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Collapsing of the Chern–Ricci flow on elliptic surfaces

Valentino Tosatti · Ben Weinkove · Xiaokui Yang

Abstract We investigate the Chern–Ricci flow, an evolution equation of Hermitian metrics generalizing the Kähler–Ricci flow, on elliptic bundles over a Riemann surface of genus greater than one. We show that, starting at any Gauduchon metric, the flow collapses the elliptic fibers and the metrics converge to the pullback of a Kähler–Einstein metric from the base. Some of our estimates are new even for the Kähler–Ricci flow. A consequence of our result is that, on every minimal non-Kähler surface of Kodaira dimension one, the Chern–Ricci flow converges in the sense of Gromov–Hausdorff to an orbifold Kähler–Einstein metric on a Riemann surface.

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1 Introduction

The Chern–Ricci flow is an evolution equation for Hermitian metrics on complex manifolds. Given a starting Hermitian metric g_0, which we represent as a real $(1, 1)$ form $\omega_0 = \sqrt{-1}(g_0)_{i\bar{j}}dz^i \wedge d\bar{z}^j$, the Chern–Ricci flow is given by

$$\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{1.1}$$

where $\text{Ric}(\omega) := -\sqrt{-1}\partial\bar{\partial} \log \det g$ is the Chern–Ricci form of ω. In the case when g_0 is Kähler, namely $d\omega_0 = 0$, (1.1) coincides with the Kähler–Ricci flow.

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The Chern–Ricci flow was first introduced by Gill [11] in the setting of manifolds with $c_1^{\mathrm{BC}}(M) = 0$, where $c_1^{\mathrm{BC}}(M)$ is the first Bott–Chern class given by

$$c_1^{\mathrm{BC}}(M) = [\mathrm{Ric}(\omega)] \in H_{\mathrm{BC}}^{1,1}(M, \mathbb{R}) = \frac{\{\text{closed real } (1, 1)\text{-forms}\}}{\sqrt{-1}\partial\bar{\partial}(C^\infty(M))},$$

for any Hermitian metric ω. Making use of an estimate for the complex Monge–Ampère equation [6,35], Gill showed that solutions to (1.1) on manifolds with $c_1^{\mathrm{BC}}(M) = 0$ exist for all time and converge to Hermitian metrics with vanishing Chern–Ricci form. Gill's theorem generalizes the convergence result of Cao [4] for the Kähler–Ricci flow (which made use of estimates of Yau [39]).

The first and second named authors investigated the Chern–Ricci flow on more general manifolds [36,37] and proved a number of further results. It was shown in particular that the maximal existence time for the flow can be determined from the initial metric; that the Chern–Ricci flow on manifolds with negative first Chern class smoothly deforms Hermitian metrics to Kähler–Einstein metrics; that when starting on a complex surface with Gauduchon initial metric ω_0 (meaning $\partial\bar{\partial}\omega_0 = 0$), the Chern–Ricci flow exists until the volume of the manifold or a curve of negative self-intersection goes to zero; and that on surfaces with nonnegative Kodaira dimension the Chern–Ricci flow contracts an exceptional curve when one exists. There are analogues of all of these results for the Kähler–Ricci flow [4,7,27,32].

For the purpose of this discussion it will be useful to make reference to the following condition:

$(*)$ M is a minimal non-Kähler complex surface and ω_0 is Gauduchon.

Surfaces which satisfy $(*)$ are of significant interest as they are not yet completely classified. Recall that a surface is minimal if it contains no (-1)-curves and every complex surface is birational to a minimal one by via a finite sequence of blow downs. Every complex surface admits a Gauduchon metric, and the Gauduchon condition is preserved by the Chern–Ricci flow.

We remark that Streets–Tian [29] earlier proposed the use of a different parabolic flow, called the Pluriclosed Flow, to study complex non-Kähler surfaces (see Sect. 2 of [36] for some discussion on how this flow differs from the Chern–Ricci flow).

The Kodaira–Enriques classification (see [1]) tells us that manifolds M satisfying $(*)$ fall into one of the following groups:

– $\mathrm{Kod}(M) = 1$. Minimal non-Kähler properly elliptic surfaces.
– $\mathrm{Kod}(M) = 0$. Kodaira surfaces.
– $\mathrm{Kod}(M) = -\infty$. Class VII surfaces which have either:
    ⋄ $b_2(M) = 0$. Hopf surfaces or Inoue surfaces by [15,16,30].
    ⋄ $b_2(M) = 1$. These are classified by [18,31].
    ⋄ $b_2(M) > 1$. Still unclassified.

Here $\mathrm{Kod}(M)$ is the Kodaira dimension of M. The result of Gill [11] shows that when $\mathrm{Kod}(M) = 0$ the Chern–Ricci flow exists for all time and converges to a Chern–Ricci flat metric.

In [37], *explicit examples* of solutions to the Chern–Ricci flow were found on all M with $\mathrm{Kod}(M) = 1$, for all Inoue surfaces and for a large class of Hopf surfaces. In particular, it was shown that for any M with $\mathrm{Kod}(M) = 1$ there exists an explicit solution $\omega(t)$ of the Chern–Ricci flow for $t \in [0, \infty)$ with the property that as $t \to \infty$ the normalized metrics $\omega(t)/t$ converge in the sense of Gromov–Hausdorff to (C, d_{KE}) where C is a Riemann surface and d_{KE} is the distance function induced by an orbifold Kähler–Einstein metric on C.

The main result of this paper is to show that this collapsing behavior on surfaces of Kodaira dimension one in the examples of [37] actually occurs for *every choice* of initial starting Gauduchon metric ω_0. Combined with Gill's theorem, our results mean that the only remaining case (presumably the most difficult!) under assumption (∗) is to understand the behavior of the Chern–Ricci flow on surfaces of negative Kodaira dimension. We believe that the results of this paper add to the growing body of evidence that the Chern–Ricci flow is a natural geometric evolution equation on complex surfaces, whose behavior reflects the underlying geometry of the manifold.

We first consider the case of elliptic bundles over a Riemann surface. Later we will see that this is sufficient to understand the behavior of the flow on all M satisfying (∗) with $\mathrm{Kod}(M) = 1$. Suppose that $\pi : M \to S$ is now an elliptic bundle over a compact Riemann surface S of genus at least 2, with fiber an elliptic curve E. We will denote by $E_y = \pi^{-1}(y)$ the fiber over a point $y \in S$ and by $\omega_{\mathrm{flat},y}$ the unique flat metric on E_y in the Kähler class $[\omega_0|_{E_y}]$. Let ω_S be the unique Kähler–Einstein metric on S with $\mathrm{Ric}(\omega_S) = -\omega_S$ and let ω_0 be a Gauduchon metric on M.

We consider the *normalized Chern–Ricci flow*

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) - \omega, \quad \omega|_{t=0} = \omega_0, \tag{1.2}$$

starting at ω_0. With this normalized flow we will see that the volume of the base Riemann surface S remains positive and bounded while the elliptic fibers collapse. One could equally well study the unnormalized flow (1.1) on M (so that our main collapsing result would apply to $\omega(t)/t$ as in [37]) but we choose this normalization to stay in keeping with the literature on the Kähler–Ricci flow [24]. From [36] we know that a smooth solution to (1.2) exists for all time (see Sect. 2 below for more details). In this paper we prove the following convergence result as $t \to \infty$.

Theorem 1.1 *Let $\pi : M \to S$ be an elliptic bundle over a Riemann surface S of genus at least 2. Let $\omega(t)$ be a solution of the normalized Chern–Ricci flow (1.2) on M starting at a Gauduchon metric ω_0. Then as $t \to \infty$,*

$$\omega(t) \to \pi^*\omega_S,$$

exponentially fast in the $C^0(M, g_0)$ topology, where ω_S is the unique Kähler–Einstein metric on S. In particular, the diameter of each elliptic fiber tends to zero uniformly exponentially fast and $(M, \omega(t))$ converges to (S, ω_S) in the Gromov–Hausdorff topology.

Furthermore, with the notation above, $e^t\omega(t)|_{E_y}$ converges to the metric $\omega_{\mathrm{flat},y}$ exponentially fast in the $C^1(E_y, g_0)$ topology, uniformly in $y \in S$.

Note that in Theorem 1.1 we do not need to assume that M is non-Kähler. On the other hand, we do assume that M is an elliptic bundle, so that the fibers are all isomorphic as elliptic curves. General elliptic surfaces may have singular fibers and in such cases, the complex structure of the smooth fibers may vary. However, we will see shortly that this does not arise for the non-Kähler surfaces that are of interest to us.

In the case that M is Kähler *and* ω_0 is Kähler, then $\omega(t)$ is a solution of the normalized Kähler–Ricci flow. There are already a number of results on this, which we now briefly discuss. On a general minimal Kähler elliptic surface, and its higher dimensional analogue, the Kähler–Ricci flow was first investigated by Song–Tian [24,25]. They showed that the flow converges at the level of potentials to a *generalized Kähler–Einstein metric* on the base Riemann surface. The generalized Kähler–Einstein equation involves the Weil–Petersson metric and singular currents. These terms arise because, unlike in our case, the fibration structure on a Kähler elliptic surface is not in general locally trivial and may have singular fibers. When the Kähler surface is a genuine elliptic bundle over a Riemann surface of genus larger than one, the results of Song–Tian give C^0 collapsing of the fibers along the Kähler–Ricci flow, as well as a uniform scalar curvature bound [26]. These convergence results were strengthened by Song–Weinkove [28] and Gill [12] in the special case of a product $E \times S$, giving C^∞ convergence of the metrics to the pull-back of a Kähler–Einstein metric on the base. Fong–Zhang [9], adapting a technique of Gross–Tosatti–Zhang [13] on Calabi–Yau degenerations, established smooth convergence for the Kähler–Ricci flow on more general elliptic bundles. In particular, the statement of Theorem 1.1 is known if the initial metric ω_0 is Kähler (with the exception of the assertion that the convergence $\omega(t) \to \pi^*\omega_S$ is exponential - as far as we know, this result is new even in the Kähler–Ricci flow case).

Of course, we are much more interested in manifolds which do *not* admit Kähler metrics. For non-Kähler elliptic surfaces, we make use of the following **key fact**:

Every minimal non-Kähler properly elliptic surface is an elliptic bundle or has a finite cover which is an elliptic bundle.

This is well-known from the Kodaira classification (see for example [3, Lemmas 1, 2] or [38, Theorem 7.4]). Then an immediate consequence of our Theorem 1.1 is that we can identify the Gromov–Hausdorff behavior of the Chern–Ricci flow on all minimal non-Kähler surfaces of Kodaira dimension one.

Corollary 1.2 *Let $\pi : M \to S$ be any minimal non-Kähler properly elliptic surface and let $\omega(t)$ be the solution of the normalized Chern–Ricci flow (1.2) starting at a Gauduchon metric ω_0. Then $(M, \omega(t))$ converges to (S, d_S) in the Gromov–Hausdorff topology.*

Here d_S is the distance function induced by an orbifold Kähler–Einstein metric ω_S on S, whose set Z of orbifold points is precisely the image of the multiple fibers of π. Furthermore, $\omega(t)$ converges to $\pi^*\omega_S$ in the $C^0(M, g_0)$ topology, and for any $y \in S\backslash Z$ the metrics $e^t\omega(t)|_{E_y}$ converge exponentially fast in the $C^1(E_y, g_0)$ topology (and uniformly as y varies in a compact set of $S\backslash Z$) to the flat Kähler metric on E_y cohomologous to $[\omega_0|_{E_y}]$.

As mentioned above, explicit examples exhibiting the behavior of Corollary 1.2 were given in [37].

We now outline the steps we need to establish Theorem 1.1, and point out some of the difficulties that arise from the non-Kählerity of the metrics.

The first parts of the proof follow quite closely the arguments used by Song–Tian [24] for the Kähler–Ricci flow. In Sect. 2, we show that the Chern–Ricci flow can be written as parabolic complex Monge–Ampère equation

$$\frac{\partial}{\partial t}\varphi = \log \frac{e^t(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi, \quad \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \quad \varphi(0) = -\rho, \quad (1.3)$$

where $\tilde{\omega} = \tilde{\omega}(t)$ is a family of reference forms (which are metrics for t large) given by

$$\tilde{\omega} = e^{-t}\omega_{\text{flat}} + (1 - e^{-t})\omega_S, \quad \text{with } \omega_{\text{flat}} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho,$$

where ρ is chosen so that ω_{flat} restricted to the fiber E_y is exactly the metric $\omega_{\text{flat},y}$ discussed above. Here Ω is a particular fixed volume form on M with the property that $\sqrt{-1}\partial\bar{\partial}\log\Omega = \omega_S$. If φ satisfies (1.3) then $\omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ satisfies the Chern–Ricci flow (1.2).

In Sect. 3 we establish uniform bounds for φ and $\dot{\varphi}$, which imply that the volume form of the evolving metric $\omega(t)$ is uniformly equivalent to the volume form of the reference metric $\omega(t)$. These follow in the same way as in the case of the Kähler–Ricci flow [24]. In addition, we prove a crucial decay estimate for φ,

$$|\varphi| \leq C(1 + t)e^{-t}, \quad (1.4)$$

using the argument of [28]. This estimate makes use of the Gauduchon assumption on ω_0, and in fact is the only place where we use this condition.

So far, the torsion terms of $\omega(t)$ and $\tilde{\omega}$ have not entered the picture. They show up in the next step of obtaining uniform bounds for the metrics $\omega(t)$. The evolution equation for $\text{tr}_{\tilde{\omega}}\omega$, essentially already computed in [36], contains terms involving the torsion and curvature of the reference metrics $\tilde{\omega}$. In Sect. 4 we prove a technical lemma giving bounds for the torsion and curvature of these metrics. In particular we show:

$$|\tilde{T}|_{\tilde{g}} \leq C, \quad |\bar{\partial}\tilde{T}|_{\tilde{g}} + |\tilde{\nabla}\tilde{T}|_{\tilde{g}} + |\widetilde{\text{Rm}}|_{\tilde{g}} \leq Ce^{t/2}.$$

To deal with these bounds of order $e^{t/2}$, our idea is to exploit the strong decay estimate (1.4) on φ to control these terms.

In Sect. 5, we evolve the quantity

$$Q = \log \text{tr}_{\tilde{\omega}}\omega - Ae^{t/2}\varphi + \frac{1}{\tilde{C} + e^{t/2}\varphi},$$

noting that $e^{t/2}\varphi$ is bounded by (1.4). The third term of Q is the "Phong–Sturm term" [20], which was used in [36] to control some torsion terms along the Chern–Ricci flow. Using the good positive terms arising from the Laplacian landing on $e^{t/2}\varphi$ we

can control the bad terms of order $e^{t/2}$ coming from the torsion and curvature of $\tilde{\omega}$. We obtain a uniform bound on Q which gives the estimate

$$C^{-1}\tilde{\omega} \leq \omega \leq C\tilde{\omega}, \tag{1.5}$$

namely, that the solution ω is uniformly equivalent to the reference metric $\tilde{\omega}$.

We point out that our argument here differs substantially from that of Song–Tian [24] where they prove first a parabolic Schwarz Lemma, namely an estimate of the type $\omega \geq C^{-1}\omega_S$ for a uniform $C > 0$. We were unable to prove this by a similar direct maximum principle argument, because of troublesome torsion terms arising in the evolution of $\mathrm{tr}_\omega \omega_S$. However, we still obtain the estimate $\omega \geq C^{-1}\omega_S$ once we have (1.5).

The next step is to improve the bound (1.5) to the stronger exponential convergence result

$$(1 - Ce^{-\varepsilon t})\tilde{\omega} \leq \omega \leq (1 + Ce^{-\varepsilon t})\tilde{\omega}, \tag{1.6}$$

for $\varepsilon > 0$. To our knowledge, this estimate is new even for the Kähler–Ricci flow on elliptic bundles. The idea is to evolve the quantity

$$Q = e^{\varepsilon t}(\mathrm{tr}_\omega \tilde{\omega} - 2) - e^{\delta t}\varphi,$$

for a carefully chosen $\delta > 1/2 + 2\varepsilon$ and again exploit the decay estimate (1.4). Showing that Q is bounded from above then gives the estimate

$$\mathrm{tr}_\omega \tilde{\omega} - 2 \leq Ce^{-\varepsilon t},$$

and a similar argument gives the same inequality with $\mathrm{tr}_\omega \tilde{\omega}$ replaced by $\mathrm{tr}_{\tilde{\omega}} \omega$. Combining these two estimates gives (1.6).

However, in order to apply the maximum principle to Q we first require an exponential decay estimate for $\dot{\varphi}$. To prove this, we observe that the evolution equation for $\dot{\varphi}$ is

$$\frac{\partial}{\partial t}\dot{\varphi} = -R - 1 - \dot{\varphi},$$

where R is the Chern scalar curvature of g. If we had a uniform bound for the Chern scalar curvature, an exponential decay estimate for $\dot{\varphi}$ would follow from this evolution equation and the decay estimate for φ. However, we are only able to prove the weaker estimate

$$-C \leq R \leq Ce^{t/2}. \tag{1.7}$$

Nevertheless, this suffices since the coefficient of t in the exponent is strictly less than 1. The bound (1.7) is the content of Sect. 6. The factor $e^{t/2}$ arises from the bounds on the curvature and torsion of the reference metrics we obtained in Sect. 4.

The idea for bounding the Chern scalar curvature from above (the lower bound is easy) is to consider the quantity $u = \varphi + \dot{\varphi}$ and bound from above $-\Delta u = R + \mathrm{tr}_\omega \omega_S \geq R$. Using an idea that goes back to Cheng–Yau [5], and is used in the

context of the Kähler–Ricci flow on Fano manifolds by Perelman (see Sesum–Tian [22]) and on elliptic surfaces by Song–Tian [24], we first bound the gradient of u by considering the quantity $|\nabla u|_g^2/(A-u)$ for a fixed large A. We then evolve the quantity $-\Delta u + 6|\nabla u|_g^2$, which is almost enough to obtain the estimate we need. There are some bad terms which we can control by adding large multiples of $\text{tr}_\omega \omega_S$ and $\text{tr}_{\tilde\omega}\omega$. We already know that these terms are bounded from (1.5). Putting this together gives the upper bound on scalar curvature and the exponential decay estimate for $\dot\varphi$ that we require.

Now that we have this exponential decay estimate on $\dot\varphi$, we carry out in Sect. 7 the argument mentioned above for the exponential convergence of the metrics (1.6).

In Sect. 8, we prove a local Calabi type estimate

$$|\hat\nabla g|_g^2 \le Ce^{2t/3}, \tag{1.8}$$

where $\hat\nabla$ is the connection associated to a local semi-flat product Kähler metric defined in a neighborhood U. Note that if we had the better estimate $|\hat\nabla g|_g^2 \le C$ (as in [9,12,28] for example) then we could immediately conclude the global convergence of the metrics $\omega(t)$ to $\pi^*\omega_S$ from the estimates (1.4) and (1.5) and the Ascoli–Arzelà Theorem. We do not know whether this stronger estimate $|\hat\nabla g|_g^2 \le C$ holds or not.

To establish (1.8), we use some arguments and calculations similar to the local Calabi estimate in [23]. However, a key difference here is that the metrics are collapsing in the fiber directions and we need to take account of the error terms that arise in this way. The local Calabi estimate is then used to establish the last part of Theorem 1.1 that $e^t\omega(t)|_{E_y}$ converges to $\omega_{\text{flat},y}$ exponentially fast in the $C^1(E_y, g_0)$ topology, uniformly in $y \in S$.

In Sect. 9 we complete the proofs of Theorem 1.1 (this essentially follows immediately) and Corollary 1.2.

2 Preliminaries

2.1 Hermitian geometry and notation

We begin with a brief recap of Hermitian geometry and the Chern connection (for more details see for example [36]).

Given a Hermitian metric g, we write $\omega = \sqrt{-1}g_{i\bar j}dz^i \wedge d\bar z^j$ for its associated $(1, 1)$ form, which we will also refer to as a metric. Write ∇ for its Chern connection, with respect to which g and the complex structure are covariantly constant. The Christoffel symbols of ∇ are given by $\Gamma_{ij}^k = g^{k\bar q}\partial_i g_{j\bar q}$. For example, if $X = X^i\partial_i$ is a vector field then its covariant derivative has components $\nabla_i X^\ell = \partial_i X^\ell + \Gamma_{ij}^\ell X^j$.

The torsion tensor of g has components $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$. We will often lower an index using the metric g, writing

$$T_{ij\bar\ell} = g_{k\bar\ell}T_{ij}^k = \partial_i g_{j\bar\ell} - \partial_j g_{i\bar\ell}.$$

Note that $T_{ij\bar{\ell}} = T'_{ij\bar{\ell}}$ if g and g' are Hermitian metrics whose (1, 1) forms ω and ω' differ by a closed form. The Chern curvature of g is defined to be $R_{k\bar{\ell}i}{}^p = -\partial_{\bar{\ell}}\Gamma^p_{ki}$, and we will raise and lower indices using the metric g. We have the usual commutation formulae involving the curvature, such as

$$[\nabla_k, \nabla_{\bar{\ell}}]X^i = R_{k\bar{\ell}j}{}^i X^j.$$

Define the Chern–Ricci curvature of g to be $R_{k\bar{\ell}} = g^{i\bar{j}}R_{k\bar{\ell}i\bar{j}} = -\partial_k\partial_{\bar{\ell}}\log\det g$, and we write

$$\mathrm{Ric}(\omega) = \sqrt{-1}R_{k\bar{\ell}}dz^k \wedge d\bar{z}^{\ell}$$

for the associated Chern–Ricci form, a real closed (1,1) form. Write $R = g^{k\bar{\ell}}R_{k\bar{\ell}}$ for the Chern scalar curvature.

We write Δ for the complex Laplacian of g, which acts on a function f by $\Delta f = g^{i\bar{j}}\partial_i\partial_{\bar{j}}f$. For functions f_1, f_2, we define $\langle\nabla f_1, \nabla f_2\rangle_g = g^{i\bar{j}}\partial_i f_1\partial_{\bar{j}}f_2$ and $|\nabla f|_g^2 = \langle\nabla f, \nabla f\rangle_g$. If $\alpha = \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a real (1, 1) form and ω a Hermitian metric we write $\mathrm{tr}_\omega\alpha$ for $g^{i\bar{j}}\alpha_{i\bar{j}}$.

A final remark about notation: we will write $C, C', C_0, \ldots$ etc. for a uniform constant, which may differ from line to line.

2.2 Elliptic bundles and semi-flat metrics

We now specialize to the setting of Theorem 1.1. Let $\pi : M \to S$ be an elliptic bundle over a compact Riemann surface S of genus at least 2, with fiber an elliptic curve E. Clearly $\pi : M \to S$ is relatively minimal, because there is no (-1)-curve contained in any fiber. We will denote by $E_y = \pi^{-1}(y)$ the fiber over a point $y \in S$. Let ω_S be the unique Kähler metric on S with $\mathrm{Ric}(\omega_S) = -\omega_S$, let ω_0 be a Gauduchon metric on M.

Since each fiber $E_y = \pi^{-1}(y)$ is a torus, we can find a function ρ_y on E_y with

$$\omega_0|_{E_y} + \sqrt{-1}\partial\bar{\partial}\rho_y = \omega_{\mathrm{flat},y},$$

the unique flat metric on E_y in the Kähler class $[\omega_0|_{E_y}]$. Furthermore we can normalize the functions ρ_y by $\int_{E_y}\rho_y\omega_0 = 0$, so that they vary smoothly in y (in general this follows from Yau's estimates [39], although in this simple case it can also be proved directly, see also [8, Lemma 2.1]), and they define a smooth function ρ on M. We then let

$$\omega_{\mathrm{flat}} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho. \tag{2.1}$$

ω_{flat} is a semi-flat form, in the sense that it restricts to a flat metric on each fiber E_y, but in general it is not positive definite on M. But note that $\omega_{\mathrm{flat}} \wedge \pi^*\omega_S$ is a strictly positive smooth volume form on M.

2.3 The canonical bundle and long time existence for the flow

In the same setting as above, we claim that $K_M = \pi^* K_S$. To see this, start from Kodaira's canonical bundle formula for relatively minimal elliptic surfaces without singular fibers [1, Theorem V.12.1]

$$K_M = \pi^*(K_S \otimes L),$$

where L is the dual of $R^1 \pi_* \mathcal{O}_M$. But since M is an elliptic bundle, it follows that the line bundle $R^1 \pi_* \mathcal{O}_M$ is trivial (see e.g. [2, Proposition 2.1]), and the claim follows.

Therefore $c_1(M) = \pi^* c_1(S)$ (an alternative more direct proof of this fact is contained in Lemma 3.2), and so there exists a unique volume form Ω with

$$\text{Ric}(\Omega) = -\omega_S \quad \text{and} \quad \int_M \Omega = 2 \int_M \omega_0 \wedge \omega_S. \tag{2.2}$$

Here and henceforth, we are abbreviating $\pi^* \omega_S$ by ω_S, and for any smooth positive volume form Ω we write $\text{Ric}(\Omega)$ for the globally defined real $(1, 1)$-form given locally by $-\sqrt{-1}\partial\bar{\partial} \log \Omega$.

It follows that the Bott–Chern class of the canonical bundle K_M, which equals $c_1^{BC}(K_M) = -c_1^{BC}(M)$, is *nef*. In general this means that given any $\varepsilon > 0$ there exists a real smooth function f_ε on M such that $-\text{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f_\varepsilon > -\varepsilon\omega_0$. Equivalently, this can be phrased by saying that for any $\varepsilon > 0$ there is a smooth Hermitian metric h_ε on the fibers of K_M with curvature form bigger than $-\varepsilon\omega_0$. The maximal existence theorem for the Chern–Ricci flow [36, Theorem 1.2] has the following immediate corollary.

Theorem 2.1 *Let (M, ω_0) be any compact Hermitian manifold. Then the Chern–Ricci flow*

$$\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{2.3}$$

has a smooth solution defined for all $t \geq 0$ if and only if the first Bott–Chern class $c_1^{BC}(K_M)$ is nef. The exact same statement holds for the normalized Chern–Ricci flow (1.2).

Since this theorem was not stated explicitly in [36], we provide the simple proof.

Proof An elementary space-time scaling argument [14] allows one to transform a solution of (1.2) into a solution of (2.3) and vice versa, and one exists for all positive time if and only if the other one does, so it is enough to consider (2.3).

In this case, we know from [36] that as long as a solution $\omega(t)$ exists, it is of the form

$$\omega(t) = \omega_0 - t\text{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\varphi(t),$$

and therefore

$$-\text{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\left(\frac{\varphi}{t}\right) > -\frac{1}{t}\omega_0,$$

so if the solution exists for all $t \geq 0$, then we see that $c_1^{BC}(K_M)$ is nef.

Conversely, if $c_1^{\mathrm{BC}}(K_M)$ is nef, then for every given $t > 0$ we can find a smooth function f_t with

$$-\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}f_t > -\frac{1}{t}\omega_0,$$

which is equivalent to

$$\omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}(tf_t) > 0,$$

and so the flow exists at least on $[0, t)$ by [36, Theorem 1.2]. $\square$

Applying this to the setting of Theorem 1.1, we obtain a smooth solution $\omega(t)$ to the normalized Chern–Ricci flow (1.2) for $t \in [0, \infty)$.

2.4 The parabolic complex Monge–Ampère equation

From now on, until we get to Sect. 9, we assume we are in the setting of Theorem 1.1. We will rewrite the normalized Chern–Ricci flow (1.2) as a parabolic complex Monge–Ampère equation. Define reference $(1, 1)$-forms $\tilde{\omega} = \tilde{\omega}(t)$ by

$$\tilde{\omega} = e^{-t}\omega_{\mathrm{flat}} + (1 - e^{-t})\omega_S,$$

where we recall that ω_{flat} is defined by (2.1). Note that $\tilde{\omega}$ may not necessarily be positive definite for all t, but there exists a time T_I such that $\tilde{\omega} > 0$ for all $t \geq T_I$. (On the other hand, observe that $\tilde{\omega} - e^{-t}\sqrt{-1}\partial\bar{\partial}\rho$ is positive definite for all $t \geq 0$). We fix this constant T_I now once and for all. By the long time existence result of [36], we have uniform C^∞ estimates on $\omega(t)$ for $t \in [0, T_I]$. Our goal is to obtain estimates on $\omega(t)$ for $t > T_I$ which are independent of t.

Define a function $\varphi(t)$ by

$$\frac{\partial}{\partial t}\varphi = \log\frac{e^t\omega(t)^2}{\Omega} - \varphi, \quad \varphi(0) = -\rho,$$

where we recall that $\omega_{\mathrm{flat}} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho$ and Ω is given by (2.2). We claim that $\omega(t) = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ holds. Indeed,

$$\frac{\partial}{\partial t}\tilde{\omega} = \omega_S - \tilde{\omega} = -\mathrm{Ric}(\Omega) - \tilde{\omega},$$

and so,

$$\frac{\partial}{\partial t}(e^t(\omega - \tilde{\omega} - \sqrt{-1}\partial\bar{\partial}\varphi)) = 0, \quad (e^t(\omega - \tilde{\omega} - \sqrt{-1}\partial\bar{\partial}\varphi))|_{t=0} = 0,$$

which implies that indeed $\omega = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$. Therefore φ also satisfies the PDE

$$\frac{\partial}{\partial t}\varphi = \log\frac{e^t(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi, \ \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \ \varphi(0) = -\rho, \quad (2.4)$$

and conversely every solution of (2.4) gives rise to a solution $\omega = \tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$ of the normalized Chern–Ricci flow (1.2).

3 Estimates on the potential and its time derivative

We now begin the proof of Theorem 1.1. We assume $\pi : M \to S$ is an elliptic bundle over a Riemann surface S of genus at least 2 and $\omega(t)$ is a solution of the normalized Chern–Ricci flow (1.2) on M starting at a Gauduchon metric ω_0.

In this section we collect some estimates on the potential function φ solving (2.4), and its time derivative $\dot{\varphi} := \partial\varphi/\partial t$. The proofs of these results are almost identical to the corresponding results for the Kähler–Ricci flow [24] (see also [28]). For the reader's convenience we include here the brief arguments. We also point out the one place where we use the Gauduchon condition.

Lemma 3.1 *There exists a uniform positive constant C such that on M,*

 (i) $|\varphi(t)| \leq C$ *for all $t \geq 0$.*
 (ii) $|\dot{\varphi}(t)| \leq C$ *for all $t \geq 0$.*
(iii) $\frac{1}{C}\tilde{\omega}^2 \leq \omega^2 \leq C\tilde{\omega}^2$ *for all $t \geq T_I$.*

Proof We follow the exposition in [28]. Since $e^t\tilde{\omega}^2 = e^{-t}\omega_{\text{flat}}^2 + 2(1 - e^{-t})\omega_{\text{flat}} \wedge \omega_S$, we have for $t \geq T_I$,

$$\frac{1}{C}\Omega \leq e^t\tilde{\omega}^2 \leq C\Omega. \quad (3.1)$$

If φ attains a minimum at a point (x_0, t_0) with $t_0 > T_I$, then at that point

$$0 \geq \frac{\partial}{\partial t}\varphi = \log\frac{e^t(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} - \varphi \geq \log\frac{e^t\tilde{\omega}^2}{\Omega} - \varphi \geq -\log C - \varphi,$$

giving $\varphi \geq -\log C$ and hence a uniform lower bound for φ. The upper bound for φ is similar. This gives (i).

For (ii), we first compute:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi} = \text{tr}_\omega(\omega_S - \tilde{\omega}) + 1 - \dot{\varphi}. \quad (3.2)$$

On the other hand, there exists a uniform constant $C_0 > 1$ such that $C_0\tilde{\omega} > \omega_S$ for $t \geq T_I$. We apply the maximum principle to $Q_1 = \dot{\varphi} - (C_0 - 1)\varphi$. Calculate for $t \geq T_I$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_1 = \operatorname{tr}_\omega(\omega_S - \tilde{\omega}) + 1 - C_0\dot{\varphi} + (C_0 - 1)\operatorname{tr}_\omega(\omega - \tilde{\omega})$$
$$\leq 1 - C_0\dot{\varphi} + 2(C_0 - 1),$$

and the maximum principle shows that Q_1 is bounded from above uniformly. This gives the upper bound for $\dot{\varphi}$.

Next consider $Q_2 = \dot{\varphi} + 2\varphi$ and compute

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_2 = \operatorname{tr}_\omega(\omega_S - \tilde{\omega}) + 1 + \dot{\varphi} - 2\operatorname{tr}_\omega(\omega - \tilde{\omega}) \geq \operatorname{tr}_\omega\tilde{\omega} + \dot{\varphi} - 3.$$

By the geometric–arithmetic means inequality, we have for $t \geq T_I$,

$$e^{-\frac{\dot{\varphi}+\varphi}{2}} = \left(\frac{\Omega}{e^t\omega^2}\right)^{\frac{1}{2}} \leq C\left(\frac{\tilde{\omega}^2}{\omega^2}\right)^{\frac{1}{2}} \leq \frac{C}{2}\operatorname{tr}_\omega\tilde{\omega}. \tag{3.3}$$

Then at a point (x_0, t_0) with $t_0 > T_I$ where Q_2 attains a minimum, $\operatorname{tr}_\omega\tilde{\omega} \leq 3 - \dot{\varphi}$ and so $e^{-\frac{\dot{\varphi}+\varphi}{2}} \leq C(3 - \dot{\varphi})$, which gives a uniform lower bound for $\dot{\varphi}$. This completes the proof of (ii).

Part (iii) follows from (i) and (ii) and the equations (2.4) and (3.1). □

Our next result is an exponential decay estimate for φ. We first need a lemma. Recall that the volume form Ω is defined by (2.2). This lemma is the only place in the paper where we make use of the Gauduchon assumption on ω_0.

Lemma 3.2 *We have that*
$$\Omega = 2\omega_{\text{flat}} \wedge \omega_S. \tag{3.4}$$

Proof Since $2\int_M \omega_{\text{flat}} \wedge \omega_S = 2\int_M \omega_0 \wedge \omega_S = \int_M \Omega$, it is enough to show that

$$\sqrt{-1}\partial\bar{\partial} \log \frac{\Omega}{\omega_{\text{flat}} \wedge \omega_S} = 0. \tag{3.5}$$

Recalling that M is an elliptic bundle with fiber E, we can fix a small ball $B \subset S$ over which π is holomorphically trivial, so $\pi^{-1}(B) \cong B \times E$. If we identify $E = \mathbb{C}/\Lambda$, for some lattice $\Lambda \subset \mathbb{C}$, and call z^1 the coordinate on $\mathbb{C}$, then dz^1 descends to a never vanishing holomorphic 1-form on E. If we call α its pullback to $B \times E$, then $\sqrt{-1}\alpha \wedge \bar{\alpha}$ is a smooth semi-flat form on $\pi^{-1}(U)$. Then there is a function $u(y)$ defined on B such that for any $y \in B$ we have

$$\omega_{\text{flat}}|_{E_y} = u(y)\sqrt{-1}\alpha \wedge \bar{\alpha}.$$

This is because both $\omega_{\text{flat}}|_{E_y}$ and $\sqrt{-1}\alpha \wedge \bar{\alpha}$ are flat volume forms on E_y, and so their ratio is a constant on E_y. Integrating this equality over E_y we get

$$\int_{E_y} \omega_{\text{flat}} = u(y)\int_{E_y} \sqrt{-1}\alpha \wedge \bar{\alpha}.$$

But on the one hand the integral $\int_{E_y} \sqrt{-1}\alpha \wedge \overline{\alpha}$ is independent of y by definition, and on the other hand the function $y \mapsto \int_{E_y} \omega_{\text{flat}}$ is also constant in y, because it equals the pushforward $\pi_* \omega_{\text{flat}}$ and we have

$$\partial \overline{\partial} \pi_* \omega_{\text{flat}} = \pi_* \partial \overline{\partial} \omega_{\text{flat}} = \pi_* \partial \overline{\partial} \omega_0 = 0.$$

Note that the last equality uses the Gauduchon condition. This implies that $\pi_* \omega_{\text{flat}}$ is constant by the strong maximum principle. Therefore u is a constant.

Fix now a point $x \in M$, call $y = \pi(x)$, and choose local bundle coordinates near x and y, so that in these coordinates the projection π is given by $\pi(z^1, z^2) = z^2$. Then write locally

$$\omega_S = \sqrt{-1} g(z^2) dz^2 \wedge d\overline{z}^2,$$
$$\Omega = G(z^1, z^2)(\sqrt{-1})^2 dz^1 \wedge d\overline{z}^1 \wedge dz^2 \wedge d\overline{z}^2,$$

and compute

$$F := \frac{\Omega}{\omega_{\text{flat}} \wedge \omega_S} = \frac{\Omega}{u\sqrt{-1}\alpha \wedge \overline{\alpha} \wedge \omega_S} = G(ug)^{-1}, \qquad (3.6)$$

and so locally on S we have

$$\sqrt{-1}\partial \overline{\partial} \log F = \omega_S + \text{Ric}(\omega_S) = 0, \qquad (3.7)$$

because $\text{Ric}(\Omega) = -\omega_S = \text{Ric}(\omega_S)$. This proves (3.5).

Incidentally the same calculation proves that the volume form $\omega_{\text{flat}} \wedge \omega_S$ satisfies

$$\text{Ric}(\omega_{\text{flat}} \wedge \omega_S) = -\omega_S,$$

which gives another proof of the fact that $c_1(M) = \pi^* c_1(S)$. $\qquad \square$

Very similar arguments can be found in the paper of Song–Tian [24], in the Kähler case (see also [25,34]).

Remark 3.3 Lemma 3.2 fails if we drop the assumption that $\partial \overline{\partial} \omega_0 = 0$. Indeed, consider the case when $M = S \times E$, and let ω_E be a flat Kähler metric on E (while ω_S is as before). If $F : S \to \mathbb{R}$ is any nonconstant positive function then

$$\omega_0 = F \omega_E + \omega_S,$$

is a Hermitian metric on M with $\partial \overline{\partial} \omega_0 \neq 0$. We have that $\omega_0 = \omega_{\text{flat}}$, because ω_0 is already semi-flat. On the other hand, $\Omega = c \cdot \omega_E \wedge \omega_S$, where c is the constant given by

$$c = \frac{2\int_M \omega_0 \wedge \omega_S}{\int_M \omega_E \wedge \omega_S}.$$

Therefore we have

$$\Omega \neq 2\omega_{\text{flat}} \wedge \omega_S = F\omega_E \wedge \omega_S.$$

We can now prove a decay estimate for φ. The analogous estimate was proved for the Kähler–Ricci flow on a product surface in [28, Lemma 6.7]. The proof in this case is almost identical, given Lemma 3.2.

Lemma 3.4 *There exists a uniform constant $C > 0$ such that on $M \times [0, \infty)$,*

$$|\varphi| \leq C(1+t)e^{-t}.$$

Proof First, we claim that for $t \geq T_I$,

$$\left| e^t \log \frac{e^t \tilde{\omega}^2}{\Omega} \right| \leq C. \tag{3.8}$$

This follows from the argument in [28, Lemma 6.7]. Indeed, using Lemma 3.2, we see that

$$e^t \log \frac{e^t \tilde{\omega}^2}{\Omega} = e^t \log \frac{2\omega_S \wedge \omega_{\text{flat}} + e^{-t}(\omega_{\text{flat}}^2 - 2\omega_{\text{flat}} \wedge \omega_S)}{2\omega_S \wedge \omega_{\text{flat}}}$$
$$= e^t \log(1 + O(e^{-t})),$$

which is bounded.

Define now $Q = e^t \varphi + At$, for A a large positive constant to be determined. Then

$$\frac{\partial Q}{\partial t} = e^t \log\left(\frac{e^t(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^2}{\Omega} \right) + A. \tag{3.9}$$

We wish to bound Q from below. Suppose that (x_0, t_0) is a point with $t_0 > T_I$ at which Q achieves a minimum. At this point we have

$$0 \geq \frac{\partial Q}{\partial t} \geq e^t \log \frac{e^t \tilde{\omega}^2}{\Omega} + A \geq -C' + A,$$

for a uniform C', thanks to (3.8). Choosing $A > C'$ gives contradiction. Hence Q is bounded from below and it follows that $\varphi \geq -C(1+t)e^{-t}$ for a uniform C. The upper bound for φ is similar. $\qquad\square$

4 Torsion and curvature of the reference metrics

This section is devoted to proving a technical lemma on estimates for the torsion and curvature of the reference metrics. These estimates will be needed later in Sects. 5 and 8.

Recall that the reference forms $\tilde{\omega} = \tilde{\omega}(t)$ are given by

$$\tilde{\omega} = e^{-t}\omega_{\text{flat}} + (1 - e^{-t})\omega_S.$$

For $t \geq T_I$, this defines a Hermitian metric which we denote by $\tilde{g}$. We will use a tilde to denote quantities with respect to $\tilde{g}$, such as $\tilde{T}_{ij}^k$ for the torsion tensor, $\tilde{\nabla}$ for the Chern connection and $\widetilde{\text{Rm}}$ for the Chern curvature tensor. We will write $\tilde{\partial}\tilde{T}$ for the tensor $\partial_{\bar{\ell}}\tilde{T}_{jk}^i = \tilde{\nabla}_{\bar{\ell}}\tilde{T}_{jk}^i$.

Denote by $(T^0)_{ij}^k$ the torsion tensor of the initial metric g_0, and $T_{ij\bar{\ell}}^0 = (T^0)_{ij}^k(g_0)_{k\bar{\ell}}$. Then since $d\tilde{\omega} = e^{-t}d\omega_0$, we have $\tilde{T}_{ij\bar{\ell}} = e^{-t}T_{ij\bar{\ell}}^0$.

Lemma 4.1 *There exists a uniform constant C such that for $t \geq T_I$,*

(i) $|\tilde{T}|_{\tilde{g}} \leq C$.
(ii) $|\tilde{\partial}\tilde{T}|_{\tilde{g}} + |\tilde{\nabla}\tilde{T}|_{\tilde{g}} + |\widetilde{\text{Rm}}|_{\tilde{g}} \leq Ce^{t/2}$.
(iii) $|\tilde{\nabla}\tilde{\partial}\tilde{T}|_g + |\tilde{\nabla}\tilde{\partial}\tilde{T}|_g \leq Ce^t$.

Proof We may choose local product holomorphic coordinates z^1, z^2, independent of t, with z^1 in the fiber direction and z^2 in the base direction. Since g_{flat} is flat in the z^1 direction, we may assume that derivatives of $(g_{\text{flat}})_{1\bar{1}}$ in the z^1 direction vanish. Now with respect to these coordinates, we may write

$$\tilde{g}_{1\bar{1}} = e^{-t}(g_{\text{flat}})_{1\bar{1}}, \quad \tilde{g}_{1\bar{2}} = e^{-t}(g_{\text{flat}})_{1\bar{2}} \tag{4.1}$$

$$\tilde{g}_{2\bar{1}} = e^{-t}(g_{\text{flat}})_{2\bar{1}}, \quad \tilde{g}_{2\bar{2}} = e^{-t}(g_{\text{flat}})_{2\bar{2}} + (1 - e^{-t})(g_S)_{2\bar{2}}, \tag{4.2}$$

where we are writing g_S for π^*g_S. Then a straightforward computation shows that there exists a uniform constant $C > 0$ so that

$$\frac{e^{-t}}{C} \leq \tilde{g}_{1\bar{1}} \leq Ce^{-t}, \quad \frac{e^t}{C} \leq \tilde{g}^{1\bar{1}} \leq Ce^t$$

$$\frac{1}{C} \leq \tilde{g}_{2\bar{2}} \leq C, \quad \frac{1}{C} \leq \tilde{g}^{2\bar{2}} \leq C$$

$$|\tilde{g}_{1\bar{2}}| \leq Ce^{-t} \quad |\tilde{g}^{1\bar{2}}| \leq C.$$

Then

$$|\tilde{T}|_{\tilde{g}}^2 = \tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{g}^{p\bar{q}}\tilde{T}_{ik\bar{q}}\overline{\tilde{T}_{j\ell\bar{p}}} = e^{-2t}\tilde{g}^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{g}^{p\bar{q}}T_{ik\bar{q}}^0\overline{T_{j\ell\bar{p}}^0}.$$

This is uniformly bounded since the only unbounded terms of type $\tilde{g}^{i\bar{j}}$ are the terms $\tilde{g}^{1\bar{1}}$, but by the skew-symmetry of $T_{ij\bar{\ell}}^0$ in i, j, there can be at most two such terms in

the above expression, and each is bounded above by Ce^t (since the components $T^0_{ij\bar{k}}$ are all uniformly bounded in the holomorphic coordinates z^1, z^2). This completes the proof of (i).

For (ii), note that we may choose the coordinates z^1, z^2 as above with the additional property that at a fixed point x say, the derivative $\partial_2 g_S$ vanishes. This implies that at x we have $\partial_i \tilde{g}_{j\bar{\ell}} = e^{-t} \partial_i (g_{\text{flat}})_{j\bar{\ell}}$ for all i, j, ℓ. Note that since our coordinates are independent of t and depend continuously on the point $x \in M$, we may allow our constants to depend on this choice of coordinate system.

We first claim that at x,

$$|\tilde{\Gamma}^p_{ik}|^2_{\tilde{g}} := \tilde{g}^{i\bar{j}} \tilde{g}^{k\bar{\ell}} \tilde{g}_{p\bar{q}} \tilde{\Gamma}^p_{ik} \overline{\tilde{\Gamma}^q_{j\ell}} \le Ce^t, \tag{4.3}$$

where $\tilde{\Gamma}^p_{ik}$ are the Christoffel symbols of the Chern connection of $\tilde{g}$. Note that since $\tilde{\Gamma}^p_{ik}$ is not a tensor, this quantity depends on our choice of coordinates.

For (4.3) compute,

$$|\tilde{\Gamma}^p_{ik}|^2_{\tilde{g}} = \tilde{g}^{i\bar{j}} \tilde{g}^{k\bar{\ell}} \tilde{g}^{p\bar{q}} \partial_i \tilde{g}_{k\bar{q}} \partial_{\bar{j}} \tilde{g}_{p\bar{\ell}} = e^{-2t} \tilde{g}^{i\bar{j}} \tilde{g}^{k\bar{\ell}} \tilde{g}^{p\bar{q}} \partial_i (g_{\text{flat}})_{k\bar{q}} \partial_{\bar{j}} (g_{\text{flat}})_{p\bar{\ell}}.$$

But this is bounded from above by Ce^t since each term of type $\tilde{g}^{i\bar{j}}$ is bounded from above by Ce^t.

Next note that, at x,

$$|\partial_{\bar{\ell}} T^0_{ij\bar{k}}|^2_{\tilde{g}} \le Ce^{3t}. \tag{4.4}$$

Indeed this follows from the skew symmetry of $\partial_{\bar{\ell}} T^0_{ij\bar{k}}$ (again, not a tensor) in the indices i and j. Then at x,

$$\begin{aligned} |\bar{\partial} \tilde{T}|^2_{\tilde{g}} &= e^{-2t} |\tilde{\nabla}_{\bar{\ell}} T^0_{ij\bar{k}}|^2_{\tilde{g}} \\ &= e^{-2t} |\partial_{\bar{\ell}} T^0_{ij\bar{k}} - \overline{\tilde{\Gamma}^r_{\ell k}} T^0_{ij\bar{r}}|^2_{\tilde{g}} \\ &\le 2e^{-2t} |\partial_{\bar{\ell}} T^0_{ij\bar{k}}|^2_{\tilde{g}} + 2e^{-2t} |\tilde{\Gamma}^r_{\ell k}|^2_{\tilde{g}} |T^0_{ij\bar{r}}|^2_{\tilde{g}} \le Ce^t \end{aligned} \tag{4.5}$$

where the last inequality follows from (4.3), (4.4) and the fact that $|T^0_{ij\bar{r}}|^2_{\tilde{g}} \le Ce^{2t}$.

The bound on $|\tilde{\nabla} \tilde{T}|_{\tilde{g}}$ is completely analogous (again we compute at x):

$$\begin{aligned} |\tilde{\nabla} \tilde{T}|^2_{\tilde{g}} &= e^{-2t} |\tilde{\nabla}_\ell T^0_{ij\bar{k}}|^2_{\tilde{g}} \\ &= e^{-2t} |\partial_\ell T^0_{ij\bar{k}} - \tilde{\Gamma}^r_{\ell i} T^0_{rj\bar{k}} - \tilde{\Gamma}^r_{\ell j} T^0_{ir\bar{k}}|^2_{\tilde{g}} \\ &\le 2e^{-2t} |\partial_\ell T^0_{ij\bar{k}}|^2_{\tilde{g}} + 4e^{-2t} |\tilde{\Gamma}^r_{\ell k}|^2_{\tilde{g}} |T^0_{ij\bar{k}}|^2_{\tilde{g}} \le Ce^t. \end{aligned} \tag{4.6}$$

For the bound on the curvature $\tilde{R}_{i\bar{j}k\bar{\ell}}$ of $\tilde{g}$, we first compute in our coordinates,

$$|\tilde{R}_{1\bar{1}1\bar{1}}| \le Ce^{-2t}, \tag{4.7}$$

where by $|\cdot|$ we mean the absolute value as a complex (or real) number. Recall that the Chern curvature of $\tilde{g}$ is given by

$$\tilde{R}_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}\tilde{g}_{k\bar{\ell}} + \tilde{g}^{p\bar{q}}\partial_i\tilde{g}_{k\bar{q}}\partial_{\bar{j}}\tilde{g}_{p\bar{\ell}}.$$

Hence from (4.1) and (4.2) and the fact that $\partial_1\partial_{\bar{1}}(g_{\text{flat}})_{1\bar{1}}$ and $\partial_1(g_{\text{flat}})_{1\bar{1}}$ vanish in our coordinate system,

$$\tilde{R}_{1\bar{1}1\bar{1}} = -\partial_1\partial_{\bar{1}}\tilde{g}_{1\bar{1}} + \tilde{g}^{p\bar{q}}\partial_1\tilde{g}_{1\bar{q}}\partial_{\bar{1}}\tilde{g}_{p\bar{1}}$$
$$= e^{-2t}\sum_{(p,q)\neq(1,1)}\tilde{g}^{p\bar{q}}\partial_1(g_{\text{flat}})_{1\bar{q}}\partial_{\bar{1}}(g_{\text{flat}})_{p\bar{1}}.$$

but since each term $\tilde{g}^{p\bar{q}}$ for $(p, q) \neq (1, 1)$ is uniformly bounded, this gives (4.7).

Next we show that

$$|\tilde{R}_{2\bar{2}2\bar{2}}| \leq C. \tag{4.8}$$

For this note that $|\partial_2\partial_{\bar{2}}\tilde{g}_{2\bar{2}}| \leq C$ and

$$\sum_{p,q}\tilde{g}^{p\bar{q}}\partial_2\tilde{g}_{2\bar{q}}\partial_{\bar{2}}\tilde{g}_{p\bar{2}} = \tilde{g}^{1\bar{1}}\partial_2\tilde{g}_{2\bar{1}}\partial_{\bar{2}}\tilde{g}_{1\bar{2}} + \sum_{(p,q)\neq(1,1)}\tilde{g}^{p\bar{q}}\partial_2\tilde{g}_{2\bar{q}}\partial_{\bar{2}}\tilde{g}_{p\bar{2}},$$

but the first term is of order $O(e^{-t})$ and the second is uniformly bounded. This proves (4.8).

Finally, we show that

$$|\tilde{R}_{i\bar{j}k\bar{\ell}}| \leq Ce^{-t}, \quad \text{for } (i, j, k, \ell) \text{ not all equal.} \tag{4.9}$$

To see this observe that, for i, j, k, ℓ not all equal, we have

$$|\partial_i\partial_{\bar{j}}\tilde{g}_{k\bar{\ell}}| \leq Ce^{-t}. \tag{4.10}$$

Indeed, this follows immediately from (4.1) and (4.2) unless $k = \ell = 2$. But then one of i or j must equal 1 and we use the fact that $\partial_1(g_S)_{2\bar{2}} = 0$.

Moreover, for i, j, k, ℓ not all equal, we claim:

$$\left|\sum_{p,q}\tilde{g}^{p\bar{q}}\partial_i\tilde{g}_{k\bar{q}}\partial_{\bar{j}}\tilde{g}_{p\bar{\ell}}\right| \leq Ce^{-t}. \tag{4.11}$$

Indeed, first assume that neither k nor ℓ is equal to 2. Then

$$|\tilde{g}^{p\bar{q}}| \leq Ce^t, \quad |\partial_i\tilde{g}_{k\bar{q}}| \leq Ce^{-t} \quad \text{and} \quad |\partial_{\bar{j}}\tilde{g}_{p\bar{\ell}}| \leq Ce^{-t},$$

and so

$$|\tilde{g}^{p\bar{q}}\partial_i\tilde{g}_{k\bar{q}}\partial_{\bar{j}}\tilde{g}_{p\bar{\ell}}| \leq Ce^{-t}.$$

Next suppose that $k = 2$ and $\ell = 1$. Then

$$\sum_{p,q} \tilde{g}^{p\bar{q}} \partial_i \tilde{g}_{2\bar{q}} \partial_{\bar{j}} \tilde{g}_{p\bar{1}} = \sum_p \tilde{g}^{p\bar{1}} \partial_i \tilde{g}_{2\bar{1}} \partial_{\bar{j}} \tilde{g}_{p\bar{1}} + \sum_p \tilde{g}^{p\bar{2}} \partial_i \tilde{g}_{2\bar{2}} \partial_{\bar{j}} \tilde{g}_{p\bar{1}}.$$

The first term on the right hand side is of order $O(e^{-t})$ by the same argument as above, and the second is of order $O(e^{-t})$ since $|\tilde{g}^{p\bar{2}}| \leq C$, $|\partial_i \tilde{g}_{2\bar{2}}| \leq C$ and $|\partial_{\bar{j}} \tilde{g}_{p\bar{1}}| \leq Ce^{-t}$. This proves (4.11) if $k = 2$ and $\ell = 1$ and the case $k = 1$, $\ell = 2$ is similar.

Finally we deal with the case $k = \ell = 2$. We wish to bound

$$\sum_{p,q} \tilde{g}^{p\bar{q}} \partial_i \tilde{g}_{2\bar{q}} \partial_{\bar{j}} \tilde{g}_{p\bar{2}}.$$

If one of p or q is equal to 1 then the summand is of order $O(e^{-t})$ by similar arguments to the ones given above. Otherwise the summand is

$$\tilde{g}^{2\bar{2}} \partial_i \tilde{g}_{2\bar{2}} \partial_{\bar{j}} \tilde{g}_{2\bar{2}},$$

and then we use that fact that one of i or j must be 1, since we are assuming that i, j, k, ℓ are not all equal. But $|\partial_1 \tilde{g}_{2\bar{2}}| \leq Ce^{-t}$ and so the summand is of order $O(e^{-t})$. This completes the proof of (4.11).

Combining (4.10) and (4.11) gives (4.9).

To complete the proof of (ii), we note that

$$|\widetilde{\mathrm{Rm}}|_{\tilde{g}}^2 = \tilde{g}^{i\bar{q}} \tilde{g}^{p\bar{j}} \tilde{g}^{k\bar{s}} \tilde{g}^{r\bar{\ell}} \tilde{R}_{i\bar{j}k\bar{\ell}} \overline{\tilde{R}_{q\bar{p}s\bar{r}}}.$$

Recall that $\tilde{g}^{a\bar{b}}$ is bounded by C if $(a, b) \neq (1, 1)$ and by Ce^t if $a = b = 1$. When $i = j = k = \ell = p = q = r = s = 1$, then we apply (4.7) to see that the summand is bounded by C. We get the same bound if $i = j = k = \ell = 2$ or $p = q = r = s = 2$ by applying (4.7), (4.8) and (4.9). Otherwise, both of the terms $\tilde{R}_{i\bar{j}k\bar{\ell}}$ and $\tilde{R}_{q\bar{p}s\bar{r}}$ are bounded by Ce^{-t} (or better) by (4.7) and (4.9) (because the term $\tilde{R}_{2\bar{2}2\bar{2}}$ does not appear). Moreover, at least one of the metric terms $\tilde{g}^{-1}$ is bounded by a uniform constant C, while the other three are each bounded from above by Ce^t. Thus in every case we obtain $|\widetilde{\mathrm{Rm}}|_{\tilde{g}}^2 \leq Ce^t$. Combining this with (4.5) and (4.6) gives (ii).

For (iii), compute

$$\tilde{\nabla}_{\bar{p}} \tilde{\nabla}_{\bar{q}} \tilde{T}_{ij\bar{k}} = \tilde{\nabla}_{\bar{p}} (\partial_{\bar{q}} \tilde{T}_{ij\bar{k}} - \overline{\tilde{\Gamma}_{qk}^{\ell}} \tilde{T}_{ij\bar{\ell}})$$

$$= \partial_{\bar{p}} \partial_{\bar{q}} \tilde{T}_{ij\bar{k}} - \tilde{T}_{ij\bar{\ell}} \partial_{\bar{p}} \overline{\tilde{\Gamma}_{qk}^{\ell}} - \overline{\tilde{\Gamma}_{qk}^{\ell}} \partial_{\bar{p}} \tilde{T}_{ij\bar{\ell}} - \overline{\tilde{\Gamma}_{pq}^{\ell}} \partial_{\bar{\ell}} \tilde{T}_{ij\bar{k}} - \overline{\tilde{\Gamma}_{pk}^{\ell}} \partial_{\bar{q}} \tilde{T}_{ij\bar{\ell}}$$

$$+ \overline{\tilde{\Gamma}_{pq}^{r} \tilde{\Gamma}_{rk}^{\ell}} \tilde{T}_{ij\bar{\ell}} + \overline{\tilde{\Gamma}_{pk}^{r} \tilde{\Gamma}_{qr}^{\ell}} \tilde{T}_{ij\bar{\ell}}.$$

For the first term we observe that

$$|\partial_{\overline{p}}\partial_{\overline{q}}\tilde{T}_{ij\overline{k}}|^2_{\tilde{g}} = e^{-2t}|\partial_{\overline{p}}\partial_{\overline{q}}T^0_{ij\overline{k}}|^2_{\tilde{g}} \leq Ce^{2t},$$ (4.12)

because of the skew symmetry in i, j. And, as in the proof of (ii),

$$|\partial_{\overline{p}}\tilde{T}_{ij\overline{\ell}}|^2_{\tilde{g}} = e^{-2t}|\partial_{\overline{p}}T^0_{ij\overline{k}}|^2_{\tilde{g}} \leq Ce^t.$$ (4.13)

We claim that

$$|\partial_p\tilde{\Gamma}^i_{jk}|^2_{\tilde{g}} := \tilde{g}^{p\overline{q}}\tilde{g}^{j\overline{a}}\tilde{g}^{k\overline{b}}\tilde{g}_{i\overline{c}}\partial_p\tilde{\Gamma}^i_{jk}\overline{\partial_q\tilde{\Gamma}^c_{ab}} \leq Ce^{2t}.$$ (4.14)

To see this note that at x we have

$$\partial_p\tilde{\Gamma}^i_{jk} = \tilde{g}^{i\overline{\ell}}\partial_p\partial_j\tilde{g}_{k\overline{\ell}} - \tilde{g}^{i\overline{s}}\tilde{g}^{r\overline{\ell}}\partial_p\tilde{g}_{r\overline{s}}\partial_j\tilde{g}_{k\overline{\ell}}$$

$$= e^{-t}\tilde{g}^{i\overline{\ell}}\partial_p\partial_j(g_{\text{flat}})_{k\overline{\ell}} + (1 - e^{-t})\tilde{g}^{i\overline{\ell}}\partial_p\partial_j(g_S)_{k\overline{\ell}}$$

$$- e^{-2t}\tilde{g}^{i\overline{s}}\tilde{g}^{r\overline{\ell}}\partial_p(g_{\text{flat}})_{r\overline{s}}\partial_j(g_{\text{flat}})_{k\overline{\ell}},$$

and so (with the obvious notation)

$$|\partial_p\tilde{\Gamma}^i_{jk}|^2_{\tilde{g}} \leq C(e^{-2t}|\partial_p\partial_j(g_{\text{flat}})_{k\overline{\ell}}|^2_{\tilde{g}} + |\partial_p\partial_j(g_S)_{k\overline{\ell}}|^2_{\tilde{g}} + e^{-4t}|\partial_p(g_{\text{flat}})_{r\overline{s}}\partial_j(g_{\text{flat}})_{k\overline{\ell}}|^2_{\tilde{g}}).$$

But the second term equals $C|\partial_2\partial_2(g_S)_{2\overline{2}}|^2|\tilde{g}^{2\overline{2}}|^4$, and so is bounded by C, while the other two terms are bounded by Ce^{2t} because each term of type $\tilde{g}^{i\overline{j}}$ is bounded above by Ce^t. This establishes the claim (4.14).

Combining (4.3), (4.12), (4.13), (4.14) and parts (i) and (ii) we have proved the bound $|\overline{\nabla}\overline{\partial}\tilde{T}|_g \leq Ce^t$.

Finally, calculate

$$\tilde{\nabla}_p\tilde{\nabla}_{\overline{q}}\tilde{T}_{ij\overline{k}} = \tilde{\nabla}_p(\partial_{\overline{q}}\tilde{T}_{ij\overline{k}} - \overline{\tilde{\Gamma}^\ell_{qk}}\tilde{T}_{ij\overline{\ell}})$$

$$= \partial_p\partial_{\overline{q}}\tilde{T}_{ij\overline{k}} - \tilde{T}_{ij\overline{\ell}}\partial_p\overline{\tilde{\Gamma}^\ell_{qk}} - \overline{\tilde{\Gamma}^\ell_{qk}}\partial_p\tilde{T}_{ij\overline{\ell}} - \tilde{\Gamma}^\ell_{pi}\partial_{\overline{q}}\tilde{T}_{\ell j\overline{k}} - \tilde{\Gamma}^\ell_{pj}\partial_{\overline{q}}\tilde{T}_{i\ell\overline{k}}$$

$$+ \tilde{\Gamma}^r_{pi}\overline{\tilde{\Gamma}^\ell_{qk}}\tilde{T}_{rj\overline{\ell}} + \tilde{\Gamma}^r_{pj}\overline{\tilde{\Gamma}^\ell_{qk}}\tilde{T}_{ir\overline{\ell}}.$$

Arguing as above we can bound the $|\cdot|_{\tilde{g}}$ of all these terms by Ce^t. Indeed, the only terms which are different are the one involving $\partial_p\overline{\tilde{\Gamma}^\ell_{qk}} = -\overline{\tilde{R}_{q\overline{p}k}{}^\ell}$, which can be bounded using part (ii) and the one involving $\partial_p\tilde{T}_{ij\overline{\ell}}$ which can be bounded by the same argument as in (4.13). This finishes the proof of (iii). □

5 Evolution of the trace of the metric

Let $\omega = \omega(t)$ solve the normalized Chern–Ricci flow (1.2) in the setting of Theorem 1.1. The main theorem we prove in this section is:

Theorem 5.1 *There exists a uniform constant $C > 0$ such that for $t \geq T_I$,*

$$\mathrm{tr}_{\tilde{\omega}}\omega \leq C.$$

Hence the metrics ω and $\tilde{\omega}$ are uniformly equivalent for $t \geq T_I$.

For the last assertion, note that we have

$$\mathrm{tr}_{\omega}\tilde{\omega} = \frac{\tilde{\omega}^2}{\omega^2}\mathrm{tr}_{\tilde{\omega}}\omega, \tag{5.1}$$

and the uniform equivalence of the volume forms ω^2 and $\tilde{\omega}^2$ (Lemma 3.1). Then an upper bound for $\mathrm{tr}_{\tilde{\omega}}\omega$ is equivalent to an upper bound for $\mathrm{tr}_{\omega}\tilde{\omega}$ and hence also the uniform equivalence of ω and $\tilde{\omega}$.

In [28] a similar estimate is proved using a direct maximum principle argument and a bound for the potential φ. Here, as discussed in the Introduction, there are new unbounded terms arising from the torsion. We will control these terms using the exponential decay estimate for φ (Lemma 3.4).

First we need the following lemma.

Lemma 5.2 *For $t \geq T_I$, the following evolution inequality holds:*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\mathrm{tr}_{\tilde{\omega}}\omega \leq \frac{2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}\mathrm{Re}(\tilde{g}^{i\bar{\ell}}g^{k\bar{q}}\tilde{T}_{ki\bar{\ell}}\partial_{\bar{q}}\mathrm{tr}_{\tilde{\omega}}\omega) + Ce^{t/2}\mathrm{tr}_{\omega}\tilde{\omega}. \tag{5.2}$$

Proof From [36, Proposition 3.1] we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\log\mathrm{tr}_{\tilde{\omega}}\omega = \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}\Bigg(&-g^{p\bar{j}}g^{i\bar{q}}\tilde{g}^{k\bar{\ell}}\tilde{\nabla}_k g_{i\bar{j}}\tilde{\nabla}_{\bar{\ell}}g_{p\bar{q}} + \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}g^{k\bar{\ell}}\partial_k\mathrm{tr}_{\tilde{\omega}}\omega\partial_{\bar{\ell}}\mathrm{tr}_{\tilde{\omega}}\omega \\
&-2\mathrm{Re}(g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ki}\tilde{\nabla}_{\bar{\ell}}g_{p\bar{j}}) - g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{j\ell}}g_{p\bar{q}} \\
&+ g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}(\tilde{\nabla}_i\overline{\tilde{T}^q_{j\ell}} - \tilde{R}_{i\bar{\ell}p\bar{j}}\tilde{g}^{p\bar{q}})g_{k\bar{q}} - g^{i\bar{j}}\tilde{\nabla}_i\overline{\tilde{T}^\ell_{j\ell}} \\
&- g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}g_{p\bar{j}}\tilde{\nabla}_{\bar{\ell}}\tilde{T}^p_{ik} + g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{j\ell}}g_{p\bar{q}} - \mathrm{tr}_{\tilde{\omega}}\omega \\
&-\tilde{g}^{i\bar{\ell}}\tilde{g}^{k\bar{j}}g_{i\bar{j}}\frac{\partial}{\partial t}\tilde{g}_{k\bar{\ell}}\Bigg).
\end{aligned}$$

Note that there are some differences from the computation in [36] since here we are evolving ω by the *normalized* Chern–Ricci flow, and our reference metrics $\tilde{\omega}$ depend on time. In particular here we have $T_{ij\bar{k}} = \tilde{T}_{ij\bar{k}}$ (instead of $T_{ij\bar{k}} = (T_0)_{ij\bar{k}}$ in [36]). Also, the last two terms above are new: the first arising from the $-\omega$ term on the right hand side of (1.2) and the second from the time derivative of $\tilde{\omega}$. Fortunately, the contribution

of these two terms has a good sign. Indeed, observe that $\frac{\partial}{\partial t}\tilde{g} = g_S - \tilde{g} \geq -\tilde{g}$ and hence

$$-\mathrm{tr}_{\tilde{\omega}}\omega - \tilde{g}^{i\bar{\ell}}\tilde{g}^{k\bar{j}}g_{i\bar{j}}\frac{\partial}{\partial t}\tilde{g}_{k\bar{\ell}} \leq 0.$$

Again from Proposition 3.1 in [36], we have

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}\bigg(-g^{p\bar{j}}g^{i\bar{q}}\tilde{g}^{k\bar{\ell}}\tilde{\nabla}_k g_{i\bar{j}}\tilde{\nabla}_{\bar{\ell}}g_{p\bar{q}} + \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}g^{k\bar{\ell}}\partial_k\mathrm{tr}_{\tilde{\omega}}\omega\partial_{\bar{\ell}}\mathrm{tr}_{\tilde{\omega}}\omega$$
$$- 2\mathrm{Re}(g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ki}\tilde{\nabla}_{\bar{\ell}}g_{p\bar{j}}) - g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{j\ell}}g_{p\bar{q}}\bigg) \leq \frac{2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}\mathrm{Re}(\tilde{g}^{i\bar{\ell}}g^{k\bar{q}}\tilde{T}_{ki\bar{\ell}}\partial_{\bar{q}}\mathrm{tr}_{\tilde{\omega}}\omega).$$

Hence to complete the proof of the lemma it remains to show that for $t \geq T_I$ we have

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}(g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}(\tilde{\nabla}_i\overline{\tilde{T}^q_{j\ell}} - \tilde{R}_{i\bar{\ell}p\bar{j}}\tilde{g}^{p\bar{q}})g_{k\bar{q}} - g^{i\bar{j}}\tilde{\nabla}_i\overline{\tilde{T}^\ell_{j\ell}} - g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{g}_{p\bar{j}}\tilde{\nabla}_{\bar{\ell}}\tilde{T}^p_{ik}$$
$$+ g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{j\ell}}\tilde{g}_{p\bar{q}}) \leq C(\mathrm{tr}_{\omega}\tilde{\omega})e^{t/2}.$$

But this follows easily from Lemma 4.1, the fact that the quantities $\mathrm{tr}_{\omega}\tilde{\omega}$ and $\mathrm{tr}_{\tilde{\omega}}\omega$ are uniformly equivalent and the inequality $\mathrm{tr}_{\tilde{\omega}}\omega \geq C^{-1} > 0$ for a uniform constant C (the geometric–arithmetic means inequality). Indeed,

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{\nabla}_i\overline{\tilde{T}^q_{j\ell}}g_{k\bar{q}}| \leq \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{-1}|_{\tilde{g}}|\tilde{g}^{-1}|_{\tilde{g}}|\overline{\tilde{\nabla}\tilde{T}}|_{\tilde{g}}|g|_{\tilde{g}} \leq C(\mathrm{tr}_{\omega}\tilde{\omega})e^{t/2},$$

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{g}^{p\bar{q}}g_{k\bar{q}}\tilde{R}_{i\bar{\ell}p\bar{j}}| \leq \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{-1}|_{\tilde{g}}|\tilde{g}^{-1}|^2_{\tilde{g}}|g|_{\tilde{g}}|\widetilde{\mathrm{Rm}}|_{\tilde{g}} \leq C(\mathrm{tr}_{\omega}\tilde{\omega})e^{t/2},$$

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{i\bar{j}}\tilde{\nabla}_i\overline{\tilde{T}^\ell_{j\ell}}| \leq \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{-1}|_{\tilde{g}}|\overline{\tilde{\nabla}\tilde{T}}|_{\tilde{g}} \leq Ce^{t/2},$$

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{g}_{p\bar{j}}\tilde{\nabla}_{\bar{\ell}}\tilde{T}^p_{ik}| \leq \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{-1}|_{\tilde{g}}|\tilde{g}^{-1}|_{\tilde{g}}|\tilde{g}|_{\tilde{g}}|\overline{\tilde{\nabla}}\tilde{T}|_{\tilde{g}} \leq Ce^{t/2},$$

$$\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}^p_{ik}\overline{\tilde{T}^q_{j\ell}}\tilde{g}_{p\bar{q}}| \leq \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}|g^{-1}|_{\tilde{g}}|\tilde{g}^{-1}|_{\tilde{g}}|\tilde{g}|_{\tilde{g}}|\tilde{T}|^2_{\tilde{g}} \leq C,$$

as required. $\qquad\square$

We can now prove Theorem 5.1, making use of the decay estimate of φ (Lemma 3.4) and the bound on $\dot{\varphi}$ (Lemma 3.1).

Proof of Theorem 5.1 We use the fact that $e^{t/2}\varphi$ is uniformly bounded, and consider the quantity

$$Q = \log\mathrm{tr}_{\tilde{\omega}}\omega - Ae^{t/2}\varphi + \frac{1}{\tilde{C} + e^{t/2}\varphi},$$

where $\tilde{C}$ is a uniform constant chosen so that $\tilde{C} + e^{t/2}\varphi \geq 1$, and A is a large constant to be determined later. The idea of adding an extra term, of the form of a reciprocal of a potential function, comes from Phong–Sturm [20] and was used in the context of the Chern–Ricci flow in [36]. Notice that

$$0 \leq \frac{1}{\tilde{C} + e^{t/2}\varphi} \leq 1.$$

We will show that at a point (x_0, t_0) with $t_0 > T_I$ at which Q achieves a maximum, we have a uniform upper bound of $\mathrm{tr}_{\tilde{\omega}}\omega$, and the theorem will follow thanks to Lemma 3.4.

First compute, using the fact that $\Delta\varphi = 2 - \mathrm{tr}_{\omega}\tilde{\omega}$ and the bounds for φ and $\dot{\varphi}$ from Lemma 3.1,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(-Ae^{t/2}\varphi + \frac{1}{\tilde{C} + e^{t/2}\varphi}\right)$$
$$= -\left(A + \frac{1}{(\tilde{C} + e^{t/2}\varphi)^2}\right)\left(e^{t/2}\dot{\varphi} + \frac{1}{2}e^{t/2}\varphi\right)$$
$$+ \left(A + \frac{1}{(\tilde{C} + e^{t/2}\varphi)^2}\right)\Delta(e^{t/2}\varphi) - \frac{2|\partial(e^{t/2}\varphi)|_g^2}{(\tilde{C} + e^{t/2}\varphi)^3}$$
$$\leq CAe^{t/2} - Ae^{t/2}\mathrm{tr}_{\omega}\tilde{\omega} - \frac{2|\partial(e^{t/2}\varphi)|_g^2}{(\tilde{C} + e^{t/2}\varphi)^3}. \tag{5.3}$$

At the point (x_0, t_0), we have $\partial_{\overline{q}}Q = 0$, which implies that

$$\frac{\partial_{\overline{q}}\mathrm{tr}_{\tilde{\omega}}\omega}{\mathrm{tr}_{\tilde{\omega}}\omega} = \left(A + \frac{1}{(\tilde{C} + e^{t/2}\varphi)^2}\right)e^{t/2}\partial_{\overline{q}}\varphi.$$

Then at this point,

$$\frac{2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}\mathrm{Re}(\tilde{g}^{i\overline{\ell}}g^{k\overline{q}}\tilde{T}_{ki\overline{\ell}}\partial_{\overline{q}}\mathrm{tr}_{\tilde{\omega}}\omega)$$
$$= \frac{2}{\mathrm{tr}_{\tilde{\omega}}\omega}\mathrm{Re}\left(\tilde{g}^{i\overline{\ell}}g^{k\overline{q}}\tilde{T}_{ki\overline{\ell}}\left(A + \frac{1}{(\tilde{C} + e^{t/2}\varphi)^2}\right)e^{t/2}\partial_{\overline{q}}\varphi\right)$$
$$\leq \frac{CA^2}{(\mathrm{tr}_{\tilde{\omega}}\omega)^2}(\tilde{C} + e^{t/2}\varphi)^3 g^{k\overline{q}}\tilde{g}^{i\overline{\ell}}\tilde{T}_{ki\overline{\ell}}\tilde{g}^{m\overline{j}}\overline{\tilde{T}_{qj\overline{m}}} + \frac{|\partial(e^{t/2}\varphi)|_g^2}{(\tilde{C} + e^{t/2}\varphi)^3}$$
$$\leq \frac{CA^2}{\mathrm{tr}_{\tilde{\omega}}\omega} + \frac{|\partial(e^{t/2}\varphi)|_g^2}{(\tilde{C} + e^{t/2}\varphi)^3}, \tag{5.4}$$

where for the last step we have used Lemma 3.4, part (i) of Lemma 4.1, and the fact that $\mathrm{tr}_{\tilde{\omega}}\omega$ and $\mathrm{tr}_{\omega}\tilde{\omega}$ are uniformly equivalent.

Combining (5.2), (5.3) and (5.4), we have, at a point at which Q achieves a maximum, for a uniform $C > 0$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq CA^2 + Ce^{t/2}\mathrm{tr}_\omega\tilde{\omega} + CAe^{t/2} - Ae^{t/2}\mathrm{tr}_\omega\tilde{\omega}$$

where we are assuming, without loss of generality, that at this maximum point of Q we have $\mathrm{tr}_{\tilde{\omega}}\omega \geq 1$. Choose a uniform A large enough so that

$$A \geq C + 1.$$

Then we obtain at the maximum of Q,

$$e^{t/2}\mathrm{tr}_\omega\tilde{\omega} \leq CA^2 + CAe^{t/2},$$

which implies that $\mathrm{tr}_\omega\tilde{\omega}$ and hence $\mathrm{tr}_{\tilde{\omega}}\omega$ is uniformly bounded from above at the maximum of Q. This establishes the estimate $\mathrm{tr}_{\tilde{\omega}}\omega \leq C$ and completes the proof of the theorem. □

6 A bound on the Chern scalar curvature

In this section, we establish the following estimate for the Chern scalar curvature.

Theorem 6.1 *There exists a uniform constant C such that along the normalized Chern–Ricci flow (1.2) we have*

$$-C \leq R \leq Ce^{t/2},$$

for all $t \geq 0$.

First note that the lower bound for the Chern scalar curvature follows from the same argument as in the Kähler–Ricci flow (see for example Theorem 2.2 in [28]). Indeed, from (1.2), we have

$$g^{k\bar{\ell}}\frac{\partial}{\partial t}g_{k\bar{\ell}} = -R - 2.$$

But $R = -g^{i\bar{j}}\partial_i\partial_{\bar{j}}\log\det g$ and hence

$$\begin{aligned}
\frac{\partial R}{\partial t} &= -g^{i\bar{j}}\partial_i\partial_{\bar{j}}\left(g^{k\bar{\ell}}\frac{\partial}{\partial t}g_{k\bar{\ell}}\right) - \left(\frac{\partial}{\partial t}g^{i\bar{j}}\right)\partial_i\partial_{\bar{j}}\log\det g \\
&= \Delta R + |\mathrm{Ric}|^2 + R \\
&\geq \Delta R + \frac{1}{2}R^2 + R
\end{aligned}$$

and then the lower bound for R follows.

We now establish the upper bound of the Chern scalar curvature. Before we start the main argument, we need a few preliminary calculations.

Lemma 6.2 *There exists a uniform constant $C > 0$ such that for $t \geq T_I$, we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\tilde{\omega}}\omega \leq -C^{-1}|\tilde{\nabla}g|_g^2 + Ce^{t/2}. \tag{6.1}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\omega}\omega_S \leq |\tilde{\nabla}g|_g^2 - C^{-1}|\nabla\mathrm{tr}_{\omega}\omega_S|_g^2 + Ce^{t/2}. \tag{6.2}$$

As a consequence, there are uniform positive constants C_0, C_1 such that for $t \geq T_I$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\mathrm{tr}_{\omega}\omega_S + C_0\mathrm{tr}_{\tilde{\omega}}\omega) \leq -|\tilde{\nabla}g|_g^2 - C_1^{-1}|\nabla\mathrm{tr}_{\omega}\omega_S|_g^2 + C_1e^{t/2}. \tag{6.3}$$

Proof For (6.1), we compute the evolution of $\mathrm{tr}_{\tilde{\omega}}\omega$. As in Lemma 5.2 we may modify [36, Proposition 3.1] to obtain, for $t \geq T_I$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\tilde{\omega}}\omega = -g^{p\bar{j}}g^{i\bar{q}}\tilde{g}^{k\bar{\ell}}\tilde{\nabla}_k g_{i\bar{j}}\tilde{\nabla}_{\bar{\ell}}g_{p\bar{q}} - 2\mathrm{Re}(g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}_{ki}^p\tilde{\nabla}_{\bar{\ell}}g_{p\bar{j}})$$

$$- g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}_{ik}^p\overline{\tilde{T}_{j\ell}^q}g_{p\bar{q}} + g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}(\tilde{\nabla}_i\overline{\tilde{T}_{j\ell}^q} - \tilde{R}_{i\bar{\ell}p\bar{j}}\tilde{g}^{p\bar{q}})g_{k\bar{q}}$$

$$- g^{i\bar{j}}\tilde{\nabla}_i\overline{\tilde{T}_{j\ell}^\ell} - g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}g_{p\bar{j}}\tilde{\nabla}_{\bar{\ell}}\tilde{T}_{ik}^p + g^{i\bar{j}}\tilde{g}^{k\bar{\ell}}\tilde{T}_{ik}^p\overline{\tilde{T}_{j\ell}^q}g_{p\bar{q}} - \mathrm{tr}_{\tilde{\omega}}\omega$$

$$- \tilde{g}^{i\bar{\ell}}\tilde{g}^{k\bar{j}}g_{i\bar{j}}((g_S)_{k\bar{\ell}} - \tilde{g}_{k\bar{\ell}}).$$

Using Theorem 5.1, Lemma 4.1, and the Cauchy–Schwarz inequality in the second term, we obtain (6.1).

The inequality (6.2) is a parabolic Schwarz Lemma calculation for the map π : $M \to S$ [24,40]. Note that we already know that $\mathrm{tr}_{\omega}\omega_S \leq C$ since the metrics ω and $\tilde{\omega}$ are uniformly equivalent.

The computation for (6.2) is similar to that of Song–Tian [24], except that of course here we need to control the extra torsion terms. Given any point $x \in M$ we choose local coordinates $\{z^i\}$ on M centered at x such that g is the identity at x, and a coordinate w on S near $\pi(x) \in N$, which we can assume is normal for the metric g_S. In these coordinates we can represent the map π as a local holomorphic function f. We will use subscripts like f_i, f_{ij}, ... to indicate covariant derivatives of f with respect to g. For example we have

$$f_i = \partial_i f, \quad f_{ij} = \partial_j f_i - (g^{k\bar{q}}\partial_j g_{i\bar{q}})f_k, \quad f_{i\bar{j}} = f_{\bar{j}i} = 0.$$

We will also write g_S for the coefficient of the metric g_S in the coordinate w, so $g_S(x) = 1$ and the pullback of the metric g_S to M is given by $f_i\overline{f_j}g_S$. We use the shorthand $h^{i\bar{j}} = g^{i\bar{\ell}}g^{k\bar{j}}f_k\overline{f_\ell}g_S$, where $h^{i\bar{j}}$ is semipositive definite and satisfies

$|h|_g^2 := h^{i\bar{j}}h^{k\bar{\ell}}g_{i\bar{\ell}}g_{k\bar{j}} \le C$. Then we have (cf. [33])

$$\Delta \mathrm{tr}_\omega \omega_S = g^{i\bar{j}}\partial_i\partial_{\bar{j}}(g^{k\bar{\ell}}f_k\overline{f_\ell}g_S) = g^{i\bar{j}}g^{k\bar{\ell}}f_{ki}\overline{f_{\ell j}}g_S + g^{i\bar{j}}h^{p\bar{q}}R_{i\bar{j}p\bar{q}}$$
$$- g^{i\bar{j}}g^{k\bar{\ell}}f_k\overline{f_\ell}f_i\overline{f_j}R_S,$$

for R_S the scalar curvature of g_S, and so

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_\omega\omega_S = \mathrm{tr}_\omega\omega_S - g^{i\bar{j}}g^{k\bar{\ell}}f_{ki}\overline{f_{\ell j}}g_S + g^{i\bar{j}}h^{p\bar{q}}(R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})$$
$$+ g^{i\bar{j}}g^{k\bar{\ell}}f_k\overline{f_\ell}f_i\overline{f_j}R_S. \tag{6.4}$$

The last term can be dropped since $R_S < 0$. Now at x we have $\partial_i(\mathrm{tr}_\omega\omega_S) = \sum_k f_{ki}\overline{f_k}$, and using the Cauchy–Schwarz inequality we have

$$|\nabla\mathrm{tr}_\omega\omega_S|_g^2 = \sum_{i,k,p} f_{ki}\overline{f_{pi}}f_p\overline{f_k}$$

$$\le \sum_{k,p}|f_k||f_p|\left(\sum_i|f_{ki}|^2\right)^{1/2}\left(\sum_j|f_{pj}|^2\right)^{1/2}$$

$$= \left(\sum_k|f_k|\left(\sum_i|f_{ki}|^2\right)^{1/2}\right)^2 \le \left(\sum_\ell|f_\ell|^2\right)\left(\sum_{i,k}|f_{ki}|^2\right)$$

$$= (\mathrm{tr}_\omega\omega_S)g^{i\bar{j}}g^{k\bar{\ell}}f_{ki}\overline{f_{\ell j}}g_S \le Cg^{i\bar{j}}g^{k\bar{\ell}}f_{ki}\overline{f_{\ell j}}g_S. \tag{6.5}$$

Next we claim that given any constant C_0 we can find a constant C such that

$$|g^{i\bar{j}}h^{p\bar{q}}(R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})| \le Ce^{t/2} + \frac{1}{2C_0}|\tilde{\nabla}g|_g^2. \tag{6.6}$$

In fact, we only need the case $C_0 = 1$ here, but the general case will be useful later. To prove this claim, we first calculate (see also [23, (2.6)]),

$$R_{i\bar{j}p\bar{q}} = -g_{r\bar{q}}\partial_{\bar{j}}\Gamma^r_{ip} = -g_{r\bar{q}}\partial_{\bar{j}}\Gamma^r_{pi} + g_{r\bar{q}}\partial_{\bar{j}}T^r_{pi} = R_{p\bar{j}i\bar{q}} + g_{r\bar{q}}\partial_{\bar{j}}T^r_{pi}$$
$$= \overline{R_{j\bar{p}q\bar{i}}} + g_{r\bar{q}}\partial_{\bar{j}}T^r_{pi} = \overline{R_{q\bar{p}j\bar{i}}} + g_{i\bar{s}}\partial_p\overline{T^s_{qj}} + g_{r\bar{q}}\partial_{\bar{j}}T^r_{pi}$$
$$= R_{p\bar{q}i\bar{j}} + g_{i\bar{s}}\partial_p\overline{T^s_{qj}} + g_{r\bar{q}}\partial_{\bar{j}}T^r_{pi}, \tag{6.7}$$

and therefore

$$g^{i\bar{j}}h^{p\bar{q}}(R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}}) = -h^{p\bar{q}}\partial_p\overline{T^j_{qj}} - g^{i\bar{j}}h^{p\bar{q}}g_{r\bar{q}}\partial_{\bar{j}}T^r_{pi}. \tag{6.8}$$

Recall that $T_{ij\bar{\ell}} = \tilde{T}_{ij\bar{\ell}}$. Differentiating this gives

$$\partial_p \overline{T_{qj}^j} = \tilde{g}_{r\bar{s}} g^{r\bar{j}} \partial_p \overline{\tilde{T}_{qj}^s} - \tilde{g}_{r\bar{s}} g^{r\bar{u}} g^{t\bar{j}} \overline{\tilde{T}_{qj}^s} \tilde{\nabla}_p g_{t\bar{u}} = g^{r\bar{j}} \tilde{\nabla}_p \overline{\tilde{T}_{qj\bar{r}}} - g^{r\bar{u}} g^{t\bar{j}} \overline{\tilde{T}_{qj\bar{r}}} \tilde{\nabla}_p g_{t\bar{u}},$$
(6.9)

$$\partial_{\bar{j}} T_{pi}^r = \tilde{g}_{s\bar{b}} g^{r\bar{b}} \partial_{\bar{j}} \tilde{T}_{pi}^s - \tilde{g}_{s\bar{b}} g^{r\bar{u}} g^{t\bar{b}} \tilde{T}_{pi}^s \tilde{\nabla}_{\bar{j}} g_{t\bar{u}} = g^{r\bar{b}} \tilde{\nabla}_{\bar{j}} \tilde{T}_{pi\bar{b}} - g^{r\bar{u}} g^{t\bar{b}} \tilde{T}_{pi\bar{b}} \tilde{\nabla}_{\bar{j}} g_{t\bar{u}}.$$
(6.10)

Putting together (6.8), (6.9), (6.10) we get

$$g^{i\bar{j}} h^{p\bar{q}} (R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}}) = -h^{p\bar{q}} g^{r\bar{j}} \tilde{\nabla}_p \overline{\tilde{T}_{qj\bar{r}}} + h^{p\bar{q}} g^{r\bar{u}} g^{t\bar{j}} \overline{\tilde{T}_{qj\bar{r}}} \tilde{\nabla}_p g_{t\bar{u}}$$
$$- h^{p\bar{q}} g^{i\bar{j}} \tilde{\nabla}_{\bar{j}} \tilde{T}_{pi\bar{q}} + h^{p\bar{q}} g^{i\bar{j}} g^{t\bar{r}} \tilde{T}_{pi\bar{r}} \tilde{\nabla}_{\bar{j}} g_{t\bar{q}}.$$
(6.11)

Using again that the metrics g and $\tilde{g}$ are equivalent and $|h|_g \leq C$, we can then bound this by

$$|g^{i\bar{j}} h^{p\bar{q}} (R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})| \leq C |\bar{\partial} \tilde{T}|_{\tilde{g}} + C |\tilde{T}|_{\tilde{g}} |\tilde{\nabla} g|_g.$$

But from Lemma 4.1 we have that $|\tilde{T}|_{\tilde{g}} \leq C$ and $|\bar{\partial} \tilde{T}|_{\tilde{g}} \leq C e^{t/2}$, and so we have (6.6) as required.

Then (6.2) follows from (6.4), (6.5) and (6.6). (6.3) follows immediately from (6.1) and (6.2). □

We can now start the main argument for the proof of Theorem 6.1. Note that since many of our inequalities require $\tilde{\omega}$ to be a metric, we will often assume (without saying it explicitly) that $t \geq T_I$, which is not a problem since R is bounded on $[0, T_I]$. As in [24], we consider the quantity $u = \varphi + \dot{\varphi} = \log \frac{e^t \omega^2}{\Omega}$. We know that $|u| \leq C$, and we have that $-\Delta u = R + \mathrm{tr}_\omega \omega_S \geq R$, so our goal is to get an upper bound for $-\Delta u$. First compute from (3.2),

$$\left(\frac{\partial}{\partial t} - \Delta \right) u = \mathrm{tr}_\omega \omega_S - 1,$$
(6.12)

and

$$\left(\frac{\partial}{\partial t} - \Delta \right) \Delta u = R^{i\bar{j}} u_{i\bar{j}} + \Delta u + \Delta \mathrm{tr}_\omega \omega_S.$$

But on the other hand $R_{i\bar{j}} = -u_{i\bar{j}} - (g_S)_{i\bar{j}}$, and so

$$\left(\frac{\partial}{\partial t} - \Delta \right) \Delta u = -|\nabla \overline{\nabla} u|_g^2 - \langle g_S, \nabla \overline{\nabla} u \rangle_g + \Delta u + \Delta \mathrm{tr}_\omega \omega_S$$
$$\geq -\frac{3}{2} |\nabla \overline{\nabla} u|_g^2 + \Delta u + \Delta \mathrm{tr}_\omega \omega_S - C,$$

using that $|g_S|_g \leq C$. From (6.2) we have

$$-\Delta \mathrm{tr}_\omega \omega_S \leq Ce^{t/2} + |\tilde{\nabla} g|_g^2 - C^{-1}|\nabla \mathrm{tr}_\omega \omega_S|_g^2 - h^{i\bar{j}}(R_{i\bar{j}} + g_{i\bar{j}})$$
$$\leq Ce^{t/2} + |\tilde{\nabla} g|_g^2 - C^{-1}|\nabla \mathrm{tr}_\omega \omega_S|_g^2 + h^{i\bar{j}}(u_{i\bar{j}} + (g_S)_{i\bar{j}})$$
$$\leq Ce^{t/2} + |\tilde{\nabla} g|_g^2 - C^{-1}|\nabla \mathrm{tr}_\omega \omega_S|_g^2 + \frac{1}{2}|\nabla \overline{\nabla} u|_g^2,$$

where we have used the fact that $\frac{\partial}{\partial t}\mathrm{tr}_\omega \omega_S = -h^{i\bar{j}}\frac{\partial}{\partial t}g_{i\bar{j}}$, with $h^{i\bar{j}}$ as in the proof of Lemma 6.2. Therefore

$$\left(\frac{\partial}{\partial t} - \Delta\right)(-\Delta u) \leq 2|\nabla \overline{\nabla} u|_g^2 - \Delta u + Ce^{t/2} + |\tilde{\nabla} g|_g^2 - C^{-1}|\nabla \mathrm{tr}_\omega \omega_S|_g^2. \quad (6.13)$$

We need a quantity whose evolution can kill the bad term $2|\nabla \overline{\nabla} u|_g^2$, and this quantity is $|\nabla u|_g^2$. Before we compute its evolution, we need formulae to commute two covariant derivatives of the same type. For any function ψ and $(0, 1)$ form $a = a_{\bar{k}}d\bar{z}^k$, a short calculation gives

$$[\nabla_{\bar{j}}, \nabla_{\bar{\ell}}]\psi = -\overline{T_{j\ell}^k}\nabla_{\bar{k}}\psi, \quad [\nabla_i, \nabla_j]a_{\bar{k}} = -T_{ij}^\ell \nabla_\ell a_{\bar{k}}.$$

We will also use the familiar formulae

$$[\nabla_i, \nabla_{\bar{j}}]a_{\bar{k}} = g^{p\bar{\ell}}R_{i\bar{j}p\bar{k}}a_{\bar{\ell}}, \quad R_{i\bar{\ell}p\bar{j}} = R_{p\bar{\ell}i\bar{j}} + g_{r\bar{j}}\partial_{\bar{\ell}}T_{pi}^r,$$

where the second equation is contained in (6.7). We then compute:

$$\Delta|\nabla u|_g^2 = g^{i\bar{j}}g^{k\bar{\ell}}(\nabla_i \nabla_{\bar{j}}\nabla_k u \nabla_{\bar{\ell}}u + \nabla_k u \nabla_i \nabla_{\bar{j}}\nabla_{\bar{\ell}}u + \nabla_i \nabla_k u \nabla_{\bar{j}}\nabla_{\bar{\ell}}u + \nabla_i \nabla_{\bar{\ell}}u \nabla_{\bar{j}}\nabla_k u)$$
$$= |\nabla \overline{\nabla} u|_g^2 + |\nabla \nabla u|_g^2 + g^{i\bar{j}}g^{k\bar{\ell}}(\nabla_i \nabla_k \nabla_{\bar{j}}u \nabla_{\bar{\ell}}u + \nabla_k u \nabla_i \nabla_{\bar{\ell}}\nabla_{\bar{j}}u$$
$$\quad - \nabla_k u \nabla_i (\overline{T_{j\ell}^p}\nabla_{\bar{p}}u))$$
$$= |\nabla \overline{\nabla} u|_g^2 + |\nabla \nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \Delta u, \nabla u\rangle_g - g^{i\bar{j}}g^{k\bar{\ell}}T_{ik}^p \nabla_p \nabla_{\bar{j}}u \nabla_{\bar{\ell}}u$$
$$\quad + g^{i\bar{j}}g^{k\bar{\ell}}g^{p\bar{q}}R_{i\bar{\ell}p\bar{j}}\nabla_{\bar{q}}u \nabla_k u - g^{i\bar{j}}g^{k\bar{\ell}}\overline{T_{j\ell}^p}\nabla_k u \nabla_i \nabla_{\bar{p}}u - g^{i\bar{j}}g^{k\bar{\ell}}\partial_i \overline{T_{j\ell}^p}\nabla_k u \nabla_{\bar{p}}u$$
$$= |\nabla \overline{\nabla} u|_g^2 + |\nabla \nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \Delta u, \nabla u\rangle_g + g^{k\bar{\ell}}g^{p\bar{q}}R_{p\bar{\ell}}\nabla_{\bar{q}}u \nabla_k u$$
$$\quad - 2\mathrm{Re}(g^{i\bar{j}}g^{k\bar{\ell}}T_{ik}^p \nabla_p \nabla_{\bar{j}}u \nabla_{\bar{\ell}}u) + g^{k\bar{\ell}}g^{p\bar{q}}\partial_{\bar{\ell}}T_{pi}^i \nabla_{\bar{q}}u \nabla_k u$$
$$\quad - g^{i\bar{j}}g^{k\bar{\ell}}\partial_i \overline{T_{j\ell}^p}\nabla_k u \nabla_{\bar{p}}u.$$

We use again (6.9) and (6.10)

$$\partial_i \overline{T_{j\ell}^p} = g^{q\bar{p}}\tilde{\nabla}_i \overline{\tilde{T}_{j\ell\bar{q}}} - g^{r\bar{p}}g^{q\bar{s}}\overline{\tilde{T}_{j\ell\bar{q}}}\tilde{\nabla}_i g_{r\bar{s}}, \quad \partial_{\bar{\ell}}T_{pi}^i = g^{i\bar{j}}\tilde{\nabla}_{\bar{\ell}}\tilde{T}_{pi\bar{j}} - g^{i\bar{s}}g^{r\bar{j}}\tilde{T}_{pi\bar{j}}\tilde{\nabla}_{\bar{\ell}}g_{r\bar{s}},$$

and Lemma 4.1 to conclude that

$$
\begin{aligned}
\Delta |\nabla u|_g^2 &\geq |\nabla \overline{\nabla} u|_g^2 + |\nabla \nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \Delta u, \nabla u\rangle_g + g^{k\bar{\ell}} g^{p\bar{q}} R_{p\bar{\ell}} \nabla_{\bar{q}} u \nabla_k u \\
&\quad - C|\nabla \overline{\nabla} u|_g |\nabla u|_g - Ce^{t/2}|\nabla u|_g^2 - C|\tilde{\nabla} g|_g |\nabla u|_g^2 \\
&\geq \frac{1}{2}|\nabla \overline{\nabla} u|_g^2 + |\nabla \nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \Delta u, \nabla u\rangle_g + g^{k\bar{\ell}} g^{p\bar{q}} R_{p\bar{\ell}} \nabla_{\bar{q}} u \nabla_k u \\
&\quad - Ce^{t/2}|\nabla u|_g^2 - C|\tilde{\nabla} g|_g |\nabla u|_g^2.
\end{aligned}
$$

Next, using (6.12), we have

$$
\frac{\partial}{\partial t}|\nabla u|_g^2 = g^{k\bar{\ell}} g^{p\bar{q}} R_{p\bar{\ell}} \nabla_k u \nabla_{\bar{q}} u + |\nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \Delta u, \nabla u\rangle_g + 2\mathrm{Re}\langle \nabla \mathrm{tr}_\omega \omega_S, \nabla u\rangle_g
$$

and hence

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|_g^2 \leq{}& -\frac{1}{2}|\nabla \overline{\nabla} u|_g^2 - |\nabla \nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \mathrm{tr}_\omega \omega_S, \nabla u\rangle_g \\
&+ Ce^{t/2}|\nabla u|_g^2 + C|\tilde{\nabla} g|_g |\nabla u|_g^2.
\end{aligned} \tag{6.14}
$$

We will use this evolution inequality to bound $|\nabla u|_g^2$.

Proposition 6.3 *There is a constant C such that*

$$
|\nabla u|_g^2 \leq Ce^{t/2}.
$$

Proof We use the method of Cheng–Yau [5], see also [22,24]. We fix a constant A such that $|u| \leq A - 1$ and use (6.14) to compute

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla u|_g^2}{A - u}\right) ={}& \frac{1}{A - u}\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla u|_g^2 + \frac{|\nabla u|_g^2}{(A - u)^2}\left(\frac{\partial}{\partial t} - \Delta\right)u \\
&- \frac{2\mathrm{Re}\langle \nabla |\nabla u|_g^2, \nabla u\rangle_g}{(A - u)^2} - 2\frac{|\nabla u|_g^4}{(A - u)^3} \\
\leq{}& \frac{1}{A - u}\left(-\frac{1}{2}|\nabla \overline{\nabla} u|_g^2 - |\nabla \nabla u|_g^2 + 2\mathrm{Re}\langle \nabla \mathrm{tr}_\omega \omega_S, \nabla u\rangle_g \right. \\
&\left. + Ce^{t/2}|\nabla u|_g^2 + C|\tilde{\nabla} g|_g |\nabla u|_g^2\right) + (\mathrm{tr}_\omega \omega_S - 1)\frac{|\nabla u|_g^2}{(A - u)^2} \\
&- \frac{2\mathrm{Re}\langle \nabla |\nabla u|_g^2, \nabla u\rangle_g}{(A - u)^2} - 2\frac{|\nabla u|_g^4}{(A - u)^3}.
\end{aligned} \tag{6.15}
$$

Note that the term $(\mathrm{tr}_\omega \omega_S - 1)\frac{|\nabla u|_g^2}{(A-u)^2}$ can be absorbed in the term $\frac{Ce^{t/2}|\nabla u|_g^2}{A-u}$. For $\varepsilon > 0$ small (to be fixed soon), write

$$\frac{2\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g}{(A-u)^2} = \varepsilon\frac{2\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g}{(A-u)^2}$$

$$+ \frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\left\langle\nabla\left(\frac{|\nabla u|_g^2}{A-u}\right), \nabla u\right\rangle_g - \frac{2(1-\varepsilon)|\nabla u|_g^4}{(A-u)^3},$$

$$(6.16)$$

and use the Cauchy–Schwarz inequality to bound

$$-\varepsilon\frac{2\mathrm{Re}\langle\nabla|\nabla u|_g^2, \nabla u\rangle_g}{(A-u)^2} \le \varepsilon 2\sqrt{2}\frac{|\nabla u|_g^2(|\nabla\overline{\nabla}u|_g^2 + |\nabla\nabla u|_g^2)^{1/2}}{(A-u)^2}$$

$$\le \frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^2} + 4\varepsilon\frac{|\nabla\overline{\nabla}u|_g^2 + |\nabla\nabla u|_g^2}{(A-u)^2}$$

$$\le \frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^2} + \frac{1}{2}\frac{|\nabla\overline{\nabla}u|_g^2 + |\nabla\nabla u|_g^2}{A-u}, \qquad (6.17)$$

provided $\varepsilon \le 1/8$ (here we fix ε). We also bound

$$\frac{C|\tilde{\nabla}g|_g|\nabla u|_g^2}{A-u} \le C|\tilde{\nabla}g|_g^2 + \frac{\varepsilon}{2}\frac{|\nabla u|_g^4}{(A-u)^2}. \qquad (6.18)$$

Putting (6.15), (6.16), (6.17), (6.18) together, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla u|_g^2}{A-u}\right) \le \frac{1}{A-u}(2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\omega_S, \nabla u\rangle_g + Ce^{t/2}|\nabla u|_g^2) + C|\tilde{\nabla}g|_g^2$$

$$- \varepsilon\frac{|\nabla u|_g^4}{(A-u)^3} - \frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\left\langle\nabla\left(\frac{|\nabla u|_g^2}{A-u}\right), \nabla u\right\rangle_g.$$

Call now

$$Q_1 = \frac{|\nabla u|_g^2}{A-u} + C_2(\mathrm{tr}_\omega\omega_S + C_0\mathrm{tr}_{\tilde{\omega}}\omega),$$

where C_0 is as in (6.3), and C_2 is a large uniform constant to be fixed soon. We can use (6.3) to get

$$\left(\frac{\partial}{\partial t} - \Delta\right)Q_1 \le \frac{1}{A-u}(2\mathrm{Re}\langle\nabla\mathrm{tr}_\omega\omega_S, \nabla u\rangle_g + Ce^{t/2}|\nabla u|_g^2)$$

$$- \varepsilon\frac{|\nabla u|_g^4}{(A-u)^3} - \frac{2(1-\varepsilon)}{A-u}\mathrm{Re}\left\langle\nabla\left(\frac{|\nabla u|_g^2}{A-u}\right), \nabla u\right\rangle_g$$

$$- \frac{C_2}{2}|\tilde{\nabla}g|_g^2 - 2|\nabla\mathrm{tr}_\omega\omega_S|_g^2 + Ce^{t/2}, \qquad (6.19)$$

provided C_2 is large enough. We now observe that $|\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_{\tilde{g}} \leq 2|\tilde{\nabla} g|^2_{\tilde{g}}$. Indeed, if we choose local coordinates such that $\tilde{g}$ is the identity at a point, then at that point we have

$$|\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_{\tilde{g}} = \sum_k \left| \sum_i \tilde{\nabla}_k g_{i\bar{i}} \right|^2 \leq 2 \sum_{i,k} |\tilde{\nabla}_k g_{i\bar{i}}|^2 \leq 2 \sum_{i,j,k} |\tilde{\nabla}_k g_{i\bar{j}}|^2 = 2|\tilde{\nabla} g|^2_{\tilde{g}}. \quad (6.20)$$

Since $\tilde{g}$ and g are uniformly equivalent, we conclude that $|\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_g \leq C|\tilde{\nabla} g|^2_g$. We can then assume that C_2 was large enough (this fixes C_2) so that

$$-\frac{C_2}{2}|\tilde{\nabla} g|^2_g \leq -|\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_g. \quad (6.21)$$

Furthermore, we can bound

$$C\frac{e^{t/2}|\nabla u|^2_g}{A - u} \leq \frac{\varepsilon}{4} \frac{|\nabla u|^4_g}{(A - u)^3} + Ce^t, \quad (6.22)$$

$$2\frac{\mathrm{Re}\langle \nabla \mathrm{tr}_\omega \omega_S, \nabla u \rangle_g}{A - u} \leq |\nabla \mathrm{tr}_\omega \omega_S|^2_g + \frac{\varepsilon}{4} \frac{|\nabla u|^4_g}{(A - u)^3} + C, \quad (6.23)$$

so combining (6.19), (6.21), (6.22), (6.23) we get

$$\left(\frac{\partial}{\partial t} - \Delta \right) Q_1 \leq -\frac{\varepsilon}{2} \frac{|\nabla u|^4_g}{(A - u)^3} - \frac{2(1 - \varepsilon)}{A - u} \mathrm{Re} \left\langle \nabla \left(\frac{|\nabla u|^2_g}{A - u} \right), \nabla u \right\rangle_g$$
$$- |\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_g - |\nabla \mathrm{tr}_\omega \omega_S|^2_g + Ce^t.$$

We can write this as

$$\left(\frac{\partial}{\partial t} - \Delta \right) Q_1 \leq -\frac{\varepsilon}{2} \frac{|\nabla u|^4_g}{(A - u)^3} - \frac{2(1 - \varepsilon)}{A - u} \mathrm{Re} \langle \nabla Q_1, \nabla u \rangle_g$$
$$+ \frac{2(1 - \varepsilon)C_2}{A - u} \mathrm{Re} \langle \nabla \mathrm{tr}_\omega \omega_S, \nabla u \rangle_g + \frac{2(1 - \varepsilon)C_0 C_2}{A - u} \mathrm{Re} \langle \nabla \mathrm{tr}_{\tilde{\omega}} \omega, \nabla u \rangle_g$$
$$- |\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_g - |\nabla \mathrm{tr}_\omega \omega_S|^2_g + Ce^t,$$

which together with the bounds

$$\frac{2(1 - \varepsilon)C_2}{A - u} \mathrm{Re}\langle \nabla \mathrm{tr}_\omega \omega_S, \nabla u \rangle_g \leq |\nabla \mathrm{tr}_\omega \omega_S|^2_g + \frac{\varepsilon}{8} \frac{|\nabla u|^4_g}{(A - u)^3} + C,$$

$$\frac{2(1 - \varepsilon)C_0 C_2}{A - u} \mathrm{Re}\langle \nabla \mathrm{tr}_{\tilde{\omega}} \omega, \nabla u \rangle_g \leq |\nabla \mathrm{tr}_{\tilde{\omega}} \omega|^2_g + \frac{\varepsilon}{8} \frac{|\nabla u|^4_g}{(A - u)^3} + C,$$

gives us

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_1 \leq -\frac{\varepsilon}{4} \frac{|\nabla u|_g^4}{(A - u)^3} - \frac{2(1 - \varepsilon)}{A - u} \mathrm{Re}\langle \nabla Q_1, \nabla u \rangle_g + Ce^t.$$

Now define $Q_2 = e^{-t/2} Q_1$, which satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_2 \leq -C^{-1} e^{-t/2} |\nabla u|_g^4 - \frac{2(1 - \varepsilon)}{A - u} \mathrm{Re}\langle \nabla Q_2, \nabla u \rangle_g + Ce^{t/2}.$$

At a maximum of Q_2 (occurring at a time $t_0 > T_I$) we conclude that $|\nabla u|_g^4 \leq Ce^t$, which implies that $Q_2 \leq C$ everywhere. This proves the proposition. $\qquad \square$

Now that we know that $|\nabla u|_g^2 \leq Ce^{t/2}$, we can go back to (6.14) and get

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla u|_g^2 \leq -\frac{1}{2} |\nabla \overline{\nabla} u|_g^2 - |\nabla \nabla u|_g^2 + |\nabla \mathrm{tr}_\omega S|_g^2 + |\tilde{\nabla} g|_g^2 + Ce^t. \quad (6.24)$$

Finally we can put everything together to prove Theorem 6.1.

Proof of Theorem 6.1 We will show that there is a constant C such that

$$-\Delta u \leq Ce^{t/2}.$$

and this will give $R \leq Ce^{t/2}$.

From (6.13) and (6.24), we see that

$$\left(\frac{\partial}{\partial t} - \Delta\right) (-\Delta u + 6|\nabla u|_g^2) \leq -|\nabla \overline{\nabla} u|_g^2 - \Delta u + C|\nabla \mathrm{tr}_\omega S|_g^2$$
$$+ C|\tilde{\nabla} g|_g^2 + Ce^t.$$

Using (6.3) we get

$$\left(\frac{\partial}{\partial t} - \Delta\right) (-\Delta u + 6|\nabla u|_g^2 + C_2(\mathrm{tr}_\omega \omega_S + C_0 \mathrm{tr}_{\tilde{\omega}} \omega)) \leq -|\nabla \overline{\nabla} u|_g^2 - \Delta u + Ce^t,$$

provided C_2 is large enough. Define now

$$Q_3 = e^{-t/2}(-\Delta u + 6|\nabla u|_g^2 + C_2(\mathrm{tr}_\omega \omega_S + C_0 \mathrm{tr}_{\tilde{\omega}} \omega)).$$

Note that $Q_3 \geq -Ce^{-t/2}$, because $-\Delta u \geq R \geq -C$. Then,

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_3 \leq -e^{-t/2}|\nabla \overline{\nabla} u|_g^2 - e^{-t/2}\Delta u + Ce^{t/2},$$

where we absorbed a term like $e^{-t/2}$ into $e^{t/2}$. From the Cauchy–Schwarz inequality we also have that

$$(-\Delta u)^2 \leq 2|\nabla\overline\nabla u|_g^2.$$

It follows that at a maximum of Q_3 (occurring at $t_0 > T_I$) we have that $(-\Delta u)^2 \leq Ce^t$, which implies that $Q_3 \leq C$ everywhere. This completes the proof of the theorem.
□

We end this section by applying our Chern scalar curvature bound to obtain an exponential decay estimate for $\dot\varphi$.

Lemma 6.4 *For any η with $0 < \eta < 1/2$ and any σ with $0 < \sigma < 1/4$, there exists a constant C such that*

$$-Ce^{-\eta t} \leq \dot\varphi \leq Ce^{-\sigma t}.$$

Proof We first prove the lower bound by refining an argument in [28]. We have

$$\frac{\partial}{\partial t}\dot\varphi = -R - 1 - \dot\varphi \tag{6.25}$$

and hence, by Theorem 6.1 and the fact that $|\dot\varphi|$ is bounded,

$$\frac{\partial}{\partial t}\dot\varphi(t) \leq C_0, \tag{6.26}$$

for a uniform C_0. Suppose for a contradiction that we do not have the bound $\dot\varphi \geq -Ce^{-\eta t}$ for any C. Then there exists a sequence $(x_k, t_k) \in M \times [0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\dot\varphi(x_k, t_k) \leq -ke^{-\eta t_k}.$$

Put $\gamma_k = \frac{k}{2C_0}e^{-\eta t_k}$. From now on we work at the point x_k. Then by (6.26), we have that

$$\dot\varphi \leq -\frac{k}{2}e^{-\eta t_k} \quad \text{on } [t_k, t_k + \gamma_k].$$

Indeed,

$$\dot\varphi(t_k + a) - \dot\varphi(t_k) = \int_{t_k}^{t_k+a} \frac{\partial}{\partial t}\dot\varphi\, dt \leq C_0\gamma_k, \quad \text{for } a \in [0, \gamma_k],$$

and hence $\dot\varphi(t_k + a) \leq \dot\varphi(t_k) + C_0\gamma_k \leq -ke^{-\eta t_k} + \frac{k}{2}e^{-\eta t_k}$.
Then, using Lemma 3.4,

$$-C(1+t_k)e^{-t_k} \leq \varphi(t_k + \gamma_k) - \varphi(t_k) = \int_{t_k}^{t_k+\gamma_k} \dot\varphi dt \leq -\gamma_k\frac{k}{2}e^{-\eta t_k} = -\frac{k^2}{4C_0}e^{-2\eta t_k}.$$

But if $2\eta < 1$ then we get a contradiction when $k \to \infty$ and we are done.

For the upper bound of φ we use the upper bound $R \leq Ce^{t/2}$ of Theorem 6.1. From (6.25), we have

$$\frac{\partial}{\partial t}\dot{\varphi}(t) \geq -C_0 e^{t/2}, \tag{6.27}$$

for a uniform C_0. Note that we may assume, by increasing C_0, that we have

$$\frac{\partial^2}{\partial t^2}\varphi(t) \geq -C_0 e^{t'/2} \quad \text{for } t \in [t', t'+1], \tag{6.28}$$

for any time t'. Suppose for a contradiction that we do not have the bound $\dot{\varphi} \leq Ce^{-\sigma t}$ for any C. Then there exists a sequence $(x_k, t_k) \in M \times [0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\dot{\varphi}(x_k, t_k) \geq ke^{-\sigma t_k}.$$

Put $\gamma_k = \frac{k}{2C_0}e^{-(\sigma+1/2)t_k}$ which we assume for the moment satisfies $\gamma_k \leq 1$. From now on we work at the point x_k. Then by (6.28), we have that

$$\dot{\varphi} \geq \frac{k}{2}e^{-\sigma t_k} \quad \text{on } [t_k, t_k + \gamma_k].$$

Indeed, this follows from (6.28) since

$$\dot{\varphi}(t_k + a) - \dot{\varphi}(t_k) = \int_{t_k}^{t_k+a} \frac{\partial}{\partial t}\dot{\varphi}\, dt \geq -\gamma_k C_0 e^{t_k/2}, \quad \text{for } a \in [0, \gamma_k],$$

and hence $\dot{\varphi}(t_k + a) \geq \dot{\varphi}(t_k) - \gamma_k C_0 e^{t_k/2} \geq ke^{-\sigma t_k} - \frac{k}{2}e^{-\sigma t_k}$.

Then from Lemma 3.4,

$$C(1 + t_k)e^{-t_k} \geq \varphi(t_k + \gamma_k) - \varphi(t_k) = \int_{t_k}^{t_k+\gamma_k} \dot{\varphi}\, dt \geq \gamma_k \frac{k}{2}e^{-\sigma t_k} = \frac{k^2}{4C_0}e^{-(2\sigma+1/2)t_k}.$$

But since $2\sigma + 1/2 < 1$ we get a contradiction when $k \to \infty$ and we are done.

It remains to check the case when $\gamma_k > 1$. But then we have $\dot{\varphi} \geq \frac{k}{2}e^{-\sigma t_k}$ on $[t_k, t_k + 1]$ since $\dot{\varphi}(t_k + a) - \dot{\varphi}(t_k) \geq -\gamma_k C_0 e^{t_k/2}$ for $a \in [0, 1]$ and so we get $\dot{\varphi}(t_k + a) \geq \frac{k}{2}e^{-\sigma t_k}$ for $a \in [0, 1]$. Then

$$C(1 + t_k)e^{-t_k} \geq \varphi(t_k + 1) - \varphi(t_k) = \int_{t_k}^{t_k+1} \dot{\varphi}\, dt \geq \frac{k}{2}e^{-\sigma t_k}$$

and we get a contradiction since $\sigma < 1$. $\qquad\square$

7 Exponential decay estimates for the metric

In this section we establish the key estimates which show that $\omega(t)$ and $\tilde{\omega}$ approach each other exponentially fast as $t \to \infty$. More precisely we prove:

Theorem 7.1 *For any ε with $0 < \varepsilon < 1/8$, there exists C such that for $t \geq T_I$*

$$(1 - Ce^{-\varepsilon t})\tilde{\omega} \leq \omega(t) \leq (1 + Ce^{-\varepsilon t})\tilde{\omega}.$$

In this section we will always assume $t \geq T_I$, without necessarily mentioning it explicitly. First, we have the following evolution inequality for $\text{tr}_\omega\tilde{\omega}$ which has the same form as the inequality for $\text{tr}_{\tilde{\omega}}\omega$ given by (6.1). We will make use of both of these inequalities to prove Theorem 7.1.

Lemma 7.2 *We have, for a uniform $C > 0$,*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\text{tr}_\omega\tilde{\omega} \leq Ce^{t/2} - C^{-1}|\tilde{\nabla}g|_g^2.$$

Proof We will use the shorthand $h^{i\bar{j}} = g^{i\bar{\ell}}g^{k\bar{j}}\tilde{g}_{k\bar{\ell}}$. Note that we know already that h, g and $\tilde{g}$ are all uniformly equivalent to each other.

We start by computing a formula for the evolution of $\text{tr}_\omega\tilde{\omega}$. First of all, we have

$$\Delta\text{tr}_\omega\tilde{\omega} = g^{i\bar{j}}\tilde{\nabla}_i\tilde{\nabla}_{\bar{j}}(g^{k\bar{\ell}}\tilde{g}_{k\bar{\ell}}) = -g^{i\bar{j}}\tilde{\nabla}_i(g^{k\bar{q}}g^{p\bar{\ell}}\tilde{g}_{k\bar{\ell}}\tilde{\nabla}_{\bar{j}}g_{p\bar{q}})$$

$$= g^{i\bar{j}}g^{k\bar{s}}g^{r\bar{q}}g^{p\bar{\ell}}\tilde{g}_{k\bar{\ell}}\tilde{\nabla}_i g_{r\bar{s}}\tilde{\nabla}_{\bar{j}}g_{p\bar{q}} + g^{i\bar{j}}g^{p\bar{s}}g^{r\bar{\ell}}g^{k\bar{q}}\tilde{g}_{k\bar{\ell}}\tilde{\nabla}_i g_{r\bar{s}}\tilde{\nabla}_{\bar{j}}g_{p\bar{q}}$$

$$- g^{i\bar{j}}g^{k\bar{q}}g^{p\bar{\ell}}\tilde{g}_{k\bar{\ell}}\tilde{\nabla}_i\tilde{\nabla}_{\bar{j}}g_{p\bar{q}}.$$

But

$$\tilde{\nabla}_i\tilde{\nabla}_{\bar{j}}g_{p\bar{q}} = \tilde{\nabla}_i(\partial_{\bar{j}}g_{p\bar{q}} - \overline{\tilde{\Gamma}^s_{jq}}g_{p\bar{s}}) = \partial_i\partial_{\bar{j}}g_{p\bar{q}} - \tilde{\Gamma}^r_{ip}\partial_{\bar{j}}g_{r\bar{q}} - g_{p\bar{s}}\partial_i\overline{\tilde{\Gamma}^s_{jq}}$$

$$- \overline{\tilde{\Gamma}^s_{jq}}\partial_i g_{p\bar{s}} + \tilde{\Gamma}^r_{ip}\overline{\tilde{\Gamma}^s_{jq}}g_{r\bar{s}} = \tilde{R}_{i\bar{j}r\bar{q}}\tilde{g}^{r\bar{s}}g_{p\bar{s}} - R_{i\bar{j}p\bar{q}} + g^{r\bar{s}}\tilde{\nabla}_i g_{p\bar{s}}\tilde{\nabla}_{\bar{j}}g_{r\bar{q}},$$

and so

$$\Delta\text{tr}_\omega\tilde{\omega} = g^{i\bar{j}}g^{k\bar{s}}g^{r\bar{q}}g^{p\bar{\ell}}\tilde{g}_{k\bar{\ell}}\tilde{\nabla}_i g_{r\bar{s}}\tilde{\nabla}_{\bar{j}}g_{p\bar{q}} + g^{i\bar{j}}g^{k\bar{q}}g^{p\bar{\ell}}\tilde{g}_{k\bar{\ell}}R_{i\bar{j}p\bar{q}} - g^{i\bar{j}}g^{p\bar{q}}\tilde{R}_{i\bar{j}p\bar{q}}.$$

On the other hand

$$\frac{\partial}{\partial t}\text{tr}_\omega\tilde{\omega} = \text{tr}_\omega\tilde{\omega} + \text{tr}_\omega(\omega_S - \tilde{\omega}) + g^{i\bar{j}}h^{p\bar{q}}R_{p\bar{q}i\bar{j}}.$$

Therefore,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\text{tr}_\omega\tilde{\omega} = \text{tr}_\omega\omega_S + g^{i\bar{j}}g^{k\bar{\ell}}\tilde{R}_{i\bar{j}k\bar{\ell}} + g^{i\bar{j}}h^{p\bar{q}}(R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})$$

$$- g^{i\bar{j}}g^{k\bar{\ell}}h^{p\bar{q}}\tilde{\nabla}_i g_{k\bar{q}}\tilde{\nabla}_{\bar{j}}g_{p\bar{\ell}}.$$

From Lemma 4.1, we conclude that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_\omega \tilde{\omega} \leq C e^{t/2} + g^{i\bar{j}} h^{p\bar{q}} (R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}}) - g^{i\bar{j}} g^{k\bar{\ell}} h^{p\bar{q}} \tilde{\nabla}_i g_{k\bar{q}} \tilde{\nabla}_{\bar{j}} g_{p\bar{\ell}}. \quad (7.1)$$

We also use the equivalence of g and h to bound

$$- g^{i\bar{j}} g^{k\bar{\ell}} h^{p\bar{q}} \tilde{\nabla}_i g_{k\bar{q}} \tilde{\nabla}_{\bar{j}} g_{p\bar{\ell}} \leq -C_0^{-1} |\tilde{\nabla} g|_g^2. \quad (7.2)$$

Next we have

$$|g^{i\bar{j}} h^{p\bar{q}} (R_{p\bar{q}i\bar{j}} - R_{i\bar{j}p\bar{q}})| \leq C e^{t/2} + \frac{1}{2C_0} |\tilde{\nabla} g|_g^2. \quad (7.3)$$

Indeed, this follows from the same argument as in the proof of (6.6), even though the tensor h there is different. Combining (7.1), (7.2), (7.3), we get the desired inequality. $\qquad\square$

Next we use the exponential decay of φ (Lemma 3.4) and $\dot\varphi$ (Lemma 6.4) to obtain an exponential decay bound from above for $\mathrm{tr}_\omega \tilde\omega - 2$ and $\mathrm{tr}_{\tilde\omega} \omega - 2$.

Proposition 7.3 *For any* $0 < \varepsilon < 1/4$ *there is a constant* $C > 0$ *such that for* $t \geq T_I$,

$$\mathrm{tr}_\omega \tilde\omega - 2 \leq C e^{-\varepsilon t} \quad (7.4)$$

and

$$\mathrm{tr}_{\tilde\omega} \omega - 2 \leq C e^{-\varepsilon t}. \quad (7.5)$$

Proof Given $0 < \varepsilon < 1/4$, choose $\eta > \varepsilon$ such that $\varepsilon + 1/2 + \eta < 1$ (in particular $0 < \eta < 1/2$), and choose δ satisfying $2\varepsilon + 1/2 < \delta < \varepsilon + 1/2 + \eta$ (in particular $0 < \delta < 1$), which we can do because $\varepsilon < \eta$. For (7.4), we compute the evolution of

$$Q_1 = e^{\varepsilon t} (\mathrm{tr}_\omega \tilde\omega - 2) - e^{\delta t} \varphi.$$

From Lemma 3.4, it suffices to obtain a uniform upper bound for Q_1. Compute using Lemma 7.2 and the fact that $\Delta\varphi = 2 - \mathrm{tr}_\omega \tilde\omega$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_1 \leq C e^{(\varepsilon+1/2)t} + \varepsilon e^{\varepsilon t} (\mathrm{tr}_\omega \tilde\omega - 2) - \delta e^{\delta t} \varphi - e^{\delta t} \dot\varphi - e^{\delta t} (\mathrm{tr}_\omega \tilde\omega - 2)$$

$$\leq C e^{(\varepsilon+1/2)t} + C e^{(\delta-\eta)t} - e^{\delta t} (\mathrm{tr}_\omega \tilde\omega - 2), \quad (7.6)$$

where in the last line we have used the lower bound $\dot\varphi \geq -C e^{-\eta t}$ from Lemma '6.4 (since $0 < \eta < 1/2$), Lemma 3.4 and the fact that $\mathrm{tr}_\omega \tilde\omega \leq C$.

But we have $\delta - \eta < \varepsilon + 1/2$ and so at a maximum point of Q_1,

$$0 \leq C e^{(\varepsilon+1/2)t} - e^{\delta t} (\mathrm{tr}_\omega \tilde\omega - 2),$$

and hence

$$e^{\varepsilon t}(\mathrm{tr}_\omega \tilde{\omega} - 2) \le Ce^{(2\varepsilon - \delta + 1/2)t} \le C.$$

since we chose δ so that $2\varepsilon - \delta + 1/2 < 0$. This implies that Q_1 is bounded from above at any maximum point, and completes the proof of (7.4).

The proof of (7.5) is slightly more complicated. First recall that (see (6.1)) for $t \ge T_I$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\tilde{\omega}}\omega \le Ce^{t/2}.$$

Fix σ with $0 < \varepsilon < \sigma < 1/4$. From Lemma 6.4 we have

$$-Ce^{-\sigma t} \le \dot{\varphi} \le Ce^{-\sigma t}. \tag{7.7}$$

Now choose δ with $1/2 + 2\varepsilon < \delta < 1/2 + \varepsilon + \sigma$, which we can do because $\varepsilon < \sigma$. Then set

$$Q_2 = e^{\varepsilon t}(\mathrm{tr}_{\tilde{\omega}}\omega - 2) - e^{\delta t}\varphi.$$

Compute

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_2 \le Ce^{(\varepsilon + 1/2)t} - e^{\delta t}\dot{\varphi} - e^{\delta t}(\mathrm{tr}_\omega \tilde{\omega} - 2)$$

$$\le Ce^{(\varepsilon + 1/2)t} - e^{\delta t}(\mathrm{tr}_\omega \tilde{\omega} - 2), \tag{7.8}$$

using (7.7) and the fact that $\delta - \sigma < 1/2 + \varepsilon$. We now wish to replace the term $\mathrm{tr}_\omega \tilde{\omega}$ by the sum of $\mathrm{tr}_{\tilde{\omega}}\omega$ and a small error term.

$$\mathrm{tr}_\omega \tilde{\omega} = \frac{\tilde{\omega}^2}{\omega^2}\mathrm{tr}_{\tilde{\omega}}\omega = \mathrm{tr}_{\tilde{\omega}}\omega + \left(\frac{\tilde{\omega}^2}{\omega^2} - 1\right)\mathrm{tr}_{\tilde{\omega}}\omega. \tag{7.9}$$

Then from (2.4), (3.8) and Lemma 3.4,

$$\dot{\varphi} = \log\frac{\omega^2}{\tilde{\omega}^2} + O(e^{-\sigma t}),$$

and so using (7.7) again

$$\left|\frac{\tilde{\omega}^2}{\omega^2} - 1\right| = |e^{O(e^{-\sigma t})} - 1| \le Ce^{-\sigma t},$$

which implies that, since $\mathrm{tr}_{\tilde{\omega}}\omega$ is uniformly bounded,

$$\left|\left(\frac{\tilde{\omega}^2}{\omega^2} - 1\right)\mathrm{tr}_{\tilde{\omega}}\omega\right| \le Ce^{-\sigma t}. \tag{7.10}$$

Then combining (7.8), (7.9), (7.10) and again using the fact that $e^{(\delta-\sigma)t} \le e^{(\varepsilon+1/2)t}$ we obtain for $t \ge T_I$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q_2 \le Ce^{(\varepsilon+1/2)t} - e^{\delta t}(\mathrm{tr}_{\tilde{\omega}}\omega - 2) - e^{\delta t}\left(\frac{\tilde{\omega}^2}{\omega^2} - 1\right)\mathrm{tr}_{\tilde{\omega}}\omega$$

$$\le Ce^{(\varepsilon+1/2)t} - e^{\delta t}(\mathrm{tr}_{\tilde{\omega}}\omega - 2).$$

Then at the maximum point of Q_2 (occurring at a time $t > T_I$) we have

$$e^{\varepsilon t}(\mathrm{tr}_{\tilde{\omega}}\omega - 2) \le Ce^{(2\varepsilon+1/2-\delta)t} \le C',$$

since we chose $\delta > 1/2 + 2\varepsilon$. This shows that Q_2 is bounded from above and completes the proof. □

To show that ω and $\tilde{\omega}$ approach each other exponentially fast we use an elementary lemma:

Lemma 7.4 *Let* $\varepsilon > 0$ *be small. Suppose that*

$$tr_{\omega}\tilde{\omega} - 2 \le \varepsilon \quad and \quad tr_{\tilde{\omega}}\omega - 2 \le \varepsilon.$$

Then

$$(1 - 2\sqrt{\varepsilon})\tilde{\omega} \le \omega \le (1 + 2\sqrt{\varepsilon})\tilde{\omega}.$$

Proof We may work at a point at which $\tilde{g}$ is the identity and g is diagonal with eigenvalues λ_1, λ_2. Then the lemma amounts to proving that if $\lambda_1, \lambda_2 > 0$ satisfy

$$\lambda_1 + \lambda_2 \le 2 + \varepsilon, \quad \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \le 2 + \varepsilon,$$

then

$$1 - 2\sqrt{\varepsilon} \le \lambda_i \le 1 + 2\sqrt{\varepsilon}, \quad for\ i = 1, 2.$$

By symmetry, we only have to prove the estimate for λ_1. We have

$$\lambda_1 \le 2 + \varepsilon - \lambda_2, \quad \frac{1}{\lambda_2} \le \frac{(2+\varepsilon)\lambda_1 - 1}{\lambda_1}, \tag{7.11}$$

which implies in particular that $(2 + \varepsilon)\lambda_1 - 1 > 0$. This last inequality implies that

$$-\lambda_2 \le \frac{-\lambda_1}{(2+\varepsilon)\lambda_1 - 1}.$$

Then in (7.11),

$$\lambda_1 \leq 2 + \varepsilon - \frac{\lambda_1}{(2+\varepsilon)\lambda_1 - 1}.$$

Multiplying this by $(2+\varepsilon)\lambda_1 - 1 > 0$ and simplifying, we get

$$\lambda_1^2 - (2+\varepsilon)\lambda_1 + 1 \leq 0.$$

Completing the square, we obtain

$$(\lambda_1 - (1 + \varepsilon/2))^2 \leq \varepsilon + \varepsilon^2/4.$$

Then, assuming $\varepsilon > 0$ is smaller than some universal constant,

$$1 - 2\sqrt{\varepsilon} \leq \lambda_1 \leq 1 + 2\sqrt{\varepsilon},$$

as required.   □

Finally, we complete the proof of Theorem 7.1.

Proof of Theorem 7.1 Combine Proposition 7.3 with Lemma 7.4.   □

8 A third order estimate

In this section we prove a "Calabi-type" estimate for the first derivative of the evolving metric. One might guess that the natural quantity to consider is $|\tilde{\nabla} g|_g^2$, following the computation in [28], say. However, we encountered difficulties in obtaining a good bound for this quantity because of the non-Kählerity of the reference metrics $\tilde{\omega}$. Our idea then is to take a *Kähler* reference metric. Of course, in general M may not admit a global Kähler metric, so we work locally on an open set where the bundle M is trivial.

We obtain a Calabi estimate on this open set, using a cut-off function and a local reference Kähler metric. Our computations are based on those in [23]. However, the situation here is complicated by the fact that the metrics are collapsing in the fiber directions, and we will need to make careful use of the bounds from Lemma 4.1.

Fix a point $y \in S$ and neighborhood B of y over which π is trivial, so $U = \pi^{-1}(B) \cong B \times E$. Over U we have $\omega_E = i\alpha \wedge \bar{\alpha}$ a d-closed semi-flat $(1,1)$-form constructed as in the proof of Lemma 3.2. Therefore, $\hat{\omega} = \omega_E + \omega_S$ is a semi-flat product Kähler metric on U. From now on we work exclusively on U, where we define $\mathcal{S} = |\hat{\nabla} g|_g^2$. Fix a smaller open set $V \subset\subset U$.

Theorem 8.1 *On V we have*

$$\mathcal{S} \leq Ce^{2t/3},$$

for all $t > 0$.

By compactness, we obtain the same bound in any such neighborhood V. Recall that $\omega_{\text{flat},y}$ denotes the unique flat metric on the fiber $E_y = \pi^{-1}(y)$ in the Kähler class $[\omega_0|_{E_y}]$. Exactly as in [28, Lemma 6.9] (see also [12], [13, Theorem 1.1], [9, Proposition 5.8]) we have that:

Corollary 8.2 *For any $y \in S$, we have on E_y,*

$$e^t \omega(t)|_{E_y} \to \omega_{\text{flat},y}$$

exponentially fast in the $C^1(E_y, g_0)$ topology. Moreover, the convergence is uniform in $y \in S$.

Proof We use an idea from [34, p. 440]. Since $g|_{E_y}$ is uniformly equivalent to $e^{-t}\hat{g}|_{E_y} = e^{-t}g_E$, we conclude that

$$|\nabla_{g_E}(e^t g|_{E_y})|^2_{g_E} = e^{-t}|\nabla_{g_E}(g|_{E_y})|^2_{e^{-t}g_E} \leq Ce^{-t}|\nabla_{\hat{g}|_{E_y}}(g|_{E_y})|^2_g$$
$$\leq Ce^{-t}\mathcal{S} \leq Ce^{-t/3},$$

using Theorem 8.1. But on E_y, $g_{\text{flat},y}$ is a constant multiple of g_E, and so

$$|\nabla_{g_E}(e^t g|_{E_y} - g_{\text{flat},y})|^2_{g_E} \leq Ce^{-t/3}.$$

The rest of the proof follows easily, and exactly as in [28, Lemma 6.9], since $e^t g|_{E_y}$ and $g_{\text{flat},y}$ lie in the same Kähler class on E_y. □

Before we start the proof of Theorem 8.1, we need some preliminary calculations. Denote by $\Psi^i_{jk} = \Gamma^i_{jk} - \hat{\Gamma}^i_{jk}$, the difference of the Christoffel symbols of g and $\hat{g}$. It is a tensor which satisfies $|\Psi|^2_g = \mathcal{S}$. The evolution of $\mathcal{S}$ is computed in [23, (3.4)], generalizing calculations of [4,19,39], which gives

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{S} = \mathcal{S} - |\overline{\nabla}\Psi|^2_g - |\nabla\Psi|^2_g$$
$$+ g^{i\bar{j}}g^{r\bar{s}}g^{a\bar{b}}(\nabla_r\overline{T_{bj\bar{a}}} + \nabla_{\bar{b}}T_{ar\bar{j}})\Psi^k_{ip}\overline{\Psi^\ell_{sq}}g^{p\bar{q}}g_{k\bar{\ell}}$$
$$+ g^{i\bar{j}}g^{r\bar{s}}g^{a\bar{b}}(\nabla_r\overline{T_{bj\bar{a}}} + \nabla_{\bar{b}}T_{ar\bar{j}})\Psi^k_{pi}\overline{\Psi^\ell_{qs}}g^{p\bar{q}}g_{k\bar{\ell}}$$
$$- g^{a\bar{b}}(\nabla_k\overline{T_{bs\bar{a}}} + \nabla_{\bar{b}}T_{ak\bar{s}})\Psi^k_{ip}\overline{\Psi^s_{jq}}g^{i\bar{j}}g^{p\bar{q}}$$
$$- 2\text{Re}[g^{r\bar{s}}(\nabla_i\nabla_p\overline{T_{s\ell\bar{r}}} + \nabla_i\nabla_{\bar{s}}T_{rp\bar{\ell}}$$
$$- T^a_{ir}R_{a\bar{s}p\bar{\ell}} + g_{k\bar{\ell}}\nabla_r\hat{R}_{i\bar{s}p}{}^k)\overline{\Psi^\ell_{jq}}g^{i\bar{j}}g^{p\bar{q}}], \tag{8.1}$$

where $\hat{R}_{i\bar{s}p}{}^k$ is the curvature tensor of $\hat{g}$. We are going to bound each of these terms separately. The key difference from the calculation in [23] is that in our case the torsion $T_{ij\bar{\ell}}$ of g does not equal the torsion of $\hat{g}$ (which here is zero), but rather the torsion $\tilde{T}_{ij\bar{\ell}}$ of $\tilde{g}$.

Therefore we let $\tilde{\Psi}^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$ and $H^i_{jk} = \tilde{\Gamma}^i_{jk} - \hat{\Gamma}^i_{jk} = \Psi^i_{jk} - \tilde{\Psi}^i_{jk}$.

Lemma 8.3 *For all $t \geq T_I$, we have that*

$$|H|_g \leq Ce^{t/2}. \tag{8.2}$$

$$|\bar{\partial}H|_g \leq Ce^{t/2}. \tag{8.3}$$

$$|\tilde{\nabla}H|_g \leq Ce^t. \tag{8.4}$$

Proof At any given point $x \in M$ we can choose local bundle coordinates (z^1, z^2) as before. Then in this coordinate system (4.3) gives

$$|\tilde{\Gamma}^i_{jk}|^2_g \leq Ce^t,$$

where we are using the fact that g and $\tilde{g}$ are uniformly equivalent. Since $\hat{g}$ is a semi-flat product Kähler metric, $\hat{\Gamma}^i_{jk}$ is zero except when $i = j = k = 2$, and so

$$|\hat{\Gamma}^i_{jk}|^2_g \leq C.$$

The bound (8.2) follows immediately from these bounds.

Next, note that

$$\partial_{\bar{\ell}} H^i_{jk} = -\tilde{R}_{j\bar{\ell}k}{}^i + \hat{R}_{j\bar{\ell}k}{}^i.$$

From Lemma 4.1, part (ii), we have that $|\tilde{R}_{j\bar{\ell}k}{}^i|_g \leq Ce^{t/2}$, while the fact that $\hat{g}$ is a semi-flat product Kähler metric implies that the only nonzero component of $\hat{R}$ is $\hat{R}_{2\bar{2}2}{}^2$, and so

$$|\hat{R}_{j\bar{\ell}k}{}^i|_g \leq C, \tag{8.5}$$

from which (8.3) follows.

We have that

$$\tilde{\nabla}_p H^i_{jk} = \partial_p H^i_{jk} - \tilde{\Gamma}^\ell_{pj} H^i_{\ell k} - \tilde{\Gamma}^\ell_{pk} H^i_{j\ell} + \tilde{\Gamma}^i_{p\ell} H^\ell_{jk}.$$

Thanks to (4.3) and (8.2), in these coordinates we can bound the $|\cdot|_g$ norm of the last three terms by Ce^t. As for the first term, we have

$$\partial_p H^i_{jk} = \partial_p \tilde{\Gamma}^i_{jk} - \partial_p \hat{\Gamma}^i_{jk},$$

and

$$\partial_p \hat{\Gamma}^i_{jk} = \hat{g}^{i\bar{\ell}} \partial_p \partial_j \hat{g}_{k\bar{\ell}} - \hat{g}^{i\bar{s}} \hat{g}^{r\bar{\ell}} \partial_p \hat{g}_{r\bar{s}} \partial_j \hat{g}_{k\bar{\ell}},$$

which is zero except when $i = j = k = p = 2$, and so

$$|\partial_p \hat{\Gamma}^i_{jk}|_g \leq C.$$

On the other hand

$$|\partial_p \tilde{\Gamma}^i_{jk}|_g \leq Ce^t,$$

thanks to (4.14). This proves (8.4).   □

We now start bounding the terms in (8.1). We have

$$\nabla_{\bar{\ell}} T_{ik\bar{j}} = \tilde{\nabla}_{\bar{\ell}} \tilde{T}_{ik\bar{j}} - \overline{\Psi^q_{\ell j}} \tilde{T}_{ik\bar{q}} + \overline{H^q_{\ell j}} \tilde{T}_{ik\bar{q}},$$

and so thanks to Lemmas 4.1 and 8.3 we have that

$$|\nabla_{\bar{\ell}} T_{ik\bar{j}}|_g \leq Ce^{t/2} + C\mathcal{S}^{1/2}.$$

Therefore we can bound

$$g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} (\nabla_r \overline{T_{bj\bar{a}}} + \nabla_{\bar{b}} T_{ar\bar{j}}) \Psi^k_{ip} \overline{\Psi^\ell_{sq}} g^{p\bar{q}} g_{k\bar{\ell}}$$
$$+ g^{i\bar{j}} g^{r\bar{s}} g^{a\bar{b}} (\nabla_r \overline{T_{bj\bar{a}}} + \nabla_{\bar{b}} T_{ar\bar{j}}) \Psi^k_{pi} \overline{\Psi^\ell_{qs}} g^{p\bar{q}} g_{k\bar{\ell}}$$
$$- g^{a\bar{b}} (\nabla_k \overline{T_{bs\bar{a}}} + \nabla_{\bar{b}} T_{ak\bar{s}}) \Psi^k_{ip} \overline{\Psi^s_{jq}} g^{i\bar{j}} g^{p\bar{q}} \leq C(e^{t/2} + \mathcal{S}^{1/2})\mathcal{S}. \tag{8.6}$$

Next, we compute

$$\nabla_a \nabla_b \overline{T_{ij\bar{k}}} = \nabla_a (\tilde{\nabla}_b \overline{\tilde{T}_{ij\bar{k}}} - \Psi^r_{bk} \overline{\tilde{T}_{ij\bar{r}}} + H^r_{bk} \overline{\tilde{T}_{ij\bar{r}}})$$
$$= \tilde{\nabla}_a \tilde{\nabla}_b \overline{\tilde{T}_{ij\bar{k}}} - \Psi^r_{ab} \tilde{\nabla}_r \overline{\tilde{T}_{ij\bar{k}}} - \Psi^r_{ak} \tilde{\nabla}_b \overline{\tilde{T}_{ij\bar{r}}}$$
$$+ H^r_{ab} \tilde{\nabla}_r \overline{\tilde{T}_{ij\bar{k}}} + H^r_{ak} \tilde{\nabla}_b \overline{\tilde{T}_{ij\bar{r}}} - (\nabla_a \Psi^r_{bk}) \overline{\tilde{T}_{ij\bar{r}}}$$
$$- \Psi^r_{bk} \tilde{\nabla}_a \overline{\tilde{T}_{ij\bar{r}}} + \Psi^r_{bk} \Psi^s_{ar} \overline{\tilde{T}_{ij\bar{s}}} - \Psi^r_{bk} H^s_{ar} \overline{\tilde{T}_{ij\bar{s}}}$$
$$+ (\tilde{\nabla}_a H^r_{bk}) \overline{\tilde{T}_{ij\bar{r}}} + H^r_{bk} \tilde{\nabla}_a \overline{\tilde{T}_{ij\bar{r}}} - \Psi^s_{ab} H^r_{sk} \overline{\tilde{T}_{ij\bar{r}}}$$
$$- \Psi^s_{ak} H^r_{bs} \overline{\tilde{T}_{ij\bar{r}}} + H^s_{ab} H^r_{sk} \overline{\tilde{T}_{ij\bar{r}}} + H^s_{ak} H^r_{bs} \overline{\tilde{T}_{ij\bar{r}}}.$$

Using again Lemmas 4.1 and 8.3 we can bound

$$|\nabla_a \nabla_b \overline{T_{ij\bar{k}}}|_g \leq C(e^t + e^{t/2}\mathcal{S}^{1/2} + \mathcal{S} + |\nabla\Psi|_g),$$

and so

$$-2\mathrm{Re}(g^{r\bar{s}} (\nabla_i \nabla_p \overline{T_{s\ell\bar{r}}}) \overline{\Psi^\ell_{jq}} g^{i\bar{j}} g^{p\bar{q}}) \leq C(e^t \mathcal{S}^{1/2} + e^{t/2}\mathcal{S} + \mathcal{S}^{3/2} + |\nabla\Psi|_g \mathcal{S}^{1/2}). \tag{8.7}$$

Similarly we have

$$\nabla_a \nabla_{\bar{b}} T_{ij\bar{k}} = \nabla_a (\tilde{\nabla}_{\bar{b}} \tilde{T}_{ij\bar{k}} - \overline{\Psi^r_{bk}} \tilde{T}_{ij\bar{r}} + \overline{H^r_{bk}} \tilde{T}_{ij\bar{r}})$$
$$= \tilde{\nabla}_a \tilde{\nabla}_{\bar{b}} \tilde{T}_{ij\bar{k}} - \Psi^r_{ai} \tilde{\nabla}_{\bar{b}} \tilde{T}_{rj\bar{k}} - \Psi^r_{aj} \tilde{\nabla}_{\bar{b}} \tilde{T}_{ir\bar{k}}$$

$$+ H_{ai}^r \tilde{\nabla}_{\overline{b}} \tilde{T}_{rj\overline{k}} + H_{aj}^r \tilde{\nabla}_{\overline{b}} \tilde{T}_{ir\overline{k}} - (\nabla_a \overline{\Psi_{bk}^r}) \tilde{T}_{ij\overline{r}}$$
$$- \overline{\Psi_{bk}^r} \tilde{\nabla}_a \tilde{T}_{ij\overline{r}} + \overline{\Psi_{bk}^r} \Psi_{ai}^s \tilde{T}_{sj\overline{r}} + \overline{\Psi_{bk}^r} \Psi_{aj}^s \tilde{T}_{is\overline{r}}$$
$$- \overline{\Psi_{bk}^r} H_{ai}^s \tilde{T}_{sj\overline{r}} - \overline{\Psi_{bk}^r} H_{aj}^s \tilde{T}_{is\overline{r}} + (\tilde{\nabla}_a \overline{H_{bk}^r}) \tilde{T}_{ij\overline{r}}$$
$$+ \overline{H_{bk}^r} \tilde{\nabla}_a \tilde{T}_{ij\overline{r}} - \Psi_{ai}^s \overline{H_{bk}^r} \tilde{T}_{sj\overline{r}} - \Psi_{aj}^s \overline{H_{bk}^r} \tilde{T}_{is\overline{r}}$$
$$+ H_{ai}^s \overline{H_{bk}^r} \tilde{T}_{sj\overline{r}} + H_{aj}^s \overline{H_{bk}^r} \tilde{T}_{is\overline{r}}.$$

Using again Lemmas 4.1 and 8.3 we can bound

$$|\nabla_a \nabla_{\overline{b}} T_{ij\overline{k}}|_g \le C(e^t + e^{t/2} \mathcal{S}^{1/2} + \mathcal{S} + |\overline{\nabla}\Psi|_g),$$

and so

$$- 2\mathrm{Re}(g^{r\overline{s}} \nabla_i \nabla_{\overline{s}} T_{rp\overline{\ell}} \overline{\Psi_{jq}^\ell} g^{i\overline{j}} g^{p\overline{q}}) \le C(e^t \mathcal{S}^{1/2} + e^{t/2} \mathcal{S} + \mathcal{S}^{3/2} + |\overline{\nabla}\Psi|_g \mathcal{S}^{1/2}). \quad (8.8)$$

Next, we have

$$\partial_{\overline{p}} \Psi_{qk}^\ell = \hat{R}_{q\overline{p}k}{}^\ell - R_{q\overline{p}k}{}^\ell,$$

and so we can bound

$$2\mathrm{Re}(g^{r\overline{s}} T_{ir}^a R_{a\overline{s}p\overline{\ell}} \overline{\Psi_{jq}^\ell} g^{i\overline{j}} g^{p\overline{q}}) \le C|\overline{\nabla}\Psi|_g \mathcal{S}^{1/2} + C|\widehat{\mathrm{Rm}}|_g \mathcal{S}^{1/2} \le C(1 + |\overline{\nabla}\Psi|_g) \mathcal{S}^{1/2}, \quad (8.9)$$

because of (8.5).

Finally, we have

$$\nabla_r \hat{R}_{i\overline{s}p}{}^k = \hat{\nabla}_r \hat{R}_{i\overline{s}p}{}^k - \Psi_{ri}^\ell \hat{R}_{\ell\overline{s}p}{}^k - \Psi_{rp}^\ell \hat{R}_{i\overline{s}\ell}{}^k + \Psi_{r\ell}^k \hat{R}_{i\overline{s}p}{}^\ell.$$

But $\hat{g}$ is a product of Kähler–Einstein metrics on Riemann surfaces, therefore $\hat{\nabla}_r \hat{R}_{i\overline{s}p}{}^k = 0$. Using (8.5) again, we conclude that

$$- 2\mathrm{Re}(g^{r\overline{s}} g_{k\overline{\ell}} \nabla_r \hat{R}_{i\overline{s}p}{}^k \overline{\Psi_{jq}^\ell} g^{i\overline{j}} g^{p\overline{q}}) \le C\mathcal{S}. \quad (8.10)$$

Putting together (8.1), (8.6), (8.7), (8.8), (8.9) and (8.10), we conclude that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{S} \le C(e^{t/2}\mathcal{S} + e^t \mathcal{S}^{1/2} + \mathcal{S}^{3/2}) - \frac{1}{2}|\overline{\nabla}\Psi|_g^2 - \frac{1}{2}|\nabla\Psi|_g^2. \quad (8.11)$$

Next, we define Kähler metrics on U by $\hat{\omega}_t = e^{-t}\omega_E + \omega_S$. These are uniformly equivalent to ω independent of t thanks to Theorem 5.1. Furthermore, the covariant derivative of $\hat{\omega}_t$ is independent of t and equal to that of $\hat{\omega}$, and we will denote it by $\hat{\nabla}$ as before. The same is true for the curvature of $\hat{\omega}_t$, which equals $\hat{R}_{i\overline{j}k}{}^p$. We use [36, Proposition 3.1] to compute

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{tr}_{\hat{\omega}_t}\omega = -g^{p\bar{j}}g^{i\bar{q}}\hat{g}_t^{k\bar{\ell}}\hat{\nabla}_k g_{i\bar{j}}\hat{\nabla}_{\bar{\ell}}g_{p\bar{q}} - g^{i\bar{j}}\hat{g}_t^{k\bar{\ell}}\hat{g}_t^{p\bar{q}}g_{k\bar{q}}\hat{R}_{i\bar{\ell}p\bar{j}}$$

$$- g^{i\bar{j}}\hat{g}_t^{k\bar{\ell}}\hat{\nabla}_i\overline{\tilde{T}_{j\ell\bar{k}}} - g^{i\bar{j}}\hat{g}_t^{k\bar{\ell}}\hat{\nabla}_{\bar{\ell}}\tilde{T}_{ik\bar{j}} - \mathrm{tr}_{\hat{\omega}_t}\omega + e^{-t}\hat{g}_t^{i\bar{\ell}}\hat{g}_t^{k\bar{j}}g_{i\bar{j}}(g_E)_{k\bar{\ell}}.$$

We have that

$$\hat{\nabla}_i\overline{\tilde{T}_{j\ell\bar{k}}} = \tilde{\nabla}_i\overline{\tilde{T}_{j\ell\bar{k}}} + H_{ik}^p\overline{\tilde{T}_{j\ell\bar{p}}},$$

and so we can use Lemma 4.1 and (8.2) to bound

$$|\hat{\nabla}_i\overline{\tilde{T}_{j\ell\bar{k}}}|_g \le Ce^{t/2}.$$

Hence, making use of (8.5), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\hat{\omega}_t}\omega \le -C^{-1}\mathcal{S} + Ce^{t/2}, \tag{8.12}$$

for a uniform $C > 0$.

We now give the proof of Theorem 8.1, along the lines of [23].

Proof Let K be a large constant such that

$$\frac{K}{2} \le K - \mathrm{tr}_{\hat{\omega}_t}\omega \le K,$$

whose value will be fixed later. Let $0 \le \rho \le 1$ be a smooth nonnegative cutoff function supported in the open ball B in S, which is identically 1 in a smaller neighborhood of y, and denote the pullback $\rho \circ \pi$ also by ρ. Consider the quantity

$$Q = \rho^2 \frac{e^{-2t/3}\mathcal{S}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega} + \mathrm{tr}_{\hat{\omega}_t}\omega \qquad \text{on } \mathrm{supp}(\rho) \subset U.$$

Our goal is to obtain an upper bound for Q, giving the bound $\mathcal{S} \le Ce^{2t/3}$ on a smaller neighborhood of U, which we may assume contains V. We will apply the maximum principle to this function Q, noting that it is equal to $\mathrm{tr}_{\hat{\omega}_t}\omega$, and hence is bounded, on the boundary of $\mathrm{supp}(\rho)$.

We start with the following observations. Since ρ is the pullback of a function from the base S, we have from the estimate $C\omega \ge \omega_S$,

$$|\nabla\rho|_g^2 \le C, \quad |\Delta\rho|_g \le C, \tag{8.13}$$

independent of t. Furthermore, we have the simple inequalities (see [23, (3.9), (3.10)]),

$$|\nabla\mathrm{tr}_{\hat{\omega}_t}\omega|_g^2 \le C\mathcal{S}, \quad |\nabla\mathcal{S}|_g^2 \le 2\mathcal{S}(|\nabla\Psi|_g^2 + |\overline{\nabla}\Psi|_g^2), \tag{8.14}$$

where the first one also follows from the argument for (6.20).

We can now compute

$$
\left(\frac{\partial}{\partial t} - \Delta\right) Q = -\frac{2\rho^2 e^{-2t/3}\mathcal{S}}{3(K - \mathrm{tr}_{\hat{\omega}_t}\omega)} + \frac{\rho^2 e^{-2t/3}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{S}
$$
$$
+ \left(1 + \rho^2 \frac{e^{-2t/3}\mathcal{S}}{(K - \mathrm{tr}_{\hat{\omega}_t}\omega)^2}\right)\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\hat{\omega}_t}\omega
$$
$$
- 2\rho^2 e^{-2t/3}\frac{\mathrm{Re}\langle\nabla\mathcal{S}, \nabla\mathrm{tr}_{\hat{\omega}_t}\omega\rangle_g}{(K - \mathrm{tr}_{\hat{\omega}_t}\omega)^2} - 2\rho^2\frac{e^{-2t/3}\mathcal{S}}{(K - \mathrm{tr}_{\hat{\omega}_t}\omega)^3}|\nabla\mathrm{tr}_{\hat{\omega}_t}\omega|_g^2
$$
$$
- \Delta(\rho^2)\left(\frac{e^{-2t/3}\mathcal{S}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\right) - 4\rho\frac{e^{-2t/3}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\mathrm{Re}\langle\nabla\rho, \nabla\mathcal{S}\rangle_g
$$
$$
- 4\rho\frac{e^{-2t/3}\mathcal{S}}{(K - \mathrm{tr}_{\hat{\omega}_t}\omega)^2}\mathrm{Re}\langle\nabla\rho, \nabla\mathrm{tr}_{\hat{\omega}_t}\omega\rangle_g .
$$

Let (x_0, t_0) be a point in $\mathrm{supp}(\rho)$ at which Q achieves a maximum. We may assume without loss of generality that $t_0 > 0$ and x_0 lies in the interior of $\mathrm{supp}(\rho)$. We have at (x_0, t_0),

$$
2\rho\nabla\rho\frac{e^{-2t/3}\mathcal{S}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega} + \rho^2\frac{e^{-2t/3}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\nabla\mathcal{S} + \rho^2\frac{e^{-2t/3}\mathcal{S}}{(K - \mathrm{tr}_{\hat{\omega}_t}\omega)^2}\nabla\mathrm{tr}_{\hat{\omega}_t}\omega + \nabla\mathrm{tr}_{\hat{\omega}_t}\omega = 0.
$$

Taking the inner product of this with $\nabla\mathrm{tr}_{\hat{\omega}_t}\omega$, we see that, at this point,

$$
0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) Q = -\frac{2\rho^2 e^{-2t/3}\mathcal{S}}{3(K - \mathrm{tr}_{\hat{\omega}_t}\omega)} + \frac{\rho^2 e^{-2t/3}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{S}
$$
$$
+ \left(1 + \rho^2\frac{e^{-2t/3}\mathcal{S}}{(K - \mathrm{tr}_{\hat{\omega}_t}\omega)^2}\right)\left(\frac{\partial}{\partial t} - \Delta\right)\mathrm{tr}_{\hat{\omega}_t}\omega
$$
$$
- \Delta(\rho^2)\left(\frac{e^{-2t/3}\mathcal{S}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\right)
$$
$$
- 4\rho\frac{e^{-2t/3}}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}\mathrm{Re}\langle\nabla\rho, \nabla\mathcal{S}\rangle_g + \frac{2|\nabla\mathrm{tr}_{\hat{\omega}_t}\omega|^2}{K - \mathrm{tr}_{\hat{\omega}_t}\omega}. \tag{8.15}
$$

We use (8.11), (8.12), (8.13), (8.14) to obtain at this point,

$$
0 \leq \frac{2\rho^2 e^{-2t/3}}{K}\left(C(e^{t/2}\mathcal{S} + e^t\mathcal{S}^{1/2} + \mathcal{S}^{3/2}) - \frac{1}{4}|\overline{\nabla}\Psi|_g^2 - \frac{1}{4}|\nabla\Psi|_g^2\right)
$$
$$
+ \left(1 + \frac{4\rho^2 e^{-2t/3}\mathcal{S}}{K^2}\right)\left(-\frac{\mathcal{S}}{C} + Ce^{t/2}\right)
$$
$$
+ \frac{Ce^{-2t/3}\mathcal{S}}{K} + \frac{\rho^2 e^{-2t/3}}{2K}(|\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2) + \frac{C\mathcal{S}}{K},
$$

where we have used the Young inequality and (8.14):

$$4\rho \frac{e^{-2t/3}}{K - \text{tr}_{\hat{\omega}_t}\omega}|\langle \nabla\rho, \nabla\mathcal{S}\rangle_g| \le \frac{Ce^{-2t/3}\mathcal{S}}{K} + \frac{\rho^2 e^{-2t/3}}{4K}\frac{|\nabla\mathcal{S}|^2}{\mathcal{S}}$$

$$\le \frac{Ce^{-2t/3}\mathcal{S}}{K} + \frac{\rho^2 e^{-2t/3}}{2K}(|\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2).$$

Suppose that at (x_0, t_0) we have $e^{-2t/3}\mathcal{S}K^{-1} \ge 1$, and hence $e^t \le \mathcal{S}^{3/2}K^{-3/2}$ (otherwise, $e^{-2t/3}\mathcal{S}$ is bounded and thus so is Q). We may also assume that $\mathcal{S}$ is much larger than K, say $\mathcal{S} \ge K^4$. Then at this point,

$$\frac{\mathcal{S}}{C} + \frac{4\rho^2 e^{-2t/3}\mathcal{S}^2}{CK^2} \le \frac{C\rho^2 e^{-2t/3}}{K}\left(\frac{\mathcal{S}^{7/4}}{K^{3/4}} + \frac{\mathcal{S}^2}{K^{3/2}} + \frac{\mathcal{S}^2}{K^2}\right)$$
$$+ \frac{C\mathcal{S}^{3/4}}{K^{3/4}} + \frac{C\rho^2 e^{-2t/3}\mathcal{S}^{7/4}}{K^{11/4}} + \frac{Ce^{-2t/3}\mathcal{S}}{K} + \frac{C\mathcal{S}}{K}$$

Choosing K to be much larger than C^2, we see that the second term on the left hand side of this inequality dominates all the terms involving ρ^2 on the right hand side. This gives

$$\mathcal{S}\left(\frac{1}{C} - \frac{C}{K^{3/4}\mathcal{S}^{1/4}} - \frac{Ce^{-2t/3}}{K} - \frac{C}{K}\right) \le 0,$$

a contradiction since we chose K to be much larger than C^2. It follows that Q is uniformly bounded from above at the point (x_0, t_0). This completes the proof of the theorem.   □

9 Proofs of the main results

In this section we give the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 The estimate proved in Theorem 7.1 immediately implies that given any $0 < \varepsilon < 1/8$ there exists a constant C such that

$$\|\omega(t) - \tilde{\omega}(t)\|_{C^0(M, g_0)} \le Ce^{-\varepsilon t}.$$

From the definition of $\tilde{\omega}(t) = e^{-t}\omega_{\text{flat}} + (1 - e^{-t})\pi^*\omega_S$ we deduce that

$$\|\omega(t) - \pi^*\omega_S\|_{C^0(M, g_0)} \le Ce^{-\varepsilon t}.$$

The Gromov–Hausdorff convergence of $(M, \omega(t))$ to (S, ω_S) follows from Lemma 9.1 below. Finally, given any $y \in S$, the exponential convergence of $e^t\omega(t)|_{E_y}$ to $\omega_{\text{flat}, y}$ in the $C^1(E_y, g_0)$ topology (uniformly in y), follows from Corollary 8.2 and the compactness of M.   □

We used the following elementary result, which is undoubtedly well-known (cf. [37, Theorem 8.1]).

Lemma 9.1 *Let $\pi : M \to S$ be a fiber bundle, where (M, g_M) and (S, g_S) are closed Riemannian manifolds. If $g(t), t \geq 0$, is a family of Riemannian metrics on M with $\|g(t) - \pi^* g_S\|_{C^0(M, g_M)} \to 0$ as $t \to \infty$, then $(M, g(t))$ converges to (S, g_S) in the Gromov–Hausdorff sense as $t \to \infty$.*

Proof For any $y \in S$ we denote by $E_y = \pi^{-1}(y)$ the fiber over y. Fix $\varepsilon > 0$, denote by L_t the length of a curve in M measured with respect to $g(t)$, and by d_t the induced distance function on M. Similarly we have L_S, d_S on S. Using the standard formulation of Gromov–Hausdorff convergence (see e.g. [37]), let $F = \pi : M \to S$ and define a map $G : S \to M$ by sending every point $y \in S$ to some chosen point in M on the fiber E_y. The map G will in general be discontinuous, and it satisfies $F \circ G = \text{Id}$, so

$$d_S(y, F(G(y))) = 0. \tag{9.1}$$

On the other hand since $g(t)|_{E_y}$ goes to zero, we have that for any t large and for any $x \in M$

$$d_t(x, G(F(x))) \leq \varepsilon. \tag{9.2}$$

Next, given two points $x_1, x_2 \in M$ let $\gamma : [0, L] \to S$ be a unit-speed minimizing geodesic in S joining $F(x_1)$ and $F(x_2)$. Since the bundle π is locally trivial, we can cover the image of γ by finitely many open sets U_j, $1 \leq j \leq N$, such that $\pi^{-1}(U_j)$ is diffeomorphic to $U_j \times E$ (where E is the fiber of the bundle) and there is a subdivision $0 = t_0 < t_1 < \cdots < t_N = L$ of $[0, L]$ such that $\gamma([t_{j-1}, t_j]) \subset U_j$. Fix a point $e \in E$, and use the trivializations to define $\tilde{\gamma}_j(s) = (\gamma(s), e)$, for $s \in [t_{j-1}, t_j]$, which are curves in M with the property that

$$|L_t(\tilde{\gamma}_j) - L_S(\gamma|_{[t_{j-1}, t_j]})| \leq \varepsilon/N,$$

as long as t is sufficiently large (because $g(t) \to \pi^* g_S$). The points $\tilde{\gamma}_j(t_j)$ and $\tilde{\gamma}_{j+1}(t_j)$ lie in the same fiber of π, so we can join them by a curve contained in this fiber with L_t-length at most $\varepsilon/2N$ (for t large). We also join x_1 with $\tilde{\gamma}_1(0)$ and x_2 with $\tilde{\gamma}_N(L)$ in the same fashion. Concatenating these "vertical" curves and the curves $\tilde{\gamma}_j$, we obtain a piecewise smooth curve $\tilde{\gamma}$ in M joining x_1 and x_2, with $\pi(\tilde{\gamma}) = \gamma$ and $|L_t(\tilde{\gamma}) - d_S(F(x_1), F(x_2))| \leq 2\varepsilon$. Therefore,

$$d_t(x_1, x_2) \leq L_t(\tilde{\gamma}) \leq d_S(F(x_1), F(x_2)) + 2\varepsilon. \tag{9.3}$$

Since $F \circ G = \text{Id}$, we also have that for all t large and for all $y_1, y_2 \in S$,

$$d_t(G(y_1), G(y_2)) \leq d_S(y_1, y_2) + 2\varepsilon. \tag{9.4}$$

Given now two points $x_1, x_2 \in M$, let γ be a unit-speed minimizing $g(t)$-geodesic joining them. If we denote by $L_{\pi^* g_S}(\gamma)$ the length of γ using the degenerate metric $\pi^* g_S$, then we have for t large,

$$d_S(F(x_1), F(x_2)) \le L_S(F(\gamma)) = L_{\pi^*g_S}(\gamma) \le L_t(\gamma) + \varepsilon = d_t(x_1, x_2) + \varepsilon, \quad (9.5)$$

where we used again that $g(t) \to \pi^*g_S$. Obviously this also implies that for all t large and for all $y_1, y_2 \in S$,

$$d_S(y_1, y_2) \le d_t(G(y_1), G(y_2)) + \varepsilon. \tag{9.6}$$

Combining (9.1), (9.2), (9.3), (9.4), (9.5) and (9.6) we get the required Gromov–Hausdorff convergence. □

Proof of Corollary 1.2 The proof is similar to [37, Theorem 8.2]. From [3, Lemmas 1, 2] or [38, Theorem 7.4] we see that there is a finite unramified covering $p : M' \to M$ (with deck transformation group Γ) which is also a minimal properly elliptic surface $\pi' : M' \to S'$ and π' is an elliptic fiber bundle with S' a compact Riemann surface of genus at least 2. Furthermore, Γ also acts on S' (so that π' is Γ-equivariant), with finitely many fixed points whose union Z is precisely the image of the multiple fibers of π, with quotient $S = S'/\Gamma$, and so that the quotient map $q : S' \to S$ satisfies $q \circ \pi' = \pi \circ p$.

Denote by $\omega_{S'}$ the orbifold Kähler–Einstein metric on S' with $\mathrm{Ric}(\omega_{S'}) = -\omega_{S'}$. From the description of M and M' as quotients of $H \times \mathbb{C}^*$, where H is the upper half plane in $\mathbb{C}$ (see e.g. [17], [37, Section 8]), it follows that $\pi'^*\omega_{S'}$ is a smooth real $(1, 1)$ form on M', which also equals $p^*\pi^*\omega_S$. Indeed, if we let $z \in H$ be the variable in the upper half plane, $w \in \mathbb{C}^*$, and $y = \mathrm{Im}z$, then from the arguments in [37, Section 8] we see that the form $\pi^*\omega_S$ on M is induced from the form $\frac{1}{2y^2}\sqrt{-1}dz \wedge d\bar{z}$ on $H \times \mathbb{C}^*$, and the exact same formula holds on M'.

Given any Gauduchon metric ω_0 on M, call $\omega(t)$ its evolution under the normalized Chern–Ricci flow on M, as before. Let $\omega'_0 = p^*\omega_0$, which is a Γ-invariant Gauduchon metric on M'. If we call $\omega'(t)$ its evolution under the normalized Chern–Ricci flow on M', then $\omega'(t)$ is also Γ-invariant, and equal to $p^*\omega(t)$. Furthermore, Γ also acts by isometries of the distance function $d_{S'}$ of $\omega_{S'}$, with quotient space (S, d_S), the distance function of the orbifold metric ω_S.

Now Theorem 1.1 applied to the elliptic bundle $\pi' : M' \to S'$ shows that $(M', \omega'(t))$ converges to $(S', \omega_{S'})$ in the Gromov–Hausdorff topology. But exactly as in [37, Theorem 8.2] we see that the convergence happens also in the Γ-equivariant Gromov–Hausdorff topology, and therefore by [10, Theorem 2.1] or [21, Lemma 1.5.4] we conclude that $(M, \omega(t))$ converges to (S, d_S) in the Gromov–Hausdorff topology.

Now we apply Theorem 1.1 again to M' to see that

$$\|\omega'(t) - \pi'^*\omega_{S'}\|_{C^0(M', p^*g_0)} \le Ce^{-\varepsilon t}.$$

Fix now an open set U of M, small enough so that $p^{-1}(U)$ is a disjoint union of finitely many copies U_j of U. Then $p : U_j \to U$ is a biholomorphism for each j and the Γ-action on $p^{-1}(U)$ permutes the U_j's. Therefore for each j, the map $p : U_j \to U$ gives an isometry between $\omega'(t)|_{U_j}$ and $\omega(t)|_U$, and also between $(\pi'^*\omega_{S'})|_{U_j}$ and $(\pi^*\omega_S)|_U$. Fixing one value of j, from $\|\omega'(t) - \pi'^*\omega_{S'}\|_{C^0(U_j, p^*g_0)} \le Ce^{-\varepsilon t}$, we

conclude that

$$\|\omega(t) - \pi^*\omega_S\|_{C^0(U, g_0)} \le Ce^{-\varepsilon t}.$$

Covering M by finitely many such open sets U shows that $\omega(t)$ converges to $\pi^*\omega_S$ in the $C^0(M, g_0)$ topology.

Finally, fix any point $y \in S \backslash Z$, and let V be a small open neighborhood of y such that $\pi^{-1}(V) \cong V \times E$ and $q^{-1}(V)$ is a disjoint union of finitely many copies V_j of V with points $y_j \in V_j$ mapping to y and with $q : V_j \to V$ a biholomorphism. Then $\pi'^{-1}(V_1) \cong V_1 \times E$, and under these identifications the biholomorphism $p : \pi'^{-1}(V_1) \to \pi^{-1}(V)$ equals $(q, \mathrm{Id}) : V_1 \times E \to V \times E$. Under this map the fiber $E'_{y_1} := \pi'^{-1}(y_1)$ is carried to the fiber E_y. Applying Theorem 1.1 to M', we see that $e^t \omega'(t)|_{E'_{y_1}}$ converges exponentially fast in the $C^1(E'_{y_1}, g'_0)$ topology to $\omega'_{\mathrm{flat}, y_1}$, the flat Kähler metric on E'_{y_1} cohomologous to $[\omega'_0|_{E'_{y_1}}]$, and the convergence is uniform when varying y_1. But the local biholomorphism p maps $\omega'(t)$ to $\omega(t)$, and $\omega'_{\mathrm{flat}, y_1}$ to $\omega_{\mathrm{flat}, y}$, and the result follows. $\qquad\square$

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