\hat{\mathbf{S}}$ in [\(4.1\)](#) is separable with the row and column partitions given in [\(4.3\)](#) and we have

$$\hat{\mathbf{S}}[\alpha_i, \beta_i] = \begin{bmatrix} \mathbf{s}_R^\top \\ \mathbf{s}_I^\top \end{bmatrix} \in \mathbb{R}^{2 \times M}, \quad i = 1, 2, \dots, n.$$

Moreover, we can see that the feasible set of [\(ESDR2-T\)](#) is in the form of [\(4.5\)](#) with the set \mathcal{A}_i given by

$$\begin{aligned} \mathcal{A}_i &= \left\{ \begin{bmatrix} 1 & \mathbf{t}^\top \\ \mathbf{t} & \text{Diag}(\mathbf{t}) \end{bmatrix} : \mathbf{t} \in \mathbb{R}^M, \sum_{j=1}^M t_j = 1, t_j \geq 0, j = 1, 2, \dots, M \right\} \\ &= \left\{ \sum_{j=1}^M t_j \mathbf{E}_j : \sum_{j=1}^M t_j = 1, t_j \geq 0, j = 1, 2, \dots, M \right\} \\ &= \text{conv}\{\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_M\}, \end{aligned}$$

where

$$\mathbf{E}_j = \begin{bmatrix} 1 \\ \mathbf{e}_j \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{e}_j \end{bmatrix}^\top, \quad j = 1, 2, \dots, M,$$

and $\mathbf{e}_j \in \mathbb{R}^M$ is the j -th unit vector. Applying [Theorem 4.2](#) to [\(ESDR2-T\)](#) gives the following equivalent formulation:

$$(4.17) \quad \begin{aligned} & \begin{bmatrix} 1 & \mathbf{y}^\top \\ \mathbf{y} & \mathbf{Y} \end{bmatrix} \in \mathbb{S}_+^{2n+1} \\ \text{s.t.} & \begin{bmatrix} 1 & y_i & y_{n+i} \\ y_i & Y_{i,i} & Y_{i,n+i} \\ y_{n+i} & Y_{n+i,i} & Y_{n+i,n+i} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_R^\top \\ \mathbf{0} & \mathbf{s}_I^\top \end{bmatrix} \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_R & \mathbf{s}_I \end{bmatrix}, \\ & \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix} \in \mathbb{S}_+^{M+1}, \\ & \begin{bmatrix} 1 & (\mathbf{t}^{(i)})^\top \\ \mathbf{t}^{(i)} & \mathbf{T}^{(i)} \end{bmatrix} \in \text{conv}\{\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_M\}, \quad i = 1, 2, \dots, n. \end{aligned}$$

Since each matrix \mathbf{E}_j is PSD, their convex hull is a subset of \mathbb{S}_+^{M+1} and hence the PSD constraints in [\(4.17\)](#) are redundant. Furthermore, note that

$$\begin{aligned} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_R^\top \\ \mathbf{0} & \mathbf{s}_I^\top \end{bmatrix} \mathbf{E}_j \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_R & \mathbf{s}_I \end{bmatrix} &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_R^\top \\ \mathbf{0} & \mathbf{s}_I^\top \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{e}_j \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{e}_j \end{bmatrix}^\top \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_R & \mathbf{s}_I \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ \mathbf{s}_{R,j} \\ \mathbf{s}_{I,j} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{s}_{R,j} & \mathbf{s}_{I,j} \end{bmatrix}, \end{aligned}$$

which is exactly the matrix \mathbf{K}_j defined in (2.6). Therefore, we can conclude that (4.17) is the same as (ESDR-Y), and hence (ESDR2-T) and (ESDR-Y) are equivalent.

4.2.2. Equivalence between (VA-SDR) and (BC-SDR). Similarly, we observe that the transformation matrix \mathbf{W} in (4.2) is separable with row and column partitions given in (4.4), and let

$$(4.18) \quad \mathbf{w}^\top := \mathbf{W}[\alpha_i, \beta_i] = [1 \quad 2 \quad 4 \quad \dots \quad 2^{q-1}].$$

The feasible set in (VA-SDR) conforms to (4.5) with the set \mathcal{A}_i given by

$$\mathcal{A}_i = \left\{ \begin{bmatrix} 1 & \mathbf{b}^\top \\ \mathbf{b} & \mathbf{B} \end{bmatrix} : \mathbf{b} \in \mathbb{R}^q, \mathbf{B} \in \mathbb{R}^{q \times q}, \text{diag}(\mathbf{B}) = \mathbf{1}_q \right\}.$$

Hence, by applying Theorem 4.2 to (VA-SDR), we get the following equivalent formulation:

$$(4.19) \quad \begin{aligned} & \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \in \mathbb{S}_+^{n+1} \\ \text{s.t. } & X_{i,i} = \mathbf{w}^\top \mathbf{B}^{(i)} \mathbf{w}, \quad x_i = \mathbf{w}^\top \mathbf{b}^{(i)}, \\ & \begin{bmatrix} 1 & (\mathbf{b}^{(i)})^\top \\ \mathbf{b}^{(i)} & \mathbf{B}^{(i)} \end{bmatrix} \in \mathbb{S}_+^{q+1}, \\ & \text{diag}(\mathbf{B}^{(i)}) = \mathbf{1}_q, \quad i = 1, 2, \dots, n. \end{aligned}$$

Next we argue that all the constraints $x_i = \mathbf{w}^\top \mathbf{b}^{(i)}$ are redundant, i.e., the set in (4.19) is equivalent to

$$(4.20) \quad \left\{ \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \in \mathbb{S}_+^{n+1} : X_{i,i} = \mathbf{w}^\top \mathbf{B}^{(i)} \mathbf{w}, \mathbf{B}^{(i)} \in \mathbb{S}_+^q, \text{diag}(\mathbf{B}^{(i)}) = \mathbf{1}_q, i = 1, 2, \dots, n \right\}.$$

To show this, we need to prove that, for any $\mathbf{x}, \mathbf{X}, \mathbf{B}^{(1)}, \dots, \mathbf{B}^{(n)}$ satisfying the constraints in (4.20), there must exist $\mathbf{b}^{(i)} \in \mathbb{R}^q$ such that

$$(4.21) \quad x_i = \mathbf{w}^\top \mathbf{b}^{(i)} \text{ and } \begin{bmatrix} 1 & (\mathbf{b}^{(i)})^\top \\ \mathbf{b}^{(i)} & \mathbf{B}^{(i)} \end{bmatrix} \succeq 0, \quad i = 1, 2, \dots, n.$$

Fix $i \in \{1, 2, \dots, n\}$. Note that the PSD constraints in (4.20) implies

$$(4.22) \quad \begin{bmatrix} 1 & x_i \\ x_i & X_{i,i} \end{bmatrix} \succeq 0 \Leftrightarrow x_i^2 \leq X_{i,i}.$$

When $X_{i,i} = 0$, we must have $x_i = 0$ and we can achieve (4.21) by simply letting $\mathbf{b}^{(i)} = \mathbf{0}$. Otherwise, we have $X_{i,i} > 0$ and hence we can let

$$(4.23) \quad \mathbf{b}^{(i)} = \frac{x_i}{X_{i,i}} \mathbf{B}^{(i)} \mathbf{w}.$$

Since $X_{i,i} = \mathbf{w}^\top \mathbf{B}^{(i)} \mathbf{w}$, we can see that $\mathbf{w}^\top \mathbf{b}^{(i)} = (\mathbf{w}^\top \mathbf{B}^{(i)} \mathbf{w}) x_i / X_{i,i} = x_i$.

To verify the PSD constraint in (4.21), it suffices to show that $\mathbf{B}^{(i)} \succeq \mathbf{b}^{(i)} (\mathbf{b}^{(i)})^\top$. Note that

$$\begin{bmatrix} X_{i,i} & \mathbf{w}^\top \mathbf{B}^{(i)} \\ \mathbf{B}^{(i)} \mathbf{w} & \mathbf{B}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{w}^\top \mathbf{B}^{(i)} \mathbf{w} & \mathbf{w}^\top \mathbf{B}^{(i)} \\ \mathbf{B}^{(i)} \mathbf{w} & \mathbf{B}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{w}^\top \\ \mathbf{I}_q \end{bmatrix} \mathbf{B}^{(i)} \begin{bmatrix} \mathbf{w} & \mathbf{I}_q \end{bmatrix} \succeq 0,$$

which implies the Schur complement is also PSD, i.e.,

$$\mathbf{B}^{(i)} - \frac{1}{X_{i,i}} \mathbf{B}^{(i)} \mathbf{w} \mathbf{w}^\top \mathbf{B}^{(i)} \succeq 0.$$

This, together with (4.22) and (4.23), shows

$$\mathbf{b}^{(i)} (\mathbf{b}^{(i)})^\top = \frac{x_i^2}{X_{i,i}^2} \mathbf{B}^{(i)} \mathbf{w} \mathbf{w}^\top \mathbf{B}^{(i)} \preceq \frac{X_{i,i}}{X_{i,i}^2} \mathbf{B}^{(i)} \mathbf{w} \mathbf{w}^\top \mathbf{B}^{(i)} \preceq \mathbf{B}^{(i)}.$$

Hence both conditions in (4.21) are satisfied.

Finally, to show that (4.20) is the same as (BC-SDR), we need the following lemma.

LEMMA 4.4. *Let $\mathbf{w} \in \mathbb{R}^q$ be the vector defined in (4.18). It holds that*

$$(4.24) \quad \{x : \exists \mathbf{B} \in \mathbb{S}_+^q \text{ s.t. } x = \mathbf{w}^\top \mathbf{B} \mathbf{w}, \text{diag}(\mathbf{B}) = \mathbf{1}_q\} = \{1 \leq x \leq (2^q - 1)^2\}.$$

Proof. See Appendix B. □

Putting all pieces together, we have proved that (VA-SDR) is equivalent to (BC-SDR) by showing the correspondence (4.2).

5. Numerical results. In this section, we present some numerical results. Following standard assumptions in the wireless communication literature (see, e.g., [31, Chapter 7]), we assume that all entries of the channel matrix \mathbf{H} are independent and identically distributed (i.i.d.) following a complex circular Gaussian distribution with zero mean and unit variance, and all entries of the additive noise \mathbf{v} are i.i.d. following a complex circular Gaussian distribution with zero mean and variance σ^2 . Further, we choose the transmitted symbols $x_1^*, x_2^*, \dots, x_n^*$ from the symbol set \mathcal{S}_M in (1.3) independently and uniformly. We define the SNR as the received SNR per symbol:

$$\text{SNR} := \frac{\mathbb{E}[\|\mathbf{H}\mathbf{x}^*\|_2^2]}{n \mathbb{E}[\|\mathbf{v}\|_2^2]} = \frac{mn}{n \cdot m\sigma^2} = \frac{1}{\sigma^2}.$$

We first consider a MIMO system where $(m, n) = (16, 10)$ and $M = 8$. To evaluate the empirical probabilities of SDRs not being tight, we compute the optimal solutions of (ESDR- \mathbf{X}), (ESDR- \mathbf{Y}), and (ESDR1- \mathbf{T}) by the general-purpose SDP solver SeDuMi [29] with the desired accuracy set to 10^{-6} . The SDR is decided to be tight if the output $\hat{\mathbf{x}}$ returned by the SDP solver⁴ satisfies $\|\hat{\mathbf{x}} - \mathbf{x}^*\|_\infty \leq 10^{-4}$. We also evaluate the empirical probabilities of conditions (1.4)–(1.6) not being satisfied. We run the simulations at 8 SNR values in total ranging from 3 dB to 24 dB. For each SNR value, 10,000 random instances are generated and the averaged results are plotted in Figure 2.

We can see from Figure 2 that our results (1.5) and (1.6) provide better characterizations than the previous tightness condition (1.4) in [16]. The empirical probability of (ESDR- \mathbf{X}) not being tight matches perfectly with our analysis given by the necessary and sufficient condition (1.5). The probability of (1.6) not being satisfied is also a good approximation to the probability of (ESDR- \mathbf{Y}) not being tight, underestimating the latter roughly by a factor of 9. Moreover, the numerical results also validate

⁴The output $\hat{\mathbf{x}}$ is directly given by the optimal solution in (ESDR- \mathbf{X}), while it is obtained from the relation (2.5) between \mathbf{x} and \mathbf{y} in (ESDR- \mathbf{Y}) and the relation (2.7) between \mathbf{x} and \mathbf{t} in (ESDR1- \mathbf{T}).

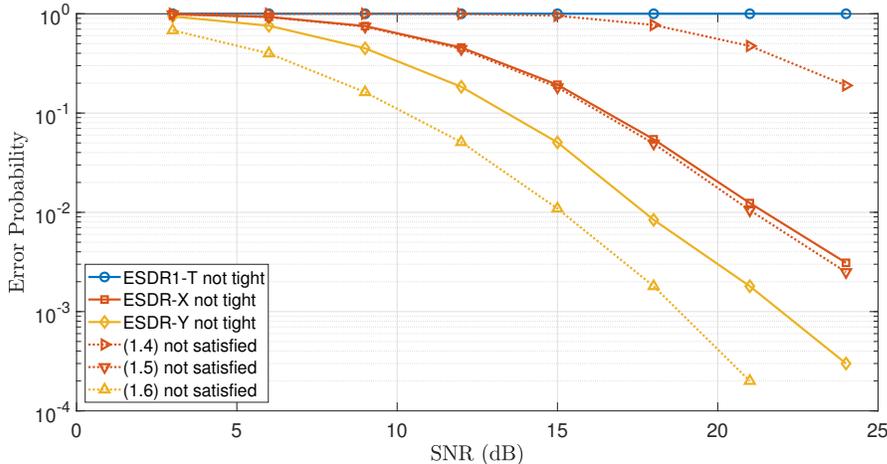


FIG. 2. Error probabilities versus the SNR in a 16×10 MIMO system with 8-PSK.

our analysis that **(ESDR1-T)** is not tight with high probability. In fact, **(ESDR1-T)** fails to recover the vector of transmitted symbols in all 80,000 instances.

Next, we compare the optimal values as well as the CPU time of solving **(ESDR-Y)** and **(ESDR2-T)**. Table 2 shows the relative difference between the optimal values of **(ESDR-Y)** (denoted as $\text{opt}_{\text{ESDR-Y}}$) and **(ESDR2-T)** (denoted as $\text{opt}_{\text{ESDR2-T}}$) averaged over 300 simulations, which is defined as $|\text{opt}_{\text{ESDR-Y}} - \text{opt}_{\text{ESDR2-T}}| / |\text{opt}_{\text{ESDR2-T}}|$. We can see from Table 2 that the difference is consistently in the order $1e-7$ in various settings, which verifies the equivalence between **(ESDR-Y)** and **(ESDR2-T)**. In Figure 3, we plot the average CPU time consumed by solving **(ESDR-Y)** and **(ESDR2-T)** in an 8-PSK system with increasing problem size n . For fair comparison, both SDRs are implemented and solved by SeDuMi and we repeat the simulations for 300 times. With the same error performance, we can see that **(ESDR-Y)** indeed solves the MIMO detection problem (1.2) more efficiently and saves roughly 90% of the computational time in our experiment.

TABLE 2
Average relative difference between **(ESDR-Y)** and **(ESDR2-T)** in optimal objective values.

SNR	Relative diff. in optimal objective values.			
	$(m, n) = (4, 4)$	$(m, n) = (6, 4)$	$(m, n) = (10, 10)$	$(m, n) = (15, 10)$
5dB	$4.62e-7$	$5.10e-7$	$6.26e-7$	$7.73e-7$
10dB	$5.67e-7$	$4.06e-7$	$6.50e-7$	$7.16e-7$
15dB	$5.94e-7$	$3.83e-7$	$7.80e-7$	$5.58e-7$

6. Conclusions. In this paper, we studied the tightness and equivalence of various existing SDR models for the MIMO detection problem (1.2). For the two SDRs **(ESDR-X)** and **(ESDR-Y)** proposed in [17], we improved their sufficient tightness condition and showed that the former is tight if and only if (1.5) holds while the latter is tight only if (1.6) holds. On the other hand, for the SDR **(ESDR1-T)** proposed in [22], we proved that its tightness probability decays to zero exponentially fast with an increasing problem size under some mild assumptions. Together with known results, our analysis provides a more complete understanding of the tightness conditions for existing SDRs. Moreover, we proposed a general theorem that unifies

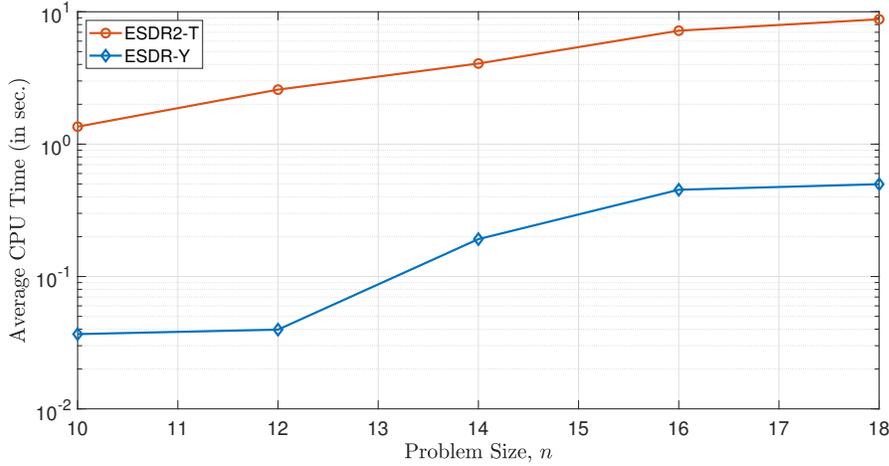


FIG. 3. Average CPU time of solving (ESDR-Y) and (ESDR2-T) when $M = 8$.

previous results on the equivalence of SDRs [21, 16]. For a subset of PSD matrices with a special “separable” structure, we showed its equivalence to another subset of PSD matrices in a potentially much smaller dimension. Our numerical results demonstrated that we could significantly improve the computational efficiency by using such equivalence.

Due to its generality, we believe that our equivalence theorem can be applied to SDPs in other domains beyond MIMO detection and we would like to put this as a future work. Additionally, we noticed that the SDRs for problem (1.2) combined with some simple rounding procedure can detect the transmitted symbols successfully even when the optimal solution has rank more than one. Similar observations have also been made in [12]. It would be interesting to extend our analysis to take the postprocessing procedure into account.

Appendix A. Simplification of (3.5) and (3.6). Fix $i \in \{1, 2, \dots, n\}$. From (3.5), we have

$$(\hat{\mathbf{H}}^T \hat{\mathbf{v}})_i = g_i + \lambda_i y_i^* + \mu_i y_{n+i}^* \text{ and } (\hat{\mathbf{H}}^T \hat{\mathbf{v}})_{n+i} = g_{n+i} + \mu_i y_i^* + \lambda_{n+i} y_{n+i}^*,$$

which can be written in a matrix form:

$$(A.1) \quad \begin{bmatrix} (\hat{\mathbf{H}}^T \hat{\mathbf{v}})_i \\ (\hat{\mathbf{H}}^T \hat{\mathbf{v}})_{n+i} \end{bmatrix} = \begin{bmatrix} g_i \\ g_{n+i} \end{bmatrix} + \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} y_i^* \\ y_{n+i}^* \end{bmatrix}.$$

Recall the definitions of $\mathbf{\Gamma}_i$ in (3.7) and \mathbf{K}_j in (2.6). Then

$$(A.2) \quad \langle \mathbf{\Gamma}_i, \mathbf{K}_j \rangle = 2 \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^T \begin{bmatrix} g_i \\ g_{n+i} \end{bmatrix} + \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^T \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}.$$

Using (A.1), we have

$$(A.3) \quad \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^T \begin{bmatrix} g_i \\ g_{n+i} \end{bmatrix} = \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^T \begin{bmatrix} (\hat{\mathbf{H}}^T \hat{\mathbf{v}})_i \\ (\hat{\mathbf{H}}^T \hat{\mathbf{v}})_{n+i} \end{bmatrix} - \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^T \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} y_i^* \\ y_{n+i}^* \end{bmatrix} \\ = \begin{bmatrix} \cos(\Delta\theta_j) \\ \sin(\Delta\theta_j) \end{bmatrix}^T \begin{bmatrix} \hat{z}_i \\ \hat{z}_{n+i} \end{bmatrix} - \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^T \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} y_i^* \\ y_{n+i}^* \end{bmatrix},$$

where $\Delta\theta_j = \theta_j - \theta_{u_i}$, $\hat{z}_i = \text{Re}(z_i)$, and $\hat{z}_{n+i} = \text{Im}(z_i)$ (cf. (3.1)). Combining (A.2) with (A.3), we get

$$\begin{aligned} \langle \mathbf{\Gamma}_i, \mathbf{K}_j \rangle &= 2 \begin{bmatrix} \cos(\Delta\theta_j) \\ \sin(\Delta\theta_j) \end{bmatrix}^\top \begin{bmatrix} \hat{z}_i \\ \hat{z}_{n+i} \end{bmatrix} - 2 \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^\top \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} y_i^* \\ y_{n+i}^* \end{bmatrix} + \\ &\quad \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}^\top \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}. \end{aligned}$$

In particular, when $j = u_i$, the above becomes

$$\langle \mathbf{\Gamma}_i, \mathbf{K}_{u_i} \rangle = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top \begin{bmatrix} \hat{z}_i \\ \hat{z}_{n+i} \end{bmatrix} - \begin{bmatrix} y_i^* \\ y_{n+i}^* \end{bmatrix}^\top \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} y_i^* \\ y_{n+i}^* \end{bmatrix}.$$

Hence, when $j \neq u_i$, (3.6) is equivalent to

$$\begin{aligned} 2 \begin{bmatrix} 1 - \cos(\Delta\theta_j) \\ -\sin(\Delta\theta_j) \end{bmatrix}^\top \begin{bmatrix} \hat{z}_i \\ \hat{z}_{n+i} \end{bmatrix} &\geq \begin{bmatrix} y_i^* - \cos(\theta_j) \\ y_{n+i}^* - \sin(\theta_j) \end{bmatrix}^\top \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} y_i^* - \cos(\theta_j) \\ y_{n+i}^* - \sin(\theta_j) \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} 1 \\ -\cot\left(\frac{\Delta\theta_j}{2}\right) \end{bmatrix}^\top \begin{bmatrix} \hat{z}_i \\ \hat{z}_{n+i} \end{bmatrix} &\geq \begin{bmatrix} \sin(\theta_{u_i} + \frac{\Delta\theta_j}{2}) \\ -\cos(\theta_{u_i} + \frac{\Delta\theta_j}{2}) \end{bmatrix}^\top \begin{bmatrix} \lambda_i & \mu_i \\ \mu_i & \lambda_{n+i} \end{bmatrix} \begin{bmatrix} \sin(\theta_{u_i} + \frac{\Delta\theta_j}{2}) \\ -\cos(\theta_{u_i} + \frac{\Delta\theta_j}{2}) \end{bmatrix}, \end{aligned}$$

which is exactly the same as (3.8).

Appendix B. Proof of Lemma 4.4. To simplify the notations, we use \mathcal{A} and \mathcal{B} to denote the left-hand side and the right-hand side in (4.24), respectively.

We first prove that $\mathcal{A} \supset \mathcal{B}$. Let $\mathcal{C} = \{\mathbf{B} \in \mathbb{S}_+^q : \text{diag}(\mathbf{B}) = \mathbf{1}_q\}$, and we can view \mathcal{A} as the image of the convex set \mathcal{C} under the affine mapping $\mathbf{B} \mapsto \langle \mathbf{B}, \mathbf{w}\mathbf{w}^\top \rangle$. Therefore, the set \mathcal{A} is also convex. Moreover, note that both the rank-one matrices $\mathbf{1}_q \mathbf{1}_q^\top$ and $\begin{bmatrix} -\mathbf{1}_{q-1} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{1}_{q-1} \\ 1 \end{bmatrix}^\top$ belong to \mathcal{C} . Direct computations show that

$$\begin{aligned} \mathbf{w}^\top \mathbf{1}_q \mathbf{1}_q^\top \mathbf{w} &= \left(\sum_{i=1}^q 2^{i-1} \right)^2 = (2^q - 1)^2, \\ \mathbf{w}^\top \begin{bmatrix} -\mathbf{1}_{q-1} \\ 1 \end{bmatrix} \begin{bmatrix} -\mathbf{1}_{q-1} \\ 1 \end{bmatrix}^\top \mathbf{w} &= \left(2^{q-1} - \sum_{i=1}^{q-1} 2^{i-1} \right)^2 = 1, \end{aligned}$$

and hence both 1 and $(2^q - 1)^2$ belong to \mathcal{A} . Finally, the convexity of \mathcal{A} implies $\mathcal{B} \subset \mathcal{A}$.

Now we prove the other direction, i.e., $\mathcal{A} \subset \mathcal{B}$. This is equivalent to showing

$$1 \leq \mathbf{w}^\top \mathbf{B} \mathbf{w} \leq (2^q - 1)^2, \quad \forall \mathbf{B} \in \mathcal{C}.$$

For the upper bound, we first note that $\mathbf{B} \in \mathbb{S}_+^q$ implies

$$(B.1) \quad |B_{i,j}| \leq \sqrt{B_{i,i} B_{j,j}} = 1, \quad i \neq j, \quad 1 \leq i, j \leq q.$$

Since every entry of the matrix $\mathbf{w}\mathbf{w}^\top$ is positive, we have

$$\mathbf{w}^\top \mathbf{B} \mathbf{w} = \langle \mathbf{B}, \mathbf{w}\mathbf{w}^\top \rangle \leq \langle \mathbf{1}\mathbf{1}^\top, \mathbf{w}\mathbf{w}^\top \rangle = (2^q - 1)^2$$

for any $\mathbf{B} \in \mathcal{C}$, and hence the upper bound holds.

For the lower bound, it clearly holds when $q = 1$. When $q > 1$, for any matrix $\mathbf{B} \in \mathcal{C}$ we partition it as

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}' & \mathbf{b}' \\ (\mathbf{b}')^\top & 1 \end{bmatrix},$$

where $\mathbf{B}' \in \mathbb{R}^{(q-1) \times (q-1)}$ and $\mathbf{b}' \in \mathbb{R}^{q-1}$. Note that we have $|b'_i| \leq 1$ for $1 \leq i \leq q-1$ (cf. (B.1)), and $\mathbf{B} \in \mathbb{S}_+^q$ implies $\mathbf{B}' \succeq \mathbf{b}'(\mathbf{b}')^\top$. Further, we let $\mathbf{w}' = [1 \ 2 \ 4 \ \dots \ 2^{q-2}]^\top \in \mathbb{R}^{q-1}$ such that $\mathbf{w} = [(\mathbf{w}')^\top \ 2^{q-1}]^\top$ (cf. (4.18)). We have

$$\begin{aligned} \mathbf{w}^\top \mathbf{B} \mathbf{w} &= (\mathbf{w}')^\top \mathbf{B} \mathbf{w}' + 2^q (\mathbf{w}')^\top \mathbf{b}' + (2^{q-1})^2 \\ &\geq (\mathbf{w}')^\top \mathbf{b}' (\mathbf{b}')^\top \mathbf{w}' + 2^q (\mathbf{w}')^\top \mathbf{b}' + (2^{q-1})^2 \\ &= ((\mathbf{w}')^\top \mathbf{b}' + 2^{q-1})^2. \end{aligned}$$

Since

$$(\mathbf{w}')^\top \mathbf{b}' = \sum_{i=1}^{q-1} 2^{i-1} b'_i \geq -\sum_{i=1}^{q-1} 2^{i-1} = -2^{q-1} + 1,$$

we immediately get $\mathbf{w}^\top \mathbf{B} \mathbf{w} \geq (-2^{q-1} + 1 + 2^{q-1})^2 = 1$, and hence the lower bound also holds.

The proof is now complete.

Acknowledgments. The authors would like to thank Professors Zi Xu and Cheng Lu for their useful discussions on an earlier version of this paper.

REFERENCES

- [1] A. S. BANDEIRA, N. BOUMAL, AND A. SINGER, *Tightness of the maximum likelihood semidefinite relaxation for angular synchronization*, Math. Program., 163 (2017), pp. 145–167.
- [2] P. BISWAS, T.-C. LIAN, T.-C. WANG, AND Y. YE, *Semidefinite programming based algorithms for sensor network localization*, ACM Trans. Sen. Netw., 2 (2006), pp. 188–220.
- [3] P. BISWAS AND Y. YE, *Semidefinite programming for ad hoc wireless sensor network localization*, in Proceedings of the 3rd International Symposium on Information Processing in Sensor Networks (IPSN'04), New York, NY, 2004, ACM, pp. 46–54.
- [4] O. DAMEN, A. CHKEIF, AND J.-C. BELFIORE, *Lattice code decoder for space-time codes*, IEEE Commun. Lett., 4 (2000), pp. 161–163.
- [5] A. DUEL-HALLEN, *Decorrelating decision-feedback multiuser detector for synchronous code-division multiple-access channel*, IEEE Trans. Commun., 41 (1993), pp. 285–290.
- [6] G. J. FOSCHINI, *Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas*, Bell Labs Tech. J., 1 (1996), pp. 41–59.
- [7] M. X. GOEMANS AND D. P. WILLIAMSON, *Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming*, J. ACM, 42 (1995), pp. 1115–1145.
- [8] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, New York, USA, 2nd ed., 2013.
- [9] J. JALDÉN, *Detection for Multiple Input Multiple Output Channels*, PhD thesis, School of Electrical Engineering, KTH, Stockholm, Sweden, 2006.
- [10] J. JALDÉN, C. MARTIN, AND B. OTTERSTEN, *Semidefinite programming for detection in linear systems - Optimality conditions and space-time decoding*, in Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP'03), Piscataway, NJ, 2003, IEEE Press, pp. 9–12.
- [11] J. JALDÉN AND B. OTTERSTEN, *On the complexity of sphere decoding in digital communications*, IEEE Trans. Signal Process., 53 (2005), pp. 1474–1484.
- [12] J. JALDÉN AND B. OTTERSTEN, *The diversity order of the semidefinite relaxation detector*, IEEE Trans. Inf. Theory, 54 (2008), pp. 1406–1422.

- [13] R. JIANG, Y.-F. LIU, C. BAO, AND B. JIANG, *A companion technical report of "tightness and equivalence of semidefinite relaxations for MIMO detection"*, tech. report, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, 2020, http://lsec.cc.ac.cn/~yafliu/Technical_Report_MIMO.pdf.
- [14] M. KISIALIOU AND Z.-Q. LUO, *Performance analysis of quasi-maximum-likelihood detector based on semi-definite programming*, in Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'05), Piscataway, NJ, 2005, IEEE Press, pp. 433–436.
- [15] Y.-F. LIU, M. HONG, AND Y.-H. DAI, *Max-Min fairness linear transceiver design problem for a multi-user SIMO interference channel is polynomial time solvable*, IEEE Signal Process. Lett., 20 (2013), pp. 27–30.
- [16] Y.-F. LIU, Z. XU, AND C. LU, *On the equivalence of semidefinite relaxations for MIMO detection with general constellations*, in Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'19), Piscataway, NJ, 2019, IEEE Press, pp. 4549–4553.
- [17] C. LU, Y.-F. LIU, W.-Q. ZHANG, AND S. ZHANG, *Tightness of a new and enhanced semidefinite relaxation for MIMO detection*, SIAM J. Optim., 29 (2019), pp. 719–742.
- [18] Z.-Q. LUO, W.-K. MA, A. M.-C. SO, Y. YE, AND S. ZHANG, *Semidefinite relaxation of quadratic optimization problems*, IEEE Signal Process. Mag., 27 (2010), pp. 20–34.
- [19] W.-K. MA, P.-C. CHING, AND Z. DING, *Semidefinite relaxation based multiuser detection for M-ary PSK multiuser systems*, IEEE Trans. Signal Process., 52 (2004), pp. 2862–2872.
- [20] W.-K. MA, T. N. DAVIDSON, K. M. WONG, Z.-Q. LUO, AND P.-C. CHING, *Quasi-maximum-likelihood multiuser detection using semi-definite relaxation with application to synchronous CDMA*, IEEE Trans. Signal Process., 50 (2002), pp. 912–922.
- [21] W.-K. MA, C.-C. SU, J. JALDÉN, T.-H. CHANG, AND C.-Y. CHI, *The equivalence of semidefinite relaxation MIMO detectors for higher-order QAM*, IEEE J. Sel. Top. Signal Process., 3 (2009), pp. 1038–1052.
- [22] A. MOBASHER, M. TAHERZADEH, R. SOTIROV, AND A. K. KHANDANI, *A near-maximum-likelihood decoding algorithm for MIMO systems based on semi-definite programming*, IEEE Trans. Inf. Theory, 53 (2007), pp. 3869–3886.
- [23] K. S. SCHNEIDER, *Optimum detection of code division multiplexed signals*, IEEE Trans. Aerosp. Electron. Syst., 15 (1979), pp. 181–185.
- [24] N. D. SIDIROPOULOS, T. N. DAVIDSON, AND Z.-Q. LUO, *Transmit beamforming for physical-layer multicasting*, IEEE Trans. Signal Process., 54 (2006), pp. 2239–2251.
- [25] A. SINGER, *Angular synchronization by eigenvectors and semidefinite programming*, Appl. Comput. Harmon. Anal., 30 (2011), pp. 20–36.
- [26] A. M.-C. SO, *Probabilistic analysis of the semidefinite relaxation detector in digital communications*, in Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'10), Philadelphia, PA, 2011, SIAM, pp. 698–711.
- [27] A. M.-C. SO AND Y. YE, *Theory of semidefinite programming for sensor network localization*, Math. Program., 109 (2007), pp. 367–384.
- [28] B. STEINGRIMSSON, Z.-Q. LUO, AND K. M. WONG, *Soft quasi-maximum-likelihood detection for multiple-antenna wireless channels*, IEEE Trans. Signal Process., 51 (2003), pp. 2710–2719.
- [29] J. F. STURM, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones*, Optim. Methods Softw., 11 (1999), pp. 625–653.
- [30] P. H. TAN AND L. K. RASMUSSEN, *The application of semidefinite programming for detection in CDMA*, IEEE J. Sel. Areas Commun., 19 (2001), pp. 1442–1449.
- [31] D. TSE AND P. VISWANATH, *Fundamentals of Wireless Communication*, Cambridge University Press, New York, 2005.
- [32] S. VERDÚ, *Computational complexity of optimum multiuser detection*, Algorithmica, 4 (1989), pp. 303–312.
- [33] S. VERDÚ, *Multiuser Detection*, Cambridge University Press, New York, 1998.
- [34] I. WALDSPURGER, A. D'ASPREMONT, AND S. MALLAT, *Phase recovery, MaxCut and complex semidefinite programming*, Math. Program., 149 (2015), pp. 47–81.
- [35] Z. XIE, R. T. SHORT, AND C. K. RUSHFORTH, *A family of suboptimum detectors for coherent multiuser communications*, IEEE J. Sel. Areas Commun., 8 (1990), pp. 683–690.
- [36] S. YANG AND L. HANZO, *Fifty years of MIMO detection: The road to large-scale MIMOs*, IEEE Commun. Surveys Tuts., 17 (2015), pp. 1941–1988.
- [37] Y. ZHONG AND N. BOUMAL, *Near-optimal bounds for phase synchronization*, SIAM J. Optim., 28 (2018), pp. 989–1016.