J. Math. Pures Appl. 128 (2019) 140–151

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

On projective varieties with strictly nef tangent bundles

Duo Li [a], Wenhao Ou [b,*], Xiaokui Yang [c,d]

[a] Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China
[b] UCLA Mathematics Department, 520 Portola Plaza, Los Angeles, CA 90095, USA
[c] Department of Mathematics and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China [1]
[d] Morningside Center of Mathematics, Institute of Mathematics, HCMS, CEMS, NCNIS, HLM, UCAS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

ARTICLE INFO

Article history:
Received 30 January 2018
Available online 2 April 2019

MSC:
14J45
14J60
14M22

Keywords:
Positivity of tangent bundles
Rational connectedness
Campana-Peternell conjecture

ABSTRACT

In this paper, we study smooth complex projective varieties X such that some exterior power $\bigwedge^r T_X$ of the tangent bundle is strictly nef. We prove that such varieties are rationally connected. We also classify the following two cases. If T_X is strictly nef, then X isomorphic to the projective space $\mathbb{P}^n$. If $\bigwedge^2 T_X$ is strictly nef and if X has dimension at least 3, then X is either isomorphic to $\mathbb{P}^n$ or a quadric $\mathbb{Q}^n$.

RÉSUMÉ

Dans ce papier, nous étudions les variétés complexes projectives lisses X telles que certain produit extérieur $\bigwedge^r T_X$ du fibré tangent est strictement nef. Nous démontrons que ces variétés sont rationnellement connexes. De plus, nous classifions les deux cas suivants. Si T_X est strictement nef, alors X est isomorphe à l'espace projective $\mathbb{P}^n$. Si $\bigwedge^2 T_X$ est strictement nef et si la dimension de X est au moins 3, alors ou bien X est isomorphe à $\mathbb{P}^n$, ou bien elle est isomorphe à une quadrique $\mathbb{Q}^n$.

1. Introduction

Throughout this paper, we will study projective varieties defined over $\mathbb{C}$, the field of complex numbers. We recall that a line bundle L over a smooth projective variety X is said to be *strictly nef* if there is

$$L \cdot C > 0$$

* Corresponding author.
E-mail addresses: duoli@math.tsinghua.edu.cn (D. Li), wenhaoou@math.ucla.edu (W. Ou), xkyang@mail.tsinghua.edu.cn (X. Yang).

[1] Current address.

https://doi.org/10.1016/j.matpur.2019.04.007
0021-7824/© 2019 Published by Elsevier Masson SAS.

for any curve $C \subset X$, while the Nakai-Moishezon-Kleiman criterion asserts that L is *ample* if and only if there is

$$L^{\dim Y} \cdot Y > 0$$

for every positive-dimensional subvariety $Y \subset X$. In particular, ampleness implies strict nefness. However, the converse is not true in general, as shown in an example of Mumford (see [1, Section 10, Chapter I]). Nevertheless, one might expect more for the canonical bundle ω_X of X. Indeed, on the one hand, since a strictly nef semi-ample line bundle must be ample, the abundance conjecture suggests that if ω_X is strictly nef, then it should be ample. On the other hand, Campana and Peternell proposed in [2, Problem 11.4] the following conjecture.

Conjecture 1.1. *Let X be a smooth projective variety. If ω_X^{-1} is strictly nef, then ω_X^{-1} is ample, that is, X is a Fano variety.*

This conjecture is only verified by Maeda for surfaces (see [3]) and by Serrano for threefolds (see [4], also [5] and [6]). In this paper, we prove the following theorem, which provides some evidence for this conjecture in all dimensions.

Theorem 1.2. *Let X be a smooth projective variety of dimension n, and let T_X be its tangent bundle. If $\bigwedge^r T_X$ is strictly nef for some $1 \leqslant r \leqslant n$, then X is rationally connected. In particular, if ω_X^{-1} is strictly nef, then X is rationally connected.*

We recall that a vector bundle E is said to be strictly nef if the tautological line bundle $\mathcal{O}_E(1)$ on the projective bundle $\operatorname{Proj} E$ of hyperplanes is strictly nef.

One of the key ingredients for the proof of Theorem 1.2 relies on the recent breakthrough of Cao and Höring on the structure theorems for projective varieties with nef anticanonical bundle (see [7] and [8]). Actually, based on their work, we prove the following result, which is essential for Theorem 1.2.

Theorem 1.3. *Let X be a smooth projective variety with nef anticanonical bundle ω_X^{-1}. Then, up to replacing X with some finite étale cover if necessary, the Albanese morphism $f : X \to A$ has a section $\sigma : A \to X$ such that $\sigma^*\omega_X$ is numerically trivial.*

As an application of Theorem 1.2, we prove the following analogue of Mori's characterization of projective spaces (see [9]).

Theorem 1.4. *Let X be a smooth projective variety of dimension n. If T_X is strictly nef, then X is $\mathbb{P}^n$.*

In another word, this theorem states that the tangent bundle T_X is strictly nef if and only if it is ample. Along the same lines as Theorem 1.4, we obtain the following characterization as well.

Theorem 1.5. *Let X be a smooth projective variety of dimension $n \geq 3$. Suppose that $\bigwedge^2 T_X$ is strictly nef, then X is isomorphic to $\mathbb{P}^n$ or a quadric $\mathbb{Q}^n$.*

There are also other characterizations of projective spaces or quadrics, see, for example, [10], [11], [12], [13], [14], [15,16], [17], [18], [19], [20], [21], [22], [23], [24], etc.

Acknowledgments

The authors would like to thank Shing-Tung Yau for his comments and suggestions on an earlier version of this paper which clarify and improve the presentations. The authors would also like to thank Yifei Chen, Yi Gu and Xiaotao Sun for some useful discussions. The first author is very grateful to Baohua Fu for his support, encouragement and stimulating discussions over the last few years. The second author would like to thank Meng Chen, Zhi Jiang and Burt Totaro for fruitful conversations. The third author wishes to thank Kefeng Liu, Valentino Tosatti and Xiangyu Zhou for helpful discussions. They would also like to thank the referee for pointing out a more general version of Lemma 3.2.

2. Basic properties of strictly nef vector bundles

In this section, we collect some basic results on strictly nef vector bundles. The following proposition is analogous to the Barton-Kleiman criterion for nef vector bundles (see for example [25, Proposition 6.1.18]).

Proposition 2.1. *Let E be a vector bundle on a smooth projective variety X. Then the following conditions are equivalent.*

1. *E is strictly nef.*
2. *For any smooth projective curve C with a finite morphism $\nu : C \to X$, and for any line bundle quotient $\nu^*(E) \to L$, one has*

$$\deg\, L > 0.$$

Proof. For a fixed smooth projective curve C, we know that any non-constant morphism $\mu : C \to \operatorname{Proj} E$ whose image is horizontal over X corresponds one-to-one to a finite morphism $\nu : C \to X$ with a line bundle quotient $\nu^* E \to L$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} C \arrow[rr, "\mu"] \arrow[dr, "\nu"'] & & \operatorname{Proj} E \arrow[dl] \\ & X & \end{tikzcd} \tag{2.1}$$

such that $L = \mu^*\mathcal{O}_E(1)$, where $\mathcal{O}_E(1)$ is the tautological line bundle of $\operatorname{Proj} E$ (see for example [26, Proposition II.7.12]).

(1) $\Longrightarrow$ (2). Let $\nu : C \to X$ be a finite morphism, where C is a smooth projective curve. Assume that there is a line bundle quotient $\nu^* E \to L$. Let $\mu : C \to \operatorname{Proj} E$ be the induced morphism as in diagram (2.1). Then we have

$$\deg\, L = \deg\, \mu^*\mathcal{O}_E(1) = \mathcal{O}_E(1) \cdot \mu(C).$$

Since $\mathcal{O}_E(1)$ is strictly nef, we obtain that $\deg\, L > 0$.

(2) $\Longrightarrow$ (1). Let B be a curve in $\operatorname{Proj} E$. Let $f : \widetilde{B} \to \operatorname{Proj} E$ be its normalization. If B is vertical over X, then we have $\mathcal{O}_E(1) \cdot B > 0$. If B is horizontal over X, then the natural morphism $g : \widetilde{B} \to X$ is finite. In this case, there is an induced line bundle quotient $g^* E \to Q$ such that $Q \cong f^*\mathcal{O}_E(1)$. We note that

$$\mathcal{O}_E(1) \cdot B = \deg\, Q,$$

which is positive by hypothesis. Thus E is strictly nef. $\square$

We also have the following list of properties of strictly nef vector bundles.

Proposition 2.2. *Let E and F be two vector bundles on a smooth projective variety X. Then we have the following assertions.*

1. *E is a strictly nef if and only if for every smooth projective curve C and for any non-constant morphism $f : C \to X$, f^*E is strictly nef.*
2. *If E is strictly nef, then any non-zero quotient bundle Q of E is strictly nef.*
3. *If $E \oplus F$ is strictly nef, then both E and F are strictly nef.*
4. *If the symmetric power $Sym^k E$ is strictly nef for some $k \geq 1$, then E is strictly nef.*
5. *Let $f : Y \to X$ be a finite morphism such that Y is a smooth projective variety. If E is strictly nef, then so is f^*E.*
6. *Let $f : Y \to X$ be a surjective morphism such that Y is a smooth projective variety. If f^*E is strictly nef, then E is strictly nef.*
7. *If E is strictly nef, then $h^0(X, E^* \otimes L) = 0$ for any numerically trivial line bundle L.*

Proof. (1) follows directly from Proposition 2.1. For (2), we note that there is a natural embedding $\iota : \operatorname{Proj} Q \to \operatorname{Proj} E$ such that $\iota^*\mathcal{O}_E(1) \cong \mathcal{O}_Q(1)$. Hence if E is strictly nef, then so is Q. (3) follows from (2). For (4), we note that there is a Veronese embedding $v : \operatorname{Proj} E \to \operatorname{Proj}(Sym^k E)$ such that $v^*\mathcal{O}_{Sym^k E}(1) \cong \mathcal{O}_E(k)$. This implies (4).

For (5), we notice that for any smooth projective curve C with a finite morphism $\nu : C \to X$, the composition $f \circ \nu : C \to X$ is also finite. The assertion then follows from Proposition 2.1.

Now we consider (6). We note that for every curve in X, there is a curve in Y which maps onto it. Hence by (1), we only need to prove the case when X and Y are smooth curves. In this case, there is a natural finite surjective morphism $g : \operatorname{Proj}(f^*E) \to \operatorname{Proj} E$ induced by f. Moreover, we have $\mathcal{O}_{f^*E}(1) \cong g^*\mathcal{O}_E(1)$. By assumption, $\mathcal{O}_{f^*E}(1)$ is a strictly nef line bundle. Since g is finite surjective, this implies that $\mathcal{O}_E(1)$ is also strictly nef. Therefore, E is strictly nef.

It remains to prove (7). We remark that $E \otimes L^{-1}$ is still strictly nef for L^{-1} is numerically trivial. Thus, by replacing E with $E \otimes L^{-1}$, we may assume that L is trivial. Assume by contradiction that $h^0(X, E^*) > 0$. By [27, Proposition 1.16], there exists a nowhere vanishing section $\sigma \in H^0(X, E^*)$. Then σ induces a subbundle $\mathcal{O}_X \to E^*$ as well as a quotient bundle $E \to \mathcal{O}_X$. This contradicts Proposition 2.1. $\square$

3. Strictly nef bundles on curves

In this section, we will look at strictly nef vector bundle E on a smooth projective curve C. If C is rational, then E is a direct sum of line bundles. Hence E is strictly nef if and only if E is ample in this case. However, on a smooth curve C of genus at least 2, there exists a strictly nef vector bundle E which is also a Hermitian flat stable vector bundle (see [1, Section 10 in Chapter I]). In particular, this bundle E is not ample. Now it remains to look at the case when C is elliptic. We observe the following fact.

Theorem 3.1. *Let E be a vector bundle on an elliptic curve C. If E is strictly nef, then E is ample.*

For the proof of this theorem, we will first prove the following lemma. Recall that a vector bundle E is called numerically flat if both E and E^* are nef, or equivalently, if both E and $\det(E^*)$ are nef (see [27, Definition 1.17]).

Lemma 3.2. *Let E be a strictly nef vector bundle on smooth projective variety X whose Kodaira dimension $\kappa(X)$ satisfies $0 \leqslant \kappa(X) < \dim X$. Then $\det E$ is not numerically trivial.*

Proof. We will first prove the case when the Kodaira dimension of X is 0. Assume by contradiction that $\det E$ is numerically trivial. Then E is numerically flat, and so is E^*. By [27, Theorem 1.18], E^* admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_p = E^*$$

of subbundles such that the quotients E_k/E_{k-1} are Hermitian flat. In particular, E_1 is Hermitian flat, and is defined by a unitary representation of the fundamental group $\pi_1(X)$. By [28, Corollary 1], such a representation splits into a direct sum of one-dimensional representations. Hence E_1 is a direct sum of flat line bundles. Let Q be one of them. Then there is a line bundle quotient $E \to L$ with $L = Q^{-1}$. Moreover, since L is also flat, it is numerically trivial. This contradicts Proposition 2.1.

Now we study the general case. Let $\varphi : X \dashrightarrow Y$ be the Iitaka fibration for ω_X. Let F be the closure of a general fiber of φ. Then the Kodaira dimension of F is 0. Moreover, F has positive dimension as $\kappa(X) < \dim X$. Hence, from the first paragraph, the restriction of $\det E$ on F is not numerically trivial. Thus $\det E$ is not numerically trivial. □

Now we can conclude Theorem 3.1.

Proof of Theorem 3.1. The vector bundle E can be decomposed as $E = \oplus E_i$ so that each E_i is an indecomposable bundle. By Proposition 2.2, each E_i is strictly nef. Hence we have $\deg(E_i) > 0$ by Lemma 3.2. This implies that E is ample (see [29, Theorem 1.3] or [30, Theorem 2.3]). □

Remark 3.3. From Proposition 2.2 and Theorem 3.1, we can obtain that if a projective variety X contains the image of an elliptic curve (or a rational curve) which is not a point, then the determinant of every strictly nef vector bundle E on X is not numerically trivial. As a consequence, if E is strictly nef over a projective variety X with pseudo-effective ω_X^{-1}, then $\det E$ is not numerically trivial. Indeed, if ω_X is not pseudo-effective, then by [31, Corollary 0.3], X is covered by rational curves. If both ω_X and ω_X^{-1} are pseudo-effective, then ω_X is numerically trivial. Then the Kodaira dimension of X is zero by Beauville's decomposition theorem (see [32, Théorème 1]), and we can apply Lemma 3.2 to conclude.

4. Sections of Albanese morphisms

In this section, we shall prove Theorem 1.3. We will divide this section into two parts. In the first one, we will prove a theorem on periodic points for group actions on projective schemes. By using this theorem, we will conclude Theorem 1.3.

4.1. Periodic points for linear actions of abelian groups

The goal of this subsection is to prove the following theorem.

Theorem 4.1. *Let G be a finitely generated abelian group. Assume that G acts on a projective scheme Z such that there is a G-equivariant ample line bundle L on Z. Then the action of G on Z has a periodic point.*

We recall that if G acts on a projective scheme Z, then a line bundle L on Z is said to be G-equivariant if there is an isomorphism $g^*L \cong L$ for any element $g \in G$ which is compatible with the group structure of G. A (closed) point $z \in Z$ is said to be a periodic point for an element $g \in G$ if $g^k.z = z$ for some positive integer k. A point z is called a periodic point for the action of G if there is a positive integer k such that $g^k.z = z$ for all elements $g \in G$.

Proof of Theorem 4.1. By replacing L with some positive power of it if necessary, we can assume that L is very ample. Then there is a nature linear action of G on $W = H^0(Z, L)$, which induces a natural action of G on $\operatorname{Proj} W$. Moreover, there is a G-equivariant embedding $Z \to \operatorname{Proj} W$. The theorem is then equivalent to the following proposition. □

Proposition 4.2. *Let G be a finitely generated abelian group. Let V be a linear representation of G. Assume that $Z \subseteq \mathbb{P}(V)$ is a closed subscheme which is stable under the induced action of G on $\mathbb{P}(V)$, where $\mathbb{P}(V)$ is the projective space of lines in V. Then the action of G on Z has a periodic point.*

We will first prove Proposition 4.2 in the case when G is generated by one element.

Lemma 4.3. *With the notations in Proposition 4.2, if we assume that G is generated by one element, then the action of G on Z has a periodic point.*

Proof. Assume that G is generated by an element g. Then it is enough to prove that some positive power of g has a periodic point in Z. Hence during the proof, we will replace g with some positive power of it if necessary. Let $(x_0, ..., x_m)$ be a coordinates system of V such that the vector with coordinates $(1, 0, ..., 0)$ in V is an eigenvector for g. Let $[x_0 : \cdots : x_m]$ be the induced homogeneous coordinates system of $\mathbb{P}(V)$.

We will prove the lemma by induction on the dimension m of $\mathbb{P}(V)$. If $m = 1$, then Z is either a finite set or the whole $\mathbb{P}(V)$. If Z is a finite set, then all of its points are periodic. If $Z = \mathbb{P}(V)$, then the point $[1 : 0]$ belongs to Z and is fixed by g. Hence the lemma is true in this case.

Assume that the lemma is true in dimensions smaller than $m \geqslant 2$. If the point y with coordinates $[1 : 0 : \cdots : 0]$ is in Z, then it is a fixed-point for g and we are done. Assume that y does not belong to Z. Let $H \subseteq \mathbb{P}(V)$ be the hyperplane of points whose 0-th coordinates are 0. Then the rational projection $\varphi : \mathbb{P}(V) \dashrightarrow H$ such that

$$\varphi : [x_0 : \cdots : x_m] \mapsto [0 : x_1 : \cdots : x_m]$$

is a well-defined morphism on Z. We also note that $\varphi|_Z$ is proper, and hence the image $Z' = \varphi(Z)$ is a closed subscheme of $H \cong \mathbb{P}^{m-1}$. Since $(1, 0, ..., 0) \in V$ is an eigenvector for g, the action of G on $\mathbb{P}(V)$ descends naturally to an action of G on H and the rational projection φ is G-equivariant. Thus Z' is stable under the action of G on H. By induction hypothesis, there is a point $z' \in Z'$ which is a periodic point for the action of G. By replacing g with some positive power if necessary, we may assume that z' is a fixed-point. Let $L \subseteq \mathbb{P}(V)$ be the line joining y and z'. Then $L \cap Z$ is non-empty and stable under the action of G. Since we have assumed that $y \notin Z$, the intersection $L \cap Z$ is a proper subset of L. Thus it is a finite set. In particular, every point in $Z \cap L$ is a periodic point for g. This completes the proof. □

Now we can conclude Proposition 4.2.

Proof of Proposition 4.2. We first note that it is enough to prove that some subgroup of G of finite index has a periodic point. Hence during the proof we may replace G by some subgroup of finite index of it if necessary. In particular, we may assume that G is torsion-free. Moreover, without loss of generality, we may assume that the representation V of G is faithful. We will prove by induction on the rank of G. If the rank is one, then the theorem follows from Lemma 4.3.

Assume that the theorem holds for ranks smaller than $k \geqslant 2$. Assume that G has rank k. Let $\{g_1, ..., g_k\}$ be a set of generators of G. Let F be the subgroup generated by g_1 and let H be the subgroup generated by $\{g_2, ..., g_k\}$. Then by Lemma 4.3, the action of F on Z has a periodic point. By replacing g_1 with some positive power of it and G by some subgroup of finite index if necessary, we may assume that the action of F

on Z has a fixed-point. In particular, the set Z^F is not empty. We note that Z^F is also a closed subscheme of $\mathbb{P}^m$. Moreover, it is stable under the actions of H since G is an abelian group. By induction hypothesis, the action of H on Z^F has a periodic point z. Then z is a periodic point for the action of G on Z. This completes the proof. $\square$

4.2. Proof of Theorem 1.3

We will finish the proof of Theorem 1.3 in this subsection. We will need the following two lemmas.

Lemma 4.4. *Let B be a smooth projective variety and let $\pi : \widetilde{B} \to B$ be the universal cover with Galois group $G = \pi_1(B)$. Let V be a linear representation of G and let E be the corresponding flat vector bundle over B. Then there is a one-to-one correspondence between the set of G-fixed-points $y \in \operatorname{Proj} V$ and the set of codimension one flat subbundles $F \to E$.*

Proof. There is a one-to-one correspondence between the set of G-fixed-points y and the set of codimension one subrepresentations $W \subseteq V$. Moreover, the set of codimension one subrepresentations $W \subseteq V$ is one-to-one correspondent to the set of codimension one flat subbundles $F \to E$. $\square$

In the next lemma, we consider the following situation. Let Y be a projective variety and let H be a very ample line bundle on Y. Let B be a smooth projective variety with fundamental group $G = \pi_1(B)$. Let $\widetilde{B} \to B$ be the universal cover. Assume that there is an action of G on Y such that H is G-equivariant. Let G act on $Y \times \widetilde{B}$ diagonally and let X be the quotient $(Y \times \widetilde{B})/G$. Then there is a natural fibration $f : X \to B$, and H descends to a line bundle L on X. Moreover, the natural linear action of G on $V = H^0(Y, H)$ induces a flat vector bundle structure on $E = f_*L$.

Lemma 4.5. *With the notation in the paragraph above, we assume that there is a G-fixed-point $y \in Y$. Then on the one hand, y induces a section $\sigma : B \to X$ of f. On the other hand, y also induces a short exact sequence of flat vector bundles*

$$0 \to F \to E \to Q \to 0$$

such that $Q \cong \sigma^* L$.

Proof. Let $P = \operatorname{Proj} E$ and let $p : P \to B$ be the natural projection. We note that there is a G-equivariant embedding $Y \to \operatorname{Proj} V$, which induces a closed embedding $X \to P$. Moreover, we have $\mathcal{O}_E(1)|_X \cong L$.

Since $y \in Y \subseteq \operatorname{Proj} V$ is a G-fixed-point, it corresponds to a codimension one flat subbundle $F \to E$ by Lemma 4.4. Let Q be the quotient E/F. Then the quotient map $E \to Q$ induces a section $\mu : B \to P$ of p such that $\mu^* \mathcal{O}_E(1) \cong Q$. Moreover, for every $b \in B$, $\mu(b) \in P_b$ corresponds exactly to $y \in \operatorname{Proj} V$. Since $y \in Y$, the section μ factors through a section $\sigma : B \to X$ of f. Since $\mathcal{O}_E(1)|_X = L$, we have $Q \cong \sigma^* L$. This completes the proof of the lemma. $\square$

Remark 4.6. We note that the section $\sigma : B \to X$ in the lemma above depends only on the fixed-point y, and is independent of the choice of the line bundle H. Indeed, $\sigma(B)$ is the quotient $(\{y\} \times \widetilde{B})/G$.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. As in [7, Corollary 4.16], by replacing X with some finite étale cover, we may assume that the fibers of f are simply connected. If A is a point or if f is an isomorphism, then there is nothing to prove. Assume that A has positive dimension and that f is not an isomorphism. Let L be an f-relatively very ample divisor and let $E = f_*L$. Assume that $\operatorname{rank} E = m + 1$ with $m \geqslant 0$. There is an isogeny $p : A' \to A$ such that $p^*(\det E)$ is divisible by $m + 1$. Since we have assumed that the fibers of f are simply connected, the natural morphism $X \times_A A' \to A'$ is still the Albanese morphism. Hence, by replacing X with $X \times_A A'$, we may assume that $\det E = N^{m+1}$ for some line bundle N. By replacing L with $L - f^*N$, we may then assume that $\det E$ is trivial.

Let $\pi : \widetilde{A} \to A$ be the universal cover with Galois group $G = \pi_1(A)$. Let $\widetilde{X}$ be the fiber product $X \times_A \widetilde{A}$ and let $p : \widetilde{X} \to X$ be the natural morphism. By [7, Lemma 4.15], $f_*(kL)$ is a numerically flat vector bundle on A for any positive integer k. In particular, E is numerically flat and hence is a flat vector bundle (see [33, Lemma 6.5 and Corollary 6.6]). Let (V, ρ) be the corresponding representation of G. Then there is a G-equivariant isomorphism

$$(\operatorname{Proj} E) \times_A \widetilde{A} \cong (\operatorname{Proj} V) \times \widetilde{A},$$

where the action of G on $\operatorname{Proj} V$ is the one induced by ρ. By [7, Proposition 2.8], there is a G-stable subvariety $Y \subseteq \operatorname{Proj} V$ such that there is a G-equivariant isomorphism $\widetilde{X} \cong Y \times \widetilde{A}$ which makes the following diagram commute

$$\begin{tikzcd} Y \times \widetilde{A} \arrow[r] \arrow[d] & (\operatorname{Proj} V) \times \widetilde{A} \arrow[d] \\ X \arrow[r] & \operatorname{Proj} E \end{tikzcd}$$

Let $H = \mathcal{O}_{\operatorname{Proj} V}(1)|_Y$. Then, on $\widetilde{X}$, we have $p^*L \cong pr_1^*H$, where $pr_1 : \widetilde{X} \to Y$ is the natural projection induced by the isomorphism $\widetilde{X} \cong Y \times \widetilde{A}$. After all, we have $X \cong (Y \times \widetilde{A})/G$, and we are in the same situation as in Lemma 4.5.

We note that G is an abelian group. By Theorem 4.1, the action of G on Y has a periodic point for H is a G-equivariant ample line bundle. Hence there is a subgroup G' of G of finite index such that the action of G' on Y has a fixed-point $y \in Y$. The quotient $\widetilde{X}/G'$ is a finite étale cover of X, and the natural morphism $\widetilde{X}/G' \to \widetilde{A}/G'$ is the Albanese morphism for we have assumed that the fibers of f are simply connected. Hence, by replacing X with $\widetilde{X}/G'$ if necessary, we may assume that the action of G on Y has a fixed-point $y \in Y$. This fixed-point induces a section $\sigma : A \to X$ of the Albanese morphism f by Lemma 4.5.

We note that the anticanonical bundle ω_Y^{-1} of Y is nef for ω_X^{-1} is nef. Moreover, ω_Y^{-1} is canonically G-equivariant. Hence $\omega_X^{-a} \otimes H$ is a G-equivariant ample line bundle for any positive integer a. On $\widetilde{X}$, we have

$$p^*(\omega_{X/A}^{-a} \otimes L) \cong \omega_{\widetilde{X}/\widetilde{A}}^{-a} \otimes p^*L \cong pr_1^*(\omega_Y^{-a} \otimes H).$$

Let r_a be some large enough positive integer such that $(\omega_Y^{-a} \otimes H)^{r_a}$ is very ample. Then there is a natural linear action of G on $H^0(Y, (\omega_Y^{-a} \otimes H)^{r_a})$, which induces a flat vector bundle structure on $E_a = f_*(\omega_{X/A}^{-a} \otimes L)^{r_a}$. By Lemma 4.5, the G-fixed-point y induces a short exact sequence of flat vector bundles

$$0 \to F_a \to E_a \to Q_a \to 0,$$

such that $Q_a \cong \sigma^*(\omega_{X/A}^{-a} \otimes L)^{r_a}$. Since Q_a is flat, it is numerically trivial. Thus $\sigma^*(\omega_{X/A}^{-a} \otimes L)$ is numerically trivial. Since this is true for all positive integer a, we obtain that $\sigma^*\omega_{X/A}$ is numerically trivial. This completes the proof of the theorem for ω_A is trivial. $\square$

5. Projective manifolds with strictly nef tangent bundles

In this section, we will prove Theorem 1.2, Theorem 1.4 and Theorem 1.5. At first, we observe the following fact.

Lemma 5.1. *Let $X = Y \times Z$ be a variety of dimension n which is a product of two smooth projective varieties of positive dimensions. If $\bigwedge^r T_X$ is strictly nef for some $1 \leqslant r \leqslant n$, then both Y and Z are uniruled.*

Proof. We have $T_X \cong E \oplus F$, where E and F are the pullbacks of the tangent bundles of Y and Z respectively. Then we have

$$\bigwedge^r T_X \cong \bigoplus_{a+b=r} \bigwedge^a E \otimes \bigwedge^b F.$$

Let s be the dimension of Y. We will first show that $r > s$. Suppose by contradiction that $r \leqslant s$. On the one hand, $\bigwedge^r E$ is a direct summand of $\bigwedge^r T_X$. On the other hand, for any $y \in Y$, the restriction of $\bigwedge^r E$ on the fiber X_y is trivial. Hence, $(\bigwedge^r T_X)|_{X_y}$ cannot be strictly nef by Proposition 2.2. This is a contradiction.

As a consequence, we obtain that $N = \bigwedge^s E \otimes \bigwedge^{r-s} F$ is a direct summand of $\bigwedge^r T_X$. In particular, N is strictly nef by Proposition 2.2. Let z be a point in Z. Since $(\bigwedge^{r-s} F)|_{X_z}$ is trivial and since $N|_{X_z}$ is still strictly nef, we conclude that $(\bigwedge^s E)|_{X_z}$ is strictly nef. This implies that ω_Y^{-1} is strictly nef for $X_z \cong Y$. Hence Y is uniruled by [34, Corollary 2]. By symmetry, we can also show that Z is uniruled. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We note that ω_X^{-1} is nef. We will first show that the augmented irregularity $\widetilde{q}(X)$ is zero. Assume the opposite. By replacing X with some finite étale cover if necessary, we may assume that the irregularity $q(X)$ is equal to $\widetilde{q}(X) > 0$. Let $f : X \to A$ be the Albanese morphism. Then $\dim A = q(X) > 0$. Thanks to Theorem 1.3, by replacing X with some finite étale cover if necessary, we may assume that there is a section $\sigma : A \to X$ such that $\sigma^*\omega_X$ is numerically trivial. On the one hand, we remark that $\det(\sigma^* \bigwedge^r T_X) \cong \sigma^*\omega_X^{-t}$ is numerically trivial, where t is the binomial number $\binom{n-1}{r-1}$. On the other hand, since $\sigma^* \bigwedge^r T_X$ is strictly nef, $\det(\sigma^* \bigwedge^r T_X)$ cannot be numerically trivial by Lemma 3.2. We obtain a contradiction.

Therefore, we have $\widetilde{q}(X) = 0$. In particular, X has finite fundamental group. Let $\widetilde{X} \to X$ be the universal cover. Then by [8, Theorem 1.2], we have $\widetilde{X} \cong Y \times F$ such that ω_Y is trivial and that F is rationally connected. By Lemma 5.1, Y must be a point for $\bigwedge^r T_{\widetilde{X}}$ is strictly nef. Hence $\widetilde{X}$ is rationally connected and so is X. □

The following corollary is a direct consequence of Theorem 1.2.

Corollary 5.2. *Let $f : Y \to X$ be a smooth surjective morphism between projective manifolds. If ω_Y^{-1} is strictly nef, then X is rationally connected.*

In the following, we will present two applications of Theorem 1.2 on characterizations of projective spaces and quadrics. We will prove Theorem 1.4 and Theorem 1.5 successively.

Proof of Theorem 1.4. By Theorem 1.2 and the structure theorem for smooth projective varieties with nef tangent bundles (see [27, Main Theorem]), we deduce that X is a Fano variety. For any rational curve $f : \mathbb{P}^1 \to X$, the bundle f^*T_X is ample by Proposition 2.1. Moreover, since there is a non-zero morphism from $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ to f^*T_X, we obtain that $\deg f^*\omega_X^{-1} \geq n + 1$. Hence $X \cong \mathbb{P}^n$ by [19, Corollary 0.3]. □

Proof of Theorem 1.5. We know that X is rationally connected from Theorem 1.2. For a rational curve $f : \mathbb{P}^1 \to X$, we can write

$$f^*T_X \cong \left(\bigoplus_{a_i>0} \mathcal{O}_{\mathbb{P}^1}(a_i)\right) \bigoplus \left(\bigoplus_{b_j\leq 0} \mathcal{O}_{\mathbb{P}^1}(b_j)\right).$$

Since $\bigwedge^2 T_X$ is strictly nef, so is $f^*(\bigwedge^2 T_X)$ by Proposition 2.1. Hence $\sharp\{b_j\} \leq 1$. If $\sharp\{b_j\}$ is 0, then f^*T_X is ample, and deg $f^*\omega_X^{-1} \geq n+1$ by the same argument as in the proof of Theorem 1.4.

If $\sharp\{b_j\}$ is 1, then we can assume that

$$f^*T_X \cong \left(\bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)\right) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-c)$$

with $0 < a_1 \leq a_2 \leq \cdots \leq a_{n-1}$ and $c \geq 0$. Since $f^*(\bigwedge^2 T_X)$ is strictly nef, we must have $a_1 - c > 0$. Moreover, since there is a natural non-zero morphism from $T_{\mathbb{P}^1}$ to f^*T_X, there exists some i such that $a_i \geq 2$. If $c > 0$, then a_1 is at least 2 and we have

$$\deg\, f^*\omega_X^{-1} = (a_1 - c) + a_2 + \cdots + a_{n-1} \geq 1 + 2(n-2) = 2n - 3 \geq n.$$

If $c = 0$, we also have

$$\deg\, f^*\omega_X^{-1} = a_1 + a_2 + \cdots + a_{n-1} \geq (n-2) + 2 = n.$$

After all, we always have deg $f^*\omega_X^{-1} \geq n$. By [23, Corollary D], X is isomorphic to $\mathbb{P}^n$, or a quadric $\mathbb{Q}^n$, or a projective bundle over some smooth curve. It remains to rule out the case of projective bundles. Assume by contradiction that $X \cong \operatorname{Proj} V$, where V is a vector bundle over a smooth projective curve B. We note that B is isomorphic to $\mathbb{P}^1$ for X is rationally connected. We may assume that $V = \bigoplus_{i=1}^n \mathcal{O}_B(d_i)$ with $0 = d_1 \leq d_2 \leq \cdots \leq d_n$. If $\pi : X \to B$ is the natural projection, then we have

$$\begin{aligned}\omega_X^{-1} &= (\pi^*\omega_B^{-1}) \otimes \omega_{X/B}^{-1} \\ &\cong (\pi^*\mathcal{O}_B(2)) \otimes (\mathcal{O}_V(n) \otimes \pi^*(\det V)^{-1}) \\ &= \mathcal{O}_V(n) \otimes \left(\pi^*\mathcal{O}_B\left(2 - \sum_{i=1}^n d_i\right)\right).\end{aligned}$$

We note that the quotient morphism $V \to \mathcal{O}_B(d_1)$ induces a section $\sigma : B \to X$ of π such that $\sigma^*\mathcal{O}_V(1) \cong \mathcal{O}_B(d_1) = \mathcal{O}_B$. Thus we have

$$\sigma^*\omega_X^{-1} \cong \mathcal{O}_B\left(2 - \sum_{i=1}^n d_i\right).$$

This shows that deg $\sigma^*\omega_X^{-1} \leq 2 < n$, which is a contradiction. □

Remark 5.3. We note that if a vector bundle E is strictly nef, then det E is not necessarily strictly nef in general (see [1, Section 10 in Chapter I]). However, inspired by Theorem 1.4 and Theorem 1.5, we expect that if $\bigwedge^r T_X$ is strictly nef for some $r > 0$, then so is $-K_X$. We then extend the conjecture of Campana and Peternell: if $\bigwedge^r T_X$ is strictly nef for some $r > 0$, then X is a Fano variety.

References

[1] R. Hartshorne, Ample Subvarieties of Algebraic Varieties, Lecture Notes in Mathematics, vol. 156, Springer-Verlag, Berlin-New York, 1970, notes written in collaboration with C. Musili.

[2] F. Campana, T. Peternell, Projective manifolds whose tangent bundles are numerically effective, Math. Ann. 289 (1) (1991) 169–187, https://doi.org/10.1007/BF01446566.

[3] H. Maeda, A criterion for a smooth surface to be Del Pezzo, Math. Proc. Camb. Philos. Soc. 113 (1) (1993) 1–3, https://doi.org/10.1017/S0305004100075721.

[4] F. Serrano, Strictly nef divisors and Fano threefolds, J. Reine Angew. Math. 464 (1995) 187–206, https://doi.org/10.1515/crll.1995.464.187.

[5] H. Uehara, On the canonical threefolds with strictly nef anticanonical divisors, J. Reine Angew. Math. 522 (2000) 81–91, https://doi.org/10.1515/crll.2000.042.

[6] F. Campana, J.A. Chen, T. Peternell, Strictly nef divisors, Math. Ann. 342 (3) (2008) 565–585, https://doi.org/10.1007/s00208-008-0248-x.

[7] J. Cao, Albanese maps of projective manifolds with nef anticanonical bundles, Ann. Sci. Éc. Norm. Supér. (2019), in press, arXiv:1612.05921.

[8] J. Cao, A. Höring, A decomposition theorem for projective manifolds with nef anticanonical bundle, J. Algebraic Geom. (2019), in press, arXiv:1706.08814.

[9] S. Mori, Projective manifolds with ample tangent bundles, Ann. Math. (2) 110 (3) (1979) 593–606, https://doi.org/10.2307/1971241.

[10] S. Kobayashi, T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ. 13 (1973) 31–47, https://doi.org/10.1215/kjm/1250523432.

[11] Y.-T. Siu, S.-T. Yau, Compact Kähler manifolds of positive bisectional curvature, Invent. Math. 59 (2) (1980) 189–204, https://doi.org/10.1007/BF01390043.

[12] Y.-T. Siu, Curvature characterization of hyperquadrics, Duke Math. J. 47 (3) (1980) 641–654, http://projecteuclid.org/euclid.dmj/1077314187.

[13] N. Mok, The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, J. Differ. Geom. 27 (2) (1988) 179–214, http://projecteuclid.org/euclid.jdg/1214441778.

[14] Y.-G. Ye, Q. Zhang, On ample vector bundles whose adjunction bundles are not numerically effective, Duke Math. J. 60 (3) (1990) 671–687, https://doi.org/10.1215/S0012-7094-90-06027-2.

[15] T. Peternell, A characterization of $\mathbf{P}_n$ by vector bundles, Math. Z. 205 (3) (1990) 487–490, https://doi.org/10.1007/BF02571257.

[16] T. Peternell, Ample vector bundles on Fano manifolds, Int. J. Math. 2 (3) (1991) 311–322, https://doi.org/10.1142/S0129167X91000181.

[17] K. Cho, E.-i. Sato, Smooth projective varieties with the ample vector bundle $\bigwedge^2 T_X$ in any characteristic, J. Math. Kyoto Univ. 35 (1) (1995) 1–33, https://doi.org/10.1215/kjm/1250518837.

[18] M. Andreatta, J.a.A. Wiśniewski, On manifolds whose tangent bundle contains an ample subbundle, Invent. Math. 146 (1) (2001) 209–217, https://doi.org/10.1007/PL00005808.

[19] K. Cho, Y. Miyaoka, N.I. Shepherd-Barron, Characterizations of projective space and applications to complex symplectic manifolds, in: Higher Dimensional Birational Geometry, Kyoto, 1997, in: Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 1–88.

[20] Y. Miyaoka, Numerical characterisations of hyperquadrics, in: Complex Analysis in Several Variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, in: Adv. Stud. Pure Math., vol. 42, Math. Soc. Japan, Tokyo, 2004, pp. 209–235.

[21] C. Araujo, S. Druel, S.J. Kovács, Cohomological characterizations of projective spaces and hyperquadrics, Invent. Math. 174 (2) (2008) 233–253, https://doi.org/10.1007/s00222-008-0130-1.

[22] J.-M. Hwang, Varieties of minimal rational tangents of codimension 1, Ann. Sci. Éc. Norm. Supér. (4) 46 (4) (2013) 629–649, https://doi.org/10.24033/asens.2197.

[23] T. Dedieu, A. Höring, Numerical characterisation of quadrics, Algebr. Geom. 4 (1) (2017) 120–135, https://doi.org/10.14231/AG-2017-006.

[24] X. Yang, Big vector bundles and complex manifolds with semi-positive tangent bundles, Math. Ann. 367 (1) (2017) 251–282, https://doi.org/10.1007/s00208-016-1382-5.

[25] R. Lazarsfeld, Positivity in Algebraic Geometry. II: Positivity for Vector Bundles, and Multiplier Ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 49, Springer-Verlag, Berlin, 2004.

[26] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977.

[27] J.-P. Demailly, T. Peternell, M. Schneider, Compact complex manifolds with numerically effective tangent bundles, J. Algebraic Geom. 3 (2) (1994) 295–345.

[28] K. Zuo, Kodaira dimension and Chern hyperbolicity of the Shafarevich maps for representations of π_1 of compact Kähler manifolds, J. Reine Angew. Math. 472 (1996) 139–156.

[29] R. Hartshorne, Ample vector bundles on curves, Nagoya Math. J. 43 (1971) 73–89, http://projecteuclid.org/euclid.nmj/1118798366.

[30] D. Gieseker, p-Ample bundles and their Chern classes, Nagoya Math. J. 43 (1971) 91–116, http://projecteuclid.org/euclid.nmj/1118798367.

[31] S. Boucksom, J.-P. Demailly, M. Păun, T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, J. Algebraic Geom. 22 (2) (2013) 201–248, https://doi.org/10.1090/S1056-3911-2012-00574-8.

[32] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Differ. Geom. 18 (4) (1983) 755–782, http://projecteuclid.org/euclid.jdg/1214438181.
[33] M. Verbitsky, Coherent sheaves on generic compact tori, in: Algebraic Structures and Moduli Spaces, in: CRM Proc. Lecture Notes, vol. 38, Amer. Math. Soc., Providence, RI, 2004, pp. 229–247.
[34] Y. Miyaoka, S. Mori, A numerical criterion for uniruledness, Ann. Math. (2) 124 (1) (1986) 65–69, https://doi.org/10.2307/1971387.