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Quantitative thermo-acoustic imaging: An exact reconstruction formula ☆

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ABSTRACT

This paper aims to mathematically advance the field of quantitative thermo-acoustic imaging. Given several electromagnetic data sets, we establish for the first time an analytical formula for reconstructing the absorption coefficient from thermal energy measurements. Since the formula involves derivatives of the given data up to the third order, it is unstable in the sense that small measurement noises may cause large errors. However, in the presence of measurement noise, the obtained formula, together with a noise regularization technique, provides a good initial guess for the true absorption coefficient. We finally correct the errors by deriving a reconstruction formula based on the least square solution of an optimal control problem and prove that this optimization step reduces the errors occurring and enhances the resolution.

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1. Introduction

Hybrid imaging modalities are based on a multi-wave concept. Different physical types of waves are combined into one tomographic process to alleviate deficiencies of each separate type of waves, while combining their strengths. Multi-wave systems are capable of high-resolution and high-contrast imaging [1,17]. Quantitative thermo-acoustic tomography is an emerging hybrid modality [14,12]. It allows to determine the absorption distribution of a tissue from boundary measurements of the pressure induced by electromagnetic heating. Other examples of hybrid modalities are acousto-electric

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tomography [3,2,6,9,13,21,32,33], magnetic resonance electrical impedance tomography [20,28,26], magnetic resonance elastography [8,25,23], impedance-acoustic tomography [18], photo-acoustic [31, 22,4], quantitative photo-acoustic tomography [5,11,27], magneto-acoustic imaging [7], and vibro-acoustography [16].

The aims of this paper are to derive an exact formula for the absorption coefficient from noiseless thermo-acoustic measurements and to correct the errors of in the presence of measurement noise. The former task is motivated by the knowledge of the ratio between two modified data. For the latter purpose, we show how to regularize the exact formula and propose an optimal control algorithm to achieve a resolved image starting from the regularized one. As far as we know, our exact formula in this paper together with the one successfully derived in [6] are among a few exact formulas in hybrid imaging. Moreover, the fine analysis of the effect of measurement noise on the image quality and the proof that an optimal control approach starting from the regularized images yields a resolved one have never been done elsewhere.

To describe our approach, we employ several notations. Let X be a smooth bounded domain in $\mathbb{R}^d$, $d=2$ or 3. Let ∂X denote the boundary of X and let ν be the outward normal at ∂X. For m a non-negative integer, we define the space $H^m(X)$ as the family of all m times weakly differentiable functions in $L^2(X)$, whose weak derivatives of orders up to m are functions in $L^2(X)$. We let $H_0^m(X)$ be the closure of $\mathcal{C}_c^\infty(X)$ in $H^m(X)$, where $\mathcal{C}_c^\infty(X)$ is the set of all infinitely differentiable functions with compact supports in X. Finally, we introduce the space $H^{1/2}(\partial X)$ of traces on ∂X of all functions in $H^1(X)$.

Let q be a positive real-valued function on X. Consider the Helmholtz problem:

$$\begin{aligned} &\left(\Delta + k^2 + ikq\right)u = 0, && x \in X,\\ &\nu \cdot \nabla u - iku = g, && x \in \partial X, \end{aligned} \tag{1.1}$$

which is the scalar approximation of Maxwell's equations. Here, $k>0$ is the wave number, g is a boundary datum, and u is the electrical field. The Robin boundary condition approximates Sommerfeld's radiation condition at high frequencies [15,19]. For simplicity, instead of considering the Helmholtz equation on the whole Euclidean space with Sommerfeld's radiation condition we focus on the Helmholtz problem with Robin boundary condition on the bounded open set X. Problem (1.1) is well-posed in $H^1(X)$ for all $g \in L^2(\partial X)$. In fact, writing a variational formulation of (1.1) shows the uniqueness of a solution to (1.1), while the existence of a solution follows from Fredholm's alternative.

The thermo-acoustic imaging problem can be formulated as the inverse problem of reconstructing the absorption coefficient q from thermo-acoustic measurements $q|u|^2$ in X. The quantity $q|u|^2$ in X is the heat energy due to the absorption distribution q. It generates an acoustic wave propagating inside the medium. Finding the initial data in the acoustic wave from boundary measurements yields the heat energy distribution. Our aim in this paper is to separate q from u. We provide an explicit formula for reconstructing q from the heat energy $q|u|^2$ in X. As far as we know, our formula is new. Indeed, it is promising since it can be used as an initial guess to achieve a resolved image of the absorption distribution in a robust way.

Our first task is to enrich the set of data. Suppose that we have measurements $q(x)|u_j|^2$ corresponding to linear combinations of boundary data g_j, for $j=1,\dots,d+1$. We show that one can construct the set of quantities:

$$\mathcal{E} = \left\{E_j(x) = q(x)u_j(x)\overline{u_1(x)},\ x \in X \;\middle|\; j = 1,\dots,d+1\right\}, \tag{1.2}$$

where u_j denotes the solution of

$$\begin{aligned} &\left(\Delta + k^2 + ikq\right)u_j = 0, && x \in X,\\ &\nu \cdot \nabla u_j - iku_j = g_j, && x \in \partial X, \end{aligned} \tag{1.3}$$

provided that $(g_j)_{j=1}^{d+1}$ is a proper set of measurements (see Definition 2.1). The construction of E_1 was completely described in [12] and that of E_j, $j = 2, \dots, d+1$, will be done using Proposition 2.6. Noting that

$$\frac{u_j}{u_1} = \frac{E_j}{E_1}, \quad j = 2, \dots, d+1,$$

we are able to establish an exact formula for q provided that $\mathcal{E} = (E_j)_{j=1}^{d+1}$ is "good" enough as in Theorem 3.3. This procedure will be described in Section 3.

As said, the collected data $\mathcal{E}$ are often corrupted by measurement noise that varies on very small length scale. This renders the aforementioned exact formula, which requires differentiating the data up to third order, completely unpractical. To solve this issue, we smooth the noise by averaging the data over a small window and apply the smoothed data to the exact formula. The resulting function is then shown to be close to the real one, provided that the width of the averaging window is properly chosen. We thus view this function as an initial guess and then perform a further step of least square optimization. The resulting reconstruction improves the initial guess in the L^2 sense.

The rest of the paper is organized as follows. In Section 2 we introduce the notion of a proper set of measurements and its role to get data $\mathcal{E}$ and some useful estimates as well. The aim of Section 3 is to provide an explicit formula for reconstructing q when a proper set of measurements is given. In Section 4 we study the Fréchet differentiability of the data with respect to variations of q and prove that the differential operator is invertible for small enough variations. In Section 5 we consider a noise model for the data and show how to regularize the exact inversion formula in order to obtain a good initial guess. We also perform a refinement of the initial guess using an optimal control approach and show that this procedure yields a resolution enhancement.

2. Preliminaries

Motivated by [6], we introduce the following concept.

Definition 2.1. The set $(g_j)_{j=1}^{d+1} \subset L^2(\partial X)$ is a **proper set of measurements** of (1.1) if and only if:

(i) $|u_1| > 0$ in X.
(ii) The matrix $[u_j, \nabla^T u_j]_{1 \leqslant j \leqslant d+1}$ is invertible for all $x \in X$.

Here, T denotes the transpose and u_j is the solution of (1.3).

The following proposition is a direct consequence of Lemma 4.1 in [12] and Proposition 3.1 in [11]. It plays an important role to prove that it is possible to find a proper set of measurements.

Proposition 2.2. *Let $\delta > 0$ and $m > d/2$. There exists a positive constant C such that for any $\xi \in \mathbb{C}^d$, $\xi \cdot \xi = 0$, and $|\xi| > \delta$, and for any $q \in H^m(X)$, the solution w of*

$$\Delta w + \xi \cdot \nabla w = -\big(k^2 + ikq\chi(X)\big)(1 + w) \quad \text{in } \mathbb{R}^d, \tag{2.1}$$

where $\chi(X)$ denotes the characteristic function of X, satisfies

$$\|w\|_{H^m(X)} \leqslant \frac{C\|q\|_{H^m(X)}}{|\xi|}. \tag{2.2}$$

Proposition 2.3. *If $q \in H^m(X)$, $m > 1 + \frac{d}{2}$, then (1.1) has a proper set of measurements.*

Proof. Let ϵ be a small number. By choosing ξ such that $\xi \cdot \xi = 0$ and $|\xi|$ is large enough, we find from the Sobolev embedding theorem and (2.2) that the solution w of (2.1) satisfies

$$\|w\|_{L^\infty(X)} + \|\nabla w\|_{L^\infty(X)} < \epsilon. \tag{2.3}$$

It is not hard to verify that the function

$$u = e^{\xi \cdot x}(1 + w)$$

is a solution of

$$\big(\Delta + k^2 + ikq\chi(X)\big)u = 0 \tag{2.4}$$

and it satisfies

$$|u| > \left|e^{\xi \cdot x}\right|(1 - \epsilon) > 0.$$

Choosing $g_1 = \nu \cdot \nabla u - iku$ on ∂X gives a solution u_1 of (1.3) satisfying part (i) of Definition 2.1.

Define

$$\begin{aligned}
&\xi_j = n(e_j + ie_{j+1}), \quad j = 1, \ldots, d-1, \\
&\xi_d = n(e_d + ie_1),
\end{aligned}$$

and

$$\xi_{d+1} = n\left(\left[\sum_{j=1}^{d-1} e_j + \sqrt{d-1}e_d\right] + i\left[\sum_{j=1}^{d-1} e_j - \sqrt{d-1}e_d\right]\right),$$

where $n \gg 1$ and e_j is the jth component of the natural basis of $\mathbb{R}^d$. Again, it is not hard to verify that

$$\xi_j \cdot \xi_j = 0$$

for all $j = 1, \ldots, d+1$, and the vectors $(1, \xi_j)_{1 \leqslant j \leqslant d+1}$ are linearly independent in $\mathbb{C}^d$. Hence,

$$\left|\det\begin{bmatrix} 1 & \xi_j^T \end{bmatrix}_{1 \leqslant j \leqslant d+1}\right| \gg 1, \tag{2.5}$$

provided that $n \gg 1$. Let w_j, $1 \leqslant j \leqslant d+1$, be the solution of

$$\Delta w_j + 2\xi_j \cdot \nabla w_j = -\big(k^2 + ikq\chi(X)\big)(1 + w_j)$$

and

$$u_j = e^{\xi_j \cdot x}(1 + w_j)$$

be the solution of (2.4). We have

$$\det\left[u_j \quad \nabla^T u_j\right]_{1\leqslant j\leqslant d+1} = e^{\xi_j\cdot x}(1+w_j)\det\left[(1+w_j) \quad \xi_j^T + \tfrac{\nabla^T w_j}{1+w_j}\right]_{1\leqslant j\leqslant d+1}.$$

Thus, (2.3), (2.5), the continuity of the map that sends a square matrix to its determinant and the choice of large n imply the second part of Definition 2.1 with

$$g_j = \nu\cdot\nabla u_j - iku_j, \quad j = 1,\dots,d,$$

on ∂X. □

Remark 2.4. The solution w of (2.1) is the so-called complex geometric optics solution of (1.1), which was introduced in [10,29]. The proof of Proposition 2.3 was partly motivated by [30].

We next construct the data $\mathcal{E}$, mentioned in Section 1. Let us for the moment accept the following proposition.

Proposition 2.5. *If g is given, then one can make some measurements to obtain $q(x)|u|^2$, $x\in X$, where u solves (1.1).*

The following proposition holds.

Proposition 2.6. *Let $g_1, g_2 \in L^2(\partial X)$. Denote by u_j the solution of*

$$\begin{aligned}&(\Delta + k^2 + ikq)u_j = 0, \quad x\in X,\\ &\nu\cdot\nabla u_j - iku_j = g_j, \quad x\in\partial X,\ j=1,2.\end{aligned} \tag{2.6}$$

Then the function $q(x)u_2(x)\overline{u_1(x)}$, $x\in X$, can be evaluated.

Proof. Applying Proposition 2.5 for g_1+g_2 and then ig_1+g_2, we obtain the knowledge of

$$q|u_1+u_2|^2 \quad\text{and}\quad q|iu_1+u_2|^2,$$

respectively. Then the desired data E_2 is given by

$$E_2 = \frac{1}{2}\left(q|u_1+u_2|^2 - q|u_1|^2 - q|u_2|^2\right) + \frac{i}{2}\left(q|iu_1+u_2|^2 - q|u_1|^2 - q|u_2|^2\right), \tag{2.7}$$

which can be easily verified. □

Let $(g_j)_{j=1}^{d+1}$ be a proper set of measurements of (1.1) and u_j be the solution of (1.1) with g replaced by g_j. From now on, we have the knowledge of

$$\mathcal{E} = (E_j)_{j=1}^{d+1}, \tag{2.8}$$

where $E_j = qu_1\bar{u}_j$, and $\mathcal{E}$ is, therefore, considered as the data to reconstruct q.

We also need the following proposition. It plays an important role to evaluate the derivative of the data with respect to q in Section 4 as well as some crucial properties.

Proposition 2.7. *Let $q \in L^\infty(X)$ be such that* $\inf q > 0$. *For all $f \in L^2(X)$, the problem*

$$\begin{aligned} &\left(\Delta + k^2 + ikq\right)u = f, && x \in X,\\ &\nu \cdot \nabla u - iku = 0, && x \in \partial X, \end{aligned} \tag{2.9}$$

has a unique solution. Moreover, the solution satisfies

$$\|u\|_{L^2(X)} \leqslant \frac{1}{k \inf q} \|f\|_{L^2(X)} \tag{2.10}$$

and

$$\|u\|_{H^1(X)} \leqslant \frac{\sqrt{(k^2+1) + k \inf q}}{k \inf q} \|f\|_{L^2(X)}. \tag{2.11}$$

Proof. The well-posedness of (2.9) is well-known. Using the test function u in (2.9) and considering the imaginary and real parts of the resulting equation, we can establish (2.10) and (2.11), respectively. □

3. The exact formula

The main aim of this section is to reconstruct q when a proper set of measurements $(g_j)_{j=1}^{d+1}$ of (1.1) and the data $\mathcal{E}$, defined in (2.8), are given.

Let

$$\alpha_j = \frac{E_j}{E_1}, \quad 2 \leqslant j \leqslant d+1. \tag{3.1}$$

Then it is not hard to see that

$$u_j = \alpha_j u_1,$$

for $2 \leqslant j \leqslant d+1$. We have the following lemma.

Lemma 3.1. *Let $\beta = \Im(\bar{u}_1 \nabla u_1)$. Then*

$$-\operatorname{div} \beta = kE_1, \quad \text{in } X. \tag{3.2}$$

Proof. Let $\varphi \in \mathcal{C}_c^\infty(X, \mathbb{R})$ be an arbitrary function. Then using $\varphi u_1 \in H_0^1(X)$ as a test function in

$$-\Delta u_1 = \left(k^2 + ikq\right)u_1$$

yields

$$\int_X \varphi |\nabla u_1|^2 \, dx + \int_X \overline{u_1} \nabla u_1 \cdot \nabla \varphi \, dx = \int_X \left(k^2 + ikq\right)|u|^2 \varphi \, dx.$$

Taking the imaginary part of the equation above gives

$$-\int_X \operatorname{div}(\Im \overline{u_1} \nabla u_1)\varphi\, dx = \int_X kq|u_1|^2 \varphi\, dx = \int_X kE_1 \varphi\, dx,$$

and (3.2) follows. □

The following lemma plays an important role in the derivation of an exact inversion formula for q.

Lemma 3.2. *For all* $2 \leqslant j \leqslant d+1$,

$$\nabla \alpha_j \cdot \left(\nabla \log \frac{q}{E_1} - \frac{2iq\beta}{E_1}\right) = \Delta \alpha_j. \tag{3.3}$$

Proof. Let us fix $j \in \{2, \dots, d+1\}$. Since u_j is a solution of the Helmholtz equation under consideration,

$$\begin{aligned}
\big(k^2 + ikq\big)\alpha_j u_1 &= -\Delta(\alpha_j u_1) \\
&= -\alpha_j \Delta u_1 - u_1 \Delta \alpha_j - 2\nabla u_1 \cdot \nabla \alpha_j \\
&= \big(k^2 + ikq\big)\alpha_j u_1 - u_1 \Delta \alpha_j - 2\nabla u_1 \cdot \nabla \alpha_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-E_1 \Delta \alpha_j &= 2q\overline{u_1}\nabla u_1 \cdot \nabla \alpha_j \\
&= 2q(\Re \overline{u_1}\nabla u_1 + i\Im \overline{u_1}\nabla u_1) \cdot \nabla \alpha_j \\
&= q\big(\nabla |u_1|^2 + 2i\Im \overline{u_1}\nabla u_1\big) \cdot \nabla \alpha_j.
\end{aligned}$$

We have proved that

$$-E_1 \Delta \alpha_j = q\big(\nabla |u_1|^2 + 2i\beta\big) \cdot \nabla \alpha_j,$$

or equivalently,

$$q\nabla |u_1|^2 \cdot \nabla \alpha_j = -E_1 \Delta \alpha_j - 2iq\beta \cdot \nabla \alpha_j. \tag{3.4}$$

On the other hand, differentiating the equation $E_1 = q|u_1|^2$ gives

$$\nabla E_1 = q\nabla |u_1|^2 + E_1 \nabla \log q.$$

This, together with (3.4), implies

$$(\nabla E_1 - E_1 \nabla \log q) \cdot \nabla \alpha_j = -E_1 \Delta \alpha_j - 2iq\beta \cdot \nabla \alpha_j,$$

and (3.3), therefore, holds. □

We claim that the set

$$(\nabla \alpha_j)_{j=2}^{d+1}$$

is linearly independent for all $x \in \overline{X}$, where α_j was defined in (3.1). We only prove this when $d = 2$. The proof when d is larger than 2 can be done in the same manner. In fact, the linear independence of $\{\nabla\alpha_2, \nabla\alpha_3\}$ comes from the following calculation:

$$\begin{aligned}
\det\begin{bmatrix} \nabla^T\alpha_2 \\ \nabla^T\alpha_3 \end{bmatrix} &= \frac{1}{u_1^4}\det\begin{bmatrix} u_1\nabla^T u_2 - u_2\nabla^T u_1 \\ u_1\nabla^T u_3 - u_3\nabla^T u_1 \end{bmatrix} \\
&= \frac{1}{u_1^4}\left(\det\begin{bmatrix} u_1\nabla^T u_2 \\ u_1\nabla^T u_3 - u_3\nabla^T u_1 \end{bmatrix} - u_2\det\begin{bmatrix} \nabla^T u_1 \\ u_1\nabla^T u_3 - u_3\nabla^T u_1 \end{bmatrix}\right) \\
&= \frac{1}{u_1^3}\left(u_1\det\begin{bmatrix} \nabla^T u_2 \\ \nabla^T u_3 \end{bmatrix} + u_3\det\begin{bmatrix} \nabla^T u_1 \\ \nabla^T u_2 \end{bmatrix} - u_2\det\begin{bmatrix} \nabla^T u_1 \\ \nabla^T u_3 \end{bmatrix}\right) \\
&= \frac{1}{u_1^3}\det\begin{bmatrix} u_1 & \nabla^T u_1 \\ u_2 & \nabla^T u_2 \\ u_3 & \nabla^T u_3 \end{bmatrix} \neq 0.
\end{aligned}$$

Here, part (ii) in Definition 2.1 has been used. Since the $d \times d$ matrix

$$A = \left[\nabla^T\alpha_{j+1}\right]_{1 \leqslant j \leqslant d}, \qquad A_{jl} = \partial_l\alpha_{j+1}, \tag{3.5}$$

is invertible, we can solve system (3.3) to get

$$\nabla\log\frac{q}{E_1} - \frac{2iq\beta}{E_1} = a, \tag{3.6}$$

where a is the vector $a = A^{-1}[(\nabla^T A^T)^T]$.

We are now ready to evaluate q. We first split the real and the imaginary parts of (3.6) to get

$$\nabla\log\frac{q}{E_1} = \frac{\nabla q}{q} - \nabla\log E_1 = \Re(a) \tag{3.7}$$

and

$$\beta = -\frac{E_1\Im(a)}{2q}. \tag{3.8}$$

Then, differentiating (3.8), we have

$$\operatorname{div}\beta = \frac{E_1\Im(a)\cdot\nabla q}{2q^2} - \frac{\operatorname{div}(E_1\Im(a))}{2q}.$$

This, together with (3.2) and (3.7), implies

$$\begin{aligned}
q &= -\frac{E_1(\Re(a) + \nabla\log E_1)\cdot\Im(a) - \operatorname{div}(E_1\Im(a))}{2kE_1} \\
&= -\frac{E_1\Re(a)\cdot\Im(a) + \nabla E_1\cdot\Im(a)}{2kE_1} + \frac{E_1\operatorname{div}\Im(a) + \nabla E_1\cdot\Im(a)}{2kE_1} \\
&= -\frac{\Re(a)\cdot\Im(a) - \operatorname{div}\Im(a)}{2k}.
\end{aligned}$$

The results above are summarized in the following theorem.

Theorem 3.3. *Given a proper set of measurements* $(g_j)_{j=1}^{d+1}$ *so that the matrix* A, *defined in* (3.5), *is known and invertible. Then,*

$$q(x) = \frac{-\Re(a) \cdot \Im(a) + \operatorname{div} \Im(a)}{2k}, \tag{3.9}$$

where $a = A^{-1}[(\nabla^T A^T)^T]$ *and* $A = (\partial_l \alpha_{j+1})_{j,l=1,\dots,d}$.

Remark 3.4. Although in the proof of Theorem 3.3, we wrote some notations requiring the first and second derivatives of $\mathcal{E}$ at a single point $x \in X$, it is not necessary to impose the smoothness conditions for $\mathcal{E}$. The reason is that one can make the arguments and establish (3.3) in the weak sense. We argued, using strong forms of differential equations, only for simplicity.

Remark 3.5. Formula (3.9) is unstable in the sense that if there are some noises occurring when we measure the data E_j, $1 \leqslant j \leqslant d+1$, then q, given by (3.9), might be far away from the actual q since the right-hand side of (3.9) depends on the derivatives of the noise (up to the third order).

4. The differentiability of the data map and its inverse

Let $0 < q_{\min} < q_{\max}$. Let

$$L_+^\infty(X) = \left\{p \in L^\infty(X)\colon q_{\min} < p < q_{\max} \text{ in } X\right\}.$$

Then, $L_+^\infty(X)$ is an open set in $L^\infty(X)$. We define the solution and the data map as

$$\begin{aligned} u : L_+^\infty(X) &\to H^1(X) \\ q &\mapsto u[q] \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} F : L_+^\infty(X) &\to L^2(X) \\ q &\mapsto F[q] = q\left|u[q]\right|^2, \end{aligned} \tag{4.2}$$

where $u[q]$ is the solution of (1.1). The map F is well-defined because of the Sobolev embedding theorems and the fact that $d = 2$ or 3, which guarantees that $u \in L^4(X)$.

The main purpose of this section is to study the differential operator, $\mathcal{D}F[q]$, of F and show that it is invertible provided that $q_{\max}$ is small enough.

Lemma 4.1. *The map* u, *defined in* (4.1), *is Fréchet differentiable in* $L_+^\infty(X)$. *Its derivative at the function* q *is given by*

$$\mathcal{D}u[q](\rho) = v(\rho), \quad \forall \rho \in B_q, \tag{4.3}$$

where $B_q \subset L^\infty(X)$ *is an open neighborhood of* q *in* $L^\infty(X)$ *and* $v(\rho)$ *is the solution of*

$$\begin{aligned} &\left(\Delta + k^2 + ikq\right)v = -ik\rho u[q], && x \in X, \\ &\nu \cdot \nabla v - ikv = 0, && x \in \partial X. \end{aligned} \tag{4.4}$$

Consequently, F *is also Fréchet differentiable and*

$$\mathcal{D}F[q]\rho = \rho\left|u[q]\right|^2 + 2q\Re\left(u[q]\bar{v}(\rho)\right), \quad \forall q \in L_+^\infty(X),\ \rho \in B_q. \tag{4.5}$$

Proof. It is sufficient to show that

$$\lim_{\|\rho\|_{L^\infty(X)}\to 0} h(\rho) = 0, \tag{4.6}$$

where

$$h(\rho) = \frac{\|u[q+\rho] - u[q] - v(\rho)\|_{L^2(X)}}{\|\rho\|_{L^\infty(X)}}.$$

In fact, since $u[q+\rho] - u[q] - v(\rho)$ solves the problem

$$\begin{aligned}
&(\Delta + k^2 + ikq)\big(u[q+\rho] - u[q] - v(\rho)\big) = -ik\rho\big(u[q+\rho] - u[q]\big), && x \in X,\\
&\nu \cdot \nabla\big(u[q+\rho] - u[q] - v(\rho)\big) - ik\big(u[q+\rho] - u[q] - v(\rho)\big) = 0, && x \in \partial X,
\end{aligned}$$

we can apply inequality (2.10) to obtain

$$\big\|u[q+\rho] - u[q] - v(\rho)\big\|_{L^2(X)} \leqslant \frac{\|\rho\|_{L^\infty(X)}\|(u[q+\rho] - u[q])\|_{L^2(X)}}{\inf q}. \tag{4.7}$$

On the other hand, since $u[q+\rho] - u[q]$ satisfies

$$\begin{aligned}
&\big(\Delta + k^2 + ik(q+\rho)\big)\big(u[q+\rho] - u[q]\big) = -ik\rho u[q], && x \in X,\\
&\nu \cdot \big(\nabla u[q+\rho] - u[q]\big) - ik\big(u[q+\rho] - u[q]\big) = 0, && x \in \partial X,
\end{aligned}$$

inequality (2.10), again, implies

$$\big\|u[q+\rho] - u[q]\big\|_{L^2(X)} \leqslant \frac{\|\rho\|_{L^\infty(X)}\|u[q]\|_{L^2(X)}}{\inf(q+\rho)}. \tag{4.8}$$

Combining (4.7) and (4.8) yields (4.6). Using the chain rule in differentiation, we readily get (4.5). □

Using regularity theory, we see that $u[q]$ belongs to $L^\infty(X)$ in the two-dimensional case. In three dimensions, we should assume that $g \in H^{1/2}(\partial X)$ in order to claim that $u[q] \in L^\infty(X)$. Hence, $\mathcal{D}F[q]$ can be extended so that its domain is $L^2(X)$. By abuse of notation, we denote the extended operator still by $\mathcal{D}F[q]$. The following key lemma of this section establishes an estimate of the $L^2(X)$ norm of $v(\rho)$, the solution to (4.4), in terms of the $L^2(X)$ norm of the source $\rho u[q]$. A corollary of this result allows us to show the invertibility of $\mathcal{D}F[q]$ from $L^2(X)$ to $L^2(X)$.

Lemma 4.2. *Assume that the origin* 0 *is included in X and define*

$$\operatorname{rad}(X) = \sup_{x\in\partial X} |x|.$$

Suppose that X is star-shaped and balanced with respect to the origin so that

$$x \cdot \nu_x \geqslant \gamma \operatorname{rad}(X)$$

for some positive number γ. If

$$\|q\|_{L^\infty}\,\mathrm{rad}(X) \leqslant \frac{1}{4},$$

and $k > 2$, then

$$\big\|v(\rho)\big\|_{L^2} \leqslant \eta \big\|\rho u[q]\big\|_{L^2}, \tag{4.9}$$

where

$$\eta = \sqrt{\frac{8(1+\gamma^{-1})^2 + 2d + 29}{(11-2d)}}\,\max\big\{\mathrm{rad}(X), 1\big\}. \tag{4.10}$$

Proof. Let us define the bilinear form

$$B[v,w] = -\int\limits_X \nabla v \cdot \nabla \overline{w}\,dx + k^2 \int\limits_X v\overline{w}\,dx + ik\int\limits_X qv\overline{w}\,dx + ik\int\limits_{\partial X} v\overline{w}\,ds, \tag{4.11}$$

and the linear form

$$G[w] = -\int\limits_X ik\rho u[q]\overline{w}\,dx. \tag{4.12}$$

Then the weak solution of (4.4) is characterized by v satisfying

$$B[v,w] = G[w], \quad \forall w \in H^1(X). \tag{4.13}$$

Using $w = v$ in (4.13) and considering the imaginary and real parts separately, we have

$$\begin{gathered}\int\limits_{\partial X} |v|^2\,ds + \int\limits_X q|v|^2\,dx \leqslant \left|\int\limits_X \rho u\bar{v}\,dx\right|,\\ \int\limits_X |\nabla v|^2\,dx - k^2\int\limits_X |v|^2\,dx \leqslant k\left|\int\limits_X \rho u\bar{v}\,dx\right|.\end{gathered} \tag{4.14}$$

It follows from these inequalities that

$$\|v\|^2_{L^2(\partial X)} \leqslant \|\rho u\|_{L^2}\|v\|_{L^2}, \tag{4.15}$$

and

$$\|\nabla v\|^2_{L^2} \leqslant \big(k^2+1\big)\|v\|^2_{L^2} + \frac{k^2}{4}\|\rho u\|^2_{L^2}. \tag{4.16}$$

To estimate $\|v\|_{L^2}$, we mimic the technique used in [24, Chapter 8]. We have

$$\Re\big(\nabla v \cdot \nabla(x\cdot\nabla\bar{v})\big) = |\nabla v|^2 + x\cdot\nabla\left(\frac{|\nabla v|^2}{2}\right), \qquad \Re\big(v(x\cdot\overline{\nabla v})\big) = x\cdot\nabla\left(\frac{|v|^2}{2}\right).$$

Integrating the first equation above gives

$$\int_X \Re\big(\nabla v\cdot\nabla(x\cdot\nabla\bar v)\big)\,dx = \int_X |\nabla v|^2\,dx + \frac{1}{2}\int_X \nabla\cdot\big(x|\nabla v|^2\big) - (\nabla\cdot x)|\nabla v|^2\,dx$$
$$= \frac{1}{2}\int_{\partial X}(\nu\cdot x)|\nabla v|^2\,ds + \left(1-\frac{d}{2}\right)\|\nabla v\|_{L^2}^2.$$

The second term above is due to the fact that $\nabla\cdot x = d$. Similarly,

$$k^2\int_X \Re\big(v(x\cdot\nabla\bar v)\big)\,dx = \frac{k^2}{2}\int_X \nabla\cdot\big(x|v|^2\big) - (\nabla\cdot x)|v|^2\,dx$$
$$= -\frac{dk^2}{2}\int_X |v|^2\,dx + \frac{k^2}{2}\int_{\partial X}(\nu\cdot x)|v|^2\,ds.$$

Consequently, taking $w = -x\cdot\nabla v$ in (4.11) we find

$$-\Re B[v, x\cdot\nabla v] = \frac{dk^2}{2}\|v\|_{L^2}^2 + \frac{1}{2}\int_{\partial X}(x\cdot\nu)|\nabla v|^2\,ds - \frac{k^2}{2}\int_{\partial X}(x\cdot\nu)|v|^2\,ds$$
$$+ \left(1-\frac{d}{2}\right)\|\nabla v\|_{L^2}^2 + \Re\left(-ik\int_X qv(x\cdot\nabla\bar v)\,dx - ik\int_{\partial X} v(x\cdot\nabla\bar v)\,ds\right).$$

Equate the above expression with the real part of $-\Re G[x\cdot\nabla v]$, *i.e.*, $\Re ik\int \rho u x\cdot\nabla\bar v\,dx$. We then obtain the estimate (using the fact that $x\cdot\nu \geqslant \gamma\,\mathrm{rad}(X)$):

$$\frac{dk^2}{2}\|v\|_{L^2}^2 + \frac{\mathrm{rad}(X)\gamma}{2}\|\nabla v\|_{L^2(\partial X)}^2$$
$$\leqslant \frac{k^2\,\mathrm{rad}(X)}{2}\|v\|_{L^2(\partial X)}^2 + \left(\frac{d}{2}-1\right)\|\nabla v\|_{L^2}^2$$
$$+ k\,\mathrm{rad}(X)\big(\|q\|_{L^\infty}\|v\|_{L^2}\|\nabla v\|_{L^2} + \|v\|_{L^2(\partial X)}\|\nabla v\|_{L^2(\partial X)} + \|\rho u\|_{L^2}\|\nabla v\|_{L^2}\big).$$

On the other hand, it follows from Young's inequality that

$$\|v\|_{L^2(\partial X)}\|\nabla v\|_{L^2(\partial X)} \leqslant \epsilon\|\nabla v\|_{L^2(\partial X)}^2 + \frac{1}{4\epsilon}\|v\|_{L^2(\partial X)}^2,$$

for all $\epsilon > 0$. We choose ϵ such that $k\epsilon = \gamma/2$ to get

$$\frac{k^2\,\mathrm{rad}(X)}{2}\|v\|_{L^2(\partial X)}^2 + k\,\mathrm{rad}(X)\|v\|_{L^2(\partial X)}\|\nabla v\|_{L^2(\partial X)}$$
$$\leqslant \frac{\gamma\,\mathrm{rad}(X)}{2}\|\nabla v\|_{L^2(\partial X)}^2 + \frac{k^2\,\mathrm{rad}(X)}{2}\frac{\gamma+1}{\gamma}\|v\|_{L^2(\partial X)}^2.$$

Recall (4.15). The left-hand side of the inequality above can be further bounded by

$$\frac{\gamma\,\mathrm{rad}(X)}{2}\|\nabla v\|^2_{L^2(\partial X)}+\frac{k^2\,\mathrm{rad}(X)}{2}\frac{\gamma+1}{\gamma}\left(\epsilon_1\|v\|^2_{L^2}+\frac{1}{4\epsilon_1}\|\rho u\|^2_{L^2}\right). \tag{4.17}$$

Applying Young's inequality to the term $\|\rho u\|_{L^2}\|\nabla v\|_{L^2}$ with $\epsilon k\,\mathrm{rad}(X)=1/8$ yields

$$k\,\mathrm{rad}(X)\|\rho u\|_{L^2}\|\nabla v\|_{L^2}\leqslant\frac{1}{8}\|\nabla v\|^2_{L^2}+2k^2\,\mathrm{rad}^2(X)\|\rho u\|^2_{L^2}. \tag{4.18}$$

Applying the same technique to the term $\|v\|_{L^2}\|\nabla v\|_{L^2}$ shows

$$k\,\mathrm{rad}(X)\|q\|_{L^\infty}\|v\|_{L^2}\|\nabla v\|_{L^2}\leqslant\frac{1}{8}\|\nabla v\|^2_{L^2}+2k^2\,\mathrm{rad}^2(X)\|q\|^2_{L^\infty}\|v\|^2_{L^2}. \tag{4.19}$$

Finally, recalling estimate (4.16) and combining the above inequalities, we have

$$\begin{aligned}\frac{d}{2}\|v\|^2_{L^2}\leqslant{}&\left(\frac{\mathrm{rad}(X)}{2}\frac{\gamma+1}{\gamma}\epsilon_1+\left(\frac{d}{2}-\frac{3}{4}\right)\left(1+k^{-2}\right)+2\|q\|^2_{L^\infty}\mathrm{rad}^2(X)\right)\|v\|^2_{L^2}\\&+\left(\frac{\mathrm{rad}(X)}{8\epsilon_1}\frac{\gamma+1}{\gamma}+2\,\mathrm{rad}^2(X)+\frac{1}{4}\left(\frac{d}{2}-\frac{3}{4}\right)\right)\|\rho u\|^2_{L^2}.\end{aligned} \tag{4.20}$$

Suppose that the wave number k is larger than 2 and the product $\|q\|_{L^\infty}\mathrm{rad}(X)$ is smaller than $1/4$. Then, if $4\epsilon_1$ is chosen to be $(\mathrm{rad}(X)(\gamma+1)/\gamma)^{-1}$, the coefficient in front of $\|v\|^2_{L^2}$ on the right is less than $5d/8-11/16$. Then $\|v\|_{L^2}$ term on the left dominates and we have

$$\left(\frac{11}{16}-\frac{d}{8}\right)\|v\|^2_{L^2}\leqslant\left(\frac{(\gamma+1)^2}{2\gamma^2}\mathrm{rad}^2(X)+2\,\mathrm{rad}^2(X)+\frac{1}{4}\left(\frac{d}{2}-\frac{3}{4}\right)\right)\|\rho u\|^2_{L^2}.$$

Estimate (4.9) follows from this immediately. □

Lemma 4.3. *Let η denote the constant (4.10). Suppose that the absorption coefficient q is such that*

$$\eta\|q\|_{L^\infty(X)}<\frac{1}{4}. \tag{4.21}$$

Suppose also that $|u[q]|$ is bounded from below by a positive number. Then the map $\mathcal{D}F[q]$, as an operator from $L^2(X)$ to $L^2(X)$, is invertible. Moreover,

$$\left\|DF[q]^{-1}\right\|_{\mathcal{L}(L^2(X))}\leqslant\frac{1}{\inf|u[q]|^2\sqrt{1-4\eta\|q\|_{L^\infty(X)}}}. \tag{4.22}$$

Proof. Define

$$T[q](\rho)=\left|u[q]\right|^{-2}\mathcal{D}F[q](\rho)-\rho.$$

It is not hard to see that T is compact since it can be decomposed as

$$\begin{aligned}T:L^2(X)\to H^1(X)&\hookrightarrow L^2(X)\to L^2(X)\\ \rho\mapsto v(\rho)&\mapsto v(\rho)\mapsto 2q\left|u[q]\right|^{-2}\Re\left(u[q]\bar{v}(\rho)\right).\end{aligned}$$

The continuity of maps in the diagram above can be deduced from Proposition 2.7 and the choice of g such that $|u[q]| > 0$ in $\overline{X}$.

On the other hand, a straightforward calculation shows that

$$\left\|\mathcal{D}F[q](\rho)\right\|_{L^2(X)}^2 \geqslant \inf\left|u[q]\right|^4 \|\rho\|_{L^2(X)}^2 \left[1 - 4\eta \|q\|_{L^\infty(X)}\right]. \tag{4.23}$$

In fact,

$$\begin{aligned}
\left\|\mathcal{D}F[q](\rho)\right\|_{L^2(X)}^2 &= \int_X \left[\rho^2 \left|u[q]\right|^4 + 4q^2 \Re^2\left(u[q]\bar{v}(\rho)\right) + 4q\rho\left|u[q]\right|^2 \Re\left(u[q]\bar{v}(\rho)\right)\right] dx \\
&\geqslant \inf\left|u[q]\right|^2 \int_X \left[\rho^2\left|u[q]\right|^2 + \frac{4q^2\Re^2(u[q]\bar{v}(\rho))}{|u[q]|^2} + 4q\Re\left(\rho u[q]\bar{v}(\rho)\right)\right] dx \\
&\geqslant \inf\left|u[q]\right|^2 \int_X \left[\rho^2\left|u[q]\right|^2 - 4\|q\|_{L^\infty(X)}\left|\rho u[q]\bar{v}(\rho)\right|\right] dx \\
&\geqslant \inf\left|u[q]\right|^2 \left[\left\|\rho\left|u[q]\right|\right\|_{L^2(X)}^2 - 4\|q\|_{L^\infty(X)}\left\|\rho u[q]\right\|_{L^2(X)}\left\|v(\rho)\right\|_{L^2(X)}\right] \\
&\geqslant \inf\left|u[q]\right|^2 \left\|\rho u[q]\right\|_{L^2(X)}^2 \left[1 - 4\eta\|q\|_{L^\infty(X)}\right].
\end{aligned}$$

Since $\eta\|q\|_{L^\infty(X)} < 1/4$, we find (4.23). It follows that the kernel of $\mathcal{D}F[q]$ is $\{0\}$. Hence, by the Fredholm theory, $\mathcal{D}F[q]$ is invertible. Moreover, (4.23) also implies (4.22). $\square$

Remark 4.4. Recall the definition of η in (4.10). When X is a ball, η is roughly three to four times the radius of X in dimensions three or two. Condition (4.21) hence requires that $\|q\|_{L^\infty}\operatorname{rad}(X)$, which can be interpreted as the typical absorption rate as signals propagate to the boundary, should be sufficiently small.

5. Measurement noise and resolution enhancement

In this section, we consider additive noise in the data set $\mathcal{E}$ given in (1.2).

5.1. Noise model

As described in Proposition 2.6, the data $\mathcal{E}$ are acquired by measuring several sets of absorbed radiations: $q|u_1 + u_j|^2$, $q|iu_1 + u_j|^2$, $q|u_1|^2$, and $q|u_j|^2$ for $j = 2, \dots, d+1$. In practice, the measurements of these absorbed energies are corrupted by additive noises. We model a typical energy measurement by

$$E^{\mathrm{m}}(x) = E(x) + \sigma W_\delta(x). \tag{5.1}$$

Here and in the sequel, the superscript "m" indicates measured quantity, and E itself is the pure quantity without noise. W_δ is a stationary random field with mean zero and covariance function of the form

$$\mathbb{E}\left[W_\delta(x)W_\delta(y)\right] = \mathbb{E}\left[W_\delta(0)W_\delta(x-y)\right] = R\left(\frac{x-y}{\delta}\right), \tag{5.2}$$

where R is an integrable function normalized so that $R(0) = 1$. In this additive noise model, σ^2 is the variance of the noise, δ is the correlation length which is related to the distance between measurement points.

The random process W_δ is assumed to be bounded almost surely by a constant independent of δ. This constant is assumed to be smaller than $E_{\min}$ which is a lower bound for the real energy. This technical hypothesis ensures that E^{m} is bounded from below by a positive constant for any $\sigma \leqslant 1$ and for any δ.

In the forthcoming analysis, both the noise variance σ and the noise correlation length δ will be supposed to be small. We assume that the measured data $\mathcal{E}^{\mathrm{m}} = (E_j^{\mathrm{m}})_{j=1}^{d+1}$ are given by

$$\begin{aligned} E_1^{\mathrm{m}}(x) &= E_1(x) + \sigma W_{\delta 1}(x), \\ E_j^{\mathrm{m}}(x) &= E_j(x) + \sigma U_{\delta j}(x) + i\sigma V_{\delta j}(x), \quad j = 2, \dots, d+1. \end{aligned} \tag{5.3}$$

According to the procedure of measuring E_j, the random fields $U_{\delta j}$ and $V_{\delta j}$ are given by $(W_{\delta 1 j} - W_{\delta 1} - W_{\delta j})/2$ and $(W_{\delta 1 j'} - W_{\delta 1} - W_{\delta j})/2$ respectively, where $W_{\delta j}$, $W_{\delta 1 j}$ and $W_{\delta 1 j'}$ correspond to the additive noises of the energy measurements $q|u_j|^2$, $q|u_1 + u_j|^2$ and $q|iu_1 + u_j|^2$, respectively. It is natural to assume that $W_{\delta 1}$, $W_{\delta j}$, $W_{\delta 1 j}$ and $W_{\delta 1 j'}$ are mutually independent and have the same statistical distribution as W_δ in (5.1). As a consequence, $U_{\delta j}$, $V_{\delta j}$ and $W_{\delta 1}$ are correlated.

5.2. *Initial guess with smoothed data*

We smooth the data $\mathcal{E}$ by using the convolution kernel

$$\varphi_\delta(x) := \frac{1}{\delta^{dp}} \varphi\left(\frac{x}{\delta^p}\right), \tag{5.4}$$

where $p \in (0, \frac{d}{d+6})$ and φ is in the Schwartz space of smooth non-negative functions that decay rapidly at infinity and that satisfy $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$. The condition $p < d/(d+6)$ will be clear later. The following lemma will be useful.

Lemma 5.1. *Let $|\gamma|$ denote the sum of all components of the multi-index γ and $\partial^\gamma \varphi$ (resp. $\partial^\gamma \varphi_\delta$) denote the usual γ-partial derivative of φ (resp. φ_δ). For any δ we have*

$$\mathbb{E}\left|W_{1\delta} * \partial^\gamma \varphi_\delta\right|^2 \leqslant \delta^{d-(d+2|\gamma|)p} \|R\|_{L^1(\mathbb{R}^d)} \left\|\partial^\gamma \varphi\right\|^2_{L^2(\mathbb{R}^d)}. \tag{5.5}$$

More precisely, for $\delta \ll 1$, we have

$$\mathbb{E}\left|W_{1\delta} * \partial^\gamma \varphi_\delta\right|^2 = \delta^{d-(d+2|\gamma|)p} \int_{\mathbb{R}^d} R(y)\,dy \int_{\mathbb{R}^d} \left|\partial^\gamma \varphi\big(y'\big)\right|^2 dy' + o\big(\delta^{d-(d+2|\gamma|)p}\big). \tag{5.6}$$

Proof. The variance (5.5) can be written as

$$\begin{aligned} \mathbb{E}\left|W_{1\delta} * \partial^\gamma \varphi_\delta\right|^2 &= \mathbb{E} \frac{1}{\delta^{2p|\gamma|+2dp}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W_{1\delta}(x-y) W_{1\delta}\big(x-y'\big) \big(\partial^\gamma \varphi\big)\left(\frac{y}{\delta^p}\right) \big(\partial^\gamma \varphi\big)\left(\frac{y'}{\delta^p}\right) dy\,dy' \\ &= \frac{1}{\delta^{2p|\gamma|+2dp}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} R\left(\frac{y-y'}{\delta}\right) \big(\partial^\gamma \varphi\big)\left(\frac{y}{\delta^p}\right) \big(\partial^\gamma \varphi\big)\left(\frac{y'}{\delta^p}\right) dy\,dy'. \end{aligned}$$

We apply the change of variable $(y-y')/\delta \to y'$ and $y/\delta^p \to y$, and take advantage of the resulting Jacobian. We verify that the variance can be written as

$$\mathbb{E}\left|W_{1\delta} * \partial^\gamma \varphi_\delta\right|^2 = \delta^{d+dp-2p|\gamma|-2dp} \int_{\mathbb{R}^d} R\left(y'\right) \int_{\mathbb{R}^d} \left(\partial^\gamma \varphi\right)(y)\left(\partial^\gamma \varphi\right)\left(y - \delta^{1-p} y'\right) dy\, dy'.$$

Using Cauchy–Schwarz inequality and the fact that $\partial^\gamma \varphi \in L^2$ and $R \in L^1$, we obtain (5.5). Since $\partial^\gamma \varphi \in L^2$, $p < 1$, and R is integrable, (5.6) is also easily verified by the dominated convergence theorem. □

Remark 5.2. The above calculation works also for $U_{j\delta}$ and $V_{j\delta}$.

We smooth the data by evaluating the convolution with the kernel φ_δ:

$$E_j^{\mathrm{s}} = E_j^{\mathrm{m}} * \varphi_\delta, \quad j = 1, \ldots, d+1, \tag{5.7}$$

which gives

$$E_1^{\mathrm{s}} = E_1 * \varphi_\delta + \sigma W_{1\delta} * \varphi_\delta, \tag{5.8}$$

$$E_j^{\mathrm{s}} = E_j * \varphi_\delta + \sigma U_{j\delta} * \varphi_\delta + i\sigma V_{j\delta} * \varphi_\delta, \quad j = 2, \ldots, d+1. \tag{5.9}$$

Here and below, the superscript "s" indicates smoothed quantities. The parameter δ^p can be interpreted as the size of the averaging window. To simplify the notation, $E_{j\delta}$ will be used as the short-hand notation for the smoothed unperturbed data $E_j * \varphi_\delta$ in the sequel.

Proposition 5.3. *If we substitute the smoothed measured data $(E_j^{\mathrm{s}})_{j=1}^{d+1}$ into the reconstruction formula (3.9):*

$$q^{\mathrm{s}}(x) = \frac{-\Re(a^{\mathrm{s}})\Im(a^{\mathrm{s}}) + \operatorname{div}\Im(a^{\mathrm{s}})}{2k}, \tag{5.10}$$

with $a^{\mathrm{s}} = (A^{\mathrm{s}})^{-1}[(\nabla^T(A^{\mathrm{s}})^T)^T]$, $A^{\mathrm{s}} = (\partial_l \alpha_{j+1}^{\mathrm{s}})_{j,l=1,\ldots,d}$, and $\alpha_j^{\mathrm{s}} = E_j^{\mathrm{s}}/E_1^{\mathrm{s}}$, then the estimate q^{s} satisfies:

$$\sup_{x\in X} \mathbb{E}\left[\left|q^{\mathrm{s}}(x) - q_\delta(x)\right|^2\right] \leqslant C\sigma^2 \delta^{d-(d+6)p}, \tag{5.11}$$

where

$$q_\delta(x) = \frac{-\Re(a^\delta)\Im(a^\delta) + \operatorname{div}\Im(a^\delta)}{2k}$$

is obtained by substituting the smoothed unperturbed data $(E_{j\delta})_{j=1}^{d+1}$ into the reconstruction formula (3.9).

Proof. We substitute the smoothed data $(E_j^{\mathrm{s}})_{j=1}^{d+1}$ into the reconstruction formula (3.9). Recall the definitions of A and α_j in (3.5) and (3.1). Then,

$$\alpha_j^{\mathrm{s}} = \frac{E_{j\delta} + \sigma U_{j\delta} * \varphi_\delta + i\sigma V_{j\delta} * \varphi_\delta}{E_{1\delta} + \sigma W_{1\delta} * \varphi_\delta}. \tag{5.12}$$

When $\sigma \ll 1$, we can linearize this term and find that

$$\alpha_j^s = \frac{E_{j\delta}}{E_{1\delta}} - \sigma \frac{W_{1\delta} * \varphi_\delta}{E_{1\delta}} \frac{E_{j\delta}}{E_{1\delta}} + \sigma \frac{U_{j\delta} * \varphi_\delta}{E_{1\delta}} + i\sigma \frac{V_{j\delta} * \varphi_\delta}{E_{1\delta}} + O\big(\sigma^2\big). \tag{5.13}$$

The coefficients of the matrix A^s are defined by $A^s_{jl} = \partial_l \alpha^s_{j+1}$ and they can be expanded from (5.12) as

$$A^s_{jl} = A^\delta_{jl} + \sigma A^{\delta(1)}_{jl} + o\big(\sigma \delta^{d/2-(d+2)p/2}\big), \quad 1 \leqslant j, l \leqslant d, \tag{5.14}$$

where

$$A^\delta_{jl} = \partial_l \frac{E_{j+1\delta}}{E_{1\delta}}, \qquad A^{\delta(1)}_{jl} = -\frac{W_{1\delta} * \partial_l \varphi_\delta}{E_{1\delta}} \frac{E_{j+1\delta}}{E_{1\delta}} + \frac{U_{j+1\delta} * \partial_l \varphi_\delta}{E_{1\delta}} + i \frac{V_{j+1\delta} * \partial_l \varphi_\delta}{E_{1\delta}}.$$

The leading-order error terms $\sigma A^{\delta(1)}_{jl}$ have zero means and their variances are of order $O(\sigma^2 \delta^{d-(d+2)p})$ according to Lemma 5.1, provided that the functions E_j's are sufficiently smooth with bounded derivatives. The following error terms like $W_{1\delta} * \varphi_\delta \partial_l (\frac{E_{j+1\delta}}{E^2_{1\delta}})$ are smaller since their square means are of order $O(\sigma^2 \delta^{d-dp})$.

Since A^δ is a smoothed version of A, which was defined in (3.5) and whose determinant can be bounded from below by a large constant (see Proposition 2.3), the inverse of A^δ is well-defined. Linearizing $(A^s)^{-1}$, we have

$$\big(A^s\big)^{-1} = \big(A^\delta\big)^{-1} + \sigma \big(A^\delta\big)^{-1} A^{\delta(1)} \big(A^\delta\big)^{-1} + o\big(\sigma \delta^{d/2-(d+2)p/2}\big).$$

Similarly, the vector $(\nabla^T A^{s^T})^T$ can be decomposed as

$$\begin{aligned}\big(\nabla^T A^{s^T}\big)_j &= \big(\nabla^T A^T\big)_j + \sigma \left(-\frac{W_{1\delta} * \Delta\varphi_\delta}{E_{1\delta}} \frac{E_{j+1\delta}}{E_{1\delta}} + \frac{U_{j+1\delta} * \Delta\varphi_\delta}{E_{1\delta}} + i \frac{V_{j+1\delta} * \Delta\varphi_\delta}{E_{1\delta}} \right) \\ &\quad + o\big(\sigma \delta^{d/2-(d+4)p/2}\big).\end{aligned}$$

Finally, we have for the vector $a^s = (A^s)^{-1} (\nabla^T A^{s^T})^T$:

$$\begin{aligned} a^s_j &= a^\delta_j + \sigma \sum_{l=1}^d \big(A^\delta\big)^{-1}_{jl} \left(-\frac{W_{1\delta} * \Delta\varphi_\delta}{E_{1\delta}} \frac{E_{l+1\delta}}{E_{1\delta}} + \frac{U_{l+1\delta} * \Delta\varphi_\delta}{E_{1\delta}} + i \frac{V_{l+1\delta} * \Delta\varphi_\delta}{E_{1\delta}} \right) \\ &\quad + o\big(\sigma \delta^{d/2-(d+4)p/2}\big), \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} a^s &= \operatorname{div} a^\delta + \sigma \sum_{j,l=1}^d \big(A^\delta\big)^{-1}_{jl} \left(-\frac{W_{1\delta} * (\partial_j \Delta\varphi)_\delta}{E_{1\delta}} \frac{E_{l+1\delta}}{E_{1\delta}} + \frac{U_{l+1\delta} * (\partial_j \Delta\varphi)_\delta}{E_{1\delta}} + i \frac{V_{l+1\delta} * (\partial_j \Delta\varphi)_\delta}{E_{1\delta}} \right) \\ &\quad + o\big(\sigma \delta^{d/2-(d+6)p/2}\big). \end{aligned}$$

The vector $a^\delta = (A^\delta)^{-1} (\nabla^T A^{\delta^T})^T$ is obtained by applying formulas (3.1) and (3.5) to the smoothed unperturbed data $(E_{j\delta})_{j=1}^{d+1}$. The leading-order error terms have zero means and their variances are of order $O(\sigma^2 \delta^{d-(d+6)p})$ according to Lemma 5.1. Our choice $p < \frac{d}{d+6}$ guarantees that the noisy data are smoothed enough so that the terms above have variance of order smaller than σ^2. To summarize, if we apply (3.9) to the smoothed data $(E^s_j)_{j=1}^{d+1}$, then we get

$$q^s(x) = q_\delta(x) - \frac{\sigma}{2k}\left\{ \sum_{j,l=1}^{d} \Im\left(A^\delta\right)_{jl}^{-1}\left(-\frac{W_{1\delta} * \partial_j \Delta\varphi_\delta}{E_{1\delta}}\frac{E_{l+1\delta}}{E_{1\delta}} + \frac{U_{l+1\delta} * \partial_j \Delta\varphi_\delta}{E_{1\delta}}\right) + \Re\left(A^\delta\right)_{jl}^{-1}\frac{V_{l+1\delta} * \partial_j \Delta\varphi_\delta}{E_{1\delta}}\right\} + o\left(\sigma\delta^{d/2-(d+6)p/2}\right), \tag{5.15}$$

from which we deduce the desired result. □

The terms q_δ can be shown to be close to the real absorption parameter q_o uniformly in x (we show this in Theorem 5.4). However, it is impossible to separate q_δ from the noise, that is the other terms in (5.15). Nevertheless, the estimate q^s is a good initial guess in the mean square sense as shown by the following theorem.

Theorem 5.4. *Suppose that the pure data $(E_j)_{j=1}^{d+1}$ belong to $C^{3,\varepsilon}$ for some positive real number ε. Then, we have*

$$\|q_\delta - q_o\|_{L^\infty(X)} \leqslant C\delta^{\varepsilon p}. \tag{5.16}$$

As a result, estimate (5.10) obtained from the smoothed data satisfies

$$\sup_{x\in X}\mathbb{E}\left[\left|q^s(x) - q_o(x)\right|^2\right] \leqslant C\left(\delta^{2\varepsilon p} + \sigma^2\delta^{d-(d+6)p}\right). \tag{5.17}$$

Proof. Under the conditions of the theorem, the inequality $|\partial^\gamma E_j(x-y) - \partial^\gamma E_j(x)| \leqslant C|y|^\varepsilon$ holds for some constant C and for any multi-index γ with $|\gamma| \leqslant 3$. As a result, we have the following estimate as an analog of Lemma 5.1:

$$\left|\partial^\gamma E_{j\delta}(x) - \partial^\gamma E_j(x)\right| = \left|\frac{1}{\delta^{dp}}\int_{\mathbb{R}^d}\left(\partial^\gamma E_j(x-y) - \partial^\gamma E_j(x)\right)\varphi\left(\frac{y}{\delta^p}\right)dy\right| \leqslant C\frac{1}{\delta^{dp}}\int_{\mathbb{R}^d}|y|^\varepsilon\left|\varphi\left(\frac{y}{\delta^p}\right)\right|dy = C\delta^{\varepsilon p}\int_{\mathbb{R}^d}|y|^\varepsilon\left|\varphi(y)\right|dy \leqslant C\delta^{\varepsilon p}. \tag{5.18}$$

Then the estimate of q_δ follows because the reconstruction formula in (3.9) depends continuously on the data and their derivatives. For the second estimate, we apply the triangle inequality and use the control of the stochastic terms in the linearization procedure. □

Remark 5.5. Estimate (5.17) is a bit over pessimistic. Indeed, it does not imply that q^s is positive, which is a physical constraint for the absorption parameter. We will exploit this remark in the next section.

5.3. The optimization step and resolution enhancement

Now we refine the above initial guess q^s by an optimal control approach. We seek for the least square estimate of the discrepancy functional

$$J[q] = \int_X \left|F[q](x) - E_1^s(x)\right|^2 dx. \tag{5.19}$$

Here, E_1^s is the smoothed data $(E_1 + \sigma W_{1\delta}) * \varphi_\delta$ and $F[q] = q|u_1[q]|^2$ is the absorbed heat energy with boundary condition g_1.

In Theorem 5.4, the initial guess q^s is shown to be close to the true absorption coefficient. This allows us to approximate the integrand in the definition of J by its linearization around q^s; that is,

$$J[q] \approx \int_X \left|\mathcal{D}F\left[q^s\right]\left(q - q^s\right) - b^s\right|^2 dx, \tag{5.20}$$

where $b^s = E_1^s - F[q^s]$ is the residue. In the case when $\mathcal{D}F[q^s]$ is invertible from L^2 to L^2, the least square solution of the approximate discrepancy functional is given by

$$q_* = q^s + \left(\mathcal{D}F\left[q^s\right]\right)^{-1} b^s. \tag{5.21}$$

The following result shows that q_* is a refinement of q^s in the mean square sense (compared to Theorem 5.4).

Theorem 5.6. *Recall that q_o denotes the true absorption coefficient and assume that the condition in Theorem 5.4 holds. We have*

$$\mathbb{E}\left[\|q_* - q_o\|_{L^2(X)}^2\right] = o\left(\delta^{2\varepsilon p} + \sigma^2 \delta^{d-(d+6)p}\right) + O\left(\delta^{2p} + \sigma^2 \delta^{d(1-p)}\right). \tag{5.22}$$

Proof. From the definition of b^s and E_1^s, the residue can be expanded as

$$b^s = E_1 - F\left[q^s\right] + (E_{1\delta} - E_1) + \sigma W_{1\delta} * \varphi_\delta.$$

Since $E_1 = F[q_o]$, the difference $F[q_o] - F[q^s]$ can be linearized as $\mathcal{D}F[q^s](q_o - q^s) + o(q_o - q^s)$. This, together with (5.21), implies

$$q_* - q_o = \left(\mathcal{D}F\left[q^s\right]\right)^{-1}\left\{\sigma W_{1\delta} * \varphi_\delta + (E_{1\delta} - E_1) + o\left(q_o - q^s\right)\right\}. \tag{5.23}$$

Lemma 5.1 shows that $\sigma W_{1\delta} * \varphi_\delta$ has mean square of order $\sigma^2 \delta^{d(1-p)}$; the calculation in (5.18) shows that $E_{1\delta} - E_1$ can be bounded uniformly by $C\delta^p$; the term $q_o - q^s$ is also controlled in (5.17). Consequently, since $\mathcal{D}F[q^s]$ has bounded inverse (see Lemma 4.3), the desired estimate holds. □

Remark 5.7. Assume that q_o is bounded from below and above by two known positive numbers $q_{\min}$ and $q_{\max}$. Let

$$\hat{q}_* = \min\left\{\max\{q_*, q_{\min}\}, q_{\max}\right\} \in [q_{\min}, q_{\max}].$$

We can see that

$$\|\hat{q}_* - q_o\|_{L^2(X)} \leqslant \|q_* - q_o\|_{L^2(X)}.$$

We note that there is no guarantee that q_* is positive, but the modified version $\hat{q}_*$ is. In addition to this advantage, the estimate above shows that $\hat{q}_*$ is a better approximation of q_o in comparison with q_*. Further, the range of $\hat{q}_*$ allows us to make iterations for further corrections.

Remark 5.8. Finally, we note that the above result also shows that the optimization step enhances the resolution. In fact, from (5.21) it follows that q_* contains higher oscillations than q^s and therefore, yields a more resolved approximation of q_o.

6. Conclusion

In this paper we have derived an exact reconstruction formula for the absorption coefficient from thermo-acoustic data associated with a proper set of measurements. Using a noise model for the data, we have regularized this formula in order to obtain a good initial guess. We have also performed a refinement of the initial guess using an optimal control approach and shown that this procedure reduces the occurring errors and yields a resolution enhancement. A challenging problem is to estimate analytically the resolution. It would be also very interesting to study the reconstruction problem in the case of incomplete measurements, where the thermal energy is known only on an open subset of the domain. The numerical implementation of our approach in this paper is the subject of forthcoming work, which will be published elsewhere.

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