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SCALAR CURVATURE ON COMPACT COMPLEX MANIFOLDS

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ABSTRACT. In this paper, we prove that, a compact complex manifold X admits a smooth Hermitian metric with positive (resp., negative) scalar curvature if and only if K_X (resp., K_X^{-1}) is not pseudo-effective. On the contrary, we also show that on an arbitrary compact complex manifold X with complex dimension ≥ 2, there exist smooth Hermitian metrics with positive *total* scalar curvature, and one of the key ingredients in the proof relies on a recent solution to the Gauduchon conjecture by G. Székelyhidi, V. Tosatti, and B. Weinkove.

CONTENTS

1. INTRODUCTION

In this paper, we investigate the relationship between the sign of the (total) scalar curvature of Hermitian metrics and the geometry of the ambient complex manifolds.

On a compact Kähler manifold, one can define the positivity of holomorphic bisectional curvature, Ricci curvature, scalar curvature, and so on. The weakest one among them is the positivity of total scalar curvature. In algebraic geometry, the Kodaira dimension can also characterize the positivity of the canonical bundles and anti-canonical bundles. In his seminal work [32], Yau proved that, on a compact Kähler manifold X, if it admits a Kähler metric with positive total scalar curvature, then the Kodaira dimension $\kappa(X) = -\infty$. Furthermore, Yau established that a compact Kähler surface is uniruled if and only if there exists a Kähler metric with positive total scalar curvature. Recently, Heier–Wong pointed out in [11] that a projective manifold is uniruled if it admits a Kähler metric with positive total scalar curvature. By using Boucksom–Demailly–Paun–Peternell's criterion

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for uniruled projective manifolds ([5, Corollary 0.3]), Chiose-Rasdeaconu-Suvaina obtained in [6] a more general characterization that a compact Moishezon manifold is uniruled if and only if it admits a smooth Gauduchon metric with positive total Chern scalar curvature. Motivated by these works ([5,6,11,32]), we investigate the total Chern scalar curvature of Gauduchon metrics on general compact complex manifolds. Let ω be a smooth Hermitian metric on the compact complex manifold X. For simplicity, we denote by $\mathscr{F}(\omega)$ the total Chern scalar curvature of ω, i.e.,

$$\mathscr{F}(\omega) = \int_X s\omega^n = n\int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1}.$$

Note that, when the manifold is not Kähler, the total Chern scalar curvature differs from the total scalar curvature of the Levi-Civita connection of the underlying Riemannian metric (e.g., [16]). Let $\mathscr{W}$ be the space of smooth Gauduchon metrics on X. We obtain a complete characterization on the image of the total scalar curvature function $\mathscr{F} : \mathscr{W} \to \mathbb{R}$ (cf. [11, pp. 761–762]).

Theorem 1.1. *The image of the total scalar function $\mathscr{F} : \mathscr{W} \to \mathbb{R}$ has exactly four different cases:*

(1) $\mathscr{F}(\mathscr{W}) = \mathbb{R}$ *if and only if neither K_X nor K_X^{-1} is pseudo-effective;*
(2) $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{>0}$ *if and only if K_X^{-1} is pseudo-effective but not unitary flat;*
(3) $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{<0}$ *if and only if K_X is pseudo-effective but not unitary flat;*
(4) $\mathscr{F}(\mathscr{W}) = \{0\}$ *if and only if K_X is unitary flat.*

One of the key ingredients in the proof of Theorem 1.1 relies on a border line case of Lamari's positivity criterion ([14]) over compact complex manifolds and Tosatti's characterizations for non-Kähler Calabi–Yau manifolds ([22]). Moreover, in Section 5 we exhibit a variety of *non-Kähler Calabi–Yau* manifolds which can distinguish all different cases in Theorem 1.1.

Remark 1.2. More generally, for a Bott–Chern class $[\alpha] \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$, we can also define the function $\mathscr{F}_{[\alpha]} : \mathscr{W} \to \mathbb{R}$ with respect to the class $[\alpha]$

$$\mathscr{F}_{[\alpha]}(\omega) = \int_X [\alpha] \wedge \omega^{n-1},$$

which is well defined since ω is Gauduchon. Analogous to Theorem 1.1, we obtain in Theorem 3.4 a criterion for the positivity of the class $[\alpha]$. It generalizes a result in [21] which only deals with compact complex surfaces.

As an application of Theorem 1.1 and Gauduchon's conformal method, we obtain a criterion for the existence of a Hermitian metric with positive or negative (total) scalar curvature on compact complex manifolds (see also [6, Theorem D] for some special cases).

Theorem 1.3. *Let X be a compact complex manifold. The following are equivalent*:

(1) K_X (*resp.*, K_X^{-1}) *is not pseudo-effective*;
(2) *X has a Hermitian metric with positive* (*resp., negative*) *scalar curvature*;
(3) *X has a Gauduchon metric with positive* (*resp., negative*) *total scalar curvature.*

On the other hand, it is well known that if a compact complex manifold X admits a smooth Hermitian metric with positive scalar curvature, then the Kodaira dimension $\kappa(X) = -\infty$. As another application of Theorem 1.1, we obtain the following.

Theorem 1.4. *Let X be a compact complex manifold. Then there exists a smooth Gauduchon metric ω_G with vanishing total scalar curvature if and only if X lies in one of the following cases*:

(1) $\kappa(X) = -\infty$ *and neither* K_X *nor* K_X^{-1} *is pseudo-effective*;
(2) $\kappa(X) = -\infty$ *and* K_X *is unitary flat*;
(3) $\kappa(X) = 0$ *and* K_X *is a holomorphic torsion, i.e.,* $K_X^{\otimes m} = \mathcal{O}_X$ *for some* $m \in \mathbb{Z}^+$.

It is easy to see that Theorem 1.4 excludes non-Kähler Calabi–Yau manifolds with $\kappa(X) \geq 1$ (e.g., Example 5.3). More generally, one has (see also [1, Proposition 2.4]) the following.

Corollary 1.5. *Let X be a compact complex manifold. Suppose* $\kappa(X) > 0$; *then for any Gauduchon metric ω, the total scalar curvature* $\mathscr{F}(\omega) < 0$.

The following result is a straightforward application of Theorem 1.4, and it appears to be new and interesting in its own right.

Corollary 1.6. *Let X be a compact Kähler manifold. If there exists a Gauduchon* (*e.g., Kähler*) *metric ω with vanishing total scalar curvature, then either*

(1) $\kappa(X) = -\infty$ *and neither* K_X *nor* K_X^{-1} *is pseudo-effective*; *or*
(2) *X is a Kähler Calabi–Yau, i.e., there exists a smooth Kähler metric $\widetilde{\omega}$ with* $\mathrm{Ric}(\widetilde{\omega}) = 0$.

One may wonder whether similar results hold for Hermitian metrics other than Gauduchon metrics. Unfortunately, one cannot replace the Gauduchon metric condition by an arbitrary Hermitian metric even if the ambient manifold is Kähler. More precisely, we obtain the following.

Theorem 1.7. *Let X be an arbitrary compact complex manifold with* $\dim X > 1$. *Then there exists a Hermitian metric ω with positive total scalar curvature. Moreover, if X is Kähler, then there exists a conformally Kähler metric ω with positive total scalar curvature* (*i.e.,* $\omega = e^f \omega_0$ *for some Kähler metric* ω_0 *and* $f \in C^\infty(X, \mathbb{R})$).

The proof of Theorem 1.7 relies on Székelyhidi-Tosatti-Weinkove's solution to the Gauduchon conjecture on compact complex manifolds ([20, Theorem 1.3], i.e., Theorem 6.1).

For more applications, we refer to [17, 29–31].

2. Preliminaries

2.1. **Curvatures on complex manifolds.** Let (X, ω_g) be a compact Hermitian manifold. The Chern connection on $(T^{1,0}X, \omega_g)$ has Chern curvature components

$$R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 g_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{p\overline{\ell}}}{\partial \overline{z}^j}. \tag{2.1}$$

The (first Chern) Ricci form $\mathrm{Ric}(\omega_g)$ of (X, ω_g) has components

$$R_{i\overline{j}} = g^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \overline{z}^j}$$

and it is well known that the Ricci form represents the first Chern class of the complex manifold X. The (Chern) scalar curvature s_g of (X, ω_g) is defined as

$$s_g = \mathrm{tr}_{\omega_g} \mathrm{Ric}(\omega_g) = g^{i\overline{j}} R_{i\overline{j}}. \tag{2.2}$$

The total scalar curvature is

$$\int_X s_g \omega_g^n = n \int \mathrm{Ric}(\omega_g) \wedge \omega_g^{n-1}, \tag{2.3}$$

where n is the complex dimension of X.

(1) A Hermitian metric ω_g is called a Gauduchon metric if $\partial\overline{\partial}\omega_g^{n-1} = 0$. It is proved by Gauduchon ([9]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to constant scaling).
(2) A compact complex manifold X is called a Calabi–Yau manifold if $c_1(X) = 0 \in H^2(X, \mathbb{R})$.
(3) A compact complex manifold X is called uniruled if it is covered by rational curves.

2.2. **Positivity of line bundles.** Let (X, ω_g) be a compact Hermitian manifold, and let $L \to X$ be a holomorphic line bundle.

(1) L is said to be *positive* (resp., *semi-positive*) if there exists a smooth Hermitian metric h on L such that the curvature form $R = -\sqrt{-1}\partial\overline{\partial} \log h$ is a positive (resp., semi-positive) $(1,1)$-form.
(2) L is said to be *nef*, if for any $\varepsilon > 0$, there exists a smooth Hermitian metric h_ε on L such that the curvature of (L, h_ε) satisfies $-\sqrt{-1}\partial\overline{\partial}\log h_\varepsilon \geq -\varepsilon\omega_g$.
(3) L is said to be *pseudo-effective* if there exists a (possibly) singular Hermitian metric h on L such that the curvature of (L, h) satisfies $-\sqrt{-1}\partial\overline{\partial}\log h \geq 0$ in the sense of currents. (See [7] for more details.)
(4) L is said to be $\mathbb{Q}$-*effective* if there exists some positive integer m such that $H^0(X, L^{\otimes m}) \neq 0$.
(5) L is called *unitary flat* if there exists a smooth Hermitian metric h on L such that the curvature of (L, h) is zero, i.e., $-\sqrt{-1}\partial\overline{\partial}\log h = 0$.
(6) The Kodaira dimension $\kappa(L)$ of L is defined to be

$$\kappa(L) := \limsup_{m \to +\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, L^{\otimes m})}{\log m}$$

and the *Kodaira dimension* $\kappa(X)$ of X is defined as $\kappa(X) := \kappa(K_X)$ where the logarithm of zero is defined to be $-\infty$.

2.3. **Bott–Chern classes and Aeppli classes.** On compact complex (especially, non-Kähler) manifolds, the Bott–Chern cohomology and the Aeppli cohomology are very useful:

$$H^{p,q}_{\mathrm{BC}}(X) := \frac{\mathrm{Ker} d \cap \Omega^{p,q}(X)}{\mathrm{Im}\partial\overline{\partial} \cap \Omega^{p,q}(X)} \quad \text{and} \quad H^{p,q}_{\mathrm{A}}(X) := \frac{\mathrm{Ker}\partial\overline{\partial} \cap \Omega^{p,q}(X)}{\mathrm{Im}\partial \cap \Omega^{p,q}(X) + \mathrm{Im}\overline{\partial} \cap \Omega^{p,q}(X)}.$$

Let $\mathrm{Pic}(X)$ be the set of holomorphic line bundles over X. Similar to the first Chern class map $c_1 : \mathrm{Pic}(X) \to H^{1,1}_{\overline{\partial}}(X)$, there is a *first Bott–Chern class* map

$$c_1^{\mathrm{BC}} : \mathrm{Pic}(X) \to H^{1,1}_{\mathrm{BC}}(X). \tag{2.4}$$

Given any holomorphic line bundle $L \to X$ and any Hermitian metric h on L, its curvature form Θ_h is locally given by $-\sqrt{-1}\partial\overline{\partial}\log h$. We define $c_1^{\mathrm{BC}}(L)$ to be the class of Θ_h in $H^{1,1}_{\mathrm{BC}}(X)$ (modulo a constant 2π). For a complex manifold X, $c_1^{\mathrm{BC}}(X)$ is defined to be $c_1^{\mathrm{BC}}(K_X^{-1})$ where K_X^{-1} is the anti-canonical line bundle $\wedge^n T^{1,0}X$. It is easy to see that $c_1^{\mathrm{BC}}(L) = 0$ if and only if L is unitary flat.

3. Total scalar curvature of Gauduchon metrics

Let X be a compact complex manifold of complex dimension n. Suppose

- $\mathscr{E}$ is the set of real $\partial\overline{\partial}$-closed $(n-1,n-1)$-forms on X;
- $\mathscr{V}$ is the set of real positive $\partial\overline{\partial}$-closed $(n-1,n-1)$-forms on X;
- $\mathscr{G} = \{\omega^{n-1} \mid \omega \text{ is a Gauduchon metric }\}$.

In [19], M. L. Michelsohn observed that the power map: $\eta \to \eta^{n-1}$, from $\Lambda^{1,1}T_x^*X$ to $\Lambda^{n-1,n-1}T_x^*X$, carries the cone of strictly positive $(1,1)$-forms bijectively onto the cone of strictly positive $(n-1,n-1)$-forms at each point $x \in X$, and obtained the following.

Lemma 3.1. *$\mathscr{V} = \mathscr{G}$.*

The proof of Theorem 1.1.

Claim 1. The canonical bundle K_X is pseudo-effective if and only if $\mathscr{F}(\omega) \leq 0$ for every Gauduchon metric ω. If, in addition, there exists some Gauduchon metric ω_0 such that $\mathscr{F}(\omega_0) = 0$, then $\mathscr{F}(\mathscr{W}) = \{0\}$ and K_X^{-1} is unitary flat.

Suppose K_X is pseudo-effective; it is well known that there exist a smooth Hermitian metric ω_1 and a real valued function $\varphi \in L^1(X,\mathbb{R})$ such that

$$\mathrm{Ric}(\omega_1) + \sqrt{-1}\partial\overline{\partial}\varphi \leq 0$$

in the sense of currents. Then for any smooth Gauduchon metric $\omega \in \mathscr{W}$,

$$\begin{aligned}\mathscr{F}(\omega) &= n\int_X \mathrm{Ric}(\omega) \wedge \omega^{n-1} \\ &= n\int_X \left(\mathrm{Ric}(\omega_1) - \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega^n}{\omega_1^n}\right)\right) \wedge \omega^{n-1} \\ &= n\int_X \mathrm{Ric}(\omega_1) \wedge \omega^{n-1} \\ &= n\int_X \left(\mathrm{Ric}(\omega_1) + \sqrt{-1}\partial\overline{\partial}\varphi\right) \wedge \omega^{n-1} \leq 0.\end{aligned}$$

Conversely, we assume $\mathscr{F}(\omega) \leq 0$ for every Gauduchon metric ω. Now we follow the strategy in [14] to show K_X is pseudo-effective. Suppose there exists some Gauduchon metric ω_0 such that $\mathscr{F}(\omega_0) = 0$; we shall show that $\mathscr{F}(\mathscr{W}) = \{0\}$ and there exists a Hermitian metric $\widetilde{\omega}$ with $\mathrm{Ric}(\widetilde{\omega}) = 0$, i.e., K_X is unitary flat.

Indeed, for any fixed $\partial\overline{\partial}$-closed $(n-1,n-1)$-form $\eta \in \mathscr{E}$, we define a real function $f : \mathbb{R} \to \mathbb{R}$ as

$$f(t) = n\int_X \mathrm{Ric}(\omega_0) \wedge \left((1-t)\omega_0^{n-1} + t\eta\right).$$

Since ω_0 has zero total scalar curvature, we have

$$f(t) = t \cdot \left(n\int_X \mathrm{Ric}(\omega_0) \wedge \eta\right). \tag{3.1}$$

Hence, $f(t)$ is linear in t. On the other hand, for small $|t|$, we have

$$(1-t)\omega_0^{n-1} + t\eta \in \mathscr{V}.$$

By Lemma 3.1, there exist a small number $\varepsilon_0 > 0$ and a family of Gauduchon metrics ω_t with $t \in (-2\varepsilon_0, 2\varepsilon_0)$ such that

$$\omega_t^{n-1} = (1-t)\omega_0^{n-1} + t\eta.$$

Then we have

$$\begin{aligned} f(t) &= n\int_X \mathrm{Ric}(\omega_0)\wedge\omega_t^{n-1} = n\int_X \left(\mathrm{Ric}(\omega_t) - \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega_0^n}{\omega_t^n}\right)\right)\wedge\omega_t^{n-1} \\ &= n\int_X \mathrm{Ric}(\omega_t)\wedge\omega_t^{n-1} = \mathscr{F}(\omega_t)\le 0 \end{aligned}$$

for $t\in(-2\varepsilon_0, 2\varepsilon_0)$. In particular, we have $f(\varepsilon_0)\le 0$ and $f(-\varepsilon_0)\le 0$. However, by (3.1), $f(t)$ is linear in t and $f(0)=0$. Hence, $f(t)\equiv 0$, i.e.,

$$n\int_X \mathrm{Ric}(\omega_0)\wedge\eta = 0.$$

Since η is an arbitrary element in $\mathscr{E}$, by Lamari's criterion [14], there exists $\varphi\in L^1(X,\mathbb{R})$ such that $\mathrm{Ric}(\omega_0)+\sqrt{-1}\partial\overline{\partial}\varphi = 0$ in the sense of currents. Hence, $\varphi\in C^\infty(X,\mathbb{R})$ and the metric $\widetilde{\omega}=e^{-\frac{\varphi}{n}}\omega_0$ is Ricci-flat, i.e.,

$$\mathrm{Ric}(\widetilde{\omega}) = -\sqrt{-1}\partial\overline{\partial}\log\widetilde{\omega}^n = 0.$$

Therefore, K_X is unitary flat. For any Gauduchon metric ω, we have

$$\mathscr{F}(\omega) = n\int_X \mathrm{Ric}(\omega)\wedge\omega^{n-1} = n\int_X\left(\mathrm{Ric}(\widetilde{\omega}) - \sqrt{-1}\partial\overline{\partial}\log\left(\frac{\omega^n}{\widetilde{\omega}^n}\right)\right)\wedge\omega^{n-1} = 0.$$

Next, we show if $\mathscr{F}(\omega)<0$ for every Gauduchon metric ω, then K_X is pseudo-effective but not unitary flat. It follows from Lemma 3.1 and [14, Lemma 3.3]. Indeed, we fix a smooth Gauduchon metric ω_G. By Lemma 3.1, for any $\partial\overline{\partial}$-closed positive $(n-1,n-1)$-form $\psi\in\mathscr{V}$, there exists a smooth Gauduchon metric ω such that $\omega^{n-1}=\psi$. It is easy to see that

$$\mathscr{F}(\omega) = n\int_X \mathrm{Ric}(\omega)\wedge\omega^{n-1} = n\int_X \mathrm{Ric}(\omega_G)\wedge\omega^{n-1} = n\int_X \mathrm{Ric}(\omega_G)\wedge\psi < 0$$

By [14, Lemma 3.3], there exists $\varphi\in L^1(X,\mathbb{R})$ such that

$$-\mathrm{Ric}(\omega_G)+\sqrt{-1}\partial\overline{\partial}\varphi\ge 0$$

in the sense of currents. That means, K_X is pseudo-effective.

Claim 2. The anti-canonical bundle K_X^{-1} is pseudo-effective if and only if $\mathscr{F}(\omega)\ge 0$ for every Gauduchon metric ω. If, in addition, there exists some Gauduchon metric ω_0 such that $\mathscr{F}(\omega_0)=0$, then $\mathscr{F}(\mathscr{W})=\{0\}$ and K_X^{-1} is unitary flat.

The proof of Claim 2 is similar to that of Claim 1.

Claim 3. $\mathscr{F}(\mathscr{W})=\mathbb{R}$ if and only if neither K_X nor K_X^{-1} is pseudo-effective.

Indeed, if there exist two Gauduchon metrics ω_1 and ω_2 such that $\mathscr{F}(\omega_1)>0$ and $\mathscr{F}(\omega_2)<0$, then there exists a smooth Gauduchon metric ω_G such that $\mathscr{F}(\omega_G)=0$, and by the scaling relation

$$\mathscr{F}(\lambda\omega)=\lambda^{n-1}\mathscr{F}(\omega), \tag{3.2}$$

we have $\mathscr{F}(\mathscr{W})=\mathbb{R}$. Actually, by Lemma 3.1, there exists a Gauduchon metric ω_G such that

$$\omega_G^{n-1} = \mathscr{F}(\omega_1)\omega_2^{n-1} - \mathscr{F}(\omega_2)\omega_1^{n-1} \tag{3.3}$$

and the total scalar curvature

$$\begin{aligned}\mathscr{F}(\omega_G) &= n\int_X \mathrm{Ric}(\omega_G)\wedge\omega_G^{n-1} = n\int_X \mathrm{Ric}(\omega_G)\wedge\left(\mathscr{F}(\omega_1)\omega_2^{n-1}-\mathscr{F}(\omega_2)\omega_1^{n-1}\right)\\ &= \mathscr{F}(\omega_1)\cdot n\int_X \mathrm{Ric}(\omega_2)\wedge\omega_2^{n-1}-\mathscr{F}(\omega_2)\cdot n\int_X \mathrm{Ric}(\omega_1)\wedge\omega_1^{n-1}\\ &= 0.\end{aligned}$$

Now Claim 3 follows from Claim 1 and Claim 2. The proof of Theorem 1.1 is completed. □

Before giving the proof of Theorem 1.3, we need the following results which follow from Gauduchon's conformal method. We refer to [1–3, 9, 10] for more details. See also an almost Hermitian version investigated in [12]. For the reader's convenience, we include a proof here.

Lemma 3.2. *Let X be a compact complex manifold. The following are equivalent*:

(1) *there exists a smooth Gauduchon metric with positive* (*resp., negative, zero*) total *scalar curvature*;
(2) *there exists a smooth Hermitian metric with positive* (*resp., negative, zero*) *scalar curvature.*

Proof. Let ω_G be a Gauduchon metric and let s_G be its Chern scalar curvature. It is well known (e.g., [9, 10]) that the following equation:

$$s_G - \mathrm{tr}_{\omega_G}\sqrt{-1}\partial\bar\partial f = \frac{\int_X s_G\omega_G^n}{\int_X \omega_G^n} \tag{3.4}$$

has a solution $f\in C^\infty(X)$ since ω_G is Gauduchon and the integration

$$\int_X\left(s_G-\frac{\int_X s_G\omega_G^n}{\int_X\omega_G^n}\right)\omega_G^n = 0.$$

Let $\omega_g = e^{\frac{f}{n}}\omega_G$. Then the (Chern) scalar curvature s_g of ω_g is

$$\begin{aligned}s_g &= \mathrm{tr}_{\omega_g}\mathrm{Ric}(\omega_g) = -\mathrm{tr}_{\omega_g}\sqrt{-1}\partial\bar\partial\log(\omega_g^n)\\ &= -f^{-\frac{1}{n}}\mathrm{tr}_{\omega_G}\sqrt{-1}\partial\bar\partial\log(e^f\omega_G^n)\\ &= f^{-\frac{1}{n}}\left(s_G-\mathrm{tr}_{\omega_G}\sqrt{-1}\partial\bar\partial f\right)\\ &= f^{-\frac{1}{n}}\frac{\int_X s_G\omega_G^n}{\int_X\omega_G^n} = \frac{f^{-\frac{1}{n}}}{\int_X\omega_G^n}\cdot\mathscr{F}(\omega_G).\end{aligned}$$

Hence, a smooth Gauduchon metric ω_G with positive (resp., negative, zero) total scalar curvature can induce a smooth Hermitian metric with positive (resp., negative, zero) scalar curvature.

Conversely, let $\omega_G = f_0^{\frac{1}{n-1}}\omega$ be a Gauduchon metric in the conformal class of ω for some strictly positive function $f_0\in C^\infty(X)$. Let s_G be the corresponding Chern scalar curvature with respect to the Gauduchon metric ω_G. Then we obtain

$$\begin{aligned}\int_X s_G\omega_G^n &= n\int_X\mathrm{Ric}(\omega_G)\wedge\omega_G^{n-1} = n\int_X\mathrm{Ric}(\omega)\wedge\omega_G^{n-1}\\ &= n\int_X f_0\mathrm{Ric}(\omega)\wedge\omega^{n-1} = \int_X f_0 s\omega^n.\end{aligned}\tag{3.5}$$

Hence, a Hermitian metric with positive (resp., negative, zero) scalar curvature can induce a Gauduchon metric with positive (resp., negative, zero) total scalar curvature. ☐

By using standard Bochner technique (e.g., [22, 26]), it is easy to show that if a compact complex manifold X admits a smooth Hermitian metric with positive scalar curvature, then the Kodaira dimension $\kappa(X) = -\infty$. Hence, by Lemma 3.2, one has the following well-known result.

Corollary 3.3. *Let X be a compact complex manifold. Suppose X has a smooth Gauduchon metric ω with positive total scalar curvature; then $\kappa(X) = -\infty$.*

The proof of Theorem 1.3. If there exists a smooth Gauduchon metric with positive total scalar curvature, by Theorem 1.1, we deduce K_X is not pseudo-effective. Conversely, if K_X is not pseudo-effective, then by Theorem 1.1, there exists a smooth Gauduchon metric with positive total scalar curvature. Now Theorem 1.3 follows from Lemma 3.2. ☐

The proof of Theorem 1.4. Suppose there exists some Gauduchon metric ω_0 such that $\mathscr{F}(\omega_0) = 0$; then by Theorem 1.1, $\mathscr{F}(\mathscr{W}) = \mathbb{R}$ or $\{0\}$. If $\mathscr{F}(\mathscr{W}) = \mathbb{R}$, then by Corollary, 3.3 $\kappa(X) = -\infty$ and by Theorem 1.1, neither K_X nor K_X^{-1} is pseudo-effective. On the other hand, if $\mathscr{F}(\mathscr{W}) = \{0\}$, by Theorem 1.1, K_X is unitary flat, i.e., $c_1^{\mathrm{BC}}(X) = 0$. If $\kappa(X) \geq 0$, then by [22, Theorem 1.4], K_X is a holomorphic torsion, i.e., $K_X^{\otimes m} = \mathcal{O}_X$ for some $m \in \mathbb{Z}^+$. Indeed, since K_X^{-1} is unitary flat and hence nef, suppose $0 \neq \sigma \in H^0(X, K_X^{\otimes m})$; then σ is nowhere vanishing ([8, Proposition 1.16]), i.e., $K_X^{\otimes m} = \mathcal{O}_X$. (Note that there exist compact complex manifolds with $\kappa(X) = -\infty$ and $c_1^{\mathrm{BC}}(X) = 0$, e.g., Example 5.2.)

Conversely, if K_X is unitary flat, or neither K_X nor K_X^{-1} is pseudo-effective, by Theorem 1.1, $0 \in \mathscr{F}(\mathscr{W})$, i.e., there exists a smooth Gauduchon metric with vanishing total scalar curvature. ☐

The proof of Corollary 1.5. Suppose $\mathscr{F}(\mathscr{W}) = \mathbb{R}$ or $\mathbb{R}^{>0}$, by Corollary 3.3, $\kappa(X) = -\infty$ which is a contradiction. Suppose $\mathscr{F}(\mathscr{W}) = \{0\}$; then $c_1^{\mathrm{BC}}(X) = 0$ and by Theorem 1.4, $\kappa(X) = -\infty$ or $\kappa(X) = 0$ which is a contradiction again. Hence we have $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{<0}$. ☐

By using similar ideas as in the proof of Theorem 1.1, we obtain the following.

Theorem 3.4. *Let X be a compact complex manifold. For a Bott–Chern class $[\alpha] \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$, we define a function $\mathscr{F}_{[\alpha]} : \mathscr{W} \to \mathbb{R}$ with respect to the class $[\alpha]$ as*

$$\mathscr{F}_{[\alpha]}(\omega) = \int_X [\alpha] \wedge \omega^{n-1}. \tag{3.6}$$

Then the image of the total scalar function $\mathscr{F}_{[\alpha]} : \mathscr{W} \to \mathbb{R}$ has four different cases:

(1) $\mathcal{F}_{[\alpha]}(\mathscr{W}) = \mathbb{R}$ *if and only if neither $[\alpha]$ nor $-[\alpha]$ is pseudo-effective;*
(2) $\mathcal{F}_{[\alpha]}(\mathscr{W}) = \mathbb{R}^{>0}$ *if and only if $[\alpha]$ is pseudo-effective but not zero;*
(3) $\mathcal{F}_{[\alpha]}(\mathscr{W}) = \mathbb{R}^{<0}$ *if and only if $-[\alpha]$ is pseudo-effective but not zero;*
(4) $\mathcal{F}_{[\alpha]}(\mathscr{W}) = \{0\}$ *if and only if $[\alpha]$ is zero.*

Remark 3.5. Since $c_1^{\mathrm{BC}}(X) = c_1^{\mathrm{BC}}(K_X^{-1})$, if we set $[\alpha] = c_1^{\mathrm{BC}}(X) \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ in Theorem 3.4, we establish Theorem 1.1.

4. Some open problems

Let X be a compact complex manifold in class $\mathcal{C}$, i.e., X is bimeromorphic to a compact Kähler manifold. Compact Kähler, Moishezon and projective manifolds are all in class $\mathcal{C}$.

The following conjectures are either well known or implicitly indicated in the literatures in some special cases, and we refer to [5, 6, 11, 32] and the references therein.

Conjecture 4.1. *$\kappa(X) = -\infty$ if and only if X is uniruled, i.e., X is covered by rational curves.*

Conjecture 4.2. *K_X is pseudo-effective if and only if K_X is $\mathbb{Q}$-effective.*

Conjecture 4.3. *$\kappa(X) = -\infty$ if and only if there exists a Gauduchon metric with positive total scalar curvature.*

Remark 4.4. In Conjecture 4.1, Conjecture 4.2, and Conjecture 4.3, the necessary condition directions are well known.

Corollary 4.5. *Conjecture* 4.2 *is equivalent to Conjecture* 4.3.

Proof. Suppose Conjecture 4.2 is valid. If $\kappa(X) = -\infty$ and for any Gauduchon metric ω the total scalar curvature is non-positive, then by Theorem 1.1 K_X is pseudo-effective. Hence by Conjecture 4.2, K_X is $\mathbb{Q}$-effective, i.e., $H^0(X, K_X^{\otimes m}) \neq 0$. That means $\kappa(X) \geq 0$ which is a contradiction.

Assume Conjecture 4.3 is true. Suppose K_X is pseudo-effective but K_X is not $\mathbb{Q}$-effective, i.e., $\kappa(X) = -\infty$. By Conjecture 4.3, there exists a Gauduchon metric with positive total scalar curvature. According to Theorem 1.1, K_X is not pseudo-effective which is a contradiction. □

Corollary 4.6. *If X is Moishezon, Conjecture* 4.1*, Conjecture* 4.2*, and Conjecture* 4.3 *are equivalent.*

Proof. It follows by Corollary 4.5 and [6, Theorem D]. Indeed, it is shown in [6, Theorem D] that, a compact Moishezon manifold X is uniruled if and only if X admits Gauduchon metric with positive total scalar curvature. Hence, Conjecture 4.1 and Conjecture 4.3 are equvialent. □

The following conjecture is of particular interest in Kähler geometry.

Conjecture 4.7. *Let X be a compact Kähler manifold. Then X is uniruled if and only if X admits a smooth Kähler metric with positive total scalar curvature.*

When X is a compact Kähler surface, Conjecture 4.7 was proved by Yau ([32, Section 1.2]). On the other hand, Conjecture 4.7 predicts that on compact Kähler manifolds, the existence of rational curves requires merely the positivity of total scalar curvature of some Kähler metric.

5. Examples of non-Kähler Calabi–Yau manifolds

In this section we give some examples of *non-Kähler Calabi–Yau* manifolds satisfying the conditions in Theorem 1.1 or Theorem 1.4. These examples also show significant differences between non-Kähler manifolds and Kähler manifolds in our setting.

5.1. Let $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ be the standard n-dimensional ($n \geq 2$) Hopf manifold. It is diffeomorphic to $\mathbb{C}^n - \{0\}/G$ where G is a cyclic group generated by the transformation $z \to \frac{1}{2}z$. On X, there is a natural induced Hermitian metric ω given by

$$\omega = \sqrt{-1}h_{i\bar{j}}dz^i \wedge d\bar{z}^j = \frac{4\delta_{i\bar{j}}}{|z|^2}dz^i \wedge d\bar{z}^j.$$

This example is studied with details in [15, 16, 22–24, 27, 28]. One has

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det\omega^n = n \cdot \sqrt{-1}\partial\bar{\partial}\log|z|^2.$$

Hence $\mathrm{Ric}(\omega)$ is semi-positive. In particular, K_X^{-1} is pseudo-effective. Moreover, for any Gauduchon metric ω_G, we have $\mathscr{F}(\omega_G) > 0$. Indeed,

$$\begin{aligned}
\mathscr{F}(\omega_G) &= n\int_X \mathrm{Ric}(\omega_G) \wedge \omega_G^{n-1} \\
&= n\int_X \left(\mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\omega_G^n}{\omega^n}\right)\right) \wedge \omega_G^{n-1} \\
&= n\int_X \mathrm{Ric}(\omega) \wedge \omega_G^{n-1} = \int_X \left(\mathrm{tr}_{\omega_G}\mathrm{Ric}(\omega)\right)\omega_G^n > 0.
\end{aligned}$$

Moreover, X contains no rational curve. Otherwise, we have a non-zero holomorphic map from $\mathbb{P}^1$ to $\mathbb{C}^n$ which is absurd. It is easy to see K_X is not pseudo-effective. In summary,

Example 5.1. On Hopf manifold $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$, we have the following properties:

(1) $\kappa(X) = -\infty$;
(2) the total scalar curvature $\mathscr{F}(\omega) > 0$ for every Gauduchon metric ω;
(3) X has no rational curve (counterexample to Conjecture 4.1 on general complex manifolds);
(4) $c_1(X) = 0$ and $c_1^{\mathrm{BC}}(X) \neq 0$;
(5) K_X^{-1} is pseudo-effective, and K_X is not pseudo-effective.

This example lies in case (2) of Theorem 1.1.

5.2. We give an example described in [18, 22]. Let α, β be the two roots of the equation $x^2 - (1+i)x + 1 = 0$. The minimal polynomial over $\mathbb{Q}$ of α (and $\bar{\beta}$) is $x^4 - 2x^3 + 4x^2 - 2x + 1$. Let Λ be the lattice in $\mathbb{C}^2$ spanned by the vectors $(\alpha^j, \bar{\beta}^j)$, $j = 0, \ldots, 3$. Let $Y = \mathbb{C}^2/\Lambda$. The automorphism of $\mathbb{C}^2$ given by multiplication by $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ descends to an automorphism f of Y. Let $C = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ be an elliptic curve, and we define a holomorphic free $\mathbb{Z}^2$-action on $Y \times \mathbb{C}$ by

$$(1,0)\cdot(x,z) = (x, z+1), \quad (0,1)\cdot(x,z) = (f(x), z+\tau).$$

Then the quotient space X is a holomorphic fiber bundle $X \to C$ with fiber Y. Following [18, 22], X is a non-Kähler manifold with $c_1^{\mathrm{BC}}(X) = 0$ and $\kappa(X) = -\infty$.

Example 5.2. On X, we have the following properties:

(1) $\kappa(X) = -\infty$;
(2) $c_1(X) = 0$, $c_1^{\mathrm{BC}}(X) = 0$, i.e., K_X is unitary flat;
(3) the total scalar curvature $\mathscr{F}(\omega) \equiv 0$ for every Gauduchon metric ω;
(4) X has no rational curve;
(5) K_X is pseudo-effective, but it is not $\mathbb{Q}$-effective (counterexample to Conjecture 4.2 on general complex manifolds).

This example lies in case (4) of Theorem 1.1 and case (2) of Theorem 1.4.

5.3. This construction follows from [22, Example 3.4]. Let $T = \mathbb{C}^n/\Lambda$ be an n-torus, let Σ be a compact Riemann surface of genus $g \geq 2$, and let $\pi : X \to \Sigma$ be any topologically non-trivial principal T-bundle over Σ. It is shown in [22, Example 3.4] that X is a non-Kähler manifold with $c_1(X) = 0$, $c_1^{\mathrm{BC}}(X) \neq 0$, and $\kappa(X) = 1$. By Corollary 1.5, we know for any Gauduchon metric ω, the total scalar curvature $\mathscr{F}(\omega) < 0$.

Example 5.3. On X, we have the following properties:

(1) $\kappa(X) = 1$;
(2) $c_1(X)$=0 and $c_1^{\mathrm{BC}}(X) \neq 0$;
(3) the total scalar curvature $\mathscr{F}(\omega) < 0$ for every Gauduchon metric ω;
(4) K_X is pseudo-effective, but $-K_X$ is not pseudo-effective.

This example lies in case (3) of Theorem 1.1.

Example 5.4. Let $X = X_2 \times X_3$ be the product manifold where X_2 and X_3 are the complex manifolds constructed in Example 5.2 and Example 5.3, respectively. It has the following properties:

(1) $\kappa(X) = -\infty$;
(2) $c_1(X) = 0$, $c_1^{\mathrm{BC}}(X) \neq 0$;
(3) the total scalar curvature $\mathscr{F}(\omega) < 0$ for every Gauduchon metric ω;
(4) K_X is pseudo-effective, but it is not $\mathbb{Q}$-effective.

This example lies in case (3) of Theorem 1.1.

Example 5.5. Let $X = X_1 \times X_2 \times X_3$ be the product manifold where X_1, X_2, and X_3 are the complex manifolds constructed in Example 5.1, Example 5.2, and Example 5.3, respectively. It has the following properties:

(1) $\kappa(X) = -\infty$.
(2) $c_1(X) = 0$, $c_1^{\mathrm{BC}}(X) \neq 0$.
(3) The total scalar curvature can be any real number. Indeed, it follows from Example 5.1 and Example 5.4 by using the scaling trick.
(4) Neither K_X nor K_X^{-1} is pseudo-effective.

This example lies in case (1) of Theorem 1.1 and case (1) of Theorem 1.4.

Example 5.6. Let X be a Kodaira surface (a non-Kähler compact complex surface with torsion canonical line bundle). It has the following properties:

(1) $\kappa(X) = 0$;
(2) $c_1(X) = 0$, $c_1^{\mathrm{BC}}(X) = 0$ and K_X is unitary flat;
(3) the total scalar curvature $\mathscr{F}(\omega) = 0$ for every Gauduchon metric ω.

This example lies in case (4) of Theorem 1.1 and case (3) of Theorem 1.4.

6. Existence of smooth Hermitian metrics with positive total scalar curvature

In this section we prove Theorem 1.7. As we pointed out before, one of the key gradients in the proof is Székelyhidi-Tosatti-Weinkove's solution to the Gauduchon conjecture on compact complex manifold, which is analogous to Yau's solution to the Calabi conjecture [33] on compact Kähler manifolds.

Theorem 6.1 ([20, Theorem 1.3]). *Let X be a compact complex manifold. Let ω_0 be a smooth Gauduchon metric, and let Φ be a closed real $(1,1)$-form on X with $[\Phi] = c_1^{\mathrm{BC}}(X) \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$. Then there exists a smooth Gauduchon metric ω satisfying $[\omega^{n-1}] = [\omega_0^{n-1}]$ in $H^{n-1,n-1}_{\mathrm{A}}(X,\mathbb{R})$ and*

$$\mathrm{Ric}(\omega) = \Phi. \tag{6.1}$$

In particular, for any smooth volume form σ on X, there exists a smooth Gauduchon metric ω such that

$$\omega^n = \sigma. \tag{6.2}$$

Theorem 6.2. *Let X be an arbitrary compact complex manifold with $\dim X > 1$. Then there exists a smooth Hermitian metric ω with positive total scalar curvature. Moreover, if X is Kähler, then there exists a conformally Kähler metric ω with positive total scalar curvature (i.e., $\omega = e^f\omega_0$ for some Kähler metric ω_0 and $f \in C^\infty(X,\mathbb{R})$).*

Proof. There exists a smooth Gauduchon metric ω such that the scalar curvature of ω is strictly positive at some point $p \in X$. Indeed, fix an arbitrary smooth Hermitian metric ω_0 on X; then there exists some smooth function F such that

$$\mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\overline{\partial}F$$

is positive definite at point p. Let $\{z^i\}$ be the holomorphic coordinates centered at point p; we can choose $F(p) = -\lambda|z|^2 + o(|z|^3)$ for some large positive constant λ (depending on $\mathrm{Ric}(\omega_0)(p)$). On the other hand, by Theorem 6.1, there exists a smooth Gauduchon metric ω such that

$$\omega^n = e^F\omega_0^n. \tag{6.3}$$

Then $\mathrm{Ric}(\omega) = \mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\overline{\partial}F$ is positive definite at point p. Hence the scalar curvature of ω is positive at p. Let $\omega_f = e^f\omega$; then the scalar curvature s_f of ω_f is

$$s_f = \mathrm{tr}_{\omega_f}\mathrm{Ric}(\omega_f) = e^{-f}\mathrm{tr}_\omega\left(\mathrm{Ric}(\omega) - n\sqrt{-1}\partial\overline{\partial}f\right) = e^{-f}(s_\omega - n\Delta_\omega f)$$

and the total scalar curvature of ω_f is

$$\begin{aligned}
\int_X s_f\omega_f^n &= \int_X e^{(n-1)f}\cdot(s_\omega - n\Delta_\omega f)\cdot\omega^n \\
&= \int_X e^{(n-1)f}\cdot s_\omega\cdot\omega^n - n^2\int_X e^{(n-1)f}\cdot\sqrt{-1}\partial\overline{\partial}f\wedge\omega^{n-1} \\
&= \int_X e^{(n-1)f}\cdot s_\omega\cdot\omega^n + n^2(n-1)\int_X e^{(n-1)f}\cdot\sqrt{-1}\partial f\wedge\overline{\partial}f\wedge\omega^{n-1} \\
&\quad -n^2\int_X e^{(n-1)f}\sqrt{-1}\overline{\partial}f\wedge\partial\omega^{n-1},
\end{aligned}$$

where we use Stokes' theorem in the last identity. It is obvious that

$$n^2(n-1)\int_X e^{(n-1)f}\cdot\sqrt{-1}\partial f\wedge\overline{\partial}f\wedge\omega^{n-1} \geq 0.$$

Moreover, we have

$$\begin{aligned}-n^2\int_X e^{(n-1)f}\sqrt{-1}\partial f\wedge\partial\omega^{n-1} &= -\frac{n^2}{n-1}\int_X\sqrt{-1}\left(\overline{\partial}e^{(n-1)f}\right)\wedge\partial\omega^{n-1}\\ &= \frac{n^2}{n-1}\int_X\sqrt{-1}e^{(n-1)f}\cdot\overline{\partial}\partial\omega^{n-1}\\ &= 0\end{aligned}$$

since ω is Gauduchon. Therefore, we obtain

$$\int_X s_f\omega_f^n \geq \int_X e^{(n-1)f}\cdot s_\omega\cdot\omega^n. \tag{6.4}$$

Since $s_\omega(p)>0$, by standard analytic techniques, there exists some smooth function f such that

$$\int_X s_f\omega_f^n \geq \int_X e^{(n-1)f}\cdot s_\omega\cdot\omega^n > 0.$$

Indeed, without loss of generality, we can assume

$$\max_X s_\omega = k_0 > 0,\quad \min_X s_\omega = -k_1 < 0\quad\text{and}\quad\int_X\omega^n = 1. \tag{6.5}$$

Let

$$X_1=\left\{q\in X\mid s_\omega(q)\in\left[\frac{k_0}{2},k_0\right]\right\},\quad X_2=\left\{q\in X\mid s_\omega(q)\in\left[\frac{k_0}{4},\frac{k_0}{2}\right)\right\},$$

and

$$X_3=\left\{q\in X\mid s_\omega(q)\in\left[-k_1,\frac{k_0}{4}\right)\right\}.$$

Let f be a smooth function such that $e^{(n-1)f}\equiv 1$ on X_3 and $e^{(n-1)f}\equiv 1+\frac{2k_1}{k_0}$ on X_1. Then

$$\begin{aligned}\int_X e^{(n-1)f}\cdot s_\omega\cdot\omega^n &\geq \int_{X_1} e^{(n-1)f}\cdot s_\omega\cdot\omega^n+\int_{X_3}e^{(n-1)f}\cdot s_\omega\cdot\omega^n\\ &\geq -k_1+\left(1+\frac{2k_1}{k_0}\right)\cdot\frac{k_0}{2}=\frac{k_0}{2}>0.\end{aligned}$$

The proof of the first part of Theorem 1.7 is completed.

Suppose X is Kähler; then by the same arguments as above, there exists a smooth Kähler metric ω such that the scalar curvature of ω is strictly positive at some point $p\in X$ where we use the Calabi–Yau theorem in (6.3). By using the conformal method and integration by parts, we obtain a conformally Kähler metric with positive total scalar curvature. □

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