



L-packets of quasisplit $GSp(2n)$ and $GO(2n)$

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Abstract In his monograph Arthur (The endoscopic classification of representations: orthogonal and symplectic groups, Colloquium Publications, American Mathematical Society, Providence, 2013) characterizes the L-packets of quasisplit symplectic groups and orthogonal groups. By extending his work, we characterize the L-packets for the corresponding similitude groups with desired properties. In particular, we show these packets satisfy the conjectural endoscopic character identities.

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1 Introduction

Let F be a local field of characteristic zero and W_F be the Weil group, then the local Langlands group is defined as follows

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times SL(2, \mathbb{C}) & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

Let G be a quasisplit connected reductive group over F and \widehat{G} be its complex dual group, the Langlands dual group ${}^L G$ is a semidirect product $\widehat{G} \rtimes W_F$, where the action of W_F on \widehat{G} factors through the absolute Galois group $\Gamma_F = \text{Gal}(\bar{F}/F)$. A local Langlands parameter ϕ is a \widehat{G} -conjugacy class of admissible homomorphisms from L_F to ${}^L G$ (see [8]). In particular, it respects the projections on W_F from both L_F and ${}^L G$. We denote a representative of ϕ by $\phi: L_F \rightarrow {}^L G$. Let $\Phi(G)$ be the set of local Langlands parameters and $\Pi(G(F))$ be the set of isomorphism classes of irreducible admissible representations of $G(F)$. The local Langlands conjecture asserts a correspondence between $\Phi(G)$ and $\Pi(G(F))$. The correspondence is not necessarily a bijection. In fact, it is conjectured that each $\phi \in \Phi(G)$ is associated with a finite set Π_ϕ of $\Pi(G(F))$, such that they give a partition of $\Pi(G(F))$

$$\Pi(G(F)) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi.$$

Such sets Π_ϕ are called L-packets. The local Langlands conjecture has been proved for $GL(N)$ by Harris-Taylor [16], Henniart [14] and Scholze [36], in which case one does get a bijection. Arthur [7] extended their results to $Sp(N)$ and $SO(N)$ through the theory of twisted endoscopy, and in his case the packets are not always singletons. By the Langlands classification of irreducible admissible representations of $G(F)$, one can reduce this correspondence to the tempered case, namely one can replace

$\Pi(G(F))$ by the subset $\Pi_{temp}(G(F))$ of tempered representations, and $\Phi(G)$ by the subset $\Phi_{bdd}(G)$ of bounded parameters (i.e., the closure of the image of $\phi|_{W_F}$ is compact). The tempered L-packets can be characterized by “stability”. To explain this concept, we need to introduce the Harish-Chandra characters. For any $\pi \in \Pi(G(F))$, the associated Harish-Chandra character is a distribution on $G(F)$ defined by

$$f_G(\pi) := \text{trace} \int_{G(F)} f(g)\pi(g)dg$$

for $f \in C_c^\infty(G(F))$. Harish-Chandra showed this distribution can be represented by a $G(F)$ -conjugate invariant locally integrable function Θ_π over $G(F)$. Moreover, Θ_π is smooth over the strongly regular semisimple elements $G_{reg}(F)$. Later on, we will simply call them characters. We say a finite linear combination Θ of Harish-Chandra characters is **stable** if it is $G(\bar{F})$ -conjugate invariant over $G_{reg}(F)$, namely $\Theta(\gamma) = \Theta(\gamma')$ for any $\gamma, \gamma' \in G_{reg}(F)$ such that $\gamma = g^{-1}\gamma'g$ for some $g \in G(\bar{F})$. Then the tempered L-packets are conjectured to be the minimal subsets of irreducible tempered representations, within which some linear combination of the Harish-Chandra characters is stable (cf. Conjecture 9.2, [38]).

Let D be a torus and \tilde{G} be a quasisplit connected reductive group over F , which is an extension of D by G

$$1 \longrightarrow G \longrightarrow \tilde{G} \xrightarrow{\lambda} D \longrightarrow 1.$$

Dual to this exact sequence, we have

$$1 \longrightarrow \hat{D} \longrightarrow \hat{\tilde{G}} \xrightarrow{\mathbf{p}} \hat{G} \longrightarrow 1.$$

The projection $\mathbf{p}: \hat{\tilde{G}} \rightarrow \hat{G}$ can be extended to an L-homomorphism, so it induces a map $\Phi_{bdd}(\tilde{G}) \rightarrow \Phi_{bdd}(G)$. Labesse ([19], Theorem 8.1) showed this map is in fact surjective. For $\tilde{\phi} \in \Phi_{bdd}(\tilde{G})$ and $\phi = \mathbf{p} \circ \tilde{\phi}$, it is believed that the restriction $\Pi_{\tilde{\phi}}|_G = \Pi_\phi$. Motivated by this, we want to construct the L-packets of \tilde{G} from that of G , when $G = Sp(2n)$ (resp. $SO(2n)$) and $\tilde{G} = GSp(2n)$ (resp. $GSO(2n)$). In fact, one can also consider the case when $G = SO(2n+1)$ and $\tilde{G} = GO(2n+1)$. Note $GO(2n+1)$ is connected. Since $GO(2n+1) \cong SO(2n+1) \times \mathbb{G}_m$, this case would be trivial. To give the precise statement of our result, we need to first recall Arthur's results about G . We fix an outer automorphism θ_0 of G , such that it is trivial when $G = Sp(2n)$, and it is induced from the conjugate action of $O(2n)$ when $G = SO(2n)$. Let $\Sigma_0 = \langle \theta_0 \rangle$, then Σ_0 acts on $\Pi(G(F))$. Note θ_0 induces a dual automorphism $\hat{\theta}_0$ on \hat{G} , so Σ_0 also acts on $\Phi(G)$ through the action of $\hat{\theta}_0$ on \hat{G} . We denote the set of Σ_0 -orbits in $\Pi_{temp}(G(F))$ by $\bar{\Pi}_{temp}(G(F))$ and the set of Σ_0 -orbits in $\Phi_{bdd}(G)$ by $\bar{\Phi}_{bdd}(G)$. The action of Σ_0 can be extended to \tilde{G} , so we can also define the analogues of these sets for \tilde{G} .

Theorem 1.1 (Arthur)

1. There is a canonical way to associate any $\phi \in \tilde{\Phi}_{bdd}(G)$ with a finite subset $\bar{\Pi}_\phi$ of $\bar{\Pi}_{temp}(G(F))$ such that

$$\bar{\Pi}_{temp}(G(F)) = \bigsqcup_{\phi \in \tilde{\Phi}_{bdd}(G)} \bar{\Pi}_\phi.$$

2. For $\phi \in \tilde{\Phi}_{bdd}(G)$,

$$\bar{\Theta}_\phi := \frac{1}{2} \sum_{[\pi] \in \bar{\Pi}_\phi} (\Theta_\pi + \Theta_{\pi^{\theta_0}})$$

is stable.

When $G = SO(2n)$, we let $\Pi_\phi^{\Sigma_0}$ be the set of all irreducible representations of $O(2n)$, whose restriction to $SO(2n)$ have irreducible constituents contained in $\bar{\Pi}_\phi$, and we call $\Pi_\phi^{\Sigma_0}$ an L-packet of $O(2n)$. In this sense, the sets $\bar{\Pi}_\phi$ really determine the L-packets of $Sp(2n)$ and $O(2n)$. But for simplicity, we will still call the sets $\bar{\Pi}_\phi$ L-packets of G in this paper.

Suppose $\tilde{\phi} \in \tilde{\Phi}_{bdd}(\tilde{G})$ and $\phi = \mathbf{p} \circ \tilde{\phi}$. Since $\bar{\Pi}_\phi$ admits a stable linear combination of Harish-Chandra characters, $\tilde{G}(F)$ acts on $\bar{\Pi}_\phi$ by conjugation. We fix a character $\tilde{\zeta}$ of the centre $Z_{\tilde{G}}(F)$ of $\tilde{G}(F)$, such that its restriction to $Z_G(F) = Z_{\tilde{G}}(F) \cap G(F)$ is the central character of $\bar{\Pi}_\phi$. Let $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ be the subset of representations of $\bar{\Pi}_{temp}(\tilde{G}(F))$ with central character $\tilde{\zeta}$, whose restriction to $G(F)$ have irreducible constituents contained in $\bar{\Pi}_\phi$. Let $X = \text{Hom}(\tilde{G}(F)/Z_{\tilde{G}}(F)G(F), \mathbb{C}^\times)$. Note X acts on $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ by twisting. In Corollary 4.2 we show there exists a subgroup $\alpha(S_\phi^{\Sigma_0})$ of X such that for any $[\tilde{\pi}] \in \tilde{\Pi}_{\phi, \tilde{\zeta}}$, $\tilde{\pi} \otimes \omega \cong \tilde{\pi}^\theta$ for some $\theta \in \Sigma_0$ if and only if $\omega \in \alpha(S_\phi^{\Sigma_0})$. Now we can state our main result.

Theorem 1.2 Suppose $\phi \in \tilde{\Phi}_{bdd}(G)$, there exists a subset $\bar{\Pi}_{\tilde{\phi}}$ of $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ unique up to twisting by X , such that it satisfies the following properties:

- 1.

$$\tilde{\Pi}_{\phi, \tilde{\zeta}} = \bigsqcup_{\omega \in X/\alpha(S_\phi^{\Sigma_0})} \bar{\Pi}_{\tilde{\phi}} \otimes \omega.$$

- 2.

$$\bar{\Theta}_{\tilde{\phi}} := \frac{1}{2} \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} (\Theta_{\tilde{\pi}} + \Theta_{\tilde{\pi}^{\theta_0}})$$

is stable.

In this paper, we call the sets $\bar{\Pi}_{\tilde{\phi}}$ in this theorem L-packets of \tilde{G} , although they really determine the L-packets of $Sp(2n)$ and $GO(2n)$ for the same reason as we have discussed above. When F is archimedean, this theorem is known due to Langlands [20] and Shelstad [39]. In fact this case could also follow from Theorem 1.1 directly. So in this paper, we will focus on the case when F is nonarchimedean. Note if $\bar{\Pi}_{\phi}$ is a singleton, then $\bar{\Pi}_{\tilde{\phi}}$ is also a singleton by part (1) of the theorem, but it is still by no means clear that part (2) will hold for such $\bar{\Pi}_{\tilde{\phi}}$. Our proof of this theorem is by global means, and it is certainly interesting to know if one can establish it by purely local methods.

The main idea of the proof is to realize the L-packet as the local component of some global L-packet. To describe the global picture, we let F be a number field and \mathbb{A}_F be the adele ring of F . We define the automorphic representations of G to be the irreducible constituents of the regular representation of $G(\mathbb{A}_F)$ on $L^2(G(F)\backslash G(\mathbb{A}_F))$. If π is an irreducible admissible representation of $G(\mathbb{A}_F)$, it can be decomposed as a restricted tensor product

$$\pi = \otimes'_v \pi_v$$

of irreducible admissible representations π_v of $G(F_v)$ over all the places v . These local representations π_v are unramified for almost all places, which is the necessary condition to form the restricted tensor product. We assume the global Langlands group L_F exists and it is equipped with embeddings $L_{F_v} \rightarrow L_F$ for all places v . Then we can define the global Langlands parameters as in the local case. We denote the set of Σ_0 -orbits of bounded global Langlands parameters by $\bar{\Phi}(G)$, for this is the set relevant in the classification of automorphic representations of G . For any $\phi \in \bar{\Phi}(G)$, we can associate a family of local Langlands parameter $\phi_v \in \bar{\Phi}_{bdd}(G_v)$ for all places by the following diagram

$$\begin{array}{ccc} L_{F_v} & \xrightarrow{\phi_v} & L_{G_v} \\ \downarrow & & \downarrow \\ L_F & \xrightarrow{\phi} & L_G. \end{array}$$

So one can define the global L-packet to be the restricted tensor product of the local L-packets

$$\bar{\Pi}_{\phi} := \otimes'_v \bar{\Pi}_{\phi_v}.$$

Theorem 1.3 (Arthur) *There exist automorphic representations in $\bar{\Pi}_{\phi}$.*

For any irreducible admissible representation π of $G(\mathbb{A}_F)$, one can associate a family of Satake parameters $c(\pi) = \{c(\pi_v)\}$ for all unramified places of π . If we define an equivalence relation on the families of Satake parameters attached to irreducible admissible representations of $G(\mathbb{A}_F)$ by requiring $c(\pi) \sim c(\pi')$ if $c(\pi_v)$ is Σ_0 -conjugate to $c(\pi'_v)$ for almost all places, then another way of characterizing $\bar{\Pi}_{\phi}$ is

through the equivalence class $c(\phi)$ of family of Satake parameters associated with the representations in $\bar{\Pi}_\phi$. If we take the standard embedding $\xi : {}^L G \rightarrow GL(N, \mathbb{C})$, where $N = 2n + 1$ (resp. $2n$) if $G = Sp(2n)$ (resp. $G = SO(2n)$), then $\xi(c(\phi))$ defines a family of Satake parameters for irreducible admissible representations of $GL(N, \mathbb{A}_F)$. By the conjectural Langlands principle of functoriality and strong multiplicity one for automorphic representations of $GL(N)$, $\xi(c(\phi))$ determines a unique automorphic representation of $GL(N)$. In practice, Arthur gets around the assumption on global Langlands group by reversing our discussion here. To be more precise, he substituted for $\bar{\Phi}(G)$ by the subset of self-dual automorphic representations of $GL(N)$, which are induced from cuspidal automorphic representations of the Levi subgroups of $GL(N)$. Then ϕ_v will correspond to the representations of $GL(N, F_v)$. Since we do not have the generalized Ramanujan conjecture, now we can only conclude $\phi_v \in \bar{\Phi}_{unit}^+(G) \supseteq \bar{\Phi}_{bdd}(G)$ (see Proposition 3.10). Nonetheless, the local packet $\bar{\Pi}_\phi$ can still be defined in this case. In this way, Theorem 1.3 should really be viewed as a statement about Langlands principle of functoriality with respect to the embedding ξ . To summarize, the global L-packet $\bar{\Pi}_\phi$ can be uniquely characterized by either an equivalence class of family of Satake parameters $c(\phi)$ or an automorphic representation of $GL(N)$ associated with $\xi(c(\phi))$. We call this the strong multiplicity one property for the global L-packets of G .

The main tool in our proof is the stabilized twisted Arthur-Selberg trace formula. The ordinary stable trace formula has been established by Arthur in [4–6]. The twisted case results from a long project of Mœglin and Waldspurger [32] which has been finished recently. All of these also rest upon Ngo's celebrated proof [34] of the Fundamental Lemma. To give some ideas of the proof of our theorem, we would like to briefly describe two typical kinds of trace formulas used in this paper. Let $\tilde{\zeta}$ be a character of $Z_{\tilde{G}}(F) \backslash Z_{\tilde{G}}(\mathbb{A}_F)$. The space of $\tilde{\zeta}$ -equivariant L^2 -functions over $\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F)$ can be decomposed into a discrete part and a continuous part:

$$L^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) = L_{disc}^2(\tilde{G}, \tilde{\zeta}) \oplus L_{cont}^2(\tilde{G}, \tilde{\zeta}).$$

If we take a $\tilde{\zeta}^{-1}$ -equivariant smooth compactly supported function $\tilde{f} = \otimes_v \tilde{f}_v$ over $\tilde{G}(\mathbb{A})$, then we can define an operator on $L_{disc}^2(\tilde{G}, \tilde{\zeta})$ by

$$\left(R_{disc}^{\tilde{G}}(\tilde{f})\varphi \right)(x) = \int_{Z_{\tilde{G}}(\mathbb{A}_F) \backslash \tilde{G}(\mathbb{A}_F)} \tilde{f}(y)\varphi(xy)dy, \quad \varphi \in L_{disc}^2(\tilde{G}, \tilde{\zeta}).$$

Müller [31] showed this operator $R_{disc}^{\tilde{G}}(\tilde{f})$ is of trace class, so we can write

$$\text{tr} R_{disc}^{\tilde{G}}(\tilde{f}) = \sum_{\tilde{\pi}} m(\tilde{\pi}) \tilde{f}_{\tilde{G}}(\tilde{\pi}).$$

where the sum is over all irreducible admissible representations of $\tilde{G}(\mathbb{A}_F)$ and $m(\tilde{\pi})$ is the multiplicity of $\tilde{\pi}$ in $L_{disc}^2(\tilde{G}, \tilde{\zeta})$. We define the discrete part of the trace formula to be the following distribution

$$I_{disc}^{\tilde{G}}(\tilde{f}) = \text{tr } R_{disc}^{\tilde{G}}(\tilde{f}) + \text{“symmetric part in } L_{cont}^2(\tilde{G}, \tilde{\zeta})\text{”},$$

where the symmetry on the continuous spectrum is given by the action of the regular elements of the relative Weyl groups (see Sect. 5.1). The stable trace formula gives a stabilization of this distribution $I_{disc}^{\tilde{G}}(\tilde{f})$, and it relates the “error terms” to the stable distributions on some smaller groups, i.e, elliptic endoscopic groups of \tilde{G} . We state it in the following theorem.

Theorem 1.4 (Arthur) *By induction, one can define a stable distribution*

$$S_{disc}^{\tilde{G}}(\tilde{f}) = I_{disc}^{\tilde{G}}(\tilde{f}) - \sum_{\tilde{G}'} \iota(\tilde{G}, \tilde{G}') S_{disc}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}), \quad (1.1)$$

where the sum is over elliptic endoscopic groups $\tilde{G}' \neq \tilde{G}$ of \tilde{G} , $\iota(\tilde{G}, \tilde{G}')$ are some constants (see (5.3)), and $\tilde{f} \rightarrow \tilde{f}^{\tilde{G}'}$ is the Langlands–Shelstad–Kottwitz transfer.

The relation between $S_{disc}^{\tilde{G}}(\tilde{f})$ and L-packets can be described in the following conjecture.

Conjecture 1.5 (Stable multiplicity formula)

$$S_{disc}^{\tilde{G}}(\tilde{f}) = \sum_{\tilde{\psi} \in \Psi(\tilde{G})} a_{\tilde{\psi}} S_{\tilde{\psi}}^{\tilde{G}}(\tilde{f}),$$

and

$$S_{\tilde{\psi}}^{\tilde{G}}(\tilde{f}) = \prod_v \tilde{f}_v(\tilde{\psi}_v),$$

where $\tilde{f}_v(\tilde{\psi}_v)$ is a linear combination of Harish-Chandra characters in some finite subset $\Pi_{\tilde{\psi}_v}$ of $\Pi(\tilde{G}(F_v))$, which defines a stable distribution on $\tilde{G}(F_v)$. Moreover, there is an explicit formula for the constants $a_{\tilde{\psi}}$.

In this conjecture, $\Psi(\tilde{G})$ is the set of so-called global Arthur parameters of \tilde{G} , which generalizes the set $\Phi(\tilde{G})$ of bounded global Langlands parameters. The global packet $\Pi_{\tilde{\psi}} = \bigotimes_v \Pi_{\tilde{\psi}_v}$ associated with the stable distribution $S_{\tilde{\psi}}^{\tilde{G}}(\tilde{f})$ is called a global Arthur packet. One can view the global L-packets as a special case of global Arthur packets, and the local L-packets that we are looking for will be the local components of some global L-packets, which contribute to $S_{disc}^{\tilde{G}}(\tilde{f})$. Before we can talk about how to isolate a global L-packet from $S_{disc}^{\tilde{G}}(\tilde{f})$, we want to introduce the twisted version of (1.1) first.

Let ω be a character of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$ and $\theta \in \Sigma_0$, we define the discrete part of the (θ, ω) -twisted trace formula to be

$$I_{disc}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = \text{tr}(R(\theta)^{-1} \circ R(\omega) \circ R_{disc}^{\tilde{G}}(\tilde{f})) \\ + \text{“}(\theta, \omega) - \text{twisted symmetric part in } L_{cont}^2(\tilde{G}, \tilde{\zeta})\text{”},$$

where $R(\theta)$ is induced by action on $\tilde{G}(\mathbb{A}_F)$ by θ , and $R(\omega)$ is induced by multiplication on $L^2_{disc}(\tilde{G}, \tilde{\zeta})$ by ω . Then the stabilization of $I^{(\tilde{G}^\theta, \omega)}_{disc}(\tilde{f})$ is given by the following theorem.

Theorem 1.6 (Mœglin and Waldspurger)

$$I^{(\tilde{G}^\theta, \omega)}_{disc}(\tilde{f}) = \sum_{\tilde{G}'} \iota(\tilde{G}, \tilde{G}') S^{\tilde{G}'}_{disc}(\tilde{f}^{\tilde{G}'}), \quad (1.2)$$

where the sum is over (θ, ω) -twisted elliptic endoscopic groups \tilde{G}' of \tilde{G} .

One application of (1.2) is it gives a multiplicity formula for the automorphic representations of \tilde{G} . Let $X = \text{Hom}(\tilde{G}(\mathbb{A}_F)/Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F), \mathbb{C}^\times)$ and $Y = \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F), \mathbb{C}^\times)$. If $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$, we write $Y(\tilde{\pi}) = \{\omega \in Y : \tilde{\pi} \cong \tilde{\pi} \otimes \omega\}$, which is finite.

Proposition 1.7 1. Suppose $\tilde{\pi}$ is a discrete automorphic representation of \tilde{G} , and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$. If $[\pi] \in \bar{\Pi}_\phi$ for $\phi \in \bar{\Phi}(G)$, then

$$m(\tilde{\pi}) = m_{\tilde{\phi}} |Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)|, \quad (1.3)$$

where $\alpha(\mathcal{S}_\phi)$ (see (2.15)) is a subgroup of $Y(\tilde{\pi})$, $m_{\tilde{\phi}} = 1$ or 2 . Moreover, $m_{\tilde{\phi}} = 2$ only when G is special even orthogonal, $\phi \notin \Phi(G^{\theta_0})$ (see Sect. 3.1), and $\tilde{\pi} \otimes \omega \cong \tilde{\pi}^{\theta_0}$ for some $\omega \in Y$.

2. Suppose $\tilde{\pi}$ and $\tilde{\pi}'$ are discrete automorphic representations of \tilde{G} , and there exists $\omega \in X$ such that $\tilde{\pi}_v$ is Σ_0 -conjugate to $\tilde{\pi}'_v \otimes \omega_v$ for all places. If π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$ and $[\pi] \in \bar{\Pi}_\phi$ for $\phi \in \bar{\Phi}(G)$, then there exists some $\omega' \in Y$ and $\theta \in \Sigma_0$ such that $\tilde{\pi}' \cong \tilde{\pi}^\theta \otimes \omega'$.

Back to the proof of Theorem 1.2, a key step is to isolate the global L-packets from the stable distribution $S^{\tilde{G}}_{disc}(\tilde{f})$ for $\tilde{f} = \otimes_v \tilde{f}_v$ such that \tilde{f}_v is Σ_0 -invariant. By the theory of multipliers, one can isolate the parts associated with different equivalence classes of families of Satake parameters. For $\phi \in \bar{\Phi}(G)$, the equivalence class $c(\phi)$ determines the packet $\bar{\Pi}_\phi$ of G uniquely from our previous discussion. But this may not be the case for \tilde{G} . In view of part (2) of Proposition 1.7, this means if the global L-packet $\bar{\Pi}_{\tilde{\phi}}$ exists for $\tilde{\phi} \in \bar{\Phi}(\tilde{G})$, there might exist $\omega \in Y$ such that $\bar{\Pi}_{\tilde{\phi}} \neq \bar{\Pi}_{\tilde{\phi}} \otimes \omega$, whereas $\bar{\Pi}_{\tilde{\phi}_v} = \bar{\Pi}_{\tilde{\phi}_v} \otimes \omega_v$ for almost all places. For our proof, we only need something weaker, that is we will fix a nonarchimedean place u , and we require if $\bar{\Pi}_{\tilde{\phi}_v} = \bar{\Pi}_{\tilde{\phi}_v} \otimes \omega_v$ for all places $v \neq u$, then $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\tilde{\phi}} \otimes \omega$. Such global parameters can be constructed using the result of Shin [41] on automorphic plancherel density. At last, we prove the following global result, which is parallel with Theorem 1.3.

Theorem 1.8 For $\phi \in \bar{\Phi}(G)$ satisfying $\mathcal{S}_{\tilde{\phi}} = 1$ (see Sect. 2.2), there exists a global L-packet

$$\bar{\Pi}_{\tilde{\phi}} = \otimes'_v \bar{\Pi}_{\tilde{\phi}_v}$$

of \tilde{G} unique up to twisting by Y , such that if $\tilde{\pi}$ is an automorphic representation of \tilde{G} whose irreducible constituents in the restriction to $G(\mathbb{A}_F)$ are contained in $\bar{\Pi}_\phi$, then $[\tilde{\pi}]$ is contained in $\bar{\Pi}_{\tilde{\phi}} \otimes \omega$ for some $\omega \in Y$.

The local and global results of this paper will be proved together by a complicated induction argument. For the purpose of giving a clear proof of the local results, we have minimized the global assumptions needed in our induction arguments by imposing very restrictive conditions on the global results (like in Theorem 1.8). In a sequel to this paper, we will prove the global results of this paper in a more general setting.

A full description of the discrete spectrum of \tilde{G} will also require the Arthur packets. Unfortunately, the technique in this paper will not be sufficient for that. This is somehow reflected by the fact that the Arthur packets of G can have more complicated structure than its L-packets. To be able to construct the Arthur packets of \tilde{G} in the nonarchimedean case, one will need to extend the works of Mœglin [24, 25] on explicit construction of the Arthur packets of G . The global case could be even more challenging, because that would require certain description of the residue spectrum for both G and \tilde{G} . So we would like to keep that as a project for the future.

This paper is organized as follows. In Sect. 2, we discuss various group theoretic properties about G and \tilde{G} . We introduce their Levi subgroups and twisted endoscopic groups. We also discuss the relation between $\Phi(G)$ and $\Phi(\tilde{G})$ both in the local and global cases. We recall some known results about restricting the local representations of \tilde{G} to G , in particular we have restriction multiplicity one in this case. In Sect. 3, we review Arthur's endoscopic classification theory for G in the tempered case. In the local theory, we describe the θ -twisted endoscopic character identities (or character relations) for G and $\theta \in \Sigma_0$. In the global theory, we give Arthur's multiplicity formula for automorphic representations of G . In Sect. 4, we state our main local theorem (Theorem 4.6). In this theorem, we formulate the (θ, ω) -twisted endoscopic character identities for \tilde{G} and $\omega \in X$, which are natural extensions of the θ -twisted endoscopic character identities for G . Similarly we also formulate the natural extensions of the twisted local intertwining relations from G to \tilde{G} . In Sect. 5, we introduce various stable trace formulas used in this paper. We prove Proposition 1.7 as an application of the twisted stable trace formula. We also state some global conjectures, whose special versions have to be proved together with our main local theorem. In particular, we give the precise statement of Conjecture 1.5 in the tempered case. In the end of this section, we make a comparison of both sides of the twisted stable trace formulas for \tilde{G} , which is analogous to what Arthur did for G . In the final section, we give the proofs of our main local theorem together with all the global theorems by an induction argument. In particular, we address the issue of lack of strong multiplicity one as we mentioned above.

Some standard notations If G is a reductive group over a field F , let G^0 be the identity component, G_{der} be the derived group of G^0 , G_{sc} be the simply connected cover of G_{der} , and G_{ad} be the adjoint group of G_{der} . We denote the centre of G by Z_G or $Z(G)$, the split connected component of Z_G by A_G . If G is connected, let $X^*(G)$ be the group of algebraic characters of G over F and $\mathfrak{a}_G = \text{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{R})$. If F is a local field, there is a homomorphism $H_G : G(F) \rightarrow \mathfrak{a}_G$ defined by $e^{(H_G(g), \chi)} = |\chi(g)|_F$ for $g \in G(F)$ and $\chi \in X^*(G)$. If G is abelian and θ is an automorphism of G , let

G^θ be the θ -invariant subgroup of G , and G_θ be the θ -coinvariant group of G , i.e., $G_\theta = G/(\theta - 1)G$. If A is a locally compact abelian group, we denote its Pontryagin dual by A^* .

2 Preliminary

2.1 Groups

2.1.1 Similitude groups

Let F be a local or global field of characteristic zero and \bar{F} be its algebraic closure. When F is global, let us denote the adèle ring over F by \mathbb{A}_F , and the idèle group by I_F . The absolute Galois group over F is written as Γ_F or Γ for abbreviation. Let G be a quasisplit connected reductive group over F and D be a torus. We denote by \tilde{G} an extension of D by G

$$1 \longrightarrow G \longrightarrow \tilde{G} \xrightarrow{\lambda} D \longrightarrow 1. \quad (2.1)$$

Let us denote the centres of G and \tilde{G} by Z_G and $Z_{\tilde{G}}$ respectively. Sometimes we need to distinguish λ for different groups, so we will also write $\lambda_G = \lambda$. The primary example that we are going to consider in this paper is when G is a special even orthogonal group or a symplectic group, and \tilde{G} is the corresponding similitude group, in which case λ is called the similitude character.

A split general symplectic group (or symplectic similitude group) is defined as follows

$$GSp(2n) = \left\{ g \in GL(2n) : g \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix}^t g = \lambda(g) \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} \right\},$$

where $J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \ddots & & \\ 1 & & & \end{pmatrix}$ and $\lambda(g)$ is a scalar. It is connected as an algebraic group.

A split general even orthogonal group (or orthogonal similitude group) is defined by

$$GO(2n) = \left\{ g \in GL(2n) : g \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix}^t g = \lambda(g) \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} \right\}.$$

Since $(\det g)^2 = \lambda(g)^{2n}$, it has two connected components depending on whether $\det g/\lambda(g)^n$ being 1 or -1 . Let us denote the identity component by $GSO(2n)$, and we call it the connected general even orthogonal group. Because $SO(2n)$ (resp. $GSO(2n)$) has an outer automorphism from the conjugate action by $O(2n)$ (resp. $GO(2n)$), let us denote an outer twist of $SO(2n)$ (resp. $GSO(2n)$) with respect to this outer automorphism and an arbitrary quadratic extension E/F by $SO(2n, \eta)$ (resp. $GSO(2n, \eta)$), where η is the quadratic (idèle class) character associated to E/F by the local (global)

class field theory. We would like to allow $E = F$ and $\eta = 1$, in which case this is the split group. If $G = SO(2n, \eta)$, we define $\eta_G = \eta$. If $G = Sp(2n)$, we define $\eta_G = 1$. These groups that we have defined above give all the quasisplit general symplectic groups and quasisplit connected general even orthogonal groups.

Another description of quasisplit general symplectic groups and quasisplit connected general even orthogonal groups is given by

$$GSp(2n) = (\mathbb{G}_m \times Sp(2n))/(\mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad GSO(2n, \eta) = (\mathbb{G}_m \times SO(2n, \eta))/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z}$ is embedded diagonally into the centre of each factor. The similitude character λ is square on \mathbb{G}_m and trivial on the other factor. More generally we can define

$$G(Sp(2n_1) \times \cdots \times Sp(2n_s) \times SO(2n_{s+1}, \eta_1) \times \cdots \times SO(2n_{s+t}, \eta_t)) \quad (2.2)$$

to be

$$(\mathbb{G}_m \times Sp(2n_1) \times \cdots \times Sp(2n_s) \times SO(2n_{s+1}, \eta_1) \times \cdots \times SO(2n_{s+t}, \eta_t))/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z}$ is again embedded diagonally. We can also generalize the similitude character λ to these groups such that it is square on \mathbb{G}_m and trivial on all the other factors. At last let us write $GSp(0) = GSO(0) = \mathbb{G}_m$ and set $\lambda = id$ in this case.

For any quasisplit connected reductive group G defined over F , we denote by \widehat{G} its complex dual group, by $Z(\widehat{G})$ the centre of \widehat{G} , and by ${}^L G$ its L -group, which is a semidirect product of \widehat{G} with the Weil group W_F , i.e., $\widehat{G} \rtimes W_F$. Then dual to the extension (2.1), we have

$$1 \longrightarrow \widehat{D} \longrightarrow \widehat{\widetilde{G}} \xrightarrow{\mathbf{p}} \widehat{G} \longrightarrow 1,$$

where all the homomorphisms can be extended to L -homomorphisms of L -groups. If \widetilde{G} is $GSp(2n)$ or $GSO(2n, \eta)$, then $\widehat{\widetilde{G}}$ is the general Spin group

$$\begin{aligned} GSpin(2n+1, \mathbb{C}) &= (\mathbb{C}^\times \times Spin(2n+1, \mathbb{C}))/(\mathbb{Z}/2\mathbb{Z}) \quad \text{or} \\ GSpin(2n, \mathbb{C}) &= (\mathbb{C}^\times \times Spin(2n, \mathbb{C}))/(\mathbb{Z}/2\mathbb{Z}), \end{aligned}$$

where $\mathbb{Z}/2\mathbb{Z}$ is embedded diagonally to the centre of each factor. Here the embedding needs to be specified. Note that the Spin group is an extension of the special orthogonal group by $\mathbb{Z}/2\mathbb{Z}$. If we denote the generator of this $\mathbb{Z}/2\mathbb{Z}$ by z , then in defining the general Spin group we want $\mathbb{Z}/2\mathbb{Z}$ to be embedded to $\langle z \rangle$ for the Spin factor. In fact, $Z(Spin(2n+1, \mathbb{C})) = \langle z \rangle$, and for $Z(Spin(2n, \mathbb{C}))$ there is an exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow Z(Spin(2n, \mathbb{C})) \longrightarrow Z(SO(2n, \mathbb{C})) \longrightarrow 1.$$

We take a preimage of the generator of $Z(SO(2n, \mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$ in $Z(Spin(2n, \mathbb{C}))$ and denote it by w , then it is well-known that $w^2 = 1$ if n is even and $w^2 = z$ if n is odd. On

the other hand, $Z(GSpin(2n+1, \mathbb{C})) \cong \mathbb{C}^\times$, and $Z(GSpin(2n, \mathbb{C})) \cong \mathbb{C}^\times \times \mathbb{Z}/2\mathbb{Z}$. This is because when n is even $u = (1, w)$ (resp. $u = (\sqrt{-1}, w)$ when n is odd) splits the exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow Z(GSpin(2n, \mathbb{C})) \longrightarrow Z(SO(2n, \mathbb{C})) \longrightarrow 1.$$

${}^L\tilde{G}$ is $GSpin(2n+1, \mathbb{C}) \times W_F$ or $GSpin(2n, \mathbb{C}) \rtimes W_F$ where the action of W_F on $GSpin(2n, \mathbb{C})$ factors through the Galois group $\Gamma_{E/F}$ of the quadratic extension E/F associated with η , and it acts trivially on \mathbb{C}^\times . It is interesting to see its action on the centre of $GSpin(2n, \mathbb{C})$. If τ is the nontrivial element in $\Gamma_{E/F}$, then τ is trivial on the factor \mathbb{C}^\times and

$$\tau(u) = (-1) \cdot u, \text{ for } -1 \in \mathbb{C}^\times. \quad (2.3)$$

If \tilde{G} is type (2.2), then $\widehat{\tilde{G}}$ is

$$(\mathbb{C}^\times \times Spin(2n_1+1, \mathbb{C}) \times \cdots \times Spin(2n_s+1, \mathbb{C}) \times Spin(2n_{s+1}, \mathbb{C}) \times \cdots \\ \times Spin(2n_{s+t}, \mathbb{C}))/(\mathbb{Z}/2\mathbb{Z})^{s+t},$$

where $(\mathbb{Z}/2\mathbb{Z})^{s+t}$ is embedded as the subgroup generated by

$$(-1, \overbrace{1, \dots, 1}^{k-1}, z, \overbrace{1, \dots, 1}^{s+t-k})$$

for $1 \leq k \leq s+t$. For ${}^L\tilde{G}$, the action of W_F on $\widehat{\tilde{G}}$ factors through the Galois group $\Gamma_{E'/F}$, where E' is the composite field $E_1 E_2 \cdots E_t$ for the quadratic extensions E_i/F associated with η_i , and it acts on each factor as in the previous case.

Lemma 2.1 *The image of λ on $GSp(2n, F)$, $GSO(2n, F)$ is F^\times , and on $GSO(2n, \eta)(F)$ is $Nm_{E/F}E^\times$, where E/F is the quadratic extension associated to η .*

Proof The cases of $GSp(2n)$ and $GSO(2n)$ are obvious, so we will only consider the case that $\tilde{G} = GSO(2n, \eta)$. If $n = 1$, $GSO(2, \eta)$ can be embedded into $GL(2)$ and λ is given by the determinant map. Since $GSO(2, \eta)(F) = E^\times$, it is easy to see that the determinant map becomes the norm map on E^\times , and the image is $Nm_{E/F}E^\times$. For general n , we can take a Borel subgroup \tilde{B} of $GSO(2n, \eta)$ with a maximal torus \tilde{T} and unipotent radical \tilde{N} . By the Bruhat decomposition,

$$\tilde{G}(F) = \bigsqcup_{w \in W(\tilde{T}(F), \tilde{G}(F))} \tilde{B}(F)\dot{w}\tilde{B}(F),$$

where \dot{w} are representatives of w in $\tilde{G}(F)$. Since $W(\tilde{T}(F), \tilde{G}(F)) \cong W(T(F), G(F))$ for $T = G \cap \tilde{T}$, one can take \dot{w} in $G(F)$. Moreover, $\tilde{N} = N$ for $N = G \cap \tilde{N}$. Therefore,

$\lambda(\tilde{G}(F)) = \lambda(\tilde{B}(F)) = \lambda(\tilde{T}(F))$. Let us write $\tilde{G} = (\mathbb{G}_m \times G)/(\mathbb{Z}/2\mathbb{Z})$, and choose $\tilde{T}(\bar{F})$ such that it consists of (x, g) modulo $\mathbb{Z}/2\mathbb{Z}$, where $x \in \bar{F}^\times$ and

$$g = \text{diag}\{z_1, \dots, z_{n-1}, y, y^{-1}, z_{n-1}^{-1}, \dots, z_1^{-1}\} \in G(\bar{F})$$

with $z_i, y \in \bar{F}^\times$. If $(x, g) \in \tilde{T}(F)$, then $(x, y) \in GSO(2, \eta)(F)$, and $\lambda(x, g) = \lambda_{SO(2, \eta)}(x, y) = x^2$. On the other hand, if $(x, y) \in GSO(2, \eta)(F)$, then by letting $z_i = x$ for $1 \leq i \leq n-1$, we have $(x, g) \in \tilde{T}(F)$. This shows $\lambda(\tilde{T}(F)) = \lambda_{SO(2, \eta)}(GSO(2, \eta)(F)) = Nm_{E/F}E^\times$. \square

This lemma can be easily generalized to groups of type (2.2).

Lemma 2.2 *Suppose \tilde{G} is of type (2.2), the image of λ on $\tilde{G}(F)$ is*

$$F^\times \cap Nm_{E_1/F}E_1^\times \cap \dots \cap Nm_{E_t/F}E_t^\times,$$

where E_i/F is the quadratic extension associated to η_i for $1 \leq i \leq t$.

Proof Let us denote by $\tilde{\tilde{G}}$ the product

$$\begin{aligned} &GSp(2n_1) \times GSp(2n_2) \times \dots \times GSp(2n_s) \\ &\times GSO(2n_{s+1}, \eta_1) \times \dots \times GSO(2n_{s+t}, \eta_t), \end{aligned} \quad (2.4)$$

then \tilde{G} is the subgroup of $\tilde{\tilde{G}}$ characterized by $\lambda_1(g_1) = \dots = \lambda_{s+t}(g_{s+t})$ for $(g_1, \dots, g_{s+t}) \in \tilde{\tilde{G}}$. In particular, $\lambda(g) = \lambda_1(g_1)$ for $g \in \tilde{G} \subseteq \tilde{\tilde{G}}$. Then the lemma follows immediately from Lemma 2.1. \square

When F is global, we have the following corollary, whose proof is obvious.

Corollary 2.3 *Suppose \tilde{G} is of type (2.2), the image of λ on $\tilde{G}(\mathbb{A}_F)$ is*

$$I_F \cap Nm_{E_1/F}I_{E_1} \cap \dots \cap Nm_{E_t/F}I_{E_t},$$

where E_i/F is the quadratic extension associated to η_i for $1 \leq i \leq t$.

Corollary 2.4 *Suppose \tilde{G} is of type (2.2), then $\lambda(\tilde{G}(\mathbb{A}_F)) \cap F^\times = \lambda(\tilde{G}(F))$ and $\lambda(Z_{\tilde{G}}(\mathbb{A}_F)) \cap F^\times = \lambda(Z_{\tilde{G}}(F))$.*

Proof For the first equality, by Lemma 2.2 and Corollary 2.3 it suffices to show $Nm_{E_i/F}I_{E_i} \cap F^\times = Nm_{E_i/F}E_i^\times$ for all $1 \leq i \leq t$, and this is a consequence of Hasse norm theorem (see [33], Corollary VI.4.5). For the second equality, note $\lambda(Z_{\tilde{G}}(\mathbb{A}_F)) = I_F^2$ and $\lambda(Z_{\tilde{G}}(F)) = F^{\times^2}$. So we need to show $F^\times \cap I_F^2 = F^{\times^2}$, and this follows from Grunwald–Wang theorem. \square

2.1.2 Levi subgroups

Let \tilde{G} , G , D and λ be defined as in Sect. 2.1.1. If we restrict λ to a Levi subgroup \tilde{M} of \tilde{G} , then its kernel will be a Levi subgroup M of G , and we have

$$1 \longrightarrow M \longrightarrow \tilde{M} \xrightarrow{\lambda} D \longrightarrow 1.$$

It is easy to see that this induces a bijection between Levi subgroups of \tilde{G} and G .

Suppose \tilde{G} is a general symplectic group or a connected general even orthogonal group of semisimple rank n , then \tilde{M} is isomorphic to

$$GL(n_1) \times \cdots \times GL(n_r) \times \tilde{G}_-, \quad (2.5)$$

where \tilde{G}_- is of the same type as \tilde{G} with semisimple rank $n_- \geq 0$ and $n = \sum_{i=1}^r n_i + n_-$. Throughout this paper we fix a Borel subgroup \tilde{B} of \tilde{G} consisting of upper-triangular matrices and we choose \tilde{M} to be contained in the group

$$\begin{pmatrix} GL(n_1) & & & & & 0 \\ & \ddots & & & & \\ & & GL(n_r) & & & \\ & & & \tilde{G}_- & & \\ & & & & GL(n_r) & \\ & & & & & \ddots \\ 0 & & & & & & GL(n_1) \end{pmatrix}.$$

In fact this gives all the standard Levi subgroups if \tilde{G} is $GSp(2n)$ or $GSO(2n, \eta)$ ($\eta \neq 1$), and $GO(2n)$ -conjugacy classes of standard Levi subgroups if \tilde{G} is $GSO(2n)$. We fix an isomorphism from (2.5) to \tilde{M} as follows

$$(g_1, \dots, g_r, g) \longrightarrow \text{diag}\{g_1, \dots, g_r, g, \lambda(g)_t g_r^{-1}, \dots, \lambda(g)_t g_1^{-1}\}$$

if $n_- > 0$, and

$$(g_1, \dots, g_r, g) \longrightarrow \text{diag}\{g_1, \dots, g_r, \lambda(g)_t g_r^{-1}, \dots, \lambda(g)_t g_1^{-1}\}$$

if $n_- = 0$. Here ${}_t g_i = J_{n_i} {}^t g_i J_{n_i}^{-1}$ for $1 \leq i \leq r$. Under this isomorphism, the Weyl group $W(\tilde{M}) = \text{Norm}(A_{\tilde{M}}, G)/\tilde{M}$ acts on \tilde{M} by permuting the general linear factors and changing some g_i to $\lambda(g)_t g_i^{-1}$ (also compositions of these). Finally, note $M \cong GL(n_1) \times \cdots \times GL(n_r) \times G_-$ and $W(M) \cong W(\tilde{M})$.

2.1.3 Twisted endoscopic groups

Let G be a quasisplit connected reductive group over F . When F is local, we have an isomorphism

$$H^1(W_F, Z(\widehat{G})) \longrightarrow \text{Hom}(G(F), \mathbb{C}^\times). \quad (2.6)$$

When F is global, we have a homomorphism

$$H^1(W_F, Z(\widehat{G})) \longrightarrow \text{Hom}(G(\mathbb{A}_F)/G(F), \mathbb{C}^\times), \quad (2.7)$$

where $\text{Hom}(G(\mathbb{A}_F)/G(F), \mathbb{C}^\times)$ denotes the quasicharacter of $G(\mathbb{A}_F)$ trivial on $G(F)$. If we let F_v be the localization of F at place v , then there is a commutative diagram

$$\begin{array}{ccc} H^1(W_F, Z(\widehat{G})) & \longrightarrow & \text{Hom}(G(\mathbb{A}_F)/G(F), \mathbb{C}^\times) \\ \downarrow & & \downarrow \\ H^1(W_{F_v}, Z(\widehat{G}_v)) & \longrightarrow & \text{Hom}(G(F_v), \mathbb{C}^\times). \end{array}$$

Let

$$\text{Ker}^1(W_F, Z(\widehat{G})) := \bigcap_v \text{Ker}\{H^1(W_F, Z(\widehat{G})) \rightarrow H^1(W_{F_v}, Z(\widehat{G}_v))\},$$

it is finite and gives the kernel of (2.7). Suppose \widetilde{G} , G , D and λ are defined as in Sect. 2.1.1, then we have the following fact.

Lemma 2.5 *Suppose $Z(\widehat{G})$ is Γ_F -invariant and D is split, then $\text{Ker}^1(W_F, Z(\widehat{G})) = 1$.*

Proof It is a consequence of Chebotarev's density theorem (see [33], Corollary VII.13.10) that

$$\text{Ker}^1(W_F, Z(\widehat{G})) = \text{Ker}^1(W_F, \widehat{D}) = 1.$$

Then we consider the exact sequence

$$1 \longrightarrow \widehat{D} \longrightarrow Z(\widehat{\widetilde{G}}) \longrightarrow Z(\widehat{G}) \longrightarrow 1,$$

it induces a commutative diagram

$$\begin{array}{ccccccc} \pi_0(Z(\widehat{G})^\Gamma) & \longrightarrow & H^1(W_F, \widehat{D}) & \longrightarrow & H^1(W_F, Z(\widehat{\widetilde{G}})) & \longrightarrow & H^1(W_F, Z(\widehat{G})) \\ \downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \\ \pi_0(Z(\widehat{G}_v)^{\Gamma_v}) & \longrightarrow & H^1(W_{F_v}, \widehat{D}_v) & \longrightarrow & H^1(W_{F_v}, Z(\widehat{\widetilde{G}}_v)) & \longrightarrow & H^1(W_{F_v}, Z(\widehat{G}_v)), \end{array}$$

with both the top and bottom rows being exact. Suppose $u \in \text{Ker}^1(W_F, Z(\widehat{G}))$, then by the commutativity of the right square and $\text{Ker}^1(W_F, Z(\widehat{G})) = 1$, u has a preimage w in $H^1(W_F, \widehat{D})$. Since $\text{Ker}^1(W_F, \widehat{D}) = 1$, the Langlands correspondence for tori allows us to identify $H^1(W_F, \widehat{D})$ with $\text{Hom}(D(\mathbb{A}_F)/D(F), \mathbb{C}^\times)$. Now without loss of generality, we can assume $w \neq 1$. By the commutativity of the left square and the fact that the left end vertical map is an isomorphism, the localization of w is determined by the localizations of those in the image of $\pi_0(Z(\widehat{G})^\Gamma)$ in $H^1(W_F, \widehat{D})$. Finally, we use Chebotarev's density theorem again to conclude that w has to lie in the image of $\pi_0(Z(\widehat{G})^\Gamma)$, and hence $u = 1$. \square

Corollary 2.6 *Suppose \widetilde{G} is of type (2.2) and λ is the generalized similitude character, then*

$$\text{Ker}^1(W_F, Z(\widehat{G})) = \text{Ker}^1(W_F, Z(\widehat{G})) = 1.$$

Proof One just needs to observe that in this case Γ_F acts trivially on $Z(\widehat{G})$, and $D = \mathbb{G}_m$. \square

Let θ be an automorphism of G , and ω be a quasicharacter of $G(F)$ if F is local, or a quasicharacter of $G(\mathbb{A}_F)$ trivial on $G(F)$ if F is global. We define a twisted endoscopic datum for (G, θ, ω) to be a triple (H, s, ξ) , where H is a quasisplit connected reductive group over F , s is a semisimple element in $\widehat{G} \rtimes \widehat{\theta}$, and ξ is an L -embedding from ${}^L H$ to ${}^L G$ satisfying the following conditions:

1. $\text{Int}(s) \circ \xi = \mathbf{a} \cdot \xi$, for a 1-cocycle \mathbf{a} of W_F in $Z(\widehat{G})$ which is mapped to ω under (2.6) or (2.7);
2. $\widehat{H} \cong \text{Cent}(s, \widehat{G})^0$ through ξ .

Here H is called a twisted endoscopic group of G and for abbreviation we will denote (H, s, ξ) by H . In this definition, we have required ξ to be an L -embedding of ${}^L H$. But in general, ξ can be an embedding of certain extension group of \widehat{H} by W_F , which may not necessarily be isomorphic to ${}^L H$. In that case, one has to consider z -pairs (see [18], 2.2). Since we do not need to deal with this general situation in this paper, we are content with the current definition.

Two twisted endoscopic data (H, s, ξ) and (H', s', ξ') are called isomorphic if there exists an element $g \in \widehat{G}$ such that $g\xi({}^L H)g^{-1} = \xi'({}^L H')$ and $gs g^{-1} \in s'Z(\widehat{G})$. Here such g is called an isomorphism. We denote by $\mathcal{E}(G^\theta, \omega)$ the set of isomorphism classes of twisted endoscopic data for (G, θ, ω) . When $\theta = id$ and $\omega = 1$, we get the ordinary endoscopic data, and we abbreviate $\mathcal{E}(G^\theta, \omega)$ to $\mathcal{E}(G)$. A twisted endoscopic datum (H, s, ξ) is called **elliptic** if $\xi(Z(\widehat{H})^{\Gamma_F})^0 \subseteq Z(\widehat{G})$, and we denote by $\mathcal{E}_{ell}(G^\theta, \omega)$ the set of isomorphism classes of twisted elliptic endoscopic data for (G, θ, ω) . When $G = GL(N)$, we write $\mathcal{E}_{ell}(N^\theta)$ for $\mathcal{E}_{ell}(GL(N)^\theta)$. One can see from the definition that a twisted endoscopic group for G can be viewed as an elliptic endoscopic group of some θ -stable Levi subgroup M (which also admits a θ -stable parabolic subgroup $P \supseteq M$) of G . On the other hand, one can obtain all the twisted endoscopic groups of G by taking the Levi subgroups of the twisted elliptic endoscopic groups of G .

If (H, s, ξ) is a twisted endoscopic datum for (G, θ, ω) , we denote the automorphism group of this twisted endoscopic datum by $\text{Aut}_G(H)$. By our definition, it is

a subgroup of \widehat{G} . We define the inner automorphism group $\text{Int}_G(H)$ of this twisted endoscopic datum to be $\widehat{H}Z(\widehat{G})^{\Gamma_F}$, and the outer automorphism group to be

$$\text{Out}_G(H) = \text{Aut}_G(H)/\text{Int}_G(H).$$

By fixing an F -splitting for H , we get a homomorphism from $\text{Out}_G(H)$ to the outer automorphism group $\text{Out}(H)$ of H . Let us denote the image by $\text{Out}(H, G)$, and define

$$C := \{z \in Z(\widehat{G}) : \sigma(z)z^{-1} \in Z(\widehat{G}) \cap \widehat{H} \text{ for } \sigma \in \Gamma_F\}.$$

Then there is an exact sequence

$$1 \longrightarrow C/C \cap \widehat{H}Z(\widehat{G})^{\Gamma_F} \longrightarrow \text{Out}_G(H) \longrightarrow \text{Out}(H, G) \longrightarrow 1. \quad (2.8)$$

When F is local, there is an action of $\text{Out}_G(H)$ on $\mathcal{H}(H)$ (or equivalently $C_c^\infty(H(F))$). For $g \in \text{Out}_G(H)$, let us denote its image in $\text{Out}(H, G)$ by τ_g and choose a representative \dot{g} in $\text{Aut}_G(H)$ such that $\text{Int}(\dot{g})$ preserves a Γ_F -splitting of \widehat{H} . Then $b_g(w) = \dot{g}^{-1}\xi(1 \rtimes w)\dot{g}\xi(1 \rtimes w)^{-1}$ defines a 1-cocycle of W_F in $Z(\widehat{H})$ and it induces a quasicharacter w_g of $H(F)$ by (2.6). So the action of $\text{Out}_G(H)$ on $\mathcal{H}(H)$ sends $f(h)$ to ${}^g f(h) = f(\tau_g(h))\omega_g(h)^{-1}$. In all the cases that we will be considering in this paper, one can always split the exact sequence (2.8) and get $\text{Out}_G(H) \cong \text{Out}(H, G) \times (C/C \cap \widehat{H}Z(\widehat{G})^{\Gamma_F})$ such that $\text{Out}(H, G)$ acts on $\mathcal{H}(H)$ through its action on $H(F)$. When G is a product of symplectic groups and special even orthogonal groups, $\text{Out}_G(H) \cong \text{Out}(H, G)$. When $G = GL(N)$, we write $\text{Out}_N(H)$ for $\text{Out}_{GL(N)}(H)$.

Suppose \widetilde{G} , G , D , λ are defined as Sect. 2.1. Let θ be an automorphism of \widetilde{G} , and we assume λ is θ -invariant, then θ also induces an automorphism of G . If $\omega_{\widetilde{G}}$ is a quasicharacter associated with \widetilde{G} as in the setup of twisted endoscopic datum, let us write ω_G for the restriction of $\omega_{\widetilde{G}}$ to $G(F)$ if F is local or to $G(\mathbb{A}_F)/G(F)$ if F is global. The following lemma describes the relation for twisted endoscopic data between \widetilde{G} and G .

Proposition 2.7 *There is a one to one correspondence between $\mathcal{E}(G^\theta, \omega_G)$ and*

$$\bigsqcup_{\omega_{\widetilde{G}}|_G = \omega_G} \mathcal{E}(\widetilde{G}^\theta, \omega_{\widetilde{G}}),$$

such that if G' corresponds to \widetilde{G}' , then there exists an exact sequence

$$1 \longrightarrow G' \longrightarrow \widetilde{G}' \xrightarrow{\lambda'} D \longrightarrow 1.$$

Moreover, the same is true for twisted elliptic endoscopic data.

Proof We say a twisted endoscopic datum $(\tilde{G}', \tilde{s}, \tilde{\xi})$ for $(\tilde{G}, \theta, \omega_{\tilde{G}})$ corresponds to a twisted endoscopic datum (G', s, ξ) for (G, θ, ω_G) if $\mathbf{p}(\tilde{s}) = s$ and they satisfy the following diagram

$$\begin{array}{ccc} L\tilde{G}' & \xrightarrow{\tilde{\xi}} & L\tilde{G} \\ \downarrow \mathbf{p} & & \downarrow \mathbf{p} \\ LG' & \xrightarrow{\xi} & LG. \end{array}$$

In [49] we have shown this gives a one to one correspondence between isomorphism classes of twisted endoscopic data when F is local. In fact, our proof also works in the global case except that we need to use a global lifting result of Labesse for homomorphisms from the global Weil group to L -groups (see the paragraph after Theorem 2.9). Moreover, it is easy to see $\tilde{\xi}(Z(\tilde{G}')^{\Gamma_F})^0 \subseteq Z(\tilde{G})$ if and only if $\xi(Z(\widehat{G}')^{\Gamma_F})^0 \subseteq Z(\widehat{G})$, so $(\tilde{G}', \tilde{s}, \tilde{\xi})$ is elliptic if and only if (G', s, ξ) is elliptic. \square

Remark 2.8 The most important case for us is when $\omega_G = 1$. Then Proposition 2.7 shows there is a one to one correspondence between the θ -twisted endoscopic data $\mathcal{E}(G^\theta)$ and the (θ, ω) -twisted endoscopic data

$$\bigsqcup_{\omega} \mathcal{E}(\tilde{G}^\theta, \omega),$$

where ω runs through quasicharacters of $\tilde{G}(F)/G(F)$ if F is local and quasicharacters of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$ if F is global. The same is true for elliptic endoscopic data.

As our most important examples, let us consider the general symplectic groups and connected general even orthogonal groups with trivial automorphisms, and we have the following table (cf. [7], 1.2 and [26], 2.1): let $n = n_1 + n_2$.

G	$\mathcal{E}_{\text{ell}}(G)$	\tilde{G}	$\mathcal{E}_{\text{ell}}(\tilde{G}, \omega)$
$Sp(2n)$	$Sp(2n_1) \times SO(2n_2, \eta)$	$GSp(2n)$	$G(Sp(2n_1) \times SO(2n_2, \eta)) \quad \omega = \eta \circ \lambda$
$SO(2n)$	$SO(2n_1, \eta) \times SO(2n_2, \eta)$	$GSO(2n)$	$G(SO(2n_1, \eta) \times SO(2n_2, \eta)) \quad \omega = \eta \circ \lambda$
$SO(2n, \eta)$	$SO(2n_1, \eta_1) \times SO(2n_2, \eta_1\eta)$	$GSO(2n, \eta')$	$G(SO(2n_1, \eta) \times SO(2n_2, \eta\eta')) \omega = \eta \circ \lambda$

Note in the cases above the isomorphism classes of twisted endoscopic data are completely determined by the twisted endoscopic groups. But that is not the case in general. For example, in the case of connected general even orthogonal groups, if we let θ_0 be the outer automorphism induced by the conjugate action of the full orthogonal group, then the isomorphism classes of θ_0 -twisted elliptic endoscopic data of $SO(2n, \eta')$ are classified by θ_0 -twisted elliptic endoscopic groups $Sp(2n_1) \times Sp(2n_2)$ ($n = n_1 + n_2 + 1$) with a pair of quadratic characters $(\eta, \eta\eta')$. Correspondingly, the isomorphism classes of (θ_0, ω) -twisted elliptic endoscopic data of $GSO(2n, \eta')$ are classified by (θ_0, ω) -twisted elliptic endoscopic groups $G(Sp(2n_1) \times Sp(2n_2))$ ($n = n_1 + n_2 + 1$) with a pair of quadratic characters $(\eta, \eta\eta')$ and $\omega = \eta \circ \lambda$.

2.2 Langlands parameters

Suppose G is a quasisplit connected reductive group over F , a Langlands parameter of G is a \widehat{G} -conjugacy class of admissible homomorphisms from the Langlands group L_F to the L -group of G (cf. [8]). We denote the set of Langlands parameters of G by $\Phi(G)$, and for any $\phi \in \Phi(G)$ we denote a representative by $\underline{\phi} : L_F \rightarrow {}^L G$. If F is local, the Langlands group is defined as follows,

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times SL(2, \mathbb{C}) & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

If F is global, the existence of Langlands group is still conjectural. Let \widetilde{G} , G , D and λ be defined as in Sect. 2.1.1. The following theorem shows the relation for local Langlands parameters between G and \widetilde{G} .

Theorem 2.9 (Labesse) *Suppose F is a local, every Langlands parameter ϕ of G can be lifted to a Langlands parameter $\tilde{\phi}$ of \widetilde{G} in the sense that the following diagram commutes*

$$\begin{array}{ccc} L_F & \xrightarrow{\tilde{\phi}} & {}^L \widetilde{G} \\ & \searrow \underline{\phi} & \downarrow p \\ & & {}^L G. \end{array}$$

In fact the global analogue of this theorem is also true if one uses the the global Weil group W_F instead of the global Langlands group L_F . Both the local and global cases are proved in ([19], Theorem 8.1). In the global case, since we do not have the global Langlands group yet, this kind of lifting theorem for global Langlands parameters is unavailable. However let us assume the existence of global Langlands group and also the same kind of lifting theorem at this moment, so that we can investigate the relation for both local and global Langlands parameters between G and \widetilde{G} in a uniform way. Moreover, the consequences of this investigation will serve as motivations for the later definitions (see Sect. 3) that complement the lack of global Langlands group.

To further simplify our discussion, we are going to assume

$$\mathrm{Ker}^1(W_F, Z(\widehat{\widetilde{G}})) = \mathrm{Ker}^1(W_F, Z(\widehat{G})) = \mathrm{Ker}^1(W_F, \widehat{D}) = 1 \quad (2.9)$$

when F is global. This assumption allows us to treat the local and global cases at the same time, and it also suffices for our purpose in view of Corollary 2.6. Let Σ be a finite abelian group of automorphisms of \widetilde{G} preserving an F -splitting of \widetilde{G} , and we assume λ is Σ -invariant, so Σ also acts on G . We denote the dual automorphisms by $\widehat{\Sigma}$ and form the semidirect products $\widehat{\widetilde{G}}^{\Sigma} := \widehat{\widetilde{G}} \rtimes \widehat{\Sigma}$ and $\widehat{G}^{\Sigma} := \widehat{G} \rtimes \widehat{\Sigma}$. Let Σ act on $\Phi(\widetilde{G})$ and $\Phi(G)$ through the action of $\widehat{\Sigma}$ on $\widehat{\widetilde{G}}$ and \widehat{G} respectively. For $\theta \in \Sigma$, we denote by $\Phi(G^{\theta})$ the set of $\phi \in \Phi(G)$ such that $\phi^{\theta} = \phi$.

Suppose F is either local or global, for any $\phi \in \Phi(G)$ we choose a representative $\underline{\phi}$. Let L_F act on \widehat{D} , \widehat{G}^Σ , and $\widehat{\widehat{G}}^\Sigma$ by conjugation through $\underline{\phi}$. We denote the corresponding group cohomology by $H_{\underline{\phi}}^*(L_F, \cdot)$. Note $H_{\underline{\phi}}^0(L_F, \widehat{D}) = \widehat{D}^\Gamma$, $H_{\underline{\phi}}^1(L_F, \widehat{D}) = H^1(W_F, \widehat{D})$, and

$$\begin{aligned} S_{\underline{\phi}}^\Sigma &:= \text{Cent}(\text{Im } \underline{\phi}, \widehat{G}^\Sigma) = H_{\underline{\phi}}^0(L_F, \widehat{G}^\Sigma), \\ S_{\underline{\phi}}^\Sigma &:= \text{Cent}(\text{Im } \underline{\phi}, \widehat{\widehat{G}}^\Sigma) = H_{\underline{\phi}}^0(L_F, \widehat{\widehat{G}}^\Sigma). \end{aligned}$$

The short exact sequence

$$1 \longrightarrow \widehat{D} \longrightarrow \widehat{\widehat{G}}^\Sigma \longrightarrow \widehat{G}^\Sigma \longrightarrow 1$$

induces a long exact sequence

$$1 \longrightarrow \widehat{D}^\Gamma \longrightarrow S_{\underline{\phi}}^\Sigma \longrightarrow S_{\underline{\phi}}^\Sigma \xrightarrow{\delta} H^1(W_F, \widehat{D}),$$

and hence

$$1 \longrightarrow S_{\underline{\phi}}^\Sigma / \widehat{D}^\Gamma \xrightarrow{\iota} S_{\underline{\phi}}^\Sigma \xrightarrow{\delta} H^1(W_F, \widehat{D}). \quad (2.10)$$

To describe δ , we can identify

$$S_{\underline{\phi}}^\Sigma = \left\{ \tilde{s} \in \widehat{\widehat{G}}^\Sigma : \tilde{s} \tilde{\phi}(u) \tilde{s}^{-1} \tilde{\phi}(u)^{-1} \in \widehat{D}, \text{ for all } u \in L_F \right\} / \widehat{D}.$$

Then $\delta(s) : u \mapsto \tilde{s} \tilde{\phi}(u) \tilde{s}^{-1} \tilde{\phi}(u)^{-1}$, where \tilde{s} is a preimage of s in $\widehat{\widehat{G}}^\Sigma$, and $\delta(s)$ factors through W_F . About (2.10) we have the following lemma.

Lemma 2.10 *The image of δ consists of $\mathbf{a} \in H^1(W_F, \widehat{D})$ such that*

$$\tilde{\phi}^\theta = \tilde{\phi} \otimes \mathbf{a}$$

for some $\theta \in \Sigma$, and in particular it is finite.

Proof We have shown the lemma when F is local in [49]. In particular the same argument applies to the global case except for the finiteness of $\text{Im } \delta$. When F is global, we need to use the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & S_{\phi}^{\Sigma} / \widehat{D}^{\Gamma} & \longrightarrow & S_{\phi}^{\Sigma} & \xrightarrow{\delta} & H^1(W_F, \widehat{D}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & S_{\phi_v}^{\Sigma} / \widehat{D}_v^{\Gamma_v} & \longrightarrow & S_{\phi_v}^{\Sigma} & \xrightarrow{\delta_v} & H^1(W_{F_v}, \widehat{D}_v).
 \end{array}$$

Since S_{ϕ}^0 is mapped into $S_{\phi_v}^0$ and δ_v is trivial on $(S_{\phi_v}^{\Sigma} / \widehat{D}_v^{\Gamma_v})^0 = S_{\phi_v}^0$, then $\delta = \prod_v \delta_v$ is trivial on S_{ϕ}^0 . This implies the image of δ is finite. \square

Take

$$1 \longrightarrow \widehat{D} \longrightarrow Z(\widehat{\widehat{G}}) \longrightarrow Z(\widehat{G}) \longrightarrow 1$$

and it induces

$$1 \longrightarrow \widehat{D}^{\Gamma} \longrightarrow Z(\widehat{\widehat{G}})^{\Gamma} \longrightarrow Z(\widehat{G})^{\Gamma} \xrightarrow{\delta} H^1(W_F, \widehat{D}) \longrightarrow H^1(W_F, Z(\widehat{\widehat{G}})).$$

So $\text{Ker} \delta|_{Z(\widehat{\widehat{G}})^{\Gamma}} = Z(\widehat{\widehat{G}})^{\Gamma} / \widehat{D}^{\Gamma}$. Let $\bar{H}^1(W_F, \widehat{D}) := H^1(W_F, \widehat{D}) / \delta(Z(\widehat{\widehat{G}})^{\Gamma})$. Taking the quotient of (2.10) by $Z(\widehat{\widehat{G}})^{\Gamma}$, we get

$$1 \longrightarrow \bar{S}_{\phi}^{\Sigma} \xrightarrow{\iota} \bar{S}_{\phi}^{\Sigma} \xrightarrow{\bar{\delta}} \bar{H}^1(W_F, \widehat{D}), \quad (2.11)$$

where $\bar{S}_{\phi}^{\Sigma} = S_{\phi}^{\Sigma} / Z(\widehat{\widehat{G}})^{\Gamma}$ and $\bar{S}_{\phi}^{\Sigma} = S_{\phi}^{\Sigma} / Z(\widehat{G})^{\Gamma}$. Since $\text{Im } \delta$ is finite, we have $\bar{S}_{\phi}^0 = \bar{S}_{\phi}^0$. After taking the quotient of (2.11) by the identity component, we get

$$1 \longrightarrow S_{\phi}^{\Sigma} \xrightarrow{\iota} S_{\phi}^{\Sigma} \xrightarrow{\bar{\delta}} \bar{H}^1(W_F, \widehat{D}), \quad (2.12)$$

where $S_{\phi}^{\Sigma} = \bar{S}_{\phi}^{\Sigma} / \bar{S}_{\phi}^0$ and $S_{\phi}^{\Sigma} = \bar{S}_{\phi}^{\Sigma} / \bar{S}_{\phi}^0$. There are natural maps from S_{ϕ}^{Σ} , \bar{S}_{ϕ}^{Σ} , and S_{ϕ}^{Σ} to $\widehat{\Sigma}$, and for $\theta \in \Sigma$, we denote the preimages of $\widehat{\theta} \in \widehat{\Sigma}$ by S_{ϕ}^{θ} , \bar{S}_{ϕ}^{θ} and S_{ϕ}^{θ} respectively.

By the Langlands correspondence for tori and the assumption $\text{Ker}^1(W_F, \widehat{D}) = 1$, we can identify $H^1(W_F, \widehat{D})$ with $\text{Hom}(D(F), \mathbb{C}^{\times})$ if F is local or $\text{Hom}(D(\mathbb{A}_F) / D(F), \mathbb{C}^{\times})$ if F is global. Then we can compose with λ to get a homomorphism from $H^1(W_F, \widehat{D})$ to $\text{Hom}(\widetilde{G}(F) / G(F), \mathbb{C}^{\times})$ if F is local or $\text{Hom}(\widetilde{G}(\mathbb{A}_F) / \widetilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^{\times})$ if F is global. Since $\delta(Z(\widehat{\widehat{G}})^{\Gamma})$ is trivial in $H^1(W_F, Z(\widehat{\widehat{G}}))$, it induces the trivial character on $\widetilde{G}(F)$ if F is local or $\widetilde{G}(\mathbb{A}_F)$ if F is global. So we have a homomorphism

$$r : \bar{H}^1(W_F, \widehat{D}) \rightarrow \text{Hom}(\widetilde{G}(F) / G(F), \mathbb{C}^{\times})$$

if F is local, and

$$r : \bar{H}^1(W_F, \widehat{D}) \rightarrow \text{Hom}(\widetilde{G}(\mathbb{A}_F) / \widetilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^{\times})$$

if F is global. In the local case, r is an isomorphism due to the fact that (2.6) is an isomorphism. For the global case, we have the following lemma.

Lemma 2.11 *If F is global and \tilde{G} is of type (2.2), then r is an isomorphism.*

Proof First we consider the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^\times) & \xleftarrow{\lambda^*} & \mathrm{Hom}(D(\mathbb{A}_F)/D(F), \mathbb{C}^\times) & & (2.13) \\ \uparrow & & \uparrow \simeq & & \\ H^1(W_F, Z(\widehat{\tilde{G}})) & \xleftarrow{\quad} & H^1(W_F, \widehat{D}) & \xleftarrow{\delta} & \pi_0(Z(\widehat{G})^\Gamma). \end{array}$$

By Corollary 2.4, we have $\mathrm{Im} \lambda^* = \mathrm{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times)$, and hence r is surjective. On the other hand, the kernel of

$$H^1(W_F, Z(\widehat{\tilde{G}})) \longrightarrow \mathrm{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^\times)$$

is $\mathrm{Ker}^1(W_F, Z(\widehat{\tilde{G}})) = 1$ by Corollary 2.6. Therefore r is also an inclusion. \square

Let us denote the composition $r \circ \bar{\delta}$ by α , then we can rewrite (2.12) as

$$1 \longrightarrow \mathcal{S}_{\underline{\phi}}^\Sigma \xrightarrow{\iota} \mathcal{S}_{\underline{\phi}}^\Sigma \xrightarrow{\alpha} \mathrm{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^\times) \quad (2.14)$$

if F is local, and

$$1 \longrightarrow \mathcal{S}_{\underline{\phi}}^\Sigma \xrightarrow{\iota} \mathcal{S}_{\underline{\phi}}^\Sigma \xrightarrow{\alpha} \mathrm{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times) \quad (2.15)$$

if F is global. Note in the global case, we only know it is exact when \tilde{G} is of type (2.2) according to the previous lemma. Sometimes, we want to distinguish the map α for different groups, so we will also write $\alpha^G = \alpha$.

Next we want to discuss the relation between lifting Langlands parameters (see Theorem 2.9) and lifting twisted endoscopic groups (see Proposition 2.7). Suppose F is local or global and $\phi \in \Phi(G)$. For any semisimple element $s \in \tilde{\mathcal{S}}_\phi^\theta$, let $\widehat{G}' := \mathrm{Cent}(s, \widehat{G})^0$ and it can be equipped with a Galois action given by $\underline{\phi}$. This determines a quasisplit connected reductive group G' , and $\underline{\phi}$ will factor through ${}^L G'$ for some θ -twisted endoscopic datum (G', s, ξ) of G , and hence we get a parameter $\phi' \in \Phi(G')$. In this way, we call (G', ϕ') corresponds to (ϕ, s) , and we denote it by $(G', \phi') \rightarrow (\phi, s)$.

By Proposition 2.7, (G', s, ξ) can be lifted to a (θ, ω) -twisted endoscopic datum $(\tilde{G}', \tilde{s}, \tilde{\xi})$ of \tilde{G} for some character ω of $\tilde{G}(F)/G(F)$ if F is local or $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$ if F is global. Then by Theorem 2.9 and the global assumption that we made after, we can have a lift $\tilde{\phi}'$ of ϕ' in $\Phi(\tilde{G}')$. All of these can be summarized in the diagram below

$$\begin{array}{ccccc} L_F & \xrightarrow{\tilde{\phi}'} & L\tilde{G}' & \xrightarrow{\tilde{\xi}} & L\tilde{G} \\ & \searrow \phi' & \downarrow & & \downarrow \\ & & L G' & \xrightarrow{\xi} & L G. \end{array}$$

Then we have the following lemma.

Lemma 2.12 $\alpha(s) = \omega$.

Proof It has been shown for the local case in [49], and the proof for the global case is the same. \square

Remark 2.13 From this lemma we see the character ω associated with the twisted endoscopic datum \tilde{G}' only depends on the image of s in \mathcal{S}_ϕ^θ . In the global case, lifting Langlands parameter is not available due to the lack of the global Langlands group. However one can always lift twisted endoscopic groups in both local and global cases, so this lemma is behind the idea of our later definition of the map α [see (2.15)] in the global case.

2.3 Representations

Let us assume F is a local field, and G, \tilde{G}, D, λ are defined as in Sect. 2.1.1. In this section, we would like to recall some results about the restriction map $\Pi(\tilde{G}(F)) \rightarrow \Pi(G(F))$ from ([49], Section 6.1).

Lemma 2.14 *If $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(F)$, then the restriction of $\tilde{\pi}$ to $G(F)$ is a direct sum of finitely many irreducible admissible representations.*

Theorem 2.15 (Adler and Prasad [1]) *Suppose \tilde{G} is a quasisplit general symplectic group or connected general even orthogonal group, and $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(F)$, then the restriction of $\tilde{\pi}$ to $G(F)$ is multiplicity free.*

Remark 2.16 This theorem can be easily extended to the groups \tilde{G} of type (2.2). To do so, we can first extend a representation of $\tilde{G}(F)$ to $\tilde{\tilde{G}}(F)$ [see (2.4)], and then restrict it to $G(F)$.

Lemma 2.17 *If π is an irreducible admissible representation of $G(F)$, then there exists a unique irreducible admissible representation $\tilde{\pi}$ of $\tilde{G}(F)$ up to twisting by $\text{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^\times)$, such that it contains π in its restriction to $G(F)$.*

If π is an irreducible admissible representation of $G(F)$, let us denote

$$\tilde{G}(\pi) = \{g \in \tilde{G}(F) : \pi^g \cong \pi\}.$$

If $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(F)$, let us denote

$$X(\tilde{\pi}) = \{\omega \in (\tilde{G}(F)/Z_{\tilde{G}}(F)G(F))^* : \tilde{\pi} \otimes \omega \cong \tilde{\pi}\}.$$

Proposition 2.18 Suppose \tilde{G} is of type (2.2), $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(F)$ and π is contained in its restriction to $G(F)$, then for $\omega \in (\tilde{G}(F)/Z_{\tilde{G}}(F)G(F))^*$, ω is in $X(\tilde{\pi})$ if and only if ω is trivial on $\tilde{G}(\pi)$. Moreover, the restriction of $\tilde{\pi}$ contains $|X(\tilde{\pi})|$ irreducible admissible representations of $G(F)$.

Lemma 2.19 Suppose $\tilde{\pi}$ is an irreducible admissible unitary representation of $\tilde{G}(F)$, then $\tilde{\pi}$ is an essentially discrete series representation of $\tilde{G}(F)$ if and only if its restriction to $G(F)$ is an essentially discrete series representation. The same is true of the tempered representations.

2.4 Langlands–Shelstad–Kottwitz transfer

Let F be a local field of characteristic zero and G be a quasisplit connected reductive group over F . Suppose θ is an automorphism of G preserving an F -splitting and ω_G is a quasicharacter of $G(F)$. We choose a quasicharacter χ on a closed subgroup Z_F of $Z_G(F)$, and define $\mathcal{H}(G, \chi)$ to be the space χ^{-1} -equivariant smooth compactly supported functions over $G(F)$ (i.e., equivariant Hecke algebra of G). Let δ be a strongly θ -regular θ -semisimple element of $G(F)$ such that ω_G is trivial on the θ -twisted centralizer group $G_{\delta}^{\theta}(F)$ of δ . We choose Haar measures on $G(F)$ and $G_{\delta}^{\theta}(F)$, and they induce a $G(F)$ -invariant measure on $G_{\delta}^{\theta}(F) \backslash G(F)$. Then we can form the (θ, ω_G) -twisted orbital integral of $f \in \mathcal{H}(G, \chi)$ over δ as

$$O_G^{\theta, \omega_G}(f, \delta) := \int_{G_{\delta}^{\theta}(F) \backslash G(F)} \omega_G(g) f(g^{-1} \delta \theta(g)) dg.$$

We also form the (θ, ω_G) -twisted stable orbital integral over δ as

$$SO_G^{\theta, \omega_G}(f, \delta) := \sum_{\{\delta'\}_{G(F)} \sim_{st} \{\delta\}_{G(F)}} O_G^{\theta, \omega_G}(f, \delta'),$$

where the sum is over θ -twisted conjugacy classes $\{\delta'\}_{G(F)}$ in the θ -twisted stable conjugacy class of δ (i.e., $\delta' = g^{-1} \delta \theta(g)$ for some $g \in G(F)$), and the Haar measure on $G_{\delta'}^{\theta}(F)$ is translated from that on $G_{\delta}^{\theta}(F)$ by conjugation. Let $\mathcal{I}(G^{\theta, \omega_G}, \chi)$ ($SI(G^{\theta, \omega_G}, \chi)$) be the space of (θ, ω_G) -twisted (stable) orbital integrals of $\mathcal{H}(G, \chi)$ over the set $G_{reg}^{\theta}(F)$ of strongly θ -regular θ -semisimple elements of $G(F)$, then by definition we have projections

$$\mathcal{H}(G, \chi) \longrightarrow \mathcal{I}(G^{\theta, \omega_G}, \chi) \longrightarrow SI(G^{\theta, \omega_G}, \chi).$$

Suppose π is an irreducible admissible representation of $G(F)$ and χ_{π} is the central character of π . Let $\chi = \chi_{\pi}|_{Z_F}$. Suppose $\pi^{\theta} \cong \pi \otimes \omega_G$, let $A_{\pi}(\theta, \omega_G)$ be the intertwining operator between $\pi \otimes \omega_G$ and π^{θ} (this is uniquely determined up to a scalar), we then define the (θ, ω_G) -twisted character of π to be the distribution

$$f_{G^\theta}(\pi, \omega_G) := \text{trace} \left(\int_{Z_F \backslash G(F)} f(g) \pi(g) dg \circ A_\pi(\theta, \omega_G) \right), \quad (2.16)$$

for $f \in \mathcal{H}(G, \chi)$. By results of Harish-Chandra [11, 13] in the non-twisted case, Bouaziz [9], Clozel [10] and Lemaire [21] in the twisted case, there exists a locally integrable function $\Theta_\pi^{G^\theta, \omega_G}$ on $G(F)$ such that for $x \in G_{reg}^\theta(F)$, $g \in G(F)$

$$\Theta_\pi^{G^\theta, \omega_G}(g^{-1}x\theta(g)) = \omega_G(g)\Theta_\pi^{G^\theta, \omega_G}(x),$$

and

$$f_{G^\theta}(\pi, \omega_G) = \int_{Z_F \backslash G(F)} f(g) \Theta_\pi^{G^\theta, \omega_G}(g) dg.$$

By the twisted Weyl integration formula, one can show this character defines a linear functional on $\mathcal{I}(G^{\theta, \omega_G}, \chi)$. A linear functional on $\mathcal{I}(G^{\theta, \omega_G}, \chi)$ is called **stable** if it factors through $\mathcal{SI}(G^{\theta, \omega_G}, \chi)$.

For a (θ, ω_G) -twisted endoscopic datum (H, s, ξ) of G , there is a map defined over F from the semisimple conjugacy classes of $H(\bar{F})$ to the θ -twisted conjugacy classes of θ -semisimple elements in $G(\bar{F})$. We call a strongly regular semisimple element $\gamma \in H(\bar{F})$ is strongly G -regular if its associated $H(\bar{F})$ -conjugacy class maps to a θ -twisted $G(\bar{F})$ -conjugacy class of strongly θ -regular θ -semisimple elements in $G(\bar{F})$. We denote the set of strongly G -regular semisimple elements of $H(F)$ by $H_{G-reg}(F)$. The transfer factor defined in [18] is a function

$$\Delta_{G,H}(\cdot, \cdot) : H_{G-reg}(F) \times G_{reg}^\theta(F) \rightarrow \mathbb{C},$$

which is nonzero only when $\gamma \in H_{G-reg}(F)$ is a norm of $\delta \in G_{reg}^\theta(F)$, i.e., the $H(\bar{F})$ -conjugacy class of γ maps to the θ -twisted $G(\bar{F})$ -conjugacy class of δ . Note if $\delta \in G_{reg}^\theta(F)$ has a norm $\gamma \in H_{G-reg}(F)$, then ω_G is trivial on $G_\delta^\theta(F)$ (see Lemma 4.4.C, [18]). In this paper, we always normalize the transfer factor with respect to some fixed θ -stable Whittaker datum (B, Λ) , and we also assume the Haar measure is preserved for any admissible embedding $T_H \xrightarrow{\sim} T_\theta$, where T_H is a maximal torus of H , T is a θ -stable maximal torus of G and $T_\theta = T/(\theta - 1)T$.

There is a canonical inclusion $(Z_G)_\theta \hookrightarrow Z_H$. Let us denote the image of Z_F in $Z_H(F)$ by Z'_F , then one can associate a quasicharacter χ' of Z'_F , depending only on χ and the twisted endoscopic embedding ξ . The Langlands-Kottwitz-Shelstad transfer map (or twisted endoscopic transfer) is a correspondence from $f \in \mathcal{H}(G, \chi)$ to $f^H \in \mathcal{H}(H, \chi')$ such that

$$SO_H(f^H, \gamma) = \sum_{\{\delta'\}_{G(F)}^\theta \sim_{st} \{\delta\}_{G(F)}^\theta} \Delta_{G,H}(\gamma, \delta') O_G^{\theta, \omega_G}(f, \delta') \quad (2.17)$$

where the sum is over θ -twisted conjugacy classes $\{\delta'\}_{G(F)}^\theta$ in the θ -twisted stable conjugacy class of δ . In particular, it descends to a surjection

$$\mathcal{I}(G^{\theta, \omega_G}, \chi) \longrightarrow \mathcal{SI}(H, \chi')^{\text{Out}_G(H)},$$

where the action of $\text{Out}_G(H)$ on $\mathcal{SI}(H, \chi')$ is independent of the choice of F -splitting for H (see Sect. 2.1.3). The existence of such a transfer has been a long standing problem. In the real case, it is now a theorem of Shelstad [40]. In the nonarchimedean case, the main obstacle is the Fundamental Lemma, which has been finally resolved by Ngo [34]. And the proof of the transfer conjecture in this case was completed by Waldspurger [48].

Now let us assume G , \tilde{G} , D and λ are defined as in Sect. 2.1.1. Let θ be an automorphism of \tilde{G} preserving an F -splitting and λ is θ -invariant. Let \tilde{Z}_F be a closed subgroup of $Z_{\tilde{G}}(F)$ such that $\tilde{Z}_F \rightarrow (Z_{\tilde{G}})_\theta(F)$ is injective and $D(F)/\lambda(\tilde{Z}_F)$ is finite (this is possible because we assume λ is θ -invariant). Let $Z_F = \tilde{Z}_F \cap G(F)$. We choose Haar measures on \tilde{Z}_F and Z_F such that the measure on $Z_F \backslash G(F)$ is the restriction of that on $\tilde{Z}_F \backslash \tilde{G}(F)$. Let $\tilde{\chi}$ be a quasicharacter of \tilde{Z}_F and we denote its restriction to Z_F by χ . For every $f \in \mathcal{H}(G, \chi)$, it can be extended to $\tilde{G}(F)$ through \tilde{Z}_F by $\tilde{\chi}$, and the extension lies in $\mathcal{H}(\tilde{G}, \tilde{\chi})$, supported on $\tilde{Z}_F G(F)$. Hence we get an inclusion map

$$\mathcal{H}(G, \chi) \hookrightarrow \mathcal{H}(\tilde{G}, \tilde{\chi}), \quad (2.18)$$

$$f \longmapsto \tilde{f}$$

and we can identify $\mathcal{H}(G, \chi)$ with its image. Let $\omega_{\tilde{G}}$ be a quasicharacter of $\tilde{G}(F)$ and $\omega_G = \omega_{\tilde{G}}|_G$. For any strongly θ -regular θ -semisimple element δ of $G(F)$ such that ω_G is trivial on the $G_\delta^\theta(F)$, we fix the Haar measure on $\tilde{G}_\delta^\theta(F) \backslash \tilde{G}(F)$, which determines the Haar measure on $G_\delta^\theta(F) \backslash G(F)$ by restriction. Then for $f \in \mathcal{H}(G, \chi)$, and $\tilde{f} \in \mathcal{H}(\tilde{G}, \tilde{\chi})$ being its extension, we have

$$SO_{\tilde{G}}(\tilde{f}, \delta) = SO_G(f, \delta),$$

and

$$O_{\tilde{G}}^{\theta, \omega_{\tilde{G}}}(\tilde{f}, \delta) = \sum_{\{\delta'\}_{G(F)}^\theta \sim_{\tilde{G}(F)} \{\delta\}_{G(F)}^\theta} O_G^{\theta, \omega_G}(f, \delta') \omega_{\tilde{G}}(g)$$

where the sum is over θ -twisted $G(F)$ -conjugacy classes $\{\delta'\}_{G(F)}^\theta$ in the θ -twisted $\tilde{G}(F)$ -conjugacy classes $\{\delta\}_{\tilde{G}(F)}^\theta$ with $\delta' = g^{-1}\delta g$ for $g \in \tilde{G}(F)$, and the Haar measure on $G_\delta^\theta(F)$ is translated from that on G_δ^θ by conjugation. Because $\tilde{Z}_F G(F)$ is θ -conjugate invariant under $\tilde{G}(F)$, the map (2.18) induces a map from $\mathcal{I}(G^{\theta, \omega_G}, \chi)$ to $\mathcal{I}(\tilde{G}^{\theta, \omega_{\tilde{G}}}, \tilde{\chi})$. Moreover $\tilde{Z}_F G(F)$ is conjugate invariant under $\tilde{G}(\bar{F})$, so it also induces a map from $\mathcal{SI}(G, \chi)$ to $\mathcal{SI}(\tilde{G}, \tilde{\chi})$.

Suppose $\tilde{G}' \in \mathcal{E}(\tilde{G}^\theta, \omega_{\tilde{G}})$ and $G' \in \mathcal{E}(G^\theta, \omega_G)$ correspond to each other according to Proposition 2.7. The natural inclusion $(Z_{\tilde{G}})_\theta \rightarrow Z_{\tilde{G}'}$ induces an inclusion on \tilde{Z}_F . So we can define $\tilde{Z}'_F \subseteq Z_{\tilde{G}'}(F)$ to be the image of \tilde{Z}_F and $Z'_F = \tilde{Z}'_F \cap Z_{G'}(F)$. The twisted endoscopic transfer sends $\mathcal{H}(\tilde{G}, \tilde{\chi})$ to $\mathcal{H}(\tilde{G}', \tilde{\chi}')$, where χ' is a quasicharacter of \tilde{Z}'_F , depending only on $\tilde{\chi}$ and the twisted endoscopic embedding. Let χ' be the restriction of $\tilde{\chi}'$ to Z'_F . Then we have

$$\mathcal{H}(G', \chi') \hookrightarrow \mathcal{H}(\tilde{G}', \tilde{\chi}'),$$

$$f \longmapsto \tilde{f}$$

The following lemma shows these inclusion maps are compatible with the twisted endoscopic transfers.

Lemma 2.20 ([49], Lemma 3.8) *Suppose $f \in \mathcal{H}(G, \chi)$, then the $(\theta, \omega_{\tilde{G}})$ -twisted endoscopic transfer of the extension \tilde{f} of f is equal to the extension of (θ, ω_G) -twisted endoscopic transfer $f^{G'}$ of f as elements in $S\mathcal{I}(\tilde{G}', \tilde{\chi}')$, i.e.*

$$\tilde{f}^{\tilde{G}'} = \widetilde{(f^{G'})} \quad (2.19)$$

Remark 2.21 The inclusion map (2.18) of Hecke algebras induces a restriction map of distributions in the opposite direction. Moreover the restriction of an invariant distribution is again invariant, and the restriction of a stable distribution is again stable. In particular, the restriction of the character of a representation is compatible with the restriction of the representation in the usual sense.

Corollary 2.22 *Suppose $S^{\tilde{G}'}(\cdot)$ is a stable distribution on \tilde{G}' , then the restriction of the pull-back of $S^{\tilde{G}'}(\cdot)$ is equal to the pull-back of the restriction of $S^{\tilde{G}'}(\cdot)$, i.e.*

$$S^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}) = S^{\tilde{G}'}(\widetilde{(f^{G'})})$$

Proof One just need to substitute (2.19) into $S^{\tilde{G}'}(\cdot)$. □

3 Arthur's classification theory: tempered case

In this section we will review Arthur's classification theory for the tempered representations of quasi-split symplectic groups and special even orthogonal groups (cf. [7]). So throughout this section, G will always be a quasisplit symplectic group or special even orthogonal group over F (if not specified). We fix an outer automorphism θ_0 of G , and a nontrivial automorphism θ_N of $GL(N)$, so that they preserve an F -splitting respectively. When G is symplectic, θ_0 is trivial. When G is special even orthogonal, we require θ_0 to be the unique outer automorphism induced from the conjugation of the full orthogonal group. We denote $\Sigma_0 = \langle \theta_0 \rangle$. When F is local, Σ_0 acts on $\Pi(G(F))$

and we denote the set of Σ_0 -orbits in $\Pi(G(F))$ by $\bar{\Pi}(G(F))$. We denote by $\bar{\mathcal{H}}(G, \chi)$ the subspace of Σ_0 -invariant functions in $\mathcal{H}(G, \chi)$, and we abbreviate $\mathcal{H}(GL(N))$ to $\mathcal{H}(N)$. We also denote the corresponding space of (stable) twisted orbital integrals by $\bar{\mathcal{I}}(G^\theta, \omega_G)$ ($\bar{\mathcal{S}}\mathcal{I}(G^\theta, \omega_G)$) for $\theta \in \Sigma_0$ and $\omega_G \in \text{Hom}(G(F), \mathbb{C}^\times)$.

3.1 Substitute Langlands parameter

First let F be a global field, we define the sets of substitute global generic (or tempered) Langlands parameters as follows,

$$\begin{aligned}\Phi_{sim}(N) &:= \{\text{isomorphism classes of irreducible unitary cuspidal automorphic} \\ &\quad \text{representations of } GL(N)\}, \\ \Phi_{sim}(N^{\theta_N}) &:= \{\phi \in \Phi_{sim}(N) : \phi = \phi^\vee\}, \\ \Phi(N^{\theta_N}) &:= \left\{ \phi = l_1 \phi_1 \boxplus \cdots \boxplus l_r \phi_r : \phi = \phi^\vee, \phi_i \in \Phi_{sim}(N_i), \text{ with } \sum_{i=1}^r l_i N_i = N \right\}.\end{aligned}$$

Here ϕ^\vee denotes the dual (or contragredient) of ϕ if $\phi \in \Phi_{sim}(N)$, and

$$\phi^\vee := l_1 \phi_1^\vee \boxplus \cdots \boxplus l_r \phi_r^\vee$$

if $\phi \in \Phi(N^{\theta_N})$. Note that $\Phi(N^{\theta_N})$ is just a set of formal sums of irreducible unitary cuspidal automorphic representations, and for every parameter $\phi \in \Phi(N^{\theta_N})$ we can assign a family of semisimple conjugacy classes in $GL(N, \mathbb{C})$ by

$$c(\phi_v) := \underbrace{c(\phi_{1,v}) \oplus \cdots \oplus c(\phi_{1,v})}_{l_1} \oplus \cdots \oplus \underbrace{c(\phi_{r,v}) \oplus \cdots \oplus c(\phi_{r,v})}_{l_r}$$

for unramified places v of ϕ , where $c(\phi_{i,v})$ is the Satake parameter of the local component $\phi_{i,v}$. Inside $\Phi_{sim}(N^{\theta_N})$ there are two types of parameters, we call ϕ is of **orthogonal type** if the symmetric square L -function $L(s, \phi, S^2)$ has a pole at $s = 1$; we call ϕ is of **symplectic type** if the skew-symmetric square L -function $L(s, \phi, \wedge^2)$ has a pole at $s = 1$. In fact every $\phi \in \Phi_{sim}(N^{\theta_N})$ will always be either one of these two types due to the fact that the Rankin–Selberg L -function

$$L(s, \phi \otimes \phi) = L(s, \phi, S^2) L(s, \phi, \wedge^2)$$

has a simple pole at $s = 1$. Moreover when N is odd, ϕ is always of orthogonal type. The following theorem proved in ([7], Theorem 1.4.1 and Theorem 1.5.3) shows how automorphic representations of $GL(N)$ are related to that of orthogonal groups and symplectic groups. If π is an automorphic representation of G , we denote by $c(\pi) = \{c(\pi_v)\}$ the family of Satake parameters of π_v at the unramified places.

Theorem 3.1 *Suppose $\phi \in \Phi_{sim}(N^{\theta_N})$, then there is a unique class of elliptic endoscopic data $(G_\phi, s_\phi, \xi_\phi)$ in $\mathcal{E}_{ell}(N^{\theta_N})$ such that $c(\phi_v) = \xi_\phi(c(\pi_v))$ for some discrete*

automorphic representation π of G_ϕ at almost all places. Moreover if ϕ is of orthogonal type, $\widehat{G}_\phi = SO(2n+1, \mathbb{C})$ when $N = 2n+1$, or $SO(2n, \mathbb{C})$ when $N = 2n$; if ϕ is of symplectic type, $\widehat{G}_\phi = Sp(2n, \mathbb{C})$ with $N = 2n$.

For $\phi = l_1\phi_1 \boxplus \cdots \boxplus l_r\phi_r \in \Phi(N^{\theta_N})$, since $\phi = \phi^\vee$, one gets an involution on the indices by letting $\phi_{i^\vee} = \phi_i^\vee$, and consequently one has $l_i = l_{i^\vee}$. This gives a disjoint decomposition of these indices

$$I_\phi \sqcup J_\phi \sqcup J_\phi^\vee,$$

where I_ϕ indexes the set of self-dual simple parameters. Let $K_\phi = I_\phi \sqcup J_\phi$, and let $I_{\phi,O}$ ($I_{\phi,S}$) indexes the self-dual simple parameters of orthogonal (symplectic) type. By Theorem 3.1, for each ϕ_i with $i \in I_\phi$, we have a twisted elliptic endoscopic group G_i of $GL(N_i)$ and we fix the twisted endoscopic embedding $\xi_i : {}^L G_i \rightarrow {}^L GL(N_i)$. For ϕ_j with $j \in J_\phi$, let us just take G_j to be $GL(N_j)$ and define an L -embedding $\xi_j : {}^L G_j \rightarrow {}^L GL(2N_j)$ by sending $g \rtimes w$ to $\text{diag}\{g, {}^t g^{-1}\} \times w$. Then Arthur defines a substitute global Langlands group by taking the fibre product

$$\mathcal{L}_\phi := \prod_{k \in K_\phi} \left\{ {}^L G_k \rightarrow W_F \right\},$$

and he also defines an L -homomorphism $\phi^\mathcal{E} : \mathcal{L}_\phi \rightarrow {}^L GL(N)$, where

$$\phi^\mathcal{E} := \bigoplus_{k \in K_\phi} l_k \xi_k.$$

By viewing G as in $\mathcal{E}_{ell}(N^{\theta_N})$, we can define the set of substitute global parameters of G as follows

$$\bar{\Phi}(G) := \{\phi \in \Phi(N^{\theta_N}) : \phi^\mathcal{E} \text{ factors through } {}^L G\}.$$

As a simple exercise, one can show for $\phi = l_1\phi_1 \boxplus \cdots \boxplus l_r\phi_r \in \Phi(N^{\theta_N})$, ϕ is in $\bar{\Phi}(G)$ if and only if l_i is even for all $i \in I_{\phi,S}$. Since $\text{Out}_N(G) \cong \Sigma_0$, the above set is really the analogue of the set of Σ_0 -conjugacy classes of global Langlands parameters for G . For $\phi \in \bar{\Phi}(G)$ and $\Sigma \subseteq \Sigma_0$, one can define

$$\begin{aligned} S_\phi^\Sigma &= \text{Cent}(\text{Im } \phi^\mathcal{E}, \widehat{G}^\Sigma), \\ \bar{S}_\phi^\Sigma &= S_\phi^\Sigma / Z(\widehat{G})^{\Gamma_F}, \\ \mathcal{S}_\phi^\Sigma &= \bar{S}_\phi^\Sigma / \bar{S}_\phi^0, \end{aligned}$$

and from here one can also define the following important subsets of $\bar{\Phi}(G)$

$$\begin{aligned}\bar{\Phi}_{sim}(G) &= \{\phi \in \bar{\Phi}(G) : \bar{S}_{\phi}^{\Sigma_0} = 1\}, \\ \bar{\Phi}_2(G) &= \{\phi \in \bar{\Phi}(G) : |\bar{S}_{\phi}| < \infty\}, \\ \bar{\Phi}(G^{\theta}) &= \{\phi \in \bar{\Phi}(G) : S_{\phi}^{\theta} \neq \emptyset\}, \\ \bar{\Phi}_{ell}(G^{\theta}) &= \{\phi \in \bar{\Phi}(G^{\theta}) : |\bar{S}_{\phi,s}^0| < \infty \text{ for some semisimple } s \in \bar{S}_{\phi}^{\theta}\},\end{aligned}$$

where $\theta \in \Sigma_0$. In fact, one can even compute S_{ϕ} very explicitly (see [7, (1.4.8)])

$$S_{\phi} = \left(\prod_{i \in I_{\phi,O}} O(l_i, \mathbb{C}) \right)_{\phi}^{+} \times \left(\prod_{i \in I_{\phi,S}} Sp(l_i, \mathbb{C}) \right) \times \left(\prod_{j \in J_{\phi}} GL(l_j, \mathbb{C}) \right), \quad (3.1)$$

where $\left(\prod_{i \in I_{\phi,O}} O(l_i, \mathbb{C}) \right)_{\phi}^{+}$ is the kernel of the character

$$\varepsilon_{\phi}^{+} : \prod_i g_i \longrightarrow \prod_i (\det g_i)^{N_i}, \quad g_i \in O(l_i, \mathbb{C}), i \in I_{\phi,O}.$$

Note G is symplectic or special even orthogonal here, so we have $I_{\phi,O} = I_{\phi}^{+}$ and $I_{\phi,S} = I_{\phi}^{-}$ in Arthur's original formula of S_{ϕ} . When G is special even orthogonal,

$$S_{\phi}^{\Sigma_0} = \left(\prod_{i \in I_{\phi,O}} O(l_i, \mathbb{C}) \right) \times \left(\prod_{i \in I_{\phi,S}} Sp(l_i, \mathbb{C}) \right) \times \left(\prod_{j \in J_{\phi}} GL(l_j, \mathbb{C}) \right). \quad (3.2)$$

As a consequence, one has the following description of those subsets of $\bar{\Phi}(G)$.

Lemma 3.2 1. $\bar{\Phi}_{sim}(G) = \Phi_{sim}(N) \cap \bar{\Phi}(G)$.

2. Suppose $\phi \in \bar{\Phi}(G)$, then ϕ is in $\bar{\Phi}_2(G)$ if and only if $K_{\phi} = I_{\phi,O}$ and $l_i = 1$ for all $i \in K_{\phi}$.
3. Suppose ϕ is in $\bar{\Phi}_{ell}(G^{\theta})$ for $\theta \in \Sigma_0$, then $K_{\phi} = I_{\phi,O}$ and $l_i \leq 2$ for all $i \in K_{\phi}$.
4. Suppose G is special even orthogonal and $\phi \in \bar{\Phi}(G)$, then ϕ is in $\bar{\Phi}(G^{\theta_0})$ if and only if there exists $i \in I_{\phi,O}$ such that N_i is odd.

The proof is a direct application of formulas (3.1) and (3.2).

Now let F be a local field, we also define the substitute local generic (or tempered) Langlands parameters similarly as follows,

$$\begin{aligned}\Phi_{sim}^{\mathcal{E}}(N) &:= \{\text{isomorphism classes of irreducible essentially discrete} \\ &\quad \text{series representations of } GL(N)\}, \\ \Phi_{sim}^{\mathcal{E}}(N^{\theta_N}) &:= \left\{ \phi^{\mathcal{E}} \in \Phi_{sim}^{\mathcal{E}}(N) : \phi^{\mathcal{E}} = (\phi^{\mathcal{E}})^{\vee} \right\}, \\ \Phi_{bdd}^{\mathcal{E}}(N^{\theta_N}) &:= \left\{ \phi^{\mathcal{E}} = l_1 \phi_1^{\mathcal{E}} \oplus \cdots \oplus l_r \phi_r^{\mathcal{E}} : \phi^{\mathcal{E}} \right. \\ &\quad \left. = (\phi^{\mathcal{E}})^{\vee}, \phi_i^{\mathcal{E}} \in \Phi_{sim}^{\mathcal{E}}(N_i) \text{ with } \sum_{i=1}^r l_i N_i = N \right\}.\end{aligned}$$

Suppose $\phi^{\mathcal{E}} \in \Phi_{sim}^{\mathcal{E}}(N^{\theta_0})$, we call $\phi^{\mathcal{E}}$ is of **orthogonal type** if the local symmetric square L -function $L(s, \phi^{\mathcal{E}}, S^2)$ has a pole at $s = 0$; we call $\phi^{\mathcal{E}}$ is of **symplectic type** if the local skew-symmetric square L -function $L(s, \phi^{\mathcal{E}}, \wedge^2)$ has a pole at $s = 0$. As in the global case, every $\phi^{\mathcal{E}} \in \Phi_{sim}^{\mathcal{E}}(N^{\theta_N})$ will be either of orthogonal type or symplectic type. We would like to state a local version of Theorem 3.1, which is proved in [7, Theorem 6.1.1 and Corollary 6.8.1]. For $\phi^{\mathcal{E}} \in \Phi_{sim}^{\mathcal{E}}(N^{\theta_N})$, let $\pi_{\phi^{\mathcal{E}}}$ be the self-dual essentially discrete series representation of $GL(N)$ defined by $\phi^{\mathcal{E}}$. We write

$$f_{N^{\theta_N}}(\phi^{\mathcal{E}}) := f_{GL(N)^{\theta_N}}(\pi_{\phi^{\mathcal{E}}}) \quad f \in \mathcal{H}(N), \quad (3.3)$$

with respect to some intertwining operator $A_{\pi_{\phi^{\mathcal{E}}}}(\theta_N)$.

Theorem 3.3 *Suppose $\phi^{\mathcal{E}} \in \Phi_{sim}^{\mathcal{E}}(N^{\theta_N})$, then there is a unique class of elliptic endoscopic data $(G_{\phi^{\mathcal{E}}}, s_{\phi^{\mathcal{E}}}, \xi_{\phi^{\mathcal{E}}})$ in $\mathcal{E}_{ell}(N^{\theta_N})$ such that*

$$f_{N^{\theta_N}}(\phi^{\mathcal{E}}) = f^{G_{\phi^{\mathcal{E}}}}(\phi^{\mathcal{E}}), \text{ for all } f \in \mathcal{H}(N)$$

for some stable distribution $f(\phi^{\mathcal{E}})$ on $G_{\phi^{\mathcal{E}}}$, where $f^{G_{\phi^{\mathcal{E}}}}$ is the twisted endoscopic transfer of f . Moreover if $\phi^{\mathcal{E}}$ is of orthogonal type, $\widehat{G}_{\phi^{\mathcal{E}}} = SO(2n+1, \mathbb{C})$ when $N = 2n+1$, or $SO(2n, \mathbb{C})$ when $N = 2n$; if $\phi^{\mathcal{E}}$ is of symplectic type, $\widehat{G}_{\phi^{\mathcal{E}}} = Sp(2n, \mathbb{C})$ with $N = 2n$.

Note $\text{Out}_N(G) \cong \Sigma_0$, so the twisted endoscopic transfer $f^{G_{\phi^{\mathcal{E}}}}$ lies in $\bar{\mathcal{H}}(G)$. As in the global case, one can define the substitute local Langlands group $\mathcal{L}_{\phi^{\mathcal{E}}}$ and substitute local parameter $\phi^{\mathcal{E}}$. One can also define the set $\bar{\Phi}_{bdd}^{\mathcal{E}}(G)$ of substitute parameters for G , various centralizer groups of parameter $\phi^{\mathcal{E}}$ in \bar{G} , and various subsets in $\bar{\Phi}_{bdd}^{\mathcal{E}}(G)$. Moreover, the formula (3.1), (3.2) and Lemma 3.2 still hold in the local case.

The link between these substitute local parameters and the genuine local Langlands parameters is through the local Langlands correspondence for $GL(N)$ proved by Harris-Taylor [16], Henniart [14] and Scholze [36]. The local Langlands correspondence for $GL(N)$ gives a bijection between $\Phi_{sim}^{\mathcal{E}}(N)$ and the set $\Phi_2(N)$ of equivalence

classes of N -dimensional irreducible unitary representations of L_F , which also induces a bijection for the self-dual ones. Later in the paper, we will identify them and use the notation $\Phi_{sim}(N)$ as in the global case. The following theorem proved in ([7], Corollary 6.8.1) shows the substitute local parameters of G correspond to the genuine local Langlands parameters of G under this bijection.

Theorem 3.4 *Suppose $\phi^\mathcal{E} \in \Phi_{sim}^\mathcal{E}(N^{\theta_N})$, then $\phi^\mathcal{E}$ is in $\bar{\Phi}_{sim}^\mathcal{E}(G)$ if and only if its corresponding Langlands parameter ϕ factors through ${}^L G$.*

Since the elements in $\Phi_{bdd}^\mathcal{E}(N^{\theta_N})$ corresponds to self-dual tempered representations of $GL(N)$ by the parabolic induction, the local Langlands correspondence also gives a bijection between $\Phi_{bdd}(N^{\theta_N})$ and $\Phi_{bdd}^\mathcal{E}(N^{\theta_N})$. And we have the following corollary.

Corollary 3.5 *1. Suppose $\phi^\mathcal{E} \in \Phi_{bdd}^\mathcal{E}(N^{\theta_N})$, then $\phi^\mathcal{E}$ is in $\bar{\Phi}_{bdd}^\mathcal{E}(G)$ if and only if its corresponding Langlands parameter ϕ factors through ${}^L G$.
2. Suppose $\phi^\mathcal{E} \in \bar{\Phi}_{bdd}^\mathcal{E}(G)$ corresponds to $\phi \in \bar{\Phi}_{bdd}(G)$, then $S_{\phi^\mathcal{E}} \cong S_\phi$ for any representative $\underline{\phi}$ of ϕ .*

For the proof, one just needs to notice for $\phi \in \bar{\Phi}_{bdd}(G)$, there is a decomposition through the twisted endoscopic embedding to $GL(N, \mathbb{C})$

$$\phi = l_1 \phi_1 \oplus \cdots \oplus l_r \phi_r,$$

where $\phi_i \in \Phi_2(N_i)$ is irreducible.

As a consequence of Theorem 3.4 and Corollary 3.5, one can identify $\bar{\Phi}_{bdd}^\mathcal{E}(G)$ with $\bar{\Phi}_{bdd}(G)$ through the twisted endoscopic embedding $\xi : {}^L G \rightarrow {}^L GL(N)$. And we also denote $S_{\phi^\mathcal{E}}$ by S_ϕ .

3.2 Local theory

Now we can state the main local result of Arthur's theory ([7], Theorem 1.5.1 and Theorem 2.2.1) for quasisplit symplectic groups and special even orthogonal groups in the tempered case. Let F be local. We fix a θ_N -stable Whittaker datum (B_N, Λ) for $GL(N)$. We also fix the twisted endoscopic embedding $\xi : {}^L G \rightarrow {}^L GL(N)$.

Theorem 3.6 *For every $\phi \in \bar{\Phi}_{bdd}(G)$, one can associate it with a finite set $\bar{\Pi}_\phi$ of $\bar{\Pi}_{temp}(G(F))$ such that it satisfies the following properties:*

1. *The distribution*

$$f(\phi) := \sum_{[\pi] \in \bar{\Pi}_\phi} f_G(\pi), \quad f \in \bar{\mathcal{H}}(G) \quad (3.4)$$

is stable.

2. *If we normalize the intertwining operator $A_{\pi_\phi}(\theta_N)$ such that it preserves the Whittaker functional on π_ϕ , then*

$$f_{N^{\theta_N}}(\phi) = f^G(\phi) \quad (3.5)$$

for $f \in \mathcal{H}(N)$ and the twisted endoscopic transfer $f^G \in \bar{\mathcal{H}}(G)$.

3. There is a disjoint decomposition

$$\bar{\Pi}_{temp}(G(F)) = \bigsqcup_{\phi \in \bar{\Phi}_{bdd}(G)} \bar{\Pi}_{\phi}.$$

Since the transfer map $\mathcal{I}(N^{\theta_N}) \rightarrow \mathcal{SI}(G)^{\text{Out}_N(G)}$ is surjective (see Sect. 2.4), ϕ determines the stable distribution (3.4) on $G(F)$ through (3.5). In this way, ϕ determines the L-packet $\bar{\Pi}_{\phi}$. If G is a product of symplectic groups and special even orthogonal groups, we define a group of automorphisms of G by taking the product of Σ_0 on each factors, and we denote this group again by Σ_0 . We denote the set of Σ_0 -orbits in $\Pi_{temp}(G(F))$ by $\bar{\Pi}_{temp}(G(F))$ and the set of Σ_0 -orbits in $\Phi_{bdd}(G)$ by $\bar{\Phi}_{bdd}(G)$. Let $\bar{\mathcal{H}}(G)$ be the Σ_0 -invariant functions in $\mathcal{H}(G)$. Then part (1) and (3) of this theorem can also be generalized to this case, in particular, the L-packets of G are formed by taking tensor products of those of each factor. If $G' \in \mathcal{E}(G^{\theta})$ for $\theta \in \Sigma_0$, then $G' \cong M_l \times G'_-$, where M_l is a product of general linear groups, and G'_- is again a product of symplectic groups and special even orthogonal groups. We can extend the action of Σ_0 to G' by letting it act trivially on M_l . Then we can define $\bar{\Phi}_{bdd}(G')$ and $\bar{\Pi}_{temp}(G'(F))$ similarly. Part (1) and (3) of this theorem can again be extended further to this case.

Theorem 3.7 1. For $\phi \in \bar{\Phi}_{bdd}(G)$, there is a canonical pairing between $\bar{\Pi}_{\phi}$ and \mathcal{S}_{ϕ} , which induces an inclusion

$$[\pi] \longrightarrow \langle \cdot, \pi \rangle, \quad [\pi] \in \bar{\Pi}_{\phi}, \quad (3.6)$$

from $\bar{\Pi}_{\phi}$ into the group $\widehat{\mathcal{S}_{\phi}}$ of characters on \mathcal{S}_{ϕ} , such that $\langle \cdot, \pi \rangle = 1$ if G and π are unramified. It becomes a bijection when F is nonarchimedean.

2. Suppose s is a semisimple element in $\bar{\mathcal{S}_{\phi}}$ and $(G', \phi') \longrightarrow (\phi, s)$ with $G' \in \mathcal{E}(G)$ and $\phi' \in \bar{\Phi}_{bdd}(G')$. The packet $\bar{\Pi}_{\phi'}$ can be defined by the generalization of the previous theorem. If x is the image of s in \mathcal{S}_{ϕ} , then

$$f^{G'}(\phi') = \sum_{[\pi] \in \bar{\Pi}_{\phi}} \langle x, \pi \rangle f_G(\pi), \quad f \in \bar{\mathcal{H}}(G). \quad (3.7)$$

When G is special even orthogonal, one could further characterize those θ_0 -stable tempered representation. It is a theorem proved in ([7], Theorem 2.2.3).

Theorem 3.8 Suppose G is a special even orthogonal group and $\phi \in \bar{\Phi}_{bdd}(G^{\theta_0})$.

1. For any $[\pi] \in \bar{\Pi}_{\phi}$, π is a θ_0 -stable representation of $G(F)$ and hence has an extension π^+ to $G^+(F) = G(F) \rtimes \langle \theta_0 \rangle$.
2. Suppose s is a semisimple element in $\bar{\mathcal{S}_{\phi}^{\theta_0}}$ and $(G', \phi') \longrightarrow (\phi, s)$ with $G' \in \mathcal{E}(G^{\theta_0})$, $\phi' \in \bar{\Phi}_{bdd}(G')$, then

$$f'(\phi') = \sum_{[\pi] \in \bar{\Pi}_{\phi}} \langle x, \pi^+ \rangle f_{G^{\theta_0}}(\pi), \quad f \in \bar{\mathcal{H}}(G), \quad (3.8)$$

where x is the image of s in $\mathcal{S}_\phi^{\theta_0}$, $\langle \cdot, \pi^+ \rangle$ is an extension of the character $\langle \cdot, \pi \rangle$ to $\mathcal{S}_\phi^+ = \langle \mathcal{S}_\phi^{\theta_0} \rangle$, and the intertwining operator $A_\pi(\theta_0) = \pi^+(\theta_0)$.

Remark 3.9 In the θ_0 -twisted character relation (3.8), both the extensions of representation π^+ and character $\langle \cdot, \pi^+ \rangle$ are not uniquely determined, but the product $\langle \cdot, \pi^+ \rangle f_{G^{\theta_0}}(\pi)$ is determined and depends only on π . Moreover, (3.5) is the analogue of (3.8) for general linear groups, where we have fixed the extension of π_ϕ using the Whittaker datum and taken the extended character $\langle \cdot, \pi^+ \rangle$ to be trivial. It is not hard to see how to generalize both Theorems 3.6 and 3.8 to products of symplectic groups and special even orthogonal groups.

Because one does not know whether all the local constituents of a unitary cuspidal automorphic representation of $GL(N)$ are tempered, i.e. the generalized Ramanujan conjecture, one has to deal with a more general set of parameters $\Phi_{unit}^+(N)$, which is defined as follows. Let v^a denote the map $|\cdot|_F^a$ of W_F for $a \in \mathbb{R}$. Then,

$$\begin{aligned} \Phi_{unit}^+(N) := & \{ \phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus (v^{a_1} \phi_{r+1} \oplus v^{-a_1} \phi_{r+1}) \cdots \\ & \oplus (v^{a_s} \phi_{r+s} \oplus v^{-a_s} \phi_{r+s}) : \\ & \phi_i \in \Phi_{sim}(N_i) \text{ for } 1 \leq i \leq r+s \text{ and } 0 < a_j < 1/2 \text{ for } 1 \leq j \leq s \}. \end{aligned}$$

From the classification of the unitary dual of $GL(N)$ (cf. [46, 47] archimedean case, [44] nonarchimedean case), we know the associated irreducible admissible representation π_ϕ for any $\phi \in \Phi_{unit}^+(N)$ is unitary. And we have the following fact.

Proposition 3.10 *If F is global and $\phi \in \Phi_{sim}(N)$, then $\phi_v \in \Phi_{unit}^+(N_v)$.*

Correspondingly, we can define

$$\bar{\Phi}_{unit}^+(G) := \bar{\Phi}(G) \cap \Phi_{unit}^+(N).$$

Theorems 3.6–3.8 can be extended to the case $\phi \in \bar{\Phi}_{unit}^+(G)$ except for the constituents of $\bar{\Pi}_\phi$ may be non-tempered. In fact, for any $\phi \in \bar{\Phi}_{unit}^+(G^\theta)$ with $\theta \in \Sigma_0$, ϕ can be regarded as $\phi_{M,\lambda} := \phi_M \otimes (\lambda \circ |\cdot|_F)$ for some θ -stable Levi subgroup M (which also admits a θ -stable parabolic subgroup $P \supseteq M$), where $\phi_M \in \bar{\Phi}_{bdd}(M^\theta)$ and $\lambda \in \mathfrak{a}_M^*$ lies in the open chamber determined by P . Let $\bar{\Pi}_{\phi_{M,\lambda}} := \bar{\Pi}_{\phi_M} \otimes e^{\langle H_M(\cdot), \lambda \rangle}$. Then one can just define $\bar{\Pi}_\phi$ to be the irreducible constituents of the parabolic induction $\mathcal{I}_P(\bar{\Pi}_{\phi_{M,\lambda}})$. Note that the θ -twisted endoscopy transfer is compatible with this parabolic induction, and also $\mathcal{S}_{\phi_M} \cong \mathcal{S}_\phi$, then it is enough to know the following proposition.

Proposition 3.11 *Suppose F is local, $\phi \in \bar{\Phi}_{unit}^+(G)$, and ϕ can be regarded as $\phi_{M,\lambda}$ where $\phi_M \in \bar{\Phi}_{bdd}(M)$ and $\lambda \in \mathfrak{a}_M^*$ lies in some open chamber determined by $P \supseteq M$. Then for any $[\pi_M] \in \bar{\Pi}_{\phi_M}$, the induced representation $\mathcal{I}_P(\pi_{M,\lambda})$ is irreducible.*

Propositions 3.10 and 3.11 are well known to experts, but for the convenience of the readers we will give their proofs in “Appendix”.

3.3 Global theory

Now let us assume F is global, and we fix the twisted endoscopic embedding $\xi : {}^L G \rightarrow {}^L GL(N)$. The global parameters and local parameters are related by the following theorem.

Theorem 3.12 *Suppose $\phi \in \bar{\Phi}_{sim}(G)$, then ϕ_v factors through ${}^L G_v$ for all places v , i.e. $\phi_v \in \bar{\Phi}_{unit}^+(G_v)$.*

This theorem is proved in ([7], Theorem 1.4.2). So for $\phi \in \bar{\Phi}(G)$, one has a commutative diagram

$$\begin{array}{ccc} L_{F_v} & \xrightarrow{\phi_v} & {}^L G_v \\ \downarrow & & \downarrow \\ \mathcal{L}_\phi & \xrightarrow{\phi^\varepsilon} & {}^L G, \end{array}$$

where $L_{F_v} \rightarrow \mathcal{L}_\phi$ is defined by ϕ_v . It gives rise to an inclusion $S_\phi \hookrightarrow S_{\phi_v}$ for any place v , which induces a homomorphism $S_\phi \rightarrow S_{\phi_v}$. One can define the global L -packet by taking the restricted tensor product

$$\bar{\Pi}_\phi := \bigotimes_v' \bar{\Pi}_{\phi_v}$$

and define the global pairing by

$$\langle x, \pi \rangle := \prod_v \langle x_v, \pi_v \rangle.$$

Note that for almost all places v , $\langle \cdot, \pi_v \rangle = 1$ by Theorem 3.6, so this product is well-defined. The main global result of Arthur's theory is to give a description of the discrete spectrum of automorphic representations of G . Here we only state it for those discrete automorphic representations parametrized by $\bar{\Phi}_2(G)$, i.e. for $\phi \in \bar{\Phi}_2(G)$, we want to describe $L_{disc,\phi}^2(G(F) \backslash G(\mathbb{A}_F))$ which consists of discrete automorphic representations π such that the Satake parameters satisfy $\xi(c(\pi_v)) = c(\phi_v)$ for almost all places. Let $\tilde{\mathcal{H}}(G) = \bigotimes_v' \tilde{\mathcal{H}}(G_v)$.

Theorem 3.13 *Suppose $\phi \in \bar{\Phi}_2(G)$, there is a decomposition as $\tilde{\mathcal{H}}(G)$ -modules*

$$L_{disc,\phi}^2(G(F) \backslash G(\mathbb{A}_F)) = m_\phi \sum_{\substack{[\pi] \in \bar{\Pi}_\phi \\ \langle \cdot, \pi \rangle = 1}} \pi$$

where $m_\phi = 1$ or 2 , and $m_\phi = 2$ only when G is special even orthogonal and $\phi \notin \bar{\Phi}(G^{\theta_0})$. Moreover,

$$L_{disc,\phi}^2(G(F) \backslash G(\mathbb{A}_F)) = 0$$

for $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$.

Remark 3.14 This theorem is a special case of ([7], Theorem 1.5.2). By Arthur's complete description of the discrete spectrum for orthogonal and symplectic groups, one can see $\bar{\Phi}_2(G)$ only contributes to the discrete spectrum of G . It is not hard to extend this to products of symplectic and special even orthogonal groups. In fact if $G = G_1 \times G_2 \times \cdots \times G_q$, then we can define $\bar{\Phi}(G)$ to be consisting of $\phi := \phi_1 \times \phi_2 \times \cdots \times \phi_q$ such that $\phi_i \in \bar{\Phi}(G_i)$ for $1 \leq i \leq q$. Moreover, we can define $\mathcal{L}_\phi := \prod_{i=1}^q \mathcal{L}_{\phi_i}$, then $S_\phi = \prod_{i=1}^q S_{\phi_i}$. And we let $m_\phi = \prod_{i=1}^q m_{\phi_i}$.

For $\phi \in \bar{\Phi}(G)$ and any subgroup $\Sigma \subseteq \Sigma_0$, let \mathcal{L}_ϕ act on \widehat{D} , \widehat{G}^Σ and \widehat{G}^Σ by conjugation through ϕ^Σ . We denote the corresponding group cohomology by $H_{\phi^\Sigma}^*(\mathcal{L}_\phi, \cdot)$. Note $H_{\phi^\Sigma}^0(\mathcal{L}_\phi, \widehat{D}) = \widehat{D}^\Gamma$, $H_{\phi^\Sigma}^0(\mathcal{L}_\phi, \widehat{G}^\Sigma) = S_\phi^\Sigma$ and $H_{\phi^\Sigma}^1(\mathcal{L}_\phi, \widehat{D}) = H^1(W_F, \widehat{D})$. We define $S_\phi^\Sigma := H_{\phi^\Sigma}^0(\mathcal{L}_\phi, \widehat{G}^\Sigma)$. Then we have the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & S_\phi^\Sigma / \widehat{D}^\Gamma & \longrightarrow & S_\phi^\Sigma & \xrightarrow{\delta} & H^1(W_F, \widehat{D}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & S_{\phi_v}^\Sigma / \widehat{D}_v^{\Gamma_v} & \longrightarrow & S_{\phi_v}^\Sigma & \xrightarrow{\delta_v} & H^1(W_{F_v}, \widehat{D}_v). \end{array}$$

Then Lemma 2.10 is still valid, and we again have the following exact sequence as in Sect. 2.2

$$1 \longrightarrow S_\phi^\Sigma \xrightarrow{\iota} S_\phi^\Sigma \xrightarrow{\alpha} \text{Hom}(\widetilde{G}(\mathbb{A}_F) / \widetilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times).$$

4 Coarse L-packet

4.1 Statement of main local theorem

Now we assume \widetilde{G} is of type (2.2), and λ is the generalized similitude character. In this case G is a product of symplectic groups and special even orthogonal groups. We also assume $\theta \in \Sigma_0$.

Lemma 4.1 ([49], Lemma 3.13) *Suppose $\phi \in \bar{\Phi}_{bdd}(G)$ and $[\pi] \in \bar{\Pi}_\phi$ then*

$$\langle x, (\pi^+)^g \rangle = \omega_x(g) \langle x, \pi^+ \rangle \quad (4.1)$$

for any $g \in \widetilde{G}(F)$ and $x \in S_\phi^\theta$, where $\omega_x = \alpha(x)$ and π^+ is an extension of π to $G^+(F) = G(F) \times \langle \theta \rangle$.

Corollary 4.2 ([49], Proposition 6.28) *Suppose $\phi \in \bar{\Phi}_{bdd}(G)$ and $[\pi] \in \bar{\Pi}_\phi$. If $\tilde{\pi}$ is an irreducible admissible representation of $\widetilde{G}(F)$ whose restriction to $G(F)$ contains π , then $\tilde{\pi}^\theta \cong \tilde{\pi} \otimes \omega$ if and only if $\omega \in \alpha(S_\phi^\theta)$. In particular,*

$$X(\tilde{\pi}) = \alpha(S_\phi).$$

Remark 4.3 In view of Proposition 3.11, Lemma 4.1 and Corollary 4.2 also hold for parameters in $\bar{\Phi}_{unit}^+(G)$, and the proofs are the same.

For $\phi \in \bar{\Phi}_{bdd}(G)$, let us fix a character $\tilde{\zeta}$ of $Z_{\tilde{G}}(F)$ such that $\tilde{\zeta}|_{Z_G(F)}$ is the central character of $\bar{\Pi}_{\phi}$. Then we define $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ to be the subset of $\bar{\Pi}_{temp}(\tilde{G})$ with central character $\tilde{\zeta}$, whose restriction to $G(F)$ have irreducible constituents contained in $\bar{\Pi}_{\phi}$. Let $X = \text{Hom}(\tilde{G}(F)/Z_{\tilde{G}}(F)G(F), \mathbb{C}^{\times})$, so X acts on $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ by twisting. We call $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ a coarse L -packet of \tilde{G} , and its structure can be described in the following proposition.

Proposition 4.4 ([49], Proposition 6.29) *Suppose $\phi \in \bar{\Phi}_{bdd}(G)$ and $\tilde{\zeta}$ is chosen as above.*

1. *The orbits in $\bar{\Pi}_{\phi}$ under the conjugate action of $\tilde{G}(F)$ all have size $|S_{\phi}/S_{\tilde{\phi}}|$. If F is nonarchimedean, there are exactly $|S_{\tilde{\phi}}|$ orbits.*
2. *There is a natural fibration*

$$X/\alpha(S_{\phi}^{\Sigma_0}) \longrightarrow \tilde{\Pi}_{\phi, \tilde{\zeta}} \xrightarrow{\text{Res}} \bar{\Pi}_{\phi}/\tilde{G}(F)$$

3. *There is a pairing*

$$\tilde{\pi} \longrightarrow \langle \cdot, \tilde{\pi} \rangle$$

from $\tilde{\Pi}_{\phi, \tilde{\zeta}}/X$ into $\widehat{S_{\tilde{\phi}}}$. It is uniquely characterized by

$$\langle x, \tilde{\pi} \rangle = \langle \iota(x), \pi \rangle,$$

where $\iota : S_{\tilde{\phi}} \hookrightarrow S_{\phi}$ and π is any irreducible representation of $G(F)$ in the restriction of $\tilde{\pi}$. Suppose \tilde{G} and $\tilde{\pi}$ are unramified, then $\langle \cdot, \tilde{\pi} \rangle = 1$. Moreover, this mapping from $\tilde{\Pi}_{\phi, \tilde{\zeta}}/X$ to $\widehat{S_{\tilde{\phi}}}$ is injective and when F is nonarchimedean it is in fact a bijection.

This proposition is also true for $\bar{\Phi}_{unit}^+(G)$. Now it is natural to ask the following question.

Question 4.5 *For any lift $\tilde{\phi}$ of $\phi \in \bar{\Phi}_{bdd}(G)$, can one assign a packet $\bar{\Pi}_{\tilde{\phi}}$ of representations of $\tilde{G}(F)$ which gives a section of $\text{Res} : \tilde{\Pi}_{\phi, \tilde{\zeta}} \rightarrow \bar{\Pi}_{\phi}/\tilde{G}(F)$, and also a stable distribution?*

The answer to this question can be formulated in the following theorem, which is our main local result.

Theorem 4.6 *Suppose $\phi \in \bar{\Phi}_{bdd}(G)$, and $\tilde{\zeta}$ is a character of $Z_{\tilde{G}}(F)$ whose restriction to $Z_G(F)$ is the central character of $\bar{\Pi}_{\phi}$. Let $\tilde{\chi} = \tilde{\zeta}|_{Z_F}$. Then there exists a subset $\bar{\Pi}_{\tilde{\phi}}$ of $\tilde{\Pi}_{\phi, \tilde{\zeta}}$ unique up to twisting by X , and it is characterized by the following properties:*

1.

$$\tilde{\Pi}_{\phi, \tilde{\zeta}} = \bigsqcup_{\omega \in X/\alpha(\mathcal{S}_{\phi}^{\Sigma_0})} \tilde{\Pi}_{\tilde{\phi}} \otimes \omega.$$

2. For $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$, the distribution

$$\tilde{f}'(\tilde{\phi}) := \sum_{[\tilde{\pi}] \in \tilde{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}}(\tilde{\pi})$$

is stable.

3. Suppose s is a semisimple element in $\tilde{S}_{\phi}^{\theta}$ with $\omega = \alpha(s)$ and $(G', \phi') \rightarrow (\phi, s)$. Fix a packet $\tilde{\Pi}_{\tilde{\phi}'}$ defined by part (1) and local Langlands correspondence for $GL(n)$, then we can choose $\tilde{\Pi}_{\tilde{\phi}}$ such that

$$\tilde{f}'(\tilde{\phi}') = \sum_{[\tilde{\pi}] \in \tilde{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}^{\theta}}(\tilde{\pi}, \omega), \quad \tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi}) \quad (4.2)$$

where $\tilde{f}_{\tilde{G}^{\theta}}(\tilde{\pi}, \omega) = \text{tr}(\tilde{\pi}(\tilde{f}) \circ A_{\tilde{\pi}}(\theta, \omega))$, and $A_{\tilde{\pi}}(\theta, \omega)$ is an intertwining operator between $\tilde{\pi} \otimes \omega$ and $\tilde{\pi}^{\theta}$, which is normalized in a way so that if f is the restriction of \tilde{f} on $G(F)$, then

$$(\tilde{f}|_{\tilde{Z}_F G(F)})_{\tilde{G}^{\theta}}(\tilde{\pi}, \omega) = \sum_{\pi \subseteq \tilde{\pi}|_G} \langle x, \pi^+ \rangle f_{G^{\theta}}(\pi) \quad (4.3)$$

where x is the image of s in $\mathcal{S}_{\phi}^{\theta}$, π^+ is an extension of π to $G^+(F) = G(F) \rtimes \langle \theta \rangle$ such that $\pi^+(\theta) = A_{\pi}(\theta)$.

Remark 4.7 1. In the notation of $\tilde{\Pi}_{\tilde{\phi}}$, one can think of $\tilde{\phi}$ as some parameter of \tilde{G} lifted from ϕ . Since $\tilde{\Pi}_{\tilde{\phi}}$ is only defined up to twisting by X , one can also take $\tilde{\phi}$ as a formal symbol built in the notation of $\tilde{\Pi}_{\tilde{\phi}}$. In this paper, we will take the second point of view.

2. The normalizations in (4.3) is a consequence of (4.1). When $\theta = id$, $\omega = 1$ and $x \in \mathcal{S}_{\tilde{\phi}}$, $A_{\tilde{\pi}}(id, 1)$ becomes a scalar and is equal to $\langle x, \tilde{\pi} \rangle$ by (4.3). So we obtain the character relation from (4.2)

$$\tilde{f}'(\tilde{\phi}') = \sum_{[\pi] \in \tilde{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}}(\tilde{\pi}, 1) = \sum_{[\pi] \in \tilde{\Pi}_{\tilde{\phi}}} \langle x, \tilde{\pi} \rangle \tilde{f}_{\tilde{G}}(\tilde{\pi}). \quad (4.4)$$

3. If F is archimedean, $\tilde{\Pi}_{\tilde{\phi}}$ is defined by Langlands [20]. In fact, we have $\tilde{\Pi}_{\tilde{\phi}} = \tilde{\Pi}_{\phi, \tilde{\zeta}}$, if $\tilde{\Pi}_{\phi}$ is not a singleton (see Proposition 6.11 and also [12], Theorem 27.1). Moreover, Part (2) and (3) can be directly reduced to (3.4), (3.7) and (3.8) (see [49], Remark 6.32).

4. For $\phi \in \bar{\Phi}_{unit}^+(G)$, since $\bar{\Pi}_\phi = \mathcal{I}_P(\bar{\Pi}_{\phi_{M,\lambda}})$, we can define $\bar{\Pi}_{\tilde{\phi}} := \mathcal{I}_{\tilde{P}}(\bar{\Pi}_{\tilde{\phi}_{M,\lambda}})$, and this theorem can be easily extended to this case.

Let us call this subset $\bar{\Pi}_{\tilde{\phi}}$ the refined L -packet of \tilde{G} , and it is the genuine L -packet that one would expect modulo the action of Σ_0 . As we can see from this theorem, the refined L -packet is uniquely determined by the character relation (4.4) up to twisting by X . As a consequence of that, we can give another characterization of the refined L -packet.

Corollary 4.8 *In the setup of the previous theorem, any stable linear combination in $\bar{\Pi}_{\phi, \tilde{\phi}}$ is given by a linear combination of $\tilde{f}(\tilde{\phi} \otimes \omega) := (\tilde{f} \otimes \omega)(\tilde{\phi})$ for $\omega \in X$.*

Proof In the archimedean case, one can deduce this from ([39], Lemma 5.3). So we will assume F is nonarchimedean, and we fix a refined L -packet $\bar{\Pi}_{\tilde{\phi}} = \{[\tilde{\pi}_i]\}_{i=1}^r$. Suppose

$$\tilde{f}(\tilde{\phi}^1) := \sum_{i,j} a_{ij} \tilde{f}_{\tilde{G}}(\tilde{\pi}_i \otimes \omega_j)$$

is also stable for distinct $\omega_j \in X/\alpha(\mathcal{S}_\phi^{\Sigma_0})$ and $a_{ij} \in \mathbb{C}$. Here $\tilde{\phi}^1$ is just a formal symbol for denoting another stable distribution. Since the map $[\tilde{\pi}] \rightarrow \langle \cdot, \tilde{\pi} \rangle$ is a bijection in the nonarchimedean case, we have

$$\sum_{x \in \mathcal{S}_{\tilde{\phi}}} \langle x, \tilde{\pi}_i \rangle \langle x, \tilde{\pi}_j \rangle = r \cdot \delta_{ij}.$$

By inverting the formula of character relation (4.4) we get

$$\tilde{f}_{\tilde{G}}(\tilde{\pi}_i) = \sum_{x \in \mathcal{S}_{\tilde{\phi}}} c(\tilde{\pi}_i, x) \tilde{f}'(\tilde{\phi}, x),$$

where $\tilde{f}'(\tilde{\phi}, x) = \tilde{f}'(\tilde{\phi}')$ for some semisimple element $s \in \bar{S}_{\tilde{\phi}}$ whose image in $\mathcal{S}_{\tilde{\phi}}$ is x , and some $\bar{\Pi}_{\tilde{\phi}'}$ with $(G', \phi') \rightarrow (\phi, s)$, and

$$c(\tilde{\pi}_i, x) = \frac{1}{r} \langle x, \tilde{\pi}_i \rangle.$$

Therefore

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}^1) = \sum_{i,j} a_{ij} c(\tilde{\pi}_i, x) \tilde{f}'(\tilde{\phi} \otimes \omega_j, x).$$

If we separate those terms with $x = 1$ from the right hand side, and move them to the left hand side, we get

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}^1) - \frac{1}{r} \sum_{i,j} a_{ij} \tilde{f}(\tilde{\phi} \otimes \omega_j) = \sum_{i,j} \sum_{x \neq 1} c(\tilde{\pi}_i, x) \tilde{f}'(\tilde{\phi} \otimes \omega_j, x) \quad (4.5)$$

Now let us consider the endoscopic transfer map

$$\mathcal{T}^\varepsilon : \tilde{\mathcal{H}}(\tilde{G}) \longrightarrow \bigoplus_{\tilde{G}' \in \mathcal{E}_{\text{ell}}(\tilde{G})} \tilde{\mathcal{SI}}(\tilde{G}')$$

$$\tilde{f} \longmapsto \bigoplus_{\tilde{G}' \in \mathcal{E}_{\text{ell}}(\tilde{G})} \tilde{f}^{\tilde{G}'}.$$

The left hand side of (4.5) can be viewed as value of $\mathcal{T}^\varepsilon(\tilde{f})$ on some stable distribution of \tilde{G} , and similarly the right hand side of (4.5) can be viewed as values of $\mathcal{T}^\varepsilon(\tilde{f})$ on stable distributions of elliptic endoscopic groups $\tilde{G}' \in \mathcal{E}_{\text{ell}}(\tilde{G}) - \{\tilde{G}\}$. It is a consequence of the main results in [3] that the image of \mathcal{T}^ε can be characterized as families of functions $(\tilde{f}^{\tilde{G}'})_{\tilde{G}' \in \mathcal{E}_{\text{ell}}(\tilde{G})}$ such that for any $\tilde{G}'_1, \tilde{G}'_2 \in \mathcal{E}_{\text{ell}}(\tilde{G})$ the parabolic descent of any two functions $\tilde{f}^{\tilde{G}'_1}$ and $\tilde{f}^{\tilde{G}'_2}$ to their common Levi subgroups \tilde{M}' of \tilde{G}'_1 and \tilde{G}'_2 coincide. Since ϕ' does not factor through any proper Levi subgroups of ${}^L G$ for $x \neq 1$, then the stable distribution associated to ϕ' is not supported on any proper Levi subgroups of G . The same is true for the stable distribution associated with $\tilde{\Pi}_{\tilde{\phi}'}$. So the right hand side of (4.5) is not valued on any stable distributions supported on the Levi subgroups of \tilde{G} . Since (4.5) holds for all functions in $\tilde{\mathcal{H}}(\tilde{G})$, then both sides of (4.5) must be zero. Therefore,

$$0 = \tilde{f}^{\tilde{G}}(\tilde{\phi}^1) - \frac{1}{r} \sum_{i,j} a_{ij} \tilde{f}(\tilde{\phi} \otimes \omega_j) = \sum_{k,j} \left(a_{kj} - \frac{1}{r} \sum_i a_{ij} \right) \tilde{f}_{\tilde{G}}(\tilde{\pi}_k \otimes \omega_j).$$

By the linear independence of characters, we have

$$a_{kj} - \frac{1}{r} \sum_i a_{ij} = 0$$

for any k, j . As we fix j and vary k , we get a system of linear equations. The solutions of this system are $a_{ij} = a_{1j}$ for $1 \leq i \leq r$. Since this is also valid when we vary j , so we can conclude

$$\tilde{f}(\tilde{\phi}^1) = \sum_j a_{1j} \tilde{f}(\tilde{\phi} \otimes \omega_j).$$

□

This corollary is also valid for $\phi \in \bar{\Phi}_{\text{unit}}^+(G)$, and the proof is the same.

4.2 Local twisted intertwining relation

The proof of our main local theorem (Theorem 4.6) requires global methods, and the existence of refined L -packet needs to be proved together with the character relations.

Before we proceed to prove the theorem, let us first consider another form of character relation, called intertwining relation. The intertwining relation in its global form comes up naturally in the trace formula and it plays an important role in Arthur's work [7]. Here we need a twisted version of the intertwining relation, which in its local form is related to the twisted character relation (4.2). In this section, we again assume \tilde{G} is of type (2.2), and $\theta \in \Sigma_0$.

Suppose $\phi \in \tilde{\Phi}_{bdd}(G)$, and we assume ϕ factors through $\phi_M \in \tilde{\Phi}_{bdd}(M)$ for some Levi subgroup M of G . Let us define

$$\bar{T}_\phi(G, M) = A_{\hat{M}} Z(\hat{G})^{\Gamma_F} / Z(\hat{G})^{\Gamma_F},$$

where $A_{\hat{M}}$ is the maximal split central torus in \hat{M} . It is a torus in \bar{S}_ϕ^0 . Then we can define its normalizer in \bar{S}_ϕ

$$\bar{N}_\phi(G, M) = \text{Norm}(\bar{T}_\phi(G, M), \bar{S}_\phi),$$

and the group of its connected components

$$\begin{aligned} \mathfrak{N}_\phi(G, M) &= \bar{N}_\phi(G, M) / \bar{N}_\phi(G, M)^0 \\ &= \text{Norm}(\bar{T}_\phi(G, M), \bar{S}_\phi) / \text{Cent}(\bar{T}_\phi(G, M), \bar{S}_\phi^0)^0. \end{aligned}$$

Notice $\mathcal{S}_\phi(M) := \mathcal{S}_{\phi_M}$ is a normal subgroup of $\mathfrak{N}_\phi(G, M)$. The quotient $\mathfrak{N}_\phi(G, M) / \mathcal{S}_\phi(M)$ is the Weyl group

$$W_\phi(G, M) = W(\bar{S}_\phi, \bar{T}_\phi(G, M)).$$

We write $W_\phi^0(G, M)$ to be the normal subgroup of automorphisms in $W_\phi(G, M)$ that are induced from the connected component \bar{S}_ϕ^0 , and let

$$R_\phi(G, M) = W_\phi(G, M) / W_\phi^0(G, M)$$

Moreover, $W_\phi^0(G, M)$ is a normal subgroup of $\mathfrak{N}_\phi(G, M)$, and we denote their quotient by $\mathcal{S}_\phi(G, M)$, which is a subgroup of \mathcal{S}_ϕ . Suppose \tilde{M} is the Levi subgroup of \tilde{G} containing M , then similarly we can define

$$\bar{T}_{\tilde{\phi}}(G, M) = A_{\hat{\tilde{M}}} Z(\hat{\tilde{G}})^{\Gamma_F} / Z(\hat{\tilde{G}})^{\Gamma_F}$$

which is a torus of $\bar{S}_{\tilde{\phi}}^0$. Since $A_{\hat{\tilde{M}}} / \hat{D} = A_{\hat{M}}$, we have $\bar{T}_{\tilde{\phi}}(G, M) = \bar{T}_\phi(G, M)$. We can also define

$$\bar{N}_{\tilde{\phi}}(G, M) = \text{Norm}(\bar{T}_{\tilde{\phi}}(G, M), \bar{S}_{\tilde{\phi}}) \subseteq \bar{N}_\phi(G, M),$$

Suppose $u \in \mathfrak{N}_\phi(G, M)$, we write w_u for the image of u in $W_\phi(G, M)$ and x_u for the image of u in $\mathcal{S}_\phi(G, M)$. Since w_u normalizes $A_{\widehat{M}}$, it also normalizes \widehat{M} , and therefore can be treated as an element in $W(\widehat{M})$. The standard parabolic subgroup P containing M allows us to identify w_u with an element in $W(M) \cong W(\widehat{M})$. We choose a representative θ_u of w_u in $G(F)$ preserving the F -splitting of M . Then $\phi_M \in \bar{\Phi}_{bdd}(M^{\theta_u})$ and u defines an element in $\mathcal{S}_\phi(M)^{\theta_u} := \mathcal{S}_{\phi_M}^{\theta_u}$. Note that

$$M \cong GL(N_1) \times \cdots \times GL(N_q) \times G_-,$$

and

$$\widetilde{M} \cong GL(N_1) \times \cdots \times GL(N_q) \times \widetilde{G}_-,$$

where G_- (resp. \widetilde{G}_-) is of the same type as G (resp. \widetilde{G}) with smaller rank. Suppose

$$\phi_M = \phi_1 \times \cdots \times \phi_q \times \phi_-,$$

where $\phi_i \in \Phi_{bdd}(N_i)$ and $\phi_- \in \bar{\Phi}_{bdd}(G_-)$. Then we can define

$$\bar{\Pi}_{\phi_M} = \pi_{\phi_1} \otimes \cdots \otimes \pi_{\phi_q} \otimes \bar{\Pi}_{\phi_-},$$

where π_{ϕ_i} is associated to ϕ_i . And any representation in this packet can be written as

$$\begin{aligned} \pi_M &= \pi_{\phi_1} \otimes \cdots \otimes \pi_{\phi_q} \otimes \pi_- \\ &= \pi_{GL} \otimes \pi_-. \end{aligned}$$

Since $\mathcal{S}_{\phi_M} \cong \mathcal{S}_{\phi_-}$, we can define a pairing between $\bar{\Pi}_{\phi_M}$ and \mathcal{S}_{ϕ_M} by

$$\langle \cdot, \pi_M \rangle := \langle \cdot, \pi_{GL} \rangle \langle \cdot, \pi_- \rangle,$$

where $\langle \cdot, \pi_{GL} \rangle$ is in fact trivial. By Theorem 3.8 and the local Langlands correspondence for $GL(n)$, we know $\pi_M^{\theta_u} \cong \pi_M$. As usual, we can take the intertwining operator $\pi_M^+(\theta_u) = \pi_{GL}^+(\theta_u) \otimes \pi_-^+(\theta_u)$, which preserves the Whittaker models on those general linear components, then the extension $\langle \cdot, \pi_{GL}^+ \rangle$ of $\langle \cdot, \pi_{GL} \rangle$ to $\mathcal{S}_\phi(M)^{\theta_u}$ is trivial (see Theorem 3.6). So the extension $\langle \cdot, \pi_-^+ \rangle$ defined in Theorem 3.8 (see also Remark 3.9) determines an extension $\langle \cdot, \pi_M^+ \rangle$. Now we define

$$f_G(\phi, u) = \sum_{[\pi_M] \in \bar{\Pi}_{\phi_M}} \langle u, \pi_M^+ \rangle \text{tr}(R_P(\theta_u, \pi_M^+, \phi) \mathcal{I}_P(\pi_M, f)), \quad f \in \bar{\mathcal{H}}(G, \chi), \quad (4.7)$$

where $R_P(\theta_u, \pi_M^+, \phi)$ is the normalized self-intertwining operator on the space $\mathcal{H}_P(\pi_M)$ of normalized induced representation $\mathcal{I}_P(\pi_M)$ (see [7], Section 2.4). If we

assume the existence of refined L -packet $\bar{\Pi}_{\tilde{\phi}_-}$ for $\tilde{\phi}_-$ as defined in Theorem 4.6, then $\bar{\Pi}_{\tilde{\phi}_M}$ can be defined in the same way as $\bar{\Pi}_{\phi_M}$. And we can also define

$$\tilde{f}_G(\tilde{\phi}, u) = \sum_{[\tilde{\pi}_M] \in \bar{\Pi}_{\tilde{\phi}_M}} \text{tr}(R_{\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) \mathcal{I}_P^\omega(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})), \quad \tilde{f} \in \mathcal{H}(\tilde{G}, \tilde{\chi}). \quad (4.8)$$

Here we need to give some explanations of this distribution. Firstly, $\omega = \alpha^M(u) = \alpha^G(x_u)$. If

$$\tilde{\pi}_M = \pi_{\phi_1} \otimes \cdots \otimes \pi_{\phi_q} \otimes \tilde{\pi}_-$$

contains π_M in its restriction to $M(F)$, then it follows from Corollary 4.2 that $\tilde{\pi}_M^{\theta_u} \cong \tilde{\pi}_M \otimes \omega$, and we let $A_{\tilde{\pi}_M}(\theta_u, \omega)$ be the intertwining operator. Secondly, the automorphism θ_u on \tilde{M} is a composition of permutations of the general linear factors and automorphisms sending g_i to $\theta_{N_i}(g_i) \cdot \lambda(g_-)$, where $g = g_1 \times \cdots \times g_q \times g_- \in \tilde{M}$. However the effect on general linear components of $\tilde{\pi}_M$ is the same as for M , so we can use the same intertwining operator for the general linear components of $\tilde{\pi}_M$. In view of (4.3), the pairing inside (4.7) is built into $A_{\tilde{\pi}_M}(\theta_u, \omega)$ and hence into the operator $R_{\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi})$. Thirdly,

$$\mathcal{I}_P^\omega(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f}) = R(\omega) \circ \mathcal{I}_P(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})$$

where $R(\omega)$ is multiplication by ω , and $R_{\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi})$ is the normalized intertwining operator between $\mathcal{H}_{\tilde{P}}(\tilde{\pi}_M)$ and $\mathcal{H}_{\tilde{P}}(\tilde{\pi}_M \otimes \omega^{-1})$. The last thing is about this normalization. Let us recall the formulation of the normalized intertwining operator

$$R_P(\theta_u, \pi_M^+, \phi) := \pi_M^+(\theta_u) \circ (r_P(w_u, \phi_M)^{-1} J_P(\theta_u, \pi_M)).$$

Here $r_P(w_u, \phi_M)$ is the normalizing factor, and $J_P(\theta_u, \pi_M)$ is the unnormalized intertwining operator between $\mathcal{H}_P(\pi_M)$ and $\mathcal{H}_P(\pi_M^{\theta_u^{-1}})$, which is defined by an integral over

$$N_P \cap w_u N_P w_u^{-1} \backslash N_P,$$

where N_P is the unipotent radical of P . The key point is to notice that

$$\text{Res}_{G(F)}^{\tilde{G}(F)} \mathcal{I}_{\tilde{P}}(\tilde{\pi}_M) \cong \mathcal{I}_P(\text{Res}_{M(F)}^{\tilde{M}(F)} \tilde{\pi}_M).$$

So we obtain isomorphisms between the following spaces as $\tilde{\mathcal{H}}(G, \chi)$ -modules

$$\begin{aligned}\mathcal{H}_{\tilde{P}}(\tilde{\pi}_M) &\cong \bigoplus_{\pi_M \subseteq \text{Res} \tilde{\pi}_M} \mathcal{H}_P(\pi_M) \cong \mathcal{H}_{\tilde{P}}(\tilde{\pi}_M \otimes \omega^{-1}), \\ \mathcal{H}_{\tilde{P}}(\tilde{\pi}_M^{\theta_u^{-1}}) &\cong \bigoplus_{\pi_M \subseteq \text{Res} \tilde{\pi}_M} \mathcal{H}_P(\pi_M^{\theta_u^{-1}}).\end{aligned}$$

Under these identifications, we can easily see from the definition of unnormalized intertwining operators that

$$J_{\tilde{P}}(\theta_u, \tilde{\pi}_M) = \bigoplus_{\pi_M \subseteq \text{Res} \tilde{\pi}_M} J_P(\theta_u, \pi_M).$$

Let $\pi_{\phi_M} = \pi_{\phi_1} \otimes \cdots \otimes \pi_{\phi_q} \otimes \pi_{\phi_-}$. The normalizing factor $r_P(w_u, \phi_M)$ for $J_P(\theta_u, \pi_M)$ is equal to the product of λ -factor $\lambda(w_u)$ (see [7], (2.3.19)) and

$$L(0, \pi_{\phi_M}, \rho_{w_u^{-1}P|P}^\vee) \varepsilon(0, \pi_{\phi_M}, \rho_{w_u^{-1}P|P}^\vee, \psi_F)^{-1} L(1, \pi_{\phi_M}, \rho_{w_u^{-1}P|P}^\vee)^{-1} \quad (4.9)$$

where the L -functions involved here are either Rankin–Selberg L -functions or (skew)-symmetric square L -functions. We can set $r_{\tilde{P}}(w_u, \tilde{\phi}_M) := r_P(w_u, \phi_M)$ for $J_{\tilde{P}}(\theta_u, \tilde{\pi}_M)$. In fact this is what one would expect according to Langlands' conjectural formula for the normalizing factors. By his conjecture, (4.9) could be replaced by

$$L(0, \rho_{w_u^{-1}P|P}^\vee \circ \phi_M) \varepsilon(0, \rho_{w_u^{-1}P|P}^\vee \circ \phi_M, \psi_F)^{-1} L(1, \rho_{w_u^{-1}P|P}^\vee \circ \phi_M)^{-1},$$

where $\rho_{w_u^{-1}P|P}^\vee$ is the contragredient of the adjoint representation of ${}^L M$ over $\widehat{\mathfrak{n}}_{w_u^{-1}P} \cap \widehat{\mathfrak{n}}_P \backslash \widehat{\mathfrak{n}}_{w_u^{-1}P}$, where $\widehat{\mathfrak{n}}_P$ is the Lie algebra of \widehat{N}_P . Since

$$\rho_{w_u^{-1}\tilde{P}|\tilde{P}}^\vee \circ \tilde{\phi}_M = \rho_{w_u^{-1}P|P}^\vee \circ \phi_M,$$

then the conjectural formulas for $r_{\tilde{P}}(w_u, \tilde{\phi}_M)$ and $r_P(w_u, \phi_M)$ are the same. Finally we can normalize $A_{\tilde{\pi}_M}(\theta_u, \omega)$ according to (4.3), so after composing with this operator we get

$$\begin{aligned}R_{\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) &:= A_{\tilde{\pi}_M}(\theta_u, \omega) \circ (r_{\tilde{P}}(w_u, \tilde{\phi}_M)^{-1} J_{\tilde{P}}(\theta_u, \tilde{\pi}_M)) \\ &= \bigoplus_{\pi_M \subseteq \text{Res} \tilde{\pi}_M} \langle u, \pi_M^+ \rangle R_P(\theta_u, \pi_M^+, \phi).\end{aligned} \quad (4.10)$$

As a result, if $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$ is supported on $\tilde{Z}_F G(F)$ and f is its restriction to $G(F)$, then

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}, u) = f_G(\phi, u).$$

Suppose s is a semisimple element in \bar{S}_ϕ , and $(G', \phi') \longrightarrow (\phi, s)$. For any lift $\bar{\Pi}_{\tilde{\phi}'}$, let us define

$$\tilde{f}'_{\tilde{G}}(\tilde{\phi}, s) = \tilde{f}^{\tilde{G}'}(\tilde{\phi}'), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}).$$

Now we can state the local ω -twisted intertwining relation.

Theorem 4.9 Suppose $\phi \in \bar{\Phi}_{bdd}(G)$, for semisimple $s \in \bar{S}_\phi$ with $(G', \phi') \longrightarrow (\phi, s)$, the following identity holds for some lifts $\bar{\Pi}_{\tilde{\phi}}$ and $\bar{\Pi}_{\tilde{\phi}'}$ that

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}, u) = \tilde{f}'_{\tilde{G}}(\tilde{\phi}, s), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}), \quad (4.11)$$

where $u \in \mathfrak{N}_\phi(G, M)$ and $s \in \bar{S}_\phi$ have the same image in S_ϕ .

The next lemma shows that for \tilde{G} , the ω -twisted intertwining relation (4.11) is equivalent to the ω -twisted character relation (see (4.2) when $\theta = \text{id}$), if one has the local intertwining relation for G ([7], Theorem 2.4.1).

Lemma 4.10 For $\phi \in \bar{\Phi}_{bdd}(G)$, we assume ϕ factors through $\phi_M \in \bar{\Phi}_2(M)$ and $\bar{\Pi}_{\tilde{\phi}_M}$ exists. We define $\bar{\Pi}_{\tilde{\phi}} := \mathcal{I}_P(\bar{\Pi}_{\tilde{\phi}_M})$. Suppose $u \in \mathfrak{N}_\phi(G, M)$ and semisimple $s \in \bar{S}_\phi$ have the same image x in S_ϕ . Then

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}, u) = \tilde{f}'_{\tilde{G}}(\tilde{\phi}, s)$$

for some $\bar{\Pi}_{\tilde{\phi}}$ and $\bar{\Pi}_{\tilde{\phi}'}$ if and only if

$$\tilde{f}'_{\tilde{G}}(\tilde{\phi}, s) = \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega),$$

where $\omega = \alpha(x)$ and $\tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi})$.

Proof By Corollary 4.2, we have $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\tilde{\phi}} \otimes \omega$. Since

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}, u) = \sum_{[\tilde{\pi}_M] \in \bar{\Pi}_{\tilde{\phi}_M}} \text{tr}(R_{\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) \mathcal{I}_P^\omega(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})),$$

we can assume

$$\tilde{f}_{\tilde{G}}(\tilde{\phi}, u) = \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega)',$$

where $\tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega)' = \text{tr}(\tilde{\pi}(\tilde{f}) \circ A_{\tilde{\pi}}(\omega)')$ for some $A_{\tilde{\pi}}(\omega)'$ intertwining $\tilde{\pi} \otimes \omega$ and $\tilde{\pi}$. Note that if f is the restriction of \tilde{f} on $G(F)$, then $(\tilde{f}|_{\tilde{Z}_F G(F)})_{\tilde{G}}(\tilde{\phi}, u) = f_G(\phi, u)$. By the local intertwining relation for G ([7], Theorem 2.4.1), we have

$$(\tilde{f}|_{\tilde{Z}_F G(F)})_{\tilde{G}}(\tilde{\phi}, u) = f_G(\phi, u) = f'_G(\phi, s) = \sum_{[\pi] \in \bar{\Pi}_{\tilde{\phi}}} \langle x, \pi \rangle f_G(\pi).$$

So $\tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega)' = \tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega)$ for \tilde{f} supported on $\tilde{Z}_F G(F)$. This means for $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$, $A_{\tilde{\pi}}(\omega)' = A_{\tilde{\pi}}(\omega)$ as defined by (4.3). Thus $\tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega)' = \tilde{f}_{\tilde{G}}(\tilde{\pi}, \omega)$ for all $\tilde{f} \in \mathcal{H}(\tilde{G}, \tilde{\chi})$, and the lemma is clear. \square

As we can see from the proof of this lemma, $\tilde{f}_{\tilde{G}}(\tilde{\phi}, u)$ only depends on the image of u in \mathcal{S}_{ϕ} . And one should expect $\tilde{f}'_{\tilde{G}}(\tilde{\phi}, s)$ only depends on the image of s in \mathcal{S}_{ϕ} as well either from the ω -twisted character relation or the ω -twisted intertwining relation. But there is a little ambiguity here for $\tilde{f}'_{\tilde{G}}(\tilde{\phi}, s)$ depends on the choice of lift $\bar{\Pi}_{\tilde{\phi}'}$. The next lemma resolves the ambiguity and establishes this property directly.

Lemma 4.11 *For $\phi \in \bar{\Phi}_{bdd}(G)$ and $x \in \mathcal{S}_{\phi}$, there is a natural way to get a family of lifts $\bar{\Pi}_{\tilde{\phi}'}$ for all semisimple $s \in \bar{\mathcal{S}}_{\phi}$ with image x in \mathcal{S}_{ϕ} and $(G', \phi') \rightarrow (\phi, s)$. And for these lifts $\tilde{f}'_{\tilde{G}}(\tilde{\phi}, s)$ are the same.*

Proof The proof is essentially the same as for $f'_G(\phi, s)$ in ([7], Section 4.5) except for one does not have any ambiguity in that case. Since it is important to clarify the ambiguity here, we will review the original proof and show how one can get rid of this ambiguity. Suppose semisimple $s \in \bar{\mathcal{S}}_{\phi}$ has image x in \mathcal{S}_{ϕ} , if s is replaced by an $\bar{\mathcal{S}}_{\phi}^0$ -conjugate s_1 , then the corresponding pair (G'_1, ϕ'_1) is isomorphic to (G', ϕ') under $\bar{\mathcal{S}}_{\phi}^0$ -conjugation. And this extends to an isomorphism between \tilde{G}'_1 and \tilde{G}' for $\bar{\mathcal{S}}_{\phi}^0 = \bar{\mathcal{S}}_{\phi}^0$. So we can simply take the lifts $\bar{\Pi}_{\tilde{\phi}'} \cong \bar{\Pi}_{\tilde{\phi}'_1}$ and it is clear that $\tilde{f}'_{\tilde{G}}(\tilde{\phi}, s) = \tilde{f}'_{\tilde{G}}(\tilde{\phi}, s_1)$.

Now if we fix a maximal torus \bar{T}_{ϕ} of $\bar{\mathcal{S}}_{\phi}^0$ and a Borel subgroup \bar{B}_{ϕ} containing it, any automorphism of the complex reductive group $\bar{\mathcal{S}}_{\phi}^0$ stabilizes a conjugate of $(\bar{T}_{\phi}, \bar{B}_{\phi})$. So we can choose a semisimple representative s_x of x in $\bar{\mathcal{S}}_{\phi}$ so that $\text{Int}(s_x)$ stabilizes $(\bar{T}_{\phi}, \bar{B}_{\phi})$, and such representatives are determined up to a \bar{T}_{ϕ} -translate. Moreover the complex torus

$$\bar{T}_{\phi, x} = \text{Cent}(s_x, \bar{T}_{\phi})^0$$

in \bar{T}_{ϕ} is uniquely determined by x . Note that $\bar{T}_{\phi, x}$ is the connected component of the kernel of the following morphism

$$\begin{aligned} \bar{T}_{\phi} &\longrightarrow \bar{T}_{\phi} \\ t &\longmapsto s_x^{-1} t s_x t^{-1} \end{aligned}$$

So any point of \bar{T}_ϕ can be written as $(s_x^{-1}t s_x t^{-1})t_x$ for $t \in \bar{T}_\phi$ and $t_x \in \bar{T}_{\phi,x}$ (see [42], Corollary 5.4.5), and hence any point in $s_x \bar{T}_\phi$ can be written as

$$s_x(s_x^{-1}t s_x t^{-1})t_x = t s_x t^{-1} t_x = t s_x t_x t^{-1}, \quad t \in \bar{T}_\phi, \quad t_x \in \bar{T}_{\phi,x}.$$

This means it is enough to consider the $\bar{T}_{\phi,x}$ -translates of s_x . Finally, we can take the centralizer \widehat{M} of $\bar{T}_{\phi,x}$ in \widehat{G} which is a Levi subgroup of \widehat{G} , and it is dual to a Levi subgroup M_x of G . So (ϕ, s_x) is the image of a pair

$$(\phi_{M_x}, s_{M_x}), \quad \phi_{M_x} \in \bar{\Phi}_{bdd}(M_x), \quad s_{M_x} \in S_{\phi_{M_x}},$$

attached to M_x under the L -embedding ${}^L M_x \subseteq {}^L G$. This pair is in turn the image of an endoscopic pair $(M'_x, \phi'_{M'_x})$, and in particular, $\phi'_{M'_x} \in \bar{\Phi}_2(M'_x)$. Note that for all $\bar{T}_{\phi,x}$ -translates $s_{x,1}$ of s_x , the corresponding ϕ' also factors through ${}^L M'_x$. And we have

$$\tilde{f}_G^L(\tilde{\phi}, s_x) = \tilde{f}^{M'_x}(\tilde{\phi}'_{M'_x}) = \tilde{f}_G^L(\tilde{\phi}, s_{x,1}).$$

Now if we reverse our argument, we see any lift $\bar{\Pi}_{\tilde{\phi}_{M'_x}}$ will give rise to a family of lifts $\bar{\Pi}_{\tilde{\phi}'}$ for all semisimple $s \in \bar{S}_\phi$ with image x in S_ϕ , such that $\tilde{f}_G^L(\tilde{\phi}, s)$ are the same. This finishes the proof. \square

In fact our discussion of the ω -twisted intertwining relation for \tilde{G} can be extended to that for $\tilde{G} \rtimes \theta$. For $\phi \in \bar{\Phi}_{bdd}(G^\theta)$, suppose it factors through $\phi_M \in \bar{\Phi}_{bdd}(M)$ for some Levi subgroup M of G . Let us define

$$\mathfrak{N}_\phi^\theta(G, M) = \text{Norm}(\bar{T}_\phi(G, M), \bar{S}_\phi^\theta) / \text{Cent}(\bar{T}_\phi(G, M), \bar{S}_\phi^0)^0, \quad W_\phi^\theta(G, M) = W(\bar{S}_\phi^\theta, \bar{T}_\phi(G, M)),$$

$$\mathfrak{N}_\phi^+(G, M) = \text{Norm}(\bar{T}_\phi(G, M), \bar{S}_\phi^+) / \text{Cent}(\bar{T}_\phi(G, M), \bar{S}_\phi^0)^0, \quad W_\phi^+(G, M) = W(\bar{S}_\phi^+, \bar{T}_\phi(G, M)),$$

and

$$\mathfrak{N}_\phi^+(G, M) = \text{Norm}(\bar{T}_\phi(G, M), \bar{S}_\phi^+) / \text{Cent}(\bar{T}_\phi(G, M), \bar{S}_\phi^0)^0, \quad W_\phi^+(G, M) = W(\bar{S}_\phi^+, \bar{T}_\phi(G, M)).$$

Then we can have a commutative diagram analogous to (4.6).

$$\begin{array}{ccccccc}
 & & & & 1 & & 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & W_\phi^0(G, M) & \xlongequal{\quad} & W_\phi^0(G, M) \\
 & & & & \downarrow & & \downarrow \\
 & & & & W_\phi^0(G, M) & \xlongequal{\quad} & W_\phi^0(G, M) \\
 & & & & \downarrow & & \downarrow \\
 1 & \longrightarrow & S_\phi(M) & \longrightarrow & \mathfrak{N}_\phi^+(G, M) & \longrightarrow & W_\phi^+(G, M) \longrightarrow 1 \\
 & \searrow & \parallel & \searrow & \searrow & \searrow & \searrow \\
 1 & \longrightarrow & S_\phi(M) & \longrightarrow & \mathfrak{N}_\phi^+(G, M) & \longrightarrow & W_\phi^+(G, M) \longrightarrow 1 \\
 & \searrow & \parallel & \searrow & \searrow & \searrow & \searrow \\
 1 & \longrightarrow & S_\phi(M) & \longrightarrow & S_\phi^+(G, M) & \longrightarrow & R_\phi^+(G, M) \longrightarrow 1 \\
 & \searrow & \parallel & \searrow & \searrow & \searrow & \searrow \\
 1 & \longrightarrow & S_\phi(M) & \longrightarrow & S_\phi^+(G, M) & \longrightarrow & R_\phi^+(G, M) \longrightarrow 1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 1 & & 1
 \end{array}
 \tag{4.12}$$

For $u \in \mathfrak{N}_\phi^\theta(G, M)$, we again write w_u for the image of u in $W_\phi^\theta(G, M)$ and x_u for the image of u in $S_\phi^\theta(G, M)$. Since w_u normalizes $A_{\widehat{M}}$, it also normalizes \widehat{M} , and therefore can be treated as an element in the Weyl set $W(\widehat{G} \rtimes \widehat{\theta}, \widehat{M})$. The standard parabolic subgroup P containing M allows us to identify w_u with an element in the Weyl set $W(G \rtimes \theta, M) \cong W(\widetilde{G} \rtimes \theta, \widetilde{M})$. We choose a representative θ_u of w_u in $G(F) \rtimes \theta$ preserving the F -splitting of M . Then $\phi_M \in \bar{\Phi}_{bdd}(M^{\theta_u})$ and u defines an element in $S_{\phi_M}^{\theta_u}$.

As in the previous case, we define

$$f_{G^\theta}(\phi, u) = \sum_{\pi_M \in \bar{\Pi}_{\phi_M}} \langle u, \pi_M^+ \rangle \operatorname{tr}(R_{P|\theta P}(\theta_u, \pi_M^+, \phi) \mathcal{I}_P^\theta(\pi_M, f)), \quad f \in \bar{\mathcal{H}}(G, \chi).
 \tag{4.13}$$

Here $R_{P|\theta P}(\theta_u, \pi_M^+, \phi)$ is the normalized intertwining operator between $\mathcal{H}_{\theta P}(\pi_M^{\theta^{-1}})$ and $\mathcal{H}_P(\pi_M)$, and

$$\mathcal{I}_P^\theta(\pi_M, f) = R(\theta)^{-1} \circ \mathcal{I}_P(\pi_M, f),$$

where $R(\theta)$ is induced from the action of θ on $G(F)$. We can also define

$$\tilde{f}_{G^\theta}(\tilde{\phi}, u) = \sum_{[\tilde{\pi}_M] \in \bar{\Pi}_{\tilde{\phi}_M}} \operatorname{tr}(R_{\tilde{P}|\tilde{\theta} \tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) \mathcal{I}_P^{\theta, \omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}).
 \tag{4.14}$$

Here $\omega = \alpha^M(u) = \alpha^G(x_u)$, $R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi})$ is the normalized intertwining operator between $\mathcal{H}_{\theta\tilde{P}}(\tilde{\pi}_M^{\theta^{-1}})$ and $\mathcal{H}_{\tilde{P}}(\tilde{\pi}_M \otimes \omega^{-1})$, and

$$\mathcal{I}_{\tilde{P}}^{\theta, \omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f}) = R(\theta)^{-1} \circ \mathcal{I}_{\tilde{P}}^{\omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f}).$$

As before, we can identify

$$\mathcal{H}_{\theta\tilde{P}}(\tilde{\pi}_M^{\theta^{-1}}) \cong \bigoplus_{\pi_M \subseteq \text{Res}\tilde{\pi}_M} \mathcal{H}_{\theta P}(\pi_M^{\theta^{-1}}),$$

and

$$\mathcal{H}_{\tilde{P}}(\tilde{\pi}_M \otimes \omega^{-1}) \cong \bigoplus_{\pi_M \subseteq \text{Res}\tilde{\pi}_M} \mathcal{H}_P(\pi_M).$$

Then under these identifications, we have

$$R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) = \bigoplus_{\pi_M \subseteq \text{Res}\tilde{\pi}_M} \langle u, \pi_M^+ \rangle R_{P|\theta P}(\theta_u, \pi_M^+, \phi).$$

Therefore, if $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$ is supported on $\tilde{Z}_F G(F)$ and f is the restriction of \tilde{f} to $G(F)$, then

$$\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi}, u) = f_{G^\theta}(\phi, u).$$

Suppose s is a semisimple element in \bar{S}_ϕ^θ , and $(G', \phi') \longrightarrow (\phi, s)$. For any lift $\bar{\Pi}_{\tilde{\phi}'}$, let us define

$$\tilde{f}_{\tilde{G}^\theta}'(\tilde{\phi}, s) = \tilde{f}_{\tilde{G}'}'(\tilde{\phi}'), \quad \tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi}).$$

Now we can state the (θ, ω) -twisted intertwining relation.

Theorem 4.12 Suppose $\phi \in \bar{\Phi}_{bdd}(G)$, for semisimple $s \in \bar{S}_\phi^\theta$ with $(G', \phi') \longrightarrow (\phi, s)$, the following identity holds for some lifts $\bar{\Pi}_{\tilde{\phi}}$ and $\bar{\Pi}_{\tilde{\phi}'}$ that

$$\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi}, u) = \tilde{f}_{\tilde{G}^\theta}'(\tilde{\phi}, s), \quad \tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi}), \quad (4.15)$$

where $u \in \mathfrak{N}_\phi^\theta(G, M)$ and $s \in \bar{S}_\phi^\theta$ have the same image in S_ϕ^θ .

Finally, it is easy to see that Lemmas 4.10 and 4.11 can be extended to this case too. Moreover, the discussion of this section can be carried out for $\phi \in \bar{\Phi}_{unit}^+(G)$ as well, and the corresponding twisted intertwining relation will follow from the tempered case.

5 Stable trace formula and multiplicity formula

5.1 General setup and stable trace formula

In this section, we will set up the means to prove the main local theorem (Theorem 4.6). The method is global and we are going to use various types of stabilized trace formulas. To be more precise it is the discrete part of the trace formula and its stabilized form that we are going to use. A detailed discussion of this can be found in ([7], Chapter 3). The stabilization of the ordinary trace formula has been established by Arthur in [4–6], and it also rests upon Ngo's proof [34] of the Fundamental Lemma. In the twisted case, this results from the long project of Mœglin and Waldspurger [32].

Let us assume F is global, and let G , \tilde{G} , D and λ be defined as in Sect. 2.1.1. Let θ be an automorphism of G preserving an F -splitting and we assume λ is θ -invariant. Let $\tilde{Z}_{\mathbb{A}_F} = \prod'_v \tilde{Z}_{F_v}$ be a closed subgroup of $Z_{\tilde{G}}(\mathbb{A}_F)$, such that \tilde{Z}_{F_v} satisfy the conditions in Sect. 2.4. We also require $\tilde{Z}_{\mathbb{A}_F} Z_G(\mathbb{A}_F) = Z_{\tilde{G}}(\mathbb{A}_F)$ and $\tilde{Z}_{\mathbb{A}_F} Z_{\tilde{G}}(F)$ is mapped to a closed and cocompact subgroup in $Z_{\tilde{G}}(\mathbb{A}_F)_\theta$. Let $\tilde{Z}_F = \tilde{Z}_{\mathbb{A}_F} \cap Z_{\tilde{G}}(F)$ and $\tilde{\chi}$ be a character of $\tilde{Z}_{\mathbb{A}_F}/\tilde{Z}_F$. Let $Z_{\mathbb{A}_F} = \tilde{Z}_{\mathbb{A}_F} \cap Z_G(\mathbb{A}_F)$ and $Z_F = \tilde{Z}_F \cap Z_G(F)$. We denote the restriction of $\tilde{\chi}$ to $Z_{\mathbb{A}_F}$ by χ . First we consider the discrete part of the θ -twisted trace formula for G . For any nonnegative real number t and $f \in \mathcal{H}(G, \chi)$, it is a distribution defined as follows

$$I_{disc,t}^{G^\theta}(f) = \sum_{\{M\}} |W(M)|^{-1} \sum_{w \in W^\theta(M)_{reg}} |\det(w - 1)|_{\mathfrak{a}_M^{G^\theta}}^{-1} \text{tr}(M_{P|\theta P,t}(w, \chi) I_{P,t}^\theta(\chi, f)). \quad (5.1)$$

Here we need to give some explanations of this formula. The outer sum is taken over $G(F)$ -conjugacy classes of Levi subgroups M of G , and the inner sum is taken over elements w in the Weyl set

$$W^\theta(M) = \text{Norm}(A_M, G \rtimes \theta)/M$$

such that $|\det(w - 1)|_{\mathfrak{a}_M^{G^\theta}}^{-1} \neq 0$, where $\mathfrak{a}_M^{G^\theta}$ is the kernel of the canonical projection of

$$\mathfrak{a}_M \rightarrow \mathfrak{a}_G \rightarrow \mathfrak{a}_{G^\theta},$$

and

$$\mathfrak{a}_{G^\theta} = \mathfrak{a}_G / \{X - \theta(X) : X \in \mathfrak{a}_G\}.$$

For any Levi subgroup M of G , we can take the direct sum of

$$L_{disc,t}^2(M(F) \backslash M(\mathbb{A}_F), \zeta_M) \subseteq L_{disc}^2(M(F) \backslash M(\mathbb{A}_F), \zeta_M)$$

such that the central character ζ_M extends χ and is invariant under some element of $W^\theta(M)_{reg}$, and the archimedean infinitesimal characters of the irreducible constituents

have norm t on their imaginary parts, then

$$I_{P,t}(\chi, f) = \int_{Z_{\mathbb{A}_F} \backslash G(\mathbb{A}_F)} f(g) I_{P,t}(\chi, g) dg$$

defines an operator on the space $\mathcal{H}_{P,t}(\chi)$ of the corresponding normalized induced representations. The operator $I_{P,t}^\theta(\chi, f)$ is the composition $R(\theta)^{-1} \circ I_{P,t}(\chi, f)$, and $M_{P|\theta P,t}(w, \chi)$ is the standard intertwining operator between $\mathcal{H}_{\theta P,t}(\chi)$ and $\mathcal{H}_{P,t}(\chi)$. Note when $\theta = id$, the term for $M = G$ in (5.1) is given by trace of $R_{disc,t}^G(f) := I_{G,t}(\chi, f)$ on the corresponding part of the discrete spectrum of G .

The discrete part of the trace formula (5.1) can be stabilized, and we get the following formula,

$$I_{disc,t}^{G^\theta}(f) = \sum_{G' \in \mathcal{E}_{ell}(G^\theta)} \iota(G, G') S_{disc,t}^{G'}(f^{G'}). \quad (5.2)$$

Here $S_{disc,t}^{G'}(f')$ are stable distributions on G' , and they are defined by induction from the stabilized formula for $I_{disc,t}^{G'}(f')$. If we denote the image of $Z_{\mathbb{A}_F}$ under the inclusion $(Z_G)_\theta \rightarrow Z_{G'}$ by $Z'_{\mathbb{A}_F}$ and let $Z'_F = Z'_{\mathbb{A}_F} \cap Z_{G'}(\mathbb{A}_F)$, then $I_{disc,t}^{G'}(f')$ is defined with respect to a character χ' of $Z'_{\mathbb{A}_F}/Z'_F$ determined by χ and the twisted endoscopic embedding ${}^L G' \rightarrow {}^L G$, and $f' \in \mathcal{H}(G', \chi')$. The coefficients $\iota(G, G')$ are given by the Kottwitz's formula,

$$\begin{aligned} \iota(G, G') &= \frac{|\pi_0(Z(\widehat{G})^\Gamma)|}{|\pi_0(Z(\widehat{G}')^\Gamma)|} \cdot \frac{|\ker^1(F, Z(\widehat{G}'))|}{|\ker^1(F, Z(\widehat{G}))|} \cdot |\pi_0(\text{Aut}_G(G'))|^{-1} \\ &\quad \times |\kappa_{G^\theta} / \kappa_{G^\theta} \cap Z(\widehat{G}')|^{-1} \end{aligned} \quad (5.3)$$

where $\kappa_{G^\theta} = A_{\widehat{G}}^\theta$. When $\theta = id$, the term $|\kappa_{G^\theta} / \kappa_{G^\theta} \cap Z(\widehat{G}')|^{-1} = 1$.

We can also write down the same trace formulas for \widetilde{G} , but in this case we also need to consider the ω -twisted version of these trace formulas. Let ω be a character of $\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)$ and $\tilde{f} \in \mathcal{H}(\widetilde{G}, \widetilde{\chi})$, the discrete part of the (θ, ω) -twisted trace formula for \widetilde{G} takes the form

$$\begin{aligned} I_{disc,t}^{(\widetilde{G}^\theta, \omega)}(\tilde{f}) &= \sum_{\{\widetilde{M}\}} |W(\widetilde{M})|^{-1} \sum_{w \in W^\theta(\widetilde{M})_{reg}} |\det(w - 1)_{\mathfrak{a}_{\widetilde{M}}^{\widetilde{G}^\theta}}|^{-1} \\ &\quad \times \text{tr}(M_{\widetilde{P}|\theta \widetilde{P},t}(w, \widetilde{\chi}) I_{\widetilde{P},t}^{\theta, \omega}(\widetilde{\chi}, \tilde{f})), \end{aligned} \quad (5.4)$$

where the operator $I_{\widetilde{P},t}^{\theta, \omega}(\widetilde{\chi}, \tilde{f})$ is the composition $R(\theta)^{-1} \circ R(\omega) \circ I_{\widetilde{P},t}(\widetilde{\chi}, \tilde{f})$. For the term corresponding to $\widetilde{M} = \widetilde{G}$, we let $R_{disc,t}^{\widetilde{G}}(\tilde{f}) := I_{\widetilde{G},t}^{\widetilde{G}}(\widetilde{\chi}, \tilde{f})$ and denote $R(\theta)^{-1} \circ R(\omega) \circ R_{disc,t}^{\widetilde{G}}(\tilde{f})$ by $R_{disc,t}^{(\widetilde{G}^\theta, \omega)}(\tilde{f})$. After stabilization, (5.4) becomes

$$I_{disc,t}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = \sum_{\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G}^\theta, \omega)} \iota(\tilde{G}, \tilde{G}') S_{disc,t}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}), \quad (5.5)$$

where the coefficients $\iota(\tilde{G}, \tilde{G}')$ are given by the same kind of formula as in (5.3).

We denote by $\mathcal{A}(\tilde{G})$ (resp. $\mathcal{A}_2(\tilde{G})$) the set of (resp. discrete) automorphic representations of \tilde{G} , and by $\mathcal{C}_{aut}(\tilde{G})$ the set of families of Satake parameters of automorphic representations of \tilde{G} , modulo the equivalence relation that $c = c' \in \mathcal{C}_{aut}(\tilde{G})$ if and only if $c_v = c'_v$ for almost all places v . More generally, we extend this notion to admissible representations of $\tilde{G}(\mathbb{A}_F)$, and we denote the corresponding set by $\mathcal{C}_{\mathbb{A}}(\tilde{G})$. For $\tilde{c} \in \mathcal{C}_{\mathbb{A}}(\tilde{G})$ and its projection c on ${}^L G$, we can write $I_{disc,t,c}^{G^\theta}(f)$ (resp. $I_{disc,t,\tilde{c}}^{(\tilde{G}^\theta, \omega)}(\tilde{f})$) and $R_{disc,t,\tilde{c}}^{(\tilde{G}^\theta, \omega)}(\tilde{f})$ for the part of $I_{disc,t}^{G^\theta}(f)$ (resp. $I_{disc,t}^{(\tilde{G}^\theta, \omega)}(\tilde{f})$ and $R_{disc,t}^{(\tilde{G}^\theta, \omega)}(\tilde{f})$), which is contributed from automorphic representations π (resp. $\tilde{\pi}$) satisfying $c(\pi) = c$ (resp. $c(\tilde{\pi}) = \tilde{c}$). Then $S_{disc,t,c}^G(f)$ can be defined inductively using (5.2) for $\theta = id$. To be more precise, let

$$S_{disc,t,c}^{G'}(f') = \sum_{c' \rightarrow c} S_{disc,t,c'}^{G'}(f'),$$

and the sum is over the preimages c' of c in $\mathcal{C}_{\mathbb{A}}(G')$ under the twisted endoscopic embedding ${}^L G' \rightarrow {}^L G$. Then we define

$$S_{disc,t,c}^G(f) = I_{disc,t,c}^G(f) - \sum_{G' \in \mathcal{E}_{ell}(G) - \{G\}} \iota(G, G') S_{disc,t,c}^{G'}(f^{G'}).$$

Similarly, we can define $S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f})$. The next lemma shows that $S_{disc,t,c}^G(f)$ (resp. $S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f})$) is stable and we get a decomposition for (5.2) [resp. (5.5)].

Lemma 5.1 1. $S_{disc,t,c}^G(f)$ [resp. $S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f})$] is stable.
2. The stabilization of the twisted trace formula (5.2) (resp. (5.5)) can be decomposed according to $c \in \mathcal{C}_{\mathbb{A}}(G)$ (resp. $\tilde{c} \in \mathcal{C}_{\mathbb{A}}(\tilde{G})$), i.e.

$$I_{disc,t,c}^{G^\theta}(f) = \sum_{G' \in \mathcal{E}_{ell}(G^\theta)} \iota(G, G') S_{disc,t,c}^{G'}(f^{G'}).$$

resp.

$$I_{disc,t,\tilde{c}}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = \sum_{\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G}^\theta, \omega)} \iota(\tilde{G}, \tilde{G}') S_{disc,t,\tilde{c}}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}). \quad (5.6)$$

The lemma is an application of the theory of multipliers, and the proof is the same as in ([7], Lemma 3.3.1).

As in the local case where we study the relation of representations between G and \tilde{G} , here we want to discuss the relation of $I_{disc,t,c}^G(f)$ (resp. $S_{disc,t,c}^G(f)$) with $I_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f})$ (resp. $S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f})$). The next lemma is the first step of studying this relation.

Lemma 5.2 Suppose $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$ and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$, then the set of Satake parameters $c(\tilde{\pi})$ is mapped to $c(\pi)$ under the projection $p : {}^L\tilde{G} \longrightarrow {}^L G$.

Proof This lemma is essentially local, and it suffices to show for any place v of F , if both π_v and $\tilde{\pi}_v$ are unramified and π_v is contained in the restriction of $\tilde{\pi}_v$ to $G(F_v)$, then $c(\tilde{\pi}_v)$ is mapped to $c(\pi_v)$. If \tilde{G}_v is a torus, this follows from the Langlands correspondence for tori. In general, $\tilde{\pi}_v$ is an irreducible constituent of $\mathcal{I}_{\tilde{B}_v}(\tilde{\chi}_v)$ for some unramified character $\tilde{\chi}_v$ on the maximal torus \tilde{T}_v with Borel subgroup $\tilde{B}_v \supseteq \tilde{T}_v$, and one has $c(\tilde{\pi}_v) = c(\tilde{\chi}_v)$. Since

$$\mathrm{Res}_{G_v}^{\tilde{G}_v} \mathcal{I}_{\tilde{B}_v}(\tilde{\chi}_v) \cong \mathcal{I}_{B_v}(\mathrm{Res}_{T_v}^{\tilde{T}_v} \tilde{\chi}_v),$$

then $c(\pi_v) = c(\tilde{\chi}_v|_{T_v})$. So again by the Langlands correspondence for tori one has $c(\tilde{\pi}_v)$ mapped to $c(\pi_v)$. \square

Now we assume \tilde{G} is of type (2.2). By Corollary 2.4, $\lambda(Z_{\tilde{G}}(\mathbb{A}_F)) \cap D(F) = \lambda(Z_{\tilde{G}}(F))$. So we have $\lambda(Z_{\tilde{G}}(\mathbb{A}_F)) \cap \lambda(\tilde{G}(F)) = \lambda(Z_{\tilde{G}}(F))$, which is equivalent to

$$G(\mathbb{A}_F) \cap \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F) = G(F)Z_G(\mathbb{A}_F).$$

Therefore,

$$G(F)Z_G(\mathbb{A}_F) \backslash G(\mathbb{A}_F) = \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F) \backslash \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F).$$

Let $\tilde{\zeta}$ be a character of $Z_{\tilde{G}}(\mathbb{A}_F)/Z_{\tilde{G}}(F)$ and ζ be the restriction of $\tilde{\zeta}$ to $Z_G(\mathbb{A}_F)$, then we have

$$L_{disc}^2(G(F) \backslash G(\mathbb{A}_F), \zeta) = L_{disc}^2(\tilde{G}(F) \backslash \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F), \tilde{\zeta}).$$

Note that right multiplication by $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$ on the right hand side induces an action on the left hand side. In fact the action by $\tilde{G}(F)$ on the left hand side is given by conjugation on $G(F) \backslash G(\mathbb{A}_F)$ and the action by $Z_{\tilde{G}}(\mathbb{A}_F)$ is through the central character $\tilde{\zeta}$. The following lemma shows that the L^2 -discrete spectrum of $\tilde{G}(\mathbb{A}_F)$ is essentially induced from the L^2 -discrete spectrum of $G(\mathbb{A}_F)$.

Lemma 5.3 $\mathrm{Ind}_{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\tilde{G}(\mathbb{A}_F)} L_{disc}^2(G(F) \backslash G(\mathbb{A}_F), \zeta) \cong L_{disc}^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta})$.

Proof First of all, there is a natural $\tilde{G}(\mathbb{A}_F)$ -equivariant isomorphism

$$\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F) \backslash \tilde{G}(\mathbb{A}_F) \cong G(F)Z_G(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \times_{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \tilde{G}(\mathbb{A}_F).$$

Here we can view $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F) \backslash \tilde{G}(\mathbb{A}_F)$ as a closed subgroup of $\lambda(Z_{\tilde{G}}(\mathbb{A}_F))D(F) \backslash D(\mathbb{A}_F)$. Since

$$\lambda(Z_{\tilde{G}}(\mathbb{A}_F))D(F) \backslash D(\mathbb{A}_F)$$

is compact (see [33], Theorem 6.1.6), then $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\backslash\tilde{G}(\mathbb{A}_F)$ is also compact. So one can define an inner product on the space of $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$ -equivariant continuous functions from $\tilde{G}(\mathbb{A}_F)$ to $L^2(G(F)\backslash G(\mathbb{A}_F), \zeta)$, i.e.,

$$[L^2(G(F)\backslash G(\mathbb{A}_F), \zeta) \otimes C(\tilde{G}(\mathbb{A}_F))]^{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}$$

by integrating over $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\backslash\tilde{G}(\mathbb{A}_F)$. Moreover, one can normalize its Haar measure such that

$$L^2(\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_F), \tilde{\zeta}) \cong \text{completion of } [L^2(G(F)\backslash G(\mathbb{A}_F), \zeta) \otimes C(\tilde{G}(\mathbb{A}_F))]^{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)},$$

which is compatible with $\tilde{G}(\mathbb{A}_F)$ -action. Note that the right hand side is nothing but

$$\text{Ind}_{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\tilde{G}(\mathbb{A}_F)} L^2(G(F)\backslash G(\mathbb{A}_F), \zeta).$$

Finally, since $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\backslash\tilde{G}(\mathbb{A}_F)$ is compact, one must have

$$\text{Ind}_{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\tilde{G}(\mathbb{A}_F)} L_{disc}^2(G(F)\backslash G(\mathbb{A}_F), \zeta) \cong L_{disc}^2(\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_F), \tilde{\zeta}).$$

□

Let X be the set of characters of $\tilde{G}(\mathbb{A}_F)/Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$, and let Y be the set of characters of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$. If π is an irreducible admissible representation of $G(\mathbb{A}_F)$, and $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$, let us define

$$\begin{aligned}\tilde{G}(\pi) &= \{g \in \tilde{G}(\mathbb{A}_F) : \pi^g \cong \pi\} \\ X(\tilde{\pi}) &= \{\omega \in X : \tilde{\pi} \cong \tilde{\pi} \otimes \omega\} \\ Y(\tilde{\pi}) &= Y \cap X(\tilde{\pi}).\end{aligned}$$

By ([15], Lemma 4.11), we know $Y(\tilde{\pi}) = (\tilde{G}(\mathbb{A}_F)/\tilde{G}(\pi)\tilde{G}(F))^*$ is finite. The following lemma is inspired by ([22], Lemma 6.2).

Lemma 5.4 *Suppose $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$, and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$. Then the multiplicities of $\tilde{\pi}$ and π in the discrete spectrum are related by the following formula*

$$\sum_{\omega \in X/YX(\tilde{\pi})} m(\tilde{\pi} \otimes \omega) = \sum_{g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(\pi)\tilde{G}(F)} m(\pi^g). \quad (5.7)$$

Proof By Lemma 5.3,

$$L_{disc}^2(\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_F), \tilde{\zeta}) \cong \text{Ind}_{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\tilde{G}(\mathbb{A}_F)} L_{disc}^2(G(F)\backslash G(\mathbb{A}_F), \zeta) \quad (5.8)$$

and we would like to expand the right hand side. First, we need to decompose $L^2_{disc}(G(F)\backslash G(\mathbb{A}_F), \zeta)$ as a representation of $\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$. Recall $Z_{\tilde{G}}(\mathbb{A}_F)$ acts through $\tilde{\zeta}$ and $\tilde{G}(F)$ acts by conjugation on $G(F)\backslash G(\mathbb{A}_F)$. Let π be any constituent in $L^2_{disc}(G(F)\backslash G(\mathbb{A}_F), \zeta)$ and

$$G_1(\pi) = \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F) \cap \tilde{G}(\pi).$$

Then $G_1(\pi)$ will act on the π -isotypic component $I(\pi)$ and we get

$$I(\pi) = \bigoplus_{\omega_1}^{m(\pi)} \pi_1 \otimes \omega_1,$$

where π_1 is an extension of π to $G_1(\pi)$ and the sum is over $m(\pi)$ characters ω_1 of $G_1(\pi)/Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$, which depend on the extension π_1 and can have multiplicities. Since $m(\pi) = m(\pi^g)$ for $g \in \tilde{G}(F)$, we have the following decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A}_F), \zeta) = \bigoplus_{\{\pi\}} \text{Ind}_{G_1(\pi)}^{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \left(\bigoplus_{\omega_1}^{m(\pi)} \pi_1 \otimes \omega_1 \right),$$

where the outer sum is taken over equivalence classes $\{\pi\}$ of constituents in $L^2_{disc}(G(F)\backslash G(\mathbb{A}_F), \zeta)$ under the action by $\tilde{G}(F)$. Taking this expression into (5.8), we get

$$\begin{aligned} L^2_{disc}(\tilde{G}(F)\backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) &\cong \bigoplus_{\{\pi\}} \text{Ind}_{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\tilde{G}(\mathbb{A}_F)} \\ &\quad \circ \text{Ind}_{G_1(\pi)}^{\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \left(\bigoplus_{\omega_1}^{m(\pi)} \pi_1 \otimes \omega_1 \right) \\ &\cong \bigoplus_{\{\pi\}} \text{Ind}_{G_1(\pi)}^{\tilde{G}(\mathbb{A}_F)} \left(\bigoplus_{\omega_1}^{m(\pi)} \pi_1 \otimes \omega_1 \right). \end{aligned}$$

Moreover,

$$\begin{aligned} L^2_{disc}(\tilde{G}(F)\backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) &\cong \bigoplus_{\{\pi\}} \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \text{Ind}_{G_1(\pi)}^{\tilde{G}(\pi)} \left(\bigoplus_{\omega_1}^{m(\pi)} \pi_1 \otimes \omega_1 \right) \\ &\cong \bigoplus_{\{\pi\}} \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \bigoplus_{\omega_1}^{m(\pi)} \left(\bigoplus_{\omega \in (\tilde{G}(\pi)/G_1(\pi))^*} \tilde{\pi}_1 \otimes \omega \right) \otimes \omega_1 \\ &\cong \bigoplus_{\{\pi\}} \bigoplus_{\omega_1}^{m(\pi)} \bigoplus_{\omega \in (\tilde{G}(\pi)/G_1(\pi))^*} \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \tilde{\pi}_1 \otimes \omega \otimes \omega_1, \end{aligned}$$

where $\tilde{\pi}_1$ is an extension of π_1 to $\tilde{G}(\pi)$ and ω_1 is extended to $\tilde{G}(\pi)$. Suppose $\pi' = \pi^g$ for some $g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi)$, we have

$$\begin{aligned} G_1(\pi') &= G_1(\pi^g) = \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F) \cap \tilde{G}(\pi^g) \\ &= \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F) \cap \tilde{G}(\pi)^g. \end{aligned}$$

Since $\tilde{G}(\pi)^g \cong \tilde{G}(\pi)$, then $G_1(\pi') = G_1(\pi)$. Hence $\pi'_1 \cong \pi_1^g \otimes \omega_g$ for some character ω_g of $G_1(\pi)/Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$. Similarly one can show $\tilde{\pi}'_1 \cong \tilde{\pi}_1^g \otimes \omega_g$ for some extension of ω_g to $\tilde{G}(\pi)$. So

$$\text{Ind}_{G_1(\pi)}^{\tilde{G}(\pi)} \pi'_1 \cong \bigoplus_{\omega \in (\tilde{G}(\pi)/G_1(\pi))^*} \tilde{\pi}_1^g \otimes \omega \otimes \omega_g,$$

and

$$\begin{aligned} \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \text{Ind}_{G_1(\pi)}^{\tilde{G}(\pi)} \pi'_1 &\cong \bigoplus_{\omega \in (\tilde{G}(\pi)/G_1(\pi))^*} \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \tilde{\pi}_1^g \otimes \omega \otimes \omega_g \\ &\cong \bigoplus_{\omega \in (\tilde{G}(\pi)/G_1(\pi))^*} \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \tilde{\pi}_1 \otimes \omega \otimes \omega_g. \end{aligned}$$

Therefore

$$\begin{aligned} L^2_{disc}(\tilde{G}(F)\backslash\tilde{G}(\mathbb{A}_F), \tilde{\zeta}) &\cong \bigoplus_{\{\pi\}^\sim} \bigoplus_{g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi)} \bigoplus_{\omega_1}^{m(\pi^g)} \bigoplus_{\omega \in (\tilde{G}(\pi)/G_1(\pi))^*} \\ &\quad \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \tilde{\pi}_1 \otimes \omega \otimes \omega_1 \otimes \omega_g, \end{aligned}$$

where the outer sum is taken over equivalence classes $\{\pi\}^\sim$ of constituents in $L^2(G(F)\backslash G(\mathbb{A}_F), \zeta)$ under the action by $\tilde{G}(\mathbb{A}_F)$. Note the characters ω_1 in this formula depend on π^g . By our definition of $G_1(\pi)$, the characters of $\tilde{G}(\pi)/G_1(\pi)$ can be extended to that of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$. If we let

$$\tilde{\pi} = \text{Ind}_{\tilde{G}(\pi)}^{\tilde{G}(\mathbb{A}_F)} \tilde{\pi}_1,$$

then from the above formula one can see easily that

$$\sum_{\omega \in X/YX(\tilde{\pi})} m(\tilde{\pi} \otimes \omega) = \sum_{g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(\pi)\tilde{G}(F)} m(\pi^g).$$

□

5.2 Multiplicity formula

It is natural to apply Arthur's multiplicity formula (cf. Theorem 3.13) to the right hand side of (5.7) for those representations parametrized by $\phi \in \bar{\Phi}_2(G)$. But we can not apply that formula directly since it only gives the multiplicity of π as an $\bar{\mathcal{H}}(G)$ -module for $[\pi] \in \bar{\Pi}_\phi$. So let us define

$$\bar{m}(\pi) = \sum_{\pi' \sim \pi} m(\pi'),$$

where $\pi' \cong \pi$ as $\bar{\mathcal{H}}(G)$ -modules. Then the multiplicity formula for $[\pi] \in \bar{\Pi}_\phi$ asserts that

$$\bar{m}(\pi) = m_\phi |\mathcal{S}_\phi|^{-1} \sum_{x \in \mathcal{S}_\phi} \langle x, \pi \rangle, \quad (5.9)$$

where m_ϕ is defined in Theorem 3.13 and Remark 3.14. For any irreducible admissible representation $\tilde{\pi}$ of $\tilde{G}(\mathbb{A}_F)$, whose restriction to $G(\mathbb{A}_F)$ contains π , let us also write

$$\bar{m}(\tilde{\pi}) = \sum_{\{\tilde{\pi}' \sim_X \tilde{\pi}\}/X} \sum_{\omega \in X/YX(\tilde{\pi}')} m(\tilde{\pi}' \otimes \omega),$$

where $\tilde{\pi}' \cong \tilde{\pi} \otimes \omega'$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules for some $\omega' \in X$, and we take such $\tilde{\pi}'$ modulo twists by X in the sum. Then we can rewrite the formula (5.7) as

$$\bar{m}(\tilde{\pi}) = \sum_{g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(\pi)\tilde{G}(F)} \bar{m}(\pi^g). \quad (5.10)$$

Now we can apply Arthur's multiplicity formula (5.9) to the right hand side of (5.10) to get the following result.

Lemma 5.5 *Suppose $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$, and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$. If $[\pi] \in \bar{\Pi}_\phi$ for $\phi \in \bar{\Phi}_2(G)$, then*

$$\bar{m}(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|} \cdot |\mathcal{S}_\phi|^{-1} \sum_{x \in \mathcal{S}_{\tilde{\phi}}} \langle x, \pi \rangle. \quad (5.11)$$

Proof First we want to rewrite the right hand side of (5.10) as an integral over $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)$. Consider the integral

$$\begin{aligned}
& \int_{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \bar{m}(\pi^g) dg \\
&= \sum_{g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi)} \int_{\tilde{G}(\pi)/\tilde{G}(\pi) \cap \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \bar{m}(\pi^{hg}) dh \\
&= \sum_{g \in \tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi)} \bar{m}(\pi^g) \cdot \text{vol}\{\tilde{G}(\pi)/\tilde{G}(\pi) \cap \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\}.
\end{aligned}$$

Since

$$\text{vol}\{\tilde{G}(\pi)/\tilde{G}(\pi) \cap \tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\} = \frac{\text{vol}\{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\}}{|\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi)|}$$

and

$$|\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi)| = |Y(\tilde{\pi})|,$$

then

$$\bar{m}(\tilde{\pi}) = \frac{|Y(\tilde{\pi})|}{\text{vol}\{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\}} \cdot \int_{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \bar{m}(\pi^g) dg. \quad (5.12)$$

Combining the multiplicity formula (5.9) and also our local formula (4.1), we can compute the integral on the right hand side of (5.12) as follows,

$$\begin{aligned}
& \int_{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \bar{m}(\pi^g) dg \\
&= \int_{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} m_\phi |\mathcal{S}_\phi|^{-1} \sum_{x \in \mathcal{S}_\phi} \langle x, \pi^g \rangle dg \\
&= m_\phi |\mathcal{S}_\phi|^{-1} \sum_{x \in \mathcal{S}_\phi} \langle x, \pi \rangle \cdot \int_{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)} \omega_x(g) dg \\
&= m_\phi |\mathcal{S}_\phi|^{-1} \sum_{x \in \mathcal{S}_\phi} \langle x, \pi \rangle \cdot \text{vol}\{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\} \\
&= m_\phi |\mathcal{S}_\phi/\mathcal{S}_{\tilde{\phi}}|^{-1} |\mathcal{S}_{\tilde{\phi}}|^{-1} \sum_{x \in \mathcal{S}_{\tilde{\phi}}} \langle x, \pi \rangle \cdot \text{vol}\{\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)Z_{\tilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)\}.
\end{aligned}$$

Substitute this into (5.12), one gets

$$\bar{m}(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|} \cdot |\mathcal{S}_{\tilde{\phi}}|^{-1} \sum_{x \in \mathcal{S}_{\tilde{\phi}}} \langle x, \pi \rangle.$$

□

Although this lemma does not give a multiplicity formula for \tilde{G} , it has a very interesting consequence.

Corollary 5.6 *Suppose $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$, and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$. If $[\pi] \in \tilde{\Pi}_\phi$ for $\phi \in \tilde{\Phi}_2(G)$, then there exists $\omega \in X$ such that $\tilde{\pi} \otimes \omega$ is isomorphic to a discrete automorphic representation as $\tilde{\mathcal{H}}(\tilde{G})$ -module if and only if $\langle \cdot, \tilde{\pi} \rangle = 1$. In particular, if $\mathcal{S}_{\tilde{\phi}} = 1$ such character always exists.*

Proof Since $\langle x, \tilde{\pi} \rangle = \langle x, \pi \rangle$ for $x \in \mathcal{S}_{\tilde{\phi}}$, it follows from the formula (5.11) that

$$\bar{m}(\tilde{\pi}) = \begin{cases} m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_{\tilde{\phi}})|} & \text{if } \langle \cdot, \tilde{\pi} \rangle = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So the first part of this corollary is clear. Next if $\mathcal{S}_{\tilde{\phi}} = 1$, then we always have

$$\bar{m}(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_{\tilde{\phi}})|},$$

and this shows the second part. \square

In fact, we can refine the result of Lemma 5.5 to get a multiplicity formula for \tilde{G} by applying the stabilized twisted trace formulas. First, we need to define an equivalence relation on $\mathcal{C}_{\mathbb{A}}(G)$ such that $c \sim c' \in \mathcal{C}_{\mathbb{A}}(G)$ if and only if c_v is Σ_0 -conjugate to c'_v for almost all places, and we denote the set of equivalence classes by $\bar{\mathcal{C}}_{\mathbb{A}}(G)$. Let $\bar{\mathcal{C}}_{aut}(G)$ be the subset of $\bar{\mathcal{C}}_{\mathbb{A}}(G)$ consisting of equivalence classes of $\mathcal{C}_{aut}(G)$.

Lemma 5.7 *Suppose $\tilde{c} \in \mathcal{C}_{\mathbb{A}}(\tilde{G})$, then*

$$I_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) = S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) = 0$$

for $\tilde{f} \in \mathcal{H}(\tilde{G}, \tilde{\chi})$, unless the projection of \tilde{c} under $\mathbf{p} : {}^L\tilde{G} \rightarrow {}^L G$ belongs to the set $\bar{\mathcal{C}}_{aut}(G)$.

Proof It follows from Lemmas 5.2 and 5.3 that

$$tr R_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) = 0, \quad (5.13)$$

unless \tilde{c} projects to $c \in \bar{\mathcal{C}}_{aut}(G)$. Suppose the projection of \tilde{c} in $\bar{\mathcal{C}}_{\mathbb{A}}(G)$ does not belong to $\bar{\mathcal{C}}_{aut}(G)$, then by the principle of functoriality (which results from Arthur's theory [7]), it neither belongs to $\bar{\mathcal{C}}_{aut}(M)$ for any Levi subgroup M of G , nor to $\bar{\mathcal{C}}_{aut}(G')$ for any endoscopic group G' of G . Then for the same reason as (5.13), one gets $tr R_{disc,t,\tilde{c}}^{\tilde{M}}(\tilde{f}_M) = 0$. So it follows from the definition [see (5.1)] that

$$I_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) = 0.$$

Since

$$S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) = I_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) - \left(\sum_{\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G}) - \{\tilde{G}\}} \iota(\tilde{G}, \tilde{G}') S_{disc,t,\tilde{c}}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}), \right),$$

we can assume $S_{disc,t,\tilde{c}}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}) = 0$ by induction, then

$$S_{disc,t,\tilde{c}}^{\tilde{G}}(\tilde{f}) = 0.$$

□

For $\phi \in \tilde{\Phi}(G)$, Arthur (cf. [7], Section 3.3) defines the ϕ -component of the discrete part of the twisted trace formula for G and its stabilized form. Note that $c(\phi)$ defines an element in $\tilde{\mathcal{C}}_{\mathbb{A}}(G)$, and ϕ also determines the norm of the imaginary part of archimedean infinitesimal character, which can be denoted by $t(\phi)$, so we can write

$$I_{disc,\phi}^{G^\theta}(f) = \sum_{c \rightarrow c(\phi)} I_{disc,t(\phi),c}^{G^\theta}(f),$$

and

$$S_{disc,\phi}^G(f) = \sum_{c \rightarrow c(\phi)} S_{disc,t(\phi),c}^G(f),$$

where these sums are all over preimages c of $c(\phi)$ in $\mathcal{C}_{\mathbb{A}}(G)$. Then the stabilization of the ϕ -component of the twisted trace formula for G is

$$I_{disc,\phi}^{G^\theta}(f) = \sum_{G' \in \mathcal{E}_{ell}(G^\theta)} \iota(G, G') S_{disc,\phi}^{G'}(f^{G'}),$$

where

$$S_{disc,\phi}^{G'}(f^{G'}) = \sum_{c' \rightarrow c(\phi)} S_{disc,t(\phi),c'}^{G'}(f^{G'}).$$

Here we want to define the ϕ -component of the discrete part of the twisted trace formula for \tilde{G} . Let us write

$$I_{disc,\phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = \sum_{\tilde{c} \rightarrow c(\phi)} I_{disc,t(\phi),\tilde{c}}^{(\tilde{G}^\theta, \omega)}(\tilde{f}),$$

and

$$S_{disc,\phi}^{\tilde{G}}(\tilde{f}) = \sum_{\tilde{c} \rightarrow c(\phi)} S_{disc,t(\phi),\tilde{c}}^{\tilde{G}}(\tilde{f}).$$

Then the stabilization of the ϕ -component of the twisted stable trace formula for \tilde{G} is

$$I_{disc,\phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = \sum_{\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G}^\theta, \omega)} \iota(\tilde{G}, \tilde{G}') S_{disc,\phi}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}), \quad (5.14)$$

where

$$S_{disc,\phi}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}) = \sum_{\tilde{c}' \rightarrow c(\phi)} S_{disc,t(\phi),\tilde{c}'}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}).$$

Similarly, we can also define $R_{disc,\phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f})$. For $\phi \in \tilde{\Phi}_2(G)$, it only contributes to the discrete spectrum of G (cf. Remark 3.14), and by Lemma 5.3 it also only contributes to the discrete spectrum of \tilde{G} . So we have

$$I_{disc,\phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = tr R_{disc,\phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}). \quad (5.15)$$

Now we can give our multiplicity formula for \tilde{G} , and we will start with the simplest case, i.e., $\tilde{G} = GSp(2n)$ or $GSO(2n, \eta)$.

Proposition 5.8 *Suppose $\tilde{G} = GSp(2n)$ or $GSO(2n, \eta)$, $\tilde{\pi}$ is a discrete automorphic representation of \tilde{G} , and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$. If $[\pi] \in \bar{\Pi}_\phi$ for $\phi \in \tilde{\Phi}_2(G)$, then*

$$m(\tilde{\pi}) = m_{\tilde{\phi}} \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}, \quad (5.16)$$

where $m_{\tilde{\phi}} = 1$ or 2 , and $m_{\tilde{\phi}} = 2$ only when G is special even orthogonal, $\phi \notin \tilde{\Phi}(G^{\theta_0})$, and $\tilde{\pi} \cong \tilde{\pi}^{\theta_0} \otimes \omega$ for some $\omega \in Y$.

Proof Since $\tilde{\pi}$ is automorphic, we can take π to be automorphic as well by Lemma 5.3, and hence $\langle \cdot, \pi \rangle = 1$. It follows from Lemma 5.5 that

$$\bar{m}(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}. \quad (5.17)$$

Since θ_0 acts on $\{\tilde{\pi}' \sim_X \tilde{\pi}\}$, we can write

$$\bar{m}_0(\tilde{\pi}) = \sum_{\{\tilde{\pi}' \sim_X \tilde{\pi}\}/X, \theta_0} \sum_{\omega \in X/YX(\tilde{\pi}')} m(\tilde{\pi}' \otimes \omega),$$

where the sum modulo twists by X and θ_0 . If $\pi \cong \pi^{\theta_0}$, then

$$\bar{m}_0(\tilde{\pi}) = \bar{m}(\tilde{\pi}) = \sum_{\omega \in X/YX(\tilde{\pi})} m(\tilde{\pi} \otimes \omega).$$

If $\pi \not\cong \pi^{\theta_0}$, $\bar{m}_0(\tilde{\pi}) = \frac{1}{2}\bar{m}(\tilde{\pi})$. Therefore, we have

$$\bar{m}_0(\tilde{\pi}) = \begin{cases} m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|} & \text{if } \pi \cong \pi^{\theta_0}, \\ \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|} & \text{if } \pi \not\cong \pi^{\theta_0}. \end{cases} \quad (5.18)$$

Note that $\alpha(\mathcal{S}_\phi) \subseteq Y(\tilde{\pi})$, so

$$\frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|} = |Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)|.$$

In particular, $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$ is a two-group. We can fix a subgroup of representatives in $Y(\tilde{\pi})$ and denote it again by $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$.

Let us first consider the case $\pi \not\cong \pi^{\theta_0}$. If $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi) = 1$, then the lemma becomes obvious. So let us assume $1 \neq \omega \in Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$, and by the stabilized ω -twisted trace formula (5.14), one gets

$$I_{disc,\phi}^{(\tilde{G},\omega)}(\tilde{f}) = \sum_{\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G},\omega)} \iota(\tilde{G}, \tilde{G}') S_{disc,\phi}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}),$$

for $\tilde{f} \in \mathcal{H}(\tilde{G}, \tilde{\chi})$. Since ω is not in $\alpha(\mathcal{S}_\phi)$, ϕ can not factor through ${}^L G'$ for any $G' \in \mathcal{E}_{ell}(G)$ such that $\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G}, \omega)$. Then by Lemma 5.7, $S_{disc,\phi}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'}) = 0$ for all $\tilde{G}' \in \mathcal{E}_{ell}(\tilde{G}, \omega)$, and hence

$$I_{disc,\phi}^{(\tilde{G},\omega)}(\tilde{f}) = 0.$$

In particular,

$$tr R_{disc,\phi}^{(\tilde{G},\omega)}(\tilde{f}) = I_{disc,\phi}^{(\tilde{G},\omega)}(\tilde{f}) = 0, \quad (5.19)$$

as $\phi \in \bar{\Phi}_2(G)$. This is true for all nontrivial $\omega \in Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$. Let $I(\tilde{\pi})$ be the $\tilde{\pi}$ -isotypic component in $R_{disc,\phi}^{\tilde{G}}$, and one observes $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$ will act on $I(\tilde{\pi})$ by multiplication. The action of $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$ does not commute with that of $\tilde{G}(\mathbb{A}_F)$, but one can take

$$G_{\mathbb{A}_F}^1 = \{g \in \tilde{G}(\mathbb{A}_F) : \omega(g) = 1 \text{ for all } \omega \in Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)\},$$

which is of finite index in $\tilde{G}(\mathbb{A}_F)$, and then it will commute with the action of $G_{\mathbb{A}_F}^1$. In fact one can have a decomposition

$$I(\tilde{\pi}) = \bigoplus_{g \in \tilde{G}(\mathbb{A}_F)/G_{\mathbb{A}_F}^1} I((\pi^1)^g)$$

by restricting to $G_{\mathbb{A}_F}^1$. The point is each summand is invariant under $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$ and has the same multiplicity as $\tilde{\pi}$. By (5.19), one has

$$\mathrm{tr}(R(\tilde{f}) \circ R(\omega))|_{I(\tilde{\pi})} = 0,$$

where $R(\omega)$ denotes the multiplication by ω . In particular, one can restrict to those \tilde{f} supported on $G_{\mathbb{A}_F}^1$, then one has

$$\mathrm{tr}(R(\tilde{f}) \circ R(\omega))|_{I((\pi^1)^g)} = 0$$

for all $g \in \tilde{G}(\mathbb{A}_F)/G_{\mathbb{A}_F}^1$. We can view $I(\pi^1)$ as a representation of $\mathcal{H}(G_{\mathbb{A}_F}^1) \times Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$ and write it as $\pi^1 \otimes W$, then

$$\mathrm{tr}(R(\tilde{f}) \circ R(\omega))|_{I(\pi^1)} = \mathrm{tr}\pi^1(\tilde{f}) \cdot \mathrm{tr}\pi_W^1(\omega) = 0,$$

where π_W^1 is the corresponding representation of $Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$ on W . Therefore,

$$\mathrm{tr}\pi_W^1(\omega) = 0$$

for $1 \neq \omega \in Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)$. We claim

$$|Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)| \text{ divides } \dim(W). \quad (5.20)$$

If that is the case, by noticing $m(\tilde{\pi}) = \dim(W)$ and comparing with (5.18) one must have

$$|Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)| = \dim(W),$$

hence $m(\tilde{\pi}) = |Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)|$.

To prove the claim (5.20), one just needs to show the following general statement. Suppose V is a finite dimensional representation over the complex numbers of a finite group A such that the trace of each nontrivial element of A is zero, then the order of A must divide the dimension of V . To see this, let χ_V and χ_{triv} be the characters of V and the trivial representation of A respectively, then the multiplicity of the trivial representation in V can be given by

$$m = \langle \chi_V, \chi_{\mathrm{triv}} \rangle = \dim(V)/|A|,$$

which is an integer. Hence $|A|$ divides $\dim(V)$.

For the case $\pi \cong \pi^{\theta_0}$ and $m_\phi = 1$, the proof is the same. So we are left with the case $\pi \cong \pi^{\theta_0}$ and $m_\phi = 2$. In this case, we have $\tilde{\pi}^{\theta_0} \cong \tilde{\pi} \otimes \omega$ for some $\omega \in X$. Let

$$X_0(\tilde{\pi}) = \{\omega \in X : \tilde{\pi} \cong \tilde{\pi} \otimes \omega \text{ or } \tilde{\pi}^{\theta_0} \cong \tilde{\pi} \otimes \omega\}.$$

If $\tilde{\pi} \otimes \omega \not\cong \tilde{\pi}^{\theta_0}$ for any $\omega \in Y$, then

$$\sum_{\omega \in X/YX_0(\tilde{\pi})} m(\tilde{\pi} \otimes \omega) = \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}$$

and the rest of the proof is again the same. If $\tilde{\pi} \otimes \omega_1 \cong \tilde{\pi}^{\theta_0}$ for some $\omega_1 \in Y$, we need to consider the action of the two-group

$$\langle (\theta_0, \omega_1) \rangle \times (Y(\tilde{\pi})/\alpha(\mathcal{S}_\phi)) \quad (5.21)$$

on $I(\tilde{\pi})$, where (θ_0, ω_1) acts by $R(\theta_0)^{-1} \circ R(\omega_1)$. It commutes with the action of (θ_0, ω_1) -invariant functions in $\mathcal{H}(G_{\mathbb{A}_F}^1)$, i.e. $\tilde{f}^{\theta_0} \otimes \omega_1 = \tilde{f}$. And as a module of such space of functions, we have

$$I(\tilde{\pi}) \cong \left(\bigoplus_{g \in \tilde{G}(\mathbb{A}_F)/G_{\mathbb{A}_F}^1} I((\pi_+^1)^g) \right) \oplus \left(\bigoplus_{g \in \tilde{G}(\mathbb{A}_F)/G_{\mathbb{A}_F}^1} I((\pi_-^1)^g) \right)$$

where the sign is according to the eigenvalues $\{\pm 1\}$ of any fixed intertwining operator between $\tilde{\pi} \otimes \omega_1$ and $\tilde{\pi}^{\theta_0}$ after we identify $I(\tilde{\pi}) \cong m(\tilde{\pi})\tilde{\pi}$. Note that the multiplicity of $\tilde{\pi}$ is the same as that of irreducible modules in $I(\pi_+^1)$ and $I(\pi_-^1)$ of the subspace of functions described above, and then the rest of the argument proceeds in the same way as before by using the stabilized (θ, ω) -twisted trace formula (5.14) for (θ_0, ω) in (5.21). \square

Corollary 5.9 *Suppose $\tilde{\pi}$ and $\tilde{\pi}'$ are discrete automorphic representations of \tilde{G} , such that $\tilde{\pi} \cong \tilde{\pi}' \otimes \omega$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules for some $\omega \in X$. If π is an irreducible constituent in the restriction of $\tilde{\pi}$ to $G(\mathbb{A}_F)$ and $[\pi] \in \tilde{\Pi}_\phi$ for $\phi \in \tilde{\Phi}_2(G)$, then there exists some $\omega' \in Y$ and $\theta \in \Sigma_0$ such that $\tilde{\pi}' \cong \tilde{\pi}^\theta \otimes \omega'$.*

Proof If $\tilde{G} = GSp(2n)$ or $GSO(2n, \eta)$, this can be seen easily by comparing (5.16) with (5.18). In general, we can first go to the product group \tilde{G} of general symplectic groups and connected general even orthogonal groups [see (2.4)], and it is clear this corollary holds in that case. Then by restricting to \tilde{G} we get the result. \square

To generalize Proposition 5.8, for $[\pi] \in \tilde{\Pi}_\phi$ with $\phi \in \tilde{\Phi}_2(G)$, we denote by $\Sigma_0(\pi, Y)$ the subgroup of Σ_0 consisting of θ such that $\tilde{\pi} \otimes \omega \cong \tilde{\pi}^\theta$ for some $\omega \in Y$, where $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(\mathbb{A}_F)$ containing π in its restriction to $G(\mathbb{A}_F)$. If we write $\Sigma_Y(\pi)$ for the quotient of Σ_0 by $\Sigma_0(\pi, Y)$, then we have an exact sequence

$$1 \longrightarrow \Sigma_0(\pi, Y) \longrightarrow \Sigma_0 \longrightarrow \Sigma_Y(\pi) \longrightarrow 1, \quad (5.22)$$

where all these groups are two-groups. We can also choose a splitting of this sequence and write $\Sigma_0 \cong \Sigma_0(\pi, Y) \times \Sigma_Y(\pi)$.

Corollary 5.10 Suppose $\tilde{\pi}$ is a discrete automorphic representation of \tilde{G} , and π is an irreducible constituent of $\tilde{\pi}$ restricted to $G(\mathbb{A}_F)$. If $[\pi] \in \bar{\Pi}_\phi$ for $\phi \in \bar{\Phi}_2(G)$, then

$$m(\tilde{\pi}) = m_{\tilde{\phi}} \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}, \quad (5.23)$$

where $m_{\tilde{\phi}} = m_\phi / |\Sigma_Y(\pi)|$.

Proof We will use the formula (5.17)

$$\bar{m}(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}.$$

It follows from Corollary 5.9 that

$$\bar{m}(\tilde{\pi}) = \sum_{\theta \in \Sigma_Y(\pi)} m(\tilde{\pi}^\theta).$$

Since $m(\tilde{\pi}^\theta) = m(\tilde{\pi})$ for $\theta \in \Sigma_0$, then we get

$$|\Sigma_Y(\pi)| \cdot m(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}.$$

So by writing $m_{\tilde{\phi}} = m_\phi / |\Sigma_Y(\pi)|$, we have proved the formula (5.23). \square

Suppose $\phi \in \bar{\Phi}_2(G)$, let $\tilde{\zeta}$ be a character of $Z_{\tilde{G}}(\mathbb{A}_F)/Z_{\tilde{G}}(F)$ such that $\zeta = \tilde{\zeta}|_{Z_G}$ is the central character of $\bar{\Pi}_\phi$. If we denote by $\bar{\Pi}_{\phi, \tilde{\zeta}}$ all equivalence classes of irreducible admissible representations of $\tilde{G}(\mathbb{A}_F)$ as $\mathcal{H}(\tilde{G})$ -modules with central character $\tilde{\zeta}$, whose restriction to $G(\mathbb{A}_F)$ have irreducible constituents contained in $\bar{\Pi}_\phi$, then by Corollary 5.6 we can always choose a representative for $[\tilde{\pi}] \in \bar{\Pi}_{\phi, \tilde{\zeta}}/X$ with $\langle \cdot, \tilde{\pi} \rangle = 1$ in the discrete spectrum of \tilde{G} . The following proposition gives a decomposition of the ϕ -component of the discrete spectrum of \tilde{G} .

Proposition 5.11 Suppose $\phi \in \bar{\Phi}_2(G)$, we have the following decomposition as $\mathcal{H}(\tilde{G})$ -modules

$$L^2_{disc, \phi}(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) = m_\phi \sum_{\omega \in Y/\alpha(\mathcal{S}_\phi)} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\phi, \tilde{\zeta}}/X \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{\pi} \otimes \omega, \quad (5.24)$$

where $\tilde{\pi}$ are taken to be the representatives of $\bar{\Pi}_{\phi, \tilde{\zeta}}/X$ in the discrete automorphic spectrum. Moreover,

$$L^2_{disc, \phi}(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) = 0$$

for $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$.

Proof For $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$, we have $L_{disc,\phi}^2(G(F) \backslash G(\mathbb{A}_F)) = 0$ (cf. Theorem 3.13). Then it follows from Lemmas 5.2 and 5.3 that $L_{disc,\phi}^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) = 0$. Next we assume $\phi \in \bar{\Phi}_2(G)$. By Lemma 5.3, $L_{disc,\phi}^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta})$ consists of discrete automorphic representations in $\tilde{\Pi}_{\phi,\tilde{\zeta}}$. Then for any automorphic representation $\tilde{\pi}'$ in $L_{disc,\phi}^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta})$, there exists a representative $\tilde{\pi}$ chosen in (5.24) such that $\tilde{\pi} \cong \tilde{\pi}' \otimes \omega$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules for some $\omega \in X$. By Corollary 5.9, $\tilde{\pi}' \cong \tilde{\pi}^\theta \otimes \omega'$ for $\theta \in \Sigma_0$ and $\omega' \in Y$. In particular, $\tilde{\pi}' \cong \tilde{\pi} \otimes \omega'$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules. Therefore, it suffices to count the multiplicity of $\tilde{\pi}$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules in $L_{disc,\phi}^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta})$. By Corollary 5.9 again,

$$\sum_{\tilde{\pi}' \sim \tilde{\pi}} m(\tilde{\pi}') = \sum_{\theta \in \Sigma_0, \omega \in \bar{Y}(\tilde{\pi})} m(\tilde{\pi}^\theta \otimes \omega) = \left| \frac{\bar{Y}(\tilde{\pi})}{Y(\tilde{\pi})} \right| \cdot |\Sigma_Y(\pi)| \cdot m(\tilde{\pi}),$$

where $\tilde{\pi}' \cong \tilde{\pi}$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules and

$$\bar{Y}(\tilde{\pi}) = \{\omega \in Y : \tilde{\pi} \otimes \omega \cong \tilde{\pi} \text{ as } \tilde{\mathcal{H}}(\tilde{G})\text{-modules}\}.$$

By Corollary 5.10, we have

$$|\Sigma_Y(\pi)| \cdot m(\tilde{\pi}) = m_\phi \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|},$$

so

$$\sum_{\tilde{\pi}' \sim \tilde{\pi}} m(\tilde{\pi}') = m_\phi \frac{|\bar{Y}(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}.$$

This is exactly the multiplicity we get from (5.24). □

Now let us get back to the multiplicity formula. Note under the assumption of Proposition 5.8, if \tilde{G} is general symplectic, then the multiplicity formula (5.16) becomes

$$m(\tilde{\pi}) = \frac{|Y(\tilde{\pi})|}{|\alpha(\mathcal{S}_\phi)|}.$$

It is an interesting question to ask when one can have multiplicity one, i.e. $|Y(\tilde{\pi})| = |\alpha(\mathcal{S}_\phi)|$. Since $\alpha(\mathcal{S}_\phi)$ is a subgroup of $Y(\tilde{\pi})$, it is the same to ask when

$$\alpha(\mathcal{S}_\phi) = Y(\tilde{\pi}).$$

By Corollary 4.2 we have the following description for $Y(\tilde{\pi})$. Let us define

$$\prod_v^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}) := \{\omega \in Y : \omega_v \in \alpha(\mathcal{S}_{\phi_v}) \text{ for all } v\},$$

$$\prod_{\text{almost all } v}^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}) := \{\omega \in Y : \omega_v \in \alpha(\mathcal{S}_{\phi_v}) \text{ for almost all } v\},$$

then

$$Y(\tilde{\pi}) = \prod_v^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}).$$

Moreover, we get a sequence of inclusions

$$\alpha(\mathcal{S}_{\phi}) \subseteq \prod_v^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}) \subseteq \prod_{\text{almost all } v}^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}).$$

Motivated by the case that G is symplectic and $\phi \in \tilde{\Phi}_2(G)$, we give the following definition for both symplectic groups and special even orthogonal groups.

Definition 5.12 Suppose $\phi \in \tilde{\Phi}(G)$, we say **multiplicity one** holds for $\tilde{\phi}$ if

$$\alpha(\mathcal{S}_{\phi}) = \prod_v^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}).$$

Definition 5.13 Suppose $\phi \in \tilde{\Phi}(G)$, we say **strong multiplicity one** holds for $\tilde{\phi}$ if

$$\prod_v^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}) = \prod_{\text{almost all } v}^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}).$$

The motivation for the first definition is now clear, while the second definition needs some explanation. But before giving the explanation, we want to give two modified definitions of the same kind. In view of Theorem 3.13, we need to deal with the group of characters ω_v such that

$$\tilde{f}_v(\tilde{\pi}_v \otimes \omega_v) = \tilde{f}_v(\tilde{\pi}_v), \quad \tilde{f}_v \in \tilde{\mathcal{H}}(\tilde{G}_v)$$

for $[\pi_v] \in \bar{\Pi}_{\phi_v}$. It follows from Corollary 4.2 that this group is isomorphic to $\alpha(\mathcal{S}_{\phi_v}^{\Sigma_0})$. Then we can similarly define a sequence of inclusions

$$\alpha(\mathcal{S}_{\phi}^{\Sigma_0}) \subseteq \prod_v^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) \subseteq \prod_{\text{almost all } v}^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}),$$

and define the concepts of multiplicity one and strong multiplicity one in the same way regarding these groups.

Definition 5.14 Suppose $\phi \in \tilde{\Phi}(G)$, we say Σ_0 -**multiplicity one** holds for $\tilde{\phi}$ if

$$\alpha(S_{\phi}^{\Sigma_0}) = \prod_v^{\text{aut}} \alpha(S_{\phi_v}^{\Sigma_0}).$$

Definition 5.15 Suppose $\phi \in \tilde{\Phi}(G)$, we say Σ_0 -**strong multiplicity one** holds for $\tilde{\phi}$ if

$$\prod_v^{\text{aut}} \alpha(S_{\phi_v}^{\Sigma_0}) = \prod_{\text{almost all } v}^{\text{aut}} \alpha(S_{\phi_v}^{\Sigma_0}).$$

Recall that we can associate a global L -packet $\bar{\Pi}_{\phi}$ to $\phi \in \tilde{\Phi}(G)$, so we can talk about strong multiplicity one for the global L -packet $\bar{\Pi}_{\phi}$, i.e. if π is automorphic, and $[\pi_v] \in \bar{\Pi}_{\phi_v}$ for almost all places v , then $[\pi]$ lies in $\bar{\Pi}_{\phi}$. As in the local case (see Theorem 4.6), we can expect to lift the global L -packet $\bar{\Pi}_{\phi}$ to some global L -packet $\bar{\Pi}_{\tilde{\phi}}$ for \tilde{G} . Obviously the lift is not unique, but as one can see from Corollary 5.9, it should be unique up to twisting by idèle class characters. Because we already have strong multiplicity one for $\bar{\Pi}_{\phi}$, so strong multiplicity one for $\bar{\Pi}_{\tilde{\phi}}$ is equivalent to the property that for any ω in Y if $\bar{\Pi}_{\tilde{\phi}_v} = \bar{\Pi}_{\phi_v} \otimes \omega_v$ for almost all places v , then $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\phi} \otimes \omega$. And it can be easily seen that this property is equivalent to the condition of Σ_0 -strong multiplicity one in our definition.

5.3 Statement of global theorem

After discussing the multiplicity question, we want to describe the ϕ -component of the discrete spectrum for \tilde{G} , which should be an analogue of Theorem 3.13. We again assume \tilde{G} is of type (2.2).

Conjecture 5.16 1. Suppose $\phi \in \tilde{\Phi}(G)$, one can associate a global packet $\bar{\Pi}_{\tilde{\phi}}$ of $\tilde{\mathcal{H}}(\tilde{G})$ -modules of irreducible admissible representations for $\tilde{G}(\mathbb{A}_F)$ satisfying the following properties:

- (a) $\bar{\Pi}_{\tilde{\phi}} = \bigotimes_v' \bar{\Pi}_{\tilde{\phi}_v}$ where $\bar{\Pi}_{\tilde{\phi}_v}$ is some lift of $\bar{\Pi}_{\phi_v}$ defined in Theorem 4.6.
- (b) there exists $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ such that that $\tilde{\pi}$ is isomorphic to an automorphic representation as $\tilde{\mathcal{H}}(\tilde{G})$ -modules.

Moreover, $\bar{\Pi}_{\tilde{\phi}}$ is unique up to twisting by characters of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$. And we can define a global character of $S_{\tilde{\phi}}$ by

$$\langle x, \tilde{\pi} \rangle := \prod_v \langle x_v, \tilde{\pi}_v \rangle \quad \text{for } \tilde{\pi} \in \bar{\Pi}_{\tilde{\phi}} \text{ and } x \in S_{\tilde{\phi}}.$$

2. Suppose $\phi \in \tilde{\Phi}_2(G)$, the ϕ -component of the discrete spectrum of $\tilde{G}(\mathbb{A}_F)$ as $\tilde{\mathcal{H}}(\tilde{G})$ -module has a decomposition.

$$L^2_{disc, \phi}(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) = m_\phi \bigoplus_{\omega \in Y/\alpha(\mathcal{S}_\phi)} \bigoplus_{\substack{[\tilde{\pi}] \in \tilde{\Pi}_{\tilde{\phi}} \otimes \omega \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{\pi}, \quad (5.25)$$

where m_ϕ is defined as in Remark 3.14.

Along with this conjecture, we need to prove some results about the stable multiplicity formula for \tilde{G} (see Conjecture 1.5). This formula has been conjectured by Arthur [2] for any quasisplit connected reductive groups, and he also proved this for special orthogonal group and symplectic group in [7]. To state the formula, we need some preparations. Suppose S is a connected complex reductive group with an automorphism θ , we denote $S^\theta = S \rtimes \theta$, which can be viewed as a connected component of the complex reductive group $S^+ := S \rtimes \langle \theta \rangle$. We fix a maximal torus T of S , and define the Weyl set

$$W^\theta(S) = \text{Norm}(T, S^\theta)/T.$$

Let $W^\theta(S)_{reg}$ be the set of Weyl elements w such that

$$\det(w - 1)|_{\mathfrak{a}_T} \neq 0.$$

Moreover, let $s^0(w)$ denote the sign $(-1)^n$, where n is the number of positive roots of (S, T) mapped by w to negative roots. Now we can assign to S^θ a real number

$$i^\theta(S) = |W(S)|^{-1} \sum_{w \in W^\theta_{reg}(S)} s^0(w) |\det(w - 1)|_{\mathfrak{a}_T}^{-1},$$

where $W(S)$ is the Weyl group of S . Next we want to define a constant $\sigma(S_1)$ associated with any connected complex reductive group S_1 . To define this we have to introduce some more notations. Still for the original S^θ , let us denote the set of semisimple elements of S^θ by S^θ_{ss} . And for any $s \in S^\theta_{ss}$, we write

$$S_s = \text{Cent}(s, S).$$

Let

$$S^\theta_{ell} = \{s \in S^\theta_{ss} : |Z(S_s)| < \infty\},$$

and $\mathcal{E}^\theta_{ell}(S)$ be the S -conjugacy classes in S^θ_{ell} . Finally the constant $\sigma(S_1)$ can be characterized by the following proposition ([7], Proposition 4.1.1).

Proposition 5.17 *There are unique constants $\sigma(S_1)$ defined for all connected complex reductive groups S_1 , such that for any connected component S^θ of a complex reductive group, the following number*

$$e^\theta(S) = \sum_{s \in \mathcal{E}_{\text{ell}}^\theta(S)} |\pi_0(S_s)|^{-1} \sigma((S_s)^0)$$

equals $i^\theta(S)$, and furthermore

$$\sigma(S_1) = \sigma(S_1/Z_1) |Z_1|^{-1},$$

for any central subgroup Z_1 of S_1 .

Now we can state the stable multiplicity formula for \tilde{G} as follows.

Conjecture 5.18 *Suppose $\phi \in \bar{\Phi}(G)$, then*

$$S_{\text{disc}, \phi}^{\tilde{G}}(\tilde{f}) = m_\phi \sum_{\omega \in Y/\alpha(\mathcal{S}_\phi)} |\mathcal{S}_{\tilde{\phi}}|^{-1} \sigma(\bar{S}_\phi^0) \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}), \quad (5.26)$$

where

$$\tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega) := \prod_v \tilde{f}_v(\tilde{\phi}_v \otimes \omega_v),$$

with respect to $\bar{\Pi}_{\tilde{\phi}}$ defined in Conjecture 5.16.

Finally, we need a twisted version of the decomposition (5.25), whose role will be clear in the next section.

Conjecture 5.19 *Suppose $\phi \in \bar{\Phi}_2(G)$ and $x \in \mathcal{S}_\phi^\theta$ with $\alpha(x) = \omega$ for $\theta \in \Sigma_0$ and some character ω of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$. For $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ with $\langle \cdot, \tilde{\pi} \rangle = 1$, the canonical intertwining operator*

$$R(\theta)^{-1} \circ R(\omega)$$

restricted to the $\tilde{\pi}$ -isotypic component $I(\tilde{\pi})$ in the discrete spectrum is equal to the product of $m(\tilde{\pi})$ and the local intertwining operators $A_{\tilde{\pi}_v}(\theta, \omega_v)$ normalized by x_v (see (4.3)), i.e.

$$I_{\text{disc}, \phi}^{\tilde{G}^\theta, \omega}(\tilde{f}) = m_\phi \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}), \quad (5.27)$$

where $\tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega) = \prod_v \tilde{f}_{\tilde{G}_v^\theta}(\tilde{\pi}_v, \omega_v)$, and it does not depend on x .

Remark 5.20 This kind of result has been proved in the cases of special even orthogonal groups (see [7], Theorem 4.2.2) and general linear groups (see [7], Lemma 4.2.3).

In this paper, we will only establish these conjectures in a special case.

Theorem 5.21 *Suppose $G = G_1 \times G_2 \times \cdots \times G_q$, such that G_i is a symplectic group or a special even orthogonal group. For $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q \in \tilde{\Phi}(G)$ with $\phi_i \in \tilde{\Phi}(G_i)$, if $S_{\tilde{\phi}_i} = 1$ for all i , then Conjecture 5.16, 5.19 hold. If we further assume $\phi \in \tilde{\Phi}_2(G)$, then Conjecture 5.18 also holds.*

5.4 Comparison of trace formulas

We assume \tilde{G} is of type 2.2 and $\theta \in \Sigma_0$. Since we are going to prove all the theorems by induction, here we would like to take a temporary induction assumption: we assume Conjecture 5.16, 5.18, 5.19 together with our main local theorem (Theorem 4.6) hold for the proper Levi subgroups and twisted endoscopic groups of \tilde{G} . Based on this assumption, we want to expand the ϕ -component of (5.4) and (5.5) in terms of local objects. Before we do the expansion, let us write $\Phi(G, \phi)$ for the set of global Langlands parameters of G giving rise to $\phi \in \tilde{\Phi}(G)$. It is clear that $|\Phi(G, \phi)| = m_\phi$. So we can write formally those formulas (5.25), (5.26) and (5.27) for $\phi_G \in \Phi(G, \phi)$ by simply setting $m_{\phi_G} = 1$. In fact these formal formulas do make sense when we associate to ϕ_G the refined global L -packet (see [7], Section 8.4). But we do not need this refinement here, for eventually we are going to sum over $\Phi(G, \phi)$. The benefit of working with these global Langlands parameters is one can imitate the computation in ([2], Section 5 and 7), where one does assume the global Langlands correspondence.

5.4.1 The spectral expansion

Let us write the ϕ -component of (5.4) as

$$I_{disc, \phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = \sum_{\{\tilde{M}\}} |W(\tilde{M})|^{-1} \sum_{w \in W^\theta(\tilde{M})_{reg}} |\det(w-1)_{\mathfrak{a}_{\tilde{M}}^{\tilde{G}^\theta}}|^{-1} \text{tr}(M_{\tilde{P}|\theta, \phi}(w, \tilde{\chi}) I_{\tilde{P}, \phi}^{\theta, \omega}(\tilde{\chi}, \tilde{f})).$$

So the key is to expand

$$\text{tr}(M_{\tilde{P}|\theta, \phi}(w, \tilde{\chi}) I_{\tilde{P}, \phi}^{\theta, \omega}(\tilde{\chi}, \tilde{f})) \quad (5.28)$$

By definition, (5.28) does not vanish only if there exists $\phi_M \in \tilde{\Phi}_2(M, \phi)$. Moreover, the $(G(F) \rtimes \Sigma_0)$ -conjugacy class of M such that $\tilde{\Phi}_2(M, \phi) \neq \emptyset$ is determined by ϕ , and the choice of $\phi_G \in \Phi(G, \phi)$ determines the $G(F)$ -conjugacy class of M such that $\Phi_2(M, \phi_G) \neq \emptyset$. So we can fix such a $G(F)$ -conjugacy class of M and assume $\phi_M \in \tilde{\Phi}_2(M, \phi)$. To apply our induction assumption we also need to assume $M \neq G$, i.e., $\phi \notin \tilde{\Phi}_2(G)$.

Note the diagram (4.12) in our discussion of the local (θ, ω) -twisted intertwining relation can be defined in the global case, and the global analogue of those groups in

the diagram will map to their local counterparts. It is not hard to show, using a similar argument in Proposition 5.8, that (5.28) vanishes unless there exists $u \in \mathfrak{N}_\phi^\theta$ such that $w_u = w$ and $\omega = \alpha(x_u)$. So $\phi_M \in \tilde{\Phi}_2(M^{\theta_u})$. Now we can apply Conjecture 5.16 to $\tilde{\phi}_M$. For any $[\tilde{\pi}_M] \in \tilde{\Pi}_{\tilde{\phi}_M}$, let us define

$$R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) := \prod_v R_{\tilde{P}_v|\theta\tilde{P}_v}(u_v, \tilde{\pi}_{M_v}, \tilde{\phi}_v).$$

In particular, if $\tilde{\pi}_M \in \mathcal{A}_2(\tilde{M})$, we can write

$$R_{\tilde{P}|\theta\tilde{P}}(w, \tilde{\pi}_M, \tilde{\phi}) := r_P(w, \phi_M)^{-1} M_{\tilde{P}|\theta\tilde{P}}(w, \tilde{\pi}_M),$$

where $r_P(w, \phi_M)$ is the global normalizing factor defined by

$$r_P(w, \phi_M) = \prod_v r_P(w_v, \phi_{M_v}).$$

It follows from Conjecture 5.19 and analogous result for $GL(N)$ (cf. Remark 5.20) that

$$R_{\tilde{P}|\theta\tilde{P}}(w, \tilde{\pi}_M, \tilde{\phi}) = R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi})$$

for any $u \in \mathfrak{N}_\phi^\theta(w, \omega)$. Here $\mathfrak{N}_\phi^\theta(w, \omega)$ consists of $u \in \mathfrak{N}_\phi^\theta$ such that $w_u = w$ and $\alpha(x_u) = \omega$. Applying Conjecture 5.16 (2) to \tilde{M} , we can write (5.28) as a double sum over $\phi_G \in \Phi(G, \phi)$ and $\phi_M \in \Phi_2(M^{\theta_u}, \phi_G)$ of

$$\sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_M})} \sum_{[\tilde{\pi}_M] \in \tilde{\Pi}_{\tilde{\phi}_M} \otimes \omega'} \delta_{\tilde{\phi}_M}(\tilde{\pi}_M) r_P(w, \phi_M) \text{tr}(R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) I_{\tilde{P}}^{\theta, \omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})),$$

where

$$\delta_{\tilde{\phi}_M}(\tilde{\pi}_M) = |\mathcal{S}_{\tilde{\phi}_M}|^{-1} \sum_{x \in \mathcal{S}_{\tilde{\phi}_M}} \langle x, \tilde{\pi}_M \rangle.$$

Moreover, we can write

$$\sum_{x \in \mathcal{S}_{\tilde{\phi}_M}} \langle x, \tilde{\pi}_M \rangle R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) = \sum_{u \in \mathfrak{N}_\phi^\theta(w, \omega)} R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}).$$

If we switch the sum over $\tilde{\pi}_M \in \tilde{\Pi}_{\tilde{\phi}_M} \otimes \omega'$ with $u \in \mathfrak{N}_\phi^\theta(w, \omega)$, and define

$$\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', u) = \sum_{[\tilde{\pi}_M] \in \tilde{\Pi}_{\tilde{\phi}_M} \otimes \omega'} \text{tr}(R_{\tilde{P}|\theta\tilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) I_{\tilde{P}}^{\theta, \omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})),$$

then (5.28) becomes a double sum over $\phi_G \in \Phi(G, \phi)$ and $\phi_M \in \Phi_2(M^{\theta_u}, \phi_G)$ of

$$\sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_M})} |\mathcal{S}_{\tilde{\phi}_M}|^{-1} \sum_{u \in \mathfrak{N}_{\phi}^{\theta}(w, \omega)} r_P(w, \phi_M) \tilde{f}_{\tilde{G}^{\theta}}(\tilde{\phi} \otimes \omega', u). \quad (5.29)$$

Since we are taking $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$, the contributions of $\phi_G \in \Phi_2(G, \phi)$ to the ϕ -component of (5.4) are the same. So the ϕ -component of (5.4) can be written as a sum over $w \in W^{\theta}(\tilde{M})_{reg}$ and $\phi_M \in \Phi_2(M^{\theta_u}, \phi_G)$ of

$$m_{\phi} |W(\tilde{M})|^{-1} |\det(w - 1)|_{\mathfrak{a}_{\tilde{M}}^{\tilde{G}^{\theta}}}^{-1}$$

multiplied with (5.29). Here we can identify $W(\tilde{M})$ with $W(M)$, and it is easy to see

$$|\det(w - 1)|_{\mathfrak{a}_{\tilde{M}}^{\tilde{G}^{\theta}}} = |\det(w - 1)|_{\mathfrak{a}_M^{G^{\theta}}}.$$

Next we want to switch the order of the double sum over $w \in W^{\theta}(\tilde{M})_{reg}$ and $\phi_M \in \Phi_2(M^{\theta_u}, \phi_G)$ to a double sum over $\phi_M \in \Phi_2(M, \phi_G)$ and $w \in W_{\phi, reg}^{\theta}$, where $W_{\phi, reg}^{\theta} = W^{\theta}(\tilde{M})_{reg} \cap W_{\phi}^{\theta}$. Since the nonzero contribution of each $\phi_M \in \Phi_2(M, \phi_G)$ is the same and

$$|\Phi_2(M, \phi_G)| = \frac{|W(M)|}{|W_{\phi}|},$$

then we get a single sum over $w \in W_{\phi, reg}^{\theta}$ of

$$m_{\phi} |W_{\phi}|^{-1} |\mathcal{S}_{\tilde{\phi}_M}|^{-1} |\det(w - 1)|_{\mathfrak{a}_M^{G^{\theta}}}^{-1}$$

multiplied with

$$\sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_M})} \sum_{u \in \mathfrak{N}_{\phi}^{\theta}(w, \omega)} r_P(w, \phi_M) \tilde{f}_{\tilde{G}^{\theta}}(\tilde{\phi} \otimes \omega', u).$$

Note the double sum over $w \in W_{\phi, reg}^{\theta}$ and $u \in \mathfrak{N}_{\phi}^{\theta}(w, \omega)$ can be rearranged as a double sum over $x \in \mathcal{S}_{\phi}^{\theta}(\omega)$ and $u \in \mathfrak{N}_{\phi, reg}^{\theta}(x)$, where

$$\mathcal{S}_{\phi}^{\theta}(\omega) = \{x \in \mathcal{S}_{\phi}^{\theta} : \alpha(x) = \omega\},$$

and

$$\mathfrak{N}_{\phi, reg}^{\theta}(x) = \left\{ u \in \mathfrak{N}_{\phi}^{\theta} : x_u = x, w_u \in W_{\phi, reg}^{\theta} \right\}.$$

So we end up with a sum over $x \in \mathcal{S}_\phi^\theta(\omega)$ of

$$m_\phi |W_\phi|^{-1} |\mathcal{S}_{\tilde{\phi}_M}|^{-1}$$

multiplied with

$$\sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_M})} \sum_{u \in \mathfrak{N}_{\phi, \text{reg}}^\theta(x)} |\det(w_u - 1)|_{\mathfrak{a}_M^{G^\theta}}^{-1} r_P(w_u, \phi_M) \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', u). \quad (5.30)$$

If we write

$$r_\phi^G(w_u) = r_P(w_u, \phi_M),$$

and define $s_\phi^0(w_u)$ to be $(-1)^n$, where n is the number of positive roots of $(\bar{S}_\phi^0, \bar{T}_\phi)$ mapped to negative roots by w_u , then by Arthur's sign lemma ([7], Lemma 4.3.1) we have

$$r_\phi^G(w_u) = s_\phi^0(w_u).$$

Moreover, by our comments after Lemma 4.10, $\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', u)$ only depends on the image of u in \mathcal{S}_ϕ^θ , so we can write

$$\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', u) = \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x)$$

Therefore, the term (5.30) becomes

$$\sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_M})} \left(\sum_{u \in \mathfrak{N}_{\phi, \text{reg}}^\theta(x)} s_\phi^0(w_u) |\det(w_u - 1)|_{\mathfrak{a}_M^{G^\theta}}^{-1} \right) \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x).$$

For $\omega' \in \alpha(\mathcal{S}_\phi)$, $\bar{\Pi}_{\tilde{\phi}} \otimes \omega' = \bar{\Pi}_{\tilde{\phi}}$ and

$$(\tilde{f}|_{\tilde{Z}_{FG(F)}})_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x) = f_{G^\theta}(\phi, x) = (\tilde{f}|_{\tilde{Z}_{FG(F)}})_{\tilde{G}^\theta}(\tilde{\phi}, x),$$

where f is the restriction of \tilde{f} to $G(F)$. So we get for $\omega' \in \alpha(\mathcal{S}_\phi)$

$$\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x) = \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi}, x).$$

Therefore we only need to take the sum over $\omega' \in Y/\alpha(\mathcal{S}_\phi)$ in (5.30), and then multiply by $|\alpha(\mathcal{S}_\phi)/\alpha(\mathcal{S}_{\phi_M})|$. Since

$$|\alpha(\mathcal{S}_\phi)/\alpha(\mathcal{S}_{\phi_M})| = \frac{|\mathcal{S}_\phi|}{|\mathcal{S}_{\tilde{\phi}}|} \cdot \frac{|\mathcal{S}_{\tilde{\phi}_M}|}{|\mathcal{S}_{\phi_M}|},$$

the resulting constant multiple is m_ϕ times

$$\begin{aligned} |W_\phi|^{-1} |\mathcal{S}_{\tilde{\phi}_M}|^{-1} \frac{|\mathcal{S}_\phi|}{|\mathcal{S}_{\tilde{\phi}}|} \cdot \frac{|\mathcal{S}_{\tilde{\phi}_M}|}{|\mathcal{S}_{\phi_M}|} &= |W_\phi|^{-1} |\mathcal{S}_{\phi_M}|^{-1} \frac{\mathcal{S}_\phi}{\mathcal{S}_{\tilde{\phi}}} \\ &= |\mathfrak{N}_\phi|^{-1} \frac{|\mathcal{S}_\phi|}{|\mathcal{S}_{\tilde{\phi}}|} \\ &= |W_\phi^0|^{-1} |\mathcal{S}_\phi|^{-1} \frac{|\mathcal{S}_\phi|}{|\mathcal{S}_{\tilde{\phi}}|} \\ &= |W_\phi^0|^{-1} |\mathcal{S}_{\tilde{\phi}}|^{-1}. \end{aligned}$$

Let

$$C_{\tilde{\phi}} = m_\phi |\mathcal{S}_{\tilde{\phi}}|^{-1},$$

and we define

$$i_\phi^\theta(x) = |W_\phi^0|^{-1} \sum_{w \in W_{\phi, \text{reg}}^\theta(x)} s_\phi^0(w) |\det(w - 1)_{\mathfrak{a}_M^{G^\theta}}|^{-1},$$

where $W_{\phi, \text{reg}}^\theta(x)$ is the image of $\mathfrak{N}_{\phi, \text{reg}}^\theta(x)$ in $W_{\phi, \text{reg}}^\theta$. Hence we have shown the following lemma.

Lemma 5.22 *Suppose $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$, $\theta \in \Sigma_0$ and $\omega \in Y$, then*

$$I_{\text{disc}, \phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} \sum_{x \in \mathcal{S}_\phi^\theta(\omega)} i_\phi^\theta(x) \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}). \quad (5.31)$$

5.4.2 The endoscopic expansion

Parallel to this (θ, ω) -twisted spectral expansion (5.31), we will proceed to expand the ϕ -component of (5.5). Note that if $\theta = id$, $\omega = 1$, we can only expand the right hand side of

$$I_{\text{disc}, \phi}^{\tilde{G}}(\tilde{f}) - S_{\text{disc}, \phi}^{\tilde{G}}(\tilde{f}) = \sum_{\tilde{G}' \in \mathcal{E}_{\text{ell}}(\tilde{G}) - \{\tilde{G}\}} \iota(\tilde{G}, \tilde{G}') S_{\text{disc}, \phi}^{\tilde{G}'}(\tilde{f}^{\tilde{G}'})$$

based on our temporary induction assumption. By Corollary 2.6 we know $\text{Ker}^1(F, Z(\tilde{G})) = \text{Ker}^1(F, Z(\tilde{G}')) = 1$, so the formula (5.3) applied to $\iota(\tilde{G}, \tilde{G}')$ can be simplified as

$$\iota(\tilde{G}, \tilde{G}') = |\bar{Z}(\tilde{G}')^\Gamma|^{-1} |\text{Out}_{\tilde{G}}(\tilde{G}')|^{-1} |\pi_0(\kappa_{\tilde{G}^\theta})|^{-1} \quad (5.32)$$

where

$$\bar{Z}(\widehat{G}')^\Gamma = Z(\widehat{G}')^\Gamma Z(\widehat{G})^\Gamma / Z(\widehat{G})^\Gamma,$$

and

$$\text{Out}_{\widetilde{G}}(\widetilde{G}') = \text{Aut}_{\widetilde{G}}(\widetilde{G}') / \widehat{G}' Z(\widehat{G})^\Gamma.$$

Note that $|\pi_0(\kappa_{\widetilde{G}^\theta})| = 1$ here, and this formula (5.32) is given in ([2], Lemma 3.2). By applying Conjecture 5.18 to \widetilde{G}' , we get

$$S_{disc, \phi'}^{\widetilde{G}'}(\tilde{f}') = \sum_{\omega' \in Y' / \alpha^{G'}(\mathcal{S}_{\phi'})} |\mathcal{S}_{\phi'}|^{-1} \sigma(\bar{S}_{\phi'}^\theta) \tilde{f}'(\tilde{\phi}' \otimes \omega'), \quad \tilde{f}' \in \bar{\mathcal{H}}(\widetilde{G}', \widetilde{\chi}')$$

for $\phi' \in \Phi(G')$. By Lemma 5.7, the ϕ -component of (5.5) is summed over $\phi_G \in \Phi(G, \phi)$ and

$$\{(\widetilde{G}', \phi') : \widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}^\theta, \omega) \text{ and } \phi' \in \Phi(G', \phi_G)\}$$

of distributions $S_{disc, \phi'}^{\widetilde{G}'}(\tilde{f}')$. Again because we are taking $\tilde{f}' \in \bar{\mathcal{H}}(\widetilde{G}', \widetilde{\chi}')$, the contributions of $\phi_G \in \Phi(G, \phi)$ to the ϕ -component of (5.5) are the same. If we fix a parameter $\phi_G \in \Phi(G, \phi)$, then the first sum collapses to be a constant multiple

$$|\Phi(G, \phi)| = m_\phi.$$

Now let us fix $\phi_G := \phi^\mathcal{E}$ as a homomorphism from \mathcal{L}_ϕ to ${}^L G$, instead of a \widehat{G} -conjugacy class, and let $\bar{S}_\phi^\theta = \text{Cent}(\text{Im } \phi_G, \widehat{G} \rtimes \theta) / Z(\widehat{G})^\Gamma$. We observe (\widetilde{G}', ϕ') will correspond to (ϕ_G, s) for $s \in \bar{S}_{\phi, ss}^\theta(\omega)$, where

$$\bar{S}_{\phi, ss}^\theta(\omega) = \{s \in \bar{S}_{\phi, ss}^\theta : \alpha(s) = \omega\}$$

by taking suitable \widehat{G} -conjugation, and s is determined up to \bar{S}_ϕ -conjugation. Let us use the convention to denote the conjugacy class of \bar{S}_ϕ in $\bar{S}_{\phi, ss}^\theta(\omega)$ by

$$\bar{S}_\phi \backslash \bar{S}_{\phi, ss}^\theta(\omega).$$

Then this correspondence gives us a map

$$\{(\widetilde{G}', \phi') : \widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}^\theta, \omega) \text{ and } \phi' \in \Phi(G', \phi_G)\} \longrightarrow \bar{S}_\phi \backslash \bar{S}_{\phi, ss}^\theta(\omega)$$

The point is the contribution of (\widetilde{G}', ϕ') only depends on its image under this map, so we want to write the double sum over (\widetilde{G}', ϕ') as a single sum over the image. To

characterize the image, it is equivalent to find $s \in \bar{S}_{\phi,ss}^\theta(\omega)$ with $(\tilde{G}'_s, \phi') \longrightarrow (\phi_G, s)$ such that \tilde{G}'_s is elliptic. If we define

$$\begin{aligned}\bar{S}_{\phi,ell}^\theta &= \left\{ s \in \bar{S}_{\phi,ss}^\theta : |Z(\bar{S}_{\phi,s}^0)| < \infty \right\} \\ \bar{S}_{\phi,ell}^\theta(\omega) &= \left\{ s \in \bar{S}_{\phi,ss}^\theta(\omega) : |Z(\bar{S}_{\phi,s}^0)| < \infty \right\},\end{aligned}$$

then for $s \in \bar{S}_{\phi,ell}^\theta(\omega)$, it is easy to see that \tilde{G}'_s is elliptic. The converse is not true, but the contribution from pairs (\tilde{G}'_s, ϕ') with $s \notin \bar{S}_{\phi,ell}^\theta(\omega)$ is zero by the stable multiplicity formula (5.26). In fact

$$\bar{S}_{\phi'}^0 = (\bar{S}_{\phi,s})^0 \bar{Z}(\widehat{G}')^\Gamma / \bar{Z}(\widehat{G}')^\Gamma$$

and $\sigma(\bar{S}_{\phi'}^0) = 0$ unless $|Z(\bar{S}_{\phi'}^0)| < \infty$. If we write

$$\bar{S}_{\phi,ell}^{\prime\theta} = \begin{cases} \bar{S}_{\phi,ell} - \{1\} & \text{if } \theta = id \\ \bar{S}_{\phi,ell}^\theta & \text{otherwise} \end{cases}$$

and

$$\bar{S}_{\phi,ell}^{\prime\theta}(\omega) = \begin{cases} \bar{S}_{\phi,ell} - \{1\} & \text{if } \theta = id, \omega = 1 \\ \bar{S}_{\phi,ell}^\theta(\omega) & \text{otherwise} \end{cases}$$

then the effective image of this map should be $\bar{S}_\phi \backslash \bar{S}_{\phi,ell}^{\prime\theta}(\omega)$. The next problem is to count the fibre of this map. Since \tilde{G}' is taken to be the isomorphism class of endoscopic data, the fibre containing (\tilde{G}', ϕ') must have the endoscopic datum isomorphic to \tilde{G}' , and hence can be obtained by the action of $\text{Aut}_G(G')$. Moreover, ϕ' is taken to be \widehat{G}' -conjugacy classes, so the fibre should be isomorphic to

$$\text{Aut}_G(G') / S_{\phi,s} \text{Int}_G(G') \cong \text{Out}_G(G') / (S_{\phi,s} \text{Int}_G(G') / \text{Int}_G(G')),$$

where $\text{Int}_G(G') = \widehat{G}' Z(\widehat{G})^\Gamma$ and $S_{\phi,s}$ is the preimage of $\bar{S}_{\phi,s}$ in S_ϕ . Moreover let us write

$$S_{\phi,s} \text{Int}_G(G') / \text{Int}_G(G') \cong S_{\phi,s} / S_{\phi,s} \cap \widehat{G}' Z(\widehat{G})^\Gamma$$

So we can turn the ϕ -component of (5.5) into a sum over $s \in \bar{S}_\phi \backslash \bar{S}_{\phi,ell}^{\prime\theta}(\omega)$, but multiplied with the size of each fibre

$$|\text{Out}_G(G')| |S_{\phi,s} / S_{\phi,s} \cap \widehat{G}' Z(\widehat{G})^\Gamma|^{-1}.$$

In fact, it is more convenient to sum over the conjugacy classes in $\bar{S}_{\phi,ell}^{\prime\theta}(\omega)$ by the group \bar{S}_{ϕ}^0 . So let us define

$$\begin{aligned}\mathcal{E}_{\phi,ell}^{\prime\theta} &= \bar{S}_{\phi}^0 \backslash \bar{S}_{\phi,ell}^{\prime\theta}, \\ \mathcal{E}_{\phi,ell}^{\prime\theta}(\omega) &= \bar{S}_{\phi}^0 \backslash \bar{S}_{\phi,ell}^{\prime\theta}(\omega),\end{aligned}$$

then changing to sum over $\mathcal{E}_{\phi,ell}^{\prime\theta}(\omega)$ amounts to multiplying by

$$|\bar{S}_{\phi}^0 / \bar{S}_{\phi,s}^0| |\bar{S}_{\phi} / \bar{S}_{\phi,s}|^{-1} = |\bar{S}_{\phi,s} / \bar{S}_{\phi,s}^0| |\bar{S}_{\phi} / \bar{S}_{\phi}^0|^{-1}.$$

Finally, we get a sum over $s \in \mathcal{E}_{\phi,ell}^{\prime\theta}(\omega)$ of the product of the following three terms

$$\begin{aligned}& |\text{Out}_{\tilde{G}}(\tilde{G}')|^{-1} |\text{Out}_G(G')|, \\ & |S_{\phi,s} / S_{\phi,s} \cap \widehat{G}' Z(\widehat{G})^{\Gamma}|^{-1} |\mathcal{S}_{\tilde{\phi}'}|^{-1} |\bar{Z}(\widehat{\tilde{G}}')^{\Gamma}|^{-1} |\bar{S}_{\phi,s} / \bar{S}_{\phi,s}^0|,\end{aligned}$$

and

$$m_{\phi} |\bar{S}_{\phi} / \bar{S}_{\phi}^0|^{-1} \sum_{\omega' \in Y' / \alpha^{G'}(\mathcal{S}_{\phi'})} \sigma(\bar{S}_{\phi'}^0) \tilde{f}^{\tilde{G}'}(\tilde{\phi}' \otimes \omega')$$

where $(G', \phi') \rightarrow (\phi_G, s)$. Note that

$$1 \longrightarrow \widehat{D} \longrightarrow \text{Aut}_{\tilde{G}}(\tilde{G}') \longrightarrow \text{Aut}_G(G') \longrightarrow 1.$$

so

$$\begin{aligned}|\text{Out}_{\tilde{G}}(\tilde{G}')|^{-1} |\text{Out}_G(G')| &= |\text{Int}_G(G') / (\text{Int}_{\tilde{G}}(\tilde{G}') / \widehat{D})| \\ &= |\widehat{G}' Z(\widehat{G})^{\Gamma} / (\widehat{\tilde{G}}' Z(\widehat{\tilde{G}})^{\Gamma} / \widehat{D})|^{-1} \\ &= |Z(\widehat{G})^{\Gamma} / Z(\widehat{G})^{\Gamma} \cap \widehat{G}' (Z(\widehat{\tilde{G}})^{\Gamma} / \widehat{D})|^{-1} \\ &= |Z(\widehat{G})^{\Gamma} / (\widehat{G}' \cap Z(\widehat{G})^{\Gamma}) (Z(\widehat{\tilde{G}})^{\Gamma} / \widehat{D})|^{-1}.\end{aligned}$$

Moreover, we have $\bar{S}_{\tilde{\phi},s}^0 = \bar{S}_{\phi,s}^0$ and $\bar{S}_{\tilde{\phi}}^0 = \bar{S}_{\phi}^0$, so we can rewrite the expansion of ϕ -component of (5.5) as a sum over $s \in \mathcal{E}_{\phi,ell}^{\prime\theta}(\omega)$ of the product of the following two terms

$$|S_{\phi,s} / S_{\phi,s} \cap \widehat{G}' Z(\widehat{G})^{\Gamma}|^{-1} |\mathcal{S}_{\phi'}|^{-1} |\bar{Z}(\widehat{G}')^{\Gamma}|^{-1} |\bar{S}_{\phi,s} / \bar{S}_{\phi,s}^0| \sigma(\bar{S}_{\phi'}^0) \quad (5.33)$$

and

$$\sum_{\omega' \in Y'/\alpha^{G'}(\mathcal{S}_{\phi'})} m_{\phi} |\mathcal{S}_{\phi}|^{-1} |\mathcal{S}_{\phi'}| |\mathcal{S}_{\tilde{\phi}'}|^{-1} |\bar{Z}(\widehat{G}')^{\Gamma}| |\bar{Z}(\widehat{\tilde{G}}')^{\Gamma}|^{-1} |Z(\widehat{G})^{\Gamma}| /$$

$$(\widehat{G}' \cap Z(\widehat{G})^{\Gamma})(Z(\widehat{\tilde{G}}')^{\Gamma}/\widehat{D})|^{-1} \tilde{f}'(\tilde{\phi}' \otimes \omega'). \quad (5.34)$$

As one can see, (5.33) is only relevant to G , and it can be simplified as in ([7], Section 4.4). So we will just repeat the simplification there. First we note that

$$|\mathcal{S}_{\phi'}| = |\pi_0(\bar{\mathcal{S}}_{\phi'})| = |\bar{\mathcal{S}}_{\phi,s} \cap \widehat{\tilde{G}}' / (\bar{\mathcal{S}}_{\phi,s})^0 \bar{Z}(\widehat{G}')^{\Gamma}|,$$

where $\widehat{\tilde{G}}'$ denotes the quotient $\widehat{G}' Z(\widehat{G})^{\Gamma} / Z(\widehat{G})^{\Gamma}$. Consequently,

$$|\mathcal{S}_{\phi,s} / \mathcal{S}_{\phi,s} \cap \widehat{G}' Z(\widehat{G})^{\Gamma}|^{-1} |\mathcal{S}_{\phi'}|^{-1}$$

$$= |\bar{\mathcal{S}}_{\phi,s} / \bar{\mathcal{S}}_{\phi,s} \cap \widehat{\tilde{G}}'|^{-1} |\bar{\mathcal{S}}_{\phi,s} \cap \widehat{\tilde{G}}' / (\bar{\mathcal{S}}_{\phi,s})^0 \bar{Z}(\widehat{G}')^{\Gamma}|$$

$$= |\bar{\mathcal{S}}_{\phi,s} / (\bar{\mathcal{S}}_{\phi,s})^0 \bar{Z}(\widehat{G}')^{\Gamma}|^{-1}.$$

The product of the first four factors of (5.33) therefore equals

$$|\bar{\mathcal{S}}_{\phi,s} / \bar{\mathcal{S}}_{\phi,s}^0 \bar{Z}(\widehat{G}')^{\Gamma}|^{-1} \cdot |\bar{\mathcal{S}}_{\phi,s}^0 \bar{Z}(\widehat{G}')^{\Gamma} / (\bar{\mathcal{S}}_{\phi,s})^0 \bar{Z}(\widehat{G}')^{\Gamma}|^{-1} \cdot |\bar{Z}(\widehat{G}')^{\Gamma}|^{-1} \cdot |\bar{\mathcal{S}}_{\phi,s} / \bar{\mathcal{S}}_{\phi,s}^0|$$

$$= |\bar{\mathcal{S}}_{\phi,s}^0 \bar{Z}(\widehat{G}')^{\Gamma} / \bar{\mathcal{S}}_{\phi,s}^0| \cdot |\bar{\mathcal{S}}_{\phi,s}^0 / (\bar{\mathcal{S}}_{\phi,s})^0|^{-1} \cdot |\bar{\mathcal{S}}_{\phi,s} \cap (\bar{\mathcal{S}}_{\phi,s})^0 \bar{Z}(\widehat{G}')^{\Gamma}|$$

$$/ |(\bar{\mathcal{S}}_{\phi,s})^0| \cdot |\bar{Z}(\widehat{G}')^{\Gamma}|^{-1}$$

$$= |\pi_0(\bar{\mathcal{S}}_{\phi,s}^0)|^{-1} \cdot |\bar{Z}(\widehat{G}')^{\Gamma} / \bar{\mathcal{S}}_{\phi,s}^0 \cap \bar{Z}(\widehat{G}')^{\Gamma}| \cdot |\bar{\mathcal{S}}_{\phi,s}^0 \cap \bar{Z}(\widehat{G}')^{\Gamma}|$$

$$/ |(\bar{\mathcal{S}}_{\phi,s})^0 \cap \bar{Z}(\widehat{G}')^{\Gamma}| \cdot |\bar{Z}(\widehat{G}')^{\Gamma}|^{-1}$$

$$= |\pi_0(\bar{\mathcal{S}}_{\phi,s}^0)|^{-1} \cdot |(\bar{\mathcal{S}}_{\phi,s})^0 \cap \bar{Z}(\widehat{G}')^{\Gamma}|^{-1}.$$

Furthermore, we can write

$$\sigma(\bar{\mathcal{S}}_{\phi'}^0) = \sigma((\bar{\mathcal{S}}_{\phi,s})^0 / (\bar{\mathcal{S}}_{\phi,s})^0 \cap \bar{Z}(\widehat{G}')^{\Gamma})$$

$$= \sigma((\bar{\mathcal{S}}_{\phi,s})^0 | (\bar{\mathcal{S}}_{\phi,s})^0 \cap \bar{Z}(\widehat{G}')^{\Gamma}|).$$

Hence the first term (5.33) is equal to

$$|\pi_0(\bar{\mathcal{S}}_{\phi,s}^0)|^{-1} \sigma((\bar{\mathcal{S}}_{\phi,s})^0).$$

For the second term (5.34), let us denote

$$\tilde{f}^{\widehat{\tilde{G}}'}(\tilde{\phi}' \otimes \omega') = \tilde{f}'_{\widehat{\tilde{G}}'}(\tilde{\phi} \otimes \omega', s), \quad \tilde{f} \in \mathcal{H}(\tilde{G}, \tilde{\chi}).$$

Also notice $|\alpha^{G'}(\mathcal{S}_{\phi'})| = |\mathcal{S}_{\phi'}/\mathcal{S}_{\tilde{\phi}'}|$ and $m_{\phi}|\mathcal{S}_{\tilde{\phi}}|^{-1} = C_{\tilde{\phi}}$, so we can write it as

$$\sum_{\omega' \in Y'/\alpha^{G'}(\mathcal{S}_{\phi'})} C_{\tilde{\phi}} |\alpha(\mathcal{S}_{\phi})|^{-1} |\alpha^{G'}(\mathcal{S}_{\phi'})| |\bar{Z}(\widehat{G}')^{\Gamma}| |\bar{Z}(\widehat{G}')^{\Gamma}|^{-1} |Z(\widehat{G})^{\Gamma}| \\ /(\widehat{G}' \cap Z(\widehat{G})^{\Gamma})(Z(\widehat{G})^{\Gamma}/\widehat{Z})|^{-1} \tilde{f}'_{\tilde{G}^{\theta}}(\tilde{\phi}' \otimes \omega', s).$$

In view of (5.31), we need to turn this into a sum over $Y/\alpha(\mathcal{S}_{\phi})$ instead of $Y'/\alpha^{G'}(\mathcal{S}_{\phi'})$. To do so we need the following two lemmas.

Lemma 5.23 *Suppose $\phi_v \in \bar{\Phi}_{unit}^+(G_v)$, and s_v is a semisimple element of \bar{S}_{ϕ_v} with $(G'_v, \phi'_v) \rightarrow (\phi_v, s_v)$. If we assume the main local Theorem 4.6 for the lift $\tilde{\phi}'_v$ of ϕ'_v , then for any $\omega'_v \in \alpha(\mathcal{S}_{\phi_v})$ we have*

$$\tilde{f}'_{\tilde{G}^{\theta}_v}(\tilde{\phi}_v \otimes \omega'_v, s_v) = \tilde{f}'_{\tilde{G}^{\theta}_v}(\tilde{\phi}_v, s_v), \quad \tilde{f}_v \in \tilde{\mathcal{H}}(\tilde{G}_v, \tilde{\chi}_v).$$

Proof Since $\tilde{f}'_{\tilde{G}^{\theta}_v}(\tilde{\phi}_v, s_v)$ only depends on the image of s_v in $\mathcal{S}_{\phi_v}^{\theta}$ (see Lemma 4.11), according to the formula (3.1) of \mathcal{S}_{ϕ_v} , we can assume s_v commutes with some $t_v \in \bar{S}_{\phi_v}$ such that $\alpha(t_v) = \omega'_v$. Note that $t_v \in \text{Aut}_{G_v}(G'_v)/Z(\widehat{G}_v)^{\Gamma_v}$. If $t_v \in \text{Int}_{G_v}(G'_v)/Z(\widehat{G}_v)^{\Gamma_v}$, then it is easy to see $\omega'_v \in \alpha^{G'}(\mathcal{S}_{\phi'_v})$, so there is nothing to prove. If $t_v \notin \text{Int}_{G_v}(G'_v)/Z(\widehat{G}_v)^{\Gamma_v}$, we denote the inducing automorphism of G'_v by θ' , and it can be extended to \tilde{G}'_v . Then it follows from Corollary 4.2 that $\bar{\Pi}_{\tilde{\phi}'_v}^{\theta'} = \bar{\Pi}_{\tilde{\phi}'_v} \otimes \omega'_v$. Since $\tilde{f}_v^{\tilde{G}'_v}$ is $\text{Out}_{\tilde{G}_v}(\tilde{G}'_v)$ -invariant, we have $\tilde{f}_v^{\tilde{G}'_v}(\tilde{\phi}'_v) = \tilde{f}_v^{\tilde{G}'_v}((\tilde{\phi}'_v)^{\theta'}) = \tilde{f}_v^{\tilde{G}'_v}(\tilde{\phi}'_v \otimes \omega'_v)$. \square

Since $Z_{\tilde{G}}(\mathbb{A}_F) = A_{\tilde{G}}(\mathbb{A}_F) \cdot Z_G(\mathbb{A}_F)$, we can identify Y with the quotient

$$\frac{\text{Ker}\{H^1(W_F, \widehat{D}) \rightarrow H^1(W_F, \widehat{A}_{\tilde{G}})\}}{\text{Ker}\{H^1(W_F, \widehat{D}) \rightarrow \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^{\times})\}},$$

where the involved homomorphisms are from the following diagram

$$\begin{array}{ccccc} H^1(W_F, \widehat{D}) & \longrightarrow & H^1(W_F, Z(\widehat{G})) & \longrightarrow & H^1(W_F, \widehat{A}_{\tilde{G}}) \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ \text{Hom}(D(\mathbb{A}_F)/D(F), \mathbb{C}^{\times}) & \longrightarrow & \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^{\times}) & \longrightarrow & \text{Hom}(A_{\tilde{G}}(\mathbb{A}_F)/A_{\tilde{G}}(F)). \end{array}$$

In the same way, we can identify Y' with

$$\frac{\text{Ker}\{H^1(W_F, \widehat{D}) \rightarrow H^1(W_F, \widehat{A}_{\tilde{G}'})\}}{\text{Ker}\{H^1(W_F, \widehat{D}) \rightarrow \text{Hom}(\tilde{G}'(\mathbb{A}_F)/\tilde{G}'(F), \mathbb{C}^{\times})\}}.$$

Note that $A_{\tilde{G}} \cong A_{\tilde{G}'}$ under the inclusion of $Z_{\tilde{G}} = (Z_{\tilde{G}})_\theta$ to $Z_{\tilde{G}'}$, and also we have

$$\begin{array}{ccccc} H^1(W_F, \hat{D}) & \longrightarrow & H^1(W_F, Z(\hat{\tilde{G}})) & \longrightarrow & H^1(W_F, \hat{A}_{\tilde{G}}) \\ \parallel & & & & \uparrow \simeq \\ H^1(W_F, \hat{D}) & \longrightarrow & H^1(W_F, Z(\hat{\tilde{G}}')) & \longrightarrow & H^1(W_F, \hat{A}_{\tilde{G}'}). \end{array}$$

So $\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow H^1(W_F, \hat{A}_{\tilde{G}})\} = \text{Ker}\{H^1(W_F, \hat{D}) \rightarrow H^1(W_F, \hat{A}_{\tilde{G}'})\}$, and we can first sum over this group. Then the rest is to determine the quotient

$$\frac{|\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow \text{Hom}(\tilde{G}'(\mathbb{A}_F)/\tilde{G}'(F), \mathbb{C}^\times)\}|}{|\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^\times)\}|}.$$

Lemma 5.24

$$\begin{aligned} & |\bar{Z}(\hat{G}')^\Gamma| |\bar{Z}(\hat{\tilde{G}}')^\Gamma|^{-1} |Z(\hat{G})^\Gamma / (\hat{G}' \cap Z(\hat{G})^\Gamma)(Z(\hat{\tilde{G}})^\Gamma / \hat{D})|^{-1} \\ &= \frac{|\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow \text{Hom}(\tilde{G}'(\mathbb{A}_F)/\tilde{G}'(F), \mathbb{C}^\times)\}|}{|\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^\times)\}|}. \end{aligned}$$

Proof From the proof of Lemma 2.11, we see

$$\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow \text{Hom}(\tilde{G}'(\mathbb{A}_F)/\tilde{G}'(F), \mathbb{C}^\times)\} \cong Z(\hat{G}')^\Gamma / (Z(\hat{\tilde{G}}')^\Gamma / \hat{D}),$$

and

$$\text{Ker}\{H^1(W_F, \hat{D}) \rightarrow \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F), \mathbb{C}^\times)\} \cong Z(\hat{G})^\Gamma / (Z(\hat{\tilde{G}})^\Gamma / \hat{D}).$$

Therefore it is enough to show

$$\begin{aligned} & |\bar{Z}(\hat{G}')^\Gamma| |\bar{Z}(\hat{\tilde{G}}')^\Gamma|^{-1} |Z(\hat{G})^\Gamma / (\hat{G}' \cap Z(\hat{G})^\Gamma)(Z(\hat{\tilde{G}})^\Gamma / \hat{D})|^{-1} \\ &= \frac{|Z(\hat{G}')^\Gamma / (Z(\hat{\tilde{G}}')^\Gamma / \hat{D})|}{|Z(\hat{G})^\Gamma / (Z(\hat{\tilde{G}})^\Gamma / \hat{D})|}. \end{aligned} \quad (5.35)$$

We start by considering the following exact sequence

$$\begin{aligned} 1 \longrightarrow (\hat{G}' \cap Z(\hat{G})^\Gamma) / (\hat{G}' \cap (Z(\hat{\tilde{G}})^\Gamma / \hat{D})) &\longrightarrow Z(\hat{G})^\Gamma / (Z(\hat{\tilde{G}})^\Gamma / \hat{D}) \\ &\longrightarrow Z(\hat{G})^\Gamma / (\hat{G}' \cap Z(\hat{G})^\Gamma)(Z(\hat{\tilde{G}})^\Gamma / \hat{D}) \longrightarrow 1. \end{aligned}$$

It follows

$$|Z(\hat{G})^\Gamma / (\hat{G}' \cap Z(\hat{G})^\Gamma)(Z(\hat{\tilde{G}})^\Gamma / \hat{D})|^{-1} = \frac{|(\hat{G}' \cap Z(\hat{G})^\Gamma) / (\hat{G}' \cap (Z(\hat{\tilde{G}})^\Gamma / \hat{D}))|}{|Z(\hat{G})^\Gamma / (Z(\hat{\tilde{G}})^\Gamma / \hat{D})|}.$$

If we write

$$(\widehat{G}' \cap Z(\widehat{G})^\Gamma) / (\widehat{G}' \cap (Z(\widehat{G})^\Gamma / \widehat{D})) = (Z(\widehat{G}')^\Gamma \cap Z(\widehat{G})^\Gamma) / ((Z(\widehat{G}') \cap Z(\widehat{G})^\Gamma) / \widehat{D}),$$

then

$$\begin{aligned} & |\widehat{G}' \cap Z(\widehat{G})^\Gamma / \widehat{G}' \cap (Z(\widehat{G})^\Gamma / \widehat{D})| |\bar{Z}(\widehat{G}')^\Gamma| |\bar{Z}(\widehat{G}')^\Gamma|^{-1} \\ &= |Z(\widehat{G}')^\Gamma / ((Z(\widehat{G}') \cap Z(\widehat{G})^\Gamma) / \widehat{D})| |\bar{Z}(\widehat{G}')^\Gamma|^{-1} \\ &= |Z(\widehat{G}') / (Z(\widehat{G}')^\Gamma / \widehat{D})|. \end{aligned}$$

Hence (5.35) holds. \square

As a consequence of Lemmas 5.23 and 5.24, we can sum over $Y/\alpha(\mathcal{S}_\phi)$ for (5.34) and get

$$\sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} C_{\tilde{\phi}} \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', s).$$

To sum up, we have shown the ϕ -component of (5.5) has an expansion

$$\begin{aligned} & \sum_{s \in \mathcal{E}_{\phi, ell}^\theta(\omega)} |\pi_0(\bar{S}_{\phi, s}^0)|^{-1} \sigma((\bar{S}_{\phi, s})^0) \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} C_{\tilde{\phi}} \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', s) \\ &= \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} C_{\tilde{\phi}} \sum_{s \in \mathcal{E}_{\phi, ell}^\theta(\omega)} |\pi_0(\bar{S}_{\phi, s}^0)|^{-1} \sigma((\bar{S}_{\phi, s})^0) \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', s). \end{aligned}$$

Finally by the same argument as in the proof of Lemma 4.11, there exists a family of global lifts $\bar{\Pi}_{\tilde{\phi}, s}$ for all $s \in \bar{S}_{\phi, ss}^\theta$ with image x in \mathcal{S}_ϕ^θ , such that the distribution $\tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', s)$ are the same. So we can write

$$\tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', s) = \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x).$$

Moreover, we can split the sum over $s \in \mathcal{E}_{\phi, ell}^\theta(\omega)$ into a double sum over $x \in \mathcal{S}_\phi^\theta(\omega)$ and $s \in \mathcal{E}_{\phi, ell}^\theta(x)$, where $\mathcal{E}_{\phi, ell}^\theta(x)$ is the subset of $\mathcal{E}_{\phi, ell}^\theta$ that mapped to x . If we define

$$e'_\phi(x) = \sum_{s \in \mathcal{E}_{\phi, ell}^\theta(x)} |\pi_0(\bar{S}_{\phi, s}^0)|^{-1} \sigma((\bar{S}_{\phi, s})^0),$$

then we get the following lemma.

Lemma 5.25 Suppose $\phi \in \bar{\Phi}(G)$, $\theta \in \Sigma_0$ and $\omega \in Y$. If $\theta = id$, $\omega = 1$ then

$$I_{disc, \phi}^{\tilde{G}}(\tilde{f}) - S_{disc, \phi}^{\tilde{G}}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} \sum_{x \in \bar{S}_\phi^\theta} e'_\phi(x) \tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \omega', x). \quad (5.36)$$

Otherwise,

$$I_{disc,\phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} \sum_{x \in \mathcal{S}_{\tilde{\phi}}^\theta(\omega)} e_{\tilde{\phi}}^{\prime\theta}(x) \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x). \quad (5.37)$$

Corollary 5.26 Suppose $\phi \in \bar{\Phi}(G)$ and $\mathcal{S}_{\tilde{\phi}} = 1$, then the distribution $I_{disc,\phi}^{\tilde{G}}(\tilde{f})$ is stable.

Proof Since $\mathcal{S}_{\tilde{\phi}} = 1$, it follows from (5.36) that

$$I_{disc,\phi}^{\tilde{G}}(\tilde{f}) = S_{disc,\phi}^{\tilde{G}}(\tilde{f}) + C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} e_{\tilde{\phi}}'(1) \tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \omega', 1).$$

Note that $\tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \omega', 1)$ is defined by inducing global L-packets of Levi subgroups of \tilde{G} , and hence is stable. Therefore $I_{disc,\phi}^{\tilde{G}}(\tilde{f})$ is stable. \square

Later on, we will compare the formulas in Lemmas 5.22 and 5.25. Note it follows from Proposition 5.17 that for $x \in \mathcal{S}_{\tilde{\phi}}^\theta$,

$$i_{\tilde{\phi}}^\theta(x) - e_{\tilde{\phi}}^{\prime\theta}(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ \sigma(\bar{S}_{\tilde{\phi}}^0) & \text{if } x = 1. \end{cases}$$

6 Refined L-packet

6.1 Beginning of proofs

In the following sections, we are going to prove the main local theorem (Theorem 4.6) along with the global theorem (Theorem 5.21). First we need to impose our induction assumptions. It consists of a local part and a global part. Let F be either local or global. We denote

$$\begin{aligned} G(n) &:= Sp(2n), SO(2n+2, \eta), \\ \tilde{G}(n) &:= GSp(2n), GSO(2n+2, \eta). \end{aligned}$$

Let

$$G = G(n_1) \times G(n_2) \times \cdots \times G(n_q)$$

and \tilde{G} be the corresponding similitude group (see (2.2)), then our induction assumptions can be stated as follows.

Local induction assumption The main local theorem (Theorem 4.6) holds for \tilde{G} , when $n_i < N$ for all $1 \leq i \leq q$.

Global induction assumption The global theorem (Theorem 5.21) hold for \tilde{G} , when $\sum_{i=1}^q n_i < N$.

Remark 6.1 When $\tilde{G} = GSp(2N)$, these assumptions imply all the local and global theorems hold for the Levi subgroups and twisted endoscopic groups of \tilde{G} . But this is not true when $\tilde{G} = GSO(2N+2, \eta)$ for it can have twisted endoscopic group of the form $G(Sp(2N_1) \times Sp(2N_2))$ with $N = N_1 + N_2$ and $N_1, N_2 \geq 0$. To fix this, we will first prove the local and global theorems for \tilde{G} based on our induction assumption, when G does not contain any factor of $SO(2N+2, \eta)$. Then we can add those results to our induction assumptions and repeat the same arguments to prove the rest of the cases.

We will first establish the main local theorem (Theorem 4.6) for $\tilde{G} = \tilde{G}(N)$, which is the most important case. In view of Remark 4.7, we can further assume F is nonarchimedean. Under our local induction assumption, we can prove a lot of cases of the main local theorem. The precise statement is formulated in the following lemma.

Lemma 6.2 Suppose $\phi \in \bar{\Phi}_{bdd}(G) - \bar{\Phi}_2(G)$, then one can assign an L-packet $\bar{\Pi}_{\tilde{\phi}}$ for any lift $\tilde{\phi}$ such that it satisfies (1) and (2) of the main local theorem (Theorem 4.6). Furthermore, the (θ, ω) -twisted character relation (4.2) holds for $\theta \in \Sigma_0$ and semisimple $s \in \bar{S}_{\phi, s}^0$ such that $|\bar{S}_{\phi, s}^0| = \infty$.

Proof Suppose $\phi \in \bar{\Phi}_{bdd}(G) - \bar{\Phi}_2(G)$, then ϕ factors through $\phi_M \in \bar{\Phi}_2(M)$ for some proper Levi subgroup M of G . Since

$$M \cong G(m) \times \prod_i GL(n_i)$$

with $m < N$, by our local induction assumption we can define a refined L-packet $\bar{\Pi}_{\tilde{\phi}_M}$ associated to $\tilde{\phi}_M$. Then we can take local packet $\bar{\Pi}_{\tilde{\phi}}$ for $\tilde{\phi}$ to be the irreducible constituents of those induced from $\bar{\Pi}_{\tilde{\phi}_M}$. Because $\bar{\Pi}_{\tilde{\phi}}$ is also obtained by induction from $\bar{\Pi}_{\phi_M}$, we can easily see that $\bar{\Pi}_{\tilde{\phi}}$ will satisfy (1) and (2) of the main local theorem. For the (θ, ω) -twisted character relation (4.2), it will follow from the usual descent argument. For $(G', \phi') \rightarrow (\phi, s)$, let $T_{\phi, s}$ be a maximal torus of $(S_{\phi, s})^0$, which is nontrivial by our assumption that $|\bar{S}_{\phi, s}^0| = \infty$. Then $\hat{M}' = \text{Cent}(T_{\phi, s}, \hat{G}')$ defines a proper Levi subgroup of \hat{G}' such that ϕ' factors through $\phi'_M \in \bar{\Phi}_2(M')$. Moreover, $M' \in \mathcal{E}_{ell}(M^\theta)$ for a proper θ -stable Levi subgroup M of G , which is determined by $\hat{M} = \text{Cent}(T_{\phi, s}, \hat{G})$. So

$$\tilde{f}'(\tilde{\phi}') = \tilde{f}^{\tilde{M}'}(\tilde{\phi}'_M) = \sum_{[\tilde{\pi}_M] \in \bar{\Pi}_{\tilde{\phi}_M}} \tilde{f}_{\tilde{M}'}(\tilde{\pi}_M, \omega) = \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}'}(\tilde{\pi}, \omega),$$

where $\omega = \alpha(s)$. □

Remark 6.3 We have not shown the uniqueness of $\bar{\Pi}_{\tilde{\phi}}$ here. In fact that will follow from the character relation (see Theorem 6.21).

The key issue in proving the main local theorem is to find a candidate for the stable distribution associated with any lift $\tilde{\phi}$ of $\phi \in \bar{\Phi}_{bdd}(G)$. As one can see from Lemma 6.2, the critical case is when $\phi \in \bar{\Phi}_2(G)$. The way to find such a distribution is to lift ϕ to a global parameter $\dot{\phi}$ and use the global stable distribution $S_{disc, \dot{\phi}}^{\tilde{G}}$ in the stabilized trace formula. Under some assumptions on this lifted global parameter $\dot{\phi}$, we can obtain the local stable distribution associated with $\tilde{\phi}$ using an argument based on stability (cf. Corollary 4.8). Let us write S_∞ for the set of archimedean places of a global field \dot{F} , and $S_\infty(u) = S_\infty \cup \{u\}$ for any nonarchimedean place u . Suppose $F = \dot{F}_u$ and $\dot{G}_u = G$. Let $\dot{X} = \text{Hom}(\tilde{G}(\mathbb{A}_{\dot{F}})/Z_{\tilde{G}}(\mathbb{A}_{\dot{F}})\dot{G}(\mathbb{A}_{\dot{F}}), \mathbb{C}^\times)$.

Theorem 6.4 *For $\phi \in \bar{\Phi}_2(G)$, suppose $\dot{\phi} \in \bar{\Phi}_2(\dot{G})$ is a global lift of ϕ with $\dot{\phi}_u = \phi$ and it also satisfies the following additional conditions:*

1. $\dot{\phi}_v \in \bar{\Phi}_{unit}^+(\dot{G}_v) - \bar{\Phi}_{ell}(\dot{G}_v)$ for all $v \notin S_\infty(u)$;
2. $S_{\dot{\phi}}^{\tilde{G}} = 1$;
3. Σ_0 -strong multiplicity one holds for $\tilde{\phi}$.

Then one can assign an L-packet $\bar{\Pi}_{\tilde{\phi}}$ to any lift $\tilde{\phi}$ of ϕ satisfying (1) and (2) of the main local theorem (Theorem 4.6).

Proof In view of Lemma 6.2, the first condition of our global lift $\dot{\phi}$ just means that the main local theorem (except for the (θ_0, ω) -twisted character relation in the even orthogonal case) holds for all $\dot{\phi}_v$ ($v \neq u$). The second condition means that

$$\text{tr } R_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) = I_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) = S_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) \neq 0$$

for $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$, which follows from Corollary 5.26 and the fact that $\dot{\phi} \in \bar{\Phi}_2(\dot{G})$ (cf. (5.15)). It follows from Proposition 5.11 that

$$I_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) = m_{\dot{\phi}} \sum_{\dot{\omega} \in \dot{Y}/\alpha(S_{\dot{\phi}})} \sum_{\tilde{\pi}} \tilde{f}_{\tilde{G}}(\tilde{\pi} \otimes \dot{\omega}) \quad (6.1)$$

for $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$, where the sum of $\tilde{\pi}$ is taken over representatives of $\tilde{\Pi}_{\dot{\phi}, \tilde{\zeta}}/\dot{X}$ inside $\mathcal{A}_2(\tilde{G})$. Here we will always view representations of $\tilde{G}(\mathbb{A}_F)$ as $\tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$ -modules. Since $I_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f})$ is stable, it is stable at every place. If we take $\tilde{f} = \bigotimes_w \tilde{f}_w$ and fix $\bigotimes_{w \neq v} \tilde{f}_w$ for $v \neq u$, then by Corollary 4.8 the coefficients of $\tilde{f}_v(\tilde{\pi}_v)$ in $I_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f})$ must be the same for all $[\tilde{\pi}_v] \in \bar{\Pi}_{\dot{\phi}_v}^{\tilde{G}}$. Moreover, if we fix a representation $\tilde{\pi} \in \mathcal{A}_2(\tilde{G})$, by varying $\bigotimes_{w \neq v} \tilde{f}_w$ and the linear independence of characters of $\bigotimes_{w \neq v} \tilde{\mathcal{H}}(\tilde{G}_w, \tilde{\chi}_w)$ -modules, we will observe that for $v \neq u$

$$[\tilde{\pi}_v] \otimes \left(\bigotimes_{w \neq v} [\tilde{\pi}_w] \right) \in \tilde{\Pi}_{\dot{\phi}, \tilde{\zeta}}$$

contributes to (6.1) if and only if

$$[\tilde{\pi}'_v] \otimes \left(\bigotimes_{w \neq v} [\tilde{\pi}_w] \right) \in \tilde{\Pi}_{\phi, \tilde{\zeta}}$$

also contributes to (6.1) for all $[\tilde{\pi}'_v] \in \bar{\Pi}_{\phi_v}^{\tilde{\zeta}}$, where $\bar{\Pi}_{\phi_v}^{\tilde{\zeta}}$ contains $[\tilde{\pi}_v]$. We still fix $\tilde{\pi}$ and hence $\bar{\Pi}_{\phi_v}^{\tilde{\zeta}}$ for all $v \neq u$. Then (6.1) will contain $\mathcal{H}(\tilde{G}, \tilde{\chi})$ -modules of the form

$$[\tilde{\pi}_u] \otimes \left(\bigotimes_{v \neq u} [\tilde{\pi}_v] \right)$$

where $[\tilde{\pi}_v]$ ranges over $\bar{\Pi}_{\phi_v}^{\tilde{\zeta}}$ for all $v \neq u$. Suppose there is a distinct $\mathcal{H}(\tilde{G}, \tilde{\chi})$ -module

$$[\tilde{\pi}'_u] \otimes \left(\bigotimes_{v \neq u} [\tilde{\pi}_v] \right)$$

in (6.1) such that $[\tilde{\pi}_v] \in \bar{\Pi}_{\phi_v}^{\tilde{\zeta}}$ for all $v \neq u$, then $[\tilde{\pi}'_u] \neq [\tilde{\pi}_u] \otimes \omega$ for any character $\omega \in X$. Otherwise, there will exist $\dot{\omega} \in \dot{Y}$ such that $[\tilde{\pi}_u] \otimes \dot{\omega}_u = [\tilde{\pi}'_u] \neq [\tilde{\pi}_u]$ and $\bar{\Pi}_{\phi_v}^{\tilde{\zeta}} = \bar{\Pi}_{\phi_v}^{\tilde{\zeta}} \otimes \dot{\omega}_v$ for all $v \neq u$. This is impossible because of the third condition, i.e. Σ_0 -strong multiplicity one holds for $\tilde{\phi}$. Therefore if we consider all $[\tilde{\pi}] \in \tilde{\Pi}_{\phi, \tilde{\zeta}}$ such that

$$[\tilde{\pi}] \otimes \left(\bigotimes_{v \neq u} \bar{\Pi}_{\phi_v}^{\tilde{\zeta}} \right)$$

is contained in (6.1), this gives a non-empty set $\bar{\Pi}_{\tilde{\phi}}$ of representatives of $\tilde{\Pi}_{\phi, \tilde{\zeta}}/X$ in $\tilde{\Pi}_{\phi, \tilde{\zeta}}$. To see why this gives all the representatives, one just needs to take the test function $\tilde{f} = \otimes_v \tilde{f}_v$ such that \tilde{f}_u is supported on $\tilde{Z}_{\dot{F}_u} \dot{G}(\dot{F}_u)$, then it is the same to consider representations of

$$\tilde{Z}_{\dot{F}_u} \dot{G}(\dot{F}_u) \times \prod_{v \neq u} \tilde{G}(\dot{F}_v).$$

By the same reasoning using stability, one can conclude that

$$\bar{\Pi}_{\phi} \otimes \left(\bigotimes_{v \neq u} \bar{\Pi}_{\phi_v}^{\tilde{\zeta}} \right)$$

is contained in (6.1), therefore $\bar{\Pi}_{\tilde{\phi}}$ must contain all representatives of $\tilde{\Pi}_{\phi, \tilde{\zeta}}/X$ in $\tilde{\Pi}_{\phi, \tilde{\zeta}}$. Moreover, it follows again from Σ_0 -strong multiplicity one and stability of (6.1) that

$$\tilde{f}(\tilde{\phi}) := \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\tilde{G}}(\tilde{\pi})$$

is stable. This shows the packet $\bar{\Pi}_{\tilde{\phi}}$ satisfies the property (1) and (2) of Theorem 4.6. \square

Remark 6.5 1. Following the proof, we can rewrite (6.1) as

$$I_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) = m_{\dot{\phi}} \sum_{\dot{\omega} \in \dot{Y}/\alpha(\mathcal{S}_{\dot{\phi}})} \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\zeta}} \otimes \dot{\omega}} \tilde{f}_{\tilde{G}}(\tilde{\pi})$$

where

$$\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\tilde{\phi}} \otimes \left(\bigotimes_{v \neq u} \bar{\Pi}_{\tilde{\phi}_v} \right).$$

If we define

$$\tilde{f}(\tilde{\phi}) := \prod_v \tilde{f}_v(\tilde{\phi}_v),$$

then we get the stable multiplicity formula for our lift $\tilde{\phi}$

$$S_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) = I_{disc, \dot{\phi}}^{\tilde{G}}(\tilde{f}) = m_{\dot{\phi}} \sum_{\dot{\omega} \in \dot{Y}/\alpha(\mathcal{S}_{\dot{\phi}})} \tilde{f}(\tilde{\phi} \otimes \dot{\omega}).$$

This identity will be used in the proof of Theorem 6.22.

- The statement of this theorem indicates that we need a lifting result for the existence of such $\dot{\phi}$. In fact, there is a standard argument using the simple invariant trace formula which provides a global lift so that one is allowed to impose some local conditions. That argument is carried out in quite detail in ([7], Sections 6.2 and 6.3), and the local conditions that Arthur imposes already take care of our first additional condition in most cases. Even though the global lift which Arthur uses does not necessarily satisfy the other two conditions, his argument is still flexible enough to leave us a lot of room to manipulate. In fact, it is not hard to impose the second condition after we give a combinatorial description of the exact sequence

$$1 \longrightarrow \mathcal{S}_{\tilde{\zeta}} \xrightarrow{\iota} \mathcal{S}_{\dot{\phi}} \xrightarrow{\alpha} \mathrm{Hom}(\tilde{G}(\mathbb{A}_{\tilde{F}})/\tilde{G}(\dot{F})\dot{G}(\mathbb{A}_{\tilde{F}}), \mathbb{C}^{\times}).$$

However, for technical reasons the third condition is not so easy to satisfy, and it seems that we have asked something too strong. By tracking the argument in our proof carefully, one will observe that it is enough to have

$$\prod_v^{\text{aut}} \alpha \left(\mathcal{S}_{\phi_v}^{\Sigma_0} \right) = \prod_{v \neq u}^{\text{aut}} \alpha \left(\mathcal{S}_{\phi_v}^{\Sigma_0} \right).$$

This condition can also be interpreted in terms of strong multiplicity one, which means that for any $\dot{\pi} \in \mathcal{A}(\tilde{G})$ such that $[\dot{\pi}_v] \in \bar{\Pi}_{\phi_v}^{\gamma}$ for all $v \neq u$, $[\dot{\pi}]$ must be also in $\bar{\Pi}_{\phi}^{\gamma}$. If this condition is satisfied, we say Σ_0 -strong multiplicity one holds for $\tilde{\phi}$ **at the place** u . And it is the most technical part of this paper to establish this property.

6.2 A combinatorial description of \mathcal{S}_{ϕ}

Now we will give a combinatorial description of the exact sequences (2.14) and (2.15). We assume F is either local or global. Suppose $G = G(n)$, $\phi \in \tilde{\Phi}(G)$ if F is global or $\tilde{\Phi}_{\text{bdd}}(G)$ if F is local, and

$$\phi = l_1 \phi_1 \boxplus \cdots \boxplus l_r \phi_r,$$

where $\phi_i \in \Phi_{\text{sim}}(N_i)$ for $1 \leq i \leq r$. From the discussion of Sect. 3, the set of indices can be written as a disjoint union of

$$I_{\phi,O} \sqcup I_{\phi,S} \sqcup J_{\phi} \sqcup J_{\phi}^{\vee}$$

where $I_{\phi,O}$ ($I_{\phi,S}$) is the set of indices that index self-dual parameters of orthogonal (symplectic) type. In particular, since we are considering G to be either a special even orthogonal group or a symplectic group, \widehat{G} will always be orthogonal and hence the multiplicities l_i must be even for $i \in I_{\phi,S}$. On the other hand, let us denote

$$\begin{aligned} I_{\phi,O}^{\text{odd}} &= \{i \in I_{\phi,O} : l_i \text{ is odd}\}, \\ I_{\phi,O}^{\text{even}} &= \{i \in I_{\phi,O} : l_i \text{ is even}\}. \end{aligned}$$

Moreover, let S and T be subsets of $I_{\phi,O}^{\text{odd}}$ and $I_{\phi,O}^{\text{even}}$ respectively, with the condition that $\sum_{i \in S \cup T} N_i$ is even if G is special even orthogonal. And we allow S and T to be empty sets. Then the pair of such sets modulo the following equivalence relation gives us the combinatorial object that we need to substitute for \mathcal{S}_{ϕ} , i.e.

$$\mathcal{P}_{\phi} = \{(S, T)\} / (S, T) \sim (S^c, T)$$

where S^c is the complement of S in $I_{\phi,O}^{\text{odd}}$. There is a natural map from \mathcal{P}_{ϕ} to $\text{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^{\times})$ if F is local (resp. $\text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^{\times})$ if F is global), which sends

$$(S, T) \mapsto \left(\prod_{i \in S \cup T} \eta_{\phi_i} \right) \circ \lambda$$

where η_{ϕ_i} is the central character of π_{ϕ_i} . Let us denote this map by $\alpha_{\mathcal{P}}$ and the kernel of this map by $\mathcal{P}_{\tilde{\phi}}$, then we get a sequence

$$1 \longrightarrow \mathcal{P}_{\tilde{\phi}} \longrightarrow \mathcal{P}_{\phi} \xrightarrow{\alpha_{\mathcal{P}}} \mathrm{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^{\times}) \quad (6.2)$$

if F is local, and

$$1 \longrightarrow \mathcal{P}_{\tilde{\phi}} \longrightarrow \mathcal{P}_{\phi} \xrightarrow{\alpha_{\mathcal{P}}} \mathrm{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^{\times}) \quad (6.3)$$

if F is global.

To compare these sequences with (2.14) and (2.15), we need a map connecting \mathcal{S}_{ϕ} and \mathcal{P}_{ϕ} . To define such a map, we consider semisimple $s \in \tilde{S}_{\phi}$, and $(G'_s, \phi') \rightarrow (\phi, s)$. In general, G'_s may not be elliptic, but it will lie in $\mathcal{E}_{\mathrm{ell}}(M)$ for some Levi subgroup M of G . Since M is a product of general linear groups with a group G_- of the same type as G with smaller rank, then G'_s contains a factor $G'_I \times G'_{II} \in \mathcal{E}_{\mathrm{ell}}(G_-)$ and ϕ' will decompose accordingly. Suppose ϕ_- is the component of ϕ contributing to G_- , and

$$\phi'_- = \phi'_I \times \phi'_{II}$$

if G is special even orthogonal, or

$$\phi'_- = (\phi'_I \otimes \eta_{\phi'_I}) \times \phi'_{II}$$

if G and G'_I are symplectic. In either case, ϕ'_I, ϕ'_{II} give a partition of simple parameters in ϕ_- . Let $S(T)$ be the subset of $I_{\phi, O}^{\mathrm{odd}}(I_{\phi, O}^{\mathrm{even}})$ parametrizing simple parameters in ϕ'_I with odd multiplicities. It is easy to see that $(S, T) \in \mathcal{P}_{\phi}$, so we get a map $\mathbf{c}^G = \mathbf{c} : \tilde{S}_{\phi} \rightarrow \mathcal{P}_{\phi}$ in this way. Moreover we have the following lemma.

Lemma 6.6 1. The map \mathbf{c} defined above will factor through \mathcal{S}_{ϕ} , and it gives a bijection between \mathcal{S}_{ϕ} and \mathcal{P}_{ϕ} .
2. If we denote the bijection in (1) still by \mathbf{c} , then we have a commutative diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{S}_{\tilde{\phi}} & \longrightarrow & \mathcal{S}_{\phi} & \xrightarrow{\alpha} & \mathrm{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^{\times}) \\ \parallel & & \downarrow & & \downarrow \mathbf{c} & & \parallel \\ 1 & \longrightarrow & \mathcal{P}_{\tilde{\phi}} & \longrightarrow & \mathcal{P}_{\phi} & \xrightarrow{\alpha_{\mathcal{P}}} & \mathrm{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^{\times}) \end{array}$$

if F is local, or

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{S}_{\tilde{\phi}} & \longrightarrow & \mathcal{S}_{\phi} & \xrightarrow{\alpha} & \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times) \\ \parallel & & \downarrow & & \downarrow \mathbf{c} & & \parallel \\ 1 & \longrightarrow & \mathcal{P}_{\tilde{\phi}} & \longrightarrow & \mathcal{P}_{\phi} & \xrightarrow{\alpha_{\mathcal{P}}} & \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times) \end{array}$$

if F is global.

Proof First we would like to show \mathbf{c} factors through \mathcal{S}_{ϕ} , i.e. for any $s \in \tilde{\mathcal{S}}_{\phi}$, $\mathbf{c}(s)$ only depends on the image x of s in \mathcal{S}_{ϕ} . Note that if s is replaced by an $\tilde{\mathcal{S}}_{\phi}^0$ -conjugate s_1 , then the corresponding pair (G'_1, ϕ'_1) is isomorphic to (G', ϕ') . So by our definition

$$\mathbf{c}(s_1) = \mathbf{c}(s).$$

Now if we fix a maximal torus \bar{T}_{ϕ} of $\tilde{\mathcal{S}}_{\phi}^0$ and a Borel subgroup \bar{B}_{ϕ} containing it, any automorphism of the complex reductive group $\tilde{\mathcal{S}}_{\phi}^0$ stabilizes a conjugate of $(\bar{T}_{\phi}, \bar{B}_{\phi})$. So we can choose a representative s_x of x in $\tilde{\mathcal{S}}_{\phi}$ so that $\text{Int}(s_x)$ stabilizes $(\bar{T}_{\phi}, \bar{B}_{\phi})$, and such representatives are determined up to a \bar{T}_{ϕ} -translate. Moreover the complex torus

$$\bar{T}_{\phi, x} = \text{Cent}(s_x, \bar{T}_{\phi})^0$$

in \bar{T}_{ϕ} is uniquely determined by x . Note that $\bar{T}_{\phi, x}$ is the connected component of the kernel of the following morphism

$$\begin{array}{ccc} \bar{T}_{\phi} & \longrightarrow & \bar{T}_{\phi} \\ & & t \longmapsto s_x^{-1} t s_x t^{-1} \end{array}$$

So any point of \bar{T}_{ϕ} can be written as $(s_x^{-1} t s_x t^{-1}) t_x$ for $t \in \bar{T}_{\phi}$ and $t_x \in \bar{T}_{\phi, x}$ (see [42], Corollary 5.4.5), and hence any point in $s_x \bar{T}_{\phi}$ can be written as

$$s_x (s_x^{-1} t s_x t^{-1}) t_x = t s_x t^{-1} t_x = t s_x t_x t^{-1}, \quad t \in \bar{T}_{\phi}, \quad t_x \in \bar{T}_{\phi, x}.$$

Therefore it will be enough to show that

$$\mathbf{c}(s_x) = \mathbf{c}(s_x t_x)$$

for any $t_x \in \bar{T}_{\phi, x}$.

The centralizer \widehat{M}_x of $\bar{T}_{\phi, x}$ in \widehat{G} is a Levi subgroup of \widehat{G} , which is dual to a Levi subgroup M_x of G . So (ϕ, s_x) is the image of a pair

$$(\phi_{M_x}, s_{M_x}), \quad \phi_{M_x} \in \bar{\Phi}(M_x), \quad s_{M_x} \in S_{\phi_{M_x}},$$

attached to M_x under the L -embedding ${}^L M_x \subseteq {}^L G$. And this pair is in turn the image of an endoscopic pair (M'_x, ϕ'_{M_x}) . Note that M_x has a component G_x of the same type as G and $\bar{S}_{\phi_{G_x}} \cong \bar{S}_{\phi_{M_x}}$. So we can define a map \mathbf{c}^{M_x} on $\bar{S}_{\phi_{M_x}}$ by \mathbf{c}^{G_x} with respect to the component ϕ_{G_x} of ϕ_{M_x} . Since M'_x can be identified with a Levi subgroup of G' , one can easily check that

$$\mathbf{c}^{M_x}(s_{M_x}) = \mathbf{c}(s_x).$$

Note that $\mathbf{c}^{M_x}(s_{M_x})$ is invariant under the translation of s_{M_x} by $\bar{T}_{\phi,x}$, so the same is true of $\mathbf{c}(s_x)$.

Secondly we need to show that \mathbf{c} is in fact a bijection between \mathcal{S}_ϕ and \mathcal{P}_ϕ . Note that we can actually compute $|\mathcal{S}_\phi|$ and $|\mathcal{P}_\phi|$ explicitly. For $|\mathcal{S}_\phi|$, we have the description from Sect. 3 that

$$\mathcal{S}_\phi = \left(\prod_{i \in I_{\phi,O}} O(l_i, \mathbb{C}) \right)_\phi^+ \times \left(\prod_{i \in I_{\phi,S}} Sp(l_i, \mathbb{C}) \right) \times \left(\prod_{j \in J_\phi} GL(l_j, \mathbb{C}) \right), \quad (6.4)$$

where $(\prod_{i \in I_{\phi,O}} O(l_i, \mathbb{C}))_\phi^+$ is the kernel of the character

$$\xi_\phi^+ : \prod_i g_i \longrightarrow \prod_i (\det g_i)^{N_i}, \quad g_i \in O(l_i, \mathbb{C}), i \in I_{\phi,O}.$$

If G is symplectic or G is special even orthogonal with $I_{\phi,O}^{odd}$ being empty, then

$$\mathcal{S}_\phi = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{|I_{\phi,O}|} & \text{if all } N_i \text{ are even for } i \in I_{\phi,O}, \\ (\mathbb{Z}/2\mathbb{Z})^{|I_{\phi,O}|-1} & \text{otherwise.} \end{cases}$$

If G is special even orthogonal and $I_{\phi,O}^{odd}$ is not empty, then $Z(\widehat{G}) \notin \bar{S}_\phi^0$ and thus

$$\mathcal{S}_\phi = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{|I_{\phi,O}|-1} & \text{if all } N_i \text{ are even for } i \in I_{\phi,O}, \\ (\mathbb{Z}/2\mathbb{Z})^{|I_{\phi,O}|-2} & \text{otherwise.} \end{cases}$$

For $|\mathcal{P}_\phi|$, it is just a combinatorial computation. Suppose G is symplectic, there is no condition on N_i , so $|\mathcal{P}_\phi| = 2^{|I_{\phi,O}|-1}$. Suppose G is special even orthogonal, it again divides into two cases. If all N_i are even for $i \in I_{\phi,O}$, then the condition on N_i is automatically satisfied and hence

$$|\mathcal{P}_\phi| = \begin{cases} 2^{|I_{\phi,O}|} & \text{if } I_{\phi,O}^{odd} \text{ is empty,} \\ 2^{|I_{\phi,O}|-1} & \text{otherwise.} \end{cases}$$

And if there exists $i \in I_{\phi, O}$ such that N_i is odd then

$$|\mathcal{P}_{\phi}| = \begin{cases} 2^{|I_{\phi, O}|-1} & \text{if } I_{\phi, O}^{\text{odd}} \text{ is empty,} \\ 2^{|I_{\phi, O}|-2} & \text{otherwise.} \end{cases}$$

Therefore one can conclude that $|\mathcal{S}_{\phi}| = |\mathcal{P}_{\phi}|$. Now it suffices to show that \mathbf{c} is surjective. In fact for any partition (S, T) , one can choose an element $s = (s_k)_{k \in K_{\phi}} \in S_{\phi}$ according to the decomposition (6.4) such that it has the form

$$s_i = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \text{ for } i \in S \cup T, \text{ and } s_k = I \text{ otherwise.}$$

When G is symplectic, we can assume $\sum_{i \in S \cup T} N_i$ is odd by possibly changing (S, T) to (S^c, T) . If $(G', \phi') \rightarrow (\phi, s)$, then G' is elliptic and $\phi' = \phi'_I \times \phi'_{II}$ (or $\phi'_- = (\phi'_I \otimes \eta_{\phi'_I}) \times \phi'_{II}$ if G' is symplectic) with the property that

$$\phi'_I = \boxplus_{i \in S \cup T} \phi_i.$$

Hence by the definition $\mathbf{c}(s) = (S, T)$.

For the second part of the lemma, it is enough to show that

$$\alpha(s) = \alpha_{\mathcal{P}}(\mathbf{c}(s)),$$

for $s \in S_{\phi}$ being such representatives chosen above. Let

$$\eta' = \eta_{\phi'_I} = \prod_{i \in S \cup T} \eta_{\phi_i},$$

then $\alpha_{\mathcal{P}}(\mathbf{c}(s)) = \eta' \circ \lambda$. Moreover, G' will be $Sp(2n_1) \times SO(2n_2, \eta')$ if $G = Sp(2n)$, and $SO(2n_1, \eta') \times SO(2n_2, \eta' \eta)$ if $G = SO(2n, \eta)$. As one can see from the table of twisted elliptic endoscopic groups in Sect. 2.1.3, G' can be lifted to $\tilde{G}' \in \mathcal{E}_{\text{ell}}(\tilde{G}, \omega)$ with $\omega = \eta' \circ \lambda$. So $\omega = \alpha_{\mathcal{P}}(\mathbf{c}(s))$. Finally we just need to notice $\alpha(s) = \omega$ by Lemma 2.12, hence $\alpha(s) = \alpha_{\mathcal{P}}(\mathbf{c}(s))$. \square

Corollary 6.7 Suppose $\phi = l_1 \phi_1 \boxplus \cdots \boxplus l_r \phi_r \in \bar{\Phi}_{\text{bdd}}(G)$ if F is local (resp. $\bar{\Phi}(G)$ if F is global), and S, T are subsets of $I_{\phi, O}^{\text{odd}}, I_{\phi, O}^{\text{even}}$ respectively. Suppose that

$$\left(\prod_{i \in S \cup T} \eta_{\phi_i} \right) \circ \lambda \neq 1$$

unless T is empty and S is either empty or equal to $I_{\phi, O}^{\text{odd}}$. Then $S_{\tilde{\phi}} = 1$.

Proof It follows from the definition that $|\mathcal{P}_{\tilde{\phi}}| = 1$. By Lemma 6.6, one has $|\mathcal{S}_{\tilde{\phi}}| = |\mathcal{P}_{\tilde{\phi}}| = 1$. Hence $\mathcal{S}_{\tilde{\phi}} = 1$. \square

The following proposition will become useful in our later proofs.

Proposition 6.8 *Suppose $\phi \in \bar{\Phi}_{ell}(G^\theta)$ for $\theta \in \Sigma_0$ and $\mathcal{S}_{\tilde{\phi}} = 1$. We assume one of the following condition is satisfied.*

1. G is symplectic,
2. G is special even orthogonal with $\eta_G \neq 1$,
3. G is special even orthogonal with $\eta_G = 1$, and $I_{\phi,O}^{odd}$ or $I_{\phi,O}^{even}$ is empty.

If ϕ factors through $\phi' \in \bar{\Phi}(G')$, where $G' = G_I \times G_{II}$ is a twisted elliptic endoscopic group of G , let $\phi' = \phi_I \times \phi_{II}$, where $\phi_i \in \bar{\Phi}(G_i)$ for $i = I, II$. Then $\mathcal{S}_{\tilde{\phi}_I} = \mathcal{S}_{\tilde{\phi}_{II}} = 1$.

Proof Since $\phi \in \bar{\Phi}_{ell}(G^\theta)$, we can view

$$I_{\phi_I,O}^{odd} \subseteq I_{\phi,O} \text{ and } I_{\phi_I,O}^{even} \subseteq I_{\phi,O}^{even}$$

after possibly twisting ϕ_I by η_{ϕ_I} . Suppose $\mathcal{S}_{\tilde{\phi}_I} \neq 1$, we can represent any nontrivial element of $\mathcal{S}_{\tilde{\phi}_I}$ by (S', T') for

$$S' \subseteq I_{\phi_I,O}^{odd} \text{ and } T' \subseteq I_{\phi_I,O}^{even}$$

such that $S' \cup T'$ is nonempty, $\sum_{i \in S' \cup T'} N_i$ is even and

$$\prod_{i \in S' \cup T'} \eta_{\phi_i} = 1.$$

Then we want to show $\mathcal{S}_{\tilde{\phi}} \neq 1$, which would lead to a contradiction.

Let us define (S, T) by

$$S = S' \cap I_{\phi,O}^{odd} \text{ and } T = (S' \cup T') \cap I_{\phi,O}^{even}.$$

Then (S, T) corresponds to a nontrivial element in $\mathcal{S}_{\tilde{\phi}}$ unless T is empty and $S = I_{\phi,O}^{odd}$. In the exceptional cases, we have

$$T' \text{ is empty and } S' = S = I_{\phi,O}^{odd}.$$

By our conditions on (S', T') , we see G has to be special even orthogonal and $\eta_G = 1$. So we only need to consider condition (3). In particular, we can assume $I_{\phi,O}^{even}$ is empty, i.e. $I_{\phi,O} = I_{\phi,O}^{odd}$. It follows $S' = I_{\phi_I,O}^{odd}$. But this is impossible for (S', T') should correspond to a nontrivial element in $\mathcal{S}_{\tilde{\phi}_I}$ by our assumption. Therefore, we see $\mathcal{S}_{\tilde{\phi}} \neq 1$. Similarly, if we assume $\mathcal{S}_{\tilde{\phi}_{II}} \neq 1$, we can also deduce $\mathcal{S}_{\tilde{\phi}} \neq 1$. This finishes the proof. \square

In the case that G is a special even orthogonal group and $\phi \in \Phi_{bdd}(G^{\theta_0})$ if F is local (resp. $\Phi(G^{\theta_0})$, if F is global), we can have a similar combinatorial description of the map from $\mathcal{S}_\phi^{\Sigma_0}$ to $\text{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^\times)$ if F is local (resp. $\text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times)$ if F is global). To do this we need to take a bigger set

$$\mathcal{P}_\phi^{\Sigma_0} = \{(S, T)\}/S \sim S^c$$

by withdrawing the condition that $\sum_{i \in S \cup T} N_i$ must be even. It is easy to see that one can extend the map $\alpha_{\mathcal{P}}$ to $\mathcal{P}_\phi^{\Sigma_0}$ and the map \mathbf{c} to $\tilde{\mathcal{S}}_\phi^{\Sigma_0}$ with its image in $\mathcal{P}_\phi^{\Sigma_0}$. As a result, we have the following lemma which is an analogue of Lemma 6.6, and the proof is similar.

Lemma 6.9 1. The extended map \mathbf{c} will factor through $\mathcal{S}_\phi^{\Sigma_0}$, and it gives a bijection between $\mathcal{S}_\phi^{\Sigma_0}$ and $\mathcal{P}_\phi^{\Sigma_0}$.

2. If we denote the bijection in (1) still by \mathbf{c} , then we have a commutative diagram.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{S}_\phi^{\Sigma_0} & \longrightarrow & \mathcal{S}_\phi^{\Sigma_0} & \xrightarrow{\alpha} & \text{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^\times) \\ & & \downarrow & & \downarrow \mathbf{c} & & \parallel \\ 1 & \longrightarrow & \mathcal{P}_\phi^{\Sigma_0} & \longrightarrow & \mathcal{P}_\phi^{\Sigma_0} & \xrightarrow{\alpha_{\mathcal{P}}} & \text{Hom}(\tilde{G}(F)/G(F), \mathbb{C}^\times) \end{array}$$

if F is local, or

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{S}_\phi^{\Sigma_0} & \longrightarrow & \mathcal{S}_\phi^{\Sigma_0} & \xrightarrow{\alpha} & \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times) \\ & & \downarrow & & \downarrow \mathbf{c} & & \parallel \\ 1 & \longrightarrow & \mathcal{P}_\phi^{\Sigma_0} & \longrightarrow & \mathcal{P}_\phi^{\Sigma_0} & \xrightarrow{\alpha_{\mathcal{P}}} & \text{Hom}(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F), \mathbb{C}^\times) \end{array}$$

if F is global.

Remark 6.10 It is a consequence of Lemmas 6.6 and 6.9 that

$$\alpha(\mathcal{S}_\phi^{\Sigma_0}) = \alpha_{\mathcal{P}}(\mathcal{P}_\phi^{\Sigma_0})$$

Here we give two applications of these combinatorial descriptions. The first one gives the refined L -packet in the archimedean case (cf. Remark 4.7).

Proposition 6.11 Suppose F is real, $\phi \in \tilde{\Phi}_{bdd}(G)$ and $\tilde{\pi}$ is an irreducible admissible representation of $\tilde{G}(F)$ whose restriction to $G(F)$ have irreducible constituents contained in $\bar{\Pi}_\phi$. If $\mathcal{S}_\phi \neq 1$, then $\tilde{\pi} \otimes \omega \cong \tilde{\pi}$ for all characters ω of $\tilde{G}(F)/Z_{\tilde{G}}(F)G(F)$. In particular, let $\tilde{\zeta}$ be the central character of $\tilde{\pi}$, then we can define $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\phi, \tilde{\zeta}}$ if $\bar{\Pi}_\phi$ is not a singleton.

Proof Notice that $\tilde{G}(\mathbb{R})/Z_{\tilde{G}}(\mathbb{R})G(\mathbb{R})$ is either 1 or $\mathbb{R}^\times/\mathbb{R}_{>0}$, and the only nontrivial character ε of $\mathbb{R}^\times/\mathbb{R}_{>0}$ is given by the sign character. Since $GL(n, \mathbb{R})$ has essentially discrete series only when $n \leq 2$, the set $I_{\phi, O}$ only parametrizes quadratic characters of F^\times and discrete series of $GL(2, \mathbb{R})$ with central character ε . Now we suppose $\tilde{\pi} \otimes \varepsilon \not\cong \tilde{\pi}$, then it is clear from Lemma 6.6 that this is only possible when $I_{\phi, O}$ parametrize quadratic characters of F^\times , i.e. ε and the trivial character ε_0 . Depending on which characters $I_{\phi, O}^{odd}$ and $I_{\phi, O}^{even}$ parametrize, we have eight cases and we can represent them as follows: $\varepsilon_0, 2\varepsilon_0; \varepsilon, 2\varepsilon; \varepsilon_0 \oplus 2\varepsilon, \varepsilon \oplus 2\varepsilon_0; \varepsilon_0 \oplus \varepsilon, 2\varepsilon_0 + 2\varepsilon$. One can see easily from Lemma 6.6 that in these cases either $\mathcal{S}_\phi = 1$ or $\varepsilon \in \alpha(\mathcal{S}_\phi)$. Therefore we get a contradiction. For the last point, one just needs to notice $\mathcal{S}_\phi \neq 1$ if $\bar{\Pi}_\phi$ is not a singleton. \square

Remark 6.12 The proof of Proposition 6.11 also shows that $X(\tilde{\pi}) = X$ for discrete series representation $\tilde{\pi}$ of $\tilde{G}(\mathbb{R})$.

The second application is on the multiplicity problem that we have discussed in Sect. 5. Now let us assume F is global again, and it turns out more convenient to ask when both Σ_0 -multiplicity one and Σ_0 -strong multiplicity one hold for $\tilde{\phi}$ together, i.e.

$$\alpha(\mathcal{S}_\phi^{\Sigma_0}) = \prod_{\text{almost all } v}^{\text{aut}} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}).$$

By our Remark 6.10, this is the same as

$$\alpha_{\mathcal{P}}(\mathcal{P}_\phi^{\Sigma_0}) = \prod_{\text{almost all } v}^{\text{aut}} \alpha_{\mathcal{P}}(\mathcal{P}_{\phi_v}^{\Sigma_0}).$$

The next lemma gives an answer for the simplest type of parameters.

Lemma 6.13 *Suppose*

$$\phi = l_1 \eta_1 \boxplus \cdots \boxplus l_r \eta_r \in \bar{\Phi}(G),$$

where η_i are quadratic idèle class characters for $1 \leq i \leq r$. Then both Σ_0 -multiplicity one and Σ_0 -strong multiplicity one hold for $\tilde{\phi}$.

Proof From Lemma 2.1, we can view $\eta_i \circ \lambda$ as quadratic idèle class characters of F' , where F' is the extension of F associated to the character

$$\eta_\phi = \prod_{i=1}^r \eta_{\phi_i}^{l_i}$$

by class field theory. Let us denote $\eta_i \circ \lambda$ by η'_i . We are going to prove this lemma by induction on the number of nontrivial characters η'_i for $1 \leq i \leq r$. Suppose there

exists some quadratic idèle class character ω of F such that $\omega' = \omega \circ \lambda$ is contained in

$$\prod_{\text{almost all } v}^{\text{aut}} \alpha_{\mathcal{P}}(\mathcal{P}_{\phi_v}^{\Sigma_0}).$$

If we assume η'_1 is nontrivial, and let E be the quadratic extension of F' associated to η'_1 , then after a base change to E , we get

$$\phi_E = l_1 \eta_{E,1} \boxplus \cdots \boxplus l_r \eta_{E,r}$$

where $\eta_{E,i} = \eta'_i \circ \text{Nm}_{E/F'}$. And we have

$$\omega_E = \omega' \circ \text{Nm}_{E/F'}$$

contained in

$$\prod_{\text{almost all } v}^{\text{aut}} \alpha_{\mathcal{P}}(\mathcal{P}_{\phi_{E,v}}^{\Sigma_0}).$$

Since $\eta_{E,1} = 1$, by induction the lemma is true for ϕ_E . Hence

$$\omega_E = \prod_{k=1}^m \eta_{E,i_k},$$

for some $1 < i_k \leq r$ and $m < r$. This implies that

$$\left(\omega' \cdot \prod_{k=1}^m \eta'_{i_k} \right) \circ \text{Nm}_{E/F'} = 1.$$

Since $|I_{F'} : \text{Nm}_{E/F'} I_E| = 2$, then

$$\omega' \cdot \prod_{k=1}^m \eta'_{i_k} = 1 \text{ or } \eta'_1.$$

Hence $\omega' \in \alpha_{\mathcal{P}}(\mathcal{P}_{\phi}^{\Sigma_0})$. □

6.3 Construction of global parameters

In this section we are going to give the global lifting result needed in Theorem 6.4. Let us assume F is a nonarchimedean local field and \hat{F} is a totally real global field with $\hat{F}_u = F$ (cf. [7], Lemma 6.2.1). Let \hat{G} be a quasisplit special even orthogonal group or symplectic group over \hat{F} such that $\hat{G}_u = G$ and \hat{G}_v has discrete series for

$v \in S_\infty$ (cf. [7], Lemma 6.2.2). For any finite set S of nonarchimedean places of \dot{F} , we denote the unitary dual of $\dot{G}(\dot{F}_S) = \prod_{v \in S} \dot{G}(\dot{F}_v)$ by $\widehat{\dot{G}(\dot{F}_S)}$, and the Plancherel measure on $\widehat{\dot{G}(\dot{F}_S)}$ by $\widehat{\mu}_S^{pl}$.

Lemma 6.14 *For $\phi \in \bar{\Phi}_{sim}(G)$ and an open subset \widehat{U} of tempered representations of $\dot{G}(\dot{F}_S)$ such that $\widehat{\mu}_S^{pl}(\widehat{U}) > 0$, one can find $\dot{\phi} \in \bar{\Phi}_{sim}(\dot{G})$ with the following properties.*

1. $\dot{\phi}_u = \phi$, and $\otimes_{v \in S} \bar{\Pi}_{\dot{\phi}_v} \subseteq \widehat{U}$.
2. If $v \notin S_\infty(u) \cup S$, then $\dot{\phi}_v$ is spherical. In particular, it can be written as a direct sum of quasicharacters of F_v^\times with at most one ramified quasicharacter.
3. If $v \in S_\infty$, $\dot{\phi}_v \in \bar{\Phi}_2(\dot{G}_v)$.

Proof This lemma is the consequence of ([41], Theorem 5.8) and ([7], Lemma 6.2.2 and Corollary 6.2.4). As one can see in the proof of ([7], Lemma 6.2.2), $\dot{\phi}_v$ has a ramified quasicharacter if and only if $\eta_{\dot{G}_v}$ is ramified. Also note that ([41], Theorem 5.8) requires G to have trivial centre, however this condition can be removed by choosing suitable discrete series in the archimedean places as in the proof of ([7], Lemma 6.2.2). In fact, the main techniques in both proofs are the same, i.e., Arthur's simple invariant trace formula. The new input in ([41], Theorem 5.8) is Harish-Chandra's Plancherel formula and Sauvageot's principle of density result (cf. [35], Theorem 7.3). \square

Lemma 6.14 serves as the building blocks of our global lifting result, and because we want to impose the condition of Σ_0 -strong multiplicity one at one place for any global lift, it is important to consider the case of simple parameters first. We will begin with another description of Σ_0 -strong multiplicity one, which is kind of dual to the original one.

6.3.1 Σ_0 -strong multiplicity one at one place

Suppose F is a global field, G is special even orthogonal or symplectic group over F and $\phi \in \bar{\Phi}(G)$. We define

$$G_{\mathbb{A}_F}^1 = Z_{\tilde{G}}(F_u)G(F_u) \times \prod_{v \neq u} \tilde{G}(F_v),$$

$$G_F^1 = \tilde{G}(F) \cap G_{\mathbb{A}_F}^1.$$

Let

$$\tilde{G}(\pi_v^{\Sigma_0}) = \{g \in \tilde{G}(F_v) : \omega_v(g) = 1 \text{ for all } \omega_v \in \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0})\},$$

for any $[\pi_v] \in \bar{\Pi}_{\phi_v}$, and

$$\tilde{G}(\pi^{\Sigma_0}) = \prod_v \tilde{G}(\pi_v^{\Sigma_0}),$$

for any $[\pi] \in \bar{\Pi}_\phi$. We define $G^1(\pi^{\Sigma_0}) = \tilde{G}(\pi^{\Sigma_0}) \cap G_{\mathbb{A}_F}^1$. As a consequence we can have the following identities

$$\prod_v^{aut} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) = (\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi^{\Sigma_0}))^*$$

$$\prod_{v \neq u}^{aut} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) = (G_{\mathbb{A}_F}^1/G_F^1 G^1(\pi^{\Sigma_0}))^*.$$

By the approximation theory for number fields (cf. [33]), we have $\tilde{G}(\mathbb{A}_F) = \tilde{G}(F)G_{\mathbb{A}_F}^1$. So

$$(\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)\tilde{G}(\pi^{\Sigma_0}))^* = (G_{\mathbb{A}_F}^1/G_F^1 \cap \tilde{G}(F)\tilde{G}(\pi^{\Sigma_0}))^*$$

Therefore the condition of Σ_0 -strong multiplicity one at the place u is equivalent to

$$\begin{aligned} & |G_{\mathbb{A}_F}^1 \cap \tilde{G}(F)\tilde{G}(\pi^{\Sigma_0}) : G_F^1 G^1(\pi^{\Sigma_0})| \\ &= |G_{\mathbb{A}_F}^1 \cap \tilde{G}(F)\tilde{G}(\pi^{\Sigma_0}) : G_F^1(\tilde{G}(\pi^{\Sigma_0}) \cap G_{\mathbb{A}_F}^1)| = 1. \end{aligned} \quad (6.5)$$

Let

$$\begin{aligned} A &= G_{\mathbb{A}_F}^1/G(\mathbb{A}_F), \quad A_F = G_F^1 G(\mathbb{A}_F)/G(\mathbb{A}_F) \\ B_F &= \tilde{G}(F)/G(F), \quad \bar{B}(\pi) = \tilde{G}(\pi^{\Sigma_0})/G(\mathbb{A}_F) \end{aligned}$$

and $\bar{B}(\pi_v) = \tilde{G}(\pi_v^{\Sigma_0})/G(F_v)$, then we can rewrite (6.5) as

$$|A \cap \bar{B}(\pi)B_F : (A \cap \bar{B}(\pi))A_F| = 1.$$

In particular,

$$(A \cap \bar{B}(\pi))A_F = A \cap \bar{B}(\pi)A_F,$$

so we has proved the following lemma.

Lemma 6.15 *Suppose $\phi \in \bar{\Phi}(G)$ and $[\pi] \in \bar{\Pi}_\phi$. Then Σ_0 -strong multiplicity one at the place u holds for $\tilde{\phi}$ if and only if*

$$|A \cap \bar{B}(\pi)B_F : A \cap \bar{B}(\pi)A_F| = 1. \quad (6.6)$$

Note that all these groups A , $\bar{B}(\pi)$ and A_F , B_F can be viewed as subgroups of I_F and F^\times respectively through the similitude character λ . Therefore we have the following equivalent statement for (6.6).

Lemma 6.16 $|A \cap \bar{B}(\pi)B_F : A \cap \bar{B}(\pi)A_F| = 1$ if and only if for any $x \in \bar{B}(\pi)$, there exists $y \in \bar{B}(\pi) \cap B_F$ such that $xy \in A$. (i.e., for any $x_u \in \bar{B}(\pi_u)/(F_u^\times)^2$, there exists $y \in \bar{B}(\pi) \cap B_F$ such that $y_u = x_u^{-1} \bmod (F_u^\times)^2$).

Proof Suppose $x \in \bar{B}(\pi)$ and $z \in B_F$ such that $xz \in A$. First let us assume (6.6), then $xz \in A \cap \bar{B}(\pi)A_F$ since $xz \in A \cap \bar{B}(\pi)B_F$. So we can write $xz = uw$ where $u \in A_F$ and $w \in \bar{B}(\pi) \cap A$. In particular $xw^{-1} = uz^{-1} \in \bar{B}(\pi) \cap B_F$. Let us set $y = wx^{-1}$ which is also in $\bar{B}(\pi) \cap B_F$, then one has $xy = w \in A$.

Conversely, let us take $y \in \bar{B}(\pi) \cap B_F$ such that $xy \in A$. Then we can write $xz = (xy)(y^{-1}z)$. Since both xz and xy lie in A , one has $y^{-1}z \in A$ and in particular, $y^{-1}z \in A \cap B_F = A_F$. Moreover, it clear that $xy \in \bar{B}(\pi)$. Hence $xz \in A \cap \bar{B}(\pi)A_F$ and the rest of the lemma should be clear. \square

6.3.2 Global lift

Now we are going to use Lemmas 6.14, 6.15 and 6.16 to produce a global lift with the intended property of Σ_0 -strong multiplicity one at one place.

Lemma 6.17 Suppose F is a nonarchimedean local field and $\phi \in \bar{\Phi}_{\text{sim}}(G)$, there exists a totally real global field \dot{F} and a group \dot{G} over \dot{F} such that $\dot{F}_u = F$ and $\dot{G}_u = G$, and one can lift ϕ to a global simple parameter $\dot{\phi} \in \bar{\Phi}_{\text{sim}}(G)$ satisfying the following properties.

1. $\dot{\phi}_u = \phi$, and $\dot{\phi}_v \in \bar{\Phi}_2(\dot{G}_v)$ for $v \in S_\infty$.
2. If $v \notin S_\infty(u)$, $\dot{\phi}_v$ is a direct sum of quasicharacters of F_v^\times with at most one ramified quasicharacter.
3. Σ_0 -strong multiplicity one holds for $\dot{\phi}$ at the place u .

Proof Let \dot{G} be a quasisplit special even orthogonal group or symplectic group over \dot{F} such that $\dot{G}_u = G$ and \dot{G}_v has discrete series for $v \in S_\infty$, and let $\dot{\eta}_\phi = \eta_{\dot{G}}$. In view of Lemma 6.14, one would like to impose restrictions over a finite set S of nonarchimedean places described by an open subset \widehat{U} of tempered representations with $\widehat{\mu}_S^{pl}(\widehat{U}) > 0$, so that the global lift $\dot{\phi}$ obtained from Lemma 6.14 has the property of Σ_0 -strong multiplicity one at the place u . To determine S and \widehat{U} , we need to use the equivalent characterization of the property of Σ_0 -strong multiplicity one at one place given by Lemmas 6.15 and 6.16. First, let us take two distinct quadratic idèle class character $\dot{\eta}_i$ ($i = 1, 2$), so that $\dot{\eta}_{i,u} = 1$ and $\dot{\eta}_{i,v} = \varepsilon_v$ the sign character of \mathbb{R}^\times for $v \in S_\infty$. This is possible for one can construct a quadratic extension of any number field with arbitrarily prescribed localizations at finitely many places. Then we can consider the following composite parameter

$$\dot{\phi}_\eta = \dot{\eta}_\phi \boxplus \dot{\eta}_1 \boxplus \dot{\eta}_2 \boxplus \dot{\eta}_1 \dot{\eta}_2 \in \bar{\Phi}(\dot{G}_\eta),$$

and let

$$\bar{B}(\dot{\phi}_{\eta,v}) = \left\{ z \in \tilde{G}_\eta(\dot{F}_v)/\dot{G}_\eta(\dot{F}_v) : \omega_v(z) = 1 \text{ for all } \omega_v \in \alpha(S_{\dot{\phi}_{\eta,v}}^{\Sigma_0}) \right\}$$

for any place v . Note that $B_{\dot{F}} \cong \tilde{G}_{\eta}(\dot{F})/\dot{G}_{\eta}(\dot{F})$ and $\tilde{G}_{\eta}(\dot{F}_v)/\dot{G}_{\eta}(\dot{F}_v) \cong \tilde{G}(\dot{F}_v)/\dot{G}(\dot{F}_v)$. Moreover if $[\pi] \in \bar{\Pi}_{\dot{\phi}}$, then $\bar{B}(\pi) = \bar{B}(\dot{\phi}_{\eta,u})$. And $\bar{B}(\dot{\phi}_{\eta,v}) = \mathbb{R}_{>0}$ if $v \in S_{\infty}$. If we apply Lemmas 6.13 and 6.16 to $\dot{\phi}_{\eta}$, we get for any $x \in \bar{B}(\pi)/(F^{\times})^2$, there exists $y \in B_{\dot{F}}$ such that $y_u = x^{-1} \bmod (F^{\times})^2$ and $y_v \in \bar{B}(\dot{\phi}_{\eta,v})$ for $v \neq u$. In particular, $y_v \in \mathbb{R}_{>0}$ if $v \in S_{\infty}$. Since we are going to use Lemma 6.14 to lift ϕ , by its properties we can conclude immediately that $y_v \in \bar{B}(\dot{\pi}_v)$ unless $v \neq u$ is nonarchimedean and $|y_v| \neq 1$. To emphasize the dependence of y on x , we also write $y = y(x)$. Let $S_{y(x)}$ be those nonarchimedean places $v \neq u$ where $|y_v| \neq 1$, and since the group $\bar{B}(\pi)/(F^{\times})^2$ is finite, we can take the union of all such sets and get

$$S = \bigcup_{x \in \bar{B}(\pi)/(F^{\times})^2} S_{y(x)}$$

which is still finite. Note that S depends on the choice of y for each x . Now we can describe $\widehat{U} = \prod_{v \in S} \widehat{U}_v$. In fact for any $v \in S$, one just needs to take \widehat{U}_v to be the union of tempered packets $\bar{\Pi}_{\phi_v}$ for spherical $\phi_v \in \bar{\Phi}_{bdd}(\dot{G}_v)$ such that $\alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) = 1$. By Lemma 6.9, this condition is equivalent to requiring no finite products of unramified quasicharacters in ϕ_v gives the nontrivial unramified quadratic character unless $\eta_{\dot{G}_v}$ is nontrivial and quadratic. Since this is an open condition, \widehat{U}_v is open with positive Plancherel measure. Note that the condition $\alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) = 1$ also guarantees $\bar{B}(\dot{\phi}_{\eta,v}) \subseteq \bar{B}(\pi_v)$ for $[\pi_v] \in \bar{\Pi}_{\phi_v} \subseteq \widehat{U}_v$. Finally, we can use Lemma 6.14 to get a global lift $\dot{\phi}$ such that $\bar{\Pi}_{\dot{\phi}_v} \subseteq \widehat{U}_v$ for $v \in S$. And it is clear that for any $x \in \bar{B}(\pi)/(F^{\times})^2$, the y chosen above will lie in $\bar{B}(\dot{\pi}) \cap B_{\dot{F}}$ for $[\dot{\pi}] \in \bar{\Pi}_{\dot{\phi}}$. This finishes the proof. \square

This lemma is the first step to overcome the lack of strong multiplicity one, next we will generalize this to composite parameters.

Lemma 6.18 *Suppose*

$$\phi = \phi_1 \oplus \cdots \oplus \phi_q \oplus 2\phi_{q+1} \oplus \cdots \oplus 2\phi_r \in \bar{\Phi}_{ell}(G^{\theta}),$$

where ϕ_i is simple for $1 \leq i \leq r$ and $\theta \in \Sigma_0$. We assume ϕ factors through $\phi_M \in \bar{\Phi}_2(M)$. Then one can choose a lift $(\dot{G}, \dot{M}, \dot{F}, \dot{\phi})$ of (G, M, F, ϕ) with the following properties:

1. \dot{F} is a totally real field and there exists a place u such $(\dot{G}_u, \dot{M}_u, \dot{F}_u, \dot{\phi}_u) = (G, M, F, \phi)$.
2. $\dot{\phi} = \dot{\phi}_1 \boxplus \cdots \boxplus \dot{\phi}_q \boxplus 2\dot{\phi}_{q+1} \boxplus \cdots \boxplus 2\dot{\phi}_r \in \bar{\Phi}_{ell}(\dot{G}^{\theta})$.
3. $\mathcal{S}_{\dot{\phi}}^{\Sigma} = 1$. Moreover, $(\dot{G}, \dot{\phi})$ satisfies the conditions in Proposition 6.8.
4. If $v \notin S_{\infty}(u)$, the local Langlands parameter

$$\dot{\phi}_v = \dot{\phi}_{1,v} \oplus \cdots \oplus \dot{\phi}_{q,v} \oplus 2\dot{\phi}_{q+1,v} \oplus \cdots \oplus 2\dot{\phi}_{r,v}$$

is a direct sum of quasicharacters of F_v^{\times} , and it contains at most one ramified quasicharacter counted without multiplicities modulo the unramified quasicharacters.

5. If $v \in S_\infty$, $\dot{\phi}_{i,v} \in \bar{\Phi}_2(\dot{G}_{\phi_{i,v}})$.
 6. Σ_0 -strong multiplicity one holds for $\tilde{\phi}$ at the place u .

Proof The idea is to apply Lemma 6.17 to each simple parameters. But one needs to be extra careful at the following points.

The first point is about property (3) and (4), they require choosing those global characters $\dot{\eta}_{\phi_i}$ in a consistent way so that the condition of Corollary 6.7 is satisfied, and also there is at most one ramified character in $\dot{\eta}_{\phi_{1,v}} \oplus \cdots \oplus \dot{\eta}_{\phi_{r,v}}$ at each place $v \notin S_\infty(u)$ counted without multiplicities modulo the unramified quasicharacters. But this can be done easily. In fact, one can fix nonarchimedean places $\{v_1, v_2, \dots, v_r\}$ distinct from u . If $q = 0$, we require for $1 \leq i \leq r$ and $1 \leq j \leq r$ that

$$\dot{\eta}_{\phi_{i,v_j}} = \begin{cases} \text{the unramified quadratic character of } \dot{F}_{v_j}^\times, & \text{when } i = j, \\ 1, & \text{otherwise.} \end{cases} \quad (6.7)$$

If $q \neq 0$, we only impose (6.7) for $2 \leq i \leq r$ and $1 \leq j \leq r$, and require for $2 \leq j \leq r$ that

$$\dot{\eta}_{\phi_{1,v_j}} = \begin{cases} \prod_{1 < i \leq q} \dot{\eta}_{\phi_{i,v_j}}, & \text{if } q > 1, \\ 1, & \text{if } q = 1. \end{cases}$$

In this case, we also require $\dot{\eta}_{\phi_{1,v_1}} \neq 1$ when G is special even orthogonal. It is easy to see that these conditions will guarantee (3). Next we can choose $\dot{\eta}_{\phi_i}$ satisfying these conditions. Moreover, we can choose them consecutively for i decreasing from r to 1 such that \dot{G}_{ϕ_i} has discrete series and $\dot{\eta}_{\phi_i}$ is unramified over the ramified places of $\dot{\eta}_{\phi_j}$ for $j > i$, except in the case $i = 1$ and G is symplectic, where we would like to assume G_{ϕ_1} is also symplectic and take

$$\dot{\eta}_{\phi_1} = \begin{cases} \prod_{1 < i \leq q} \dot{\eta}_{\phi_i}, & \text{if } q > 1, \\ 1, & \text{if } q = 1. \end{cases}$$

The second point is about choosing the set S of nonarchimedean places as in the proof of Lemma 6.17. It is tempting to think that it should be the union of all such sets defined in Lemma 6.17 for each simple parameter ϕ_i . In fact, one should choose this set S for ϕ as a whole. Let $\dot{\eta}_1$ and $\dot{\eta}_2$ again be two distinct quadratic idèle class characters defined as in Lemma 6.17. And here we consider

$$\dot{\phi}_\eta = \dot{\eta}_{\phi_1} \boxplus \cdots \boxplus \dot{\eta}_{\phi_q} \boxplus 2\dot{\eta}_{\phi_{q+1}} \boxplus \cdots \boxplus 2\dot{\eta}_{\phi_r} \boxplus \dot{\eta}_1 \boxplus \dot{\eta}_2 \boxplus \dot{\eta}_1 \dot{\eta}_2 \in \bar{\Phi}(\dot{G}_\eta)$$

when q is odd; or

$$\dot{\phi}_\eta = \dot{\eta}_{\phi_1} \boxplus \cdots \boxplus \dot{\eta}_{\phi_q} \boxplus 2\dot{\eta}_{\phi_{q+1}} \boxplus \cdots \boxplus 2\dot{\eta}_{\phi_r} \boxplus 2\dot{\eta}_1 \in \bar{\Phi}(\dot{G}_\eta)$$

when q is even. By applying Lemmas 6.13 and 6.16 to $\dot{\phi}_\eta$, we can get a set S of nonarchimedean places using the same argument as in Lemma 6.17. Finally, one can

choose the open set of tempered representations $\widehat{U}_i = \prod_{v \in S} \widehat{U}_{i,v}$ for each simple parameter ϕ_i as in Lemma 6.17 and make them smaller enough so that $\alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) \subseteq \alpha(\mathcal{S}_{(\phi_\eta)_v}^{\Sigma_0})$. This is possible again by Lemma 6.9. Note that if ϕ_i is a quadratic character, there is no need to impose any local conditions on S . This finishes the proof. \square

Remark 6.19 1. In view of Lemma 6.9, the second property about this global lift $\dot{\phi}$ implies the natural inclusion $S_{\dot{\phi}}^{\Sigma_0} \hookrightarrow S_{\phi}^{\Sigma_0}$ is an isomorphism here. So we can identify $S_{\dot{\phi}}^{\Sigma_0}$ with $S_{\phi}^{\Sigma_0}$.

2. In later proofs, we would like to apply the discussions in Sect. 5.4 to such global parameters $\dot{\phi}$. In fact, by our induction assumption and Proposition 6.8 we can replace Conjecture 5.16, 5.18, 5.19 by Theorem 5.21 for the proper Levi subgroups and twisted endoscopic groups of \tilde{G} . It is then clear that Lemma 5.22 is still valid for $\dot{\phi}$. Since we are only going to establish the stable multiplicity formula for discrete parameters in Theorem 5.21, we need to require

$$\bar{S}_{\dot{\phi}, ell}^{\theta}(\dot{\omega}) = \bar{S}_{\dot{\phi}, ss}^{\theta}(\dot{\omega})$$

when $\dot{\phi}$ is not a discrete parameter in Lemma 5.25. However, in our application this will always be satisfied by our choice of $\dot{\omega}$ and the fact that $S_{\dot{\phi}}^{\Sigma_0} = 1$.

6.4 Proof of main local theorem

With all these refined lifting results, we can start to prove the main local theorem. Let F be a nonarchimedean local field,

$$\phi = \phi_1 \oplus \cdots \oplus \phi_q \oplus 2\phi_{q+1} \oplus \cdots \oplus 2\phi_r \in \bar{\Phi}_{ell}(G^{\theta}), \quad (6.8)$$

where ϕ_i is simple for $1 \leq i \leq r$ and $\theta \in \Sigma_0$. The simplest cases are when ϕ_i are quadratic characters η_i for $1 \leq i \leq r$, and one can see not all of these cases will follow from induction. So we have to treat the exceptional cases differently. In fact regarding property (4) of Lemma 6.18, one only needs to consider the cases when $r \leq 4$, i.e., the trivial character ε_0 , the unramified quadratic character ε , a ramified quadratic character η , and also $\eta \cdot \varepsilon$. In fact, when $r = 4$ and G is special even orthogonal, it can be further reduced to the case $r \leq 3$ by our induction argument as one can see from the proof of Lemma 6.18.

Lemma 6.20 *For ϕ shown in (6.8), if $\phi_i = \eta_i$, and $r \leq 3$ (or $r \leq 4$ when G is symplectic), then the main local theorem (Theorem 4.6) holds for $\tilde{\phi}$.*

Proof There are two types of parameters which lead to nontrivial cases here.
Type I:

$$\phi = \eta_1 \oplus \eta_2 \oplus \eta_3 \in \bar{\Phi}_{bdd}(G)$$

Type II:

$$\phi \in \bar{\Phi}_{ell}(G^\theta) - \bar{\Phi}_2(G)$$

For type I, $\tilde{G} = GL(2)$ and $S_{\tilde{\phi}} = 1$ by Lemma 6.6, so the refined L -packet $\bar{\Pi}_{\tilde{\phi}}$ is a singleton and hence determined by $\bar{\Pi}_{\phi}$. Since the character of any irreducible admissible representation of $GL(2, F)$ is already stable, the packet $\bar{\Pi}_{\tilde{\phi}}$ is then stable. Therefore the only thing we need to prove is the twisted character relation (4.2) for $\omega \in \alpha(S_{\phi})$ and $\theta = id$. To prove this, we use the stabilized $\dot{\omega}$ -twisted trace formula for $\dot{\omega} \in \alpha(S_{\dot{\phi}})$ and some global lift $\dot{\phi}$.

Assume $\dot{\phi} = \dot{\eta}_1 \boxplus \dot{\eta}_2 \boxplus \dot{\eta}_3$ is a global lift of ϕ . Because the global L -packet for $GL(2)$ should also be a singleton and multiplicity one holds for $GL(2)$, the spectral side of the discrete part of the $\dot{\omega}$ -twisted trace formula becomes

$$I_{disc, \dot{\phi}}^{(\tilde{G}, \dot{\omega})}(\tilde{f}) = tr R_{disc, \dot{\phi}}^{(\tilde{G}, \dot{\omega})}(\tilde{f}) = \sum_{\dot{\omega}' \in Y/\alpha(S_{\dot{\phi}})} m(\tilde{\pi} \otimes \dot{\omega}', \dot{\omega}) \tilde{f}_{\tilde{G}}(\tilde{\pi} \otimes \dot{\omega}', \dot{\omega}),$$

where $\tilde{\pi}$ is taken to be any representation in $\mathcal{A}_2(\tilde{G})$, whose restriction to $\dot{G}(\mathbb{A}_F)$ are contained in $\bar{\Pi}_{\dot{\phi}}$, and

$$\tilde{f}_{\tilde{G}}(\tilde{\pi} \otimes \dot{\omega}', \dot{\omega}) = \prod_v \tilde{f}_{\tilde{G}_v}(\tilde{\pi}_v \otimes \dot{\omega}'_v, \dot{\omega}_v),$$

defined by (4.3). In particular, $m(\tilde{\pi} \otimes \dot{\omega}', \dot{\omega}) = \pm 1$. For the endoscopic side, we can apply Lemma 5.25 and get

$$I_{disc, \dot{\phi}}^{(\tilde{G}, \dot{\omega})}(\tilde{f}) = \sum_{\dot{\omega}' \in Y/\alpha(S_{\dot{\phi}})} \tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}),$$

where $\alpha(\dot{x}) = \dot{\omega}$ and $S_{\tilde{\phi}} = 1$. Therefore we have an identity

$$\sum_{\dot{\omega}' \in Y/\alpha(S_{\dot{\phi}})} m(\tilde{\pi} \otimes \dot{\omega}', \dot{\omega}) \tilde{f}_{\tilde{G}}(\tilde{\pi} \otimes \dot{\omega}', \dot{\omega}) = \sum_{\dot{\omega}' \in Y/\alpha(S_{\dot{\phi}})} \tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}).$$

Note that strong multiplicity one also holds for $\tilde{\phi}$. This either can be seen from Lemma 6.13 or from the fact that $\tilde{G} = GL(2)$ here. Then one can use the Satake parameters of representations of $\tilde{G}(\mathbb{A}_F)$ to distinguish the summands on both sides of the identity (cf. Lemma 5.1). As a result, we get

$$m(\tilde{\pi}, \dot{\omega}) \tilde{f}_{\tilde{G}}(\tilde{\pi}, \dot{\omega}) = \tilde{f}'_{\tilde{G}}(\tilde{\phi}, \dot{x}),$$

where we may need to change $\tilde{\pi}$ by twisting with some $\dot{\omega}' \in Y/\alpha(\mathcal{S}_{\dot{\phi}})$ to get this equality. Therefore for any place v one has

$$m(\tilde{\pi}_v, \dot{\omega}_v) \tilde{f}_{\tilde{G}_v}(\tilde{\pi}_v, \dot{\omega}_v) = \tilde{f}'_{\tilde{G}_v}(\tilde{\phi}_v, \dot{x}_v),$$

where $m(\tilde{\pi}_v, \dot{\omega}_v)$ are some constants in \mathbb{C}^\times . If we take \tilde{f}_v supported on $\tilde{Z}_{\dot{F}_v} \dot{G}(\dot{F}_v)$, then $\tilde{f}_{\tilde{G}_v}(\tilde{\pi}_v, \dot{\omega}_v) = \tilde{f}'_{\tilde{G}_v}(\tilde{\phi}_v, \dot{x}_v)$ by character relation for \dot{G}_v , hence $m(\tilde{\pi}_v, \dot{\omega}_v) = 1$ and so is $m(\tilde{\pi}, \dot{\omega})$. In particular we have shown the twisted character relation for $\tilde{\phi}$ of type I.

For type II, it also suffices to show the twisted character relation regarding Lemma 6.2. In fact its proof is similar to the proof of the twisted character relation for a general parameter

$$\phi = \phi_1 \oplus \cdots \oplus \phi_q \oplus 2\phi_{q+1} \oplus \cdots \oplus 2\phi_r \in \bar{\Phi}_{ell}(G^\theta) - \bar{\Phi}_2(G).$$

So here we will carry out the general strategy. First we apply Lemma 6.18 to ϕ and get a global lift $\dot{\phi}$ such that $\dot{\phi}_u = \phi$, $\mathcal{S}_{\dot{\phi}} = 1$, and Σ_0 -strong multiplicity one holds for $\tilde{\phi}$ at the place u . In particular, for the case of type II parameters considered here, it is true that both Σ_0 -multiplicity one and Σ_0 -strong multiplicity one hold according to Lemma 6.13. Next we would like to apply Lemmas 5.22 and 5.25 to get an identity of the spectral expansion and endoscopic expansion of the stabilized $(\theta, \dot{\omega})$ -twisted trace formula. In view of Lemma 6.2, we can assume semisimple $s \in \bar{S}_{\dot{\phi}}^\theta$ satisfies $|\bar{S}_{\dot{\phi}, s}^0| < \infty$. This implies $s \in \bar{S}_{\dot{\phi}, ell}^\theta$ and we denote its preimage in $\bar{S}_{\dot{\phi}, ell}^\theta$ by \dot{s} . Let \dot{x} be the image of \dot{s} in $\mathcal{S}_{\dot{\phi}, ell}^\theta$. Since $\mathcal{S}_{\dot{\phi}} = 1$, we have $\dot{\omega} \neq 1$ and

$$C_{\tilde{\phi}} \sum_{\dot{\omega}' \in Y/\alpha(\mathcal{S}_{\dot{\phi}})} i_{\dot{\phi}}^\theta(\dot{x}) \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}) = C_{\tilde{\phi}} \sum_{\dot{\omega}' \in Y/\alpha(\mathcal{S}_{\dot{\phi}})} e_{\dot{\phi}}^{\prime\theta}(\dot{x}) \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}),$$

It follows from Proposition 5.17 and the fact that $\dot{s} \in \bar{S}_{\dot{\phi}, ell}^\theta$,

$$i_{\dot{\phi}}^\theta(\dot{x}) = e_{\dot{\phi}}^{\prime\theta}(\dot{x}) \neq 0.$$

Therefore

$$\sum_{\dot{\omega}' \in Y/\alpha(\mathcal{S}_{\dot{\phi}})} \tilde{f}_{\tilde{G}^\theta}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}) = \sum_{\dot{\omega}' \in Y/\alpha(\mathcal{S}_{\dot{\phi}})} \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}). \quad (6.9)$$

In case Σ_0 -strong multiplicity one holds, we can again use the Satake parameters of admissible representations of $\tilde{G}(\mathbb{A}_{\dot{F}})$ to distinguish the summands on both sides of the identity. As a result, we get

$$\tilde{f}_{\tilde{G}^\theta}(\tilde{\phi}, \dot{x}) = \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi}, \dot{x})$$

and hence there exist constants n_v for all places such that

$$\tilde{f}_{\tilde{G}_v}^{\tilde{\phi}}(\tilde{\phi}_v, \dot{x}_v) = n_v \cdot \tilde{f}'_{\tilde{G}_v}(\tilde{\phi}_v, \dot{x}_v).$$

If we take \tilde{f}_v supported on $\tilde{Z}_{\tilde{F}_v} \dot{G}(\dot{F}_v)$, then $\tilde{f}_{\tilde{G}_v}^{\tilde{\phi}}(\tilde{\phi}_v, \dot{x}_v) = \tilde{f}'_{\tilde{G}_v}(\tilde{\phi}_v, \dot{x}_v)$ by the twisted local intertwining relation for \dot{G}_v . Therefore $n_v = 1$ and we have shown the twisted local intertwining relation for $\tilde{\phi}$. By Lemma 4.10, this implies the twisted character relation for $\tilde{\phi}$. So we have finished the proof for the parameters of type II. \square

Theorem 6.21 *Suppose F is a nonarchimedean local field and $\phi \in \bar{\Phi}_{\text{ell}}(G^\theta) - \bar{\Phi}_2(G)$, then the twisted character relations (4.2) holds for $\tilde{\phi}$.*

Proof We first get (6.9) following the general strategy in the second part of the proof of Lemma 6.20. In this general case, we only know Σ_0 -strong multiplicity one holds at the place u for $\tilde{\phi}$. But now we can assume the twisted character relation for all places except u . This is because of the property of our lift $\dot{\phi}$ shown in Lemma 6.18 and the fact that we have shown the twisted character relation for the exceptional cases considered in Lemma 6.20. Under these assumptions, we can conclude from the linear independence of twisted characters of $\otimes_{v \neq u} \tilde{\mathcal{H}}(\tilde{G}_v, \tilde{\chi}_v)$ -modules (see [21], A.4.1) that

$$\left(\tilde{f}_{\tilde{G}_u}^{\tilde{\phi}}(\tilde{\phi}_u, \dot{x}_u) - \tilde{f}'_{\tilde{G}_u}(\tilde{\phi}_u, \dot{x}_u) \right) \prod_v \tilde{f}_{\tilde{G}_v}^{\tilde{\phi}}(\tilde{\phi}_v, \dot{x}_v) = 0,$$

and hence

$$\tilde{f}_{\tilde{G}_u}^{\tilde{\phi}}(\tilde{\phi}_u, \dot{x}_u) = \tilde{f}'_{\tilde{G}_u}(\tilde{\phi}_u, \dot{x}_u).$$

This proves the twisted local intertwining relation, which implies the twisted character relation according to Lemma 4.10. \square

Now we can deal with the discrete parameters $\phi \in \bar{\Phi}_2(G)$.

Theorem 6.22 *Suppose F is a nonarchimedean local field and $\phi \in \bar{\Phi}_2(G)$, then the main local theorem (Theorem 4.6) holds for $\tilde{\phi}$.*

Proof We can apply Lemma 6.18 to ϕ , and because of Lemma 6.20 and Theorem 6.21, we can use the argument in Theorem 6.4 to show part (1) and (2) of the main local theorem. At the same time we can deduce the stable multiplicity formula for the global lift $\tilde{\phi}$ (cf. Remark 6.5), i.e.

$$I_{\text{disc}, \tilde{\phi}}^{\tilde{G}}(\tilde{f}) = S_{\text{disc}, \tilde{\phi}}^{\tilde{G}}(\tilde{f}) = m_{\tilde{\phi}} \sum_{\dot{\omega}' \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \dot{\omega}').$$

Hence the only thing left is to show the twisted character relations (4.2). In order to deduce the (θ, ω) -twisted character relation we use the stabilized $(\theta, \dot{\omega})$ -twisted trace formula. Note that

$$I_{disc, \dot{\phi}}^{(\tilde{G}^\theta, \dot{\omega})}(\tilde{f}) = \text{tr} R_{disc, \dot{\phi}}^{(\tilde{G}^\theta, \dot{\omega})}(\tilde{f}) = \sum_{\dot{\omega}'} \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\dot{\phi} \otimes \dot{\omega}'}} m(\tilde{\pi}, \dot{\omega}) \prod_v \tilde{f}_{\tilde{G}_v}^{\tilde{\pi}_v, \dot{\omega}_v},$$

where $\tilde{f}_{\tilde{G}_v}^{\tilde{\pi}_v, \dot{\omega}_v}$ is normalized according to (4.3), the sum of $\dot{\omega}'$ is taken over

$$Y / \prod_v^{aut} \alpha \left(\mathcal{S}_{\dot{\phi}_v}^{\Sigma_0} \right),$$

and $|m(\tilde{\pi}, \dot{\omega})|$ is some integer not larger than the multiplicity

$$m(\tilde{\phi}) := m_{\dot{\phi}} \left| \prod_v^{aut} \alpha(\mathcal{S}_{\dot{\phi}_v}^{\Sigma_0}) \right| |\alpha(\mathcal{S}_{\dot{\phi}})|^{-1}$$

of $\tilde{\pi}$ as $\mathcal{H}(\tilde{G}, \tilde{\chi})$ -module. By Lemma 5.25, we get

$$I_{disc, \dot{\phi}}^{(\tilde{G}^\theta, \dot{\omega})}(\tilde{f}) = m_{\dot{\phi}} \sum_{\dot{\omega}' \in Y / \alpha(\mathcal{S}_{\dot{\phi}})} \tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \dot{\omega}', \dot{x}),$$

where $\alpha(\dot{x}) = \dot{\omega}$. By Lemma 6.20 and Theorem 6.21, we can assume the twisted character relations for all places $v \neq u$. Then it follows from the linear independence of twisted characters of $\otimes_{v \neq u} \mathcal{H}(\tilde{G}_v, \tilde{\chi}_v)$ -modules and Σ_0 -strong multiplicity one for $\tilde{\phi}$ at the place u that

$$\begin{aligned} & \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} m(\tilde{\pi}, \dot{\omega}) \tilde{f}_{\tilde{G}_u}^{\tilde{\pi}_u, \dot{\omega}_u} \prod_{v \neq u} \tilde{f}_{\tilde{G}_v}^{\tilde{\pi}_v, \dot{\omega}_v} \\ &= m(\tilde{\phi}) \tilde{f}'_{\tilde{G}_u}(\tilde{\phi}_u, x) \prod_{v \neq u} \left(\sum_{[\tilde{\pi}_v] \in \bar{\Pi}_{\tilde{\phi}_v}} \tilde{f}_{\tilde{G}_v}^{\tilde{\pi}_v, \dot{\omega}_v} \right). \end{aligned} \quad (6.10)$$

Now we choose $\tilde{f} = \otimes_v \tilde{f}_v$ with \tilde{f}_u supported on $\tilde{Z}_F G(F)$ and thus

$$\tilde{f}'_{\tilde{G}_u}(\tilde{\phi}_u, x_u) = \sum_{[\tilde{\pi}_u] \in \bar{\Pi}_{\tilde{\phi}_u}} \tilde{f}_{\tilde{G}_u}^{\tilde{\pi}_u, \dot{\omega}_u}.$$

Substitute such test functions into the identity (6.10), one deduces that

$$m(\tilde{\pi}, \dot{\omega}) = m(\tilde{\phi}).$$

Therefore

$$m(\tilde{\phi})(\tilde{f}'_{\tilde{G}_u}(\tilde{\phi}_u, x) - \sum_{[\tilde{\pi}_u] \in \bar{\Pi}_{\tilde{\phi}_u}} \tilde{f}_{\tilde{G}_u}(\tilde{\pi}_u, \dot{\omega}_u)) \prod_{v \neq u} \left(\sum_{[\tilde{\pi}_v] \in \bar{\Pi}_{\tilde{\phi}_v}} \tilde{f}_{\tilde{G}_v}(\tilde{\pi}_v, \dot{\omega}_v) \right) = 0$$

for all $\tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$. So we must have

$$\tilde{f}'_{\tilde{G}_u}(\tilde{\phi}_u, x) = \sum_{[\tilde{\pi}_u] \in \bar{\Pi}_{\tilde{\phi}_u}} \tilde{f}_{\tilde{G}_u}(\tilde{\pi}_u, \dot{\omega}_u),$$

and this finishes the proof of the (θ, ω) -twisted character relation. \square

At this point, we have proved our main local theorem (Theorem 4.6) for $\tilde{G} = \tilde{G}(N)$, and the general case is just a corollary of that.

Corollary 6.23 *Suppose*

$$G = G(n_1) \times G(n_2) \times \cdots \times G(n_q),$$

with $n_i \leq N$ for $1 \leq i \leq q$ and $\phi \in \bar{\Phi}_{bdd}(G)$, then the main local theorem (Theorem 4.6) holds for $\tilde{\phi}$.

Proof Let us write $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q$ such that $\phi_i \in \bar{\Phi}_{bdd}(G(n_i))$ for $1 \leq i \leq q$. Note that

$$\tilde{G} \subseteq \tilde{G}(n_1) \times \tilde{G}(n_2) \times \cdots \times \tilde{G}(n_q)$$

form a pair of groups satisfying the assumption in Sect. 2.1.1. Since $\bar{\Pi}_{\tilde{\phi}_i}$ is well defined now, we can define $\bar{\Pi}_{\tilde{\phi}}$ to be the restriction of $\bigotimes_{i=1}^q \bar{\Pi}_{\tilde{\phi}_i}$ to $\tilde{G}(F)$. It is clear that $\bar{\Pi}_{\tilde{\phi}}$ satisfies part (1) and (2) of Theorem 4.6. Moreover, the twisted endoscopic groups of \tilde{G} lift to twisted endoscopic groups of $\tilde{G}(n_1) \times \tilde{G}(n_2) \times \cdots \times \tilde{G}(n_q)$ by Proposition 2.7. Then it is a consequence of Corollary 2.22 that the twisted character relations of \tilde{G} follow from that of $\tilde{G}(n_1) \times \tilde{G}(n_2) \times \cdots \times \tilde{G}(n_q)$ again by restriction. This completes the proof in the general case. \square

6.5 Proof of global theorem

In this section, we are going to prove the global theorem, i.e., Theorem 5.21. We will keep the notations as in Sect. 6.1. Suppose F is global,

$$G = G(n_1) \times G(n_2) \times \cdots \times G(n_q),$$

with $\sum_{i=1}^q n_i = N$. Now we can assume the main local theorem for \tilde{G} thanks to Sect. 6.4. We will first prove the corresponding statement of Conjecture 5.16 for \tilde{G} .

Theorem 6.24 *We assume $\phi \in \bar{\Phi}(G)$ satisfies the assumption in Theorem 5.21.*

1. *One can associate a global packet $\bar{\Pi}_{\tilde{\phi}}$ of irreducible admissible representations of $\tilde{G}(\mathbb{A}_F)$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules, satisfying the following properties:*
 - (a) $\bar{\Pi}_{\tilde{\phi}} = \bigotimes'_v \bar{\Pi}_{\tilde{\phi}_v}$, where $\bar{\Pi}_{\tilde{\phi}_v}$ is some lift of $\bar{\Pi}_{\phi_v}$ defined in Theorem 4.6.
 - (b) *there exists $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$, so that $\tilde{\pi}$ is isomorphic to an automorphic representation as $\tilde{\mathcal{H}}(\tilde{G})$ -modules.*

Moreover, $\bar{\Pi}_{\tilde{\phi}}$ is unique up to twisting by characters of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$. And we can define a global character of $\mathcal{S}_{\tilde{\phi}}$ by

$$\langle x, \tilde{\pi} \rangle := \prod_v \langle x_v, \tilde{\pi}_v \rangle \quad \text{for } [\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \text{ and } x \in \mathcal{S}_{\tilde{\phi}}.$$

2. *If $\phi \in \bar{\Phi}_2(G)$, the ϕ -component of the $\tilde{\zeta}$ -equivariant discrete spectrum of $\tilde{G}(\mathbb{A}_F)$ as $\tilde{\mathcal{H}}(\tilde{G})$ -modules can be decomposed as follows*

$$L_{disc,\phi}^2(\tilde{G}(F) \backslash \tilde{G}(\mathbb{A}_F), \tilde{\zeta}) = m_{\phi} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{\pi},$$

where m_{ϕ} is defined as in Remark 3.14.

Proof If ϕ factors through $\phi_M \in \bar{\Phi}_2(M)$ for some proper Levi subgroup M of G , then by our induction assumption, we have a global L -packet $\bar{\Pi}_{\tilde{\phi}_M}$ for \tilde{M} , and we can define $\bar{\Pi}_{\tilde{\phi}}$ to be the irreducible constituents induced from $\bar{\Pi}_{\tilde{\phi}_M}$. So it is enough to consider the case $\phi \in \bar{\Phi}_2(G)$. Note that one can always define a global packet $\bar{\Pi}_{\tilde{\phi}}$ as follows. If $\tilde{\pi} \in \mathcal{A}_2(\tilde{G})$ and its restriction to $G(\mathbb{A}_F)$ have irreducible constituents contained in $\bar{\Pi}_{\phi}$, then we can take the local lift $\bar{\Pi}_{\tilde{\phi}_v}$ of $\bar{\Pi}_{\phi_v}$ to be the one containing $[\tilde{\pi}_v]$. We form a global packet

$$\bar{\Pi}_{\tilde{\phi}} := \prod_v \bar{\Pi}_{\tilde{\phi}_v},$$

and the uniqueness property should follow from Corollary 5.9 and the decomposition in Part (2).

To show Part (2), we can apply Lemma 5.25 to get

$$I_{disc,\phi}^{\tilde{G}}(\tilde{f}) = S_{disc,\phi}^{\tilde{G}}(\tilde{f}) + C_{\tilde{\phi}} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} \sum_{x \in \mathcal{S}_{\tilde{\phi}} - \{1\}} \tilde{f}_{\tilde{G}}^{\omega}(\tilde{\phi} \otimes \omega, x). \quad (6.11)$$

By the local character relation, one can define a global packet $\bar{\Pi}_{\tilde{\phi}_x}$ transferred from $\bar{\Pi}_{\tilde{\phi}'}$ for any $x \in \mathcal{S}_{\tilde{\phi}} - \{1\}$. Next we would like to add

$$2 \cdot C_{\tilde{\phi}} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} \sum_{x \in \mathcal{S}_{\tilde{\phi}} - \{1\}} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_x} \\ \langle x, \tilde{\pi} \rangle = -1}} \tilde{f}_{\tilde{G}}(\tilde{\pi} \otimes \omega) \quad (6.12)$$

to both sides of (6.11). Note that this sum does not include $x \in \mathcal{S}_{\tilde{\phi}} - \{1\}$ such that $\langle x, \tilde{\pi} \rangle = 1$ for all $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_x}$. For those x which does not contribute to (6.12), we have $\tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \omega, x) = \tilde{f}_{\tilde{G}}(\tilde{\phi}_x \otimes \omega)$ which is defined by $\bar{\Pi}_{\tilde{\phi}_x}$ and is stable. Then the right hand side becomes

$$S_{disc, \phi}^{\tilde{G}}(\tilde{f}) + C_{\tilde{\phi}} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} \sum_{x \in \mathcal{S}_{\tilde{\phi}} - \{1\}} \tilde{f}_{\tilde{G}}(\tilde{\phi}_x \otimes \omega),$$

which is again stable. So the left hand side

$$I_{disc, \phi}^{\tilde{G}}(\tilde{f}) + 2 \cdot C_{\tilde{\phi}} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} \sum_{x \in \mathcal{S}_{\tilde{\phi}} - \{1\}} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_x} \\ \langle x, \tilde{\pi} \rangle = -1}} \tilde{f}_{\tilde{G}}(\tilde{\pi} \otimes \omega) \quad (6.13)$$

is also stable. By (5.15),

$$I_{disc, \phi}^{\tilde{G}}(\tilde{f}) = \text{tr } R_{disc, \phi}^{\tilde{G}}(\tilde{f}).$$

Let $\bar{\Pi}_{\tilde{\phi}}$ be the global packet defined in the beginning with respect to some fixed $\tilde{\pi}^0 \in \mathcal{A}_2(\tilde{G})$. Since (6.13) is stable, it is stable at every place. So we can take $\tilde{f} = \otimes_w \tilde{f}_w$ and fix $\otimes_{w \neq v} \tilde{f}_w$ for any place v , then by Corollary 4.8 the coefficient of $\tilde{f}_v(\tilde{\pi}_v)$ in (6.13) must be the same for all $\tilde{\pi}_v \in \bar{\Pi}_{\tilde{\phi}_v}$. By varying $\otimes_{w \neq v} \tilde{f}_w$ and the linear independence of characters of $\otimes_{w \neq v} \tilde{\mathcal{H}}(\tilde{G}_w, \tilde{\chi}_w)$ -modules, we have that

$$[\tilde{\pi}^0] = [\tilde{\pi}_v^0] \otimes (\otimes_{w \neq v} [\tilde{\pi}_w^0])$$

contributes to (6.13) if and only if all elements in

$$\bar{\Pi}_{\tilde{\phi}_v} \otimes (\otimes_{w \neq v} [\tilde{\pi}_w^0])$$

also contribute to (6.13). By repeating this kind of argument, one can show all elements in $\bar{\Pi}_{\tilde{\phi}}$ contribute to (6.13). Note for any $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ such that $\langle \cdot, \tilde{\pi} \rangle = 1$, it can only contribute to $I_{disc, \phi}^{\tilde{G}}(\tilde{f})$, which means it belongs to $\mathcal{A}_2(\tilde{G})$. Then the decomposition in Part (2) will follow from Proposition 5.11 immediately. \square

Remark 6.25 Following the proof, we can also apply the same argument to elements in $\bar{\Pi}_{\tilde{\phi}_x}$ which contributes to (6.13). It follows all elements in $\bar{\Pi}_{\tilde{\phi}_x}$ contributes to (6.13). For $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_x}$ such that $\langle \cdot, \tilde{\pi} \rangle = 1$, it can only come from $I_{disc, \phi}^{\tilde{G}}(\tilde{f})$. So $\bar{\Pi}_{\tilde{\phi}_x} = \bar{\Pi}_{\tilde{\phi}}$ up to twisting by some character in Y . Note this is only true for $x \in \mathcal{S}_{\tilde{\phi}} - \{1\}$ in the sum (6.12). As a result, (6.13) becomes

$$m_{\phi} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega),$$

where

$$\tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega) = \prod_v \tilde{f}_v(\tilde{\phi}_v \otimes \omega_v).$$

Moreover, we have

$$S_{disc, \phi}^{\tilde{G}}(\tilde{f}) = m_{\phi} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega) - C_{\tilde{\phi}} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} \sum_{x \in \mathcal{S}_{\tilde{\phi}} - \{1\}} \tilde{f}^{\tilde{G}}(\tilde{\phi}_x \otimes \omega).$$

Suppose $\bar{\Pi}_{\tilde{\phi}_x} = \bar{\Pi}_{\tilde{\phi}}$ up to twisting by some character in Y for all $x \in \mathcal{S}_{\tilde{\phi}} - \{1\}$, then

$$S_{disc, \phi}^{\tilde{G}}(\tilde{f}) = m_{\phi} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} |\mathcal{S}_{\tilde{\phi}}|^{-1} \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega).$$

This is the stable multiplicity formula in Conjecture 5.18. We will come back to this identity in Theorem 6.30.

Next, we will prove the corresponding statement of Conjecture 5.19 for \tilde{G} .

Theorem 6.26 Suppose $\phi \in \bar{\Phi}_2(G)$ satisfying the assumption in Theorem 5.21 and $x \in \mathcal{S}_{\phi}^{\theta}$ with $\alpha(x) = \omega$ for $\theta \in \Sigma_0$ and some character ω of $\tilde{G}(\mathbb{A}_F)/\tilde{G}(F)G(\mathbb{A}_F)$. For $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ with $\langle \cdot, \tilde{\pi} \rangle = 1$, the canonical intertwining operator

$$R(\theta)^{-1} \circ R(\omega)$$

restricted to the $\tilde{\pi}$ -isotypic component $I(\tilde{\pi})$ is equal to the product of $m(\tilde{\pi})$ and the local intertwining operators $A_{\tilde{\pi}_v}(\theta, \omega_v)$ determined by x_v (see (4.3)), i.e.

$$I_{disc, \phi}^{(\tilde{G}^{\theta}, \omega)}(\tilde{f}) = m_{\phi} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi})} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{f}_{\tilde{G}^{\theta}}(\tilde{\pi}, \omega), \quad \tilde{f} \in \tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi}), \quad (6.14)$$

where $\tilde{f}_{\tilde{G}^{\theta}}(\tilde{\pi}, \omega) = \prod_v \tilde{f}_{\tilde{G}_v^{\theta}}(\tilde{\pi}_v, \omega_v)$, and it does not depend on x .

Proof It follows from Theorem 6.24 that

$$I_{disc,\phi}^{(\tilde{G}^\theta,\omega)}(\tilde{f}) = tr R_{disc,\phi}^{(\tilde{G}^\theta,\omega)}(\tilde{f}) = \sum_{\omega'} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} m(\tilde{\pi}, \theta, \omega) \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega)$$

where the sum of ω' is taken over

$$Y / \prod_v^{aut} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}),$$

and $|m(\tilde{\pi}, \theta, \omega)|$ is some integer less than or equal to

$$m(\tilde{\phi}) := m_\phi \left| \prod_v^{aut} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}) \right| |\alpha(\mathcal{S}_\phi)|^{-1}.$$

By Lemma 5.25, we have

$$I_{disc,\phi}^{(\tilde{G}^\theta,\omega)}(\tilde{f}) = |\mathcal{S}_{\tilde{\phi}}|^{-1} \sum_{\omega'} \sum_{x' \in \mathcal{S}_{\tilde{\phi}}^\theta(\omega)} m(\tilde{\phi}) \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x'),$$

where the sum of ω' is again over

$$Y / \prod_v^{aut} \alpha(\mathcal{S}_{\phi_v}^{\Sigma_0}).$$

Therefore

$$\sum_{\omega'} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} m(\tilde{\pi}, \theta, \omega) \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega) = |\mathcal{S}_{\tilde{\phi}}|^{-1} \sum_{\omega'} \sum_{x' \in \mathcal{S}_{\tilde{\phi}}^\theta(\omega)} m(\tilde{\phi}) \tilde{f}'_{\tilde{G}^\theta}(\tilde{\phi} \otimes \omega', x').$$

By the twisted character relation, one can define a global packet $\bar{\Pi}_{\tilde{\phi}_{x'}}$ transferred from $\bar{\Pi}_{\tilde{\phi}}$ for any $x' \in \mathcal{S}_{\tilde{\phi}}^\theta(\omega)$. Note $\mathcal{S}_{\tilde{\phi}}^\theta(\omega) = x \cdot \mathcal{S}_{\tilde{\phi}}$, then

$$\begin{aligned} \sum_{\omega'} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} m(\tilde{\pi}, \theta, \omega) \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega) &= |\mathcal{S}_{\tilde{\phi}}|^{-1} \sum_{\omega'} \sum_{y \in \mathcal{S}_{\tilde{\phi}}} m(\tilde{\phi}) \\ &\quad \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_{xy}} \otimes \omega'} \langle y, \tilde{\pi} \rangle \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega), \quad (6.15) \end{aligned}$$

where $\tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega)$ is normalized by x (cf. (4.3)). This implies

$$\sum_{\omega'} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} m(\tilde{\pi}, \theta, \omega) \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega) = |\mathcal{S}_{\tilde{\phi}}|^{-1} \sum_{\omega'} \sum_{y \in \mathcal{S}_{\tilde{\phi}}} m(\tilde{\phi}) \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_{xy}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega).$$

It follows from the linear independence of twisted characters of $\tilde{\mathcal{H}}(\tilde{G}, \tilde{\chi})$ -modules that we can choose $\bar{\Pi}_{\tilde{\phi}_{x'}} = \bar{\Pi}_{\tilde{\phi}}$ for all $x' \in \mathcal{S}_{\tilde{\phi}}^\theta(\omega)$. Then

$$\sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \\ \langle \cdot, \tilde{\pi} \rangle = 1}} m(\tilde{\pi}, \theta, \omega) \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega) = m(\tilde{\phi}) \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega).$$

So $m(\tilde{\pi}, \theta, \omega) = m(\tilde{\phi})$. Hence

$$I_{disc, \phi}^{(\tilde{G}^\theta, \omega)}(\tilde{f}) = m_{\phi} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\tilde{\phi}})} \sum_{\substack{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}} \otimes \omega' \\ \langle \cdot, \tilde{\pi} \rangle = 1}} \tilde{f}_{\tilde{G}^\theta}(\tilde{\pi}, \omega).$$

□

Now we are only left with proving the corresponding statement of Conjecture 5.18. From Remark 6.25, we see the key is to prove the functoriality of endoscopic transfer. Here we would like to consider a more general notion of that, i.e., **functoriality of twisted endoscopic transfer**, and we formulate it in our context as follows. For $\phi \in \tilde{\Phi}(G)$ and semisimple $s \in \tilde{S}_{\phi}$, let $(G', \phi') \rightarrow (\phi, s)$ and $\tilde{G}' \in \mathcal{E}(\tilde{G}, \omega)$ be the lift of G' , the functoriality of twisted endoscopic transfer means the global L -packet $\bar{\Pi}_{\tilde{\phi}'}$ transfers to a global L -packet $\bar{\Pi}_{\tilde{\phi}}$ through the local twisted character relation (4.2). By the same argument in the proof of Lemma 4.11, we see the transfer of $\bar{\Pi}_{\tilde{\phi}'}$ only depends on the image x of s in \mathcal{S}_{ϕ} . So we can denote the transfer image by $\bar{\Pi}_{\tilde{\phi}_x}$.

Lemma 6.27 Suppose $\tilde{G} = \tilde{G}(n)$ for $n \leq N$, $\phi \in \tilde{\Phi}(G)$ such that $\mathcal{S}_{\tilde{\phi}} = 1$, then $\bar{\Pi}_{\tilde{\phi}_x} = \bar{\Pi}_{\tilde{\phi}}$ up to twisting by some character in Y for any $x \in \mathcal{S}_{\phi}$.

Proof For semisimple $s \in \tilde{S}_{\phi}$, let $(G', \phi') \rightarrow (\phi, s)$. Suppose $|\tilde{S}_{\phi, s}^0| = \infty$, let $T_{\phi, s}$ be a maximal torus of $(\mathcal{S}_{\phi, s})^0$, then $\widehat{M}' = \text{Cent}(T_{\phi, s}, \widehat{G}')$ defines a proper Levi subgroup of \widehat{G}' such that ϕ' factors through $\phi'_M \in \tilde{\Phi}_2(M')$. Moreover, $M' \in \mathcal{E}_{ell}(M)$ for a proper Levi subgroup M of G , which is determined by $\widehat{M} \equiv \text{Cent}(T_{\phi, s}, \widehat{G})$. So ϕ factors through $\phi_M \in \tilde{\Phi}(M)$, and we can reduce this case to \tilde{M} .

Next we assume $|\tilde{S}_{\phi, s}^0| < \infty$, then $\phi \in \tilde{\Phi}_{ell}(G)$. In particular, $s \in \tilde{S}_{\phi, ell}$ and we let x be its image in \mathcal{S}_{ϕ} . We can also assume $x \neq 1$, then $\mathcal{S}_{\tilde{\phi}} = 1$ implies $\alpha(x) = \omega \neq 1$. By Lemma 5.25, we have

$$I_{disc, \phi}^{(\tilde{G}, \omega)}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi})} e'_{\phi}(x) \tilde{f}'_{\tilde{G}}(\tilde{\phi} \otimes \omega', x).$$

By Lemma 5.22 and Theorem 6.26, we have

$$I_{disc, \phi}^{(\tilde{G}, \omega)}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi})} i_{\phi}(x) \tilde{f}_{\tilde{G}}(\tilde{\phi} \otimes \omega', x).$$

Note $e'_{\phi}(x) = i_{\phi}(x) \neq 0$. Then by the linear independence of twisted characters, we have $\bar{\Pi}_{\tilde{\phi}_x} = \bar{\Pi}_{\tilde{\phi}}$ up to twisting by some character in Y . \square

It is not hard to extend this result to the general case.

Lemma 6.28 *Suppose $G = G(n_1) \times G(n_2) \times \cdots \times G(n_q)$, and $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q \in \bar{\Phi}(G)$ with $\phi_i \in \bar{\Phi}(G(n_i))$ for $1 \leq i \leq q$. If $\mathcal{S}_{\tilde{\phi}_i} = 1$ for all i , then $\bar{\Pi}_{\tilde{\phi}_x} = \bar{\Pi}_{\tilde{\phi}}$ up to twisting by some character in Y for any $x \in \mathcal{S}_{\phi}$.*

Proof If we write the image of x in \mathcal{S}_{ϕ_i} by x_i , then by Lemma 6.27, $\bigotimes_{i=1}^q \bar{\Pi}_{\tilde{\phi}_{x_i}}$ is a global L-packet of $\tilde{G}(n_1) \times \tilde{G}(n_2) \times \cdots \times \tilde{G}(n_q)$, and $\bar{\Pi}_{\tilde{\phi}_x}$ is its restriction to \tilde{G} . Since the restriction of a global L-packet is again a global L-packet, then $\bar{\Pi}_{\tilde{\phi}_x} = \bar{\Pi}_{\tilde{\phi}}$ up to twisting by some character in Y . \square

Remark 6.29 We would like to point out in the case of this lemma, $\mathcal{S}_{\tilde{\phi}}$ can be nontrivial even though $\mathcal{S}_{\tilde{\phi}_i} = 1$ for all i . For example, let $G = Sp(2n) \times Sp(2n)$ and $\phi = (\phi_1 \boxplus \phi_2) \times (\phi_1 \boxplus \phi_2) \in \bar{\Phi}_2(G)$. We assume the central characters satisfy $\eta_{\phi_1} = \eta_{\phi_2} \neq 1$, then $\mathcal{S}_{\tilde{\phi}} \cong \mathbb{Z}/2\mathbb{Z}$.

Now we can prove the corresponding statement of Conjecture 5.18.

Theorem 6.30 *Suppose $G = G(n_1) \times G(n_2) \times \cdots \times G(n_q)$, and $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q \in \bar{\Phi}_2(G)$ with $\phi_i \in \bar{\Phi}(G(n_i))$ for $1 \leq i \leq q$. If $\mathcal{S}_{\tilde{\phi}_i} = 1$ for all i , then*

$$S_{disc, \phi}^{\tilde{G}}(\tilde{f}) = m_{\phi} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} |\mathcal{S}_{\tilde{\phi}}|^{-1} \sigma(\bar{S}_{\phi}^0) \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\tilde{G}, \tilde{\chi}).$$

Proof It follows from Remark 6.25 and Lemma 6.28 that

$$S_{disc, \phi}^{\tilde{G}}(\tilde{f}) = m_{\phi} \sum_{\omega \in Y/\alpha(\mathcal{S}_{\phi})} |\mathcal{S}_{\tilde{\phi}}|^{-1} \tilde{f}^{\tilde{G}}(\tilde{\phi} \otimes \omega).$$

Note in this case $\sigma(\bar{S}_{\phi}^0) = 1$. This finishes the proof. \square

Up to now, we have proved the local and global theorems for \tilde{G} under our induction assumptions, when G does not contain any factor of $SO(2N+2, \eta)$ (cf. Remark 6.1). By adding these results to our induction assumptions, we can prove the general case by repeating the previous arguments in Sects. 6.4 and 6.5 without any change.

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7 Appendix: An irreducibility result

The aim of this appendix is to prove Propositions 3.10 and 3.11. We should point out that neither these results nor their proofs are new, and just for the convenience of the readers we would like to present their proofs here. In this section G will always denote a symplectic group or special even orthogonal group. First, let us restate Proposition 3.10, and for its proof we will follow the same argument in [27].

Proposition 7.1 *If F is global and $\phi \in \Phi_{sim}(N)$, then $\phi_v \in \Phi_{unit}^+(N_v)$.*

Proof Suppose π_ϕ is the unitary cuspidal automorphic representation of $GL(N)$ associated to ϕ . By [37], $\pi_{\phi,v}$ is generic for all places v . And one knows from [17] that any irreducible generic representation of $GL(N)$ over a local field is a fully induced representation

$$\mathcal{I}_P(v^{a_1}\sigma_1 \otimes v^{a_2}\sigma_2 \otimes \cdots \otimes v^{a_r}\sigma_r)$$

where P is some standard parabolic subgroup of $GL(N)$, σ_i are unitary essentially discrete series representations and $a_i \in \mathbb{R}$ for $1 \leq i \leq r$. So for our ϕ , we have

$$\phi_v = v^{a_1}\phi_{v,1} \oplus v^{a_2}\phi_{v,2} \oplus \cdots \oplus v^{a_s}\phi_{v,s}$$

where $\phi_{v,j} \in \Phi_{sim}(N_{v,j})$ for $1 \leq j \leq s$. Since $\pi_{\phi,v}$ is also unitary, by the classification of unitary dual of $GL(N)$ (archimedean case in [46] and nonarchimedean case in [44]), ϕ_v must belongs to $\Phi_{unit}^+(N)$. In particular, $|a_j| < 1/2$ for $1 \leq j \leq s$. \square

Next we restate Proposition 3.11 as follows.

Proposition 7.2 *Suppose F is local, $\phi \in \bar{\Phi}_{unit}^+(G)$, and ϕ can be regarded as $\phi_{M,\lambda}$ where $\phi_M \in \bar{\Phi}_{bdd}(M)$ and $\lambda \in \mathfrak{a}_M^*$ lies in some open chamber of $P \supseteq M$. Then for any $[\pi_M] \in \bar{\Pi}_{\phi_M}$, the induced representation $\mathcal{I}_P(\pi_{M,\lambda})$ is irreducible.*

Before starting the proof, we want to introduce some notations for the parabolic induction. Suppose π_1 and π_2 are representations of $GL(N_1)$ and $GL(N_2)$ respectively, we will write $\pi_1 \times \pi_2$ for the parabolic induction of $\pi_1 \otimes \pi_2$ by viewing $GL(N_1) \times GL(N_2)$ as the Levi component of a maximal parabolic subgroup in $GL(N_1 + N_2)$. And similarly, suppose π and σ are representations of $GL(N)$ and G , we will write $\pi \rtimes \sigma$ for the parabolic induction of $\pi \otimes \sigma$ by viewing $GL(N) \times G$ as the Levi component of a maximal parabolic subgroup in G_+ which is of same type as G .

The proof of this proposition breaks down to several steps. First notice $\phi \in \bar{\Phi}_{unit}^+(G)$, so we can write

$$\phi = \phi_{G_-} \oplus (v^{a_1}\phi_1 \oplus v^{-a_1}\phi_1^\vee) \oplus \cdots \oplus (v^{a_m}\phi_m \oplus v^{-a_m}\phi_m^\vee),$$

where G_- is of the same type as G , $\phi_{G_-} \in \bar{\Phi}_{bdd}(G_-)$, and $\phi_i \in \Phi_{sim}(N_i)$ for $1 \leq i \leq m$ with $0 < a_m \leq \dots \leq a_1 < 1/2$. Then

$$\phi_M = \phi_{G_-} \oplus \phi_1 \oplus \dots \oplus \phi_m \in \bar{\Phi}_{bdd}(M),$$

and $\lambda = (a_1, a_2, \dots, a_m)$. Let π_{ϕ_i} be the corresponding essentially discrete series representation of $GL(N_i)$ associated to ϕ_i for $1 \leq i \leq m$, then any $\pi_M \in \bar{\Pi}_{\phi_M}$ can be written as $\pi_M = \pi_{G_-} \otimes \pi_{\phi_1} \otimes \dots \otimes \pi_{\phi_m}$, for some $\pi_{G_-} \in \bar{\Pi}_{\phi_{G_-}}$. And $\mathcal{I}_P(\pi_{M,\lambda}) = v^{a_1} \pi_{\phi_1} \times \dots \times v^{a_m} \pi_{\phi_m} \rtimes \pi_{G_-}$. Next, we have two reduction steps. These steps are due to Speh and Vogan [43], and they are also presented very clearly in [30], so we will state them without proofs. The first reduction is given by the following lemma.

Lemma 7.3 *$\mathcal{I}_P(\pi_{M,\lambda})$ is irreducible if and only if $v^{a_i} \pi_{\phi_i} \times v^{a_j} \pi_{\phi_j}$, $v^{a_i} \pi_{\phi_i} \times v^{-a_j} \pi_{\phi_j}^\vee$ are irreducible for all $i \neq j$, and also $v^{a_i} \pi_{\phi_i} \rtimes \pi_{G_-}$ is irreducible for all i .*

To make the second reduction, we need to write $\phi_{G_-} = \phi_{G'_-} \oplus (\phi'_1 \oplus \phi_1^\vee) \oplus \dots \oplus (\phi'_n \oplus \phi_n^\vee)$, where G'_- is of the same type as G_- and $\phi_{G'_-} \in \bar{\Phi}_2(G'_-)$, $\phi'_i \in \Phi_{sim}(N'_i)$. Then it is clear that π_{G_-} is a subrepresentation of $\pi_{\phi'_1} \times \dots \times \pi_{\phi'_n} \rtimes \pi_{G'_-}$ for some $\pi_{G'_-} \in \bar{\Pi}_{\phi_{G'_-}}$. And we can state the second reduction as follows.

Lemma 7.4 *For $1 \leq i \leq m$, $v^{a_i} \pi_{\phi_i} \rtimes \pi_{G_-}$ is irreducible if $v^{a_i} \pi_{\phi_i} \times \pi_{\phi'_k}$, $v^{a_i} \pi_{\phi_i} \times \pi_{\phi'_k}^\vee$ are irreducible for $1 \leq k \leq n$, and also $v^{a_i} \pi_{\phi_i} \rtimes \pi_{G'_-}$ is irreducible.*

Since $0 < a_i < 1/2$ for $1 \leq i \leq m$, by the theory of Zelevinsky [50] and its archimedean analogue [46], we can see easily that $v^{a_i} \pi_{\phi_i} \times v^{a_j} \pi_{\phi_j}$, $v^{a_i} \pi_{\phi_i} \times v^{-a_j} \pi_{\phi_j}^\vee$ are irreducible for all $i \neq j$, and $v^{a_i} \pi_{\phi_i} \times \pi_{\phi'_k}$, $v^{a_i} \pi_{\phi_i} \times \pi_{\phi'_k}^\vee$ are irreducible for all k . So it reduces to show $v^{a_i} \pi_{\phi_i} \rtimes \pi_{G'_-}$ is irreducible for $1 \leq i \leq m$. And this is the consequence of the following proposition.

Proposition 7.5 *Suppose π and σ are discrete series representations of $GL(N)$ and G respectively, then the induced representation $v^a \pi \rtimes \sigma$ is irreducible if $0 < a < 1/2$.*

The proof of this proposition is divided into two cases: archimedean and nonarchimedean. The archimedean case follows from the result of Speh and Vogan directly, and we refer the readers to [43] for precise statement of their theorems. For the nonarchimedean case, the story is not that straightforward, and we will concentrate on this case. So now we assume F is nonarchimedean, and we can write the essentially discrete series $v^a \pi$ of $GL(N)$ as segments according to the classification theory of Zelevinsky [50], i.e. $v^a \pi = \delta(v^{-l_1} \rho, v^{l_2} \rho)$ where ρ is a cuspidal representation of $GL(d_\rho)$, d_ρ is defined by ρ , and $l_1 + l_2 \in \mathbb{Z}_{\geq 0}$ with $a = (l_2 - l_1)/2$. Note that $2l_2 + 1 \notin \mathbb{Z}$ for $0 < a < 1/2$. Then it is easy to see that the nonarchimedean case follows from the following theorem due to Tadić ([29], Theorem 2.2).

Theorem 7.6 *Suppose F is nonarchimedean, $\delta = \delta(v^{-l_1} \rho, v^{l_2} \rho)$ and σ is a discrete series of G . If $\rho \neq \rho^\vee$ or $2l_2 + 1 \notin \mathbb{Z}$, then $\delta \rtimes \sigma$ is irreducible.*

The proof of this theorem again involves several reduction steps. It is clear that there is no harm to assume $l_2 > l_1$. The simplest case is when $\delta = v^l \rho$ and σ is a cuspidal representation, and this has been settled by Mœglin in [23]. In fact, Mœglin has proved the following result.

Proposition 7.7 *Suppose F is nonarchimedean, ρ and σ are cuspidal representations of $GL(N)$ and G respectively. If $\rho = \rho^\vee$ and $v^\alpha \rho \rtimes \sigma$ is reducible, then $\alpha \in \frac{1}{2}\mathbb{Z}$, and $v^\beta \rho \rtimes \sigma$ is irreducible for all $\beta \in \mathbb{R} \setminus \{\pm\alpha\}$; If $\rho \neq \rho^\vee$, then $v^l \rtimes \sigma$ is irreducible for all $l \in \mathbb{R}$.*

We should point out the original proof of Proposition 7.7 in [23] is based on the assumption of functorial lifting from classical groups to general linear groups. Now it is unconditional by Arthur's result [7]. For the second reduction step of the proof of Theorem 7.6, Tadić shows the theorem is true for σ being cuspidal in [45], and he uses an induction argument which is quite different from the proof of Proposition 7.7. Finally, we can consider the general case, i.e. σ is a discrete series of G . Suppose σ is not cuspidal, then by Mœglin-Tadić's classification of discrete series of classical groups [28], one has an inclusion

$$\sigma \hookrightarrow v^{\alpha_1} \pi_1 \times v^{\alpha_2} \pi_2 \times \cdots \times v^{\alpha_m} \pi_m \rtimes \sigma'$$

where π_i are self-dual essentially discrete series of $GL(N_i)$, $\alpha_i \in \frac{1}{2}\mathbb{Z}_{>0}$, and σ' is a discrete series of G' , where G' is of the same type as G . So we can do induction on the rank of G , and assume $\delta \rtimes \sigma'$ is irreducible, i.e. the standard intertwining operator $\delta \rtimes \sigma' \xrightarrow{\sim} \delta^\vee \rtimes \sigma'$ is a bijection by the theory of Langlands quotient. Note that $\delta \times v^{\alpha_i} \pi_i$ and $\delta^\vee \times v^{\alpha_i} \pi_i$ are irreducible, so the normalized intertwining operators

$$\delta \times v^{\alpha_i} \pi_i \xrightarrow{\sim} v^{\alpha_i} \pi_i \times \delta$$

$$v^{\alpha_i} \pi_i \times \delta^\vee \xrightarrow{\sim} \delta^\vee \times v^{\alpha_i} \pi_i$$

are bijections. Now consider the following composition of normalized intertwining operators.

$$\begin{aligned} \delta \rtimes \sigma &\hookrightarrow \delta \times v^{\alpha_1} \pi_1 \times v^{\alpha_2} \pi_2 \times \cdots \times v^{\alpha_m} \pi_m \rtimes \sigma' \xrightarrow{\sim} v^{\alpha_1} \pi_1 \\ &\quad \times \delta \times v^{\alpha_2} \pi_2 \times \cdots \times v^{\alpha_m} \pi_m \rtimes \sigma' \\ &\xrightarrow{\sim} \cdots \xrightarrow{\sim} v^{\alpha_1} \pi_1 \times v^{\alpha_2} \pi_2 \times \cdots \times v^{\alpha_m} \pi_m \rtimes \delta \\ &\quad \times \sigma' \xrightarrow{\sim} v^{\alpha_1} \pi_1 \times v^{\alpha_2} \pi_2 \times \cdots \times v^{\alpha_m} \pi_m \rtimes \delta^\vee \times \sigma' \\ &\xrightarrow{\sim} v^{\alpha_1} \pi_1 \times v^{\alpha_2} \pi_2 \times \cdots \times \delta^\vee \times v^{\alpha_m} \pi_m \rtimes \sigma' \xrightarrow{\sim} \cdots \xrightarrow{\sim} \delta^\vee \\ &\quad \times v^{\alpha_1} \pi_1 \times v^{\alpha_2} \pi_2 \times \cdots \times v^{\alpha_m} \pi_m \rtimes \sigma' \end{aligned}$$

This shows the standard intertwining operator

$$\delta \rtimes \sigma \xrightarrow{\simeq} \delta^\vee \rtimes \sigma$$

is a bijection, and hence $\delta \rtimes \sigma$ is irreducible. This finishes the proof of Theorem 7.6.

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