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On Euler characteristic and fundamental groups of compact manifolds

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Abstract

Let M be a compact Riemannian manifold, $\pi : \widetilde{M} \to M$ be the universal covering and ω be a smooth 2-form on M with $\pi^*\omega$ cohomologous to zero. Suppose the fundamental group $\pi_1(M)$ satisfies certain radial quadratic (resp. linear) isoperimetric inequality, we show that there exists a smooth 1-form η on $\widetilde{M}$ of linear (resp. bounded) growth such that $\pi^*\omega = d\eta$. As applications, we prove that on a compact Kähler manifold (M, ω) with $\pi^*\omega$ cohomologous to zero, if $\pi_1(M)$ is CAT(0) or automatic (resp. hyperbolic), then M is Kähler non-elliptic (resp. Kähler hyperbolic) and the Euler characteristic $(-1)^{\frac{\dim_{\mathbb{R}} M}{2}}\chi(M) \geq 0$ (resp. > 0).

Contents

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1 Introduction

In differential geometry, there is a well-known conjecture due to H. Hopf (e.g. [32, Problem 10]):

Conjecture 1.1 (Hopf) *Let M be a compact, oriented and even dimensional Riemannian manifold of negative sectional curvature $K < 0$. Then the signed Euler characteristic $(-1)^{\frac{n}{2}}\chi(M) > 0$, where n is the real dimension of M.*

For $n = 4$, Conjecture 1.1 was proven by Chern [8] (who attributed it to Milnor). Not much has been known in higher dimensions. This conjecture can not be established just by use of the Gauss–Bonnet–Chern formula (see [13,22]). Singer suggested that in view of the L^2-index theorem an appropriate vanishing theorem for L^2-harmonic forms on the universal covering of M would imply the conjecture (e.g. [10]). In the work [18], Gromov introduced the notion of Kähler hyperbolicity for Kähler manifolds which means the Kähler form on the universal cover is the exterior differential of some bounded 1-form. He established that the L^2-cohomology groups of the universal covering of a Kähler hyperbolic manifold are not vanishing only in the middle dimension. Combining this result and the covering index theorem of Atiyah, Gromov showed $(-1)^{\frac{n}{2}}\chi(M) > 0$ for a Kähler hyperbolic manifold M. One can also show that a compact Kähler manifold homotopic to a compact Riemannian manifold of negative sectional curvature is Kähler hyperbolic and the canonical bundle is ample [7].

When the sectional curvature of the manifold is non-positive, it is natural to consider whether $(-1)^{\frac{n}{2}}\chi(M) \geq 0$ holds. It should be noted that when the sectional curvature of a compact Kähler manifold is nonpositive [9,20] proved independently that the vanishing theorem of Gromov type still holds and the Euler characteristic satisfies $(-1)^{\frac{n}{2}}\chi(M) \geq 0$. Actually, they proved that the results also hold if the pulled-back Kähler form on the universal covering is the exterior differential of some 1-form with linear growth, and such manifolds are called Kähler non-elliptic [20] and Kähler parabolic [9] respectively. In the sequel, for simplicity, we shall use one of these notions, e.g., Kähler non-elliptic. Moreover, Jost and Zuo proposed an interesting question in [20, p. 4] that whether there are some topological conditions to ensure the manifolds to be Kähler non-elliptic. One can also propose the following

Question 1.2 Let M be a compact Riemannian manifold, $\pi : \hat{M} \to M$ a Galois covering and G the group of covering transformations. Let ω be a closed q-form ($q \geq 2$) on M such that $[\pi^*\omega] = 0$ in $H^q_{\mathrm{dR}}(\hat{M})$. Find a $(q-1)$-form η on $\hat{M}$ of least growth order (in terms of the distance function on $\hat{M}$) such that $\pi^*\omega = d\eta$.

It is clear that the growth order of η does not depend on the choices of the metrics on M, and it should depend on the geometry of the covering transformation group G. Recall that by a theorem of Gromov, a discrete group G is hyperbolic if and only if it satisfies a linear isoperimetric inequality. An expected answer for Question 1.2 would be a relation between certain isoperimetric inequality of the covering transformation group G and the least growth order of η.

We need some well-known notions of discrete groups to formulate an isoperimetric inequality in our setting. Suppose $G = \langle S|R\rangle$ is a finitely presented group, where S is a finite symmetric set generating G, $S = S^{-1}$, and R is a finite set (relator set) in the free group F_S generated by S. The word metric on G with respect to S is defined as

$$d_S(a, b) = \min\{n : b^{-1}a = s_1 s_2 \ldots s_n, s_i \in S\}. \tag{1.1}$$

For a word $w = s_1 s_2 \ldots s_n$, its length $L(w)$ is defined to be n. If the word $w = s_1 \ldots s_n \in F_S$ representing the identity e in G, there are reduced words $v_1, \ldots, v_k$ on S such that

$$w = \prod_{i=1}^{k} v_i r_i v_i^{-1}, \quad r_i \quad \text{or} \quad r_i^{-1} \in R \tag{1.2}$$

as elements in F_S. The *combinatorial area* $\mathrm{Area}(w)$ of w is the smallest possible k for Eq. (1.2).

Definition 1.3 We say a finitely presented group $G = \langle S|R\rangle$ satisfies a *radial isoperimetric inequality* of degree p, if there is a constant $C > 0$ such that for any word $w = s_1 \ldots s_n \in F_S$ of length $L(w) = n$ representing the identity e in G, we have

$$\mathrm{Area}(w) \leq C \sum_{i=1}^{n} (d_S(\bar{w}(i), e) + 1)^{p-1}, \tag{1.3}$$

where $w(i) = s_1 \ldots s_i$ is the i-th subword of w and $\bar{w}(i) \in G$ is the natural image (from F_S to G) of the word $w(i)$.

It is easy to see that the above definition is independent of the choice of the generating set S. For $p = 1$, this definition is the same as the usual linear isoperimetric inequality

$$\mathrm{Area}(w) \leq CL(w).$$

For $p > 1$, Definition 1.3 is stronger than the usual isoperimetric inequality. Actually, the radial isoperimetric inequality (1.3) can imply

$$\mathrm{Area}(w) \leq C(\mathrm{diam}(w) + 1)^{p-1} L(w) \leq CL(w)^p. \tag{1.4}$$

We obtain in Theorem 3.1 a complete answer to Question 1.2 for $q = 2$. For simplicity, we only formulate the polynomial growth case which has many important applications.

Theorem 1.4 *Let M be a compact Riemannian manifold, $\pi : \hat{M} \to M$ a Galois covering with covering transformation group G and $H^1_{\mathrm{dR}}(\hat{M}) = 0$. Let ω be a closed 2-form on M such that $[\pi^*\omega] = 0 \in H^2_{\mathrm{dR}}(\hat{M})$. Assume that the group G satisfies the radial isoperimetric inequality (1.3) of degree $p \geq 1$. Then there exists a smooth 1-form η on $\hat{M}$ such that $\pi^*\omega = d\eta$ and*

$$|\eta|(x) \leq C(d_{\hat{M}}(x, x_0) + 1)^{p-1} \tag{1.5}$$

for all $x \in \hat{M}$ where C is a positive constant and $x_0 \in \hat{M}$ is a fixed point.

Let's explain briefly the key ingredients in the proof of Theorem 1.4 and demonstrate the significance of the radial isoperimetric inequality (1.3). Let $G = \langle S|R\rangle$ be the finitely presented covering transformation group. The condition $[\pi^*\omega] = 0$ implies $\pi^*\omega = d\eta$ for some smooth 1-form η on $\widetilde{M}$.

(i) We show that there exists a constant C such that for any closed curve α on $\widetilde{M}$, we can construct a word $w \in F_S$ representing the identity and approximating α such that

$$\left|\int_\alpha \eta\right| \leq C(L(\alpha) + \mathrm{Area}(w)). \tag{1.6}$$

(ii) By using the radial isoperimetric inequality (1.3), we prove

$$\int_\alpha \eta \leq C \int_\alpha (d_{\hat{M}}(x, x_0) + 1)^{p-1} ds \tag{1.7}$$

for any closed curve α.

(iii) η can be regarded as a "bounded linear functional" L on the space of closed curves with a suitably defined norm. Then we use the Hahn–Banach theorem and Whitney's local flat norm [31] to find another bounded linear functional $\widetilde{L}$ whose restriction on the space of closed curves is L. Moreover, $\widetilde{L}$ is represented by a differential form $\widetilde{\eta}$ with measurable coefficients with $\pi^*\omega = d\widetilde{\eta}$ (in the current sense) and

$$|\widetilde{\eta}|(x) \leq C(d_{\hat{M}}(x, x_0) + 1)^{p-1}, \quad \text{a.e.} \tag{1.8}$$

(iv) We use the heat equation method to deform $\widetilde{\eta}$ to a smooth one with the desired bound in (1.5).

The radial isoperimetric inequality (1.3) is the key ingredient in step (ii), and it could not work for usual isoperimetric inequality for degree $p > 1$. With the inequality (1.7) in hand, step (iii) is classical (e.g. [16,19,29,30]). The smoothing process in step (iv) is natural in the view point of PDE since $\omega = d\eta$ is preserved under the specified heat equation (3.16) and the estimate (1.5) follows from standard apriori estimate of heat equations.

As an application of Theorem 1.4, we obtain

Theorem 1.5 *Let (M, ω) be a compact Kähler manifold and $\pi : \widetilde{M} \to M$ be the universal covering. Suppose $[\pi^*\omega] = 0 \in H^2_{\mathrm{dR}}(\widetilde{M})$. If $\pi_1(M)$ satisfies the radial isoperimetric inequality (1.3) of degree $p = 2$, then M is Kähler non-elliptic and $(-1)^{\frac{\dim_{\mathbb{R}} M}{2}}\chi(M) \geq 0$.*

It is a natural question to ask whether Theorem 1.5 can hold for the usual quadratic isoperimetric inequality. It is well-known that the automatic groups and CAT(0) groups are important examples satisfying the usual quadratic isoperimetric inequality. Recall that CAT(0)-groups are groups which can act properly discontinuously and cocompactly by isometries on proper geodesic CAT(0)-metric spaces. Typical examples of CAT(0)-groups are fundamental groups of compact manifolds with non-positive sectional curvature. Automatic groups are finitely generated groups whose operations are governed by automata. For instances, hyperbolic groups and mapping class groups are automatic [27]. These groups have been studied extensively in geometric group theory (e.g. [3]). We prove in Theorems 4.1 and 4.2 that CAT(0) groups and automatic groups do satisfy the radial isoperimetric inequality (1.3) for $p = 2$. Hence, we obtain the following result, which also gives an answer to the aforementioned question proposed by Jost-Zuo:

Theorem 1.6 *Let (M, ω) be a compact Kähler manifold and $\pi : \widetilde{M} \to M$ be the universal covering. Suppose $[\pi^*\omega] = 0 \in H^2_{\mathrm{dR}}(\widetilde{M})$. If $\pi_1(M)$ is* CAT(0) *or automatic, then M is Kähler non-elliptic and the Euler characteristic $(-1)^{\frac{\dim_{\mathbb{R}} M}{2}} \chi(M) \geq 0$.*

One can also deduce the following result easily from the special case for $p = 1$ in Theorem 1.4.

Corollary 1.7 *Let (M, ω) be a compact Kähler manifold and $\pi : \widetilde{M} \to M$ be the universal covering. Suppose $[\pi^*\omega] = 0 \in H^2_{\mathrm{dR}}(\widetilde{M})$. If $\pi_1(M)$ is a hyperbolic group, then M is Kähler hyperbolic and the Euler characteristic $(-1)^{\frac{\dim_{\mathbb{R}} M}{2}} \chi(M) > 0$.*

Remark 1.8 Note that the condition $[\pi^*\omega] = 0$ is equivalent to the fact that $\pi^*\omega = d\eta$ for some 1-form η, which holds trivially on Kähler hyperbolic or Kähler non-elliptic manifolds. It is not hard to see that this condition holds if $\pi_2(M)$ is torsion. In particular, one gets the following fact pointed out by Gromov [18, p. 266]: if $\pi_1(M)$ is hyperbolic and $\pi_2(M) = 0$, then M is Kähler hyperbolic.

Remark 1.9 When (M, ω) is a symplectic manifold, one can obtain the corresponding hyperbolicity or non-ellipticity from Theorem 1.4 for $p = 1$ or 2 as in the Kähler case. See [21, Corollary 1.12] for the former case.

Remark 1.10 It worths to point out that, every finitely generated nilpotent group of class p satisfies the usual isoperimetric inequality of degree $p + 1$ [15]. On the other hand, Cheeger–Gromov proved in [5] that if the fundamental group $\pi_1(M)$ of a compact aspherical manifold M contains an infinite normal amenable subgroup, then the Euler characteristic $\chi(M) = 0$. Hence it would be very interesting to ask: whether $\pi_1(M)$ satisfies the radial isoperimetric inequality for $p = 2$ if $\pi_1(M)$ does not contain an infinite normal amenable subgroup. If this were true, then any compact Kähler aspherical manifold M has non-negative signed Euler characteristic, i.e. $(-1)^{\frac{\dim_{\mathbb{R}} M}{2}} \chi(M) \geq 0$. This is a special case of more general conjecture proposed by Singer :

Conjecture: any compact aspherical manifold M has non-negative signed Euler characteristic.

When X is an algebraic manifold, there is an algebraic surface Y such that $\pi_1(X) = \pi_1(Y)$ by Lefschetz hyperplane theorem. It is not hard to see that, in this case we only need to consider the question for algebraic surfaces of general type thanks to the Enriques-Kodaira classification (e.g. [2, Chapter V]).

In some sense, the present paper is to build up one relationship between the geometry of the fundamental groups $\pi_1(X)$ and the geometry/topology of the space X itself. There are many important works when the fundamental group $\pi_1(X)$ admits a finite dimensional representation with infinite image. We refer to [1,4,11,18,20,23,26,33,34] and the references therein.

The paper is organized as follows. In Sect. 2, to exhibit the key ingredients in the proof of Theorem 1.4, we show a special case when the covering transformation group G is hyperbolic. In Sect. 3, we deal with more general G in Theorem 3.1 and establish Theorems 1.4 and 1.5. In Sect. 4, we show that CAT(0) groups and automatic groups satisfy the radial isoperimetric inequality (1.3) for $p = 2$ and prove Theorem 1.6.

2 Hyperbolic fundamental groups

Let G be a finitely generated group with a finite set S generating G, where $S = S^{-1}$. The *Cayley graph* $\Gamma(G, S)$ of G with respect to S is a graph satisfying

(1) The vertices of $\Gamma(G, S)$ are the elements of G;
(2) Two vertices $x, y \in G$ are joined by an edge if and only if there exists an element $s \in S$ such that $x = ys$.

We define a metric $d_{\Gamma(G,S)}$ on a Cayley graph $\Gamma(G, S)$ by letting the length of every edge be 1 and defining the distance between two points to be the minimum length of arcs joining them. This metric $d_{\Gamma(G,S)}$ is actually the word metric d_S on G defined in (1.1). We say the group G is hyperbolic if (G, d_S) is a hyperbolic metric space in the sense of Gromov ([17], or [28, Proposition 2.6]), i.e. there is a positive number $\delta > 0$ such that for any four points $x_1, x_2, x_3, x_4 \in G$, we have

$$d_S(x_1, x_2)+d_S(x_3, x_4)\leq \max\{d_S(x_1, x_3)+d_S(x_2, x_4); d_S(x_1, x_4) + d_S(x_2, x_3)\} + 2\delta.$$

One can check that the hyperbolicity of G is independent of the finite generating set S. It can be shown that if the group (G, d_S) is hyperbolic, then G is finitely presented, i.e. $G = \langle S|R\rangle$ where $R \subset F_S$ is a finite relator set. It is well-known [17] that G is hyperbolic if and only if it satisfies the linear isoperimetric inequality:

$$\mathrm{Area}(w) \leq CL(w), \tag{2.1}$$

for any word w in the free group F_S representing the identity e in G. For more comprehensive introductions, we refer to [17,28] and the references therein.

Theorem 2.1 *Let M be a compact manifold and $\pi : \widetilde{M} \to M$ be a Galois covering such that $H^1_{\mathrm{dR}}(\widetilde{M}) = 0$. Let ω be a smooth closed 2-form on M such that $[\pi^*\omega] = 0$ in $H^2_{\mathrm{dR}}(\widetilde{M})$. If the covering transformation group G is hyperbolic, then there exists a bounded smooth 1-form $\widetilde{\eta}$ on $\widetilde{M}$ such that $\pi^*\omega = d\widetilde{\eta}$.*

Fix a smooth reference metric on M and endow $\widetilde{M}$ with the induced metric. Since $[\pi^*\omega] = 0$ in $H^2_{\mathrm{dR}}(\widetilde{M})$, there is a smooth 1-form η on $\widetilde{M}$ such that $\pi^*\omega = d\eta$. If η is bounded, we are done. If it is not bounded, we shall show that there exists another bounded smooth 1-form $\widetilde{\eta}$ on $\widetilde{M}$ such that $\pi^*\omega = d\widetilde{\eta} = d\eta$. The proof of Theorem 2.1 is divided into the following steps.

Lemma 2.2 *For any $L > 0$, there is a constant $D = D(L)$ depending on L such that for any smooth closed curve α in $\widetilde{M}$ with length $L(\alpha) \leq L$, we have*

$$\left| \int_\alpha \eta \right| \leq D. \tag{2.2}$$

Proof Let $\Omega \subset \widetilde{M}$ be a fundamental domain of the group of deck transformations G. If α passes through Ω, then the inequality (2.2) can be derived from the smoothness of η. If α does not pass though Ω, there exists some $g \in G$ such that $g^{-1}(\alpha)$ passes through Ω. Hence,

$$\left| \int_{g^{-1}(\alpha)} \eta \right| \leq D$$

as in the previous case. Since $g^*(\omega) = \omega$, we know

$$d(g^*\eta - \eta) = 0.$$

On the other hand, since $H^1_{\mathrm{dR}}(\widetilde{M}) = 0$, there exists a smooth function f such that $g^*\eta - \eta = df$. Now (2.2) follows from

$$\left| \int_\alpha \eta \right| = \left| \int_{g^{-1}(\alpha)} g^*\eta \right| = \left| \int_{g^{-1}(\alpha)} \eta + df \right| = \left| \int_{g^{-1}(\alpha)} \eta \right| \leq D$$

since α is closed and $\int_{g^{-1}(\alpha)} df = 0$. $\square$

Lemma 2.3 *There exists a constant $C > 0$ such that for any smooth closed curve α on $\widetilde{M}$, we have*

$$\left| \int_\alpha \eta \right| \leq C \cdot L(\alpha). \tag{2.3}$$

Note that in the sequel the constant C can vary from line to line.

Proof By perturbations, we may assume $\alpha : [0, L] \to \widetilde{M}$ is parameterized by the arclength, where $L = L(\alpha)$ and $|\alpha'(s)| \equiv 1$. Without loss of generality, we assume $L \geq 1$. Fix a base point x_0 in the fundamental domain Ω. Let $[L]$ be the integer part of L. For each $i = 0, 1, 2 \ldots, [L]$, we choose $g_i \in G$ such that $\alpha(i) \in g_i(\Omega)$. If L is an integer, we choose $g_L = g_0$. Otherwise, we set $g_{[L]+1} = g_0$. For simplicity, we set N to be L if L is an integer, otherwise N is $[L] + 1$. Connecting $g_i(x_0)$ and $g_{i+1}(x_0)$ with a geodesic segment $\widetilde{\gamma}_i$, and $\widetilde{\gamma} = \widetilde{\gamma}_0 \cup \widetilde{\gamma}_1 \cup \cdots \cup \widetilde{\gamma}_{N-1}$ is a closed curve of length

$$L(\widetilde{\gamma}) \leq CL(\alpha)$$

for some C independent of the curve α. We claim

$$\left| \int_\alpha \eta - \int_{\widetilde{\gamma}} \eta \right| \leq CL(\alpha). \tag{2.4}$$

Connecting $\alpha(i)$ and $g_i(x_0)$ with a geodesic segment β_i, we get N closed curves $\hat{\gamma}_i = \alpha\mid_{[i,i+1]} \cup \beta_{i+1} \cup \widetilde{\gamma}_i^{-1} \cup \beta_i^{-1}$, $i = 0, 1, \ldots, N-1$. Hence,

$$\int_\alpha \eta - \int_{\widetilde{\gamma}} \eta = \sum_{i=0}^{N-1} \int_{\hat{\gamma}_i} \eta. \tag{2.5}$$

By our construction, there is a constant C such that $L(\hat{\gamma}_i) \leq C$. The claim (2.4) follows from the estimate (2.2) and (2.5).

Let $G = \langle S|R\rangle$, where the generator set $S = S^{-1}$ and relator set R are all finite. Let $\Gamma(G, S)$ be the Cayley graph of G. Then it is a standard fact that $\Gamma(G, S)$ is quasi-isometric to $\widetilde{M}$, and this implies that there are constants $C_1, C_2 > 0$ such that

$$C_1^{-1} d_{\widetilde{M}}(a(x_0), b(x_0)) - C_2 \leq d_{\Gamma(G,S)}(a, b) \leq C_1 d_{\widetilde{M}}(a(x_0), b(x_0)) + C_2 \tag{2.6}$$

for any $a, b \in G$. From this, we know $d_{\Gamma(G,S)}(g_i, g_{i+1})$ are uniformly bounded by some constant C. Hence, for each i, $g_{i+1} = g_i s_1^i \ldots s_{n_i}^i$ where $s_j^i \in S$, $n_i \leq C$. Connecting $g_i(x_0), (g_i s_1^i)(x_0), \ldots, (g_i s_1^i \ldots s_{n_i}^i)(x_0)$ with geodesic segments $\gamma_1^i, \ldots, \gamma_{n_i}^i$ successively. It is clear the closed curve $\widetilde{\gamma}_i \cup (\gamma_1^i \cup \cdots \cup \gamma_{n_i}^i)^{-1}$ has uniformly bounded length. By similar arguments as in (2.4) and (2.5), we conclude that

$$\left| \int_\alpha \eta - \int_\gamma \eta \right| \leq CL, \tag{2.7}$$

where $\gamma = \left(\gamma_1^0 \cup \cdots \cup \gamma_{n_0}^0\right) \cup \cdots \cup \left(\gamma_1^{N-1} \cup \cdots \cup \gamma_{n_{N-1}}^{N-1}\right)$. Consider the word

$$w = s_1^0 \ldots s_{n_0}^0 \ldots s_1^{N-1} \ldots s_{n_{N-1}}^{N-1}$$

representing the identity in G. It has length

$$L(w) = n_0 + \cdots + n_{N-1} \leq CL. \tag{2.8}$$

Let $k_0 = \mathrm{Area}(w)$ be the combinatorial area of w and w_0 be a word equal to w as elements of F_S with the form

$$w_0 = \prod_{i=1}^{k_0} v_i r_i v_i^{-1} \tag{2.9}$$

for reduced words $v_1, \ldots, v_{k_0}$ on S and r_i or $r_i^{-1} \in R$, $i = 1, \ldots, k_0$. Note that w_0 is obtained from w by inserting several subwords of the form aa^{-1}, where a is a word on S. According to the previous construction of the curve γ from the word w, one can construct a curve γ_0 from the word w_0 by inserting several loops into γ of the form $\delta\delta^{-1}$ corresponding to the inserted subwords aa^{-1}. It is clear that

$$\int_{\gamma_0} \eta = \int_\gamma \eta.$$

Note that each subword $v_i r_i v_i^{-1}$ in w_0 represents the identity in the group G. In the correspondence of γ_0 and the word w_0, the subword $v_i r_i v_i^{-1}$ represents a closed curve δ_i based on $g_0(x_0)$, and r_i represents a closed curve ϵ_i based on the end points of a curve η_i which corresponds to v_i. Then $\delta_i = \eta_i \cup \epsilon_i \cup \eta_i^{-1}$ and

$$\int_{\delta_i} \eta = \int_{\epsilon_i} \eta.$$

Since the relator set R is finite, by Lemma 2.2, there is a constant $C > 0$ such that $|\int_{\epsilon_i} \eta| \leq C$ and

$$\left|\int_{\gamma} \eta\right| = \left|\sum_{i=1}^{k_0} \int_{\delta_i} \eta\right| \leq C\mathrm{Area}(w). \tag{2.10}$$

Therefore, by (2.7) and (2.10), we obtain

$$\left|\int_{\alpha} \eta\right| \leq C(L(\alpha) + \mathrm{Area}(w)). \tag{2.11}$$

From the linear isoperimetric inequality (2.1), there is a constant $C_1 > 0$ such that

$$\mathrm{Area}(w) \leq C_1 \cdot L(w) \leq \widetilde{C} \cdot L(\alpha), \tag{2.12}$$

where the second inequality follows from (2.8). By (2.11), we obtain the estimate (2.3). $\square$

Note that, from (2.3), we can not deduce the norm of η is bounded by C, since (2.3) holds only for *closed curves*. If we regard η as a linear functional on the space of general curves, η is not necessarily bounded. We shall use the Hahn–Banach theorem to find another bounded $\widetilde{\eta}$ whose restriction on the space of closed curves is η. The following result is inspired by the classical results [19, Proposition 4.35] and [12, Section 4.10] which are based on [31].

Lemma 2.4 *There exists a 1-form $\widetilde{\eta}$ on $\widetilde{M}$ with bounded measurable coefficients such that $\pi^*\omega = d\widetilde{\eta}$ in the sense of currents.*

Proof We follow the framework of [31, Chapter V and Theorem 5A]. By the classical triangulation theorem (e.g. [31, Chapter IV]), there is a simplicial complex K and a homomorphism $T : |K| \to \widetilde{M}$ with the following property. For each n-simplex σ of K (n is the real dimension of $\widetilde{M}$), there is a coordinate system χ in M such that χ^{-1} is defined in a neighborhood of $T(\sigma)$ in $\widetilde{M}$ and $\chi^{-1}T$ is affine in σ. Let $K^{(1)}$ be the first barycentric subdivision of K, and $K^{(2)}$ be the second, ... For each $p > 0$, consider the chain groups (over $\mathbb{R}$), $C_p(K)$, $C_p(K^{(1)})$, $C_p(K^{(2)})$, and etc. We regard a simplex σ as the sum of its subdivisions. Then there are natural inclusions $C_p(K) \subset C_p(K^{(1)}) \subset \cdots$. Denote $C_p(K^{\infty}) = \cup C_p(K^{(i)})$, and equip $C_p(K^{\infty})$ a norm (mass) in the following manner: if $c = \sum a_i \sigma_i \in C_p(K^{(j)})$ is a p-chain such that σ_i are mutually disjoint (i.e. the interiors of σ_i are mutually disjoint), then

$$\|c\| = \sum_i |a_i| \times \mathrm{Area}_{\widetilde{M}}(T(\sigma_i)).$$

Now $(C_p(K^\infty), \|\cdot\|)$ is a normed vector space. It can be shown that for $c \in C_p(K^\infty)$,

$$\|c\| = \sup\left\{\left|\int_c \theta\right| : \theta \text{ is a} p - \text{form with } \|\theta\| \le 1\right\}.$$

The norm (comass) of a differential form θ of degree p is defined as

$$\|\theta\| = \sup_{\widetilde{M}} \sup |\theta(e_1, \ldots, e_p)| \tag{2.13}$$

where the inside supremum is taken over all the orthonormal frames $(e_1, \ldots, e_p, \ldots, e_n)$ in $T\widetilde{M}$. See [19, p. 245–246] for more details. To invoke the main result in [31, Theorem 5A], we need to recall the flat norm $|\cdot|^\flat$ defined over $C_1(K^\infty)$ by

$$|c|^\flat = \inf\{\|c - \partial D\| + \|D\|\} \tag{2.14}$$

where D ranges over all 2-chains. For any differential 1-form θ, one can show

$$|\theta|^\flat = \sup\left\{\left|\int_c \theta\right| : |c|^\flat \le 1\right\} = \max\left\{\|\theta\|, \|d\theta\|\right\}. \tag{2.15}$$

Let $Z_1(K^\infty) = \cup Z_1(K^i) \subset C_1(K^\infty)$ be the subspace of all closed 1-chains (cycles). We define a linear functional $\widetilde{L}$ on $Z_1(K^\infty)$ in the following way:

$$\widetilde{L}(c) = \int_c \eta. \tag{2.16}$$

According to Lemma 2.3, we have $|\widetilde{L}(c)| \le C\|c\|$ for any $c \in Z_1(K^\infty)$. Moreover, by using

$$\int_c \eta = \int_{c-\partial D} \eta + \int_D \pi^*\omega,$$

we get

$$|\widetilde{L}(c)| \le C(\|c - \partial D\| + \|D\|) \tag{2.17}$$

for any 2-chain D since ω is bounded. We conclude that

$$|\widetilde{L}(c)| \le C|c|^\flat$$

for any $c \in Z_1(K^\infty)$.

By Hahn–Banach theorem, $\widetilde{L}$ can be extended to a linear functional L on $C_1(K^\infty)$ satisfying

$$|L|^\flat = |\widetilde{L}|^\flat_{Z_1(K^\infty)} \le C. \tag{2.18}$$

Since the flat norm of the cochain L is bounded, by [31, Theorem 5A], we conclude that L and dL can be represented by differential forms $\widetilde{\eta}$ and $\widetilde{\omega}$ with measurable coefficients, i.e.

$$L(c) = \int_c \widetilde{\eta}, \quad (dL)(D) = \int_D \widetilde{\omega}.$$

According to the explanation in [31, p. 260–261], the integration $\int_c \widetilde{\eta}$ is well-defined since $\widetilde{\eta}$ is measurable with respect to the 1-dimensional measure on c. For the same reason, $\int_D \widetilde{\omega}$ is also well-defined. Actually, [31, Theorem 5A] is stated in a domain of the Euclidean space, and we can apply the theorem on each coordinate chart and find a set of bounded differential forms coinciding on the intersections of these coordinate charts, which give us global bounded forms $\widetilde{\eta}$ and $\widetilde{\omega}$. Note that

$$\int_D \pi^*\omega = \int_{\partial D} \eta = \int_{\partial D} \widetilde{\eta} = L(\partial D) = (dL)(D) = \int_D \widetilde{\omega}$$

holds for any 2-chain D, and it follows that $\widetilde{\omega} = \pi^*\omega$. Hence, we find a bounded 1-form $\widetilde{\eta}$ such that $\pi^*\omega = d\widetilde{\eta}$ in the sense of currents. □

To complete the proof of Theorem 2.1, we shall deform $\widetilde{\eta}$ in Lemma 2.4 to a bounded smooth one so that $\pi^*\omega = d\widetilde{\eta}$ still holds. Indeed, such a form can be obtained by using the heat equation method. Consider the following heat equation for differential forms on $\widetilde{M}$

$$\begin{cases} \left(\frac{\partial}{\partial t} - \triangle\right) \widetilde{\eta}(t) = \pi^* d^*\omega, \\ \widetilde{\eta}(0) = \widetilde{\eta} \end{cases} \tag{2.19}$$

where $-\triangle = (dd^* + d^*d)$ is the Hodge Laplacian operator. Since $\widetilde{M}$ has bounded geometry, the initial data $\widetilde{\eta}$ is bounded, and $\pi^*d^*\omega$ is a bounded smooth form on $\widetilde{M}$, it is well-known that Eq. (2.19) admits a unique bounded short time solution $\widetilde{\eta}(t)$, $t \in [0, T_0]$, $T_0 > 0$. Moreover, $\widetilde{\eta}(t)$ is smooth when $t > 0$. This can be done by solving the equation on a sequence of bounded domains exhausting $\widetilde{M}$ and obtaining a priori estimates via applying the maximum principle. Actually, the solution of (2.19) exists for all time $[0, \infty)$, but we do not need this. We claim $d\widetilde{\eta}(t) \equiv \omega$ for all $t \in [0, T_0]$ and so we can take $\widetilde{\eta}(T_0)$ as the desired bounded smooth form $\widetilde{\eta}$ in Theorem 2.1.

By applying the standard Bernstein trick and the maximum principle (localized version), one can show $|\nabla\widetilde{\eta}(t)| \le \frac{C}{\sqrt{t}}$ for all $t \in (0, T_0]$. Taking exterior differential of (2.19), we get

$$\begin{cases} \left(\frac{\partial}{\partial t} - \triangle\right) \bar{\omega}(t) = 0, \\ \bar{\omega}(t) \mid_{t=0} = 0 \end{cases} \tag{2.20}$$

where $\bar{\omega}(t) := d\widetilde{\eta}(t) - \pi^*\omega$. Let $\hat{\omega}(t) = \int_0^t \bar{\omega}(s)ds$, then by straightforward computations, we obtain

$$\begin{cases} \left(\frac{\partial}{\partial t} - \triangle\right)\hat{\omega}(t) = 0, \\ \hat{\omega}(t)\mid_{t=0} = 0. \end{cases} \tag{2.21}$$

A simple calculation shows the estimate $|\nabla\widetilde{\eta}(t)| \leq \frac{C}{\sqrt{t}}$ implies that $\hat{\omega}(t)$ is uniformly bounded on $\widetilde{M} \times [0, T_0]$. By standard Bochner formula, we obtain

$$\left(\frac{\partial}{\partial t} - \triangle\right)\left(e^{-Ct}|\hat{\omega}|^2\right) \leq 0$$

for some large C depending on the curvature bound of $\widetilde{M}$. Since the maximum principle of the heat equation on such manifolds holds when the subsolution grows not faster than $Ce^{Cd(x,x_0)^2}$ ([24, Theorem 15.2] or [6, Lemma 2.5]), we obtain $\hat{\omega}(t) \equiv 0$, i.e., $d\widetilde{\eta}(t) \equiv \pi^*\omega$. The proof of Theorem 2.1 is completed.

3 General fundamental groups

In this section, we deal with a Galois covering $\pi : \widetilde{M} \to M$ of a compact manifold M with a finitely presented covering transformation group $G = \langle S|R\rangle$.

Theorem 3.1 *Let M be a compact manifold, $\pi : \widetilde{M} \to M$ a Galois covering with $H^1_{\mathrm{dR}}(\widetilde{M}) = 0$. Let ω be a smooth closed 2-form on M such that $[\pi^*\omega] = 0$ in $H^2_{\mathrm{dR}}(\widetilde{M})$. Let $f : \mathbb{R}_+ \to [1, +\infty)$ be a non-decreasing function. Suppose the covering transformation group G is finitely presented and satisfies the following radial isoperimetric inequality: for any word $w = s_1 \ldots s_m$ in the free group F_S representing the identity e in G, we have*

$$\mathrm{Area}(w) \leq C\sum_{i=1}^{m} f(d_S(\bar{w}(i), e)) \tag{3.1}$$

where $w(i) = s_1 \ldots s_i$ and $\bar{w}(i)$ is its image in G. Then there exists a 1-form $\widetilde{\eta}$ on $\widetilde{M}$ with measurable coefficients satisfying

$$\begin{cases} \pi^*\omega = d\widetilde{\eta} \\ |\widetilde{\eta}|(x) \leq Cf(Cd(x, x_0)) \end{cases} \tag{3.2}$$

for some constant $C > 0$ where x_0 is a fixed point in $\widetilde{M}$. Moreover, if $f(\lambda) \leq Ce^{C\lambda^2}$, one can choose $\widetilde{\eta}$ to be smooth.

Proof We shall use a similar strategy as in the proof of Theorem 2.1. Let $\pi^*\omega = d\eta$ hold for some smooth 1-form η on $\widetilde{M}$. As in the proof of Lemma 2.3, there exists a constant C such that for any closed curve α on $\widetilde{M}$, we can construct a word $w \in F_S$ representing the identity and approximating α such that

$$\left|\int_\alpha \eta\right| \leq C(L(\alpha) + \mathrm{Area}(w)). \tag{3.3}$$

Since $f \geq 1$ and it is non-decreasing, by (3.1), we have

$$\begin{aligned}
\mathrm{Area}(w) &\leq C \sum_i f(d_S(\bar{w}(i), e)) \\
&\leq C \sum_i f(Cd(x_0, \bar{w}(i)(x_0)) + C) \\
&\leq C \int_0^{L(\alpha)} f(Cd(x_0, \alpha(s)))ds
\end{aligned}$$

where the second inequality follows from the quasi-isometry between G and $\widetilde{M}$, and the last inequality follows from the construction in the proof of Lemma 2.3. Note also that the constant C varies from line to line. As in Lemma 2.4, we define the same linear functional $\widetilde{L}$ on the subspace of closed 1-chains $Z_1(K^\infty)$ by (2.16). However, $\widetilde{L}$ is not necessarily a bounded linear functional in the original norm. We shall define a new norm $||\cdot||_{\breve{g}}$ on $C_1(K^\infty)$ by choosing a conformal Riemannian metric $\breve{g}(x) = f(Cd(x_0, x))^2 g(x)$ on $\widetilde{M}$. In terms of this new norm,

$$\|\alpha\|_{\breve{g}} = \int_0^L f(Cd(x_0, \alpha(s)))ds \geq \frac{1}{C}\mathrm{Area}(w). \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\left|\int_\alpha \eta\right| \leq C||\alpha||_{\breve{g}}. \tag{3.5}$$

Hence,

$$|\widetilde{L}(c)| \leq C(\|c - \partial D\|_{\breve{g}} + \|D\|_g) \tag{3.6}$$

for any 2-chain D and closed 1-chain c. Now define a new flat norm

$$|c|^\flat_{\breve{g}} = \inf_D \|c - \partial D\|_{\breve{g}} + \|D\|_g$$

on $C_1(K^\infty)$. It is not difficult to show that the dual flat norm $|\theta|^\flat = \max\{\|\theta\|_{\breve{g}}, \|d\theta\|_g\}$ for any 1-form θ on $\widetilde{M}$. We have shown $\|\widetilde{L}\|^\flat_{Z_1(K^\infty)} \leq C$ from (3.6). As before, $\widetilde{L}$ can be extended to a bounded linear functional L on the whole space $C_1(K^\infty)$. By [31, Theorem 5A] again, L is represented by a differential form $\widetilde{\eta}$ with measurable coefficients such that $\pi^*\omega = d\widetilde{\eta}$ and

$$\|\widetilde{\eta}\|_{\breve{g}} \leq C$$

which is equivalent to

$$|\widetilde{\eta}|_g(x) \le Cf(Cd(x, x_0)).$$

This completes the first part of Theorem 3.1.

When $f(\lambda) \le Ce^{C\lambda^2}$, we use the heat equation method again to deform the 1-form $\widetilde{\eta}$ to a smooth one. We first construct a solution $v(x, t)$ to the following equation:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) v = 0 \\ v|_{t=0} = d^*\widetilde{\eta}. \end{cases} \tag{3.7}$$

It is well-known that the heat kernel $H(x, y, t)$ (for functions) on $\widetilde{M}$ exists for all $t \ge 0$, and it satisfies the Gaussian type estimates

$$\begin{cases} H(x, y, t) \le Ct^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{Ct}} \\ |\nabla_y H|(x, y, t) \le Ct^{-\frac{n+1}{2}} e^{-\frac{d(x,y)^2}{Ct}} \end{cases} \tag{3.8}$$

for all $t \in [0, 1]$. Combining with (3.8), there exists a small $T_0 > 0$ such that for all $x \in \widetilde{M}$, $0 < t \le T_0$, the function

$$v(x, t) = \int_{\widetilde{M}} \langle d_y H(x, y, t), \widetilde{\eta}(y)\rangle dy \tag{3.9}$$

solves (3.7) and satisfies

$$|v|(x, t) \le Ct^{-\frac{1}{2}} f(Cd(x, x_0))). \tag{3.10}$$

Indeed, for $d(x, x_0) > 1$, we have

$$\begin{aligned} |v|(x, t) \le &\int_{B(x,2d(x,x_0))} Ct^{-\frac{n+1}{2}} e^{-\frac{d(x,y)^2}{Ct}} f(C(d(x_0, y))dy \\ &+ \sum_{k=1}^{\infty} \int_{B(x,2^{k+1}d(x,x_0))\setminus B(x,2^k d(x,x_0))} Ct^{-\frac{n+1}{2}} e^{-\frac{d(x,y)^2}{Ct}} f(C(d(x_0, y))dy. \end{aligned} \tag{3.11}$$

The first term is dominated by

$$Cf(Cd(x_0, x))t^{-\frac{1}{2}} \int_{B(x,2d(x,x_0))} t^{-\frac{n}{2}} e^{-\frac{d(x,y)^2}{Ct}} \le Ct^{-\frac{1}{2}} f(Cd(x_0, x)).$$

Note also that $f(C(d(x_0, y))) \le Ce^{C4^k d(x_0,x)^2}$ on $B(x, 2^{k+1}d(x, x_0))$ and

$$\mathrm{vol}(B(x, 2^{k+1}d(x, x_0))) \le e^{C2^{k+1}d(x,x_0)}$$

by volume comparison theorem since the Ricci curvature is bounded from below. Hence, the second term of (3.11) can be dominated by

$$Cd(x_0,x)^{-(n+1)}\left(\frac{d(x,x_0)}{\sqrt{t}}\right)^{n+1}\sum_{k=1}^{\infty}e^{-4^k\frac{d(x_0,x)^2}{Ct}}\cdot e^{C4^k d(x,x_0)^2}\leq Cd(x_0,x)^{-(n+1)}\leq C$$

when $t<T_0$, where T_0 is a suitable small constant. If $d(x,x_0)\leq 1$, we have $d(x_0,y)\leq d(x,y)+1$, and

$$|v|(x,t)\leq\int_{\widetilde{M}}Ct^{-\frac{n+1}{2}}e^{-\frac{d(x,y)^2}{Ct}}e^{Cd(x,y)^2}dy\leq Ct^{-\frac{1}{2}},\tag{3.12}$$

which completes the proof of (3.10).

Let $u(x,t)=\int_0^t v(x,t-s)ds$, then by (3.7), it is easy to verify that u is a weak solution to the following heat equation on $\widetilde{M}$:

$$\begin{cases}(\frac{\partial}{\partial t}-\triangle)u=d^*\widetilde{\eta}\\ u\mid_{t=0}=0\end{cases}\tag{3.13}$$

and u satisfies

$$|u|(x,t)\leq C\sqrt{t}f(Cd(x,x_0)).$$

Multiply both sides of the above equation with $u\xi$, where ξ is a standard cutoff function which is 1 on $B(x,1)$ and 0 outside of $B(x,2)$. An integration by part shows

$$\int_0^{T_0}\int_{B(x,1)}|du|(y,t)^2dydt\leq C\int_0^{T_0}\int_{B(x,2)}u^2(y,t)+|\widetilde{\eta}|(y)^2dydt\tag{3.14}$$

which implies

$$\int_0^{T_0}\int_{B(x,1)}|du|(y,t)^2+|\widetilde{\eta}|^2(y)dydt\leq Cf^2(C(d(x,x_0)).\tag{3.15}$$

It is easy to verify that $\widetilde{\eta}(t)=\widetilde{\eta}-du$ is a weak solution of the following equation on $\widetilde{M}\times(0,T_0]$:

$$\begin{cases}\left(\frac{\partial}{\partial t}-\triangle\right)\widetilde{\eta}(t)=\pi^*d^*\omega\\ \widetilde{\eta}(0)=\widetilde{\eta}.\end{cases}\tag{3.16}$$

Similar to Eq. (2.19) in the proof of Theorem 2.1, we know $\widetilde{\eta}(x,t)$ is a smooth solution on $\widetilde{M}\times(0,T_0]$. A standard computation shows

$$\left(\frac{\partial}{\partial t}-\triangle\right)|\widetilde{\eta}(t)|^2\leq -2|\nabla\widetilde{\eta}|(t)^2+C|\widetilde{\eta}|(t)^2+C.\tag{3.17}$$

Hence, we obtain

$$\left(\frac{\partial}{\partial t}-\triangle\right)\left(|\widetilde{\eta}(t)|^2+1\right)e^{-Ct}\leq 0.$$

By the well-known result [25, Theorem 1.2] for non-negative subsolution of heat equation, we obtain

$$\begin{aligned}|\widetilde{\eta}|^2(x,t)&\leq 2\sup_{y\in B(x,1)}|\widetilde{\eta}|^2(y,0)+C\int_0^{T_0}\int_{B(x,1)}|\widetilde{\eta}|^2(y,t)dydt+C\\&\leq Cf^2(Cd(x,x_0)),\end{aligned}$$

where the second inequality follows from (3.15). On the other hand, a local gradient estimate on $\widetilde{\eta}$ implies

$$|\nabla\widetilde{\eta}|(x,t)\leq Ct^{-\frac{1}{2}}f(Cd(x,x_0)). \tag{3.18}$$

As in the previous section, let $\hat{\omega}(t)=\int_0^t(d\widetilde{\eta}(s)-\pi^*\omega)ds$, then $\hat{\omega}(t)$ satisfies

$$\begin{cases}\left(\frac{\partial}{\partial t}-\triangle\right)\hat{\omega}(t)=0,\\ \hat{\omega}(0)=0.\end{cases} \tag{3.19}$$

The estimate (3.18) implies $|\hat{\omega}|(x,t)\leq Ce^{Cd^2(x,x_0)}$. As in the proof of Theorem 2.1, by maximum principle ([24, Theorem 15.2] or [6, Lemma 2.5]), we obtain $\hat{\omega}(t)\equiv 0$. Therefore $\widetilde{\eta}(T_0)$ is the desired smooth 1-form. The proof of Theorem 3.1 is completed.

The proof of Theorem 1.4. For $p\geq 1$, we choose $f(t)=(1+t)^{p-1}$ in Theorem 3.1. Hence, there exists a smooth 1-form η on $\hat{M}$ such that $\pi^*\omega=d\eta$ and

$$|\eta|(x)\leq C(d_{\hat{M}}(x,x_0)+1)^{p-1}$$

for all $x\in\hat{M}$ where C is a positive constant and $x_0\in\hat{M}$ is a fixed point. □

The proof of Theorem 1.5. By Theorem 1.4, there exists a smooth 1-form η on the universal covering $\widetilde{M}$ with linear growth such that $\pi^*\omega=d\eta$, i.e., M is Kähler non-elliptic. By [9,20], the Euler characteristic $(-1)^{\frac{\dim_{\mathbb{R}}M}{2}}\chi(M)\geq 0$. □

4 Quadratic radial isoperimetric inequality

In this section, we prove that automatic groups and CAT(0) groups satisfy the radial isoperimetric inequality (1.3) for $p=2$ and establish Theorem 1.6.

Theorem 4.1 *Automatic groups satisfy the quadratic radial isoperimetric inequality* (1.3) *for* $p=2$.

The notion of automatic groups was first introduced by Epstein. Automatic groups are finitely generated groups whose operations are governed by automata. Many interesting groups are automatic, e.g. hyperbolic groups and mapping class groups [27]. We only give a brief description of the necessary notions used in the proof of Theorem 4.1, and for more details, we refer to [3,14,28] and the references therein.

We shall follow the presentation in [14, Section 8]. Let S be a finite set and $S^* = \{w = s_1 \dots s_n \mid s_i \in S\}$ be the set of finite sequences of letters over S. A subset $L \subset S^*$ is called a language. A finite state automaton $\mathcal{A}$ over S is a 5-tuple $\mathcal{A} = (\Gamma, i_o, S, \lambda, Y)$ satisfying

(1) Γ is a finite directed graph with vertex set $V(\Gamma)$(states) and edge set $E(\Gamma)$ (transitions);
(2) i_o is a distinguished vertex of Γ called "initial state";
(3) $\lambda : E(\Gamma) \to S$ is a map labelling $E(\Gamma)$ by S;
(4) $Y \subset V(\Gamma)$ is a set of states, called "final states".

For a finite state automaton $\mathcal{A}$, the language $L(\mathcal{A})$ is the set of labels $\lambda(p)$ of all directed paths p of Γ which begin at i_o and end at any state of Y. A language $L \subset S^*$ is said to be regular if there is a finite state automaton $\mathcal{A}$ with alphabet S such that $L = L(\mathcal{A})$.

Let G be a group with a finite symmetric generator set $S = S^{-1}$. Let $\mu : S^* \to G$ be the evaluation map. A pair (S, L) is called an automatic structure on G if $L \subset S^*$ is a regular language satisfying

(1) $\mu : L \to G$ is surjective;
(2) there exists a number $k > 0$ such that if $w, w' \in L$ satisfy

$$d_{\Gamma(G,S)}(\mu(w), \mu(w')) \leq 1,$$

then w, w' satisfy the k-fellow traveler property:

$$d_{\Gamma(G,S)}(\mu(w(i)), \mu(w'(i))) \leq k \tag{4.1}$$

for all i where $w(i) = s_1 \dots s_i$ is the i-th subword of $w = s_1 \dots s_n$.
A group is called automatic if it admits an automatic structure. It is well-known that (e.g. [3, Theorem 2.5.1] or [14, Section 9]), for any automatic structure (S, L) on G, there exists another automatic structure (S, L'), where $L' \subset L$ can be mapped bijectively onto G by the evaluation map μ and L' is also a regular language for a possibly different finite state automaton. Moreover, there exists a constant $K > 0$ such that that for any $w, w' \in L'$ with $d_S(\mu(w), \mu(w')) = 1$ one has

$$|L(w) - L(w')| \leq K, \tag{4.2}$$

where $L(w)$ and $L(w')$ are the word lengths.

It is well-known that the automatic groups satisfy the classical quadratic isoperimetric inequality (e.g. [14, Theorem 9.7]). In the following, we shall use a similar strategy to prove that automatic groups also satisfy the stronger radial isoperimetric inequality (1.3) for $p = 2$.

The proof of Theorem 4.1. Let (S, L) be an automatic structure on G such that the evaluation map $\mu : L \to G$ is bijective. Let $w = s_1 s_2 \ldots s_n$ be a reduced word in the free group F_S representing the identity e of G. For $i = 1, \ldots, n$, let $w(i) = s_1 \ldots s_i$ and $\bar{w}(i) = \mu(w(i))$. Let p_i be the element in L satisfying $\mu(p_i) = \bar{w}(i)$. We claim

$$L(p_i) \leq K d_S(\bar{w}(i), e). \tag{4.3}$$

Indeed, let $a_1 a_2 \ldots a_{\ell(i)}$ be a reduced word representing a minimal geodesic connecting e and $\bar{w}(i)$ where $a_j \in S$ and $\ell(i) = d_S(\bar{w}(i), e)$. Let $p'_j \in L$ satisfy $\mu(p'_j) = \mu(a_1 \ldots a_j)$ for $j = 1, \ldots, \ell(i)$. From (4.2), we know $|L(p'_j) - L(p'_{j+1})| \leq K$. This implies

$$L(p'_{\ell(i)}) \leq K\ell(i) = K d_S(\bar{w}(i), e).$$

Note that $p'_{\ell(i)} = p_i$ since $\mu(p'_{\ell(i)}) = \mu(p_i) = \bar{w}(i)$ and μ is bijective from L to G. Hence, we obtain (4.3).

From the k-fellow traveler property (4.1), we get $d_S(\mu(p_{i+1}(t)), \mu(p_i(t))) \leq k$ for all t where $p_i(t)$ is the t-th subword of p_i. For fixed i and $t \leq \max\{L(p_i), L(p_{i+1})\}$, connecting the five points

$$\mu(p_i(t)), \quad \mu(p_i(t+1)), \quad \mu(p_{i+1}(t+1), \quad \mu(p_{i+1}(t), \quad \mu(p_i(t))$$

successively with 4 minimal geodesics, we get a closed loop γ_t^i of length $\leq 2k + 2$. The total number of the loops γ_t^i is bounded by

$$\sum_{i=1}^{L(w)} (\max\{L(p_i), L(p_{i+1})\} + 1) \leq 2K \sum_{i=1}^{L(w)} (d_S(\bar{w}(i), e) + 1).$$

We assume that the set R of relators consists of all words in F_S representing e with length $\leq 2k + 2$. We obtain the radial isoperimetric inequality (1.3) for $p = 2$, i.e.

$$\mathrm{Area}(w) \leq 2K \sum_{i=1}^{L(w)} [d_S(\bar{w}(i), e) + 1]. \tag{4.4}$$

The proof of Theorem 4.1 is complete. □

Theorem 4.2 CAT(0)*-groups satisfy the quadratic radial isoperimetric inequality* (1.3) *for* $p = 2$.

Recall that a CAT(0)-group is a group which can act properly discontinuously and cocompactly by isometries on a proper geodesic CAT(0)-metric space (X, d). A geodesic metric space (X, d) is called a CAT(0)-metric space if it satisfies the following properties. For any geodesic triangle $\triangle_{ABC}$ in X, let $\triangle_{\bar{A}\bar{B}\bar{C}}$ be the comparison triangle in the Euclidean plane with the same side lengths as $\triangle_{ABC}$, i.e. $d(A, B) = d(\bar{A}, \bar{B})$, $d(B, C) = d(\bar{B}, \bar{C})$, $d(C, A) = d(\bar{C}, \bar{A})$. The triangle $\triangle_{ABC}$

is at least as thin as $\triangle_{\bar{A}\bar{B}\bar{C}}$, i.e., for any point D on the side BC, let $\bar{D}$ be the corresponding point on $\bar{B}\bar{C}$, with $d(B, D) = d(\bar{B}, \bar{D})$, we must have $d(A, D) \leq d(\bar{A}, \bar{D})$. Typical examples of CAT(0)-groups are fundamental groups of compact manifolds with non-positive sectional curvatures.

The proof of Theorem 4.2. Let (X, d) be the proper geodesic CAT(0)-metric space on which G acts properly discontinuously and cocompactly. Fix a point $p \in X$ and let $\Omega = B(p, 4D)$ be an open ball of radius $4D$ centered at p, where D is the diameter of the quotient space X/G. It is clear that for any $q \in X$, there exists some $g \in G$ such that $g(q) \in B(p, 2D)$. Let $S = \{g \in G \mid g(\Omega) \cap \Omega \neq \emptyset\}$. It is obvious that S is a finite symmetric subset of G. Moreover, S is a generating set of G. Indeed, for any $g \in G$, let

$$\gamma : [0, d(p, g(p))] \to X$$

be a minimal geodesic connecting p and $g(p)$. We may assume $d(g(p), p) > 2D$, otherwise $g \in S$. For each $i = 1, 2, \dots, k = [\frac{d(g(p), p)}{D}]$, we can choose $p_i \in B(p, 2D)$ and $g_i \in G$ such that $g_i(p_i) = \gamma(iD)$. Since $d(g_i^{-1} g_{i+1}(p_{i+1}), p_i) \leq D$, we know $g_i^{-1} g_{i+1}(p_{i+1}) \in B(p, 3D) \subset \Omega$ and so $s_i := g_i^{-1} g_{i+1} \in S$, and $s_k := g_k^{-1} g \in S$. Then $g = g_k s_k = g_{k-1} s_{k-1} s_k = \cdots = g_1 s_1 s_2 \dots s_k$. It is obvious $g_1 \in S$. Hence G is generated by S.

Let $\Gamma(G, S)$ be the Cayley graph of G over S. For any $g_1, g_2 \in G$, we have

$$\frac{d(g_1(p), g_2(p))}{12D} \leq d_S(g_1, g_2) \leq \left[\frac{d(g_1(p), g_2(p))}{D}\right] + 1. \tag{4.5}$$

The second inequality has been proved by the above argument. The first inequality follows from the triangle inequality: let $g_2^{-1} g_1 = s_1 \dots s_k$ with $s_i \in S$, then

$$\begin{aligned} d(s_1 \dots s_k(p), p) &\leq d(s_1 \dots s_k(p), s_1 \dots s_{k-1}(p)) + \cdots + d(s_1(p), p) \\ &= d(s_k(p), p) + \cdots + d(s_1(p), p) \\ &\leq k \cdot 12D, \end{aligned} \tag{4.6}$$

where the last inequality follows from the fact that $d(s_i(p), p) \leq 4D + 4D + 4D = 12D$ for each i. Let $w = s_1 s_2 \dots s_k \in F_S$ be a word representing the identity e in G, $w(i) = s_1 \dots s_i$ be the i-th subword of w, $i = 1, 2, \dots, k$ and $\bar{w}(i)$ the natural image in G. Connecting p and $\bar{w}(i)(p)$ with a minimal geodesic

$$\gamma_i(s) : [0, d(p, \bar{w}(i)(p)] \to X.$$

We choose $\left[\frac{d(p, \bar{w}(i)p}{D}\right] + 1$ points on γ_i as above. Let $p_i^j = \gamma_i(jD)$ for $j \leq \left[\frac{d(p, \bar{w}(i)(p))}{D}\right]$, and $p_i^{\left[\frac{d(p, w(i)(p))}{D}\right]+1} = \bar{w}(i)(p)$. Note that

$$d(\bar{w}(i)(p), \bar{w}(i+1)(p)) = d(s_1 \dots s_i(p), s_1 \dots s_{i+1}(p)) = d(p, s_{i+1}(p)) \leq 12D.$$

Applying CAT(0) inequality, we obtain

$$d(p_i^j, p_{i+1}^j) \leq d(\bar{w}(i)(p), \bar{w}(i+1)(p)) \leq 12D \tag{4.7}$$

for all $j > 0$. Actually, this can be seen from the following simple fact: for any triangle $\triangle_{ABC}$ on the plane $\mathbb{R}^2$ with $|AB| \leq |AC|$, let E and E' lie on the sides AB and AC respectively and $|AE| = |AE'|$, then $|EE'| \leq |BC|$. Note that if $E = B$ and $|AB| \leq |AE'|$, we still have $|BE'| \leq |BC|$. If we choose $g_i^j \in G$ such that $d(g_i^j(p), p_i^j) \leq 2D$, then by (4.7), we have

$$d(g_i^j(p), g_{i+1}^j(p)) \leq 16D.$$

By (4.5), we have $d_S(g_i^j, g_{i+1}^j) \leq 17$. Similarly, one has $d_S(g_i^j, g_i^{j+1}) \leq 6$. On the Cayley graph $\Gamma(G, S)$, connecting the five points

$$g_i^j, \quad g_i^{j+1}, \quad g_{i+1}^{j+1}, \quad g_{i+1}^j, \quad g_i^j$$

successively with 4 minimal geodesics, we get a closed loop σ_i^j of length ≤ 46. Let w' be the word representing the union $\sigma_1^1 \cup \sigma_1^2 \ldots$ of closed loops, then $w = w'$ as elements in F_S. Define the relator set $R = \{r \in F_S \mid \bar{r} = e, L(w) \leq 46\}$, then R is a finite set. From the above argument, we obtain

$$\mathrm{Area}(w) \leq 2 \sum_{i=1}^{L(w)} \left(\left[\frac{d(p, \bar{w}(i)(p))}{D} \right] + 1 \right) \leq 24 \sum_{i=1}^{L(w)} [d_S(\bar{w}(i), e) + 1]$$

where the second inequality follows from (4.5). This completes the proof of Theorem 4.2. □

The proof of Theorem 1.6. If $\pi_1(G)$ is automatic or CAT(0), then by Theorems 4.1 and 4.2, we know G satisfies the radial isoperimetric inequality (1.3) for $p = 2$. Hence, by Theorem 1.5, M is Kähler non-elliptic and the Euler characteristic $(-1)^{\frac{\dim_{\mathbb{R}} M}{2}} \chi(M) \geq 0$. □

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