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Fluctuation theory for radiative transfer in random media

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ABSTRACT

We consider the effect of small scale random fluctuations of the constitutive coefficients on boundary measurements of solutions to radiative transfer equations. As the correlation length of the random oscillations tends to zero, the transport solution is well approximated by a deterministic, averaged, solution. In this paper, we analyze the random fluctuations to the averaged solution, which may be interpreted as a central limit correction to homogenization.

With the inverse transport problem in mind, we characterize the random structure of the singular components of the transport measurement operator. In regimes of moderate scattering, such components provide stable reconstructions of the constitutive parameters in the transport equation. We show that the random fluctuations strongly depend on the decorrelation properties of the random medium.

1. Introduction

Radiative transfer [8,10,13] are used in many practical applications in medical and geophysical imaging [1,17]. There, one is interested in reconstructing the optical properties of a domain of interest from measurements typically collected at the boundary of the domain. If we denote by $[\sigma]$ the set of unknown coefficients and by $[D]$ the collected data, then the *measurement operator* Φ is defined as

$$\Phi : [\sigma] \mapsto \Phi[\sigma] = [D].$$

Inverse transport theory is applied to the aforementioned imaging techniques; there $[\sigma]$ consists of $\{a,k\}$, where a is the attenuation coefficient and k the scattering coefficient, while $[D]$ is the density of particles (photons) $u(x,v)$ for v an outgoing direction at a point x on the boundary of a domain. The operator Φ is often a smoothing operator, in which small scale structures are typically lost, either because detectors are separated by a small distance, or because each detector has a finite numerical aperture. Let $\varepsilon \ll 1$ measure such a small scale. In each of these settings, the structure of $[\sigma]$ at the scale ε cannot be reconstructed. Yet, in many settings, the latter small scale structure still has an influence on the measurements $[D]$. In this paper, we consider such an influence in the setting of the radiative transfer equation.

Since the small scale structure cannot be reconstructed, it is reasonable to model it as a spatially rescaled random field $\delta a(x/\varepsilon,\omega)$ in (1) below, where the unscaled random field $\delta a(x,\omega)$ has correlation length of order one. Consequently, $\delta a(x/\varepsilon,\omega)$ has correlation length of order ε, hence modeling the small scale structure of the attenuation parameter. We also assume that the scattering cross-section $k \sim 1$, and the domain of interest has a typical scale $L \sim 1$, so that the typical *mean free path* $\lambda \sim L/k \sim 1$. Here we use $b \sim c$ to say b and c have the same order. Viewing the above small structure as random variations in the absorption section a, the small parameter ε can also be interpreted as the ratio between the length scale of this random variation and the mean free path λ. The influence of such small scale structure thus becomes a problem of characterizing the effect of random heterogeneities on the solution to a transport equation. We consider here the

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setting of detectors with very fine spatial and angular resolution that are placed on a lattice with grid size ε. Variations at scales smaller than ε are then not present in the measurements. For such a configuration, we characterize the asymptotic randomness in the measurements $[D]$ as ε tends to 0.

With the above modeling, we obtain radiative transfer equations with random coefficients that vary on a small scale ε. Such equations have been analyzed in several settings; we refer the reader to, e.g., [12,15,16,18]. Well-known homogenization results state that as ε decreases to zero, solutions of the random equations converge to solutions of averaged (homogenized) equations. The theory of random correctors to homogenization, which provides the asymptotic characterization of random fluctuations in available measurements, is not as developed. In [5], we considered the theory of correctors to homogenization for the transport equation. The main result of that paper is a characterization of the random fluctuations as approximately a Gaussian process in the limit $\varepsilon \to 0$. However, such a result requires that the transport solution be sufficiently averaged in the spatial and angular variables. That is, the data $[D]$ in those results are spatially and angularly averaged, not point-wise. We found that such data have variances of order ε^d where d is the spatial dimension of the domain of interest. The main objective of this paper is to show that the situation is modified when point-wise (in space and direction) measurements are considered. In particular, our measurements are the *ballistic* part and *single scattering* part of the transport solution, i.e., particles that are not scattered or scattered once along their paths, which are the essential contributions used in inverse transport; see Section 2.3. We find that such data have much larger variances of order ε independent of the spatial dimension d when the random field decorrelates sufficiently fast, and even larger variances otherwise; see Section 3.

The rest of the paper is structured as follows. The radiative transfer equation with random attenuation is presented in Section 2. We assume here that the scattering coefficient is deterministic. It turns out that the influence of randomness on the measurements strongly depends on the decorrelation properties of the random coefficients. To quantify this property, we introduce the (auto-)correlation function $R(x)$ of the random field $\delta a(x)$. The random field is said to have *short range* correlation if $R(x)$ is integrable along any one-dimensional subspace of $\mathbb{R}^d$, and *long range* correlation otherwise. We consider here both cases. As we mentioned above, we are interested in the influence of randomness on the measurement operator Φ. In transport theory, the reconstruction of $[\sigma]$ is stable when the singular components of Φ can be measured. The decomposition of the measurement operator into singular components is recalled in Section 2.3.

Our main results on the influence of randomness on the singular components of the measurement operator are presented in Section 3. We primarily consider the setting of an array of detectors separated by a distance ε and capable of measuring the density of particles point-wise (i.e., at a spatial and angular scale negligible compared to ε). In such a setting, we observe a much larger influence on the measured data when the correlation function of the random attenuation decays slowly. We also describe in some detail how the measurements are cross-correlated.

The methodology used to describe the measurement fluctuations can be generalized to other settings, such as, e.g., the setting of detectors that have a spatial aperture comparable to ε. We briefly present how the results are modified in such a setting. A sketch of the proof of the results is proposed in Section 4. To a large extent, the derivation of the results follows from the techniques presented, e.g., in [2,4].

2. Radiative transfer equation with random attenuation

As described above, in some inverse problem applications, gaps between detectors introduce a small scale ε, and structures in the absorption coefficients that vary on this scale cannot be stably reconstructed. We hence model this part of the absorption as a scaled random field and aim to characterize its effect on the measurements. Let us model the total absorption as

$$a_\varepsilon\left(x,\frac{x}{\varepsilon},\omega\right)=a(x)+\delta a\left(\frac{x}{\varepsilon},\omega\right), \tag{1}$$

where the deterministic function $a(x)$ is assumed to be smooth and slowly varying (on a scale much larger than ε), and where δa is a stationary random field defined on some abstract probability space $(\Omega,\mathcal{F},\mathbb{P})$, with Ω the space of realizations, $\mathcal{F}$ the space of (measurable) subsets of Ω and $\mathbb{P}$ the probability measure on $\mathcal{F}$. Thus $\delta a_\varepsilon(x) := \delta a(x/\varepsilon)$ models the high frequency structures mentioned above.

Then we need to consider the following steady-state radiative transfer equation with random attenuation coefficient:

$$\begin{aligned} &v\cdot\nabla_x u_\varepsilon + a_\varepsilon\left(x,\frac{x}{\varepsilon},\omega\right)u = k(x)\int_V u_\varepsilon(x,v')\,dv', \quad (x,v)\in X\times V,\\ &u_\varepsilon(x,v)=g(x,v), \quad (x,v)\in\Gamma_-. \end{aligned} \tag{2}$$

Here, X is an open, bounded, subset in $\mathbb{R}^d$ for $d=2,3$ spatial dimension, and V is the velocity space, which here is chosen as the unit sphere $V=S^{d-1}$ to simplify the presentation. The sets $\Gamma_\pm$ are the sets of outgoing and incoming conditions, defined by

$$\Gamma_\pm := \{(x,v)|x\in\partial X,\ \pm v_x\cdot v>0\}, \tag{3}$$

where ∂X is the boundary of X, assumed to be smooth, and the normal vector to X at $x\in\partial X$ is denoted by v_x. See Fig. 1 for an illustration of these definitions and a few more to come.

We assume here that scattering is isotropic to simplify the presentation. We also assume that $k(x)$ is deterministic while the attenuation coefficient a_ε is assumed to be random. This assumption is not unrealistic in medical imaging applications, where the absorption coefficients of tissues vary more rapidly than the scattering coefficients. Also, it may be justified from the point of view of an inverse problem as follows.

From measurements formalized by the "albedo" operator defined in Section 2.3 below, it is possible [6,9]

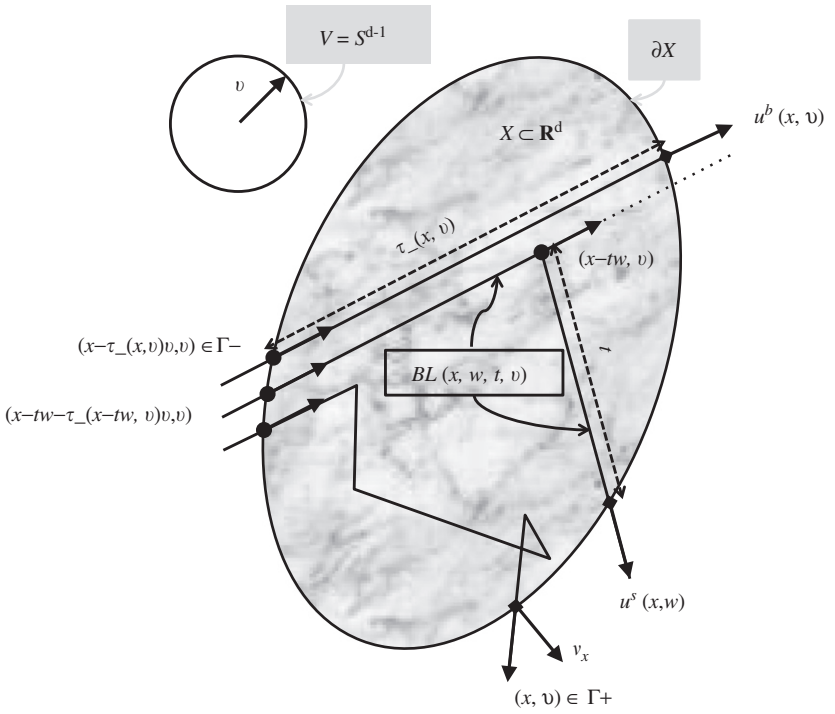

Fig. 1. Schematic of forward radiative transfer problem for a medium with high-frequency variability. Three boundary points on the left-hand side are collimated sources. On the right-hand side are direction-specific point-wise detectors. Because of the singularities of the albedo operator, these detectors read the ballistic part u^b in (13) and (in dimension $d \geq 3$) the single scattering parts u^s in (14). The two former trajectories form the contributions $\alpha_j(x,v;y,w)$ in (12) with $j = 0,1$, respectively, to the albedo operator. Here *BL* stands for the single scattering broken line used in (14).

to extract the "singular" components of $u_\varepsilon(x,v)$, cf. (12) below. Such components provide explicit reconstructions for the attenuation and scattering coefficients. The reconstruction of the attenuation coefficient involves the inversion of an X-ray transform, while the inversion of the scattering coefficient is local in space. More precisely, knowledge of $\alpha_1(x,v;y,w)$ in (12) provides reconstruction of $k(z)$ at the point z (when it exists) given by the intersection of the lines $\{x+tv\}$ and $\{y+sw\}$. A delta source emitted at (x,v) with a detector at (y,w) will thus provide a reconstruction for k at a unique point z. No statistical averaging (for instance by integrating over several values of z) occurs in our model with detectors that are assumed to have a spatial and angular resolution below the scale ε.

However, the reconstruction of a_ε from knowledge of $\alpha_0(x,v;y,w)$ in (12) involves line integrals of a_ε. The reconstruction of k from α_1 also involves line integrals of a_ε. Since our detectors are assumed to be separated from each other by spatial distances of order ε, the small scale structure of a_ε cannot be reconstructed. We thus model it as random and aim at understanding the influence of such undetectable small scale structures on the available measurements.

Existence and uniqueness of solutions of (2) has been investigated in, e.g., [6,9,10]. A sufficient condition for existence and uniqueness is that the intrinsic attenuation be non-negative; that is, the total absorption a_ε subtracted by the scattering contribution $\int_V k(x,v,v')\,dv' = c_d k(x)$ is non-negative; here c_d is the volume of the unit sphere S^{d-1}. We assume here that this condition is satisfied by (a_ε,k) almost surely with respect to probability measure $\mathbb{P}$ ($\mathbb{P}$−a.s.), so that our transport solution is well-defined $\mathbb{P}$−a.s.

To develop a homogenization and corrector theory for the random transport equation (2), we need to impose additional conditions on the random coefficients.

We first assume that the random field δa is *stationary*, which means that the joint distribution of $\{\delta a(x_1),\ldots,\delta a(x_n)\}$ is conserved under translation for any $n \in \mathbb{N}$ and any n-tuple $(x_1,\ldots,x_n)$. In particular, let $\mathbb{E}$ denote the mathematical expectation with respect to the measure $\mathbb{P}$; then $\mathbb{E}\delta a(x) = c$ for some constant independent of x. By absorbing this deterministic constant into $a(x)$ if necessary, we may assume that δa is mean-zero. The correlation function R of the stationary random field δa defined by

$$R(x) := \mathbb{E}\{\delta a(y)\delta a(y+x)\}, \tag{4}$$

measures the two-point correlation of the field. As defined in the introduction, we say δa has short range correlation if R is integrable along any line $L(v) := \{tv | t \in \mathbb{R}\}$, that is along one-dimensional subspaces of $\mathbb{R}^d$ with direction $v \in S^{d-1}$; and we say δa has long range correlation if R fails to satisfy these conditions.

Homogenization theory for (2), which is in fact an *averaging* theory since the coefficients are replaced in the limit $\varepsilon \to 0$ by their ensemble average, holds under the

general condition that δa is stationary and ergodic; see [12]. Henceforth, Eq. (2) with a_ε replaced by its average a will be referred to as the *homogenized* equation.

The random corrector to homogenization, however, is much more complicated and strongly depends on the structure of randomness. In this paper, we develop a corrector theory for two types of random coefficients: (i) strongly mixing coefficients with short range correlations; and (ii) functionals of Gaussian processes with long-range correlations. We will show that the size of the random corrector is very different for short-range and long-range coefficients.

For similar works on random correctors, we refer the reader to, e.g., [2,4,5].

2.1. Random fields with short range correlation

Our main results of this paper consider two types of random field models. In the first case, we use the following assumption which implies that random field has short range correlations.

Assumption A (*Short range correlation*). The random field a_ε is defined as in (1) and δa is stationary, mean-zero, and $\rho-$mixing with mixing coefficient $\rho(r)$ that is integrable as a function on $\mathbb{R}$. Further, $\delta a < \max_{x\in X}\{a-c_d k\}$ for almost every realization.

The last requirement implies that (a_ε,k) is admissible, so that Eq. (2) is well-posed.

A process $q(x,\omega)$ on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is said to be $\rho-$mixing if for any Borel sets $A,B\subset\mathbb{R}^d$, the sub-$\sigma-$algebras $\mathcal{F}_A$ and $\mathcal{F}_B$ generated by the process restricted on A and B respectively decorrelate rapidly in the sense that there exists some function ρ such that for any square integrable random variables ξ and η that are $\mathcal{F}_A-$measurable and $\mathcal{F}_B-$measurable respectively, we have

$$\frac{|\mathbb{E}\{(\xi-\mathbb{E}\xi)(\eta-\mathbb{E}\eta)\}|}{\sqrt{\mathrm{Var}\{\xi\}\mathrm{Var}\{\eta\}}} \le \rho(d(A,B)). \tag{5}$$

Here $d(A,B)$ is the distance between the sets A and B. What this means is that (functionals of) the random fields restricted on disjoint spatial domains A and B become more and more independent as the distance between the sets A and B increases. The function ρ quantifies that decay.

We verify that under assumption **A**, δa has short range correlation. Indeed, from (5) we see $|R(tv)|\le C\rho(|t|)$ where C is a bound for the variance of δa. Hence, we define

$$\sigma_a^2(v) := \int_{L(v)} R(x)\,dx = \int_{\mathbb{R}} R(tv)\,dt. \tag{6}$$

Since $R|_{L(v)}$ can be viewed as the correlation of the random process $\delta a|_{L(v)}$, by Bochner's theorem [19] $\sigma_a(v)$ is non-negative. We call Assumption **A** the case of short range correlation because R is integrable in all directions.

Examples. There are many examples of processes that satisfy Assumption **A**. We refer the reader to [5] for a detailed discussion of a model based on spatial Poisson point process. In the next section, we will consider a model based on a functional of a Gaussian random field that will be parameterized by a coefficient α which quantifies the decay of its correlation function. The case $\alpha>1$ in (7) below corresponds to a case that satisfies Assumption **A**.

Note that Gaussian fields do not satisfy the positivity constraint $0<\max_{x\in X}\{a-c_d k\}-\delta a$. The analysis of mean ballistic transport effects of Gaussian fields with short-range correlations (white- and blue-noise cases) is investigated in [11].

2.2. Random fields with long range correlation

In order to display the different behaviors of the random fluctuations in a transport solution, we consider the following case of random coefficients with long-range correlations:

Assumption B (*Long range correlation*). The random field a_ε is defined as in (1) and $\delta a=\Phi\circ g(x,\omega):=\Phi(g(x,\omega))$ where $g(x,\omega)$ is a real-valued mean-zero variance-one stationary *Gaussian random field* on $(\Omega,\mathcal{F},\mathbb{P})$ with correlation function R_g. For simplicity we assume R_g is a radial function and has the following asymptotic behavior:

$$R_g(r)\sim\kappa_g r^{-\alpha},\quad 0<\alpha<1,\quad \text{as}\quad r\to\infty. \tag{7}$$

The function $\Phi:\mathbb{R}\to\mathbb{R}$ is bounded and

$$\int_{\mathbb{R}}\Phi(g)e^{-g^2/2}\,dg=0. \tag{8}$$

Furthermore, we assume that $\sup_{x\in\mathbb{R}}|\Phi(x)|<\max\{a-c_d k\}$. The latter requirement implies that (2) is well-posed.

Remark 2.1. We observe that assumption **A** is relatively general comparing with **B**. As we will see in the proof of the main theorems, the fast decorrelation property in **A** allows us to invoke *central limit theorem* conveniently and processes required there are relatively general. This is no longer the case for random fields with long range correlation. We choose the above model so that we can calculate the main estimates based on our knowledge of Gaussian fields. A uniform approach for general random fields with long range correlations is still unavailable and the results we derived under assumption **B** do not necessarily hold in general.

Examples. Bounded, odd, functions satisfy the condition (8) above. With $\Phi=\mathrm{sgn}$, the process $\Phi\circ g$ models a two-component composite medium. If we take $\Phi=\tanh$ or arctan, then $\Phi\circ g$ models a continuous medium with bounded variations. This model was used in [4]. By choosing Φ bounded, we can ensure that the intrinsic attenuation coefficient is non-negative $\mathbb{P}-$a.s.

We need to analyze the random field δa restricted to lines $L(v)$. Let us denote $\varphi(t)=\delta a(tv)$, then it is a one-parameter random process of the type $\Phi\circ g$. Such random fields were discussed in details in [4]. In particular, if we define

$$V_1=\mathbb{E}\{g_0\Phi(g_0)\}=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g\Phi(g)e^{-g^2/2}\,dg, \tag{9}$$

$$V_2 = \mathbb{E}\{\Phi^2(g_0)\} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \Phi^2(g) e^{-g^2/2}\, dg, \tag{10}$$

and $\kappa = \kappa_g V_1^2$, then it is shown in [4] that φ is a stationary mean-zero process with variance V_2. Moreover, the correlation function of φ, still denoted as R, behaves like $\kappa r^{-\alpha}$ as $r\to\infty$.

Note that the constraint $\sup_{x\in\mathbb{R}}|\Phi(x)| < \max\{a-c_d k\}$ precludes the use of Gaussian fields to model δa. The analysis of long-range correlations of Gaussian fields (pink-noise case) on the mean ballistic transport term is investigated in [11].

2.3. Albedo operator of transport equation

When $k\equiv 0$ in (2), the free transport equation can be solved explicitly using the method of characteristics. When scattering is non-zero, the solution to (2) can be expressed as a Neumann series expansion. Let $\mathcal{A}$ denote the *albedo operator*, which maps the incoming boundary condition $u|_{\Gamma_-} = g$ on Γ_- to the outgoing solution on Γ_+:

$$\mathcal{A}: u|_{\Gamma_-} \mapsto \mathcal{A}(u|_{\Gamma_-})(x,v) = u|_{\Gamma_+}(x,v), \quad (x,v)\in\Gamma_+. \tag{11}$$

This operator can be decomposed into three terms $\mathcal{A} = \sum_{j=0}^{2}\mathcal{A}_j$ where the operators $\mathcal{A}_j$ have different singularities as shown in [3,6,9]; it is this difference of singularities that allows one to uniquely reconstruct the optical parameters (a,k) in inverse transport. The operator $\mathcal{A}_0$ denotes the part that does not depend on scattering. It is called the ballistic part. The operator $\mathcal{A}_1$ denotes the part that is linear in the scattering coefficient and is called the single scattering part. The operator $\mathcal{A}_2$ denotes the part that is higher order in scattering and is called the *multiple scattering* part.

The ballistic part $\mathcal{A}_0$ is always more singular than the other parts and can be extracted form the measured data [3]. Knowledge of $\mathcal{A}_0$ implies that of the Radon transform of a, which uniquely determines a. In dimension $d\geq 3$ (or in dimension $d=2$ in the time-dependent setting), $\mathcal{A}_1$ is more singular than the multiple scattering part and can also be separated from the latter for sufficiently accurate detectors. Knowledge of $\mathcal{A}_1$ and a allows us to uniquely reconstruct k. We are therefore interested in the contributions $\mathcal{A}_0$ and $\mathcal{A}_1$ in the inverse problem setting. Their distributional kernels are given by

$$\alpha_0(x,v;y,w) = \delta_v(w)\delta_{\{x-\tau_-z(x,v)v\}}(y)\exp\left(-\int_0^{\tau_-(x,v)} a(x-sv)\,ds\right),$$

$$\alpha_1(x,v;y,w) = \int_0^{\tau_-(x,v)} \exp\left(-\int_0^{t} a(x-sv)\,ds - \int_0^{\tau_-(x-tv,w)} a(x-tv-sw)\,ds\right)$$
$$k(x-tv)\delta_{\{x-tv-\tau_-(x-tv,w)w\}}(y)\,dt. \tag{12}$$

Here we used the standard notation $\tau_\pm(x,v)$ for the traveling times from x to ∂X along direction $\pm v$, and they are given by

$$\tau_\pm(x,v) = \sup\{t>0 : x\pm tv\in X\},$$

see illustrations of $\tau_-(x,v)$ in Fig. 1. The above means that

$$\mathcal{A}_j\phi(x,v) = \int_{\Gamma_-}\alpha_j(x,v;y,w)\phi(y,w)\,d\sigma(y)\,dw, \quad j=0,1,2,$$

where $d\sigma(y)$ is the surface measure on ∂X.

Let $\mathcal{A}_\varepsilon$ and α_ε denote the corresponding albedo operator and its kernel for the stochastic transport equation (2), and let $\mathcal{A}$ and α be those for the homogenized equation. This paper is devoted to the analysis of the ballistic and single scattering contributions of the random corrector $\mathcal{A}_\varepsilon-\mathcal{A}$. More specifically, we want to understand the random fluctuations in the coefficients $\alpha_k(x,v;y,w)$ for $k=0,1$.

It turns out that the random fluctuations in $\alpha_k(x,v;y,w)$ depend on the scale at which the latter quantities are observed. In the setting of random coefficients with short-range correlations, so that the correlation functions of the random coefficients are integrable on $\mathbb{R}^d$, we analyzed in [5] the random fluctuations for integrals (in all variables $(x,v;y,w)$) of the above kernels. By an application of the central limit theorem in d-dimensional spaces, we obtain that such integrals are asymptotically Gaussian with variance proportional to ε^d. Point-wise, we observe that $\alpha_k(x,v;y,w)$ for $k=0,1$ involve integrals of a_ε along lines, and this independent of dimension. As a consequence, we expect point-wise measurements to have variances of order $O(\varepsilon)$ as an application of the central limit theorem independent of dimension d. An intermediate case with measurements that are point-wise in space and averaged in angle turns out to have variance of order $O(\varepsilon^2|\log\varepsilon|)$ in dimension $d=2$ and of order $O(\varepsilon^2)$ in dimension $d\geq 3$; we refer to calculations presented in the Appendix for details.

In this paper, as we mentioned in the introduction, we are interested in the setting of point-wise measurements. In the following section, we indeed show that point-wise measurements have a variance of order $O(\varepsilon)$ as an application of the central limit theorem when the random fluctuations have short-range correlations. In the setting of long-range correlations, we observe much larger random fluctuations, which is consistent with the results obtained in [4] for elliptic equations.

3. Random fluctuations of point-wise measurements

In this section, we state our main results on the asymptotic random fluctuations in the measurements $\alpha_0(x,v;y,w)$ and $\alpha_1(x,v;y,w)$. We show that the size of the random fluctuations depends on the decorrelation properties of the random attenuation coefficient. We consider the cases of short range and long range correlations. The measurements at different values of $(x,v;y,w)$ also may be correlated. We present the limiting joint distribution for some families of measurements. Finally, we briefly mention how the results need to be modified when point-wise measurements are replaced by measurements on detectors with a spatial resolution comparable to the small scale ε. The derivation of the results is presented in Section 4.

We first observe that $\alpha_0(x,v;y,w)$ is of the form $u^b(x,v)\delta_v(w)\delta_{\{x-\tau_-(x,v)v\}}(y)$. Thus, u^b can be thought of as the solution of the free transport equation at (x,v) with a unit source located at $(x-\tau_-(x,v)v,v)$. It is given for the deterministic part by

$$u^b(x,v) = \exp\left(-\int_0^{\tau_-(x,v)} a(x-sv)\,ds\right). \tag{13}$$

Since the random coefficients affect only the function u^b_ε, it suffices to investigate the corrector $u^b_\varepsilon - u^b$.

The systematic impact of broad band fluctuations of a modeled as a Gaussian random field (with a positive probability of being (non-physically) negative) or as random fields preserving the positivity of a (such as gamma-distributed random fields) on the mean ballistic transmission $\mathbb{E}\{u^b\}$ is investigated, e.g., in [11] and the references therein.

Similarly for the single-scattering part, the kernel α_1 can be written as

$$\int_0^{\tau_-(x,v)} u^s(x,v;t,w)k(x-vt)\delta_{\{x-tv-\tau_-(x-tv,w)w\}}(y)\,dt,$$

with $u^s(x,v;t,w)$ defined by

$$u^s(x,v;t,w) := \exp\left(-\int_{BL(x,v,t,w)} a(\ell)\,d\ell\right) = \exp\left(-\int_0^t a(x-sv)\,ds - \int_0^{\tau_-(x-tv,w)} a(x-tv-sw)\,ds\right). \tag{14}$$

Here $BL(x,v,t,w)$ denotes the broken line between $x-tv-\tau_-(x-tv,w)w$, $x-tv$ and x, as illustrated in Fig. 1 for $d=2$. The random corrector of $\mathcal{A}_1$ is therefore obtained by analyzing $u^s_\varepsilon - u^s$.

Random media with short range correlations: We have the following results regarding to the limiting distribution of a point-wise/direction-specific measurement.

Theorem 3.1. *Let a_ε satisfy the short range correlation assumption* **A**. *Let (x,v) be any point in Γ_+ or in $X\times V$, and t be any real number, so that $x-tv\in X$. Then for any fixed (x,v,t) the following results hold.*

1. *For the ballistic part, we have*

$$\frac{u^b_\varepsilon - u^b}{\sqrt{\varepsilon}}(x,v) \xrightarrow{\mathcal{D}} u^b(x,v)\sqrt{\tau_-(x,v)}\sigma_a(v)\mathcal{N}(0,1). \tag{15}$$

2. *For the single scattering part, we have*

$$\frac{u^s_\varepsilon - u^s}{\sqrt{\varepsilon}}(x,v;t,w) \xrightarrow{\mathcal{D}} u^s(x,v;t,w)\left(\sqrt{\tau_-(x-tv,w)}\sigma_a(w)\mathcal{N}_1(0,1) + \sqrt{\tau_+(x-tv,v)}\sigma_a(v)\mathcal{N}_2(0,1)\right). \tag{16}$$

In both equations, $\mathcal{N}(0,1)$ denotes the centered normal variable, and the convergence is interpreted as convergence in distribution of random variables. The real numbers σ_a's are defined in (6). In the second equation, $\mathcal{N}_i$, $i=1,2$, are independent variables with normal distribution.

Remark 3.2. In particular, the above result shows that the variance of point-wise measurements is of order $O(\varepsilon)$ unlike what happens for angularly or spatially averaged measurements. We refer the reader to Appendix A and [5] for some results in the latter cases.

Solving the inverse transport problem requires many measurements [3]. In order to minimize errors in the reconstructions (see, e.g., [7]), it is useful to understand how such measurements are correlated. We do not consider the most general case of an arbitrary finite number of measurements. Rather, we consider the following families of measurements and present their joint limiting behavior as $\varepsilon\to 0$. In all cases considered here, the limiting distributions are centered and Gaussian and thus characterized by their covariance matrix.

We first introduce some notation. For $(x,v)\in\overline{X}\times V$, let $L(x,v)$ denote the line $\{x-sv, s\in\mathbb{R}\}$ and $L_X(x,v)$ be the intersection of $L(x,v)$ with X. Let us suppose we have a finite collection of ballistic measurements for sources at (x_i,v_β) where $i\in[N]$, $\beta\in[M]$ and $[n]:=\{1,2,\ldots,n\}$. Given a pair of points (x_i,v_β) and (x_j,v_γ), let $d(L_X(x_i,v_\beta),L_X(x_j,v_\gamma))$ denote the distance between the segments (as the distance between two sets), and let $\theta_{\beta\gamma}$ denote the non-oriented angle between v_β and v_γ which is always non-negative. We define the function $\mathrm{d}\colon\{(x_i,v_\beta)\}\times\{(x_j,v_\gamma)\}\to\mathbb{R}_+$ as

$$\mathrm{d}((x_i,v_\beta),(x_j,v_\gamma)) = d(L_X(x_i,v_\beta),L_X(x_j,v_\gamma)) + \theta_{\beta\gamma}. \tag{17}$$

Theorem 3.3. *Let a_ε satisfy the short range correlation assumption* **A**; *the following results hold.*

1. *Consider a collection of points $\{(x_i,v_\beta)\}$ in Γ_+, $i\in[N]$, $\beta\in[M]$. Suppose that for any i,j,β,γ, we have $\mathrm{d}((x_i,v_\beta),(x_j,v_\gamma))\gg\varepsilon$. Denote $(u^b_\varepsilon-u^b)(x_i,v_\beta)$ by $\delta u^{b,i,\beta}_\varepsilon$. Then the $N\times M$-dimensional random vector $\varepsilon^{-1/2}(\delta u^{b;1,1}_\varepsilon,\ldots,\delta u^{b;N,M}_\varepsilon)$ converges in distribution to an $N\times M$-variate normal vector with diagonal covariance matrix. That is,*

$$\frac{(\delta u^{b;1,1}_\varepsilon,\ldots,\delta u^{b;N,M}_\varepsilon)}{\sqrt{\varepsilon}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\Sigma_{N\times M}(x_i,v_\beta)), \tag{18}$$

and $\Sigma_{N\times M}$ is a diagonal matrix whose (i,β)-th diagonal entry is $(u^b(x_i,v_\beta)\sigma_a(v_\beta))^2\tau_-(x_i,v_\beta)$.

2. *Consider $(x_i,v),(x_j,v)\in\Gamma_+$. Let $v^\perp$ be the perpendicular direction of v in the two-dimensional plane determined by the lines $L_X(x_i,v)$ and $L_X(x_j,v)$; let ℓ_{ij} be the length of their common segment when one is projected into the other, and let d_{ij} be the distance betwen these parallel lines. Then we have*

$$\mathrm{Cov}\left(\frac{\delta u^b_\varepsilon(x_i,v)}{\sqrt{\varepsilon}},\frac{\delta u^b_\varepsilon(x_j,v)}{\sqrt{\varepsilon}}\right) = \ell_{ij}u^b(x_i,v)u^b(x_j,v)\int_{\mathbb{R}} R\left(\frac{d_{ij}v^\perp}{\varepsilon}+tv\right)dt + o(1). \tag{19}$$

In particular, when $d_{ij}=\zeta\varepsilon$ for $\zeta>0$, using $\tilde{R}$ to denote the Radon transform of R in the two-dimensional plane, the correlation is $\ell_{ij}u^b(x_i,v)u^b(x_j,v)\tilde{R}(\zeta,v)$.

3. *Consider the two broken lines that start at $(y,w)\in\Gamma_-$ and end at (x_i,v_β) and (x_j,v_γ), respectively. Let t_1, t_2 be such that $x_i-t_1v_\beta$ and $x_j-t_2v_\gamma$ are the points where the broken lines break. Assume that $\mathrm{d}((x_i-t_1v_\beta,v_\beta),(x_j-t_2v_\gamma,v_\gamma))\gg\varepsilon$. Then*

$$\mathrm{Cov}\left(\frac{\delta u^s_\varepsilon(x_i,v_\beta)}{\sqrt{\varepsilon}u^s(x_i,v_\beta)},\frac{\delta u^s_\varepsilon(x_j,v_\gamma)}{\sqrt{\varepsilon}u^s(x_j,v_\gamma)}\right) = \min(\tau_-(x_i-t_1v_\beta,w),\tau_-(x_j-t_2v_\gamma,w))\sigma_a^2(w). \tag{20}$$

Remark 3.4. We therefore obtain that lines that are far apart compared to ε generate measurements that are asymptotically uncorrelated. When the lines are separated by a distance of order $O(\varepsilon)$ as in case 2, then we

obtain that the measurements are correlated. We have considered here a family of two measurements although joint distributions for measurements along a finite number of parallel lines can also easily be shown to be jointly Gaussian (and correlated) in the limit $\varepsilon\to 0$. Case 3 shows that single scattering measurements generated by a given source display larger correlations than ballistic measurements because they visit the same ballistic line of propagation during a positive distance. Other collection of data with parameter (x_i,v_β) can be considered similarly.

So far, we have assumed that the detectors were point-wise and could capture particles exiting X at x with a velocity v. In practice, this models detectors with a resolution in space and angle that is much better than ε. We now consider the case of thicker detectors with a spatial resolution comparable to ε. The main features of Theorem 3.1 remain valid. However, for detectors whose spatial resolution is much worse than ε, then additional averaging would occur and the measurement would have significantly smaller variance; see [5].

For simplicity, we assume that ∂X is (a part of) a hyperplane and that detectors have perfect resolution in the angular variable. Consider the data obtained at $(x,v)\in\Gamma_+$. Let us use a coordinate system centered at x with v being the first coordinate axis while the other axes span the boundary ∂X. Then the measured "ballistic" part is

$$I_\varepsilon(x,v)=\int_{\mathbb{R}^{d-1}}\frac{1}{\varepsilon^{d-1}}\phi\Big(\frac{y}{\varepsilon}\Big)u^b(y,v)\,dy. \tag{21}$$

Here, the function $\phi(y)$ is a weight function assumed to be non-negative and compactly supported on the unit ball and integrating to one over $\mathbb{R}^{d-1}$.

Theorem 3.5. *Let a_ε satisfy the short range correlation assumption* **A**. *Let (x,v) be a point in Γ_+ and assume that ∂X is flat at x in the above sense and v is the outward normal direction. Then we have*

$$\frac{I_\varepsilon(x,v;a_\varepsilon)-I_\varepsilon(x,v;a)}{\sqrt{\varepsilon}}(x,v)\xrightarrow{\mathcal{D}}u^b(x,v)\sqrt{\tau_-(x,v)}\mathcal{N}(0,\sigma_s^2), \tag{22}$$

with σ_s^2 defined by

$$\sigma_s^2=\int_{\mathbb{R}\times\mathbb{R}^{2(d-1)}}\phi(y)\phi(z)R(y-z+tv)\,dt\,dy\,dz, \tag{23}$$

where R is the correlation function of the process δa defined in (4).

We have considered only the limiting distribution for one thick detector. Joint distributions of a finite number (independent of ε) of measurements can be dealt with as in the setting of point-wise measurements.

The case of long range correlations: Finally, we generalize the result obtained in Theorem 3.1 to the case of random fields with long range correlations. The limit of joint distributions for several measurements involves more complicated calculations that are not considered here.

For a single point-wise/direction-specific measurement, the following result holds.

Theorem 3.6. *Let a_ε satisfy the long range correlation assumption* **B**. *Let (x,v) be a point in Γ_+ or in $X\times V$ and t be a real number, so that $x-tv\in X$. Then we have*

1. *For the ballistic part, we have*

$$\frac{u_\varepsilon^b-u^b}{\varepsilon^{\alpha/2}}(x,v)\xrightarrow{\mathcal{D}}u^b(x,v)\mathcal{N}(0,\sigma_H^2),\quad \text{where}$$
$$\sigma_H^2=\frac{2\kappa(\tau_-(x,v))^{2-\alpha}}{(1-\alpha)(2-\alpha)}. \tag{24}$$

2. *For the single scattering part, we have*

$$\frac{u_\varepsilon^s-u^s}{\varepsilon^{\alpha/2}}(x,v;t,w)\xrightarrow{\mathcal{D}}u^s(x,v;t,w)(N_1+N_2), \tag{25}$$

where the random vector $(N_1,\ N_2)$ has a two-variate normal distribution $\mathcal{N}(0,\Sigma)$ with

$$\Sigma_{11}=\frac{2\kappa(\tau_-(x-tv,w))^{2-\alpha}}{(1-\alpha)(2-\alpha)},\quad \Sigma_{22}=\frac{2\kappa(\tau_+(x-tv,v))^{2-\alpha}}{(1-\alpha)(2-\alpha)},$$

and

$$\Sigma_{12}=\Sigma_{21}=\int_0^{\tau_-(x-tv,w)}\int_0^{\tau_+(x-tv,v)}\frac{\kappa}{|tw+sv|^\alpha}\,dt\,ds. \tag{26}$$

This result shows that point-wise measurements with long-range correlations have variances that are much larger than in the short range case. Moreover, the correlation of two crossing lines that are separated by a distance of order $O(1)$ is as large as the variance of each of them, unlike what was obtained for short-range correlations.

More generally, the competition between the slow decay modeled by α and the spatial dimension determines the order of the cross-correlations of the singular components of measured data. In any m-dimensional subspace, the cross-correlation will be of order $\varepsilon^{\min(m,\alpha)}$, which is smaller than the variance of order $O(\varepsilon)$ as soon as $\alpha>1$ in dimension $m\geq 2$.

4. Sketch of proofs

In this section, we sketch the proofs of our main results and also refer the reader to [5] for additional details on the analysis of transport equations with random coefficients. Using a Taylor expansion to order 2, we have

$$e^{M_\varepsilon}-1=M_\varepsilon+\frac{1}{2}M_\varepsilon^2\int_0^1(1+t)e^{tM_\varepsilon}\,dt,$$
$$M_\varepsilon(x,v)=-\int_0^{\tau_-(x,v)}\delta a_\varepsilon(x-sv)\,ds. \tag{27}$$

To simplify notation, we denote the second term in the expansion of $e^{M_\varepsilon}-1$ by $M_\varepsilon^2H_\varepsilon$. Note that, although the integral M_ε needs to be $\ll 1$, its (zero-mean) integrand $\delta a_\varepsilon(x)$ can be $O(1)$ since oscillations will indeed cancel very effectively in variability regimes dominated by very high frequencies.

Since M_ε is bounded from below, we verify that $|H_\varepsilon|$ is bounded by some constant C uniformly in ε. Now we have the following expressions:

$$(u_\varepsilon^b-u^b)=u^b(M_\varepsilon+M_\varepsilon^2H_\varepsilon),$$
$$(u_\varepsilon^s-u^s)=u^s(M_\varepsilon^s+(M_\varepsilon^s)^2H_\varepsilon^s). \tag{28}$$

Here $M^s_\varepsilon = M_\varepsilon(x-tv,w)+M_\varepsilon(x-tv,-v)$ and H^s_ε is again bounded uniformly.

4.1. Proof of Theorems 3.1, 3.3 and 3.5

The proof of these theorems stems essentially from the analysis of the moments of M_ε up to fourth order (Lemma 4.2), the asymptotic independence of $M_\varepsilon(x_i,v_\beta)$ and $M_\varepsilon(x_j,v_\gamma)$ when the lines are not close for the function (17) (Lemma 4.4), and the following central limit theorem type result proved, e.g., in [2].

Theorem 4.1 (*Bal [2]*). *Let D be a bounded interval in $\mathbb{R}$, and q a bounded, stationary, mean-zero, ρ-mixing random process with ρ integrable. Assume $m \in L^2(D)$. Then we have*

$$\int_D m(y)\frac{1}{\sqrt{\varepsilon}}q\left(\frac{y}{\varepsilon}\right)dy \xrightarrow{\mathcal{D}} \int_D m(y)\sigma\, dW_y. \tag{29}$$

Here, W_y is the standard Wiener process on $\mathbb{R}$, and $\sigma^2 = \int_{\mathbb{R}}\mathbb{E}\{q(0)q(y)\}\,dy$.

Using formula (28), we need to control the term that is nonlinear in M_ε.

Lemma 4.2. *Let δa satisfies conditions in assumption* **A**. *Then the following estimates hold*

$$\mathbb{E}M^2_\varepsilon(x,v) = \varepsilon\tau_-(x,v)\sigma_a^2(v)+o(\varepsilon), \quad \mathbb{E}M^4_\varepsilon \le C\varepsilon^2. \tag{30}$$

Proof. Let τ_- denote $\tau_-(x,v)$. We have

$$\begin{aligned}\mathbb{E}M^2_\varepsilon &= \mathbb{E}\int\int_0^{\tau_-}\delta a_\varepsilon(x-tv)\delta a_\varepsilon(x-sv)\,dt\,ds\\ &= \int_{\mathbb{R}^2}R\left(\frac{t-s}{\varepsilon}v\right)F_{x,v}(t)F_{x,v}(s)\,dt\,ds.\end{aligned} \tag{31}$$

Here $F_{x,v}(t)=\chi_X(x-tv)$ is a cut-off function where χ_X is the indicator function of the domain X. Let R_v denote the function of R restricted on the line $L(v)$ and let $R_{\varepsilon v}$ denote the function $R_v(\frac{\cdot}{\varepsilon})$. Then we have

$$\begin{aligned}\mathbb{E}M^2_\varepsilon &= \int_{\mathbb{R}}(R_{\varepsilon v}*F_{x,v})(t)F_{x,v}(t)\,dt\\ &= \int_{\mathbb{R}}\hat{R}_{\varepsilon v}(\xi)\hat{F}_{x,v}(\xi)\overline{\hat{F}_{x,v}(\xi)}\,d\xi = \varepsilon\int_{\mathbb{R}}\hat{R}_v(\varepsilon\xi)|\hat{F}_{x,v}(\xi)|^2\,d\xi.\end{aligned}$$

Here, $\hat{R}_v$ is the Fourier transform of R_v and is real according to Bochner's theorem, and $\hat{R}(\varepsilon\xi)$ converges to $\hat{R}(0)$. Hence, the right-hand side of the last equation converges to

$$\varepsilon\hat{R}_v(0)\int_{\mathbb{R}}F^2_{x,v}(t)\,dt = \varepsilon\tau_-(x,v)\int_{\mathbb{R}}R(tv)\,dt.$$

Recall that $\sigma_a^2(v)$ is defined to be the last integral. This completes the proof of the first part. The second part can be proved by the same lines using an estimate for the fourth order moments of the strong mixing random process derived in [2]. The detail is omitted here. □

Remark 4.3. If we require $tR(tv)$ to be integrable in t, then we obtain the more accurate estimate: $\mathbb{E}M^2_\varepsilon = \varepsilon\tau_-(x,v)\hat{R}_v(0)+O(\varepsilon^2)$.

This can be seen from

$$\int_{\mathbb{R}}F_{x,v}(t)F_{x,v}(t-\varepsilon s)\,dt = \tau_-(x,v)-\varepsilon s.$$

Lemma 4.4 (*Asymptotic independence*). *Let δa satisfy the conditions in assumption* **A**. *Let $\{x_j,v_\beta\}$, $j\in[N],\beta\in[M]$ be a finite collection of sampling points, so that for any two pairs of indices j,k,β,γ, the corresponding lines are not close for the function* (17), *i.e.*, $\mathrm{d}((x_j,v_\beta),(x_k,v_\gamma)) \gg \varepsilon$, *then the random variables $\{(\sqrt{\varepsilon})^{-1}M_\varepsilon(x_j,v_\beta)\}$ are independent in the limit.*

Proof. By definition of independence, we need to show that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\mathbb{E}\left\{\exp\left(i\sum_{j=1}^N\sum_{\beta=1}^M\xi_{j\beta}\frac{M_\varepsilon}{\sqrt{\varepsilon}}(x_j,v_\beta)\right)\right\}\\ &\quad=\prod_{j=1}^N\prod_{\beta=1}^M\mathbb{E}\left\{\exp\left(i\xi_{j\beta}\frac{M_\varepsilon}{\sqrt{\varepsilon}}(x_j,v_\beta)\right)\right\}.\end{aligned} \tag{32}$$

Here $i=\sqrt{-1}$ should not be confused with the indices. By induction and by breaking crossing lines into broken lines with common breaking point, this boils down to the case of N=2, M=1 and the case of N=1, M=2. In the first case, we apply the strong mixing property (5) and the result follows directly. The second case is a little more involved.

The two segments are $L_X(x_s,v)$ and $L_X(x_s,w)$ where x_s is the breaking point. Let us break the latter into two parts,

$$\begin{aligned}&\frac{1}{\sqrt{\varepsilon}}M_\varepsilon(x_s,w)\\ &\quad=-\frac{1}{\sqrt{\varepsilon}}\left(\int_0^{\eta\tau_1}\delta a_\varepsilon(x_s-sw)\,ds+\int_{\eta\tau_1}^{\tau_1}\delta a_\varepsilon(x_s-tw)\,dt\right),\end{aligned}$$

where $\tau_1=\tau_-(x_s,w)$ and η is a positive real number smaller than one. Call them $(1/\sqrt{\varepsilon})Q^\eta_\varepsilon$ and $(1/\sqrt{\varepsilon})P^\eta_\varepsilon$, respectively. Then we have

$$\begin{aligned}&\mathbb{E}\{e^{i\xi_1(1/\sqrt{\varepsilon})M_\varepsilon(x_s,w)+i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}\\ &\quad-\mathbb{E}\{e^{i\xi_1(1/\sqrt{\varepsilon})M_\varepsilon(x_s,w)}\}\mathbb{E}\{e^{i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}\\ &=\mathbb{E}\{[e^{i\xi_1(1/\sqrt{\varepsilon})Q^\eta_\varepsilon}-1]e^{i\xi_1(1/\sqrt{\varepsilon})P^\eta_\varepsilon+i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}\\ &\quad-\mathbb{E}\{[e^{i\xi_1(1/\sqrt{\varepsilon})Q^\eta_\varepsilon}-1]e^{i\xi_1(1/\sqrt{\varepsilon})P^\eta_\varepsilon}\}\mathbb{E}\{e^{i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}\\ &\quad+\mathbb{E}\{e^{i\xi_1(1/\sqrt{\varepsilon})P^\eta_\varepsilon+i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}\\ &\quad-\mathbb{E}\{e^{i\xi_1(1/\sqrt{\varepsilon})P^\eta_\varepsilon(x_s,w)}\}\mathbb{E}\{e^{i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}.\end{aligned} \tag{33}$$

Now for the last two term, we use the mixing condition (5) and get

$$\begin{aligned}&|\mathbb{E}\{e^{i\xi_1(1/\sqrt{\varepsilon})P^\eta_\varepsilon+i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}\\ &\quad-\mathbb{E}\{e^{i\xi_1(1/\sqrt{\varepsilon})P^\eta_\varepsilon(x_s,w)}\}\mathbb{E}\{e^{i\xi_2(1/\sqrt{\varepsilon})M_\varepsilon(x_s,v)}\}|\le\rho\left(\frac{\eta\sin\beta}{\varepsilon}\right),\end{aligned}$$

where $\eta\sin\beta$ is the distance of the segments $L_X(x_s,v)$ and $L_X(x_s-\eta\tau_1,w)$.

For the first term (and similarly the second term), we use the fact that $|e^{ix}-1|\le|x|$ and conclude that it is bounded by

$$|\xi_1|\left\{\mathbb{E}\left(\frac{1}{\sqrt{\varepsilon}}Q^\eta_\varepsilon\right)^2\right\}^{1/2}=|\xi_1|\frac{1}{\varepsilon}\int_0^{\eta\tau_-}\int_0^{\eta\tau_-}R\left(\frac{t-s}{\varepsilon}v\right)dt\,ds\le|\xi_1|\eta.$$

Hence, by choosing η properly, so that both η and $\rho(\eta\sin\beta/\varepsilon)$ goes to zero; for instance, take $\eta=\sqrt{\varepsilon}$. Then we see the term in (33) goes to zero,which completes the proof. □

Proof of Theorem 3.1. 1. *Ballistic part*: From the expansion (28) and the control of M_ε^2 in Lemma 4.2, we have

$$\mathbb{E}\left|\frac{u_\varepsilon^b-u^b}{\sqrt{\varepsilon}}(x,v)-u^b(x,v)\frac{M_\varepsilon}{\sqrt{\varepsilon}}\right|\le C\frac{\mathbb{E}M_\varepsilon^2}{\sqrt{\varepsilon}}\le C\sqrt{\varepsilon}\longrightarrow 0.$$

This shows that the scaled corrector δu_ε^b converges to $\varepsilon^{-1/2}u^bM_\varepsilon$, for any fixed (x,v), in $L^1(\Omega)$ and hence in distribution. Therefore, the limiting distribution of $\varepsilon^{-1/2}\delta u_\varepsilon^b$ is given by that of $\varepsilon^{-1/2}u^bM_\varepsilon$. This term is an oscillatory integral of the form

$$\varepsilon^{-1/2}u^b(x,v)\int_{\mathbb{R}}\delta a\left(\frac{x-tv}{\varepsilon}\right)F_{x,v}(t)\,dt.$$

First observe that we can ignore the x in δa_ε thanks to stationarity. Then we apply Theorem 4.1 to the process δa restricted on the line $L(v)$ and have

$$\frac{M_\varepsilon}{\sqrt{\varepsilon}}\xrightarrow{\mathcal{D}}\int_0^{\tau_-(x,v)}\sigma_a\,dW_t. \tag{34}$$

Here, $\sigma_a^2=\int_{\mathbb{R}}\mathbb{E}\{\delta a(0)\delta a(tv)\}\,dt$. Finally we observe that the stochastic integral above is simply a Gaussian random variable with mean-zero and variance $\sigma_a^2\tau_-(x,v)$. This completes the first part of the proof.

2. *Single scattering part*: As before, we only need to capture the asymptotic distribution of the term linear in M_ε, that is $u^s(M_\varepsilon(x_s,w)+M_\varepsilon(x_s,-v)$ where $x_s=x-tv$. For each of them, we apply part one to obtain their asymptotic law. Jointly, they are independent in the limit thanks to Lemma 4.4. □

Proof of Theorem 3.3. 1. *The first item* follows directly from Theorem 3.1 and Lemma 4.4.

2. *The second item*: We need to control the term that is a product of two terms linear in M_ε. Let $x_i-x_j=d_{ij}v^\perp+\eta v$. We calculate

$$\begin{aligned}\mathbb{E}M_\varepsilon(x_i,v)M_\varepsilon(x_j,v)&=\int_{\mathbb{R}^2}R\left(\frac{d_{ij}v^\perp+(\eta-t+s)v}{\varepsilon}\right)F_{x_i,v}(t)F_{x_j,v}(s)\,dt\,ds\\&=\varepsilon\int_{\mathbb{R}^2}R\left(\frac{d_{ij}v^\perp}{\varepsilon}+tv\right)F_{x_i,v}(\eta+s)F_{x_j,v}(s)\,dt\,ds+o(\varepsilon).\end{aligned} \tag{35}$$

In the second equality, we have changed variable $(\eta-t+s)/\varepsilon\to t$. Then integrate over s to obtain the conclusion.

3. *The third item* is again a simple combination of Theorem 3.1 and Lemma 4.4. Namely, the parts after scattering will be independent in the limit, and the only correlation comes from the common ballistic part, for which we can apply (15). □

Proof of Theorem 3.5. The asymptotic distribution of the corrector will be determined by that of the term that is linear in δa. Let us denote this term by δI_ε expressed by

$$\begin{aligned}\delta I_\varepsilon(x,v;a)&=-\int_{\mathbb{R}^{d-1}}u^b(y,v)\frac{1}{\varepsilon^d}\phi\left(\frac{y}{\varepsilon}\right)\int_0^{\tau_-}\delta a\left(\frac{y-tv}{\varepsilon}\right)dt\,dy\\&=-u^b(x,v)\int_0^{\tau_-}\delta b\left(\frac{t}{\varepsilon}\right)dt+o(\varepsilon),\end{aligned} \tag{36}$$

where $\delta b(t)$ is defined to be the "vertically" averaged process

$$\delta b(t):=\int_{\mathbb{R}^{d-1}}\phi(y)\delta a(y-tv)\,dy. \tag{37}$$

Clearly δb inherits stationarity and strong mixing properties from δa. Therefore, part one of Theorem 3.1 can be applied. It suffices to verify that

$$\begin{aligned}\sigma_b^2(v)&=\int_{\mathbb{R}}\mathbb{E}\{\delta b(0)\delta b(t)\}\,dt\\&=\int_{\mathbb{R}\times\mathbb{R}^{2(d-1)}}\phi(y)\phi(z)R(y-z+tv)\,dt\,dy\,dz.\end{aligned} \tag{38}$$

This completes the proof. □

4.2. Proof of Theorem 3.6

The proof in the case of long range correlations also relies on the control of moments of M_ε up to fourth order and the asymptotic law of M_ε point-wise. In both analyses, we essentially follow the procedure in [4], namely replacing the random field $\Phi\circ g$ by the underlying Gaussian random field g. We will need

Theorem 4.5 (*Bal et al. [4]*). *Let δa satisfies the conditions in assumption* **B** *with correlation function decaying like $|x|^{-\alpha}$. For any function $F\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, we have*

$$\varepsilon^{-\frac{\alpha}{2}}\int_{\mathbb{R}}\varphi_\varepsilon(t)F(t)\,dt\xrightarrow{\mathcal{D}}\sqrt{\frac{\kappa}{H(2H-1)}}\int_{\mathbb{R}}F(t)\,dW_t^H, \tag{39}$$

where W_t^H is a fractional Brownian motion with Hurst index $H=1-\alpha/2$.

Proof of Theorem 3.6. 1. *Ballistic part*: From the expansion (28) and control of M_ε^2, we have

$$\mathbb{E}\left|\frac{u_\varepsilon^b-u^b}{\varepsilon^{\alpha/2}}(x,v)-u^b(x,v)\varepsilon^{-\alpha/2}M_\varepsilon\right|\le C\varepsilon^{\alpha/2}\longrightarrow 0.$$

Hence, we only need to capture the limit distribution of $u^b(x,v)\varepsilon^{-\alpha/2}M_\varepsilon$, which can be written as product of $u^b(x,v)$ with the following oscillatory integral:

$$I_\varepsilon=\varepsilon^{-\alpha/2}\int_{\mathbb{R}}\delta a\left(\frac{x-tv}{\varepsilon}\right)F_{x,v}(t)\,dt. \tag{40}$$

Here $F_{x,v}$ is the cut-off function introduced earlier. Then applying Theorem 4.5 for the process δa restricted on the line $L(v)$ (ignoring x), we have

$$I_\varepsilon\xrightarrow{\mathcal{D}}\sqrt{\frac{\kappa}{H(2H-1)}}\int_{\mathbb{R}}F_{x,v}(t)\,dW_t^H. \tag{41}$$

Finally, the variance of the stochastic integral above is given by

$$\int_{\mathbb{R}^2}\frac{\kappa}{|t-s|^\alpha}F(t)F(s)\,dt\,ds=\int\int_0^{\tau_-(x,v)}\frac{\kappa}{|t-s|^\alpha}\,dt\,ds.$$

This integral can be calculated explicitly as given in (26).

2. *Single scattering part*: As before, only the term that is linear in M_ε^s matters. Using the same method as in [4], we need to find the limiting distribution of $\varepsilon^{-\alpha/2}u^sV_1^2M_\varepsilon^{s,g}$ with

$$M_\varepsilon^{s,g}(x,v,y,w) := \int_{BL(x,v,y,w)} g_\varepsilon(\ell)\,d\ell. \tag{42}$$

This is a Gaussian random variable and it suffices to find the limit of its variance, which is precisely

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{\varepsilon^{\alpha/2}}\int_{\mathbb{R}} F_{x_s,-v}(t)g\left(\frac{x_s+tv}{\varepsilon}\right)dt\right)^2\\ &\quad+\mathbb{E}\left(\frac{1}{\varepsilon^{\alpha/2}}\int_{\mathbb{R}} F_{x_s,w}(s)g\left(\frac{x_s-sw}{\varepsilon}\right)ds\right)^2\\ &\quad+\frac{2}{\varepsilon^{\alpha}}\mathbb{E}\int_{\mathbb{R}^2} F_{x_s,w}(s)F_{x_s,-v}(t)g\left(\frac{x_s+tv}{\varepsilon}\right)g\left(\frac{x_s-sw}{\varepsilon}\right)dt\,ds.\end{aligned} \tag{43}$$

The first two terms are exactly as part 1. By the asymptotic behavior of R_g, the third one is given by

$$\frac{2}{\varepsilon^{\alpha}}\int_{\mathbb{R}^2} F_{x_s,-v}(t)F_{x_s,w}(s)\frac{\varepsilon^\alpha\kappa_g}{|tv+sw|^\alpha}\,dt\,ds.$$

This completes the proof of the result. □

5. Conclusions

The highly oscillatory part of the absorption coefficient in the radiative transfer equation, which varies on a scale of $\varepsilon \ll 1$ and cannot be stably reconstructed using inverse transport technique, is modeled as a random field. The scale ε may be introduced in practice in several ways accounting for different experimental settings, for instance as the gap between detectors as considered in this paper. Furthermore, the effects of the above random fluctuations in the absorption coefficient on the point-wise measurements of particle densities at the boundary are described asymptotically. More precisely, we found that the fluctuations in the measurements are asymptotically Gaussian and of size $\varepsilon^{1/2}$ (resp. $\varepsilon^{\alpha/2}$ for $\alpha<1$) when the random fluctuations in the absorption have integrable (resp. non-integrable) correlation function.

Such results are useful for inverse transport as we explain now in more detail. Since we can only hope to reconstruct stably the low frequency part of the absorption a, we view the measurement u_ε as the "true" measurement u corresponding to a plus "noise".

In standard notations of the generic inverse problem, $[\sigma]$ denotes the unknown coefficient to reconstruct, $[D]$ the measured data and e the error. The collected data can be written as

$$[D]=\Phi([\sigma])+e. \tag{44}$$

The measurement operator Φ is typically determined by equations modeling the physics and is usually smoothing. Noise e is therefore typically amplified during the reconstruction of $[\sigma]$ from knowledge of $[D]$ [14]. Knowing its statistical structure allows us to mitigate the influence of noise in an optimal manner [7].

In our problem, the noise term e is precisely given by $u_\varepsilon-u$. Our results on the statistics of the corrector $u_\varepsilon-u$ provide statistics for this noise that are derived from physical principles, which is often better than assuming a "standard" noise model. We refer the reader to [7] for an application of corrector analysis to an inverse spectral problem.

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Appendix A. Variance for angularly averaged measurements

In this appendix, we show that measurements that are point-wise in space but averaged angularly have variance of order $\varepsilon^2|\log\varepsilon|$ in dimension $d=2$ and of order ε^2 when $d\geq 3$. To simplify the presentation, we present the calculations only for the ballistic part $u^b(x,v)$, knowing that the other contributions to the transport equation have variances of the same order. In the rest of the appendix, we drop the superscript ^{b} to simplify notation.

Fixing x, a point in X or on its boundary ∂X, the angularly averaged measurement at this point is obtained by integrating $u(x,v)$ over v. That is,

$$J(x)=\int_V u(x,v)\,dv, \tag{45}$$

for $x\in X$. For measurements performed on the boundary, the integration is taken over $\{v : v_x\cdot v>0\}$. Then the random corrector of such measurements is simply

$$\delta J_\varepsilon := J_\varepsilon - J = \int_V u(x,v)(\exp(M_\varepsilon(x,v))-1)\,dv. \tag{46}$$

We have the following result regarding the order of its variance.

Lemma A.1 (*Variance for angularly averaged measurements*). *Let δa be a stationary random field with correlation function R* decaying faster than $|x|^{-d-\delta}$ at infinity for some $\delta>0$. *Then*

$$\mathrm{Var}\{\delta J_\varepsilon\}\leq\begin{cases} C\varepsilon^2|\log\varepsilon|, & d=2\\ C\varepsilon^2, & d\geq 3.\end{cases} \tag{47}$$

Proof. Thanks to the control of the fourth moments of M_ε, we only need to control the term that is linear in M_ε. Using the change of variables

$$\int_V\int_0^{\tau_-(x,v)} f(x-tv,v)\,dt\,dv=\int_X \frac{f(y,v)}{|x-y|^{d-1}}\bigg|_{v=(x-y)/(|x-y|)} dy, \tag{48}$$

the linear term in δJ_ε can be written as

$$-\int_X \frac{u(x,v)\delta a_\varepsilon(y)}{|x-y|^{d-1}}\bigg|_{v=(x-y)/(|x-y|)} dy.$$

Call this term I. We have

$$\mathrm{Var}\{I\}=\mathbb{E}\{I^2\}=\int_{X^2}\frac{u(x,v)u(x,w)R\left(\frac{y-z}{\varepsilon}\right)}{|x-y|^{d-1}|x-z|^{d-1}}\bigg|_{v=(x-y)/(|x-y|),w=(x-z)/(|x-z|)} dy\,dz.$$

Now, change variables $(y-z)/\varepsilon \to z$ and $x-y \to y$. Then we have

$$\mathbb{E}I^2 = \varepsilon^d \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{u(x,v)u(x,w'(\varepsilon))R(z)}{|y|^{d-1}|y+\varepsilon z|^{d-1}} \chi_X(x-y)\chi_X(x-y-\varepsilon z)\, dz\, dy.$$

Here, $w'(\varepsilon)$ is the direction of the vector $x-y-\varepsilon z$ and as ε goes to zero it converges to that of $x-y$. Assuming that u is bounded, and then integrating in y (over X) and using the estimate that

$$\int_X \frac{1}{|y|^{d-1}|y+z|^{d-1}}\, dy \le \begin{cases} C(|\log|z|| + 1), & d = 2, \\ C\dfrac{1}{|z|^{d-2}}, & d \ge 3, \end{cases} \tag{49}$$

we obtain that in dimension two that

$$\mathbb{E}I^2 \le C\varepsilon^2 |\log\varepsilon| \int_{\mathbb{R}^d} R(z)|\log|z||\, dz,$$

and in dimension three that

$$\mathbb{E}I^2 \le C\varepsilon^2 \int_{\mathbb{R}^d} R(z) \frac{1}{|z|^{d-2}}\, dz.$$

In both cases, the integral converges due to the fact that R decays sufficiently fast at infinity and $\log|z|$ (in dimension two) and $|z|^{d-2}$ (in dimension three) are integrable near the origin. For a proof of estimate (49), we refer the reader, e.g., to the appendix of [5]. This completes the proof. □

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