Anomalous Symmetry Protected Topological States in Interacting Fermion Systems

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(Received 18 February 2019; revised manuscript received 14 July 2019; published 15 November 2019)

The classification and construction of symmetry protected topological (SPT) phases have been intensively studied in interacting systems recently. To our surprise, in interacting fermion systems, there exists a new class of the so-called anomalous SPT (ASPT) states which are only well defined on the boundary of a trivial fermionic bulk system. We first demonstrate the essential idea by considering an anomalous topological superconductor with time-reversal symmetry $T^2 = 1$ in 2D. The physical reason for this is that the fermion parity might be changed locally by certain symmetry action, but it is conserved if we introduce a bulk. Then we discuss the layer structure and systematical construction of ASPT states in interacting fermion systems in 2D with a total symmetry $G_f = G_b \times \mathbb{Z}_2^f$. Finally, potential experimental realizations of ASPT states are also addressed.

DOI: 10.1103/PhysRevLett.123.207003

Introduction.—The bulk-boundary correspondence is an essential concept in the study of topological phases. In recent years, the short-range-entangled symmetry protected topological (SPT) phases [1], e.g., topological insulators [2–6], topological superconductors [5–8], topological crystalline insulators [9], and bosonic SPT (BSPT) phases [10–12], have been studied intensively. A hallmark of these SPT states is the existence of gapless boundary states [13] that cannot be gapped out without breaking the relevant symmetries (spontaneously or explicitly). The nonexistence of a symmetric gapped boundary (without topological orders) can be regarded as a consequence of a boundary anomaly: the symmetry action on the boundary is anomalous and cannot be realized locally (on site) by any lattice model in the same dimension. Such an anomaly is in a one-to-one correspondence with the classification of bulk SPT states [14–23]. For example, in bosonic SPT states, both the boundary anomalies and bulk SPT states are classified by (generalized) group-cohomology theory [10,11,24,25].

Very recently, the concept of equivalent class of finite depth fermionic symmetric local unitary (FSLU) transformation allows us to classify and construct very general fermionic SPT (FSPT) states. In particular, it has been shown that the FSPT states have a layered structure [26–33]: they can be constructed by decorating (subject to certain obstructions) 2D $(p + ip)$ topological superconductors to 2D symmetry domain walls, 1D Majorana chains to 1D symmetry domain walls or intersection lines of domain walls, and complex-fermion modes to 0D symmetry domain walls or intersection points of domain walls, in addition to the bosonic SPT layer.

These layers not only present a way to organize the mathematical structure describing FSPT classifications but also distinguish physically different types of FSPT states. A signature phenomenon in this layered structure is the existence of the so-called anomalous SPT (ASPT) states that can only live on the boundary of a trivial bulk FSPT state. Anomalous surface states have been widely studied in the correspondence between 3D bulk SPT states and 2D long-range-entangled surface symmetry-enriched topological (SET) states with anomalous symmetry fractionalization [34–36]. However, here both the bulk and the boundary are short-range-entangled states.

The existence of ASPT is a direct consequence of the layered structure of FSPT. In fact, if we simply treat the bulk FSPT classification as one additive group, the bulk should be regarded as a trivial state because its boundary can be realized as a symmetric gapped state (without topological order). Correspondingly, naively it seems that the boundary state is not anomalous as well. Nevertheless, the combination becomes nontrivial once we take into account the layered structure in FSPT classification. The anomalous boundary FSPT states are always built on a lower layer than its bulk. For example, the ASPT states studied below are built by decorating 1D Majorana chains [37] to symmetry domain walls, where its 3D bulk does not contain any Majorana-chain decoration.

In this Letter, we mainly consider ASPT, which is related to fermion-parity symmetry violation of the FSLU transformation on the boundary. In the following, we will show how to construct this class of ASPT states systematically in 2D interacting fermion systems with a total symmetry $G_f = Z_2^T \times \mathbb{Z}_2^f$.

A simple example of 2D $T^2 = 1$ ASPT state.—It is well known that there is a nontrivial 2D topological superconductor of class DIII with symmetry $T^2 = -1$ ($G_f = \mathbb{Z}_4^{Tf}$). In strongly interacting systems, this state can also be constructed in the Majorana-chain decoration picture as the ground state of a commuting projector Hamiltonian [38]. However, if one wants to construct a similar state for $T^2 = 1$ ($G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$), there are some inconsistencies between the Kasteleyn orientation [39] (fermion parity) and the symmetry action [38]. Nevertheless, we will show that the $T^2 = 1$ case with the Majorana-chain decoration, although not well defined in pure 2D, can actually be constructed on the boundary of a 3D bulk as an ASPT state. The essential difference is that, although the fermion parity of the 2D symmetric state is not conserved under the FSLU transformation, the total fermion parity is conserved if we introduce additional degrees of freedom in the 3D bulk. As there is a gapped, symmetric boundary state *without* topological order, we conclude that the bulk 3D $T^2 = 1$ "FSPT" state constructed using the special group supercohomology [26] will be trivialized. Thus there is no nontrivial FSPT for this symmetry class.

Below, we will discuss the scheme of constructing a fixed-point 2D ASPT state with a total symmetry $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$ on arbitrary triangulation, and we will show how to introduce the 3D bulk fermion degrees of freedom to cancel the anomaly [40]. We first try to construct a symmetric fixed-point state in pure 2D. Let us consider the Majorana-chain decoration following the procedure of Ref. [31]. In addition to the Ising spin $|\sigma_i\rangle$ ($\sigma_i = \pm 1$ or $\uparrow/\downarrow$) on each vertex i of a given triangulation $\mathcal{T}$, each link $\langle ij\rangle$ has two Majorana fermions (γ_{ijA} and γ_{ijB}) on its two sides, an arrangement that is equivalent to spinless complex fermion a_{ij}, where we can split the complex fermion as $a_{ij} = (\gamma_{ijA} + i\gamma_{ijB})/2$. (See the red dots in Fig. 1 for these degrees of freedom.) We further require a_{ij} to be invariant under the time-reversal symmetry (we note that $i \to -i$ under the T action), so the Majorana fermions transform as

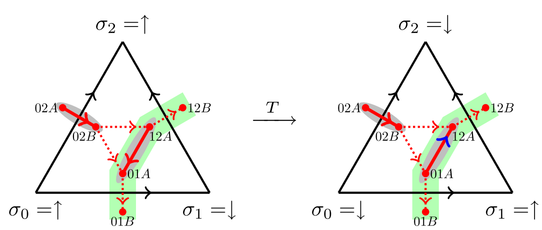


FIG. 1. Majorana-chain decorations. The Ising spins $\sigma_i = \pm 1$ or $\uparrow/\downarrow$ are on the vertices of the (black) triangulation lattice. Majorana fermions (red dots) are on the vertices of (red) lattice $\tilde{\mathcal{P}}$. They are paired up (gray ellipse) nontrivially along the domain wall (green belt). Time-reversal symmetry would flip the Ising spin and change the pairing directions (blue arrows) of the AA or BB type Majorana fermions.

$$T\colon \begin{cases} \gamma_{ijA} \to \gamma_{ijA}, \\ \gamma_{ijB} \to -\gamma_{ijB}. \end{cases}$$

And the bosonic spins transform as Ising variables under $\mathbb{Z}_2^T$ action $T\colon \sigma_i \to -\sigma_i$.

Given a 2D spacial manifold with arbitrary triangulation $\mathcal{T}$ associated with a branching structure (a branching structure is an assignment of link arrows such that the three arrows never form a closed loop for an arbitrary triangle of the lattice [41]), one can construct the dual trivalent lattice denoted by $\mathcal{P}$. In order to decorate Majorana chains, we resolve each vertex of $\mathcal{P}$ by a small triangle. The new resolved lattice is called $\tilde{\mathcal{P}}$ (see the red color lattice in Fig. 1). We also add arrows to the links of $\tilde{\mathcal{P}}$ (see the red arrows in Fig. 1) such that there is always an odd number of clockwise arrows for each small loop around a vertex. The red arrows, which are called Kasteleyn orientations, are discussed in more detail in the Supplemental Material [42].

For convenience, we define the *domain wall function*,

$$n_1(\sigma_i\sigma_j) \coloneqq \frac{1}{2}(1 - \sigma_i\sigma_j) = \begin{cases} 0, & \text{if } \sigma_i = \sigma_j, \\ 1, & \text{if } \sigma_i \neq \sigma_j, \end{cases} \tag{1}$$

which indicates whether or not there is an Ising domain wall between vertices i and j. It is in fact the nontrivial 1-cocycle in $H^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$. In the construction of the Majorana-chain decoration [31,38,43,44], we put nontrivial Majorana chains along the domain walls of the Ising spins (see the green belt in Fig. 1). To be more specific, we put Majorana fermions on vertices of $\tilde{\mathcal{P}}$ (the red dots in Fig. 1) into three different types of pairings, according to the Ising spin configuration $\{\sigma_i\}$:

If $n_1(\sigma_i\sigma_j) = 0$, the two Majorana fermions on the two sides of link $\langle ij\rangle$ (see link $\langle 02\rangle$ in Fig. 1, for example) are in the trivial vacuum pairing $-i\gamma_{ijA}\gamma_{ijB} = 1$. This is equivalent to $a_{ij}^\dagger a_{ij} = 0$ in terms of complex fermions.

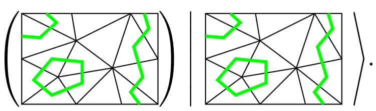
For triangle $\langle 012\rangle$ with $\sigma_0 = +1$ and a domain wall going through, the Majorana fermions along the domain wall are paired up nontrivially. For example, we have $-i\gamma_{12A}\gamma_{01A} = 1$ inside the triangle in the left panel of Fig. 1. The pairing direction is specified by the Kasteleyn orientation (red arrow) of the pairing link.

For triangle $\langle 012\rangle$ with $\sigma_0 = -1$, a time-reversal symmetry action on case (ii) above would give us the pairing $-i\gamma_{12A}\gamma_{01A} = -1$, which means that the Kasteleyn orientation is reversed (see the blue arrow in the right panel of Fig. 1). From the transformation rules of A/B type Majorana fermions and $i \to -i$, we conclude that the pairing direction is reversed if the Majorana pairing is of AA or BB type and remains the same if the pairing is of AB type.

We note that the first two pairing rules are the same as in Ref. [31]. And the third rule is designed to make the

Majorana-chain decoration time-reversal symmetric. Thus the 2D symmetric fixed-point state can be constructed as a superposition (subject to the proper algebraic conditions discussed below) of those basis states with all possible triangulations $\mathcal{T}$ and spin configurations $\{\sigma_i\}$:

$$|\Psi\rangle = \sum_{\text{all conf.}} \Psi$$



(2)

Here the spins are on the vertices of the triangulation lattice and green lines indicate the Majorana chains on the Ising domain walls using the rules above.

It is known that, for a lattice with Kasteleyn orientations, the decorated Majorana chains on the Ising domain wall (using the first two rules above) always have even fermion parity [31,43,44]. However, rule (iii) violates the Kasteleyn orientations of the lattice. As noted above, the orientation is changed if and only if $\sigma_0 = -1$ and the Majorana pairing is of AA or BB type. Therefore, the right-hand side of Fig. 1 with $\sigma_0 = \sigma_2 = -1$ and $\sigma_1 = +1$ [such that $n_1(\sigma_0\sigma_1) = n_1(\sigma_1\sigma_2) = 1$)] is the only Ising spin configuration in which the Majorana pairing direction (blue arrow) is reversed. Compared to the Kasteleyn orientated decorations, the fermion parity of triangle $\langle 012\rangle$ is changed by rule (iii) as

$$P_f^\gamma(\langle 012\rangle) = (-1)^{s_1(\sigma_0)n_1(\sigma_0\sigma_1)n_1(\sigma_1\sigma_2)}, \tag{3}$$

where we define another function $s_1(\sigma) := n_1(\sigma)$ related to antiunitary symmetry [45].

Now we can discuss the first type of algebraic condition arising from fermion-parity conservation for the fixed-point wave function (2). For the whole 2D system, the fermion-parity change (compared to the vacuum state without Majorana-chain decorations) is the product of Eq. (3) for all triangles. We can first consider the smallest 2D lattice with four triangles (triangulation of a two-sphere) on the boundary of a 3D solid tetrahedron (see Fig. 2). For the spin configuration $(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (+1, -1, +1, -1)$, there is a Majorana chain along the Ising domain wall (see the green line in Fig. 2). According to rule (iii) and Eq. (3), only the pairing direction inside triangle $\langle 123\rangle$ is reversed, resulting in a Majorana chain with odd fermion parity. Therefore, the desired wave function

$$|\Psi\rangle_{2D} = \sum_{\{\sigma_i\}} \psi(\{\sigma_i\})|\{\sigma_i\}\rangle \otimes |\gamma(n_1)\rangle_{2D} (\text{not well defined})$$

is not legitimate for a pure 2D system, as the basis states $|\{\sigma_i\}\rangle \otimes |\gamma(n_1)\rangle_{2D}$ have different fermion parities.

To evade this problem, we can add a 3D bulk and decorate a complex fermion $(c_{0123}^\dagger)^{n_3(\sigma_0\sigma_1,\sigma_1\sigma_2,\sigma_2\sigma_3)}$ at the center of the tetrahedron (the blue ball in Fig. 2). We can choose n_3 such that the resulting 3D wave function,

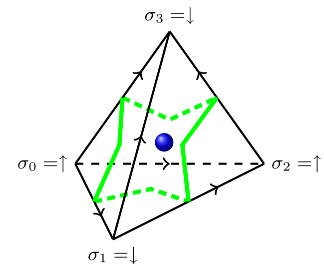


FIG. 2. 2D ASPT on the smallest lattice—boundary of a 3D solid tetrahedron. There is a complex-fermion mode $(c_{0123}^\dagger)^{n_3(\sigma_0\sigma_1,\sigma_1\sigma_2,\sigma_2\sigma_3)}$ (blue ball) at the center of the tetrahedron. One Majorana chain (green line) is decorated on the 2D surface. One can add more and more vertices in the bulk or on the boundary from this smallest lattice to obtain a larger fine lattice using Pachner (the fundamental retriangulation) moves.

$$|\Psi\rangle_{3D} = \sum_{\{\sigma_i\}} \psi(\{\sigma_i\})|\{\sigma_i\}\rangle \otimes |\gamma(n_1)\rangle_{2D} \otimes |c(n_3)\rangle_{3D},$$

is $\mathbb{Z}_2^T$ symmetric and has even total fermion parity. From the product of Eq. (3) for the four triangles, one can show that the total Majorana fermion parity for a given Ising spin configuration is

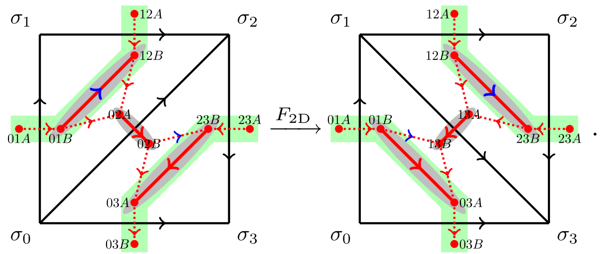
$$P_f^\gamma(\langle 0123\rangle) = (-1)^{s_1(\sigma_0\sigma_1)n_1(\sigma_1\sigma_2)n_1(\sigma_2\sigma_3)}. \tag{4}$$

Thus we require the complex-fermion number to be

$$n_3 = s_1 n_1 n_1 \tag{5}$$

such that the total fermion parity $P_f = P_f^\gamma P_f^c$ is fixed. This equation relates the complex-fermion decoration in the 3D bulk and the Majorana-chain decoration on the 2D boundary. One can further show that this n_3 is the nontrivial 3-cocycle in $H^3(\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2$. So the 3D bulk is in fact the special group-supercohomology state with symmetry $T^2 = 1$ [26].

Despite the fact that the above state is defined on one tetrahedron, we can add more and more vertices in the 3D bulk or on the 2D boundary by Pachner moves [46], and we finally obtain a larger fine lattice. The only things that we need to check are that each Pachner move is symmetric and that the fermion parity is even. There are two types of Pachner moves. The first type is the well-defined genuine 3D Pachner move (without touching the boundary) for the 3D bulk state in the special group-supercohomology theory [26]. The second type is the 2D boundary Pachner moves with the standard one:



(6)

The total Majorana fermion-parity change under the F_{2D} move is

$$\Delta P_f^{\gamma}(F_{2D}) = (-1)^{s_1(\sigma_0\sigma_1)n_1(\sigma_1\sigma_2)n_1(\sigma_2\sigma_3)}, \tag{7}$$

which is obtained as in Eq. (4). Suppose the four vertices on the boundary are connected to the bulk vertex labeled by σ_*; then the 3D bulk complex-fermion-parity change under this F_{2D} move is

$$\begin{aligned}\Delta P_f^{c}(F_{2D}) &= (-1)^{n_3(*012)+n_3(*023)+n_3(*013)+n_3(*123)} \\ &= (-1)^{n_3(0123)},\end{aligned} \tag{8}$$

where we have used $dn_3 = 0$ (mod 2), and abbreviated $n_3(\sigma_0\sigma_1, \sigma_1\sigma_2, \sigma_2\sigma_3)$ as $n_3(0123)$, and so on. Since $\Delta P_f^{\gamma}(F_{2D}) = \Delta P_f^{c}(F_{2D})$ by Eq. (5), we see that the 2D boundary F move does not change the total fermion parity $P_f = P_f^{\gamma}P_f^{c}$ either. We can also consider the $(2-0)/(0-2)$ moves changing the number of vertices, and it is easy to verify that both P_f^{γ} and P_f^{c} are conserved. Similar to the FSLU approach to FSPT states, the fixed-point condition for the (2-2) move will give rise to a second type of algebraic condition—a pentagon equation—that allows us to compute the amplitude $\psi(\{\sigma_i\})$. It turns out that we can choose a simple solution with $\psi(\{\sigma_i\}) = (1/\sqrt{2})^{N_v}$, where N_v is the total number of vertices for a given triangulation $\mathcal{T}$. For realistic systems with a fixed lattice geometry, it would be straightforward to project the above fixed-point wave function onto that particular lattice, e.g., a triangular lattice.

Thus we have constructed an ASPT state with $T^2 = 1$ on the 2D boundary of a 3D trivial FSPT system with an arbitrary triangulation lattice consistently (to be both symmetric and total fermion-parity fixed). One may wonder whether the bulk complex-fermion degrees of freedom can be moved to the 2D boundary such that this state is defined purely in 2D. For example, for the system with only one complex-fermion mode (blue ball) in the bulk in Fig. 2, we can move the complex fermion to the boundary. However, since the complex-fermion mode is used to compensate for the fermion-parity changes for all of the boundary triangles, the entanglement between them would introduce nonlocal interactions of the 2D system. So the 3D bulk is an intrinsic feature of this ASPT state.

Physical properties of the ASPT state after gauging fermion parity.—In fact, after gauging the fermion parity, the above ASPT state becomes a $\mathbb{Z}_2$ topologically ordered state, and all of the above physics can be understood as a so-called H^3 anomaly, which was discussed in the context of classifying 2D SET states [47]. The $\mathbb{Z}_2$ topological order has four types of anyons: the trivial anyon 1 representing bosonic excitations in the ungauged model, the fermionic anyon f representing fermionic excitations in the ungauged model, and two bosonic anyons e and m, representing two types of $\mathbb{Z}_2^f$ vortices. The two types of vortices have opposite fermion parities, indicated by the fusion rule $m = e \times f$. Since the ASPT state has $\mathbb{Z}_2^T$ symmetry in addition to fermion-parity symmetry, the resulting state has a $\mathbb{Z}_2^T$-symmetry-enriched $\mathbb{Z}_2$ topological order. Correspondingly, $n_1 \in H^1(\mathbb{Z}_2^T, \mathbb{Z}_2)$ becomes a piece of data describing how $\mathbb{Z}_2^T$ permutes the anyons [28,43]. In particular, the nontrivial Majorana-chain decoration $n_1(T) = 1$ is translated into the nontrivial symmetry action in which T exchanges e and m anyons. In other words, the time-reversal symmetry flips the fermion parity of the $\mathbb{Z}_2^f$ vortex. On the other hand, the group structure $G_f = \mathbb{Z}_2^T \times \mathbb{Z}_2^f$ translates into the requirement that the f anyon carries a trivial symmetry fractionalization $T^2 = +1$.

It is well known that this symmetry action is not compatible with the requirement that f carries $T^2 = +1$ [34,36], and this incompatibility can be understood as the result of an obstruction in $H^3(\mathbb{Z}_2^T, \mathbb{Z}_2)$. To see this, we recall that a symmetry-fractionalization pattern is represented by a 2-cocycle $n_2 \in H^2(\mathbb{Z}_2^T, \mathcal{A})$ [34], where the coefficients $\mathcal{A}$ are the fusion group of the four anyons in the $\mathbb{Z}_2$ topological order. Here the choice of n_2 representing f carrying $T^2 = +1$ is $n_2(T, T) = e$ or m [34]. However, neither choice satisfies the cocycle equation because they both have the same nontrivial "coboundary" $\tilde{d}n_2$, indicated by the following:

$$\tilde{d}n_2(T, T, T) \equiv \rho_T(n_2(T, T)) - n_2(T, T) = f, \tag{9}$$

where ρ_T satisfying $\rho_T(e) = m$ and $\rho_T(m) = e$ denotes the nontrivial time-reversal action on the anyons. This violation of the cocycle equation indicates that this 2D SET state has an H^3 obstruction given by $n_3 = \tilde{d}n_2$, and it can be realized on the surface of a 3D SET bulk only with the corresponding symmetry fractionalization given by n_3 [47]. It is straightforward to check that the cocycle $n_3 = \tilde{d}n_2$ computed in Eq. (9) is exactly the same as the n_3 value computed previously using Eq. (5). Therefore, the required 3D SET bulk is the same as the result of gauging the fermion parity in the 3D SPT bulk, which is a 3D $\mathbb{Z}_2$ topological order with pointlike $\mathbb{Z}_2$ charges f carrying fermionic statistics. The n_3 data, describing the complex-fermion decoration in the SPT model, become the H^3 symmetry-fractionalization data in the SET model [48]. Therefore, the bulk-boundary correspondence between the surface and bulk SETs after gauging $\mathbb{Z}_2^f$ provides an alternative way to understand the correspondence between the surface ASPT and the bulk trivial FSPT state.

Classification of ASPT states in 2D with a total symmetry $G_f = G_b \times \mathbb{Z}_2^f$.—The above construction for the ASPT state can be generalized to an arbitrary $G_f = G_b \times \mathbb{Z}_2^f$ straightforwardly, and the relation between the 2D boundary ASPT with Majorana decoration [characterized by $n_1 \in H^1(G_b, \mathbb{Z}_2)$, which actually describes all possible $\mathbb{Z}_2$ subgroups of G_b], and the 3D bulk FSPT with complex-fermion decoration [characterized by $n_3 \in H^3(G_b, \mathbb{Z}_2)$] still turns out to be $n_3 = s_1 n_1 n_1 \equiv s_1 \smile n_1 \smile n_1$. Here we introduce the so-called cup product $(s_1 \smile n_1 \smile n_1)(a,b,c) \equiv s_1(a) n_1(b) n_1(c)$ to manifest that $s_1(a) n_1(b) n_1(c)$ is actually a cohomology operation from $H^1(G_b, \mathbb{Z}_2)$ to $H^3(G_b, \mathbb{Z}_2)$. Here $s_1 \in H^1(G_b, \mathbb{Z}_2)$ indicates whether g is a unitary or antiunitary group element.

In addition to the ASPT phases constructed from the Majorana-chain decoration, the next layer of ASPT is known as the complex-fermion decoration, which leads to trivialization of some BSPT when embedded in interacting fermion systems [26]. The trivialized cocycles ν_{d+1} form a group $\Gamma^{d+1} = \{(-1)^{Sq^2(n_{d-1})} \in H^{d+1}(G_b, U(1)) | n_{d-1} \in H^{d-1}(G_b, \mathbb{Z}_2)\}$. Only the cocycles in the quotient group $H^{d+1}(G_b, U(1))/\Gamma^{d+1}$ correspond to different FSPT phases. From the perspective of ASPT states, we can use a FSLU to transform the state constructed by cocycles in Γ^{d+1} to a product state. On a space manifold with boundary, there is an ASPT state of one lower dimension on the boundary. The simplest example in 2D is again the $G_b = \mathbb{Z}_2^T$ case since $H^2(G_b, \mathbb{Z}_2) = \mathbb{Z}_2$ and Γ^4 is a nontrivial cocycle in $H^4(\mathbb{Z}_2^T, U(1))$. After gauging fermion parity, the corresponding anomalous SET state is the well-known $eTmT$ state, which could not be realized as a 2D SET either [15].

Conclusion and discussion.—In this Letter, we systematically construct ASPT phases for 2D interacting fermion systems with total symmetry $G_f = G_b \times \mathbb{Z}_2^f$. Experimentally, 3D superconductivities with coplanar spin order can realize the $T^2 = 1$ symmetry. The surface of the ^{3}He B phase could also be a potential venue for finding such an ASPT state.

We would like to thank Chenjie Wang for the insightful comments and suggestions. Z.-C. G. acknowledges Direct Grant No. 4053300 from The Chinese University of Hong Kong and funding from Hong Kong's Research Grants Council (ECS No. 24301516). Y. Q. acknowledges support from Ministry of Science and Technology of China under Grant No. 2015CB921700, and from the National Science Foundation of China under Grant No. 11874115.

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$$s_1(g) = \begin{cases} 0, & g\text{is unitary}, \\ 1, & g\text{is antiunitary}. \end{cases}$$

In the special case of $G_b = \mathbb{Z}_2^T$, we identify the s_1 and n_1 defined in Eq. (1) as the nontrivial 1-cocycle in $H^1(\mathbb{Z}_2, \mathbb{Z}_2)$.
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