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A UNIFIED HOMOGENIZATION APPROACH FOR THE DIRICHLET PROBLEM IN PERFORATED DOMAINS*

WENJIA JING†

Abstract. We revisit the periodic homogenization of Dirichlet problems for the Laplace operator in perforated domains and establish a unified proof that works for different regimes of hole-cell ratios, which is the ratio between the scaling factor of the holes and that of the periodic cells. The approach is then made quantitative and yields correctors and error estimates for vanishing hole-cell ratios. For a positive volume fraction of holes, the approach is just the standard oscillating test function method; for a vanishing volume fraction of holes, we study asymptotic behaviors of properly rescaled cell problems and use them to build oscillating test functions. Our method reveals how the different regimes are intrinsically connected through the cell problems and the connection with periodic layer potentials.

Key words. periodic homogenization, perforated domain, strange term coming from nowhere, adaptive oscillating test function method, Newtonian capacity, large box limit

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1. Introduction.

In this article we revisit the periodic homogenization problem in perforated domains that are formed by removing a periodic array of small holes from a fixed open bounded and connected domain with regular boundary. For simplicity, we restrict the analysis to the Dirichlet problem for the Laplace operator on such domains. Our goal is to develop an approach that yields both qualitative homogenization and also error estimates and that is adaptive with respect to the ratios between the size of the holes and the periodicity of the array. The problem of interest is, hence, to study the large-scale behavior of $u^{\varepsilon,a_\varepsilon}$ that solves the following problem:

$$\begin{cases} -\Delta u^{\varepsilon,a_\varepsilon}(x) = g(x), & x \in D^{\varepsilon,a_\varepsilon}, \\ u^{\varepsilon,a_\varepsilon}(x) = 0, & x \in \partial D^{\varepsilon,a_\varepsilon}. \end{cases} \tag{1.1}$$

Here $D^{\varepsilon,a_\varepsilon} \subset \mathbb{R}^d$, $d \geq 2$, denotes the perforated domain as described above, and ε and a_ε are small positive numbers which represent roughly the sizes of the periodicity and the holes, respectively. The source term g is assumed to be sufficiently regular, say, in $L^2(D)$. Homogenization then amounts to finding an effective problem posed on the regular domain D, which catches the behavior of the problem above as $\varepsilon \to 0$, and to the quantification of the errors. Such problems find natural applications in the study of conductivity behaviors of porous media and hence have attracted many modeling and analysis efforts in the applied mathematics community.

We now set up the mathematical model for $D^{\varepsilon,a_\varepsilon}$ here and use it throughout the paper. Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded and connected domain, with $C^{1,\alpha}$ boundary ∂D for some $\alpha \in (0,1)$. To obtain $D^{\varepsilon,a_\varepsilon}$, we remove from D a periodic array of small sets. Those sets are rescaled from unit ones, so we start with the scenario at

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†Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, People's Republic of China (wjjing@tsinghua.edu.cn).

the unit scale. Let $Q := (-\frac{1}{2}, \frac{1}{2})^d$ be the unit cube centered at the origin with side length one. Let $T \subset Q$ be an open set satisfying

$$\overline{B}_{\frac{1}{16}}(0) \subset T \subset \overline{T} \subset B_{\frac{1}{4}}(0), \tag{1.2}$$

and with $C^{1,\alpha}$ boundary ∂T. For each $\eta \in (0,1]$, we set $Y_{f,\eta} := Q \setminus \eta\overline{T}$. Then $Y_{f,\eta}$ is the part of Q with the set $\eta\overline{T}$ removed; for $\eta = 1$, we use Y_f for $Y_{f,1}$. The subscript "f" refers to the "fluid" or material part of the domain.

We then take copies of $Y_{f,\eta}$ and glue them together to form the perforated whole space

$$\mathbb{R}^d_{f,\eta} := \cup_{k\in\mathbb{Z}^d} (k + Y_{f,\eta} \cup \partial Q) = \mathbb{R}^d \setminus \cup_{k\in\mathbb{Z}^d} \left(k + \eta\overline{T}\right).$$

The second representation above says that $\mathbb{R}^d_{f,\eta}$ is constructed by removing from $\mathbb{R}^d$ the $\mathbb{Z}^d$-translated copies of the model hole $\eta\overline{T}$. We then rescale $\mathbb{R}^d_{f,\eta}$ by a small positive number ε and get $\varepsilon\mathbb{R}^d_{f,\eta}$. In this scaled perforated whole set, the holes are rescaled from T by a factor of $\eta\varepsilon$, and each hole is enclosed by a periodic cell rescaled from Q by a factor of ε.

The perforated domain $D^{\varepsilon,a_\varepsilon}$ of this paper is then defined by

$$D^{\varepsilon,a_\varepsilon} := D \cap (\varepsilon\mathbb{R}^d_{f,\eta_\varepsilon}), \qquad \text{with} \quad \eta_\varepsilon := a_\varepsilon/\varepsilon.$$

In other words, $D^{\varepsilon,a_\varepsilon}$ is obtained by cutting the piece of $\varepsilon\mathbb{R}^d_{f,\eta_\varepsilon}$ that is inside the set D fixed earlier. The boundary of $D^{\varepsilon,a_\varepsilon}$ hence consists of ∂D and the union of the boundaries of the holes. To avoid non-Lipschitz domains we modify the above definition a little bit: for those $k \in \mathbb{Z}^d$ such that $\partial D \cap \varepsilon(k + \partial Q)$ is nonempty, we add the set $\varepsilon(k+\eta\overline{T})$ back to $D^{\varepsilon,a_\varepsilon}$, so the holes inside (the modified) $D^{\varepsilon,a_\varepsilon}$ are separated from the outer boundary ∂D by ε.

Through out the paper, we assume the following:

(P1) For each $\varepsilon \in (0,1)$, a_ε is chosen in $(0,\varepsilon)$, the reference hole T satisfies (1.2), ∂D and ∂T are of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, and $D^\varepsilon = D^{\varepsilon,a_\varepsilon}$ is constructed as above, where $\eta_\varepsilon = \frac{a_\varepsilon}{\varepsilon}$ is the ratio between the scaling factors of the holes and the periodic cells.

(P2) The source term g belongs to $L^2(D)$.

In the rest of the paper, we simplify the notation $D^{\varepsilon,a_\varepsilon}$ and $u^{\varepsilon,a_\varepsilon}$ to D^ε and u^ε.

Our goal is to find the asymptotic behavior of u^ε as ε goes to 0. Let us first consider the setting where $a_\varepsilon = \varepsilon$. In this case, the domain D^ε has two distinguished scales: the outer boundary ∂D has length scale one, and the boundary of holes has scale ε. We use the formal two-scale expansion method and consider the ansatz

$$u^\varepsilon = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

where $u_i : \mathbb{R}^d \times Y_f \to \mathbb{R}$, $i = 0, 1, 2, \ldots$, and moreover, for each fixed x, the function $y \mapsto u_i(x,y)$ satisfies periodic condition at the outer boundary $\partial Y_f \cap \partial Q$ and the Dirichlet boundary condition $u_i = 0$ at $\partial Y_f \cap Q$. Apply this ansatz in (1.1), replace ∇ by $\nabla_x + \frac{1}{\varepsilon}\nabla_y$, and identify terms of the same order in ε. We get

$$\begin{aligned}
&-\Delta_y u_0(x,y) = 0,\\
&-\Delta_y u_1(x,y) + 2\nabla_x \cdot \nabla_y u_0(x,y) = 0,\\
&-\Delta_y u_2(x,y) + 2\nabla_x \cdot \nabla_y u_1(x,y) = g(x).
\end{aligned}$$

We view those equations as posed on Y_f with periodic boundary condition at ∂Q and Dirichlet condition at $\partial Y_f \cap Q$, and the x-variable as a parameter. Then u_0 has to be zero, and so does u_1. The equation for u_2 reduces to $-\Delta_y u_2(x,y) = g(x)$. By linearity and uniqueness, we get $u_2(x,y) = g(x)\chi(y)$, where $\chi : \mathbb{R}^d_f \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic and solves the *cell problem*:

$$\begin{cases} -\Delta_y \chi(y) = 1, & y \in Y_f, \\ \chi(y) = 0, & y \in \partial T. \end{cases} \tag{1.3}$$

Elliptic theory shows that χ is uniquely determined. Then formally we have $u^\varepsilon(x) \approx \varepsilon^2 g(x)\chi(\frac{x}{\varepsilon})$, and it follows that $u^\varepsilon/\varepsilon^2$ converges weakly in L^2 to $c_* g$, where $c_* = \int_{Y_f} \chi$. We conclude that u^ε is of order ε^2 and the limit of $u^\varepsilon/\varepsilon^2$ is given by an algebraic equation. In some sense, this is analogous to the homogenization derivation of Darcy's law for the Stokes problem in periodically perforated domains. The formal two-scale expansion derivations for the Stokes system goes back at least to Keller [23] and Sanchez-Palencia [27], and the rigorous proof was first obtained by Tartar [28]. Tartar's proof is based on oscillating test functions built from rescalings of the cell problem. The approach of Tartar was then generalized to the evolutional setting by Mikelić [25] and to the compressible setting by Masmoudi [24]. We also refer the reader to the books [7] and [19] for more comprehensive introductions to the homogenization theory.

As long as different scalings for the holes and cells are considered, Cioranescu and Murat were the first to identify the critical scaling ratio between a_ε and ε below and above which u^ε has quite distinguished behaviors; see [9, 10]. The critical relative order for a_ε is given by

$$a^\varepsilon_* \sim \varepsilon^{\frac{d}{d-2}}, \quad d \ge 3, \qquad \text{and} \qquad \log a^\varepsilon_* \sim -\frac{1}{\varepsilon^2}, \quad d = 2. \tag{1.4}$$

Under those critical scalings, say, when equality holds above, the zero extension of u^ε is of order one and converges weakly in $H^1(D)$ to u, the solution of the effective problem

$$\begin{cases} -\Delta u(x) + \mu_* u(x) = g(x), & x \in D, \\ u(x) = 0, & x \in \partial D, \end{cases} \tag{1.5}$$

where μ_* is a positive constant. In fact, μ_* is the Newtonian capacity of the set T for $d \ge 3$ and equals 2π (which is the logarithmic capacity) for $d = 2$; see section 2. On the other hand, for $a_\varepsilon \ll a^\varepsilon_*$, the limit of u^ε is the solution to

$$\begin{cases} -\Delta u(x) = g(x), & x \in D, \\ u(x) = 0, & x \in \partial D. \end{cases} \tag{1.6}$$

In other words, in the setting of $a_\varepsilon \ll a^\varepsilon_*$, the homogenized problem does not see the holes. Due to this comparison, Cioranescu and Murat named the term $\mu_* u$ in the critical scaling the "*strange term coming from nowhere*."

Scaling regimes. The critical scaling a^ε_* comes from the application of the Poincaré inequality (A.1) that is recorded in the appendix. Given the scaling factors ε and a_ε, let $\eta = \eta(\varepsilon) = a_\varepsilon/\varepsilon$. We then follow Allaire [1] and define the parameter

$$\sigma_\varepsilon = \varepsilon \eta^{-\frac{d-2}{2}}, \quad d \ge 3, \qquad \text{and} \qquad \sigma_\varepsilon = \varepsilon |\log \eta|^{\frac{1}{2}}, \quad d = 2. \tag{1.7}$$

The critical scaling a_*^ε is determined by setting σ_ε to be of order one. In the rest of the paper, by *positive (limiting) volume fraction* regime we mean $a_\varepsilon \sim \varepsilon$, and *vanishing volume fraction* refers to the situation of $a_\varepsilon/\varepsilon \to 0$. For the latter case, we consider three subcases. If $a_\varepsilon \sim a_*^\varepsilon$, or equivalently $\sigma_\varepsilon \sim 1$, we call it the *"critical hole-cell ratio"* setting; similarly, *"subcritical hole-cell ratio"* refers to the situation of $a_\varepsilon \ll a_*^\varepsilon$ (note that $\sigma_\varepsilon \gg 1$), and *"supercritical hole-cell ratio"* refers to the case when $a_*^\varepsilon \ll a_\varepsilon$, or $\sigma_\varepsilon \ll 1$.

In the pioneering work of Cioranescu and Murat [9, 10], a general framework was developed for the setting of vanishing volume fractions of holes for the Dirichlet problem of the Laplace operator. Moreover, a corresponding framework for quantification was established by Kacimi and Murat [21]. The framework was later extended by Allaire to Stokes and Navier–Stokes problems [1, 2], to the obstacle problems by Caffarelli and Mellet [8], and to random settings by Hoàng [18, 17]; see also [13], and see [16, 15] for randomly perforated domains based on Poisson point processes.

The framework in [9] was set for holes with vanishing volume fractions, and there is no reference to the cell problems. The latter, on the other hand, is central to the framework of Tartar for the more classical setting of $a_\varepsilon \sim \varepsilon$. A natural question arises to find the connection between the two frameworks. As far as we know, Allaire was the first to consider such a relation for the Stokes problem in perforated media. In [3] he found that the strange term in the critical setting (which corresponds to so-called Brinkman's law) is the inverse of the permeability tensor in the supercritical setting with vanishing volume fraction of holes (which corresponds to Darcy's laws), and this permeability tensor is the limit of the permeability tensors associated to the cell problems posed on a large periodic cell with a model hole removed.

Inspired by this work of Allaire, we develop in this paper a unified homogenization approach for (1.1) that is based on the oscillating test function method and on the analysis of cell problems, and it works for all regimes of hole-cell ratios. For vanishing volume fraction of holes, we need to consider rescaled cell problems posed in a large periodic box whose side length is the inverse of the hole-cell ratio, and with the model hole T removed. The large box limit of the cell problems is obtained by a compactness method very similar to that of Allaire [3]. A major profit of this new adaptive approach is that it can be easily quantified. The key tool we use for the quantifications is the large box limit of periodic potential theory; our results on this are interesting in their own right.

Before concluding this introduction, we remark that the Dirichlet condition at the holes is crucial to make the scaling differences mentioned above. If, for instance, the Neumann boundary $\nu\cdot\nabla u^\varepsilon = 0$ is used instead, the homogenization procedure will not distinguish a critical scaling for a_ε. This is because u^ε will not have a natural extension that has a priori smallness, and the Poincaré inequality (A.1) will be irrelevant; see, e.g., [4] for a detailed treatment of the Neumann problem. We also refer the reader to [26, 5, 20] for homogenization with transmission conditions across the boundaries of the holes.

The rest of the paper is organized as follows: In section 2 we state the main theorems, which include not only qualitative homogenization but also error estimates for vanishing volume fractions. The key large box limit of the cell problem is studied in section 3, following a weak compactness method outlined by Allaire in [3]. In section 4 we prove the homogenization results by an adaptive oscillating test function method that works for various regimes of hole-cell ratios. In section 5 we quantify the homogenization and obtain error estimates. Although the unified approach for homogenization works for all spatial dimensions $d \geq 2$, the two-dimensional setting is different in terms of technical details. Hence, the proofs in sections 4 and 5 are restricted to $d \geq 3$, and in section 6 we present the main modifications that are needed for $d = 2$. In the appendix, we record facts on a Poincaré inequality and on potential theory of the Laplace operator; they are the key tools in our analysis.

Notation. We list some special notation that will be used in the paper. Given $a, b \in [0, \infty)$, we use $a \wedge b$ for the minimum of the two numbers, and $a \vee b$ for the maximum. We use $\mathscr{D}^{1,2}(\mathbb{R}^d)$ for the completion of smooth and compactly supported functions with respect to their norm $\|\nabla\varphi\|_{L^2(\mathbb{R}^d)}$. Let E be an open set of $\mathbb{R}^d$ with smooth boundary; if f is in the Sobolev space $H^1(E)$ and the trace of f on ∂E vanishes, we denote by $\widetilde{f}$ the zero extension of f; that is, $\widetilde{f} : \mathbb{R}^d \to \mathbb{R}$ with $\widetilde{f} = f$ on E and $\widetilde{f} = 0$ in $\mathbb{R}^d \setminus \overline{E}$. For a measurable function $f \in L^1(E; \mathbb{R})$ where E is bounded, we denote by $\langle f \rangle_E$ or $⨍_E f$ the average $\frac{1}{|E|}\int_E f$; similarly, $\langle f \rangle_S$ and $⨍_S f$ both denote the average of f over a hypersurface S in $\mathbb{R}^d$, where the induced surface measure is used in the integral. We denote cubes with unit side length by Q, and by rQ or Q_r the dilation of Q with respect to its center by a factor of r; similarly, B, B_r, and rB are the corresponding notation for balls.

2. Preliminaries and the main results.

We start this section by the energy estimates for (1.1). Throughout the paper we only consider the L^2 setting. Under the usual assumptions, for each fixed ε and a_ε, the basic elliptic PDE theory yields that (1.1) admits a unique solution $u^\varepsilon \in H^1_0(D^\varepsilon)$. Let $\widetilde{u}^\varepsilon$ be the extension of u^ε to D by zero. Then the following estimates hold.

LEMMA 2.1. *Assume* (P1) *and* (P2). *Then there exists a positive constant C depending only on the model hole T, the whole set D, and the spatial dimension d, such that for all ε and a_ε, the unique solution u^ε of* (1.1) *satisfies*

$$\|\widetilde{u}^\varepsilon\|_{L^2(D)} \leq C(1 \wedge \sigma_\varepsilon^2)\|g\|_{L^2(D)}, \quad \|\nabla\widetilde{u}^\varepsilon\|_{L^2(D)} \leq C(1 \wedge \sigma_\varepsilon)\|g\|_{L^2(D)}. \tag{2.1}$$

Proof. Integrating (1.1) against u^ε itself, we get

$$\int_D |\nabla\widetilde{u}^\varepsilon|^2 = \int_{D^\varepsilon} |\nabla u^\varepsilon|^2 = \int_{D^\varepsilon} g u^\varepsilon = \int_D g\widetilde{u}^\varepsilon.$$

On the one hand, since $\widetilde{u}^\varepsilon \in H^1_0(D)$ and D has bounded diameter, by the Poincaré inequality we can find some constant C depending only on D and d such that

$$\|\widetilde{u}^\varepsilon\|_{L^2(D)} \leq C\|\nabla\widetilde{u}^\varepsilon\|_{L^2(D)}.$$

We then deduce that

$$\|\nabla\widetilde{u}^\varepsilon\|_{L^2(D)} \leq C\|g\|_{L^2(D)} \qquad \text{and} \qquad \|\widetilde{u}^\varepsilon\|_{L^2(D)} \leq C^2\|g\|_{L^2(D)}.$$

On the other hand, the integral of $|\widetilde{u}^\varepsilon|^2$ and $|\nabla\widetilde{u}^\varepsilon|^2$ over the whole space D is a sum of their integrals over cubes in $\{Q^k_\varepsilon = \varepsilon(k + Q) : Q^k_\varepsilon \cap D \neq \emptyset, k \in \mathbb{Z}^d\}$. Inside each Q^k_ε, in view of the fact that $\overline{B}_{1/16} \subset T$, we check that $\widetilde{u}^\varepsilon = 0$ in a ball whose radius is of order a_ε. Hence, $\widetilde{u}^\varepsilon$ satisfies the conditions of the Poincaré inequality in (A.1). Applying the inequality, we get

$$\|\widetilde{u}^\varepsilon\|^2_{L^2(D)} = \sum_{\{k : Q^k_\varepsilon \cap D \neq \emptyset\}} \|\widetilde{u}^\varepsilon\|^2_{L^2(Q^k_\varepsilon)} \leq \sum_k C\sigma_\varepsilon^2 \|\nabla\widetilde{u}^\varepsilon\|^2_{L^2(Q^k_\varepsilon)} = C\sigma_\varepsilon^2\|\nabla\widetilde{u}^\varepsilon\|^2_{L^2(D)}.$$

Applying this inequality to the integral at the beginning of the proof, we obtain

$$\|\nabla \widetilde{u}^\varepsilon\|_{L^2(D)} \le C\sigma_\varepsilon \|g\|_{L^2(D)} \qquad \text{and} \qquad \|\widetilde{u}^\varepsilon\|_{L^2(D)} \le C^2\sigma_\varepsilon^2 \|g\|_{L^2(D)}.$$

The constant C above depends only on T and d. The desired estimate (2.1) then follows. ∎

2.1. Homogenization results. Lemma 2.1 provides estimates of $\widetilde{u}^\varepsilon$ that depend on the hole-cell ratios. The critical scaling a_*^ε for the holes is determined so that the bounds in Poincaré inequality (A.1) (with $R = \varepsilon$ and $a = a_*^\varepsilon$) become of order one, and $\|\widetilde{u}^\varepsilon\|_{L^2(D)}$ is no longer of small order.

We expect that the homogenized behavior of $\widetilde{u}^\varepsilon$ also depends on the hole-cell ratios. Define

$$c_* = \begin{cases} \langle \chi \rangle_Q & \text{if } a_\varepsilon = \varepsilon, \\ \dfrac{1}{\mathrm{Cap}(T)} & \text{if } \lim\limits_{\varepsilon\to 0} a_\varepsilon/\varepsilon = 0 \text{ and } d \ge 3 \\ \dfrac{1}{2\pi} & \text{if } \lim\limits_{\varepsilon\to 0} a_\varepsilon/\varepsilon = 0 \text{ and } d = 2. \end{cases} \tag{2.2}$$

Here, $\mathrm{Cap}(T)$ denotes the Newtonian capacity of the set $T \subset \mathbb{R}^d$, $d \ge 3$, which is an important concept in potential theory; see (3.5) below and Appendix A.2. For $d = 2$, 2π is the Logarithmic capacity of T. It will be clear in sections 3 and 5 how the capacity of T enters our analysis.

The first main theorem of the paper is the qualitative homogenization of (1.1); the scaling factors for $\widetilde{u}^\varepsilon$ come from the energy estimates.

THEOREM 2.2. *Assume* (P1) *and* (P2), *let* σ_ε *be defined as in* (1.7), *and let* c_* *be defined as in* (2.2). *Then we have the following results:*

(1) *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, *i.e.,* $a_*^\varepsilon \ll a_\varepsilon \le \varepsilon$, *then* $\widetilde{u}^\varepsilon/\sigma_\varepsilon^2$ *converges weakly in* $L^2(D)$ *to* $u := c_* g$.

(2) *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 1$, *i.e.,* $a_\varepsilon \sim a_*^\varepsilon$, *then* $\widetilde{u}^\varepsilon$ *converges weakly in* $H^1(D)$ *and strongly in* $L^2(D)$ *to the unique solution of* (1.5), *where* $\mu_* = c_*^{-1}$.

(3) *If* $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \infty$, *i.e.,* $a_\varepsilon \ll a_*^\varepsilon$, *then* $\widetilde{u}^\varepsilon$ *converges weakly in* $H^1(D)$ *and strongly in* $L^2(D)$ *to the unique solution of* (1.6).

Remark 2.3. Item (2) above corresponds to the critical setting in [10], and item (3) is the subcritical setting there. The first item is an analogue of the supercritical setting for the Stokes problem in [28, 1, 2]; here $\widetilde{u}^\varepsilon$ is of order σ_ε in H^1 and converges strongly to zero, and a proper rescaling of it has a limit given by an algebraic equation, just like Darcy's law for the Stokes problems.

We emphasize that the coefficient μ_* in the strange term in the critical setting equals the inverse of the multiplier in the limiting algebraic equation in the supercritical setting. The multiplier c_* for positive volume fraction of holes is apparently different, and the relation between those multipliers is made clear in section 5.

Theorem 2.2 is proved in section 4 by an adaptive oscillating test function method that relies on asymptotic behaviors of rescaled cell problems, which are studied in section 3.

2.2. Correctors and error estimates. The second main theorem of the paper is the quantification of the homogenization results in the previous theorem. This is done only for the setting of vanishing volume fractions of holes.

THEOREM 2.4. *Assume* (P1), (P2), $d \geq 3$, *and* $a_\varepsilon \ll \varepsilon$. *Let* $\eta = a_\varepsilon/\varepsilon$ *and set* $v^\varepsilon = \chi^\eta(\cdot/\varepsilon\eta)$ *with* χ^η *defined by the rescaled cell problem* (3.2) *in section* 3. *Then the following hold:*

(1) *If* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$, *assume further that* $u = c_* g \in W_0^{2,d}(D)$. *Then there exists* C *depending only on* d, T, *and* D, *and we have*

$$\|\nabla(\widetilde{u}^\varepsilon/\sigma_\varepsilon^2 - v^\varepsilon g)\|_{L^2} \leq C(\sigma_\varepsilon + \eta^{\frac{d-2}{2}})\|u\|_{W^{2,d}(D)},$$
$$\|\widetilde{u}^\varepsilon/\sigma_\varepsilon^2 - v^\varepsilon g\|_{L^2} \leq C(\sigma_\varepsilon^2 + \varepsilon)\|u\|_{W^{2,d}(D)}, \quad \textit{and}$$
$$\|\widetilde{u}^\varepsilon/\sigma_\varepsilon^2 - c_* g\|_{L^2} \leq C(\sigma_\varepsilon^2 + \varepsilon + \eta^{\frac{d-2}{2}})\|u\|_{W^{2,d}(D)}.$$

(2) *If* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 1$, *let* u *be the solution to* (1.5) *and assume further that* $u \in W_0^{2,d}(D)$. *Then there exists* C *depending only on* d, T, *and* D *such that*

$$\|\widetilde{u}^\varepsilon - \mathrm{Cap}(T)v^\varepsilon u\|_{H^1} \leq C\varepsilon\|u\|_{W^{2,d}(D)} \quad \textit{and}$$
$$\|\widetilde{u}^\varepsilon - u\|_{L^2} \leq C\varepsilon\|u\|_{W^{2,d}(D)}.$$

(3) *If* $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \infty$, *let* u *be the solution to* (1.6) *and assume further that* $u \in W_0^{2,d}(D)$. *Then there exists* C *depending only on* d, T ,*and* D *such that*

$$\|\widetilde{u}^\varepsilon - \mathrm{Cap}(T)v^\varepsilon u\|_{H^1} \leq C(\sigma_\varepsilon^{-2} + \eta^{\frac{d-2}{2}})\|u\|_{W^{2,d}(D)} \quad \textit{and}$$
$$\|\widetilde{u}^\varepsilon - u\|_{L^2} \leq C(\sigma_\varepsilon^{-2} + \eta^{\frac{d-2}{2}})\|u\|_{W^{2,d}(D)}.$$

For the setting of $d = 2$, *the above results hold with* $\mathrm{Cap}(T)$ *replaced by* 2π, *with* $\eta^{\frac{d-2}{2}}$ *replaced by* $|\log\eta|^{-\frac{1}{2}}$, *and under the stronger condition that* $u \in W_0^{2,\infty}(D)$.

Remark 2.5. In each item above, the last inequality quantifies the error for the convergence of $\widetilde{u}^\varepsilon$; we only hope to get a convergence rate in L^2. The inequalities that precede the error estimates should be viewed as *corrector* results. They provide necessary corrections that should be added to the homogenization limits to get stronger convergences and estimates. Taking the first item, for instance, the corrector to the limit function $c_* g$ is given by $\phi^\varepsilon := (v^\varepsilon - c_*)g$, and with this corrector we have that $\widetilde{u}^\varepsilon/\sigma_\varepsilon^2 - (c_* g + \phi^\varepsilon)$ strongly converges to zero in $H^1(D)$ with explicit error bounds.

The above error estimates in the critical setting was implied by the work of Kacimi and Murat [21], where they provided a quantitative version of the framework of Cioranescu and Murat [9]; the above results in the other settings are new as far as the author knows. In the present paper, we establish those corrector and error estimates by a natural and straightforward quantification of the oscillating test function method. As a result, we obtain correctors that are naturally built upon cell problems; it seems that those results are new for all settings. Moreover, the requirement that $u \in W_0^{2,d}$ in our results, for the $d \geq 3$ setting, seems an improvement over those of [21, 1].

3. Asymptotic analysis for the cell problems.

We propose a unified approach to homogenization that is based on the standard energy method of Tartar with oscillating test functions built from cell problems. The new element is that when the hole-cell ratio vanishes in the limit, we need to consider cell problems with a parameter η that stands for this ratio, and study the limit of the cell problem as $\eta \to 0$. This is the main objective of this section. Some earlier results on this can be found in [3].

3.1. The cell problems.

When the scaling factors for the holes and cells are the same, i.e., $a_\varepsilon = \varepsilon$, we have derived the cell problem (1.3) through formal two-scale expansions. For the setting of vanishing volume fractions, we introduce a parameter $\eta \in (0,1]$ which represents the ratio of the rescaling factors of the holes against that of the cells, and we consider the cell problem

$$\begin{cases} -\Delta_y \chi_\eta(y) = 1, & y \in Y_{f,\eta} = Q \setminus (\eta \overline{T}), \\ \chi_\eta(y) = 0, & y \in \partial(\eta T), \\ y \mapsto \chi_\eta(y) \text{ is } Q\text{-periodic.} \end{cases} \tag{3.1}$$

Clearly, if $\eta = 1$, we come back to (1.3). For each fixed η, the standard elliptic PDE theory yields a unique solution χ_η of the above problem. We aim to study the behavior of χ_η as $\eta \to 0$.

For a reason that will be made clear, we consider a rescaled version of the cell problem above. Dilate $Y_{f,\eta}$ by $1/\eta$, define $\chi^\eta \in H^1(\eta^{-1}Q)$ by

$$\chi^\eta(x) = \eta^{d-2}\chi_\eta(\eta x), \qquad x \in \eta^{-1}Q \setminus \overline{T},$$

and extend it by zero inside T. We also extend χ^η to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ periodically (with respect to $\eta^{-1}\mathbb{Z}^d$).

Remark 3.1. Throughout the paper, for a function f in a cube rQ satisfying periodic conditions at the boundary, we identify f with its periodic extension in the whole space, and we also identify $rQ_{\mathfrak{p}}$ with a flat torus $r\mathbb{R}^d \setminus (r\mathbb{Z}^d)$ and view f as a function on that torus, where the subscript $\mathfrak{p}$ indicates the identification of opposite sides of $\partial(rQ)$.

The function χ^η is then the unique solution to the rescaled cell problem:

$$\begin{cases} -\Delta_x \chi^\eta(x) = \eta^d, & x \in \eta^{-1}Q \setminus T, \\ \chi^\eta(x) = 0, & x \in \partial T, \\ x \mapsto \chi^\eta(x) \text{ is } (\eta^{-1}Q)\text{-periodic.} \end{cases} \tag{3.2}$$

In the following, by an abuse of notation we still denote by χ^η the zero extension of the solution to the problem above. That is, χ^η is defined above on $\eta^{-1}Q \setminus \overline{T}$ and $\chi^\eta = 0$ in $\overline{T}$.

3.2. The scaling limit of cell problems. In this subsection, we study the limit of χ^η as η goes to zero. We restrict ourselves to the case of $d \geq 3$; the two-dimensional setting is a bit different and is left to section 6. The advantage of treating χ^η instead of χ_η is the following uniform estimate.

LEMMA 3.2. *Assume* (P1) *and $d \geq 3$. Then there exists a constant C that depends only on T and d, such that*

$$\|\nabla \chi^\eta\|_{L^2(\eta^{-1}Q)} \leq C, \qquad c_\eta := \langle \chi^\eta \rangle_{\eta^{-1}Q} = \frac{1}{|\eta^{-1}Q|}\int_{\eta^{-1}Q} \chi^\eta \in (0, C]. \tag{3.3}$$

This lemma can be proved easily: we only need to integrate the cell problems against χ^η and then use the Poincaré inequality (A.1). The sign of c_η is due to the maximum principle, which yields $\chi^\eta > 0$ in $\eta^{-1}Q \setminus \overline{T}$. With those uniform estimates, we can explore weak compactness and prove the following key lemma which characterizes the asymptotic behavior of χ^η.

LEMMA 3.3. *Assume* (P1) *and* $d \geq 3$. *Let* w *be the unique function in* $\mathscr{D}^{1,2}(\mathbb{R}^d)$ *that satisfies*

$$\begin{cases} -\Delta w(x) = 0, & x \in \mathbb{R}^d \setminus \overline{T}, \\ w(x) = 1, & x \in \overline{T}. \end{cases} \tag{3.4}$$

Then as $\eta \to 0$, c_η *converges to* $c_* = (\mathrm{Cap}(T))^{-1}$; χ^η *converges weakly in* $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $p \in [1, \frac{2d}{d-2}]$, *to* $c_*(1-w)$, *and* $\nabla\chi^\eta$ *converges weakly in* $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ *to* $-c_*\nabla w$.

In the statement of the theorem, $\mathscr{D}^{1,2}(\mathbb{R}^d)$ is the so-called homogeneous Sobolev space, and it is the completion of smooth and compactly supported functions with norm $\|\nabla\varphi\|_{L^2(\mathbb{R}^d)}$. We refer the reader to [11] for more information regarding this space. Formula (3.4) is the exterior problem that determines the Newtonian capacity of T. In fact, by definition,

$$\mathrm{Cap}(T) := \int_{\mathbb{R}^d\setminus\overline{T}} |\nabla w|^2. \tag{3.5}$$

The weak formulation of problem (3.4) reads

$$\int_{\mathbb{R}^d\setminus\overline{T}} \nabla w \cdot \nabla\varphi + \int_{\partial T} \varphi\frac{\partial w}{\partial\nu} = 0 \qquad \forall\varphi \in \mathscr{D}^{1,2}(\mathbb{R}^d).$$

The requirement that $w \in \mathscr{D}^{1,2}(\mathbb{R}^d)$ essentially imposes that w decays to zero at infinity. In the proof of Lemma 3.3 below, we provide a proof for the existence and uniqueness of the solution to the exterior problem; another way to solve (3.4) is by the potential theory.

Proof. Observe that χ^η is in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, vanishes in T, and is $\eta^{-1}\mathbb{Z}^d$-periodic. We introduce some cut-off functions. Let $\rho : [0,\infty) \to [0,1]$ be smooth and satisfy $\rho \equiv 1$ in $[0, \frac{1}{2}]$ and $\rho \equiv 0$ in $[1,\infty)$, with derivative $\rho' \in [-4, 0]$. Then we define $\xi_\eta(x) = \rho(\eta|x|_\infty)$, where $|x|_\infty := \max_{i=1}^d |x^i|$ is the infinity norm of $x = (x^1, \dots, x^d)$. Then there exists a constant C depending only on d and ρ, and

$$\xi_\eta \equiv 1 \text{ on } \frac{1}{2\eta}Q, \quad \|\nabla\xi_\eta\|_{L^\infty(\eta^{-1}Q)} \leq C\eta, \quad \text{and} \quad \|\nabla\xi_\eta\|_{L^d(\eta^{-1}Q)} \leq C. \tag{3.6}$$

By the Sobolev embedding, we can find a constant C_d independent of η, such that

$$\|\chi^\eta - c_\eta\|_{L^{\frac{2d}{d-2}}(\eta^{-1}Q)} \leq C_d\|\nabla\chi^\eta\|_{L^2(\eta^{-1}Q)} \leq C. \tag{3.7}$$

We emphasize that for $L^{\frac{2d}{d-2}}$, the embedding above is scaling invariant and hence C_d is independent of η. To summarize, we can find a C still independent of η, and we have

$$\|\nabla[(\chi^\eta - c_\eta)\xi_\eta]\|_{L^2(\mathbb{R}^d)} \leq \|\xi_\eta\nabla\chi^\eta\|_{L^2(\eta^{-1}Q)} + \|\chi^\eta - c_\eta\|_{L^{\frac{2d}{d-2}}(\eta^{-1}Q)}\|\nabla\xi_\eta\|_{L^d(\eta^{-1}Q)} \leq C.$$

This shows that $(\chi^\eta - c_\eta)\xi_\eta \in \mathscr{D}^{1,2}(\mathbb{R}^d)$ and satisfies

$$\|(\chi^\eta - c_\eta)\xi_\eta\|_{\mathscr{D}^{1,2}(\mathbb{R}^d)} \leq C.$$

By the Banach–Alaoglu theorem, there is a subsequence still denoted by $\eta \to 0$, a function $v \in \mathscr{D}^{1,2}(\mathbb{R}^d)$, and a constant $\hat{c} \in (0, C]$, such that $c_\eta \to \hat{c}$, $(\chi^\eta - c_\eta)\xi_\eta$

converges weakly in $\mathscr{D}^{1,2}$ to v. We also have that $\chi^\eta - c_\eta$ converges to v weakly in $L^{\frac{2d}{d-2}}$ on any bounded set.

We show next that the possible limit pair $(\hat{c}, v)$ is uniquely determined and, as a result, the whole sequence converges. To characterize the pair $(\hat{c}, v)$, we first pass to the limit in the weak formulation of (3.2) and get

$$-\Delta v = 0 \text{ in } \mathbb{R}^d \setminus \overline{T}, \quad v|_{\partial T} = -\hat{c},$$

in the sense that

$$\int_{\mathbb{R}^d\setminus\overline{T}} \nabla v \cdot \nabla \varphi + \int_{\partial T} \varphi \frac{\partial v}{\partial \nu} = 0 \qquad \forall \varphi \in \mathscr{D}^{1,2}(\mathbb{R}^d).$$

By taking the cut-off function $\xi_2 = \rho(2|x|_\infty)$ as the test function for (3.2), we find

$$\int_{\partial T} \frac{\partial v}{\partial \nu}\, d\sigma = \lim_{\eta\to 0} \int_{\partial T} \frac{\partial \chi^\eta}{\partial \nu}.$$

To compute the right-hand side above, we integrate the first equation in (3.2) on both sides and get

$$\lim_{\eta\to 0} \int_{\partial T} \frac{\partial \chi^\eta}{\partial \nu} d\sigma = \lim_{\eta\to 0} \int_{\eta^{-1}Q\setminus\overline{T}} \eta^d\, dx = 1.$$

Let $w = -\frac{v}{\hat{c}}$; then $w \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ solves the problem (3.4).

On the other hand, the solution of (3.4) must be unique. Indeed, if w_1 and w_2 are two solutions, then set $u = w_1 - w_2$; this satisfies

$$-\Delta u = 0 \text{ in } \mathbb{R}^d \setminus T, \quad u|_{\partial T} = 0.$$

Integrating the equation above against u itself, we get

$$\int_{\mathbb{R}^d\setminus T} |\nabla u|^2 = 0.$$

Since $u \in \mathscr{D}^{1,2}(\mathbb{R}^d)$, we get $u = w_1 - w_2 = 0$ in $\mathbb{R}^d$.

To conclude, we proved that c_η converges to $\hat{c}$, χ^η converges weakly in $L^{2d/(d-2)}_{\mathrm{loc}}$ to $\hat{c}(1-w)$, and $\nabla\chi^\eta$ converges weakly in L^2_{loc} to $-\hat{c}\nabla w$. Finally, $\hat{c}$ is determined by the identity

$$\int_{\mathbb{R}^d\setminus\overline{T}} |\nabla w|^2 = -\int_{\partial T} \frac{\partial w}{\partial \nu} = \frac{1}{\hat{c}} \int_{\partial T} \frac{\partial v}{\partial \nu} = \frac{1}{\hat{c}}. \tag{3.8}$$

In view of the defining identity of capacity (3.5), we deduce that the limit of c_η is indeed c_* defined in (2.2). The proof is now complete. ∎

Remark 3.4. The proof above followed some ideas of Allaire outlined in [3] in the setting of the Stokes problems. In section 4, we provide a quantitative analysis for this convergence result using layer potentials. It is clear there that the above results can be proved directly by layer potentials as well.

Next we rescale χ^η further to obtain a function that oscillates in the same scale of D^ε. We define

$$v^\varepsilon(x) := \chi^\eta\left(\frac{x}{\varepsilon\eta}\right) = \eta^{d-2}\chi_\eta\left(\frac{x}{\varepsilon}\right), \quad x \in \varepsilon\mathbb{R}^d_{f,\eta}.$$

Then v^ε belongs to $H^1_{\rm loc}(\mathbb{R}^d)$, vanishes inside $\varepsilon(k+\eta T)$, $k\in\mathbb{Z}^d$, and is $\varepsilon\mathbb{Z}^d$-periodic. We check that v^ε is the unique solution to the rescaled cell problem:

$$\begin{cases} -\Delta v^\varepsilon(x) = \dfrac{\eta^{d-2}}{\varepsilon^2} = \dfrac{1}{\sigma_\varepsilon^2}, & x\in \varepsilon\mathbb{R}^d_{f,\eta},\\ v^\varepsilon(x) = 0, & x\in \cup_{k\in\mathbb{Z}^d}\,\varepsilon(k+\eta T). \end{cases} \tag{3.9}$$

We will use v^ε to construct oscillating test functions in the proof of homogenization, and the following convergence results will be the key.

Lemma 3.5. *Assume* (P1) *and $d\ge 3$. For each ε and a_ε, let $\eta = a_\varepsilon/\varepsilon \in (0,1]$ and let v^ε be defined as above. Let K be an open and bounded set in $\mathbb{R}^d$ with smooth boundary. Then there exists a positive constant C depending only on K, T, and d, such that*

$$\|\nabla v^\varepsilon\|_{L^2(K)} \le C\varepsilon^{-1}\eta^{\frac{d-2}{2}} = C\sigma_\varepsilon^{-1}. \tag{3.10}$$

Moreover, let c_* be defined as in (2.2)*; then we have*

$$\lim_{\varepsilon\to 0}\|v^\varepsilon - c_*\|_{L^2(K)} = 0. \tag{3.11}$$

Finally, for the critical hole-cell ratio, i.e., $a_\varepsilon = a_*^\varepsilon$, we have

$$\nabla v^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(K). \tag{3.12}$$

Remark 3.6. The estimate (3.10) shows that ∇v^ε converges strongly to zero in the supercritical setting. For the critical ratio, (3.10) says that $\nabla v^\varepsilon \to 0$ holds but weakly. We remark also that in the setting of vanishing volume fractions of holes, i.e., $a_\varepsilon \ll \varepsilon$, the convergence in (3.11) holds in L^p for any $p\in[1,\frac{2d}{d-2}]$; this is a fact proved below.

Proof. We first prove the bound (3.10). By definition, we have the expression

$$\nabla v^\varepsilon(x) = \frac{1}{\varepsilon\eta}\nabla\chi^\eta\left(\frac{x}{\varepsilon\eta}\right).$$

The desired estimate can then be proved by breaking down the integral of $|\nabla v^\varepsilon|^2$ into each ε-square inside K, rescaling the ε-squares to $\frac{1}{\eta}$-squares, and then applying (3.3).

Next, we prove (3.11). In the setting of holes with positive volume fraction, i.e., $a_\varepsilon = \varepsilon$, it follows directly from the Riemann–Lebesgue lemma. We hence assume that the holes have vanishing volume fraction. The main idea is to use periodicity of v^ε and Lemma 3.3. Let $\mathcal{I}_\varepsilon = \mathcal{I}_\varepsilon(K)$ denote the integer points in $k\in\mathbb{Z}^d$ such that $\varepsilon(k+Q)$ has nonempty intersection with K. The cardinality of $\mathcal{I}_\varepsilon$ is of order ε^{-d}. Then, with 2^* representing the exponent $2d/(d-2)$, we compute

$$\begin{aligned} \|v^\varepsilon - c_*\|^{2^*}_{L^{2^*}(K)} &= \sum_{k\in\mathcal{I}_\varepsilon}\int_{\varepsilon(k+Q)}\left|\chi^\eta\left(\frac{x}{\varepsilon\eta}\right) - c_*\right|^{2^*}dx\\ &= \sum_{k\in\mathcal{I}_\varepsilon}\varepsilon^d\eta^d\int_{\frac{1}{\eta}(k+Q)}|\chi^\eta(y)-c_*|^{2^*}dx\\ &\le \sum_{k\in\mathcal{I}_\varepsilon}C\varepsilon^d\left(|c_\eta - c_*|^{2^*} + \eta^d\int_{\frac{1}{\eta}Q}|\chi^\eta(y)-c_\eta|^{2^*}dx\right)\\ &\le C\left(|c_\eta-c_*|^{2^*}+\eta^d\right). \end{aligned} \tag{3.13}$$

In the third line, we used Hölder's inequality and the constant depends only on d. In the last line, we used (3.6). In view of $\eta = a_\varepsilon/\varepsilon \to 0$ and Lemma 3.3, we conclude that the right-hand side above vanishes in the limit.

Now we prove (3.12). This is essentially the Riemann–Lebesgue lemma, although there is another parameter η_ε that goes to zero with ε. We note that $\|\nabla v^\varepsilon\|_{L^2(K)}$ is bounded in view of (3.10), since $a_\varepsilon = a_*^\varepsilon$ implies $\sigma_\varepsilon = 1$. Fix an arbitrary smooth test function ψ compactly supported in D. Then

$$\begin{aligned}\int_D \partial_j v^\varepsilon \psi &= \sum_{k\in\mathcal{I}_\varepsilon} \int_{\varepsilon(k+Q)} \frac{1}{\varepsilon\eta}(\partial_j\chi^\eta)\left(\frac{x}{\varepsilon\eta}\right)\psi(x)dx\\ &= \sum_{k\in\mathcal{I}_\varepsilon} \int_{\eta^{-1}(k+Q)} (\varepsilon\eta)^{d-1}(\partial_j\chi^\eta)(y)\psi(\varepsilon k+\varepsilon\eta y)dy\\ &= \sum_{k\in\mathcal{I}_\varepsilon}\left(\psi(\varepsilon k)\int_{\eta^{-1}(k+Q)}(\varepsilon\eta)^{d-1}(\partial_j\chi^\eta)(y)dy + r_\varepsilon(k)\right).\end{aligned}$$

In the last line we add and subtract $\psi(\varepsilon k)$ in each ε-cube. Then the first term vanishes due to periodicity. For the remainder $r_\varepsilon(k)$, we use the mean value theorem on ψ and get

$$\begin{aligned}|r_\varepsilon(k)| &= \left|\int_{\eta^{-1}(k+Q)}(\varepsilon\eta)^{d-1}\partial_j\chi^\eta(y)[\psi(\varepsilon k+\varepsilon\eta y)-\psi(\varepsilon k)]dy\right|\\ &\le (\varepsilon\eta)^d\|D\psi\|_{L^\infty}\|\nabla\chi^\eta\|_{L^2(\eta^{-1}(k+Q))}|\eta^{-1}(k+Q)|^{\frac12}\\ &\le C\varepsilon^d\eta^{\frac{d}{2}}\|D\psi\|_{L^\infty}.\end{aligned}$$

Note that the bound above is uniform in k. Plugging this into the summation above, we get

$$\lim_{\varepsilon\to 0}\int_D \partial_j v^\varepsilon\psi\,dx = 0.$$

This completes the proof of the theorem. ∎

4. A unified proof for periodic homogenizations. In this section, we prove Theorem 2.2 by a unified approach based on the oscillating test function method. The oscillatory test functions are constructed using v^ε of the last section. Again, we focus on the case of $d \ge 3$ first.

Here is an outline of the method: The starting point is to conclude from the uniform estimates (2.1) that $\nabla\widetilde{u}^\varepsilon/(1\wedge\sigma_\varepsilon)$ and $\widetilde{u}^\varepsilon/(1\wedge\sigma_\varepsilon^2)$ converge weakly in $L^2(D)$ and in $H^1(D)$, respectively. Then homogenization is proved by showing that the possible limits are uniquely determined.

To determine the limits, fix an arbitrary smooth function $\varphi\in C^\infty(D)$ with compact support in D, consider the oscillating function φv^ε which belongs to $H_0^1(D^\varepsilon)$, and test it against the equation of u^ε, i.e., (1.1). We then get

$$\int_D \varphi\nabla\widetilde{u}^\varepsilon\cdot\nabla v^\varepsilon + \int_D v^\varepsilon\nabla\widetilde{u}^\varepsilon\cdot\nabla\varphi = \int_D g\varphi v^\varepsilon.$$

On the other hand, the oscillatory function $\varphi\widetilde{u}^\varepsilon$ also belongs to $H_0^1(D)$, and we test it against the equation of v^ε, that is, (3.9). Then we get

$$\int_D \varphi \nabla v^\varepsilon \cdot \nabla \widetilde{u}^\varepsilon + \int_D \widetilde{u}^\varepsilon \nabla v^\varepsilon \cdot \nabla \varphi = \frac{1}{\sigma_\varepsilon^2} \int_D \widetilde{u}^\varepsilon \varphi.$$

Subtracting the two identities above, one obtains

$$\int_D v^\varepsilon \nabla \widetilde{u}^\varepsilon \cdot \nabla \varphi - \int_D \widetilde{u}^\varepsilon \nabla v^\varepsilon \cdot \nabla \varphi = \int_D \varphi g v^\varepsilon - \int_D \varphi \frac{\widetilde{u}^\varepsilon}{\sigma_\varepsilon^2}. \tag{4.1}$$

We then pass to the limit $\varepsilon \to 0$ in (4.1) and characterize the limits of each term according to different regimes of σ_ε. Let us number the integrals there by I_1, I_2, I_3, and I_4 according to the order of their appearance. We study them case by case for all regimes of hole-cell ratios.

4.1. The case of supercritical hole-cell ratios. In this setting, $\sigma_\varepsilon \to 0$. We assume that through a subsequence still denoted by $\widetilde{u}^\varepsilon$, the function $\widetilde{u}^\varepsilon/\sigma_\varepsilon^2$ converges weakly to $u \in L^2(D)$. To pass to the limit in (4.1) and identify u, we distinguish two subcases: positive volume fraction of holes (for simplicity $a_\varepsilon = \varepsilon$) and vanishing volume fraction ($a_\varepsilon \ll \varepsilon$).

Proof of part (1) *of Theorem* 2.2. Consider the case $a_\varepsilon = \varepsilon$. We have $\eta = 1$, $\sigma_\varepsilon = \varepsilon$, and v^ε is ε-periodic. This is the classical homogenization setting, and we have recalled the formal two-scale expansion argument in the introduction. Here we give a rigorous proof. In view of Lemma 3.5 and the estimate (2.1), we have

$$|I_1| \le \|v^\varepsilon\|_{L^2(D)} \|\nabla \widetilde{u}^\varepsilon\|_{L^2(D)} \|\nabla \varphi\|_{L^\infty} \le C\varepsilon \|\nabla \varphi\|_{L^\infty},$$

which converges to zero as $\varepsilon \to 0$; similarly, we have

$$|I_2| \le \|\widetilde{u}^\varepsilon\|_{L^2(D)} \|\nabla v^\varepsilon\|_{L^2(D)} \|\nabla \varphi\|_{L^\infty} \le C\varepsilon^2 \varepsilon^{-1} \|\nabla \varphi\|_{L^\infty},$$

which converges to zero as well. For I_3, by the Riemann–Lebesgue lemma we have

$$\lim_{\varepsilon \to 0} I_3 = \langle \chi \rangle_Q \int_D g\varphi = c_* \int_D g\varphi, \tag{4.2}$$

where χ is the unique solution of the standard cell problem (1.3), and we have used the definition $c_* = \langle \chi \rangle_Q$ of this setting. Finally, since $\widetilde{u}^\varepsilon/\sigma_\varepsilon^2$ converges weakly in L^2, we have

$$\lim_{\varepsilon \to 0} I_4 = \int_D \varphi u.$$

Hence, we showed that the limit of (4.1) reads

$$\int_D (c_* g - u)\varphi = 0. \tag{4.3}$$

Since φ is arbitrary, we conclude that $u = c_* g$.

Consider the case $a_\varepsilon \ll \varepsilon$. Then the supercritical condition $\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0$ implies that $a_*^\varepsilon \ll a_\varepsilon$. By Lemma 3.5 and the remark there, we know v^ε is uniformly bounded in $L^{2^*}(D)$, where $2^* = \frac{2d}{d-2}$. Hence,

$$|I_1| \le \|v^\varepsilon\|_{L^{2^*}(D)} \|\nabla \widetilde{u}^\varepsilon\|_{L^2(D)} \|\nabla \varphi\|_{L^d(D)} \le C\sigma_\varepsilon \|\nabla \varphi\|_{L^d}.$$

On the other hand, by (2.1) and (3.10), we have

$$|I_2| \le \|\widetilde{u}^\varepsilon\|_{L^2(D)} \|\nabla v^\varepsilon\|_{L^2(D)} \|\nabla\varphi\|_{L^\infty(D)} \le C\sigma_\varepsilon \|\nabla\varphi\|_{L^\infty}.$$

Both terms vanish in the limit. For I_3, we apply Lemma 3.5, in particular the fact that v^ε converges strongly in $L^{2^*}(D)$ (hence also in $L^2(D)$) to $c_* = 1/\mathrm{Cap}(T)$. This shows that

$$\lim_{\varepsilon\to 0} I_3 = \frac{1}{\mathrm{Cap}(T)} \int_D g\varphi. \tag{4.4}$$

Convergence of I_4 is straightforward as before. Then (4.3) follows again and we get $u = c_* g$ with $c_* = 1/\mathrm{Cap}(T)$. This completes the proof of the first part of Theorem 2.2. ∎

4.2. The case of critical hole-cell ratios. Here a_ε is critical and comparable to a_*^ε. For simplicity, assume $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 1$. In view of (2.1), we assume that, along a subsequence still denoted by $\widetilde{u}^\varepsilon$, the function $\widetilde{u}^\varepsilon$ converges weakly in H^1 to $u \in H_0^1(D)$. Then $\nabla\widetilde{u}^\varepsilon$ converges weakly in L^2 to ∇u.

Proof of part (2) *of Theorem* 2.2. It remains to study the limit of (4.1) to identify u. For I_1, we use the strong convergence (3.11) and the weak convergence of $\nabla\widetilde{u}^\varepsilon$ to conclude that

$$\lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0} c_* \int_D \nabla\widetilde{u}^\varepsilon \cdot \nabla\varphi = c_* \int_D \nabla u \cdot \nabla\varphi.$$

For I_2, we use the weak convergence of ∇v^ε in (3.12) and the strong convergence of $\widetilde{u}^\varepsilon$ and get

$$\lim_{\varepsilon\to 0} I_2 = \lim_{\varepsilon\to 0} \int_D \nabla v^\varepsilon \cdot \nabla\varphi\, u = 0.$$

The terms I_3 and I_4 are trivial in this setting, and we have

$$\lim_{\varepsilon\to 0} I_3 = \int_D c_* g\varphi, \qquad \lim_{\varepsilon\to 0} I_4 = \int_D gu.$$

Hence, passing to the limit $\varepsilon \to 0$ on (4.1), we obtain

$$\int_D c_* \left(\nabla u \cdot \nabla\varphi - g\varphi + \frac{1}{c_*} u\varphi \right) = 0. \tag{4.5}$$

Note that we used the positivity of c_*, which is clear.

Finally, since φ is arbitrary, the above implies that the limit function $u \in H_0^1(D)$ is a solution to

$$-\Delta u + \frac{1}{c_*} u = g \quad \text{in } D.$$

Note that $1/c_* = \mathrm{Cap}(T)$ is positive. The solution to this Dirichlet problem is unique. This completes the proof of the second item of Theorem 2.2. ∎

4.3. The case of subcritical hole-cell ratios. Now we consider the remaining situation where $a_\varepsilon \ll a_*^\varepsilon$ or $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \infty$. Again, by the uniform estimates (2.1) we take a subsequence, still denoted by $\widetilde{u}^\varepsilon$, for which $\widetilde{u}^\varepsilon$ converges weakly in H^1 and strongly in L^2 to $u \in H_0^1(D)$.

Proof of part (3) *of Theorem* 2.2. We study the limit of (4.1) to identify the limit u. The terms I_1 and I_3 can be treated exactly as in the case of critical hole-cell size ratios. The limits there still hold. The limit of I_4 vanishes since $\widetilde{u}^\varepsilon$ is bounded while $\sigma_\varepsilon \to \infty$. It remains to study I_2. We use (3.10) to conclude that

$$|I_2| \le \|\nabla v^\varepsilon\|_{L^2}\|\widetilde{u}^\varepsilon\|_{L^2}\|\nabla\varphi\|_{L^\infty} \le C\sigma_\varepsilon^{-1} \to 0.$$

Combine those results; we see that (4.1) becomes

$$\int_D c_*\left(\nabla u\cdot\nabla\varphi - g\varphi\right) = 0 \tag{4.6}$$

in the limit. By the positivity of c_* and the arbitrariness of φ, we conclude that the only possible limit $u \in H_0^1(D)$ is the unique solution to the problem (1.6). This completes the proof of item (3) of Theorem 2.2. ∎

5. On the correctors and error estimates. In this section, we quantify the unified proof of the last section to get the error estimates in Theorem 2.2, for cases where the holes have vanishing volume fractions. It is remarkable that this quantification is quite easy to do using tools from potential theory.

It is almost clear from the proof of Theorem 2.2 that, to quantify the homogenization error, we need to find the convergence rate of $v^\varepsilon \to c_*$ in L^2. This is done in the next key lemma for $d \ge 3$.

LEMMA 5.1. *Assume* (P1) *and $d \ge 3$. For each a_ε and ε, let $\eta = a_\varepsilon/\varepsilon$, let v^ε be defined as in* (3.9)*, and let c_* be defined as in* (2.2)*. Let K be an open and bounded set in $\mathbb{R}^d$ with smooth boundary. Then there exists a positive constant C depending only on K, T, and d, such that*

$$\|v^\varepsilon - c_*\|_{L^{2^*}(K)} \le C\eta^{\frac{d-2}{2}}.$$

Our proof below relies heavily on the potential theory for the Laplace operator. We refer the reader to the appendix and the references therein for more details.

Proof. Step 1*: The convergence rate of $c_\eta = \fint_{\eta^{-1}Q}\widetilde{\chi}^\eta \to c_*$.* Here χ^η is the solution to (3.1). Let G^η be the Green's function of the Laplace operator in the flat torus $\eta^{-1}Q_{\mathfrak{p}}$; see (A.5). Our plan is to explore the explicit formula of χ^η given by the potential theory. We refer the reader to Appendix A.2 for the terminology and notation. Let $\chi^\eta = G^\eta + \phi^\eta$. Then by definition ϕ^η satisfies

$$-\Delta\phi^\eta = 0 \quad \text{on } \frac{1}{\eta}Q_{\mathfrak{p}}\setminus\overline{T}, \qquad \phi^\eta|_{\partial T} = g^\eta := -G^\eta|_{\partial T}.$$

In view of (A.6), g^η is smooth on ∂T. By Theorem A.6, $\mathrm{Ran}(-\frac{1}{2}I + \mathcal{K}_{T,\mathfrak{p}}^{(\eta)})$ is the whole space and, hence, ϕ^η can be represented by a double-layer potential. To find the structure of the potential, we decompose g^η according to Corollary A.5: Let X denote the subspace $\mathrm{Ran}(-\frac{1}{2}I + \mathcal{K}_T)$ of $L^2(\partial T)$; then

$$g^\eta = g_1^\eta + r^\eta, \quad g_1^\eta \in X, \quad \text{and} \quad r^\eta = \langle g^\eta, \varphi_*\rangle,$$

where φ_* is defined in Theorem A.4. As a result, ϕ^η can be represented by

$$\phi^\eta = \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi_1^\eta] + r^\eta, \quad \text{where} \quad \psi_1^\eta = \left(-\frac{1}{2}I + \mathcal{K}^{(\eta)}_{T,\mathfrak{p}}\right)^{-1} g_1^\eta. \tag{5.1}$$

Back to the function $\widetilde{\chi}^\eta$, its average c_η in $\frac{1}{\eta}Q$ can be written as

$$c_\eta = r^\eta(1-\eta^d|T|) + \frac{1}{|\eta^{-1}Q|}\int_{\frac{1}{\eta}Q\setminus\overline{T}} G^\eta + \frac{1}{|\eta^{-1}Q|}\int_{\frac{1}{\eta}Q\setminus\overline{T}} \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi_1^\eta]. \tag{5.2}$$

We estimate those numbers on the right-hand side one by one. By Corollary A.5, we have

$$\begin{aligned} r^\eta &= -\langle G^\eta, \varphi_*\rangle_{L^2(\partial T), L^2(\partial T)} \\ &= -\langle \Gamma\big|_{\partial T}, \varphi_*\rangle - \eta^{d-2}\langle R(\eta\cdot), \varphi\rangle = \frac{1}{\mathrm{Cap}(T)} - \eta^{d-2}\int_{\partial T} R(\eta x)\varphi_*(x). \end{aligned}$$

Here, R is defined in formula (A.6). This leads to

$$|r^\eta - c_*| \le \eta^{d-2}\|R\|_{L^2(\partial T)}. \tag{5.3}$$

Second, for the average of G^η, we calculate

$$\left|\frac{1}{|\eta^{-1}Q|}\int_{\frac{1}{\eta}Q\setminus\overline{T}} \Gamma(x)dx\right| \le \eta^d \int_{B_{\frac{1}{2\eta}}} \Gamma + \eta^d \int_{\frac{1}{\eta}(Q\setminus\overline{B}_{1/2})} \Gamma \le C_d\,\eta^{d-2}.$$

Then since $G^\eta(x) = \Gamma(x) + \eta^{d-2}R(\eta x)$ for all $x \in \frac{1}{\eta}Q\setminus\overline{T}$, we have

$$\left|\langle G^\eta\rangle_{\frac{1}{\eta}Q}\right| \le \langle \Gamma\rangle_{\frac{1}{\eta}Q} + \eta^{d-2}\|R\|_{L^\infty} \le C\eta^{d-2}, \tag{5.4}$$

where C depends only on d and T.

For the third term on the right-hand side of (5.2), we note that

$$\mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi_1^\eta] = \eta^d\int_{\partial T} \nu_y\cdot\nabla\Gamma(x-y)\psi_1^\eta(y)d\sigma + \eta^{d-1}\int_{\partial T}\nu_y\cdot(\nabla R)(\eta(x-y))\psi_1^\eta(y)d\sigma.$$

Using the explicit form of $\nabla\Gamma$ and by the same method as above, we can average in x in the first integral and estimate the result. For the second integral above, we note that $\|\nabla R\|_{L^\infty}$ is finite and depends only on T. We hence get

$$\left|\frac{1}{|\eta^{-1}Q|}\int_{\eta^{-1}Q\setminus\overline{T}} \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi_1^\eta]\right| \le C\eta^{d-1}(1+\|\nabla R\|_{L^\infty})\|\psi_1^\eta\|_{L^2(\partial T)} \tag{5.5}$$

for some constant C that depends only on d and T. We conclude that the rate of $c_\eta \to c_*$ can be characterized once we get a quantitative estimate of $\|\psi_1^\eta\|_{L^2(\partial T)}$ in terms of η.

Step 2: The estimate of $\|\psi_1^\eta\|_{L^2(\partial T)}$. By the decomposition (A.14), the function ψ_1^η is the unique solution to the equation

$$\left(-\frac{1}{2}I + \mathcal{K}_T + \mathcal{R}^\eta\right)[\psi_1^\eta] = g_1^\eta,$$

where $\mathcal{R}^\eta = \eta^{d-1}\mathcal{R}_2$ and $\mathcal{R}_2$ is defined in (A.9). We seek a solution of the form $\psi_1^\eta = \xi_1 + \theta$, where $\xi_1 \in L_0^2(\partial T)$ and $\theta \in \mathbb{R}$. Plugging this ansatz into the equation above, we find

$$\left(-\frac{1}{2}I + \mathcal{K}_T\right)[\xi_1] + \mathcal{R}^\eta[\xi_1] - \eta^d|T|\theta = g_1^\eta. \tag{5.6}$$

Project the above equation to X, and let Π_X denote this projection operator. Recalling that g_1^η is in the range, we get

$$\left(-\frac{1}{2}I + \mathcal{K}_T\right)[\xi_1] + \Pi_X\mathcal{R}^\eta[\xi_1] = g_1^\eta.$$

We know that $-\frac{1}{2}I + \mathcal{K}_T$ is invertible from $L_0^2(\partial T)$ to X with bounded operator norm; see Theorem A.4. On the other hand, $\|\mathcal{R}^\eta\|_{L^2\to L^2}$ is of order η^{d-1}, and $\|\Pi_X\mathcal{R}^\eta\|_{L_0^2\to X}$ is of the same order. By standard perturbation theory (see, e.g., [22, Theorem IV.1.16]), for η sufficiently small, the operator norm $\|-\frac{1}{2}I + \mathcal{K}_T + \Pi_X\mathcal{R}^\eta\|_{L_0^2\to X}$ can be bounded by C independent of η. We hence get

$$\|\xi_1\|_{L^2(\partial T)} \le C\|g_1^\eta\|_{L^2(\partial T)} \le C. \tag{5.7}$$

Now the projection of (5.6) to the kernel of $-\frac{1}{2}I + \mathcal{K}_T$ reads

$$\eta^d\theta|T| = \langle\mathcal{R}^\eta[\xi_1], \varphi_*\rangle = \langle\xi_1, (\mathcal{R}^\eta)^*\varphi_*\rangle.$$

We then get the estimate

$$|\theta||T| \le \frac{1}{\eta^d}\|\xi_1\|_{L^2(\partial T)}\|(\mathcal{R}^\eta)^*\varphi_*\|_{L^2} \le \frac{1}{\eta}\|\xi_1\|_{L^2(\partial T)}\|\varphi_*\|_{L^2(\partial T)} \le C\eta^{-1}. \tag{5.8}$$

Here we used the fact that $\|(\mathcal{R}^\eta)^*\|_{L^2\to L^2} \le C\eta^{d-1}\|\nabla R\|_{L^\infty}$. Combining (5.7) with (5.8), we get $\|\psi_1^\eta\|_{L^2(\partial T)} \le C\eta^{-1}$. Use this estimate in (5.5) and then combine the result with (5.3) and (5.4); we finally obtain, for $d \ge 3$,

$$|c_\eta - c_*| \le C\eta^{d-2}, \tag{5.9}$$

where the constant C depends only on d and T.

Step 3*: Convergence rate for* $\|v^\varepsilon - c_*\|_{L^{2^*}(K)} \to 0$. In the proof of Lemma 3.5, we apply (5.9) in the last line of (3.13) and get

$$\|v^\varepsilon - c_*\|_{L^{2^*}(K)}^{2^*} \le C\left((\eta^{(d-2)/2})^{2^*} + \eta^d\right) \le C\eta^d,$$

where the constant C depends only on d, T, and K. This completes the proof of Lemma 5.1. ∎

5.1. Proof of Theorem 2.4. Now we are ready to prove the corrector and error estimates in Theorem 2.4. We emphasize that those results are only for the setting with vanishing volume fraction of holes, i.e., $a_\varepsilon \ll \varepsilon$.

Proof of Theorem 2.4. *Case* 1*: Supercritical hole-cell ratios.* Here, $a_*^\varepsilon \ll a_\varepsilon \ll \varepsilon$ and $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$ hold. Moreover, the assumption that $u = \langle\chi\rangle_Q g \in W_0^{2,d}(D)$ implies that the source term in (1.1) satisfies $g \in W_0^{2,d}(D)$.

Let $\eta = a_\varepsilon/\varepsilon$, and let v^ε be defined as in (3.9). Define $\xi^\varepsilon := u^\varepsilon/\sigma_\varepsilon^2 - v^\varepsilon g$; then $\xi^\varepsilon \in H_0^1(D^\varepsilon)$. Direct computation shows that

$$-\Delta\xi^\varepsilon = v^\varepsilon\Delta g + 2\nabla v^\varepsilon\cdot\nabla g \qquad \text{in } D^\varepsilon.$$

Integrating the above equation against ξ^ε, we get

$$\int_{D^\varepsilon}|\nabla\xi^\varepsilon|^2 = \int_{D^\varepsilon}\xi^\varepsilon v^\varepsilon\Delta g + 2\int_{D^\varepsilon}\xi^\varepsilon\nabla(v^\varepsilon - c_*)\cdot\nabla g.$$

Note that in the last integral above, we inserted $-c_*$ in a gradient term, which does not change the integral. We estimate

$$\begin{aligned}\left|\int_{D^\varepsilon}\xi^\varepsilon v^\varepsilon\Delta g\right| &\le \|\xi^\varepsilon\|_{L^2(D^\varepsilon)}\|v^\varepsilon\|_{L^{2^*}(D^\varepsilon)}\|\Delta g\|_{L^d(D)}\\ &\le \sigma_\varepsilon\|\nabla\xi^\varepsilon\|_{L^2(D^\varepsilon)}\|v^\varepsilon\|_{L^{2^*}(D^\varepsilon)}\|\Delta g\|_{L^d(D)}.\end{aligned}\tag{5.10}$$

In the first line, we applied Hölder's inequality and noted that v^ε is bounded in $L^{2^*}(D^\varepsilon)$, as indicated by (3.11) and the remark there. In the second line, we used the Poincaré inequality. For the other integral, we compute

$$\int_{D^\varepsilon}\xi^\varepsilon\nabla(v^\varepsilon - c_*)\cdot\nabla g = \int_{D^\varepsilon}\nabla\cdot[\xi^\varepsilon(v^\varepsilon - c_*)\nabla g] - (v^\varepsilon - c_*)\nabla\xi^\varepsilon\cdot\nabla g - \xi^\varepsilon(v^\varepsilon - c_*)\Delta g.$$

We view the right-hand side as three integrals. Then the first one is zero since ξ^ε vanishes on ∂D^ε. The third integral can be treated as in (5.10). To estimate the second one, we have

$$\left|\int_{D^\varepsilon}(v^\varepsilon - c_*)\nabla\xi^\varepsilon\cdot\nabla g\right| \le \|v^\varepsilon - c_*\|_{L^{2^*}(D^\varepsilon)}\|\nabla\xi^\varepsilon\|_{L^2(D^\varepsilon)}\|\nabla g\|_{L^d(D)}.$$

Combining the estimates above and applying Lemma (5.1), we get

$$\|\nabla\widetilde{\xi^\varepsilon}\|_{L^2(D)} \le C\|g\|_{W^{2,d}(D)}[\sigma_\varepsilon + \|v^\varepsilon - c_*\|_{L^2(D^\varepsilon)}] \le C(\sigma_\varepsilon + \eta^{\frac{d-2}{2}})\|g\|_{W^{2,d}(D)}.$$

By the Poincaré inequality, we also have

$$\|\widetilde{\xi^\varepsilon}\|_{L^2(D)} \le C(\sigma_\varepsilon^2 + \varepsilon)\|g\|_{W^{2,q}(D)}.$$

Here, we used the relation that $\eta^{\frac{d-2}{2}} = \varepsilon/\sigma_\varepsilon$. We hence proved the first two estimates in part (1) of Theorem 2.4. The third inequality follows from the relation that

$$\frac{\widetilde{u}^\varepsilon}{\sigma_\varepsilon^2} - c_*g = \frac{\widetilde{u}^\varepsilon}{\sigma_\varepsilon^2} - v^\varepsilon g + (v^\varepsilon - c_*)g,$$

the result of Lemma 5.1, and the triangle inequality.

Case 2: Critical hole-cell ratio. Now we consider the case of $a_\varepsilon \sim a_*^\varepsilon$, which also implies that $\sigma_\varepsilon \sim 1$. For simplicity, we assume that $\sigma_\varepsilon = 1$ for the sequence of ε that converges to 0. Then $\eta^{\frac{d-2}{2}} = \varepsilon$.

Similarly to the previous case, let $\xi^\varepsilon = u^\varepsilon - c_*^{-1}v^\varepsilon u$. Then $\xi^\varepsilon \in H_0^1(D^\varepsilon)$, solving the equation

$$-\Delta\xi^\varepsilon = \frac{1}{c_*}(v^\varepsilon - c_*)\Delta u + \frac{2}{c_*}\nabla v^\varepsilon\cdot\nabla u \qquad \text{in } D^\varepsilon.$$

Integrating against ξ^ε, we have

$$\begin{aligned}\int_{D^\varepsilon} |\nabla\xi^\varepsilon|^2 &= \frac{1}{c_*}\int_{D^\varepsilon}(v^\varepsilon - c_*)\xi^\varepsilon\Delta u + \frac{2}{c_*}\int_{D^\varepsilon}\xi^\varepsilon\nabla(v^\varepsilon - c_*)\cdot\nabla u\\ &= -\frac{1}{c_*}\int_{D^\varepsilon}(v^\varepsilon - c_*)\xi^\varepsilon\Delta u - \frac{2}{c_*}\int_{D^\varepsilon}(v^\varepsilon - c_*)\nabla\xi^\varepsilon\cdot\nabla u.\end{aligned}$$

We then note that $\widetilde{\xi} \in H^1_0(D)$, and by the Poincaré inequality we get

$$\|\nabla\widetilde{\xi^\varepsilon}\|^2_{L^2(D)} \le C\|v^\varepsilon - c_*\|_{L^{2^*}(D^\varepsilon)}\|\nabla\widetilde{\xi^\varepsilon}\|_{L^2}(\|\Delta u\|_{L^d(D)} + \|\nabla u\|_{L^d(D)}).$$

By Lemma 5.1 and by the Poincaré inequality again we finally get

$$\|\xi^\varepsilon\|_{H^1(D)} \le C\eta^{\frac{d-2}{2}}\|u\|_{W^{2,d}(D)} = C\varepsilon\|u\|_{W^{2,d}(D)}.$$

This finishes the proof of the first estimate in item (2) of Theorem 2.4; the other estimates follows, again, by adding the correction $(c_*^{-1}v^\varepsilon - 1)u$ to u and by the triangle inequality.

Case 3*: Subcritical hole-cell ratios.* Finally we consider the case of $a_\varepsilon \ll a_*^\varepsilon$, which implies that $\lim_{\varepsilon\to0}\sigma_\varepsilon = \infty$. We still let ξ^ε denote the distance function $u^\varepsilon - c_*^{-1}v^\varepsilon u$. Then

$$-\Delta\xi^\varepsilon = -\frac{1}{c_*}\frac{u}{\sigma_\varepsilon^2} + \frac{1}{c_*}(v^\varepsilon - c_*)\Delta u + \frac{2}{c_*}\nabla v^\varepsilon\cdot\nabla u \qquad \text{in } D^\varepsilon.$$

By exactly the same arguments as in the previous cases and by the usual Poincaré inequality $\|\widetilde{\xi^\varepsilon}\|_{L^2(K)} \le C(K)\|\nabla\xi^\varepsilon\|_{L^2(K)}$, we get

$$\begin{aligned}\|\nabla\widetilde{\xi^\varepsilon}\|_{L^2(D)} &\le C(\sigma_\varepsilon^{-2}\|u\|_{L^2(D)} + \|u\|_{W^{2,\infty}(D)}\|v^\varepsilon - c_*\|_{L^2(D)})\\ &\le C(\sigma_\varepsilon^{-2} + \eta^{\frac{d-2}{2}})\|u\|_{W^{2,d}(D)}.\end{aligned}$$

This finishes the proof of the first inequality in item (3) of Theorem 2.4; the other one can be treated as before. ☐

6. The two-dimensional setting. In this section we study the two-dimensional setting. It is clear from the proofs in sections 4 and 5 that to get qualitative homogenization, we only need to prove results similar to those of Lemma 3.5, and to get error and corrector estimates, we only need a quantification like that of Lemma 5.1. Those estimates are obtained in Lemma 6.2 below. With those results, the homogenization and corrector results can be proved by the oscillating test function method and its quantification in the earlier sections; we will not repeat those arguments here.

The technical difference for $d = 2$ comes from the energy estimates. In view of the Poincaré inequality (A.1), the solution to the cell problem (3.2) does not satisfy (3.3); instead, we only have

$$\|\nabla\chi^\eta\|_{L^2(\eta^{-1}Q)} \le C|\log\eta|^{\frac{1}{2}}, \qquad c_\eta = \fint_{\eta^{-1}Q}\chi^\eta \le C|\log\eta|. \tag{6.1}$$

Since $|\log\eta|$ blows up as $\eta \to 0$, the analysis of the last two sections cannot be repeated.

Inspired by Allaire [3], we set $a_\eta = \frac{1}{2\pi}(1-\eta^2|T|)$, for each $\eta < 1/4$, and define the function

$$\Phi^\eta(x) = \begin{cases} 0, & x \in B_1, \\ a_\eta \log|x|, & x \in B_{1/2\eta} \setminus \overline{B}_1, \\ a_\eta |\log \eta/2|, & x \in \frac{1}{\eta}Q \setminus \overline{B}_{1/2\eta}. \end{cases} \tag{6.2}$$

The reason for this specific definition of a_η will be made clear later. Since $\overline{T} \subset B_{1/2}$, we have $\Phi^\eta = 0$ on T. Extend Φ^η to be $\eta^{-1}\mathbb{Z}$-periodic. The difference function $w^\eta := \chi^\eta - \Phi^\eta$ then satisfies

$$\begin{cases} -\Delta w^\eta = \eta^2 - 2\eta a_\eta \delta_{S_{1/2\eta}} + a_\eta \delta_{S_1}, & x \in \eta^{-1}Q \setminus \overline{T}, \\ w^\eta = 0, & x \in \partial T. \end{cases} \tag{6.3}$$

Here, δ_{S_r}, $r = 1/2\eta$ or 1, is the uniform measure concentrated on the sphere $S_r = \partial B_r(0)$. In other words, we have

$$\delta_{S_r}(\varphi) := \int_{S_r} \varphi(y)\, d\sigma_y \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^d).$$

Those measures belong to $H^{-1}(\eta^{-1}Q)$, i.e., the dual space of $H^1(\eta^{-1}Q)$. Set $\nu^\eta := \eta^2 - 2\eta a_\eta \delta_{S_{1/2\eta}}$. The specific choice of a_η yields the following identity:

$$\langle \nu^\eta, 1\rangle_{H^{-1}(\eta^{-1}Q\setminus\overline{T}), H^1(\eta^{-1}Q\setminus\overline{T})} = \int_{\eta^{-1}Q\setminus\overline{T}} \eta^2 - 2\eta a_\eta \delta_{S_{1/2\eta}}\, dy = 0. \tag{6.4}$$

We check that $\|\nabla w^\eta\|_{L^2(\eta^{-1}Q)}$ can be controlled uniformly in η.

Lemma 6.1. *Assume* (P1) *and* $d = 2$. *For each* $\eta \in (0, 1/4]$, *let* χ^η *be as in* (3.2) *and* Φ^η *as in* (6.2), *and set* $w^\eta = \chi^\eta - \Phi^\eta$. *Then there exists* $C > 0$ *that is independent of* η *such that*

$$\|\nabla w^\eta\|_{L^2(\eta^{-1}Q)} \le C. \tag{6.5}$$

Proof. Integrating on both sides of (6.3) against w^η, we get

$$\|\nabla w^\eta\|^2_{L^2(\eta^{-1}Q)} = a_\eta \delta_{S_1}(w^\eta) + \nu^\eta(w^\eta). \tag{6.6}$$

For the first term above, we find

$$|\delta_{S_1}(w^\eta)| \le C\|w^\eta\|_{L^2(S_1)} \le C\|\nabla w^\eta\|_{L^2(B_2)}, \tag{6.7}$$

where we used the usual trace inequality for H^1 functions and the Poincaré inequality (A.1), and noted that the constants involved are uniform in η. For the second term, recall that $Y_{f,\eta}$ refers to the set $Q \setminus \eta\overline{T}$. Then, in view of (6.4), we have

$$\begin{aligned} \langle \nu^\eta, w^\eta\rangle_{H^{-1}(\eta^{-1}Y_{f,\eta}), H^1(\eta^{-1}(Y_{f,\eta}))} &= \left\langle \nu^\eta, w^\eta - \fint_{\eta^{-1}Y_{f,\eta}} w^\eta \right\rangle_{H^{-1}(\eta^{-1}Y_{f,\eta}), H^1(\eta^{-1}Y_{f,\eta})} \\ &= \int_{Y_{f,\eta}} \left(1 - 2a_\eta \delta_{S_{1/2}}\right) \left[w_\eta - \fint_{Y_{f,\eta}} w_\eta \right]. \end{aligned}$$

It is clear that $1 - 2a_\eta \delta_{S_{1/2}}$ is an element in $H^{-1}(Y_{f,\eta})$ and its norm can be bounded uniformly in η. It follows from the Poincaré–Wirtinger inequality that

$$|\langle \nu^\eta, w^\eta \rangle_{H^{-1},H^1}| \le C \left\| w_\eta - \fint_{Y_{f,\eta}} w_\eta \right\|_{H^1(Y_{f,\eta})} \le C \|\nabla w^\eta\|_{L^2(\frac{1}{\eta} Y_{f,\eta})}. \tag{6.8}$$

The last step is due to scaling invariance of the $\|\nabla \varphi\|_{L^2}$ norm for $d = 2$.

Combine (6.6), (6.7), and (6.8), and recall that w^η is extended to T by zero. We finally obtain

$$\|\nabla w^\eta\|_{L^2(\eta^{-1}Q)} \le C, \tag{6.9}$$

where C is independent of η. The proof is hence complete. ∎

To build oscillating test functions, we consider a rescaled version of χ^η and define

$$v^\varepsilon(x) := \frac{1}{|\log \eta|} \chi^\eta \left(\frac{x}{\varepsilon \eta} \right), \qquad x \in \varepsilon(Q \setminus \eta \overline{T}).$$

Again, we extend v^ε to zero in $\eta \overline{T}$ and extend it further periodically in each cube of $\varepsilon \mathbb{Z}^d$. Then v^ε solves the rescaled cell problem

$$\begin{cases} -\Delta v^\varepsilon(x) = \dfrac{1}{|\log \eta| \varepsilon^2} = \dfrac{1}{\sigma_\varepsilon^2}, & x \in \varepsilon \mathbb{R}^2_f, \\ v^\varepsilon(x) = 0, & x \in \partial(\varepsilon \mathbb{R}^2_f). \end{cases} \tag{6.10}$$

We have the following properties for v^ε.

LEMMA 6.2. *Assume* (P1), *$d = 2$, and $a_\varepsilon \ll \varepsilon$, i.e., $\eta(\varepsilon) = a_\varepsilon/\varepsilon \ll 1$. Let v^ε be as in* (6.10). *Then for each set K compactly supported in $\mathbb{R}^2$, there exists $C > 0$ that depends only on d, T, and K such that*

$$\|\nabla v^\varepsilon\|_{L^2(K)} \le C(\varepsilon |\log \eta|^{\frac{1}{2}})^{-1} = C\sigma_\varepsilon^{-1}. \tag{6.11}$$

Moreover, we have

$$\|v^\varepsilon - 1/2\pi\|_{L^2(K)} \le C|\log \eta(\varepsilon)|^{-\frac{1}{2}}. \tag{6.12}$$

Finally, if $\alpha \sim \alpha_*^\varepsilon$, i.e., $\lim_{\varepsilon \to 0} \sigma_\varepsilon \in (0, \infty)$, then

$$\nabla v^\varepsilon \rightharpoonup 0 \qquad \text{weakly in } L^2(K). \tag{6.13}$$

Proof. We first observe that (6.11) and (6.13) can be proved as in the $d \ge 3$ setting, using the two-dimensional versions of the Poincaré inequality in (A.1) and the energy estimates (6.1). Considering the proof of (6.13), for instance, the control r_ε term in the proof of Lemma 3.5 becomes, in a typical cell $\varepsilon \square$,

$$\begin{aligned} |r_\varepsilon| &\le |\log \eta|^{-1} (\varepsilon \eta)^2 \|D\psi\|_{L^\infty} \|\nabla \chi^\eta\|_{L^2(\eta^{-1}\square)} |\eta^{-1}\square|^{\frac{1}{2}} \\ &\le C\varepsilon^2 \left(\eta |\log \eta|^{-\frac{1}{2}} \right). \end{aligned}$$

This is sufficient for the proof.

The proof of (6.12) needs major modifications in $d = 2$. Since a_η converges to $1/2\pi$ with error of order η^2, we only need to estimate $\|v^\varepsilon - a_\eta\|_{L^2(K)}$. As in the proof of (3.9), we use periodicity and reduce the estimates essentially to a rescaled reference cell εQ.

In the ε-cell εQ, extend v^ε by zero inside $\varepsilon\eta T$. Recall the definitions of Φ^η and w^η in (6.2). Then direct computation shows that

$$v^\varepsilon(x) - a_\eta = \zeta_{\eta,\varepsilon} + \frac{1}{|\log\eta|}\left[\left(w^\eta\left(\frac{x}{\varepsilon\eta}\right) - \overline{w}_{\eta,\varepsilon}\right) + \overline{w}_{\eta,\varepsilon}\right]. \tag{6.14}$$

Here, $\zeta_{\eta,\varepsilon}$ is an εQ-periodic function defined by

$$\zeta_{\eta,\varepsilon}(x) := \frac{1}{|\log\eta|}\Phi^\eta\left(\frac{x}{\varepsilon\eta}\right) - a_\eta = \begin{cases} -a_\eta, & x \in \overline{B}_{\varepsilon\eta}, \\ a_\eta|\log\eta|^{-1}\log\left|\frac{x}{\varepsilon}\right|, & x \in \varepsilon\eta(B_{1/2\eta}\setminus\overline{B}_1), \\ a_\eta|\log 2||\log\eta|^{-1}, & x\in\varepsilon(\square\setminus B_{1/2}), \end{cases}$$

and $\overline{w}_{\eta,\varepsilon}$ is shorthand notation for the average $⨍_{\varepsilon Q} w^\eta(\cdot/\varepsilon\eta)$ in a typical ε-cell. Moreover, by a simple scaling check it is clear that $\overline{w}_{\eta,\varepsilon}$ is the same as the average $⨍_{\frac{1}{\eta}Q} w^\eta$ in the $\frac{1}{\eta}$-cell. Hence, $\overline{w}_{\eta,\varepsilon}$ satisfies the estimate

$$|\overline{w}_{\eta,\varepsilon}| = \left|⨍_{\eta^{-1}Q} w^\eta(y)\,dy\right| \le \frac{\|w^\eta\|_{L^2(\eta^{-1}Q)}}{|\eta^{-1}Q|^{\frac12}} \le \frac{(1/\eta)|\log\eta|^{\frac12}\|\nabla w^\eta\|_{L^2(\eta^{-1}Q)}}{|\eta^{-1}Q|^{\frac12}}.$$

In the last step, we used the Poincaré inequality (A.1). It follows that, in a typical ε-cell, we have

$$\|\overline{w}_{\eta,\varepsilon}\|^2_{L^2(\varepsilon Q)} = |\varepsilon Q||\overline{w}_{\eta,\varepsilon}|^2 \le \varepsilon^2|\log\eta|\|\nabla w^\eta\|^2_{L^2(\eta^{-1}Q)} \le C\varepsilon^2|\log\eta|. \tag{6.15}$$

To get the last inequality, we used Lemma 6.1. This controls the last item on the right-hand side of (6.14). Hence, $\||\log\eta|^{-1}\overline{w}_{\eta,\varepsilon}\|^2_{L^2(\varepsilon Q)}$ is of order $C\varepsilon^2|\log\eta|^{-1}$.

To control the oscillations of $w^\eta(\cdot/\varepsilon\eta)$, we apply the Poincaré–Wirtinger inequality and obtain

$$\begin{aligned} \|w^\eta(\cdot/\varepsilon\eta) - \overline{w}_{\eta,\varepsilon}\|^2_{L^2(\varepsilon Q)} &= (\varepsilon\eta)^d\|w^\eta(\cdot) - ⨍_{\eta^{-1}Q} w^\eta\|^2_{L^2(\eta^{-1}Q)} \\ &\le (\varepsilon\eta)^d\eta^{-2}\|\nabla w^\eta\|^2_{L^2(\eta^{-1}Q)} \le C\varepsilon^2. \end{aligned} \tag{6.16}$$

Hence, $\||\log\eta|^{-1}(w^\eta(\cdot/\varepsilon\eta) - \overline{w}_{\eta,\varepsilon})\|^2_{L^2(\varepsilon Q)}$ is of order $\varepsilon^2|\log\eta|^{-2}$.

Finally, for the $\zeta_{\eta,\varepsilon}$ term, we write $\|\zeta_{\eta,\varepsilon}\|^2_{L^2(\varepsilon Q)}$ as the following sum:

$$\|a_\eta\|^2_{L^2(B_{\varepsilon\eta})} + \|a_\eta|\log\eta|^{-1}\log(|\cdot/\varepsilon|)\|^2_{L^2(B_{\varepsilon/2}\setminus\overline{B}_{\varepsilon\eta})} + \|a_\eta|\log 2||\log\eta|^{-1}\|^2_{L^2(\varepsilon Q\setminus\overline{B}_{\varepsilon/2})}.$$

Then the first term is of order $\eta^2\varepsilon^2$, and the third term is of order $\varepsilon^2|\log\eta|^{-2}$. For the second term of the above summation, we compute directly and get

$$\|\log(|\cdot/\varepsilon|)\|^2_{L^2(\varepsilon Q)} \le \varepsilon^2\|\log(|x|+1)\|^2_{L^2(Q)} \le C\varepsilon^2.$$

Hence, $\|a_\eta|\log\eta|^{-1}\log(|\cdot/\varepsilon|)\|^2_{L^2(B_{\varepsilon/2}\setminus B_{\varepsilon\eta})}$ is of order $\varepsilon^2|\log\eta|^{-2}$.

The above estimates hold uniformly for all ε-cubes that have nonempty intersections with K, and the total number of such cubes is of order ε^{-2}. We hence obtain

$$\|v^\varepsilon - a_\eta\|^2_{L^2(K)} \le C|\log\eta|^{-1}. \tag{6.17}$$

The constant C depends only on T and K. This completes the proof of (6.12). ◻

Appendix A. Some useful lemmas.

A.1. A Poincaré inequality. We first record a Poincaré inequality for H^1 functions that vanishes in a set. This inequality is used heavily in our analysis.

THEOREM A.1. *Assume that* $0 < a < R$ *and* $\overline{B}_a \subset Q_R$. *Then there exists a positive constant* C *depending only on* d, *such that for any* $u \in H^1(Q_R)$ *satisfying* $u = 0$ *in* B_a, *we have*

$$\|u\|_{L^2(Q_R)} \le \begin{cases} CR\left(\frac{R}{a}\right)^{\frac{d-2}{2}} \|\nabla u\|_{L^2(Q_R)}, & d \ge 3, \\ CR(\log(R/a))^{\frac{1}{2}} \|\nabla u\|_{L^2(Q_R)}, & d = 2. \end{cases} \tag{A.1}$$

Clearly, the theorem still holds if the cube is changed to a ball and/or the ball is changed to a cube. This theorem is standard and a proof can be found in [2]. We record the proof here for the convenience of the reader.

Proof. We prove the theorem with Q_R is replaced by B_R. By density of smooth functions in H^1, we may assume that u is C^1. For any point $x \in Q_R \setminus \overline{B}_a$, we have

$$u(x) = u(re) = \int_a^r e \cdot \nabla u(se) ds.$$

Using Hölder's inequality, we then have the estimate

$$|u(x)|^2 \le \left(\int_a^r |\nabla u(se)|^2 s^{d-1} ds \right) \int_a^r \frac{1}{s^{d-1}} ds.$$

For $d = 2$, the explicit computation

$$\int_a^r \frac{1}{s^{d-1}} ds = \log r - \log a \tag{A.2}$$

facilitates the estimates below:

$$\begin{aligned} \|u\|^2_{L^2(B_R \setminus \overline{B}_a)} &\le \int_{S^{d-1}} d\sigma_e \int_a^R r^{d-1} \left[\left(\int_a^r |\nabla u(se)|^2 s^{d-1} ds \right) \log\left(\frac{r}{a}\right) \right] dr \\ &= \int_{S^{d-1}} d\sigma_e \int_a^R \left(\int_s^R r \log\left(\frac{r}{a}\right) dr \right) |\nabla u(se)|^2 s^{d-1} ds \\ &\le \int_{S^{d-1}} d\sigma_e \int_a^R \left(\frac{R^2}{2} \log \frac{R}{a} - \frac{s^2}{2} \log \frac{s}{a} \right) |\nabla u(se)|^2 s^{d-1}\, ds \\ &\le \left(\frac{R^2}{2} \log \frac{R}{a} \right) \int_{S^{d-1}} d\sigma_e \int_a^R |\nabla u(se)|^2 s^{d-1}\, ds \\ &= \left(\frac{R^2}{2} \log \frac{R}{a} \right) \|\nabla u\|^2_{L^2(B_R \setminus \overline{B}_a)}. \end{aligned}$$

This completes the proof for $d = 2$.

For $d \ge 3$, the explicit computation (A.2) is replaced by

$$\int_a^r \frac{1}{s^{d-1}} ds = \frac{1}{d-2} \frac{1}{r^{d-2}} \left[\left(\frac{r}{a}\right)^{d-2} - 1 \right] \le \frac{1}{d-2} \frac{1}{r^{d-2}} \left(\frac{r}{a}\right)^{d-2}.$$

We then have

$$
\begin{aligned}
\|u\|_{L^2}^2 \leq & \int_{S^{d-1}} d\sigma_e \int_a^R \left(\int_s^R \frac{r}{d-2} \left(\frac{r}{a}\right)^{d-2} dr \right) |\nabla u(se)|^2 s^{d-1} ds \\
\leq & \int_{S^{d-1}} d\sigma_e \int_a^R \frac{a^2}{d(d-2)} \left(\frac{R}{a}\right)^{d} |\nabla u(se)|^2 s^{d-1} ds \\
= & \frac{1}{d(d-2)} R^2 \left(\frac{R}{a}\right)^{d-2} \|\nabla u\|_{L^2}^2.
\end{aligned}
$$

This completes the proof for $d \geq 3$. □

Remark A.2. An easy consequence of the above theorem is, if $v \in H_0^1(D^\varepsilon)$, and D^ε is the perforated domain satisfying assumption (P1), then one has

$$
\|\widetilde{u}^\varepsilon\|_{L^2(D)} \leq C\sigma_\varepsilon \|\nabla \widetilde{u}^\varepsilon\|_{L^2(D)}.
$$

This inequality is most frequently used in this paper and is verified in the proof of Lemma 2.1.

A.2. Periodic potential theory. In section 5, we used *periodic layer potentials* to solve the cell problem (3.2) that is posed on the rescaled cube $\frac{1}{\eta}Q$, and we quantified the convergence of its mean value as $\eta \to 0$. Those potentials are natural generalizations of the classical layer potential operators associated to the Laplace operator in the whole space.

The starting point is to specify the fundamental solution to the Laplace operator in the unit periodic cell, or equivalently in the flat torus $Q_\mathfrak{p} = \mathbb{R}^d/\mathbb{Z}^d$. We seek a solution to

$$
\Delta G(x,y) = \delta_y(x) - 1 \qquad \text{in } Q_\mathfrak{p}
$$

in the distributional sense. Note that the volume of $Q_\mathfrak{p}$ is subtracted on the right-hand side; this is necessary for solving the problem, since $Q_\mathfrak{p}$ is a compact manifold without boundary. Clearly, the solution is unique only up to an additive constant. We hence impose the condition

$$
\int_Q G(x,y)dx = 0. \tag{A.3}
$$

The existence and uniqueness of G are then classical. As in Remark 3.1, we may view G either as defined on the torus or periodically over the whole space—in fact, the product of them with the diagonal is removed. It is easy to check that $G(x;y) = G_0(x-y)$, where $G_0 = G(\cdot;0)$. For notational simplicity, we still use G for G_0. It is well known that $G(x)$ is a smooth function in x except at the origin, and G has the following form:

$$
G(x) = \Gamma(x) + R(x), \qquad x \in Q \setminus \{0\}.
$$

Here, $\Gamma : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is the fundamental solution of the Laplace operator in the whole space given by the formulae

$$
\Gamma(x) = \begin{cases} \frac{1}{(n-2)c_d} \frac{1}{|x|^{d-2}}, & d \geq 3, \\ -\frac{1}{2\pi} \log|x|, & d = 2, \end{cases} \tag{A.4}
$$

where c_d is the volume of the unit sphere in $\mathbb{R}^d$. Clearly, R is the solution to the equation

$$\Delta R(x) = -1 \quad \text{in } Q,$$

with proper boundary conditions so that $R+\Gamma$ is periodic. Standard PDE theory then shows that R is smooth in $\overline{Q}$. Note that R itself does not satisfy periodic conditions at ∂Q.

To form periodic potentials in the η^{-1}-periodic cell $\eta^{-1}Q_{\mathfrak{p}}$, we rescale G and define $G^\eta(x;y) = \eta^{d-2}G(\eta(x-y))$ for $x, y \in \frac{1}{\eta}Q$ and $x \neq y$. It is the unique periodic fundamental solution to

$$\text{(A.5)} \qquad \Delta G^\eta(x;y) = \delta_y(x) - \eta^d \quad \text{in } \frac{1}{\eta}Q_{\mathfrak{p}} \qquad \text{and} \qquad \int_{\frac{1}{\eta}Q} G^\eta(x;y) = 0.$$

Then G^η has the decomposition formula

$$\text{(A.6)} \qquad G^\eta(x) = \Gamma(x) + \eta^{d-2}R(\eta x), \qquad x \in \eta^{-1}Q \setminus \{0\}.$$

For each fixed $\eta > 0$, we define the periodic single-layer potential operator on $\frac{1}{\eta}Q_{\mathfrak{p}}$ by

$$\text{(A.7)} \qquad \mathcal{S}^{(\eta)}_{T,\mathfrak{p}}[\varphi](x) := \int_{\partial T} G^\eta(x-y)\varphi(y)d\sigma(y), \qquad x \in \frac{1}{\eta}Q_{\mathfrak{p}} \setminus \partial T,$$

and define the periodic double-layer potential by

$$\text{(A.8)} \qquad \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\varphi](x) := \int_{\partial T} \nu_y \cdot \nabla G^\eta(x-y)\varphi(y)d\sigma(y), \qquad x \in \frac{1}{\eta}Q_{\mathfrak{p}} \setminus \partial T,$$

where $\varphi \in L^2(\partial T)$. In this paper, we only treat the case where the set T has regular boundary, i.e., under the following assumption.

(T1) T is open bounded and *connected*, and ∂T is $C^{1,\alpha}$ for some $\alpha \in (0,1)$.

In view of (A.6), the integrals above are regular integrals. It is easy to check that $\mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\varphi]$ is harmonic in T and in $\eta^{-1}Q_{\mathfrak{p}} \setminus \overline{T}$ for any $\varphi \in L^2(\partial T)$, and the same holds for $\mathcal{S}^{(\eta)}_{T,\mathfrak{p}}[\psi]$, but only if the further condition $\int_{\partial T}\psi = 0$ is imposed. We denote by $L^2_0(\partial T)$ the mean zero subspace of $L^2(\partial T)$.

In view of the decomposition formula (A.6), we find that, in some sense, the periodic potentials are "perturbations" to the classical potentials associated to the Laplace operator in the whole space. More precisely,

$$\mathcal{S}^{(\eta)}_{T,\mathfrak{p}}[\varphi] = \mathcal{S}_T[\varphi] + \eta^{d-2}\mathcal{R}_1[\varphi], \qquad \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\varphi] = \mathcal{D}_T[\varphi] + \eta^{d-1}\mathcal{R}_2[\varphi] \quad \text{in} \quad \frac{1}{\eta}Q_{\mathfrak{p}} \setminus \partial T.$$

where $\mathcal{S}_T$ and $\mathcal{D}_T$ are the classical single- and double-layer potentials, respectively, defined similarly to (A.7) and (A.8) with G^η replaced by Γ. The operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are well-defined by

(A.9)

$$\mathcal{R}_1[\varphi](x) = \int_{\partial T} R(\eta(x-y))\varphi(y)d\sigma_y \quad \text{and} \quad \mathcal{R}_2[\varphi](x) = \int_{\partial T} \nu_y \cdot \nabla R(\eta(x-y))\varphi(y)d\sigma_y.$$

In view of the smoothness of R, the operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are smoothing.

We refer the reader to [14, 6] for some practical guide to layer potential techniques in solving Laplace equations, and to [12, 29] for some original developments. Many results there for the classical layer potentials can be carried out almost in parallel for periodic potentials. Here are some basic results.

THEOREM A.3. *Assume* (T1). *Then for any* $\psi \in L^2(\partial T)$, *the following hold:*

(1) *The traces of the double-layer potential* $\mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\varphi]$ *from the outer and inner sides of* ∂T *exist and satisfy the jump relation*

$$\text{(A.10)}\quad \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi]\Big|_{\pm}(x) = \lim_{t\to 0+} \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi](x+t\nu_x) = \mp\frac{1}{2}\psi(x) + \mathcal{K}^{(\eta)}_{T,\mathfrak{p}}[\psi](x), \qquad x\in\partial T,$$

where $\mathcal{K}^{(\eta)}_{T,\mathfrak{p}}$ *is the generalized Neumann–Poincaré operator defined by*

$$\text{(A.11)}\qquad \mathcal{K}^{(\eta)}_{T,\mathfrak{p}}[\psi](x) = \int_{\partial T} \nu_x \cdot \nabla_y G^\eta(x-y)\psi(y)dy.$$

For the single-layer potential, we have

$$\text{(A.12)}\qquad \mathcal{S}^{(\eta)}_{T,\mathfrak{p}}[\psi]\big|_{+}(x) = \mathcal{S}^{(\eta)}_{T,\mathfrak{p}}[\psi]\big|_{-}(x), \qquad x\in\partial T.$$

(2) *The inner and outer normal derivatives, that is,*

$$\frac{\partial}{\partial\nu}\Big|_{\pm} \mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi](x) = \lim_{t\to 0+} \nu_x\cdot\nabla\mathcal{D}^{(\eta)}_{T,\mathfrak{p}}[\psi](x+t\nu_x), \qquad x\in\partial T,$$

exist and agree on ∂T. *On the other hand, for the single-layer potential, we have*

$$\text{(A.13)}\qquad \frac{\partial}{\partial\nu}\Big|_{\pm} \mathcal{S}^{(\eta)}_{T,\mathfrak{p}}[\psi](x) = \pm\frac{1}{2}\psi(x) + (\mathcal{K}^{(\eta)}_{T,\mathfrak{p}})^*[\psi](x), \qquad x\in\partial T.$$

Here and in what follows, the adjoint of an operator $\mathcal{A}$ is denoted by $\mathcal{A}^*$. For simplicity, we use the shorthand notation $\mathcal{K}^\eta_T$ for $\mathcal{K}^{(\eta)}_{T,\mathfrak{p}}$; the shorthand $\mathcal{S}^\eta_T$ and $\mathcal{D}^\eta_T$ are understood similarly. The theorem above can be proved easily following the usual arguments; see, in particular, [6, section 2.8]. In view of (A.6) again, $\mathcal{K}^\eta_T$ can be viewed as a perturbation to the classical Neumann–Poincaré operator $\mathcal{K}_T$, which is defined as in (A.10) with G^η replaced by Γ. More precisely, we have

$$\text{(A.14)}\qquad \mathcal{K}^\eta_T[\varphi] = \mathcal{K}_T[\varphi] + \eta^{d-1}\mathcal{R}_2[\varphi] \qquad \text{on } \partial T.$$

Here, $\mathcal{R}_2$ is defined as before but with $x\in\partial T$; it is well-defined in view of the smoothness of ∂T. It is a different operator, but we abuse notation and still denote it by $\mathcal{R}_2$.

Thanks to the regularity of ∂T, the integrals in $\mathcal{K}_T$ and $\mathcal{K}^\eta_T$ are regular. Those operators map $L^2(\partial T)$ to $H^1(\partial T)$ and hence are compact in $L^2(\partial T)$. It follows that $\pm\frac{1}{2}I + \mathcal{K}_T$ is Fredholm; the same statement holds for $-\frac{1}{2}I + \mathcal{K}^\eta_T$ and for their adjoint operators. Those results can be found in, e.g., [14, Chapters 1 and 3].

Finally, we present some further mapping properties regarding the Neumann–Poincaré operators.

THEOREM A.4. *Assume* (T1) *and* $d\ge 3$. *Then the following results hold:*

(1) $\mathrm{Ker}(-\frac{1}{2}I + \mathcal{K}_T)$ *is one-dimensional and spanned by the constant function* $\psi_* \equiv 1$.

(2) $\mathrm{Ker}(-\frac{1}{2}I+\mathcal{K}_T^*)$ *is one-dimensional and spanned by the unique function* $\varphi_* \in L^2(\partial T)$ *which satisfies the following:* (φ_*, a_*) *is the unique pair of elements in* $L^2(\partial T)\times\mathbb{R}$ *verifying*

$$\mathcal{S}_T[\varphi_*] + a_* = 0 \quad \text{in } \partial T \qquad \text{and} \qquad \int_{\partial T} \varphi_* = 1.$$

Here, $\mathcal{S}_T$ *is understood as the trace of the single-layer potentials from the exterior of* T.

(3) *The decomposition* $L^2(\partial T) = \mathrm{Ran}(-\frac{1}{2}I+\mathcal{K}_T) \oplus \mathrm{Ker}(-\frac{1}{2}I+\mathcal{K}_T)$ *holds.*

(4) *The operator* $-\frac{1}{2}I+\mathcal{K}_T : L_0^2(\partial T) \to \mathrm{Ran}(-\frac{1}{2}I+\mathcal{K}_T)$ *is invertible.*

The results above are classical: item (1) is proved in [14, Chapter 3]; the characterization in item (2) can be found in [6, Theorem 2.26]; item (3) is proved in [14, Corollary 3.39]. Using Fredholm theory, the fourth term can be easily proved by showing the injectivity of the operator. We remark that the dimensions determined above are due to the assumption that T is connected, which is imposed only for simplicity. If T has multiple connected components, the above theorem can be generalized accordingly.

COROLLARY A.5. *Assume* (T1) *and* $d \geq 3$. *Then, for any* $h \in L^2(\partial T)$, *it can be written as*

$$h = h_1 + h_0, \qquad h_0 := \langle h, \varphi_* \rangle_{L^2,L^2} \in \mathbb{R}, \quad \text{and} \quad h_1 \in \mathrm{Ran}\left(-\frac{1}{2}I + \mathcal{K}_T\right).$$

In particular, $\langle \Gamma\big|_{\partial T}, \varphi_* \rangle = -\frac{1}{\mathrm{Cap}(T)}$.

Proof. The decomposition of h follows from item (2) of the previous theorem. To find the constant component h_0, we use the fact that $\mathrm{Ran}(-\frac{1}{2}I+\mathcal{K}_T)$ is closed (by the Fredholm theory) and that it is orthogonal to $\mathrm{Ker}(-\frac{1}{2}I+\mathcal{K}_T^*)$. By item (3) of the previous theorem, we get the result.

To find $\langle \Gamma\big|_{\partial T}, \varphi_* \rangle$, we first recall the fact that $a_* = 1/\mathrm{Cap}(T)$; see, e.g., [6]. Also, by the continuity of $\mathcal{S}_T[\varphi_*]$ across ∂T and that $\mathcal{S}_T[\varphi_*]$ is harmonic in T, we conclude that $\mathcal{S}_T[\varphi_*] = -a_*$ in T. Then by definition and the symmetry of $\Gamma(\cdot,\cdot)$, we have

$$\langle \Gamma\big|_{\partial T}, \varphi_* \rangle = \int_{\partial T} \Gamma(0-y)\varphi_*(y)d\sigma_y = \mathcal{S}_T[\varphi_*](0) = -a_* = -\frac{1}{\mathrm{Cap}(T)}.$$

This completes the proof. ∎

Finally, we prove the following mapping property for the periodic double-layer potential.

THEOREM A.6. *Assume* (T1) *and* $d \geq 3$. *Then* $\mathrm{Ran}(-\frac{1}{2}I+\mathcal{K}_T^\eta) = L^2(\partial T)$. *In particular,*

$$\left(-\frac{1}{2}I + \mathcal{K}_T^\eta\right)^{-1}[1] = (-\eta^d|T|)^{-1}.$$

Proof. First, by direct computation we check that $\mathcal{K}_T^\eta[1] = \frac{1}{2} - \eta^d|T|$, so we only need to prove the first statement of the theorem. By the compactness of $\mathcal{K}_T^\eta$ in $L^2(\partial T)$, the range of $-\frac{1}{2}I+\mathcal{K}_T^\eta$ is closed and equals the orthogonal complement of

$\mathrm{Ker}(-\frac{1}{2}I + (\mathcal{K}_T^\eta)^*)$. Taking an arbitrary element φ from this kernel, we then have

$$0 = \int_{\partial T} \left(-\frac{1}{2}I + (\mathcal{K}_T^\eta)^*\right)[\varphi] = \left\langle \left(-\frac{1}{2}I + \mathcal{K}_T^\eta\right)[1], \varphi \right\rangle_{L^2,L^2} = -\eta^d |T| \int_{\partial T} \varphi.$$

We conclude that $\varphi \in L^2_0(\partial T)$. It follows that $\mathcal{S}_T^\eta[\varphi]$ is harmonic in $\frac{1}{\eta}Q_{\mathfrak{p}} \setminus \overline{T}$ and in T. Let u_i and u_e denote the restriction of $\mathcal{S}_T^\eta[\varphi]$ in and outside T. We then have

$$\int_T |\nabla u_i|^2 = \int_{\partial T} u_i \left.\frac{\partial u_i}{\partial \nu}\right|_- = \int_{\partial T} u_i \left(-\frac{1}{2}I + (\mathcal{K}_T^\eta)^*\right)[\varphi] = 0.$$

Hence, $u_i = C$ is a constant function. Similarly, we have

$$\begin{aligned}\int_T |\nabla u_e|^2 &= -\int_{\partial T} u_e \left.\frac{\partial u_i}{\partial \nu}\right|_+ = -\int_{\partial T} u_e \left(\frac{1}{2}I + (\mathcal{K}_T^\eta)^*\right)[\varphi] \\ &= -\int_{\partial T} u_e \varphi = -\int_{\partial T} u_i \varphi = -C \int_{\partial T} \varphi = 0.\end{aligned}$$

In the second line above, we used the continuity of $\mathcal{S}_T^\eta[\varphi]$ across ∂T, and that $\varphi \in L^2_0$. It follows that $u_e = u_i = C$, and by the jump relation in Theorem A.3 we get $\varphi = \partial_\nu u_e\big|_+ - \partial_\nu u_i\big|_- = 0$. We conclude that $\mathrm{Ker}(-\frac{1}{2}I + (\mathcal{K}_T^\eta)^*)$ contains only 0 and that $\mathrm{Ran}(-\frac{1}{2}I + \mathcal{K}_T^\eta)$ is the whole space. ∎

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