

Lévy-Khintchin Theorem for best simultaneous Diophantine approximations

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Abstract

We extend two results about the ordinary continued fraction expansion to best simultaneous Diophantine approximations of vectors or matrices. The first result is the Lévy-Khintchin theorem about the almost sure growth rate of the denominators of the convergents. The second result is a theorem of Doeblin and of Bosma, Jager and Wiedijk about the almost sure limit distribution of the sequence of products $q_n d(q_n \theta, \mathbb{Z})$ where the q_n 's are the denominators of the convergents associated with the real number θ by the ordinary continued fraction algorithm. Besides these two main results, we show that when $d \geq 2$, for almost all vectors $\theta \in \mathbb{R}^d$, $\liminf_{n \rightarrow \infty} q_{n+k} d(q_n \theta, \mathbb{Z}^d)^d = 0$ for all positive integers k , where $(q_n)_{n \in \mathbb{N}}$ is the sequence of best approximation denominators of θ .

Résumé

Nous étendons deux résultats sur le développement en fraction continue ordinaire aux meilleures approximations diophantiennes simultanées de vecteurs ou de matrices. Le premier résultat est le théorème de Lévy-Khintchin sur le taux de croissance presque sûr des dénominateurs des réduites. Le second est un théorème de Doeblin et de Bosma, Jager et Wiedijk sur la distribution presque sûre de la suite des produits $q_n d(q_n \theta, \mathbb{Z})$ où les q_n sont les dénominateurs des réduites associées au nombre réel θ par l'algorithme des fractions continues ordinaires. En dehors de ces deux résultats principaux, nous montrons que lorsque $d \geq 2$, pour presque tous les vecteurs $\theta \in \mathbb{R}^d$, $\liminf_{n \rightarrow \infty} q_{n+k} d(q_n \theta, \mathbb{Z}^d)^d = 0$ pour tous les entiers positifs k , où $(q_n)_{n \in \mathbb{N}}$ est la suite des dénominateurs des meilleures approximations de θ .

Keywords – Diophantine approximations, Ergodic theory, diagonal flow

Mots-Clés – approximations diophantienne, theorie ergodique, flot diagonal

MSC-2020 – 11J13, 11J83, 11K55, 37A17

1 Introduction

In 1936, Aleksandr Khintchin showed that there exists a constant γ such that the denominators $(q_n)_{n \geq 0}$ of the convergents of the continued fraction expansions of almost all real numbers θ satisfy

$$\lim_{n \rightarrow \infty} q_n^{1/n} = \gamma$$

(see [29]). Soon afterward, in [38], in the footnote page 289, Paul Lévy gave the explicit value of the constant,

$$\gamma = \exp \frac{\pi^2}{12 \ln 2}.$$

In [17], published in 1940, among many other results about the ordinary continued fraction expansion, Wolfgang Doeblin stated the following result: for almost all real numbers θ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{card}\{0 \leq k < n : q_k d(q_k \theta, \mathbb{Z}) \leq t\} = g(t)$$

for all $t \in [0, 1]$, where

$$g(t) = \int_0^t \frac{1}{2 \ln 2} \frac{1 - |1 - 2s|}{s} ds.$$

Doeblin only sketched the proof of this result and it is difficult to reconstitute a complete proof from his paper. In [27], Iosifescu gave a very interesting account of Doeblin's paper. Doeblin's outstanding article was not immediately noticed, so the above result was first called the Lenstra conjecture, now it is often called the Doeblin-Lenstra conjecture. A complete proof of the Doeblin-Lenstra conjecture was given in 1983 by Wieb Bosma, Hendrik Jager and Freek Wiedijk, see [5]. Later, Jager proved variants of this result, in particular with the quantity $q_{k+1} d(q_k \theta, \mathbb{Z})$ instead of $q_k d(q_k \theta, \mathbb{Z})$ (see [28] for more results).

The aim of our work is to extend to the best simultaneous Diophantine approximations, both the Lévy-Khintchin result and the Jager version of the Doeblin, Bosma, Jager and Wiedijk result. We choose the Jager version because one of the striking difference between one-dimensional and multidimensional Diophantine approximations is that the classic one-dimensional lower bound $q_{k+1} d(q_k \theta, \mathbb{Z}) \geq 1/2$ no longer holds in the multidimensional setting; this explains for example that the only singular vectors are the rational numbers in dimension one.

Let d and c be two positive integers. Suppose \mathbb{R}^d and \mathbb{R}^c are endowed with the standard Euclidean norms $\|\cdot\|_{\mathbb{R}^d}$ and $\|\cdot\|_{\mathbb{R}^c}$. We prove

Theorem 1. *There exists a constant $L_{d,c}$ such that for almost all matrices $\theta \in M_{d,c}(\mathbb{R})$,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|Q_n(\theta)\|_{\mathbb{R}^c} &= L_{d,c}, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \ln(d(\theta Q_n(\theta), \mathbb{Z}^d)) &= -\frac{c}{d} L_{d,c} \end{aligned}$$

where $Q_n(\theta) \in \mathbb{Z}^c$, $n \geq 0$, is the sequence of best Diophantine approximation denominators of θ associated with the norms $\|\cdot\|_{\mathbb{R}^d}$ and $\|\cdot\|_{\mathbb{R}^c}$.

(See Section 2.1 for the definition of best Diophantine approximation denominators).

Theorem 1 is stated for standard Euclidean norms but it holds for all pairs of Euclidean norms on \mathbb{R}^d and \mathbb{R}^c with the same constant $L_{d,c}$ (see Theorem 6.1). We are convinced that Theorem 1 is valid for general norms but we do not know whether the constants $L_{d,c}$ depend on the norms (see Section 10).

For a matrix θ in $M_{d,c}(\mathbb{R})$, let us denote $\beta_n(\theta) = \|Q_{n+1}(\theta)\|_{\mathbb{R}^c}^c d(\theta Q_n(\theta), \mathbb{Z}^d)^d$ and for a real number a , let us denote δ_a the Dirac measure at a .

Theorem 2. 1. *There exists a probability measure $\nu_{d,c}$ on \mathbb{R} such that, for almost all matrices $\theta \in M_{d,c}(\mathbb{R})$, $\nu_{d,c}$ is the weak limit of the sequence of probability measures*

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{\beta_k(\theta)}.$$

2. *The support of the measure $\nu_{d,c}$ is included in a bounded interval, and contains 0 provided that $c + d \geq 3$.*

The Lévy-Khintchin result has already been extended to multidimensional settings. For instance, for almost all θ in \mathbb{R}^d , the denominators $(J_n(\theta))_{n \geq 0}$ of the Jacobi-Perron expansion of θ satisfy $\lim_{n \rightarrow \infty} \frac{1}{n} \ln J_n(\theta) = c_d$ for some constant c_d (see [6]). The common proofs of such results use ergodic theory. The one-dimensional Lévy-Khintchin result can be proven with the Birkhoff ergodic theorem, while the growth rate of the Jacobi-Perron denominators can be derived from the Oseledec multiplicative ergodic theorem. In both cases, the proof depends on the existence of an underlying dynamical system: the Gauss map or the Jacobi-Perron map (see [52] for many examples of these kinds of maps). However, no such map associated with best Diophantine approximations is known when $d + c \geq 3$. One way to circumvent this problem is to use the left action of the diagonal flow

$$g_t = \begin{pmatrix} e^{ct} I_d & 0 \\ 0 & e^{-dt} I_c \end{pmatrix} \in \mathrm{SL}(d + c, \mathbb{R}), t \in \mathbb{R},$$

on the space of unimodular lattices $\mathcal{L}_{d+c} = \mathrm{SL}(d + c, \mathbb{R}) / \mathrm{SL}(d + c, \mathbb{Z})$. This idea can be traced back to Hermite [25] and has been used extensively in Diophantine approximation since the work of Dani [15, 16]; see for instance [1, 19, 24]. In the same vein, the parametric geometry of numbers introduced recently by Schmidt and Summerer [50, 51] and complemented by Roy [46], studies the evolution of the successive minima of a lattice under the action of the diagonal flow. This led to many results on Diophantine exponents; see for instance [39, 47].

In [13], the diagonal flow is used to prove that the sequence of best Diophantine approximation denominators of almost all θ in $M_{d,c}(\mathbb{R})$ satisfies

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \|Q_n(\theta)\|_{\mathbb{R}^c} \leq K_{d,c}$$

for some constant $K_{d,c}$. When $c = 1$, it is also possible to derive this inequality from a theorem of W. M. Schmidt (see [12]).

As in the aforementioned works, the diagonal flow (g_t) is the main tool. However, unlike e.g. [13], we introduce an important new ingredient: a co-dimension one submanifold transverse to the flow. It should be noticed that many works on continued fraction algorithms also use transversals, we mention two such works.

Firstly, the transformation induced on a well-chosen sub-interval by an interval exchange transformation T is an interval exchange transformation \hat{T} defined with the same number of intervals. The Rauzy-Veech continued fraction algorithm is the map $R : T \mapsto \hat{T}$ where T is an interval exchange transformation defined on an interval of length 1 and \hat{T} is renormalized so that it is defined on an interval of length 1 (see [42, 43, 55]). In [55], W. Veech proved that R admits a unique absolutely continuous invariant measure up to a scalar multiple, using a

bimeasureably invertible extension \hat{R} of the map R . In turn, this extension is constructed as the first return map of a “diagonal flow” on a transversal. It should be noticed that at the same time, H. Masur proposed a very similar construction in [40].

Secondly, A. Haas proved analogs of the result of Doebelin, Bosma, Jager and Wiedijk and of the Lévy-Khintchin result in the setting of excursions of geodesic into the neighborhood of a cusp on a finite area hyperbolic 2-orbifold, see [21, 22]. A. Haas used the ergodicity of the geodesic flow and a transversal in the unit tangent bundle that projects on a finite union of geodesic segments of the hyperbolic plane.

In our case, up to a negligible set of flow trajectories, the transversal is the set of unimodular lattices whose first two minima are equal (see Section 3.1). The first return map of the flow in the transversal play the role of an invertible extension of the missing Gauss map. It is crucial to observe that the visiting times of the transversal by the flow are given by a simple formula involving best simultaneous Diophantine approximations; see Lemma 4.9. Making use of the Birkhoff ergodic theorem, this observation leads to a Lévy-Khintchin result in the space of lattices and to a formula close to the Arnoux-Nogueira interpretation of the Lévy’s constant (see [2]):

$$L_{d,c} = \frac{d}{\mu_S(S)} \int_S \tau d\mu_S = \frac{d \times \mu(\mathcal{L}_{d+c})}{\mu_S(S)} \quad (1)$$

where μ is the invariant measure in the space of lattices, μ_S the invariant measure induced by the flow on the transversal S and τ the return time to S , see Theorem 6.1 and Corollary 6.4.

The second step of the proof of Theorem 1 consists in converting an almost sure result in the space of lattices \mathcal{L}_{d+c} into an almost sure result in $M_{d,c}(\mathbb{R})$. To achieve this goal, we prove a general result, Theorem 8.1, which might be of independent interest. At first sight, this result might appear as an easy consequence of the following standard fact: the set of lattices associated with the matrices θ in $M_{d,c}(\mathbb{R})$, is the expanding direction of the flow g_t . However, an example shows that Theorem 8.1 depends on some properties of the transversal; see Section 8.

When $d = 1$ or 2 and $c = 1$, the submanifold S and the measure μ_S can be entirely calculated (see Section 7). When $d = c = 1$, thanks to Siegel formula giving the volume of the modular space $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$, computing Lévy’s constant $L_{1,1} = \ln \gamma$ is easy; it is even possible to determine the first return map to the transversal. It turns out that this first return map is a 2-fold extension of the natural extension of the Gauss map (see Subsection 7.3). However, when $d = 2$ and $c = 1$, the calculation of $L_{2,1}$ leads to a seven-tuple integral and we have only succeeded in reducing it to a triple integral which can be evaluated numerically (see Subsection 7.4).

When $d = c = 1$, the double inequality $\frac{1}{2} \leq q_{n+1} d(q_n \theta, \mathbb{Z}) \leq 1$ shows that the behaviors of the two sequences $(\frac{1}{n} \ln q_n)_n$ and $(\frac{-1}{n} \ln d(q_n \theta, \mathbb{Z}))_n$ are the same and each of the limits in Theorem 1 implies the other. When d is greater than or equal to two, no such double inequality exists. Indeed, it has been proven in [12] that when $c = 1$ and $d \geq 2$,

$$\liminf_{n \rightarrow \infty} q_{n+1} d(q_n \theta, \mathbb{Z}^d)^d = 0$$

for almost all θ in \mathbb{R}^d . Observe that Theorem 2 implies this latter result; it is an immediate consequence of the fact that 0 is in the support of the measure $\nu_{d,c}$. Hopefully for the proof of

Theorem 1, the weaker inequality

$$d(\theta Q_n(\theta), \mathbb{Z}^d) \geq \frac{1}{\|Q_n(\theta)\|_{\mathbb{R}^c}^{c/d} \ln \|Q_n(\theta)\|}$$

which holds almost surely by the convergence part of the Khintchin-Groshev theorem (see [4]), is enough to link both of the limits in Theorem 1.

In Section 9, we extend the aforementioned result of [12] to best simultaneous approximations of matrices. Our proof leads to the stronger result

$$\liminf_{n \rightarrow \infty} \|Q_{n+k}(\theta)\|_{\mathbb{R}^c}^c d(\theta Q_n(\theta), \mathbb{Z}^d)^d = 0$$

for almost all θ in $M_{d,c}(\mathbb{R})$ and all $k \in \mathbb{N}$.

Obviously, if the matrix θ is badly approximable, then

$$\liminf_{n \rightarrow \infty} \|Q_{n+k}(\theta)\|_{\mathbb{R}^c}^c d(\theta Q_n(\theta), \mathbb{Z}^d)^d > 0$$

because by definition, $\liminf_{n \rightarrow \infty} \|Q_n(\theta)\|_{\mathbb{R}^c}^c d(\theta Q_n(\theta), \mathbb{Z}^d)^d > 0$. When $d = 2$ and $c = 1$, we prove that the set of θ such that, $\liminf_{n \rightarrow \infty} q_{n+1} d(q_n \theta, \mathbb{Z})^2 > 0$, is not reduced to the set of badly approximable vectors; see Proposition 9.1.

Overview of the paper. The notations are collected in Section 2. The third section is devoted to best Diophantine approximations, to minimal vectors and to the relations between these two notions. The fourth section is devoted to the definition of the transversal S and to another transversal S' . The key point of this section is the expression of visiting times of the transversals by the flow trajectories with the minimal vectors and the best approximations (Lemmas 4.8 and 4.9). The induced measure on the transversal is defined in Section 5. In Section 6, Theorem 6.1 is an important result of the paper, it is a version of Theorem 1 in the space of lattices. In Section 7, when $c = 1$, a parametrization of the transversal S is described together with a formula for the induced measure. Furthermore, the two cases $d = 1$ and 2 are considered. Section 7 is not necessary for the proof of Theorems 1 and 2. In Section 8, we give the general result, Theorem 8.1, which allows to convert the almost sure convergence in the space of lattices proved in Theorem 6.1, to an almost sure convergence in the space of matrices $M_{d,c}(\mathbb{R})$ needed in Theorems 1 and 2. In Section 9, we study the behavior of the quantities $q_{n+k}^c(\theta) r_n^d(\theta)$ and finish the proof of Theorem 2. At last, we ask a few miscellaneous questions in Section 10.

Historical Note: Lévy's proof does not rely on the ergodic theorem, which was not known for non-invertible maps at that time. A proof of the Birkhoff ergodic theorem for non-invertible maps was given by Frédéric Riesz in 1945 (see [44]) and then a proof of Lévy's theorem using the ergodic theorem was given in [48]. The authors would like to thank Vitaly Bergelson for bringing [48] to their attention.

2 Notation

Let d and c be two positive integers.

2.1 Vectors and distances

For a positive integer n , let $\|\cdot\|_{\mathbb{R}^n}$ denote the usual Euclidean norm on \mathbb{R}^n and let $B_{\mathbb{R}^n}(u, r)$ denote the ball of center u and radius r associated with the norm $\|\cdot\|_{\mathbb{R}^n}$.

Let us denote $\|\cdot\|_{d,c}$ the norm defined on \mathbb{R}^{d+c} by $\|(u, h)\|_{d,c} = \max\{\|u\|_{\mathbb{R}^d}, \|h\|_{\mathbb{R}^c}\}$ and let $B_{d,c}(X, r)$ denote the ball of center X and radius r associated with the norm $\|\cdot\|_{d,c}$.

For $X = (u, h)$ in \mathbb{R}^{d+c} , let $|X|_- = \|h\|_{\mathbb{R}^c}$ denote the *height* of the vector X and let $|X|_+ = \|u\|_{\mathbb{R}^d}$ denote the norm of the projection of X in the horizontal space. We also denote $X_+ = u$ and $X_- = h$ the horizontal and vertical components of X .

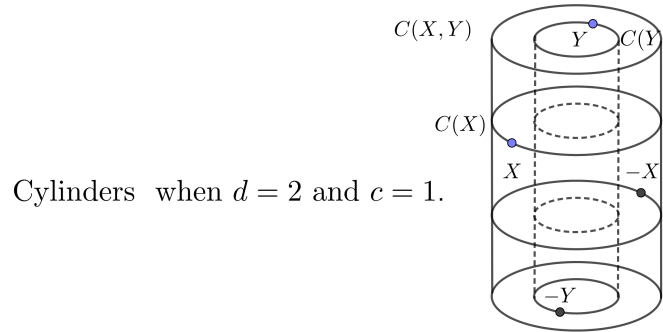
For a vector X in \mathbb{R}^{d+c} , let $C(X)$ denote the closed cylinder in \mathbb{R}^{d+c}

$$C(X) = B_{\mathbb{R}^d}(0, |X|_+) \times B_{\mathbb{R}^c}(0, |X|_-),$$

and if Y is another vector, let $C(X, Y)$ denote the closed cylinder

$$C(X, Y) = B_{\mathbb{R}^d}(0, |X|_+) \times B_{\mathbb{R}^c}(0, |Y|_-).$$

Observe that in the basic case $d = 2$ and $c = 1$, $C(X)$ is a usual cylinder with height $2 \times |X|_-$



and radius $|X|_+$.

In all cases, we denote by $d(x, y)$ the distance associated with the underlying norm between the two points x and y and by $d(x, A)$ the distance between the point x and the set A .

2.2 Matrices

We fix a norm on $M_{d+c}(\mathbb{R})$. All the distances and balls in the space of matrices are associated with this norm. This norm will be used as a tool in the proofs and when the choice of this norm is important we will specify it. When E is a subset of $M_{d+c}(\mathbb{R})$, let $B_E(x, r)$ denote the set of matrices y in E such that $d(x, y) \leq r$.

Let I_n denote the identity matrix in $M_n(\mathbb{R})$. Let

$$g_t = \begin{pmatrix} e^{ct} I_d & 0 \\ 0 & e^{-dt} I_c \end{pmatrix} \in \text{SL}(d+c, \mathbb{R}),$$

$t \in \mathbb{R}$, denote the standard diagonal flow, $E_- = \{0\} \times \mathbb{R}^c$ denote the contracting direction of the flow and $E_+ = \mathbb{R}^d \times \{0\}$ denote the expanding direction of the flow. We also refer to E_+ as the horizontal subspace and to E_- as the vertical subspace.

Let $\mathcal{H}_{>0}$ denote the subgroup of all matrices $M_\theta = \begin{pmatrix} I_d & -\theta \\ 0 & I_c \end{pmatrix}$, $\theta \in M_{d,c}(\mathbb{R})$, and let $\mathbb{T}_{d,c}$ denote its image in the space of unimodular lattices \mathcal{L}_{d+c} . In the same manner, let $\mathcal{H}_{<}$ denote

the subgroup of $\mathrm{SL}(d+c, \mathbb{R})$ of matrices of the form

$$\begin{pmatrix} I_d & 0 \\ B & I_c \end{pmatrix}$$

and let \mathcal{H}_{\leq} denote the subgroup of $\mathrm{SL}(d+c, \mathbb{R})$ of matrices of the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

where $A \in \mathrm{GL}(d, \mathbb{R})$, $B \in M_{c,d}(\mathbb{R})$ and $C \in \mathrm{GL}(c, \mathbb{R})$.

Observe that for any $M \in \mathcal{H}_{<}$, $g_t M g_{-t} \rightarrow I_{d+c}$ when t tends to $+\infty$ and that the distance from $g_t M g_{-t}$ to I_{d+c} is bounded when $t \in \mathbb{R}_+$ for all $M \in \mathcal{H}_{\leq}$.

2.3 Lattices

Let $\mathcal{L} = \mathcal{L}_{d+c}$ denote the space of $(d+c)$ -dimensional unimodular lattices in \mathbb{R}^{d+c} . We identify \mathcal{L}_{d+c} with $\mathrm{SL}(d+c, \mathbb{R})/\mathrm{SL}(d+c, \mathbb{Z})$.

For $\theta \in M_{d,c}(\mathbb{R})$, let Λ_{θ} be the lattice in \mathbb{R}^{d+c} generated by the columns of the matrix M_{θ} , i.e., $\Lambda_{\theta} = M_{\theta} \mathbb{Z}^{d+c}$.

Suppose \mathbb{R}^n is equipped with a norm $\|\cdot\|$. For a lattice Λ and an integer $i \in \{1, \dots, n\}$, let $\lambda_i(\Lambda, \|\cdot\|)$ denote the i -th minimum of the lattice Λ with respect to the norm $\|\cdot\|$, i.e.,

$$\lambda_i(\Lambda, \|\cdot\|) = \inf\{\lambda > 0 : B(0, \lambda) \cap \Lambda \text{ contains at least } i \text{ independent vectors}\}.$$

Observe that $\lambda_1(\Lambda)$ is the length of the shortest nonzero vector in Λ . When there is no ambiguity about the norm, we write $\lambda_i(\Lambda)$ instead of $\lambda_i(\Lambda, \|\cdot\|)$.

We fix a measure μ on \mathcal{L}_{d+c} invariant by the $\mathrm{SL}(d+c, \mathbb{R})$ left action. Recall that μ is unique up to a multiplicative constant.

3 Best approximations

3.1 Best Diophantine approximations

Best approximation vectors have been introduced since a long time inside proofs in a non-explicit form. To our knowledge, C. A. Rogers in 1951 [45] was the first to define best Diophantine approximations. In fact, he defined the best Diophantine approximations associated with the sup norm.

Lagarias was the first to have undertaken a systematic study of the best Diophantine approximations, see [31, 32, 33, 34, 35]. For instance, Lagarias proved that it is not possible to conciliate the unimodularity and the best approximation property (see [34] and [41]). This explains why a multidimensional continued fraction algorithm never satisfies the best approximation property when this algorithm is defined by a piecewise unimodular homographic generalized Gauss map.

The connection between best simultaneous Diophantine approximations and the diagonal flow in the space of lattices goes back at least to Hermite, see [25]. In [31], based on Hermite's idea, Lagarias defined the Minkowski multidimensional continued fraction expansion which

gives all the ‘‘Hermite best approximations’’ to a given $\theta \in \mathbb{R}^d$ (see also [20] for a deep study of the case $d = c = 1$).

For more details and the history of best Diophantine approximations see the surveys [41, 14].

Definition 3.1. *Let $\theta \in M_{d,c}(\mathbb{R})$.*

1. *A nonzero vector $Q \in \mathbb{Z}^c$ is a best simultaneous Diophantine approximation denominator of θ (with respect to the norms $\|\cdot\|_{\mathbb{R}^d}$ and $\|\cdot\|_{\mathbb{R}^c}$), if for all nonzero U in \mathbb{Z}^c ,*

$$\begin{aligned} \|U\|_{\mathbb{R}^c} < \|Q\|_{\mathbb{R}^c} &\Rightarrow d(\theta Q, \mathbb{Z}^d) < d(\theta U, \mathbb{Z}^d) \text{ and} \\ \|U\|_{\mathbb{R}^c} = \|Q\|_{\mathbb{R}^c} &\Rightarrow d(\theta Q, \mathbb{Z}^d) \leq d(\theta U, \mathbb{Z}^d). \end{aligned}$$

2. *An element (P, Q) in $\mathbb{Z}^d \times \mathbb{Z}^c$ is a best Diophantine approximation vector of θ if Q is a best simultaneous Diophantine approximation denominator of θ and if*

$$\|\theta Q - P\|_{\mathbb{R}^d} = d(\theta Q, \mathbb{Z}^d).$$

If $Q = 0$ is the only solution in \mathbb{Z}^d of the equation $\theta Q = 0 \pmod{\mathbb{Z}^d}$, the set of best Diophantine approximation denominators of θ is infinite. Numbering the set of best approximation denominators in ascending order of the norm $q = \|Q\|_c$, we obtain two sequences

$$q_0 = q_0(\theta) = \|Q_0(\theta)\|_c = \lambda_1(\mathbb{Z}^c) < q_1 = q_1(\theta) = \|Q_1(\theta)\|_c < \dots < q_n = q_n(\theta) = \|Q_n(\theta)\|_c < \dots$$

and

$$r_0 = r_0(\theta) = d(\theta Q_0, \mathbb{Z}^d) > r_1 = r_1(\theta) = d(\theta Q_1, \mathbb{Z}^d) > \dots > r_n = r_n(\theta) = d(\theta Q_n, \mathbb{Z}^d) > \dots$$

When $d = c = 1$, by the best approximation property of the ordinary continued fraction expansion (see [23] Theorem 182), the integers $q_0, q_1, \dots, q_n, \dots$ are the denominators of the convergents of the ordinary continued fraction expansion of θ . The only slight difference is that in the ordinary continued fraction expansion, it can happen that $q_0 = q_1 = 1$. Unless $d = c = 1$, the sequence $(V_n = (P_n, Q_n))_n$ of best approximation vectors does not satisfy a linear recurrence relation of the form $V_n = a_{n,1}V_{n-1} + \dots + a_{n,d+c}V_{n-d-c}$, but satisfies weak forms of well-known properties of the one-dimensional continued fraction expansion, see Lemmas 3.6 and 3.7 below.

3.2 Minimal vectors in lattices

The notion of minimal vector goes back to Voronoi. He used minimal vectors to find units in cubic fields (see [56] and also [7, 8]). Minimal vectors are a key ingredient of the proofs of Theorems 1 and 2, they allow to convert statements about best simultaneous Diophantine approximations of vectors or matrices into statements about lattices.

Definition 3.2. *Let $M \in \text{SL}(d+c, \mathbb{R})$ and let $\Lambda = M\mathbb{Z}^{d+c} \in \mathcal{L}_{d+c}$. A nonzero vector $X \in \Lambda$ is a minimal vector in Λ (with respect to the norms $\|\cdot\|_{\mathbb{R}^d}$ and $\|\cdot\|_{\mathbb{R}^c}$) if the only nonzero vectors $Y \in \Lambda$ in the cylinder $C(X)$ are such that $C(X) = C(Y)$, i.e., such that*

$$|X|_+ = |Y|_+, \quad |X|_- = |Y|_-.$$

Two minimal vectors X and Y are equivalent if they define the same cylinder.

Remark 3.3. *The connection between minimal vectors and shortest vectors of a lattice in \mathbb{R}^{d+c} with respect to the norm $\|\cdot\|_{d,c}$ is simple: in a given lattice there is at least one shortest vector of the lattice that is minimal, and if $X = (u, h)$ is minimal in the lattice, then $g_t X$ is a shortest vector of the lattice $g_t \Lambda$ with $t(X) = \frac{1}{d+c} \ln \frac{\|h\|_{\mathbb{R}^c}}{\|u\|_{\mathbb{R}^d}}$ (if $u = 0$ or $h = 0$, it is to understand that X is a shortest vector of $g_t \Lambda$ when t is in a neighborhood of $+\infty$ or $-\infty$). Moreover if a vector $X = (u, h)$ is a “robust” shortest vector, i.e., if $g_t X$ is a shortest vector in $g_t \Lambda$ for all t in a neighborhood of $t(X)$, then X is a minimal vector.*

Remark 3.4. *The connection between minimal vectors and shortest vectors of a lattice in \mathbb{R}^{d+c} with respect to the standard Euclidean norm is not that simple: in a given lattice, if $g_t X$ is a shortest vector in the lattice $g_t \Lambda$ for some t , then X is a minimal vector in Λ . However some minimal vectors are not of this shape even when $d = c = 1$, this was already observed by Humbert in 1916 (see [26] and [31]).*

The connection between minimal vectors and best Diophantine approximations is given by:

Lemma 3.5. *([9, 14]) For all θ in $M_{d,c}(\mathbb{R})$, a vector $X = M_\theta V^T$ where $V = (P, Q) \in \mathbb{Z}^{d+c}$, is a minimal vector with nonzero height in the lattice Λ_θ iff V is a best approximation vector of θ such that*

$$\|\theta Q - P\|_{\mathbb{R}^d} < d(0, \mathbb{Z}^d \setminus \{0\}).$$

Proof. Let $V = (P, Q)$ be a non zero vector in \mathbb{Z}^{d+c} and $X = M_\theta V^T$. Suppose that V is a best approximation vector of θ such that $\|\theta Q - P\|_{\mathbb{R}^d} < d(0, \mathbb{Z}^d \setminus \{0\})$. Let $W = (u, h)$ be a non zero vector in \mathbb{Z}^{d+c} . If $Y = M_\theta W^T$ is in $C(X)$ then $\|h\|_{\mathbb{R}^c} \leq \|Q\|_{\mathbb{R}^c}$ and $\|\theta h - u\|_{\mathbb{R}^d} \leq \|\theta Q - P\|_{\mathbb{R}^d}$. The latter inequality implies $h \neq 0$. Since V is a best approximation vector the two previous inequalities must be two equalities which shows X is minimal with non zero height. Conversely suppose that X is a minimal vector in Λ_θ with non zero height. If a vector $W = (h, u)$ in \mathbb{Z}^{d+c} is such that $\|h\|_{\mathbb{R}^c} < \text{or } \leq \|Q\|_{\mathbb{R}^c}$, then by minimality, $\|\theta h - u\|_{\mathbb{R}^d} > \text{or } \geq \|\theta Q - P\|_{\mathbb{R}^d}$ which shows that V is a best approximation vector of θ . Moreover, since X is minimal with nonzero height, $\|\theta Q - P\|_{\mathbb{R}^d} < d(0, \mathbb{Z}^d \setminus \{0\})$. \square

3.2.1 The sequence of minimal vectors

Given a lattice Λ in \mathcal{L}_{d+c} , we select one minimal vector in each equivalent class of minimal vectors. We number these vectors in ascending order of heights. Such a numbering exists because 0 is the only possible limit point of the set of heights of minimal vectors (see the proof of Lemma 3.7). We obtain a sequence

$$\dots X_{-n}(\Lambda), \dots, X_{-1}(\Lambda), X_0(\Lambda), X_1(\Lambda), \dots$$

This sequence might be finite, infinite one-sided or two-sided. The sequence $(|X_n(\Lambda)|_+)_n$ is decreasing while the sequence $(|X_n(\Lambda)|_-)_n$ is increasing. Such a numbering is not unique but two such numberings differ only by a constant. Although this is not essential, we will set such a numbering in the Section 4.6 about return times.

We shall use the following notations

$$q_n(\Lambda) = |X_n(\Lambda)|_- \quad \text{and} \quad r_n(\Lambda) = |X_n(\Lambda)|_+.$$

Lemma 3.5 above is very important. It shows that for $\theta \in M_{d,c}(\mathbb{R})$, the sequences $(q_n(\Lambda_\theta))_n$ and $(q_n(\theta))_n$ are deduced one another by a shift. Therefore, if one of the two limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\theta), \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\Lambda_\theta)$$

exists, then the other exists and have the same value. The same results holds with the sequences $(r_n(\theta))_n$ and $(r_n(\Lambda_\theta))_n$.

The classical inequality

$$q_{n+1}r_n \leq 1$$

which holds for the one-dimensional continued fraction expansion of a real number, can be extended to minimal vectors of lattices or to best approximation vectors. This fact is well known but it is worth stating it.

Lemma 3.6. *There is a constant $C_{d,c}$ depending only on c and d such that for all lattice $\Lambda \in \mathcal{L}_{d+c}$, all matrices $\theta \in M_{d,c}$ and all integers n , we have*

$$q_{n+1}^c(\Lambda)r_n^d(\Lambda) \leq C_{d,c} \text{ and } q_{n+1}^c(\theta)r_n^d(\theta) \leq C_{d,c}.$$

Proof. Just use the Minkowski convex body theorem with the cylinder defined by two consecutive minimal vectors or best approximation vectors. \square

The classical inequality

$$q_{n+2} \geq 2q_n$$

which holds for the denominators of the one-dimensional continued fraction expansion of a real number, can be extended to minimal vectors of lattices. This inequality has already been extended to best simultaneous Diophantine approximations, see [33], [34] and [14]. The extension to minimal vectors of lattices is straightforward.

Lemma 3.7. *There is a positive integer constant $A = A(d, c)$ such that for any Λ in \mathcal{L}_{d+c} and any $n \in \mathbb{Z}$, if $X_n(\Lambda)$, $X_{n+1}(\Lambda)$, ..., $X_{n+A}(\Lambda)$ exist, then*

$$\begin{aligned} q_{n+A}(\Lambda) &\geq 2q_n(\Lambda), \\ r_{n+A}(\Lambda) &\leq \frac{1}{2}r_n(\Lambda). \end{aligned}$$

Proof. Let A be an integer constant such that if A points $(x_1, y_1), \dots, (x_A, y_A)$ are in the product of balls $B_{\mathbb{R}^d}(0, r_1) \times B_{\mathbb{R}^c}(0, r_2)$ with $r_1, r_2 \geq 0$ then there exist two indices $i \neq j$ such that $\|x_i - x_j\|_{\mathbb{R}^d} \leq \frac{1}{2}r_1$ and $\|y_i - y_j\|_{\mathbb{R}^c} \leq \frac{1}{2}r_2$. With this choice of the constant A , if $k \geq A$ is a positive integer such that $q_{n+k}(\Lambda) \leq 2q_n(\Lambda)$, then there are two integers $0 \leq i < j \leq k$ such that the vector $X_{n+j}(\Lambda) - X_{n+i}(\Lambda)$ satisfies both conditions

$$\begin{cases} |X_{n+j}(\Lambda) - X_{n+i}(\Lambda)|_+ \leq \frac{1}{2}r_n(\Lambda) \\ |X_{n+j}(\Lambda) - X_{n+i}(\Lambda)|_- \leq \frac{1}{2}2q_n(\Lambda) \end{cases}$$

which contradicts the definition of $X_n(\Lambda)$. The same way of reasoning leads to the other inequality. \square

We shall use several times the following very simple lemma which is a consequence of the following observation. For any minimal vector X of a lattice Λ in \mathbb{R}^{d+c} and any $t \in \mathbb{R}$, $g_t X$ is a minimal vector of the lattice $g_t \Lambda$. It follows that

Lemma 3.8. *Let Λ be in \mathcal{L}_{d+c} and let t be in \mathbb{R} . The sequence of minimal vectors of the lattice $g_t \Lambda$ is $(g_t(X_n(\Lambda)))_n$.*

4 The transversals S and S'

We assume that \mathbb{R}^{d+c} is endowed with the norm

$$\|(x, y)\|_{d,c} = \max\{\|x\|_{\mathbb{R}^d}, \|y\|_{\mathbb{R}^c}\}.$$

The main idea of the proofs of Theorems 1 and 2 is to induce the flow g_t on the co-dimension one subset

$$S = \{\Lambda \in \mathcal{L}_{d+c} : \lambda_2(\Lambda) = \lambda_1(\Lambda)\}$$

where $\lambda_1(\Lambda)$ and $\lambda_2(\Lambda)$ are the first two minima of the lattice Λ associated with the above norm. The reason for introducing S is that for a generic lattice Λ , one can use the flow to transform any cylinder $C(X, Y)$ defined by two consecutive minimal vectors X and Y , into a cylinder with equal horizontal and vertical sizes which brings the lattice Λ into S (see Subsection 4.6). And then, to use the Birkhoff ergodic theorem with the first return map associated with the flow g_t . For technical reason it is better to slightly restrict the definition of S . For instance the above set is not a submanifold of \mathcal{L}_{d+c} . It could have some ‘‘branching points’’ while a slightly smaller set is clearly a submanifold, see Lemma 4.3. It will be convenient to use two transversals S and S' for the proof of Theorem 1. These two transversals are very similar; we state all the results we need for both transversals but we only perform the proofs for the first transversal S .

4.1 Definition of S

The transversal $S = S_{d,c}$ is the set of lattices Λ in \mathcal{L}_{d+c} such that there exist two independent vectors $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ in Λ such that:

- $|v_1^S(\Lambda)|_+$ and $|v_0^S(\Lambda)|_-$ are $< |v_1^S(\Lambda)|_- = |v_0^S(\Lambda)|_+$,
- the only nonzero points of Λ in the ball $B_{d,c}(0, \lambda_1(\Lambda))$ are $\pm v_0^S(\Lambda)$ and $\pm v_1^S(\Lambda)$.

Observe that for Λ in S , $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ are unique up to sign and are consecutive minimal vectors of Λ .

Since $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ are linearly independent and are both in the ball $B_{d,c}(0, \lambda_1(\Lambda))$, S is included in the set

$$\{\Lambda \in \mathcal{L}_{d+c} : \lambda_1(\Lambda) = \lambda_2(\Lambda)\}.$$

4.2 Definition of S'

The transversal S' is the set of lattices in \mathcal{L}_{d+c} such that there exists a vector $w_0^{S'}(\Lambda)$ in Λ such that:

- the only nonzero points of Λ in the ball $B_{d,c}(0, \lambda_1(\Lambda))$ are $\pm w_0^{S'}(\Lambda)$,
- the ball $B_{d,c}(0, \lambda_1(\Lambda))$ is equal to the cylinder $C(w_0^{S'}(\Lambda))$, i.e., $|w_0^{S'}|_+ = |w_0^{S'}|_-$.

Observe that $w_0^{S'}(\Lambda)$ is unique up to sign and is a minimal vector of Λ .

We gather some technical results in the next three subsections. Then, the subsection 4.6 about the visiting times and the subsection 4.7 about two functions defined on S , explain the relations between the transversal S and the sequences of minimal vectors of a lattice.

4.3 Lattices bases and minima

We shall need the following results about lattices. Recall that a sub-lattice Γ of a lattice Λ is primitive if $\Gamma_{\mathbb{R}} \cap \Lambda = \Gamma$ where $\Gamma_{\mathbb{R}}$ is the vector subspace generated by Γ .

Lemma 4.1. *Suppose \mathbb{R}^n is equipped with any norm $\|\cdot\|$. Let Λ be a lattice in \mathbb{R}^n and let v_1, v_2 be two independent vectors of Λ such that $\|v_1\| = \lambda_1(\Lambda)$ and $\|v_2\| = \lambda_2(\Lambda)$. Then $\mathbb{Z}v_1 + \mathbb{Z}v_2$ is a primitive sub-lattice of Λ unless $\frac{1}{2}(v_1 + v_2) \in \Lambda$ and $\|v_1\| = \|v_2\| = \|\frac{1}{2}(v_1 + v_2)\|$.*

Proof. Consider the parallelogram \mathcal{P} defined by the vectors v_1 and v_2 . Let v be an element of Λ that belongs to the interior of \mathcal{P} . If v is not in the segment joining v_1 and v_2 then the distance from v to either 0 or $v_1 + v_2$ is of the form $\|t_1v_1 + t_2v_2\|$ for some positive real numbers t_1 and t_2 with $t_1 + t_2 < 1$. Hence this distance is $\leq t_1\|v_1\| + t_2\|v_2\| < \lambda_2(\Lambda)$ which contradicts the definition of $\lambda_2(\Lambda)$. If v is in the segment joining v_1 and v_2 but is not the point $\frac{1}{2}(v_1 + v_2)$ then the distance from v to either v_1 or v_2 is of the form $\|t(v_1 - v_2)\|$ with $t < \frac{1}{2}$ which implies that this distance is $< \lambda_2(\Lambda)$, again a contradiction. If $v = \frac{1}{2}(v_1 + v_2)$, we have $\|v\| \leq \frac{1}{2}(\|v_1\| + \|v_2\|)$ which is $< \lambda_2(\Lambda)$ unless $\|v_1\| = \|v_2\| = \|v\|$. \square

It follows that when the norm is strictly convex, the sub-lattice $\mathbb{Z}v_1 + \mathbb{Z}v_2$ is always primitive. In our setting despite that the norm is not strictly convex it is possible to use the above lemma. With our choice of the norm on \mathbb{R}^{d+c} , the triangle inequality is strict for two vectors one inside the ‘‘top’’ of the cylinder $B_{d,c}(0, r)$ and one inside the lateral side of $B_{d,c}(0, r)$. Therefore,

Corollary 4.2. *Let Λ be in S . Then the vectors $v_0^S(\Lambda)$ and $v_1^S(\Lambda)$ associated with Λ are the first two vectors of a basis of Λ .*

4.4 Geometric properties of S and S'

Lemma 4.3. *S and S' are submanifolds of \mathcal{L}_{d+c} of dimension $(d+c)^2 - 2$, transverse to the diagonal flow g_t .*

Proof. Let Λ_0 be in S and call $v_0^S(\Lambda_0)$ and $v_1^S(\Lambda_0)$ the two vectors provided by the definition of S . By Corollary 4.2, $v_0^S(\Lambda_0)$ and $v_1^S(\Lambda_0)$ are the first two vectors of a basis (b_1, \dots, b_{d+c}) of Λ_0 . We can find a small enough positive real number ε such that for any (v_1, \dots, v_{d+c}) in the open set

$$W = B_{\mathbb{R}^{d+c}}(b_1, \varepsilon) \times \dots \times B_{\mathbb{R}^{d+c}}(b_{d+c}, \varepsilon),$$

- the matrix $M = M(v_1, \dots, v_{d+c})$, the columns of which are the v_i , is in $\text{GL}(d+c, \mathbb{R})$ and the sets $WP, P \in \text{SL}(d+c, \mathbb{Z})$ are disjoint,
- the vectors $\pm v_1$ and $\pm v_2$ are the only nonzero vectors of the lattice $\Lambda = M\mathbb{Z}^{d+c}$ in the cylinder $C(v_1, v_2)$,
- $\|v\|_{d,c} > \|v_1\|_{d,c}$ and $\|v_2\|_{d,c}$ for all v in $\Lambda \setminus \{0, \pm v_1, \pm v_2\}$,
- $|v_1|_+ > |v_1|_-$ and $|v_2|_- > |v_2|_+$.

Consider the map

$$\begin{aligned} f : W &\rightarrow \mathbb{R}^2 \\ &: M = (v_1, \dots, v_{d+c}) \rightarrow (f_1(M) = \det M, f_2(M) = |v_1|_+^2 - |v_2|_-^2). \end{aligned}$$

Then a lattice $\Lambda = M\mathbb{Z}^{d+1}$ with $M \in W$, is in S iff $f(M) = (1, 0)$. To prove that S is a submanifold, it is enough to show that the differential $Df(M)$ is onto at every point M in W . The differential of f_2 is given by

$$Df_2(M).(w_1, \dots, w_{d+c}) = 2v_1^+.w_1^+ - 2v_2^-.w_2^-.$$

The linear map $Df_2(M)$ depends only on w_1 and w_2 and since $v_1^+ \neq 0$ for all M in W , $Df_2(M)$ is never the zero map. The differential of f_1 is given by

$$Df_1(M).(w_1, \dots, w_{d+c}) = \sum_{i,j=1}^{d+c} (-1)^{i+j} \Delta_{i,j} w_{i,j}$$

where $w_j = (w_{i,j})_{i=1, \dots, d+c}$, $j = 1, \dots, d+c$ and $\Delta_{i,j}$ is the (i, j) -minor of the matrix M . Since $\det M \neq 0$, one at least of the minors $\Delta_{i,3}$, $i \leq d+c$, is not zero. Therefore the linear $Df_1(M)$ is not zero and depends on w_3 . It follows that the two linear maps $Df_1(M)$ and $Df_2(M)$ are linearly independent for all M in W which implies that S is a submanifold of \mathcal{L}_{d+c} . To show that the flow is transverse to S , we have to check that for a matrix $M = M(v_1, \dots, v_{d+c})$ in W such that $f(M) = 0$ we have $Df(M).(w_1, \dots, w_{d+c}) \neq 0$ when $w_i = (cv_i^+, -dv_i^-)$. Now, for such w_i , $Df_2(M).(w_1, \dots, w_{d+c}) = 2c|v_1^+|^2 + 2d|v_2^-|^2 > 0$, hence $Df(M).(w_1, \dots, w_{d+c})$ is not zero. \square

4.5 Negligible sets

An important ingredient of the proof of Theorem 1 is that, for a given lattice Λ , the visiting times t , i.e. the times t such that $g_t\Lambda \in S$, can be read from the sequence $(X_n(\Lambda))_n$ of minimal vectors. However, this reading is straightforward only for generic lattices, a small subset of lattices has to be avoided.

4.5.1 A negligible set \mathcal{N} in the space of lattices

Let $\mathcal{N} = \mathcal{N}_{d,c}$ be the set of lattices Λ in \mathcal{L}_{d+c} such that either

- there exist two vectors v_1 , and v_2 , such that $v_1 \neq \pm v_2$ and $|v_1|_+ = |v_2|_+ > 0$ or $|v_1|_- = |v_2|_- > 0$,
- or there exists a nonzero vector in Λ lying in the vertical subspace $\{0\} \times \mathbb{R}^c$, or in the horizontal subspace $\mathbb{R}^d \times \{0\}$.

Remark 4.4. Any lattice $\Lambda_\theta = M_\theta\mathbb{Z}^{d+c}$ (see Section 2.3) is in \mathcal{N} because it contains $\mathbb{Z}^d \times \{0\}$.

Recall that we have fixed a measure μ on \mathcal{L}_{d+c} invariant by the $\mathrm{SL}(d+c, \mathbb{R})$ action.

Lemma 4.5. \mathcal{N} is μ -negligible and g_t invariant.

Proof. Clearly \mathcal{N} is g_t invariant and the set of lattices with a nonzero vector in the vertical subspace or in the horizontal subspace is negligible. So we are reduced to prove that if $X \neq \pm Y$ are two nonzero vectors in \mathbb{Z}^{d+c} the set of matrices M in $\mathrm{SL}(d+c, \mathbb{R})$ satisfying one of the equations

$$|MX|_+^2 - |MY|_+^2 = 0$$

or

$$|MX|_-^2 - |MY|_-^2 = 0$$

is of zero measure. Firstly, by symmetry, it is enough to deal with one of the equations, say the first one. Secondly, by homogeneity it is equivalent to prove that the set of matrices in $M_{d+c}(\mathbb{R})$ that satisfy this equation is of zero measure. Since this is an algebraic equation, it is enough to prove that there exists at least one matrix M such that $|MX|_+^2 - |MY|_+^2 \neq 0$. If X and Y are proportional just choose a matrix M such that $|MX|_+ \neq 0$. Otherwise, first choose a vector Z in the subspace spanned by X and Y that is orthogonal to X . Observe that $Z \cdot Y \neq 0$. Next choose a d -dimensional subspace V of \mathbb{R}^{d+c} containing Z and orthogonal to X . A matrix M , the first d rows of which are a basis of V , is such that $|MX|_+ = 0$ and $|MY|_+ \neq 0$. \square

Remark 4.6. *A lattice Λ that is not in \mathcal{N} has a bi-infinite sequence of minimal vectors and is in S iff $\lambda_1(\Lambda) = \lambda_2(\Lambda)$. In fact if u and v are two linearly independent shortest vectors in Λ then $|u|_+ \neq |v|_+$ and $|u|_- \neq |v|_-$. Therefore $\Lambda \in S$.*

4.5.2 A negligible set \mathcal{M} in the space of matrices $M_{d,c}(\mathbb{R})$

Let C be a positive real constant and let $\mathcal{M} = \mathcal{M}_{d,c} = \mathcal{M}_{d,c}(C)$ be the set of matrices $\theta \in M_{d,c}(\mathbb{R})$ such that either

- there exist two nonzero vectors $X \neq \pm Y$ in \mathbb{Z}^{d+c} with nonzero heights such that $|M_\theta X|_+ = |M_\theta Y|_+$
- or there exist infinitely many pairs $X \neq \pm Y$ in \mathbb{Z}^{d+c} such that $|X|_- = |Y|_- \neq 0$ and $|M_\theta X|_+, |M_\theta Y|_+ \leq C |X|_-^{-\frac{c}{d}}$.

Observe that the second item only occurs when $c > 1$. The set \mathcal{M} depends on the constant C . Actually, we will only use the value $C = C_{d,c}$ where $C_{d,c}$ is given by Lemma 3.6.

Lemma 4.7. *\mathcal{M} is negligible with respect to the Lebesgue measure on $M_{d,c}(\mathbb{R})$.*

Proof. We prove that \mathcal{M} is included in a countable union of negligible sets.

Given $X \neq \pm Y$ two nonzero vectors in \mathbb{Z}^{d+c} with nonzero heights, consider the set $\mathcal{M}(X, Y)$ of matrices $\theta \in M_{d,c}(\mathbb{R})$ such that $|M_\theta X|_+ = |M_\theta Y|_+$. In order to show that $\mathcal{M}(X, Y)$ has zero measure it is enough to show that the polynomial

$$\begin{aligned} f(\theta) &= |M_\theta X|_+^2 - |M_\theta Y|_+^2 \\ &= \|X_+\|_{\mathbb{R}^d}^2 - \|Y_+\|_{\mathbb{R}^d}^2 - 2(X_+ \cdot \theta X_- - Y_+ \cdot \theta Y_-) + \|\theta X_-\|_{\mathbb{R}^d}^2 - \|\theta Y_-\|_{\mathbb{R}^d}^2 \end{aligned}$$

is not the zero polynomial.

If $X_- \neq \pm Y_-$, we can choose θ_0 such that $\|\theta_0 X_-\|_{\mathbb{R}^d}^2 - \|\theta_0 Y_-\|_{\mathbb{R}^d}^2 \neq 0$. With this choice, the one variable polynomial $P(t) = f(t\theta_0)$ has a nonzero degree two monomial which implies that the polynomial f is not the zero polynomial.

If $X_- = Y_-$ (the case $X_- = -Y_-$ is similar), $f(\theta) = |X|_+^2 - |Y|_+^2 - 2(X_+ - Y_+) \cdot \theta X_-$. Since $X_- \neq 0$, the map $\varphi : \theta \in M_{d,c}(\mathbb{R}) \rightarrow \theta X_- \in \mathbb{R}^d$ is onto. It follows that we can choose θ such that $\theta X_- = X_+ - Y_+$. With this value of θ , we obtain $f(\theta) - f(-\theta) = -4|X_+ - Y_+|^2 \neq 0$ which implies that the polynomial f is not the zero polynomial. It follows that $\mathcal{M}(X, Y)$ is

negligible.

Consider now, for a positive integer n , the set \mathcal{M}_n of matrices $\theta \in M_{d,c}(\mathbb{R})$ such that there is a pair of linearly independent vectors (X, Y) in $\mathbb{Z}^{d+c} \times \mathbb{Z}^{d+c}$ such that $n \leq |X|_-, |Y|_- < n+1$ and

$$|M_\theta X|_+, |M_\theta Y|_+ \leq C |X|_-^{-\frac{c}{d}}.$$

We want to prove that the set of matrices θ that are in infinitely many \mathcal{M}_n is negligible. We can move into the space $M_{d,c}(\mathbb{R}/\mathbb{Z})$ and consider instead the set \mathcal{T}_n of $\theta \in M_{d,c}(\mathbb{R}/\mathbb{Z})$ such that there exist $q, q' \in \mathbb{Z}^c$ linearly independent with $n \leq \|q\|_{\mathbb{R}^c}, \|q'\|_{\mathbb{R}^c} < n+1$ and

$$d(\theta q, \mathbb{Z}^d), d(\theta q', \mathbb{Z}^d) \leq C n^{-\frac{c}{d}}.$$

For q fixed, the measure of the set of $\theta \in M_{d,c}(\mathbb{R}/\mathbb{Z})$ such that $d(\theta q, \mathbb{Z}^d) \leq C n^{-\frac{c}{d}}$ is $a_{d,c} n^{-c}$ where the constant $a_{d,c}$ depends only on C and the dimensions d and c . When the inequality holds simultaneously for two linearly independent integer vectors q and q' , the measure is bounded above by the square of $a_{d,c} n^{-c}$. It follows that the measure of \mathcal{T}_n is bounded above by

$$u_n = \text{card}\{(q, q') \in \mathbb{Z}^c \times \mathbb{Z}^c : n \leq \|q\|_{\mathbb{R}^c}, \|q'\|_{\mathbb{R}^c} < n+1\} \times a_{d,c}^2 n^{-2c}.$$

By Borel-Cantelli, it is enough to prove that $\sum_n u_n < \infty$. Now, $\text{card}\{q \in \mathbb{Z}^c : n \leq \|q\|_{\mathbb{R}^c} < n+1\} \ll n^{c-1}$, hence

$$u_n \ll n^{-2}$$

and we are done. □

4.6 Visiting times and return maps

Let Λ be in \mathcal{L}_{d+c} . By definition of S , when $g_t \Lambda$ is in S , $v_0^S(g_t \Lambda)$ and $v_1^S(g_t \Lambda)$ are two consecutive minimal vectors of $g_t \Lambda$. Therefore,

$$\begin{cases} v_0^S(g_t \Lambda) = g_t X_k(\Lambda) \\ v_1^S(g_t \Lambda) = g_t X_{k+1}(\Lambda) \end{cases}$$

for an integer k . Hence $e^{ct} |X_k(\Lambda)|_+ = e^{-dt} |X_{k+1}(\Lambda)|_-$ which implies

$$t = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)}.$$

It follows that the set of real numbers t such that $g_t \Lambda \in S$ is included in the discrete set

$$V_\Lambda(S) = \{t_k = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)} : k \in \mathbb{Z}\}.$$

It can happen that some values t_k are skipped, but in that case, Λ must be in \mathcal{N} . So, when Λ is not in \mathcal{N} , $g_t \Lambda \in S$ iff $t \in V_\Lambda(S)$. For the transversal S' , the same results hold with

$$V_\Lambda(S') = \{t'_k = \frac{1}{d+c} \ln \frac{q_k(\Lambda)}{r_k(\Lambda)} : k \in \mathbb{Z}\}.$$

It follows that for almost all Λ , both the backward trajectory $(g_t\Lambda)_{t \leq 0}$ and the forward trajectory $(g_t\Lambda)_{t \geq 0}$ visit the two transversals S and S' infinitely often. Therefore the first return/entrance times in S and S' ,

$$\begin{aligned}\tau(\Lambda) &= \inf\{t > 0 : g_t(\Lambda) \in S\} \in \mathbb{R}_{>0} \cup \{\infty\}, \\ \tau'(\Lambda) &= \inf\{t > 0 : g_t(\Lambda) \in S'\} \in \mathbb{R}_{>0} \cup \{\infty\},\end{aligned}$$

are finite almost everywhere and the first return/entrance maps

$$\begin{aligned}R(\Lambda) &= g_{\tau(\Lambda)}\Lambda, \\ R'(\Lambda) &= g_{\tau'(\Lambda)}\Lambda\end{aligned}$$

are defined for all Λ that are not in \mathcal{N} .

For an integer $n \geq 1$, let us denote τ_n the n -th return (or entrance) time in S , i.e.

$$\tau_n(\Lambda) = \sum_{k=0}^{n-1} \tau(R^k(\Lambda))$$

($R^0(\Lambda) = \Lambda$ for all Λ in \mathcal{L}_{d+1}). It will be convenient to choose the numbering of the sequence of minimal vectors $(X_n(\Lambda))$ in order to have simple formulas for the return time and the return map.

Numbering convention: For a lattice $\Lambda \in \mathcal{L}_{d+c}$, $n = 0$ is the smallest integer $n \in \mathbb{Z}$ such that

$$|X_{n+1}(\Lambda)|_- \geq |X_n(\Lambda)|_+$$

when the set of such integers is non empty.

With this numbering convention, for all Λ in S , we have

$$X_0(\Lambda) = v_0^S(\Lambda)$$

and for all Λ in S' , we have

$$X_0(\Lambda) = w_0^{S'}(\Lambda).$$

Moreover when $\Lambda \notin S$ is not in \mathcal{N} ,

$$\begin{aligned}|X_1(\Lambda)|_- - |X_0(\Lambda)|_+ &> 0, \\ \tau(\Lambda) &= \frac{1}{d+c} \ln \left(\frac{|X_1(\Lambda)|_-}{|X_0(\Lambda)|_+} \right),\end{aligned}$$

and

$$g_{\tau(\Lambda)}X_0(\Lambda) = \pm v_0^S(R(\Lambda)) = \pm X_0(R(\Lambda)).$$

Let us summarize the above.

Lemma 4.8. *Let Λ be a lattice in \mathcal{L}_{d+c} .*

1. *The set of visiting times in S is included in $V_\Lambda(S) = \{t_k = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)} : k \in \mathbb{Z}\}$ and the set of visiting times in S' is included in $V_\Lambda(S') = \{t'_k = \frac{1}{d+c} \ln \frac{q_k(\Lambda)}{r_k(\Lambda)} : k \in \mathbb{Z}\}$.*
2. *Suppose Λ is not in \mathcal{N} . The set of visiting times in S is equals to $V_\Lambda(S)$ and the set of visiting times in S' is equals to $V_\Lambda(S')$.*

For θ in $M_{d,c}(\mathbb{R})$, we also need to connect the visiting times of the transversal S with the best approximation vectors of θ .

Lemma 4.9. *Let θ be in $M_{d,c}(\mathbb{R}) \setminus \mathcal{M}$. Then for all large enough integers n ,*

$$t_n(\theta) = \frac{1}{d+c} \ln \frac{q_{n+1}(\theta)}{r_n(\theta)}$$

and

$$t'_n(\theta) = \frac{1}{d+c} \ln \frac{q_n(\theta)}{r_n(\theta)}$$

are visiting times for the transversals S and S' respectively.

Proof. Let θ be in $M_{d,c}(\mathbb{R}) \setminus \mathcal{M}$. Consider the sequence of all best approximation vectors $(Y_n(\theta))_{n \in \mathbb{N}}$ of θ . By Lemma 3.5, there are integers n_1 and k such that $X_{n+k}(\Lambda_\theta) = M_\theta Y_n(\theta)$ for all $n \geq n_1$. Since θ is not in \mathcal{M} , by Lemma 3.6 there is another integer n_2 such that for all $n \geq n_2$, the only nonzero vector of Λ_θ in the box $\mathcal{C}(X_{n+k}(\theta), X_{n+k+1}(\theta))$ are $\pm X_{n+k}(\theta)$ and $\pm X_{n+k+1}(\theta)$. This means that for all n large enough, the times

$$t_n(\theta) = \frac{1}{d+c} \ln \frac{q_{n+1}(\theta)}{r_n(\theta)} \text{ and } t'_n(\theta) = \frac{1}{d+c} \ln \frac{q_n(\theta)}{r_n(\theta)}$$

are visiting times for the transversals S and S' . □

4.7 Functions defined on S

Let Λ be in S . By definition of S , the functions

$$\rho, \rho^* : \Lambda \in S \rightarrow \ln \frac{|v_1^S(\Lambda)|_-}{|v_0^S(\Lambda)|_-}, \ln \frac{|v_0^S(\Lambda)|_+}{|v_1^S(\Lambda)|_+} \in \mathbb{R}_{>0} \cup \{+\infty\}$$

are well defined on S . The next lemma is easy, its proof is close to the beginning of the proof of Lemma 4.3 and is omitted.

Lemma 4.10. *The functions $\Lambda \in S \rightarrow |v_0^S(\Lambda)|_-, |v_0^S(\Lambda)|_+, |v_1^S(\Lambda)|_-, |v_1^S(\Lambda)|_+$ are continuous and thus the functions ρ and ρ^* are continuous.*

The following lemma is important. On the one hand, it will imply that the functions ρ and ρ^* are integrable. On the other hand, it will explain the connection between the Lévy's constant $L_{d,c}$ and the average return times on S .

Lemma 4.11. *Let Λ be a lattice in $S \setminus \mathcal{N}$. Then*

$$\tau(\Lambda) = \frac{1}{d+c} (\rho(R(\Lambda)) + \rho^*(\Lambda)).$$

Proof. Let Λ be in $S \setminus \mathcal{N}$. By definition of S , $q_1(\Lambda) = r_0(\Lambda)$. Hence, by Lemma 4.8,

$$\begin{aligned} (d+c)\tau(\Lambda) &= \ln \frac{q_2(\Lambda)}{r_1(\Lambda)} \times \frac{r_0(\Lambda)}{q_1(\Lambda)} \\ &= \ln \frac{q_2(\Lambda)}{q_1(\Lambda)} + \ln \frac{r_0(\Lambda)}{r_1(\Lambda)} \\ &= \rho(R(\Lambda)) + \rho^*(\Lambda). \end{aligned}$$

□

5 Induced measures on S and S'

5.1 Definition of the induced measures μ_S and $\mu_{S'}$

Recall that we have fixed a measure μ on \mathcal{L}_{d+c} invariant by the action of $\mathrm{SL}(d+c, \mathbb{R})$. Let us explain how the flow g_t and the measure μ induce a measure μ_S on S which is invariant by the first return map R . The construction of such a measure is standard but we have not found any reference suitable for our case.

Let $X = (S \setminus \mathcal{N}) \times \mathbb{R}$, let $\pi : X \rightarrow \mathcal{L}_{d+c} \setminus \mathcal{N}$ be the map defined by $\pi(\Lambda, t) = g_t \Lambda$, let $g_t^X : X \rightarrow X$ be the flow defined by $g_t^X(\Lambda, s) = (\Lambda, s + t)$, let $\bar{R} : X \rightarrow X$ be the map defined by $\bar{R}(\Lambda, t) = (R(\Lambda), t - \tau(\Lambda))$ and let $D = \{(\Lambda, t) \in X : 0 \leq t < \tau(\Lambda)\}$.

- Since the first return map R is a bijection from $S \setminus \mathcal{N}$ onto itself, \bar{R} is a bijection with inverse $\bar{R}^{-1}(\Lambda, t) = (R^{-1}(\Lambda), t + \tau(R^{-1}(\Lambda)))$. It is easy to check that the sets $\bar{R}^n(D)$, $n \in \mathbb{Z}$, are disjoint. Since the infinite sequence $\tau_n(\Lambda)$ strictly increases from $-\infty$ to $+\infty$, $X = \cup_{n \in \mathbb{Z}} \bar{R}^n(D)$.
- We can lift the measure μ from \mathcal{L}_{d+c} to X : we define the measure $\bar{\mu}$ on X by

$$\bar{\mu}(B) = \sum_{n \in \mathbb{Z}} \mu(\pi(D \cap \bar{R}^{-n} B))$$

for all Borel sets B in X . By definition $\bar{\mu}$ is \bar{R} -invariant.

- Let us see that $\bar{\mu}$ is g_t^X -invariant for all $t \in \mathbb{R}$. If B is a Borel set in X , we have the partition $B = \cup_{m,n \in \mathbb{Z}} B_{m,n}$ where $B_{m,n} = B \cap \bar{R}^m D \cap g_{-t}^X \bar{R}^n D$. On the one hand, since $B_{m,n} \subset \bar{R}^m D$, we have $\bar{\mu}(B_{m,n}) = \mu(\pi(D \cap \bar{R}^{-m} B_{m,n})) = \mu(\pi \bar{R}^{-m} B_{m,n})$, hence $\bar{\mu}(B_{m,n}) = \mu(\pi B_{m,n})$ because $\pi \circ \bar{R} = \pi$. On the other hand, since $g_t^X B_{m,n} \subset \bar{R}^n D$,

$$\bar{\mu}(g_t^X B_{m,n}) = \bar{\mu}(\bar{R}^{-n} g_t^X B_{m,n}) = \mu(\pi \bar{R}^{-n} g_t^X B_{m,n}) = \mu(\pi g_t^X \bar{R}^{-n} B_{m,n}) = \mu(g_t \pi \bar{R}^{-n} B_{m,n}),$$

and since μ is g_t -invariant, we obtain, $\bar{\mu}(g_t^X B_{m,n}) = \mu(B_{m,n})$.

- Since $\bar{\mu}$ is g_t^X -invariant, there exists a measure μ_S on $S \setminus \mathcal{N}$ such that

$$\bar{\mu} = \mu_S \otimes \text{Lebesgue}_{\mathbb{R}}.$$

- By definition for any Borel set $B \subset S$ and any $\delta > 0$ such that $B \subset \{\Lambda : \tau(\Lambda) > \delta\}$,

$$\mu_S(B) \times \delta = \bar{\mu}(B \times [0, \delta]) = \mu(\cup_{t \in [0, \delta]} g_t B).$$

- Let us show that μ_S is R -invariant. Let B be a Borel set in $S \setminus \mathcal{N}$. Since $\bar{\mu}$ is \bar{R} -invariant,

$$\begin{aligned} \mu_S(B) &= \bar{\mu}(B \times [0, 1]) = \bar{\mu}(\bar{R}(B \times [0, 1])) \\ &= \int_{R(B)} d\mu_S(\Lambda) \int_{-\tau(R^{-1}\Lambda)}^{1-\tau(R^{-1}\Lambda)} dt = \mu_S(R(B)). \end{aligned}$$

- We extend μ_S to S by $\mu_S(S \cap \mathcal{N}) = 0$.

With our definition of μ_S , and since $\pi(D) = \mathcal{L}_{d,c} \setminus \mathcal{N}$, the Kac return-time theorem reduces to

$$\mu(\mathcal{L}_{d+c}) = \mu(\pi(D)) = \bar{\mu}(D) = \int_S \tau(\Lambda) d\mu_S(\Lambda).$$

Finally, the measure μ_S is R -ergodic because any R -invariant set of positive measure in S gives rise to a set of flow trajectories of positive measure and it is known that the action of the flow (g_t) on (\mathcal{L}_{d+c}, μ) is ergodic (see [3]).

The flow induces a measure $\mu_{S'}$ on S' as well.

5.2 Finiteness of the induced measures

Let us prove that the induced measures μ_S and $\mu_{S'}$ are finite. This is a simple consequence of the next lemma which will be very important in the proof of Theorem 8.1 about the almost sure convergence in $M_{d,c}(\mathbb{R})$.

Lemma 5.1. *1. There exists an integer constant A such that $\tau_A(\Lambda) \geq 1$ for all Λ in S and $\tau'_A(\Lambda) \geq 1$ for all Λ in S' .*

Proof. Let A be the constant given by Lemma 3.7 about the growth rate of the sequences $(q_n(\Lambda))_n$ and $(r_n(\Lambda))_n$. For all integers k , we have

$$\begin{aligned} \frac{1}{d+c} \left(\ln \frac{q_{k+A+1}(\Lambda)}{r_{k+A}(\Lambda)} - \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)} \right) &\geq \frac{1}{d+c} \ln \frac{q_{k+A+1}(\Lambda)}{q_{k+1}(\Lambda)} \\ &\geq \frac{1}{d+c} \ln 2. \end{aligned}$$

Since by Lemma 4.8, the set of visiting times of Λ is included in

$$V_\Lambda(S) = \left\{ t_k = \frac{1}{d+c} \ln \frac{q_{k+1}(\Lambda)}{r_k(\Lambda)} : k \in \mathbb{Z} \right\},$$

$\tau_A(\Lambda) \geq \frac{1}{d+c} \ln 2$. Multiplying A by the smallest integer larger than $\frac{d+c}{\ln 2}$ we are done. \square

Proposition 5.2. *$\mu_S(S)$ and $\mu_{S'}(S')$ are finite and nonzero.*

Proof. By Lemma 4.3, S is submanifold of co-dimension one, transverse to the flow, and nonempty, hence $\mu_S(S) > 0$.

Since μ_S is R -invariant, for all k

$$\int_S \tau(R^k \Lambda) d\mu_S(\Lambda) = \int_S \tau(\Lambda) d\mu_S(\Lambda),$$

and by the Kac return time theorem,

$$\int_S \tau(\Lambda) d\mu_S(\Lambda) = \mu(\mathcal{L}_{d+1}),$$

therefore,

$$\int_S \tau_A(\Lambda) d\mu_S(\Lambda) = \int_S \sum_{k=0}^{A-1} \tau(R^k \Lambda) d\mu_S(\Lambda) = A\mu(\mathcal{L}_{d+c}).$$

By the above lemma, $\tau_A \geq 1$ on S , hence $\mu_S(S) \leq A\mu(\mathcal{L}_{d+c})$ which is finite by Siegel's theorem. \square

6 Almost sure convergence in the space of lattices

6.1 Consequence of the Birkhoff ergodic theorem

Recall that for a lattice $\Lambda \in \mathcal{L}_{d+c}$, $q_n(\Lambda) = |X_n(\Lambda)|_-$ and $r_n(\Lambda) = |X_n(\Lambda)|_+$ where $(X_n(\Lambda))_n$ is the sequence of minimal vectors of Λ with the numbering convention of Section 4.6. Recall also that $\tau_n(\Lambda)$ is the n -th return time in S .

Theorem 6.1. *There exist two positive constants $L_{d,c}$ and $L_{d,c}^*$ such that for almost all lattices Λ in \mathcal{L}_{d+c} ,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\Lambda) &= \frac{1}{\mu_S(S)} \int_S \rho d\mu_S = L_{d,c} > 0, \\ \lim_{n \rightarrow \infty} \frac{-1}{n} \ln r_n(\Lambda) &= \frac{1}{\mu_S(S)} \int_S \rho^* d\mu_S = L_{d,c}^* > 0, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \tau_n(\Lambda) &= \frac{1}{d+c} (L_{d,c} + L_{d,c}^*) = \frac{\mu(\mathcal{L}_{d+c})}{\mu_S(S)}. \end{aligned}$$

Moreover, given two new Euclidean norms on the horizontal space $\mathbb{R}^d \times \{0\}$ and the vertical space $\mathbb{R}^c \times \{0\}$, if we define the minimal vectors and the transversal S with these two new Euclidean norms, then the conclusion of the theorem remains valid with the same constant $L_{d,c}$ and $L_{d,c}^*$.

The above theorem is an important result in the direction of Theorem 1. The big difference between the two theorems is the reference measure. However, the second statement in Theorem 6.1 explains why Theorem 1 remains valid with the same constant $L_{d,c}$ and $L_{d,c}^*$ for any pair of Euclidean norms.

Proof. Let Λ be in $S \setminus \mathcal{N}$. By Lemma 4.11, $\tau(\Lambda) = \frac{1}{d+c}(\rho(R(\Lambda)) + \rho^*(\Lambda))$. Because the space of lattices has finite measure, the return time τ is in $\mathcal{L}^1(S)$ and since the non negative functions $\rho \circ R$ and ρ^* are bounded above by $(d+c)\tau$, they are also in $\mathcal{L}^1(S)$. Making use of the Birkhoff ergodic theorem with the functions ρ and ρ^* , we obtain the almost everywhere convergence of the sums

$$\frac{1}{N} \sum_{n=0}^{N-1} \rho \circ R^n, \quad \frac{1}{N} \sum_{n=0}^{N-1} \rho^* \circ R^n$$

on S to R -invariant functions. Now the ergodicity of the flow g_t (see [3]) implies the ergodicity of the return map R . Therefore $\frac{1}{N} \sum_{k=0}^{N-1} \rho \circ R^k$ and $\frac{1}{N} \sum_{k=0}^{N-1} \rho^* \circ R^k$ converge almost everywhere on S to the constants

$$L_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho d\mu_S \quad \text{and} \quad L_{d,c}^* = \frac{1}{\mu_S(S)} \int_S \rho^* d\mu_S.$$

We would like to see that the Birkhoff sums converge to the same limits almost everywhere in the whole space of lattices. Let Λ be in $\mathcal{L}_{d+c} \setminus (S \cup \mathcal{N})$ and let n be a positive integer. By

Lemma 4.8 and the numbering convention, for all $n \geq 1$,

$$\begin{aligned}\rho \circ R^n(\Lambda) &= \ln \frac{|X_1(R^n(\Lambda))|_-}{|X_0(R^n(\Lambda))|_-} \\ &= \ln \frac{|g_{\tau_n(\Lambda)}(X_n(\Lambda))|_-}{|g_{\tau_n(\Lambda)}(X_{n-1}(\Lambda))|_-} \\ &= \ln \frac{|X_n(\Lambda)|_-}{|X_{n-1}(\Lambda)|_-}\end{aligned}$$

and $\rho^* \circ R^n(\Lambda) = \ln \frac{|X_0(R^n(\Lambda))|_+}{|X_1(R^n(\Lambda))|_+} = \ln \frac{|X_{n-1}(\Lambda)|_+}{|X_n(\Lambda)|_+}$ as well. It follows that if the Birkhoff sums $\frac{1}{N} \sum_{n=1}^N \rho \circ R^n(\Lambda)$ and $\frac{1}{N} \sum_{n=1}^N \rho^* \circ R^n(\Lambda)$ converge to $L_{d,c}$ and $L_{d,c}^*$, then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln q_N(\Lambda) = L_{d,c} \text{ and } \lim_{N \rightarrow \infty} \frac{-1}{N} \ln r_N(\Lambda) = L_{d,c}^*.$$

Now the image by the map $R : \mathcal{L}_{d+c} \rightarrow S$ of a subset of nonzero measure in \mathcal{L}_{d+c} is a set of nonzero measure in S , therefore the sums $\frac{1}{N} \sum_{n=1}^N \rho \circ R^n$ and $\frac{1}{N} \sum_{n=1}^N \rho^* \circ R^n$ converge almost everywhere in \mathcal{L}_{d+c} to $L_{d,c}$ and $L_{d,c}^*$.

By Lemma 3.7, we know that the sequences $(q_n(\Lambda))_n$ and $(r_n(\Lambda)^{-1})_n$ have at least exponential growth rate; therefore, the constants $L_{d,c}$ and $L_{d,c}^*$ are > 0 .

By Lemma 4.11, for all Λ in $S \setminus \mathcal{N}$ and $k \in \mathbb{N}$,

$$\tau_{k+1}(\Lambda) - \tau_k(\Lambda) = \frac{1}{d+c} (\rho(R^{k+1}(\Lambda)) + \rho^*(R^k(\Lambda))),$$

hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \tau_n(\Lambda) = \frac{1}{d+c} (L_{d,c} + L_{d,c}^*)$$

almost everywhere.

Finally, let us proof that the constants $L_{d,c}$ and $L_{d,c}^*$ do not depend on the Euclidean norm on \mathbb{R}^d and \mathbb{R}^c . For a matrix M_d in $\text{SL}(d, \mathbb{R})$ and a matrix M_c in $\text{SL}(c, \mathbb{R})$, let us denote M the matrix

$$M = \begin{pmatrix} M_d & 0 \\ 0 & M_c \end{pmatrix} \in \text{SL}(d+c, \mathbb{R}).$$

Since the action of M on \mathcal{L}_{d+c} is measure preserving,

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(A\Lambda) &= L_{d,c} \\ \lim_{n \rightarrow \infty} \frac{-1}{n} \ln r_n(A\Lambda) &= L_{d,c}^*\end{aligned}$$

almost everywhere in \mathcal{L}_{d+c} . Now, a vector AX in the lattice $A\Lambda$ is minimal iff X is a minimal vector of the lattice Λ with respect to the new Euclidean norms $\|\cdot\|_{A_d, \mathbb{R}^d}$ and $\|\cdot\|_{A_c, \mathbb{R}^c}$ where

$$\|u\|_{M_d, \mathbb{R}^d} = \|M_d u\|_{\mathbb{R}^d}, \text{ and } \|v\|_{M_c, \mathbb{R}^c} = \|M_c v\|_{\mathbb{R}^c}.$$

Since up to multiplicative constants, all the Euclidean norms are of the above form, the constants $L_{d,c}$ and $L_{d,c}^*$ do not depend on the Euclidean norms on \mathbb{R}^d and \mathbb{R}^c . \square

6.2 A consequence of the Borel-Cantelli lemma

For any lattice $\Lambda \in \mathcal{L}_{d,c} \setminus \mathcal{N}$ and any integer n , we have $q_{n+1}(\Lambda)r_n(\Lambda) = \lambda_1(g_{t_n}\Lambda)\lambda_2(g_{t_n}\Lambda)$ where $t_n = \tau_n(\Lambda) = \frac{1}{d+c} \ln \frac{q_{n+1}(\Lambda)}{r_n(\Lambda)}$ is the n -th return time in S . When $d = c = 1$, $\lambda_1(g_{t_n}\Lambda)\lambda_2(g_{t_n}\Lambda) \asymp 1$ by the Minkowski second convex body theorem. It implies that $L_{1,1} = L_{1,1}^*$. When d or $c \geq 2$, the product of the first two minima of a lattice in $\mathcal{L}_{d,c}$ can be arbitrarily small, so it is not clear whether there is a relation between $L_{d,c}$ and $L_{d,c}^*$. However,

Proposition 6.2. *With the notation of Theorem 6.1, we have*

$$cL_{d,c} = dL_{d,c}^*.$$

Lemma 6.3. *The function $\Lambda \in \mathcal{L}_{d+c} \rightarrow 1/\lambda_1(\Lambda)$ is in $L^1(\mu)$.*

Proof. By Siegel formula (see [53]) used with the function $f = 1_{B_{\mathbb{R}^{d+c}}}(0, r)$, we have for some positive constants C_1 and C_2 depending on the dimension $d+c$ and on the normalization of the Haar measure,

$$C_1 r^{d+c} = \int_{\mathbb{R}^{d+c}} f(x) dx = C_2 \int_{\mathcal{L}_{d+c}} \sum_{x \in \Lambda \setminus \{0\}} f(x) d\mu(\Lambda) \geq C_2 \mu(\{\Lambda : \lambda_1(\Lambda) \leq r\}).$$

Therefore $\int_{\mathcal{L}_{d+c}} \frac{1}{\lambda_1} d\mu = \int_0^\infty \mu(\{\Lambda : \frac{1}{\lambda_1(\Lambda)} \geq t\}) dt \leq \mu(\{\mathcal{L}_{d+c}\}) + \frac{C_1}{C_2} \int_1^\infty \frac{1}{t^{d+c}} dt < \infty$. \square

Proof of the proposition. The inequality $cL_{d,c} \leq dL_{d,c}^*$ is easy to prove. By Lemma 3.6, for all lattices Λ in \mathcal{L}_{d+c} and all n , we have $q_{n+1}^c(\Lambda)r_n^d(\Lambda) \leq C_{d,c}$. Hence for a lattice Λ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\Lambda) = L_{d,c} \text{ and } \lim_{n \rightarrow \infty} \frac{-1}{n} \ln r_n(\Lambda) = L_{d,c}^*,$$

we have,

$$cL_{d,c} - dL_{d,c}^* = c \lim_{n \rightarrow \infty} \frac{\ln q_n(\Lambda)}{n} + d \lim_{n \rightarrow \infty} \frac{\ln r_n(\Lambda)}{n} = \lim_{n \rightarrow \infty} \frac{\ln q_n^c(\Lambda)r_n^d(\Lambda)}{n} \leq 0.$$

The converse inequality uses the Borel-Cantelli lemma. Let $\varphi :]0, \infty[\rightarrow]0, \infty[$ be a decreasing function such that $\sum_{n \geq 1} \varphi(n) < \infty$, for instance $\varphi(t) = \frac{1}{t^\alpha}$ with $\alpha > 1$. Since for such a function φ , $\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \varphi(n) = 0$, the inequality $cL_{d,c} - dL_{d,c}^* \geq 0$ holds, provided that for almost all lattices Λ , we have $q_n^c(\Lambda)r_n^d(\Lambda) \geq \varphi(n)^{d+c}$ for n large enough. Let K be a constant that will be chosen later. For each integer $n \geq 1$, consider the set A_n of lattices Λ in \mathcal{L}_{d+c} such that

$$\lambda_1(\Lambda) \leq K\varphi(n)$$

and the set $B_n = g_{t_n}A_n$ where $t_n = \frac{1}{d}(\ln \varphi(n) - n)$. By the above lemma $\|\lambda_1^{-1}\|_1 < \infty$, hence using Markov inequality, we obtain

$$\mu(B_n) = \mu(A_n) \leq \frac{\|\lambda_1^{-1}\|_1}{\frac{1}{K\varphi(n)}} \ll \varphi(n).$$

Therefore, by the Borel-Cantelli lemma, the set \mathcal{B} of lattices Λ in \mathcal{L}_{d+c} such that $\Lambda \in B_n$ for infinitely many integers n , is negligible. Suppose now that Λ is a lattice such that $q_n^c(\Lambda)r_n^d(\Lambda) \leq$

$\varphi(n)^{d+c}$ for infinitely many n . For each integer $n \geq 1$, let $k_n = k_n(\Lambda) = \lfloor \ln q_n(\Lambda) \rfloor$. By Theorem 6.1, for almost all lattices, we have $k_n \leq (L_{d,c} + 1)n$ for n large enough. Therefore, for almost all lattices Λ , for n large enough, if $q_n^c(\Lambda)r_n^d(\Lambda) \leq \varphi(n)^{d+c}$ then the vector $g_{t_{k_n}}^{-1}(X_n(\Lambda))$ satisfies both

$$\begin{aligned} \left| g_{t_{k_n}}^{-1}(X_n(\Lambda)) \right|_+ &= r_n(\Lambda) e^{\frac{c}{d}(k_n - \ln \varphi(k_n))} \\ &\leq r_n(\Lambda) q_n^{\frac{c}{d}}(\Lambda) \varphi(k_n)^{-\frac{c}{d}} \\ &\leq \varphi(n)^{\frac{d+c}{d}} \varphi(k_n)^{-\frac{c}{d}} \\ &= \varphi(k_n) \times \left(\frac{\varphi(n)}{\varphi(k_n)} \right)^{\frac{d+c}{d}} \\ &\leq \varphi(k_n) \times \left(\frac{\varphi(n)}{\varphi((L_{d,c} + 1)n)} \right)^{\frac{d+c}{d}} \leq K \varphi(k_n) \end{aligned}$$

for some constant K depending only on φ (we use that $\frac{\varphi(t)}{\varphi((L_{d,c}+1)t)}$ is bounded above which is obviously true when $\varphi(t) = \frac{1}{t^\alpha}$) and

$$\left| g_{t_{k_n}}^{-1}(X_n(\Lambda)) \right|_- = q_n(\Lambda) e^{-k_n + \ln \varphi(k_n)} \leq e \varphi(k_n) \leq K \varphi(k_n).$$

Thus there are infinitely many n such that $\Lambda \in B_{k_n}$. Since the sequence $(k_n)_n$ goes to infinity, $\Lambda \in \mathcal{B}$. It follows that for almost all lattices Λ , $q_n^c(\Lambda)r_n^d(\Lambda) \geq \varphi(n)$ for n large enough and we are done. \square

As an immediate consequence of the previous proposition and of Theorem 6.1, we have

Corollary 6.4.

$$L_{d,c} = \frac{d}{\mu_S(S)} \int_S \tau \, d\mu_S = \frac{d \times \mu(\mathcal{L}_{d+c})}{\mu_S(S)}.$$

7 Parametrization of S when $c = 1$

This section is not necessary neither for the proofs of Theorems 1 and 2, nor for Sections 8 and 9. The aim is to show that the computation of the constant $L_{d,c}$ is theoretically feasible in the case $c = 1$. However, if the case $d = 1$ is easy (see below), the case $d = 2$ is already difficult. It is possible to give an integral formula for $L_{2,1}$. But, we are not able to compute the integral, only a numerical estimation of the integral has been carried out. An exact description of S when $d \geq 3$ seems to be rather difficult. In this section we assume $c = 1$.

7.1 rkN decomposition

In this subsection we give a parametrization of a set of lattices that contains S .

Let Λ be a lattice in S and let $u_1 = v_0^S(\Lambda)$ and $u_2 = v_1^S(\Lambda)$ be the two vectors associated with Λ by the definition of S (see the definition of S in Section 4.1). When $d \geq 2$, we suppose these two vectors have non negative heights and when $d = 1$, we only suppose that u_2 has a non negative height. Since u_1 and u_2 are independent shortest vectors, by Corollary 4.2, they

are the first two vectors of a basis of Λ . Thus, there is a matrix $M \in \text{SL}(d+1, \mathbb{R})$ defining Λ the first two columns of which are the vectors u_1 and u_2 .

When $d = 1$, using the scaling factor $r = |u_1|_+ = |u_2|_- > 0$, we can write $M = rN$, where N is in the set U_1 of 2×2 matrices such that

$$n_{1,1} = n_{2,2} = 1 \text{ and } |n_{2,1}|, |n_{1,2}| < 1.$$

When $d \geq 2$, let us denote $(e_1, e_2, \dots, e_{d+1})$ the standard basis of \mathbb{R}^{d+1} . Using the same scaling factor $r = |u_1|_+ = |u_2|_- > 0$ and an orthogonal matrix k that fixes e_{d+1} and sends e_1 to $\frac{1}{r}u_{1,+}$, we can find a matrix $N = (n_{i,j})_{1 \leq i,j \leq d+1}$ such that $M = rkN$, $\det N > 0$ and

$$n_{1,1} = n_{d+1,2} = 1 > n_{d+1,1} = |u_1|_- \geq 0, \tag{2}$$

$$\|(n_{1,2}, \dots, n_{d,2})\|_{\mathbb{R}^d} < 1, \tag{3}$$

$$n_{2,1} = \dots = n_{d,1} = 0. \tag{4}$$

When $d \geq 2$, k is chosen in the group

$$K_d = \{k \in \text{SO}(d+1) : ke_{d+1} = e_{d+1}\}.$$

and using the decomposition of a $d \times d$ matrix in a product of an orthogonal matrix with positive determinant and of an upper triangular matrix, we can even suppose that

$$n_{i,j} = 0, \text{ for all } 1 \leq j < i \leq d. \tag{5}$$

For $d \geq 2$, let us denote U_d the set of $(d+1) \times (d+1)$ -matrices such that (2), (3), and (5) hold ((5) implies (4)).

Since $\det M = 1$, the scaling factor r must be equal to $(\det N)^{-\frac{1}{d+1}}$. Putting $K_1 = \{I_2\}$, for all $d \geq 1$, the map

$$(k, N) \in K_d \times U_d \rightarrow (\det N)^{-\frac{1}{d+1}} kN$$

provides a natural parametrization of a subset Σ in $\text{SL}(d+1, \mathbb{R})$ whose projection in \mathcal{L}_{d+1} contains S . The main problem is now to find which of these couples (k, N) are such that $rkN\mathbb{Z}^{d+1} \in S$ and to select a fundamental domain in this set of couples. This problem reduces to

- finding the set of matrices $N \in U_d$ such that the first two columns u_1 and u_2 of N are in the unit ball $B_{d,1}(0, 1)$ and are the only nonzero vectors of the lattice $N\mathbb{Z}^{d+1}$ in this ball,
- then select a fundamental domain in this set of matrices N .

This is easy when $d = 1$ and doable when $d = 2$. When $d = 1$, it is even possible to find the first return map R .

Another issue is to find the measure μ_S on S induced by the flow g_t and the invariant measure μ of \mathcal{L}_{d+1} . This comparatively easier issue can be performed for all d without knowing explicitly S .

7.2 The induced measure μ_S

Consider the manifold $V_d = \mathbb{R}_{>0} \times \mathbb{R} \times K_d \times U_d$ and the submanifold

$$W = \{(\Delta, t, k, N) \in V_d : \Delta = 1, t = 0\} = \{1\} \times \{0\} \times K_d \times U_d$$

together with the map $F : V_d \rightarrow \mathrm{GL}(d+1, \mathbb{R})$ defined by

$$F(\Delta, t, k, N) = \left(\frac{\Delta}{\det N} \right)^{\frac{1}{d+1}} g_t k N$$

and the map $\bar{F} : V_d \rightarrow \mathrm{GL}(d+1, \mathbb{R}) / \mathrm{SL}(d+1, \mathbb{Z})$ defined by

$$\bar{F}(\Delta, t, k, N) = F(\Delta, t, k, N) \mathbb{Z}^{d+1}.$$

By the discussion of the previous subsection, \bar{F} provides a parametrization of S : $S \subset \bar{F}(W)$. We would like to compute the measure μ_S in the coordinates $(1, 0, k, N)$. The submanifold W is equipped with the reference measure $\mu_{K_d} \otimes \lambda_{U_d}$ where λ_{U_d} is the Lebesgue measure on U_d and μ_{K_d} is the invariant measure on K_d associated with the invariant volume form γ on K_d that is dual to the exterior product of the invariant vector fields generated by the standard skew symmetric matrices $(A_{i,j} = E_{ij} - E_{ji})_{1 \leq j < i \leq d}$.

We have to precise the normalization of the Haar measure that we use on $\mathrm{SL}(d+1, \mathbb{R})$. Consider the standard volume form ω on $M_{d+1}(\mathbb{R})$ seen as $\mathbb{R}^{(d+1)^2}$ and the vector fields X and Y on $M_{d+1}(\mathbb{R})$ defined by

$$Y(M) = M \text{ and } X(M) = \left. \frac{\partial(g_t M)}{\partial t} \right|_{t=0}.$$

The invariant measure in $\mathrm{SL}(d+1, \mathbb{R})$ is given by the volume form $\frac{1}{(d+1)^2} i_Y \omega$. With this normalization of the volume form we have the beautiful Siegel formula

$$\mu(\mathcal{L}_{d+1}) = \mathrm{vol}(\mathrm{SL}(d+1, \mathbb{R}) / \mathrm{SL}(d+1, \mathbb{Z})) = \frac{1}{d+1} \prod_{k=2}^{d+1} \zeta(k)$$

(see [54], p. 152). The induced measure μ_S on S is given by the restriction to S of the projection in the space of lattices of the differential form

$$\frac{1}{(d+1)^2} i_X i_Y \omega.$$

Using this differential form it is possible to compute the induced measure μ_S with the parametrization \bar{F} , we give without proof an explicit formula in next proposition.

Proposition 7.1. *Assume $d \geq 2$. Suppose that \mathcal{D} is an open subset of W such that $\bar{F}(\mathcal{D}) \subset S$ and the restriction of \bar{F} to \mathcal{D} is one to one. Then the image by \bar{F} of the measure*

$$1_{\mathcal{D}} \left(\frac{1}{\det N} \right)^{d+1} \left(\frac{1}{d+1} \prod_{j=2}^{d-1} n_{j,j}^{d-j} \right) \mu_{K_d} \otimes \lambda_{U_d}$$

is the restriction of μ_S to $\bar{F}(\mathcal{D})$.

7.3 Determination of the transversal S , $c = 1$, $d = 1$

We already have a map

$$\begin{aligned} U_1 &\rightarrow \mathcal{L}_2 \\ N &\rightarrow (\det N)^{-\frac{1}{2}} N\mathbb{Z}^2 \end{aligned}$$

that sends U_1 onto a set that contains S . By definition of S , the image of a matrix

$$N = \begin{pmatrix} 1 & n_{1,2} \\ n_{2,1} & 1 \end{pmatrix} \in U_1$$

is in S iff the only nonzero vectors of the lattice $\Lambda = N\mathbb{Z}^2$ in the ball $B_{1,1}(0, 1)$ are the two columns of N up to sign. We obtain that $\Lambda \in S$ iff

- $0 < |n_{2,1}|, |n_{1,2}| < 1$,
- the signs of $n_{1,2}$ and $n_{2,1}$ are opposite.

So the map F defined on $]0, 1[^2 \times \{-1, 1\}$ defined by

$$(x, y, \varepsilon) \rightarrow \frac{1}{(1 + xy)^{1/2}} \begin{pmatrix} 1 & -\varepsilon x \\ \varepsilon y & 1 \end{pmatrix} \mathbb{Z}^2$$

provide a parametrization of S and it is easy to see that F is a bijection. An easy computation of a 4×4 determinant gives that

$$i_X i_Y \frac{1}{4} \omega\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = \frac{1}{4} \omega(X, Y, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}) = \frac{1}{2(1 + xy)^2},$$

hence

$$f(x, y, \varepsilon) = \frac{1}{2(1 + xy)^2}$$

is the density the measure μ_S with respect to the Lebesgue measure. Therefore $\mu_S(S) = \ln 2$. With the Siegel formula ([54], p. 152) and Corollary 6.4, we obtain the Lévy's constant

$$L_{1,1} = \frac{\mu(\mathcal{L}_{d+c})}{\mu_S(S)} = \frac{\zeta(2)}{2 \ln 2} = \frac{\pi^2}{12 \ln 2}.$$

7.3.1 Determination of the first return map, $c = 1$, $d = 1$

Let

$$\Lambda = F(x, y, \varepsilon) = \frac{1}{(1 + xy)^{1/2}} \begin{pmatrix} 1 & -\varepsilon x \\ \varepsilon y & 1 \end{pmatrix} \mathbb{Z}^2$$

be in S . By Lemma 4.8, to find the first return map $R(\Lambda)$ in S , it is enough to find the minimal vector $X_2(\Lambda)$. Then $R(\Lambda)$ is given by $g_{\tau(\Lambda)}(\Lambda)$ with

$$\tau(\Lambda) = \frac{1}{2} \ln \left(\frac{|X_2(\Lambda)|_-}{|X_1(\Lambda)|_+} \right).$$

By corollary 4.2, the first minimal vectors $X_0(\Lambda)$ and $X_1(\Lambda)$ form a basis of Λ . The minimal vector $X_2(\Lambda)$ is the vector of the form $X = aX_0(\Lambda) + bX_1(\Lambda)$ in the strip $|X|_+ < |X_1(\Lambda)|_+ = x$ with $a, b \in \mathbb{Z}$, and with the smallest height. It is not difficult to see that

$$X_2(\Lambda) = \varepsilon X_0(\Lambda) + \lfloor \frac{1}{x} \rfloor X_1(\Lambda).$$

So we obtain $R(\Lambda) = F(x', y', \varepsilon')$ where

$$\varepsilon' = -\varepsilon, \quad x' = \left\{ \frac{1}{x} \right\} \quad \text{and} \quad y' = \frac{1}{y + \lfloor \frac{1}{x} \rfloor}.$$

It follows that the return map R is a two-fold extension of the natural extension of the Gauss map.

7.4 Value of Lévy's constant when $d = 2$ and $c = 1$,

An exact description of S is possible when $d = 2$ and $c = 1$. Together with the expression of the measure μ_S in Proposition 7.1, this leads to a closed formula for Lévy's constant as a seven-tuple integral of an algebraic function over a union of domains the boundaries of which are algebraic surfaces of degree at most two. We are not able to compute this seven-tuple integral. However using Octave, Seraphine Xieu (see [57],[11]) has computed a numerical approximation of Levy's constant

$$L_{2,1} = 1.135256974 \dots$$

This can be compared with the one dimensional Levy's constant

$$L_{1,1} = 1.186569111 \dots$$

8 Almost sure convergence in $M_{d,c}(\mathbb{R})$

We want to prove that $\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\Lambda_\theta) = L_{d,c}$ for Lebesgue-almost all $\theta \in M_{d,c}(\mathbb{R})$. By Theorem 6.1, we already know that $\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\Lambda) = L_{d,c}$ for μ -almost all lattices $\Lambda \in \mathcal{L}_{d+c}$, but we have to change the reference measure. Furthermore, the set $\mathcal{H}_>$ of lattices of the shape Λ_θ , $\theta \in M_{d,c}(\mathbb{R})$, is μ -negligible. Actually, we will first prove a general result, Theorem 8.1 (see below): the Birkhoff sums $\frac{1}{N} \sum_{n=0}^{N-1} \varphi \circ R^n(\Lambda_\theta)$ converge for almost all θ when the function φ satisfies some regularity assumptions. The main ingredient of the proof of this general result is that $\mathcal{H}_>$ contains the expanding direction of the flow g_t . However, Theorem 8.1 also requires that Lemma 8.2 below holds for the transversal. This is proven in Subsections 8.1.2 and 8.1.3 with an example. These two subsections are not necessary for the proof of Theorem 8.1.

8.1 A general result

Recall that \mathcal{H}_\leq is the subgroup of $\text{SL}(d+c, \mathbb{R})$ defined by

$$\mathcal{H}_\leq = \left\{ h \in \text{SL}(d+c, \mathbb{R}) : h = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \right\}$$

with $A \in \text{GL}(d, \mathbb{R})$, $B \in M_{c,d}(\mathbb{R})$ and $C \in \text{GL}(c, \mathbb{R})$. We say that a function $f : \mathcal{L}_{d+c} \rightarrow \mathbb{R}$ is *uniformly continuous in the \mathcal{H}_\leq -direction* if for all ε there exists $\beta > 0$ such that for all $\Lambda \in \mathcal{L}_{d+c}$ and all $h \in B_{\mathcal{H}_\leq}(I_{d+c}, \beta)$, $|f(h\Lambda) - f(\Lambda)| \leq \varepsilon$.

Theorem 8.1. 1. Let $\varphi : S \rightarrow \mathbb{R}$ be a function continuous almost everywhere on S . Suppose there exists a non negative function $f : \mathcal{L}_{d+c} \rightarrow \mathbb{R}_{\geq 0}$ that is continuous, uniformly continuous in the \mathcal{H}_{\leq} -direction, integrable and such that $|\varphi| \leq f$ on S . Then,

$$\int_S f d\mu_S < +\infty$$

and for almost all θ in $M_{d,c}(\mathbb{R})$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) = \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S.$$

2. The same result holds for S' instead of S

We can formulate Theorem 8.1 for a general transversal S . The assumptions about S are:

- S is a co-dimension one submanifold transverse to the flow,
- the number of visiting times in a time interval of length 1 is bounded above by a universal constant A (Lemma 5.1),
- Lemma 8.2 below holds for S .

The other assumptions and the conclusion are the same as in Theorem 8.1.

For a compact subset K of the submanifold S and $\delta > 0$, let us denote

$$U(K, \delta) = \{g_t h \Lambda : t \in [0, 1], h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \delta), \Lambda \in S \setminus K\}.$$

Lemma 8.2. For all $\varepsilon > 0$, there exist a compact subset K in S and $\delta > 0$ such that $\mu(U(K, \delta)) \leq \varepsilon$.

This is an important lemma because it explains that the part of S near its “boundary” is not relevant. This lemma holds for S' as well but it will be proven below only for S .

We shall prove first a particular case of Theorem 8.1 which also requires Lemma 8.2:

Proposition 8.3. Let $\varphi : S \rightarrow \mathbb{R}$ be a bounded continuous function. Then for almost all θ in $M_{d,c}(\mathbb{R})$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) = \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S.$$

8.1.1 Auxiliary lemmas

The proof of Theorem 8.1 needs three more lemmas. The first one only uses that S and S' are transverse to the flow together with the inverse function theorem. In the remaining of the section 8, the balls $B(I_{d+c}, r)$ are in $\text{SL}(d+c, \mathbb{R})$.

Lemma 8.4. For all compact subset K in S (or in S'), there exist α and $\eta > 0$ such that

- the map $(t, \Lambda) \rightarrow g_t \Lambda$ is one to one on $[-\alpha, \alpha] \times K$,
- for all $h \in B(I_{d+c}, \eta)$ and all Λ in K , there exists an unique $t = t(h, \Lambda) \in [-\alpha, \alpha]$ such that $g_{-t} h \Lambda \in S$.
- the maps $\sigma : (h, \Lambda) \rightarrow t = t(h, \Lambda)$ and $\pi : (h, \Lambda) \rightarrow g_{-t} h \Lambda$ are continuous on $B(I_{d+c}, \eta) \times K$ and the values of σ are in $[-\alpha/4, \alpha/4]$.

Proof. Let $\psi : (t, \Lambda) \in \mathbb{R} \times S \rightarrow g_t \Lambda \in \mathcal{L}_{d+c}$. Since S is a submanifold and since the flow is transverse to S , thanks to the inverse function theorem, we see that for each $\Lambda \in S$, there exists an open neighborhood W_Λ of Λ in S and $a_\Lambda > 0$ such that ψ is a diffeomorphism from $] - a_\Lambda, a_\Lambda[\times W_\Lambda$ onto the open set $\psi(] - a_\Lambda, a_\Lambda[\times W_\Lambda)$ in \mathcal{L}_{d+c} and such that $\psi(] - a_\Lambda, a_\Lambda[\times W_\Lambda) \cap S = W_\Lambda$. Since K can be covered by finitely many W_Λ , we can find an open set W in S containing K and $\alpha > 0$ such that

$$\psi(] - 2\alpha, 2\alpha[\times W) \cap S = W.$$

We can suppose that W is relatively compact w.l.o.g.. Since ψ is one to one on the compact set $\{0\} \times \overline{W}$, reducing α , we can suppose that ψ is one to one on $] - 2\alpha, 2\alpha[\times \overline{W}$. It follows that ψ is a diffeomorphism from $U =] - 2\alpha, 2\alpha[\times W$ onto $\psi(U)$. We can choose $\eta > 0$ such that $B(I_{d+c}, \eta)K \subset \psi(] - \alpha/4, \alpha/4[\times W)$. With this choice of η , for each $(h, \Lambda) \in B(I_{d+c}, \eta) \times K$, there exists a pair $(t, \Gamma) \in] - \alpha/4, \alpha/4[\times W$ such that $g_t \Gamma = \psi(t, \Gamma) = h\Lambda$. If $(t', \Gamma') \in] - \alpha, \alpha[\times S$ is such that $g_{t'} \Gamma' = h\Lambda = g_t \Gamma$, then $\Gamma' = g_{t-t'} \Gamma \in \psi(] - 2\alpha, 2\alpha[\times W) \cap S = W$. Hence $\Gamma' \in W$ and since ψ is one to one, $t' - t = 0$ and $\Gamma' = \Gamma$. Finally, $(\sigma(h, \Lambda), \pi(h, \Lambda)) = (\psi|_U)^{-1}(h\Lambda)$ is a continuous function of $(h, \Lambda) \in B(I_{d+c}, \eta) \times K$. \square

The second lemma is a purely measure-theoretic result.

Lemma 8.5. *Let X and Y be locally compact second countable metric spaces. Let μ_X and μ_Y be two measures on X and Y finite on compact subsets. Suppose $\psi : X \rightarrow Y$ is a continuous map such that every y in Y has at most N preimages and such that for all x in X there exists a compact neighborhood ω_x of x with the following property:*

- ψ is one to one on ω_x ,
- the image by ψ of the measure $1_{\omega_x} \mu_X$ is the measure $1_{\psi(\omega_x)} \mu_Y$.

Then for all nonnegative measurable functions $f : Y \rightarrow \mathbb{R}$,

$$\int_X f \circ \psi d\mu_X \leq N \int_Y f d\mu_Y.$$

Proof. Since X is a countable union of compact sets, making use of the monotone convergence theorem, we can reduce to X compact. Next, since for any positive Radon measure μ on Y and for any Borel set $B \subset Y$, $\mu(B) = \sup\{\mu(K) : K \text{ compact } \subset B\}$, we can assume that $f = 1_K$ is the indicator function of a compact subset K of Y . With these assumptions, $L = \psi^{-1}(K)$ is a compact subset of X and therefore is included in a finite union of ω_x , $L \subset \cup_{i=1}^n \omega_{x_i}$. For each i , let $\omega'_{x_i} = \omega_{x_i} \setminus \cup_{j < i} \omega_{x_j}$, $L_i = \omega'_{x_i} \cap L$ and $K_i = \psi(L_i)$. Since L is the disjoint union of the L_i , $\mu_X(L) = \sum_{i=1}^n \mu_X(L_i)$. Since L_i is included in ω_{x_i} , $\mu_X(L_i) = \mu_Y(K_i)$. Since the L_i are disjoint, the number of ψ -preimage of any y is at least $F(y) = \sum_{i=1}^n 1_{K_i}(y)$ therefore $F \leq N$. It follows that

$$N\mu_Y(K) \geq \int_K F(y) d\mu_Y(y) = \sum_{i=1}^n \mu_Y(K_i) = \sum_{i=1}^n \mu_X(L_i) = \mu_X(L).$$

\square

The last lemma is an easy consequence of the previous lemma and of the definition of the induced measure μ_S .

Lemma 8.6. *Let U be a Borel subset in \mathcal{L}_{d+c} such that for all Λ in U , $g_t\Lambda \in U$ for all t in a time interval I_Λ of length 1 containing 0. Then*

$$\mu_S(U \cap S) \leq 4A\mu(U)$$

where A is the maximum number of entrance times in S of a flow trajectory during a time interval of length 1 (see Lemma 5.1).

The constant $4A$ is certainly not the best one.

Proof. Take $X = [-1, 1] \times S$, $Y = \mathcal{L}_{d+c}$, $\psi : X \rightarrow Y$ the map defined by $\psi(t, \Lambda) = g_t\Lambda$, the measures $\mu_X = dt \otimes \mu_S$ and $\mu_Y = \mu$ and the function $f = 1_U$. By definition of the induced measure, we know that the image of the restriction of $dt \otimes \mu_S$ to a small enough compact neighborhood ω of any $x \in X$ is the restriction to $\psi(\omega)$ of the invariant measure μ on \mathcal{L}_{d+c} . Therefore by Lemma 8.5,

$$N\mu(U) \geq \int_S \int_{-1}^1 1_U(g_t\Lambda) dt d\mu_S(\Lambda)$$

where N is the maximum number of preimages. By assumption $\int_{-1}^1 1_U(g_t\Lambda) dt \geq 1$ for all Λ in U and it is easy to see that $N \leq 4A$. Therefore $4A\mu(U) \geq \mu_S(U \cap S) \times 1$. \square

8.1.2 An example

We want to construct a co-dimension one submanifold V in \mathcal{L}_{d+c} transverse to the flow g_t together with a bounded continuous function $\varphi : V \rightarrow \mathbb{R}$ such that for a set of $\theta \in M_{d,c}(\mathbb{R})$ of positive measure, the sequence $\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_V^k(\Lambda_\theta)$ does not converge to $\frac{1}{\mu_V(V)} \int_V \varphi d\mu_V$ where R_V is the first return map in V and μ_V is the invariant measure induced by the flow. The idea is the following. Take V an open set in S . Then μ_V is the restriction of μ_S to V . Suppose that the open set V can be chosen in order that for all $\theta \in M_{d,c}(\mathbb{R})$, and all $k \geq 1$,

$$R_V^k(\Lambda_\theta) = R_S^k(\Lambda_\theta) (= R^k(\Lambda_\theta)).$$

Then if $\varphi : S \rightarrow \mathbb{R}$ is a non negative continuous function not identically zero with support included in V , the sequences

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_V^k(\Lambda_\theta), \quad \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_S^k(\Lambda_\theta)$$

converge to the same limit which cannot be equal to both $\frac{1}{\mu_S(S)} \int_S \varphi d\mu_S$ and $\frac{1}{\mu_V(V)} \int_V \varphi d\mu_V = \frac{1}{\mu_S(V)} \int_S \varphi d\mu_S$ provided that $\mu_S(V) < \mu_S(S)$. So we are reduced to constructing V .

Observe that Theorem 8.1 implies that for such a V , $\varphi = 1_V$ is not almost everywhere continuous on S which means that the boundary of V in S has positive measure. Moreover, it shows that the assumption about the continuity of the function φ in Theorem 8.1, cannot be dropped.

8.1.3 Construction of V

Consider the set \mathbb{T} of lattices Λ_θ such that the coefficients of θ are all in $[0, 1]$. It is a compact subset in \mathcal{L}_{d+c} containing all the lattices Λ_θ . Denote W_ε the open ball $B(I_{d+c}, \varepsilon)$ in $\mathrm{SL}(d+c, \mathbb{R})$. We consider the open sets

$$U_n(\varepsilon) = \bigcup_{t \in [n, n+1]} g_t(W_\varepsilon \mathbb{T})$$

and for a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers, we consider the open set

$$U = U((\varepsilon_n)_{n \in \mathbb{N}}) = \bigcup_{n \in \mathbb{N}} U_n(\varepsilon_n).$$

Take $V = S \cap U$. For all $t \geq 0$ and all θ , $g_t \Lambda_\theta = g_t I_{d+c} \Lambda_\theta$ is in U , hence for all $k \in \mathbb{N}$, we have

$$R_S^k(\Lambda_\theta) = R_V^k(\Lambda_\theta).$$

So we are reduced to showing that when the sequence $(\varepsilon_n)_n$ is small enough,

$$\mu_S(V) < \mu_S(S).$$

By definition of $U_n(\varepsilon)$, if $\Lambda = g_t g \Lambda_\theta$ with $t \in [n, n+1]$, $g \in W_\varepsilon$ and $\Lambda_\theta \in \mathbb{T}$, then

$$g_s g_t g \Lambda_\theta \in U_n(\varepsilon)$$

for all s in the interval $[n-t, n+1-t]$. So U satisfies the assumption of the Lemma 8.6 and therefore

$$\mu_S(U \cap S) \leq 4A \sum_{n \in \mathbb{N}} \mu(U_n(\varepsilon_n)).$$

Using that $g_t g \Lambda = g_n (g_{t-n} g g_{-(t-n)}) g_{t-n} \Lambda$, we see that

$$\begin{aligned} U_n(\varepsilon) &= \{g_t \Lambda : t \in [n, n+1], \Lambda \in W_\varepsilon \mathbb{T}\} \\ &\subset g_n W_{\varepsilon'} \{g_s \Lambda : s \in [0, 1], \Lambda \in \mathbb{T}\} \end{aligned}$$

where ε' is such that $g_s W_\varepsilon g_{-s} \subset W_{\varepsilon'}$ for all s in $[0, 1]$. Furthermore, the compact set $\{g_s \Lambda : s \in [0, 1], \Lambda \in \mathbb{T}\}$ has zero measure because it has dimension $cd + 1$ which is $< (c+d)^2 - 1$. Therefore

$$\lim_{\varepsilon' \rightarrow 0} \mu(W_{\varepsilon'} \{g_s \Lambda : s \in [0, 1], \Lambda \in \mathbb{T}\}) = 0,$$

which implies

$$\lim_{\varepsilon \rightarrow 0} \mu(U_n(\varepsilon)) = 0.$$

So there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that

$$\sum_{n \in \mathbb{N}} \mu(U_n(\varepsilon_n)) < \frac{1}{4A} \mu(S)$$

and for such a sequence, we have $\mu_S(V) < \mu_S(S)$.

8.2 Proof of Proposition 8.3

Let $\varphi : S \rightarrow \mathbb{R}$ be a bounded continuous function.

Let ε be a positive real number, let K and δ be associated with ε by Lemma 8.2, and α and η associated with K by Lemma 8.4.

Preliminary observation. Let $(a_n)_{n \in \mathbb{N}}$ be a decreasing sequence of real numbers in $]0, \eta[$ tending to zero and let $L_{a_n} = B(I_{d+c}, a_n) \times K$. Since the intersection of all the compact sets L_{a_n} , $n \in \mathbb{N}$, is $L_0 = \{I_{d+c}\} \times K$ and since the map $\psi(g, \Lambda) = \varphi(\pi(g, \Lambda)) - \varphi(\Lambda)$ is continuous, we have

$$\bigcap_{n \geq 0} \psi(L_{a_n}) = \psi(\bigcap_{n \geq 0} L_{a_n}) = \psi(L_0) = \{0\}.$$

Therefore, for n large enough, $\psi(L_{a_n}) \subset]-\varepsilon, \varepsilon[$ which implies there exists $\beta > 0$ such that for all $\Lambda \in K$ and all $g \in B(I_{d+c}, \beta)$,

$$|\varphi(\pi(g, \Lambda)) - \varphi(\Lambda)| \leq \varepsilon. \quad (6)$$

Finally, let $\gamma > 0$ be such that for all $s \geq 0$ and all $h \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \gamma) B_{\mathcal{H}_{\leq}}^{-1}(I_{d+c}, \gamma)$,

$$d(g_s h g_{-s}, I_{d+c}) \leq \min(\delta, \beta, \eta).$$

For $T \geq 0$, $\Lambda \in \mathcal{L}_{d+c}$, and $E \subset S$, denote

$$I(T, \Lambda, E) = \{t \in [0, T] : g_t \Lambda \in E\}.$$

For almost all $\theta \in M_{d,c}(\mathbb{R})$, we can fix $h_\theta \in B_{\mathcal{H}_{\leq}}(I_{d+c}, \gamma)$ such that the conclusion of the Birkhoff ergodic theorem holds for the flow g_t or the first return map in S and the lattice $h_\theta \Lambda_\theta$. Observe that $h_\theta = h_{\theta, \varepsilon}$ depends on ε . It is understood that we shall use the Birkhoff ergodic theorem in countably many situations. We fix a sequence (ε_n) going to zero and for each $\varepsilon = \varepsilon_n$, we use three times Birkhoff ergodic theorem and the ergodicity of the flow: for almost all θ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T 1_{U(K, \delta)}(g_t h_\theta \Lambda_\theta) dt = \frac{1}{\mu(\mathcal{L}_{d+c})} \mu(U(K, \delta)) \leq \varepsilon, \quad (7)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{card } I(T, h_\theta \Lambda_\theta, S) = \frac{1}{\mu_S(S)} \int_S \tau_S d\mu_S = \frac{\mu(\mathcal{L}_{d+c})}{\mu_S(S)}, \quad (8)$$

where τ_S is the first return time in S , and

$$\lim_{T \rightarrow \infty} \frac{1}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \sum_{t \in I(T, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta) = \frac{1}{\mu_S(S)} \mu_S(\varphi). \quad (9)$$

Let T be positive and let $s_1 < \dots < s_m$ be the elements of $I(T, \Lambda_\theta, S \setminus K)$, we have

$$g_t h_\theta \Lambda_\theta = g_{t-s_i} (g_{s_i} h_\theta g_{-s_i}) g_{s_i} \Lambda_\theta \in U(K, \delta)$$

for all $t \in [s_i, s_i + 1]$ and we can extract a subsequence s_{n_1}, \dots, s_{n_p} defined by $n_1 = 1$ and $n_{i+1} = \min\{j : s_j \geq s_{n_i} + 1\}$. Now by Lemma 5.1, there is an absolute constant A such that there are at most A elements of $I(T, \Lambda_\theta, S \setminus K)$ ($\subset I(T, \Lambda_\theta, S)$) in an interval of length 1, hence $Ap \geq m$. Therefore, by (7)

$$\frac{m}{A} \leq p \leq \int_0^{T+1} 1_{U(K, \delta)}(g_t h_\theta \Lambda_\theta) dt \leq 2(T+1)\varepsilon$$

and hence

$$\text{card } I(T, \Lambda_\theta, S \setminus K) \leq 3AT\varepsilon \quad (10)$$

for T large enough: $T \geq T(\Lambda_\theta, \varepsilon)$. We will also need to bound above the number of elements in $I(T, h_\theta\Lambda_\theta, S \setminus K)$. Making use of (7), the same way of reasoning leads to the same result

$$I(T, h_\theta\Lambda_\theta, S \setminus K) \leq 3AT\varepsilon \quad (11)$$

for $T \geq T(\Lambda_\theta, \varepsilon)$.

Heart of the proof. We want to compare

$$\Sigma_1 = \frac{1}{\text{card } I(T, \Lambda_\theta, S)} \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t\Lambda_\theta)$$

with

$$\Sigma_2 = \frac{1}{\text{card } I(T, h_\theta\Lambda_\theta, S)} \sum_{t \in I(T, h_\theta\Lambda_\theta, S)} \varphi(g_t h_\theta\Lambda_\theta)$$

because by (9), this latter sum tends to $\frac{1}{\mu_S(S)} \int_S \varphi d\mu_S$ when T goes to infinity. We split $\sum_{t \in I(T, \Lambda_\theta, S)}$ into two sums $\sum_{t \in I(T, \Lambda_\theta, K)}$ and $\sum_{t \in I(T, \Lambda_\theta, S \setminus K)}$. Observe that for $t \in I(T, \Lambda_\theta, K)$, $g_t h_\theta\Lambda_\theta = (g_t h_\theta g_{-t}) g_t \Lambda_\theta$ is of the form $g\Lambda$ with $g \in B(I_{d+c}, \eta)$ and $\Lambda \in K$, this allows to use Lemma 8.4. We use the notations of Lemma 8.4 and for t in $I(T, \Lambda_\theta, K)$, we denote $t' = \sigma(g_t h_\theta g_{-t}, g_t \Lambda_\theta)$. By (6), we have

$$\left| \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_t \Lambda_\theta) - \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(\pi(g_t h_\theta g_{-t}, g_t \Lambda_\theta)) \right| \leq \varepsilon \text{card } I(T, \Lambda_\theta, K).$$

Now,

$$\pi(g_t h_\theta g_{-t}, g_t \Lambda_\theta) = g_{-t'} g_t h_\theta g_{-t} g_t \Lambda_\theta = g_{t-t'} h_\theta \Lambda_\theta$$

hence

$$\left| \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_t \Lambda_\theta) - \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_{t-t'} h_\theta \Lambda_\theta) \right| \leq \varepsilon \text{card } I(T, \Lambda_\theta, K).$$

Observe that the map $t \in I(T, \Lambda_\theta, K) \rightarrow t - t'$ is one to one because $t' \in [-\frac{\alpha}{4}, \frac{\alpha}{4}]$ and the gap between two visiting times of K is $\geq \alpha$. Observe also that $t - t' \in I(T, h_\theta\Lambda_\theta, S)$ except possibly for the first and the last element of $t \in I(T, h_\theta\Lambda_\theta, S)$. On the one hand, it follows that

$$\begin{aligned} & \left| \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t \Lambda_\theta) - \sum_{t \in I(T, h_\theta\Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta) \right| \\ & \leq \varepsilon \text{card } I(T, \Lambda_\theta, K) \\ & + \|\varphi\|_\infty (\text{card } I(T, \Lambda_\theta, K \setminus S) + \text{card } I(T, h_\theta\Lambda_\theta, S) - \text{card } I(T, \Lambda_\theta, K) + 2). \end{aligned}$$

On the other hand, it follows that

$$\text{card } I(T, h_\theta\Lambda_\theta, S) \geq \text{card } I(T, \Lambda_\theta, K) - 2$$

and the same way of reasoning leads to

$$\text{card } I(T, \Lambda_\theta, S) \geq \text{card } I(T, h_\theta \Lambda_\theta, K) - 2.$$

Making use of (10) and (11), we obtain

$$\begin{aligned} -2 &\leq \text{card } I(T, h_\theta \Lambda_\theta, S) - \text{card } I(T, \Lambda_\theta, K) = \\ &\text{card } I(T, h_\theta \Lambda_\theta, S) - \text{card } I(T, h_\theta \Lambda_\theta, K) \\ &+ \text{card } I(T, h_\theta \Lambda_\theta, K) - \text{card } I(T, \Lambda_\theta, S) \\ &+ \text{card } I(T, \Lambda_\theta, S) - \text{card } I(T, \Lambda_\theta, K) \\ &\leq 3AT\varepsilon + 2 + 3AT\varepsilon = 6AT\varepsilon + 2, \end{aligned}$$

hence (using (10) once again)

$$\left| \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t \Lambda_\theta) - \sum_{t \in I(T, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta) \right| \leq \varepsilon \text{card } I(T, \Lambda_\theta, K) + (9A\varepsilon T + 2) \|\varphi\|_\infty$$

for $T \geq T(\Lambda_\theta, \varepsilon)$. We obtain

$$\begin{aligned} |\text{card } I(T, h_\theta \Lambda_\theta, S) - \text{card } I(T, \Lambda_\theta, S)| &\leq |\text{card } I(T, h_\theta \Lambda_\theta, S) - \text{card } I(T, \Lambda_\theta, K)| \\ &+ |\text{card } I(T, \Lambda_\theta, S \setminus K)| \\ &\leq 9AT\varepsilon + 2 \end{aligned}$$

as well. Relation (8) implies that $\text{card } I(T, h_\theta \Lambda_\theta, S) \geq aT$ for $T \geq T(\Lambda_\theta, \varepsilon)$ where $a = \frac{1}{2} \frac{\mu(\mathcal{L})}{\mu_S(S)}$. All together, for $T \geq T(\Lambda_\theta, \varepsilon)$, we obtain

$$\begin{aligned} |\Sigma_2 - \Sigma_1| &\leq \left| \Sigma_2 - \frac{\text{card } I(T, \Lambda_\theta, S)}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \Sigma_1 \right| + \left| \frac{\text{card } I(T, \Lambda_\theta, S) - \text{card } I(T, h_\theta \Lambda_\theta, S)}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \right| |\Sigma_1| \\ &\leq \frac{1}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \left| \sum_{t \in I(T, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta) - \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t \Lambda_\theta) \right| \\ &+ \left| \frac{\text{card } I(T, \Lambda_\theta, S) - \text{card } I(T, h_\theta \Lambda_\theta, S)}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \right| \|\varphi\|_\infty \\ &\leq \frac{\varepsilon \text{card } I(T, \Lambda_\theta, K) + (9AT\varepsilon + 2) \|\varphi\|_\infty}{\text{card } I(T, h_\theta \Lambda_\theta, S)} + \frac{(9AT\varepsilon + 2) \|\varphi\|_\infty}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \\ &\leq \varepsilon \left(1 + \frac{18AT \|\varphi\|_\infty}{aT} \right) + \frac{4 \|\varphi\|_\infty}{aT} \end{aligned}$$

which is $\ll \varepsilon$ when T is large enough.

8.3 Proof of Theorem 8.1

Step 1. Let us show that the restriction of f to S is integrable with respect to μ_S .

We use Lemma 8.5 with $X =]0, 1[\times S$, $Y = \mathcal{L}_{d+c}$, the map $\psi :]0, 1[\times S \rightarrow \mathcal{L}$ defined by $\psi(t, \Lambda) = g_t \Lambda$, the measures $\mu_X = dt \otimes \mu_S$ and $\mu_Y = \mu$, and the function f . By definition of

the induced measure, we know that the image of the restriction of $dt \otimes \mu_S$ to any small enough open subset ω is the restriction to $\psi(\omega)$ of the invariant measure μ on \mathcal{L}_{d+c} . Now, by Lemma 5.1, each element of \mathcal{L}_{d+c} has at most A ψ -preimages, therefore by Lemma 8.5,

$$\int_0^1 \int_S f(g_t \Lambda) d\mu_S dt \leq A \int_{\mathcal{L}_{d+c}} f d\mu.$$

Since f is uniformly continuous in the \mathcal{H}_{\leq} direction, there exists $\Delta > 0$ such that for all Λ , and all $t \in [0, \Delta]$, $f(g_t \Lambda) \geq f(\Lambda) - 1$. Therefore

$$\int_0^\Delta \int_S (f(\Lambda) - 1) d\mu_S dt \leq A \int_{\mathcal{L}_{d+c}} f d\mu,$$

which implies $\int_S f(\Lambda) d\mu_S \leq \mu_S(S) + \frac{A}{\Delta} \int_{\mathcal{L}_{d+c}} f d\mu < +\infty$.

Step 2. It is enough to prove the theorem for non negative continuous functions φ .

Indeed, since φ is continuous almost everywhere and since $|\varphi| \leq f$ with f continuous and in $L^1(\mu_S)$, for all positive integer p , there exist two continuous functions φ_p^- and φ_p^+ such that

$$-f \leq \varphi_p^- \leq \varphi \leq \varphi_p^+ \leq f$$

and

$$\int_S \varphi d\mu_S - \frac{1}{p} \leq \int_S \varphi_p^- d\mu_S \leq \int_S \varphi_p^+ d\mu_S \leq \int_S \varphi d\mu_S + \frac{1}{p}.$$

Therefore, if the convergence holds for almost every θ , we have for all the functions φ_p^- and φ_p^+ ,

$$\begin{aligned} \int_S \varphi_p^- d\mu_S &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_p^- \circ R^k(\Lambda_\theta) \leq \lim_{n \rightarrow \infty} \inf \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \\ &\leq \limsup_{p \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_p^+ \circ R^k(\Lambda_\theta) = \int_S \varphi_p^+ d\mu_S \end{aligned}$$

which implies that for almost all θ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) = \int_S \varphi d\mu_S.$$

Thus, we are reduced to proving the theorem for φ continuous. Writing $\varphi = \varphi^+ - \varphi^-$, we can also suppose $0 \leq \varphi \leq f$.

Step 3. We prove that $\lim_{n \rightarrow \infty} \inf \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \geq \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S$ for almost all θ .

Using Proposition 8.3 with the minimum of φ and of a constant M , we obtain for almost all θ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \min(\varphi, M) \circ R^k(\Lambda_\theta) = \frac{1}{\mu_S(S)} \int_S \min(\varphi, M) d\mu_S,$$

hence, letting M going to infinity, we obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta) \geq \frac{1}{\mu_S(S)} \int_S \varphi d\mu_S.$$

Step 4. Main step, we bound above the sums $\sum_{k=0}^{n-1} \varphi \circ R^k(\Lambda_\theta)$.

Since f is in $L^1(\mu)$, there exists $\varepsilon' > 0$ such that for any measurable subset B in \mathcal{L}_{d+c} , we have

$$\mu(B) \leq \varepsilon' \Rightarrow \int_B f d\mu \leq \varepsilon.$$

This allows to strengthen Lemma 8.2:

Lemma 8.7. *For all $\varepsilon > 0$, there exists a compact subset K in S and $\delta > 0$ such that $\frac{1}{\mu(\mathcal{L}_{d+c})} \int_{U(K,\delta)} f d\mu$ and $\frac{1}{\mu(\mathcal{L}_{d+c})} \mu(U(K,\delta))$ are $\leq \varepsilon$.*

We keep all the choices and the notations of the proof of Proposition 8.3, and we use the Birkhoff ergodic theorem with one more function:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(g_t h_\theta \Lambda_\theta) 1_{U(K,\delta)}(g_t h_\theta \Lambda_\theta) dt = \frac{1}{\mu(\mathcal{L}_{d+c})} \int_{U(K,\delta)} f d\mu \leq \varepsilon \quad (12)$$

so that (7), (8), (9) and (12) hold for almost all θ (for (9) observe that $0 \leq \varphi \leq f \in L^1(\mu_S)$).

Since the function f is uniformly continuous in the \mathcal{H}_\leq -direction, there exists $\kappa > 0$ such that $f(h\Lambda) \geq f(\Lambda) - \frac{1}{2}$ for all Λ and all $h \in B_{\mathcal{H}_\leq}(I_{d+c}, \kappa)$. By choosing γ small enough we can suppose that $(g_s h g_{-s}) \in B_{\mathcal{H}_\leq}(I_{d+c}, \kappa)$ for all $s \geq 0$ and all $h \in B_{\mathcal{H}_\leq}(I_{d+c}, \gamma)$. Furthermore, there exists a positive constant $\Delta = \Delta(\kappa)$ such that $g_t \in B_{\mathcal{H}_{\geq 0}}(I_{d+c}, \kappa)$ for all $t \in [0, \Delta]$. Therefore, for all lattices Λ , all non negative real number s , all $h \in B_{\mathcal{H}_\leq}(I_{d+c}, \gamma)$ and all $t \in [0, \Delta]$, we have

$$f(g_t(g_s h g_{-s})g_s \Lambda) \geq f((g_s h g_{-s})g_s \Lambda) - \frac{1}{2} \geq f(g_s \Lambda) - 1$$

and hence

$$f(g_{t+s} h \Lambda) = f(g_t(g_s h g_{-s})g_s \Lambda) \geq f(g_s \Lambda) - 1. \quad (13)$$

As in the proof of Proposition 8.3, let $s_1 < \dots < s_m$ be the elements of $I(T, \Lambda_\theta, S \setminus K)$. On the one hand $g_t h_\theta \Lambda_\theta \in U(K, \delta)$ for all $t \in [s_i, s_i + 1]$, and on the other hand, for almost all θ , (10) and (11) hold for $T \geq T(\Lambda_\theta, \varepsilon)$. We can suppose $\Delta < 1$ and since there are at most A elements of $I(T, \Lambda_\theta, S \setminus K) \subset I(T, \Lambda_\theta, S)$ in an interval of length 1, by (13) we obtain

$$\begin{aligned} \sum_{s \in I(T, \Lambda_\theta, S \setminus K)} \varphi(g_s \Lambda_\theta) &\leq \sum_{s \in I(T, \Lambda_\theta, S \setminus K)} f(g_s \Lambda_\theta) \\ &\leq \sum_{i=1}^m \frac{1}{\Delta} \int_{s_i}^{s_i + \Delta} (1 + 1_{U(K,\delta)}(g_t h_\theta \Lambda_\theta) f(g_t h_\theta \Lambda_\theta)) dt \\ &\leq m + \frac{A}{\Delta} \int_0^{T+1} 1_{U(K,\delta)}(g_t h_\theta \Lambda_\theta) f(g_t h_\theta \Lambda_\theta) dt \end{aligned}$$

and with (10) and (12), this gives

$$\sum_{s \in I(T, \Lambda_\theta, S \setminus K)} f(g_s \Lambda_\theta) \leq 3AT\varepsilon + \frac{A}{\Delta} 3T\varepsilon \leq 6\frac{A}{\Delta} T\varepsilon$$

for all $T \geq T(\Lambda_\theta, \varepsilon)$.

We want to bound above

$$\Sigma_1(T) = \frac{1}{\text{card } I(T, \Lambda_\theta, S)} \sum_{t \in I(T, \Lambda_\theta, S)} \varphi(g_t \Lambda_\theta)$$

with

$$\Sigma_2(T) = \frac{1}{\text{card } I(T, h_\theta \Lambda_\theta, S)} \sum_{t \in I(T, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta)$$

because this latter sum tends to $\frac{1}{\mu_S(S)} \int_S \varphi d\mu_S$ when T goes to infinity. We split $\sum_{t \in I(T, \Lambda_\theta, S)}$ into two sums $\sum_{t \in I(T, \Lambda_\theta, K)}$ and $\sum_{t \in I(T, \Lambda_\theta, S \setminus K)}$. As in the proof of Proposition 8.3, for T large enough, we have

$$\begin{aligned} |I(T, h_\theta \Lambda_\theta, S) - I(T, \Lambda_\theta, S)| &\leq 9AT\varepsilon + 2, \\ I(T, h_\theta \Lambda_\theta, S) &\geq aT, \end{aligned}$$

and

$$\left| \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_t \Lambda_\theta) - \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_{t-t'} h_\theta \Lambda_\theta) \right| \leq \varepsilon \text{card } I(T, \Lambda_\theta, K)$$

where $t' = \sigma(g_t h_\theta g_{-t}, g_t \Lambda_\theta)$ is defined in Lemma 8.4. Taking into account the first element t_{min} and the last element in $I(T, \Lambda_\theta, K)$, the latter inequality implies that

$$\begin{aligned} \sum_{t \in I(T, \Lambda_\theta, K)} \varphi(g_t \Lambda_\theta) &\leq \varphi(g_{t_{min}} \Lambda_\theta) + \sum_{t \in I(T, \Lambda_\theta, K) \setminus \{t_{min}\}} \varphi(g_{t-t'} h_\theta \Lambda_\theta) + \varepsilon \text{card } I(T, \Lambda_\theta, K) \\ &\leq \varphi(R\Lambda_\theta) + \sum_{t \in I(T+1, h_\theta \Lambda_\theta, S)} \varphi(g_t h_\theta \Lambda_\theta) + \varepsilon \text{card } I(T, \Lambda_\theta, K). \end{aligned}$$

All together, we obtain (recall that $\varphi \geq 0$)

$$\begin{aligned} \Sigma_1(T) &\leq \frac{1}{\text{card } I(T, \Lambda_\theta, S)} \varphi(R\Lambda_\theta) + \frac{\text{card } I(T+1, h_\theta \Lambda_\theta, S)}{\text{card } I(T, \Lambda_\theta, S)} \Sigma_2(T+1) + \varepsilon \\ &\quad + \frac{1}{\text{card } I(T, \Lambda_\theta, S)} \sum_{s \in I(T, \Lambda_\theta, S \setminus K)} f(g_s \Lambda_\theta) \\ &\leq \frac{1}{\text{card } I(T, \Lambda_\theta, S)} \varphi(R\Lambda_\theta) + \left(1 + \left| \frac{\text{card } I(T+1, h_\theta \Lambda_\theta, S) - \text{card } I(T, \Lambda_\theta, S)}{\text{card } I(T, \Lambda_\theta, S)} \right| \right) \Sigma_2(T+1) + \varepsilon \\ &\quad + \frac{1}{\text{card } I(T, \Lambda_\theta, S)} 6 \frac{A}{\Delta} T \varepsilon \\ &\leq \Sigma_2(T+1) + \frac{1}{aT - 9AT\varepsilon - 2} \varphi(R\Lambda_\theta) + \frac{9AT\varepsilon + 2 + A}{aT - 9AT\varepsilon - 2} \Sigma_2(T+1) + \left(1 + \frac{6A}{(aT - 9AT\varepsilon - 2)\Delta} \right) \varepsilon \end{aligned}$$

and we are done. \square

8.4 Proofs of Lemma 8.2

We need an auxiliary lemma.

Lemma 8.8. *Let $E(\lambda, \eta)$ be the set of lattices Λ in $\mathcal{L}_{d,c}$ such that there exist two nonzero vectors $X \neq \pm X'$ of Λ in the open ball $\overset{\circ}{B}_{d,c}(0, \lambda)$ with $\frac{1}{1+\eta} < \frac{|X|_{\pm}}{|X'|_{\pm}} < 1 + \eta$ or a nonzero vector X in the open ball $\overset{\circ}{B}_{d,c}(0, \lambda)$ with $|X|_{\pm} < \eta$. For all $\lambda > 0$, we have $\lim_{\eta \rightarrow 0} \mu(E(\lambda, \eta)) = 0$.*

Proof. Since $\lim_{\rho \rightarrow 0} \mu(\{\lambda_1(\Lambda) \leq \rho\}) = 0$, it is enough to show that for all $\rho > 0$, $\mu(E(\lambda, \eta) \cap \{\lambda_1(\Lambda) \geq \rho\}) \rightarrow 0$ when η goes to 0. Choose a Siegel reduction domain $\mathcal{S} \subset \text{SL}(d+c, \mathbb{R})$ (see [3]). There is a constant $c = c(\mathcal{S}) > 0$ such that for all matrices M in \mathcal{S} and all vectors Y in \mathbb{R}^{d+c} , we have

$$\|MY\|_{d,c} \geq c\lambda_1(\Lambda) \|Y\|_{d,c}.$$

where $\Lambda = M\mathbb{Z}^{d+c}$ (this inequality holds for all norms with a constant c depending only on the norm, just use the norm equivalence). It follows that we can find a finite subset F_{ρ} of \mathbb{Z}^{d+c} such that for all matrices M in \mathcal{S} with $\lambda_1(M\mathbb{Z}^{d+c}) \geq \rho$, the only Y in \mathbb{Z}^{d+c} such that $\|MY\|_{d,c} \leq \lambda$, are in F_{ρ} . Therefore, if a matrix M in \mathcal{S} is such that $\Lambda = M\mathbb{Z}^{d+c}$ belongs to $E(\lambda, \eta) \cap \{\lambda_1(\Lambda) \geq \rho\}$, then there exist a nonzero Y in F_{ρ} or two nonzero vectors $Y \neq \pm Y'$ in F_{ρ} with

$$|MY|_{\pm} \leq \eta$$

or

$$\frac{1}{1+\eta} < \frac{|MY|_{\pm}}{|MY'|_{\pm}} < 1 + \eta.$$

For a fixed Y or a fixed pair $Y \neq \pm Y'$ of nonzero vectors in F_{ρ} , the measure of the set of M in \mathcal{S} for which the above inequalities hold, goes to 0 as η goes to 0. Since F_{ρ} is finite and since a Siegel domain contains a fundamental domain, we are done. \square

Proof of Lemma 8.2. Consider the set $V(\lambda, \eta, \rho) = E(\lambda, \eta) \cup \{\lambda_1(\Lambda) < \rho\}$.

Step 1: *The complementary V^C of $V(\lambda, \eta, \rho)$ is a closed subset of $\mathcal{L}_{d,c}$.*

Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of points of V^C which converges to Γ in $\mathcal{L}_{d,c}$. Firstly, since λ_1 is continuous, $\lambda_1(\Gamma) = \lim \lambda_1(\Lambda_n) \geq \rho$. Secondly, there is a sequence of matrices $(M_n)_{n \in \mathbb{N}}$ such that $\Lambda_n = M_n\mathbb{Z}^{d+c}$ for all $n \in \mathbb{N}$ and such that $(M_n)_{n \in \mathbb{N}}$ converges to M with $\Gamma = M\mathbb{Z}^{d+c}$. We have to show that Γ is not in $E(\lambda, \eta)$. Let $X = MY$ and $X' = MY'$ be two nonzero vectors in Γ with $X \neq \pm X'$ and $\|X\|_{d,c}, \|X'\|_{d,c} < \lambda$. When n is large enough, $X_n = M_n Y$ and $X'_n = M_n Y'$ are in the open ball $\overset{\circ}{B}_{d,c}(0, \lambda)$ and since $\Lambda_n = M_n\mathbb{Z}^{d+c}$ is not in $E(\lambda, \eta)$, we have both

$$|M_n Y|_{\pm} \geq \eta \quad \text{and} \quad \frac{|M_n Y|_{\pm}}{|M_n Y'|_{\pm}} \notin \left] \frac{1}{1+\eta}, 1+\eta \right[.$$

Passing through the limit we obtain

$$|X|_{\pm} \geq \eta \quad \text{and} \quad \frac{|X|_{\pm}}{|X'|_{\pm}} = \frac{|MY|_{\pm}}{|MY'|_{\pm}} \notin \left] \frac{1}{1+\eta}, 1+\eta \right[.$$

Therefore Γ is not in $E(\lambda, \eta)$.

Step 2: $F = S \setminus V(\lambda, \eta, \rho)$ is a compact subset of S when $\lambda \geq 2 \max\{\lambda_1(\Lambda) : \Lambda \in \mathcal{L}_{d,c}\}$.

Thanks to the Malher compactness theorem, it is enough to prove that F is a closed subset of $\mathcal{L}_{d,c}$. Let $(\Lambda_n)_{n \in \mathbb{N}}$ be a sequence of points in F which converges to Γ in $\mathcal{L}_{d,c}$. We want to prove that Γ is in F . By the first step, it is enough to prove that Γ is in S . Choose a Siegel domain \mathcal{S} . There is a sequence of matrices $M_n \in \mathcal{S}$ such that $\Lambda_n = M_n \mathbb{Z}^{d+c}$ for all $n \in \mathbb{N}$ and such that $(M_n)_{n \in \mathbb{N}}$ converges to $M \in \mathcal{S}$. For each n , there are two vectors $Y_{1,n}$ and $Y_{2,n}$ in \mathbb{Z}^{d+c} such that $X_{1,n} = M_n Y_{1,n}$ and $X_{2,n} = M_n Y_{2,n}$ are the two vectors associated with Λ_n by the definition of S . Since the matrices M_n are all in the Siegel domain \mathcal{S} and since $\|M_n Y_{i,n}\|_{d,c} = \lambda_1(\Lambda_n)$, $i = 1, 2$, the sequences $(Y_{i,n})_{n \in \mathbb{N}}$, $i = 1, 2$, are bounded sequences in \mathbb{Z}^{d+c} . Therefore, extracting subsequences, we can suppose that the two sequences $(Y_{i,n})_{n \in \mathbb{N}}$ are constant: $Y_{i,n} = Y_i$ for all n , $i = 1, 2$. It follows that $\|MY_i\|_{d,c} = \lim_{n \rightarrow \infty} \|M_n Y_i\|_{d,c} = \lim_{n \rightarrow \infty} \lambda_1(\Lambda_n) = \lambda_1(\Gamma)$. Moreover

$$|MY_1|_+ = \lim_{n \rightarrow \infty} |M_n Y_1|_+ = \lim_{n \rightarrow \infty} \lambda_1(\Lambda_n) = \lambda_1(\Gamma)$$

and

$$|MY_2|_- = \lim_{n \rightarrow \infty} |M_n Y_2|_- = \lim_{n \rightarrow \infty} \lambda_1(\Lambda_n) = \lambda_1(\Gamma).$$

Suppose now that $\lambda \geq 2 \max\{\lambda_1(\Lambda) : \Lambda \in \mathcal{L}_{d,c}\}$, then making use of the first step, we conclude that Γ is in S .

Step 3. For a neighborhood W of I_{d+c} in $\text{SL}(d+c, \mathbb{R})$, let

$$U_W = \{g_t h \Lambda : t \in [0, 1], h \in W, \Lambda \in V(\lambda, \eta, \rho)\}.$$

Let us show that we can choose W in order that $U_W \subset V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$. It will finish the proof of Lemma 8.2. Indeed, we first fix $\lambda \geq 2 \max\{\lambda_1(\Lambda) : \Lambda \in \mathcal{L}_{d,c}\}$, next we take η and ρ such that $\mu(E(2e^{d+c}\lambda, 5e^{d+c}\eta)) \leq \frac{\varepsilon}{2}$ and $\mu(\{\lambda_1 < 2e^{d+c}\rho\}) \leq \frac{\varepsilon}{2}$, then we take W such that $U_W \subset V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$ and δ such that $B(I_{d+c}, \delta) \subset W$. Now by the second step $K = S \setminus V(\lambda, \eta, \rho)$ is compact and since $U(K, \delta) \subset U_W \subset V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$, we have $\mu(U(K, \delta)) \leq \varepsilon$.

Let Λ be in $V(\lambda, \eta, \rho)$, h in W and $t \in [0, 1]$. We explain how to successively reduce W in order to obtain the above inclusion.

Case 1. Suppose $\lambda_1(\Lambda) < \rho$. We can choose W small enough in order that $\|hX\|_{d,c} \leq 2\|X\|_{d,c}$ for all h in W and all X in \mathbb{R}^{d+c} . This implies that $\lambda_1(g_t h \Lambda) \leq 2e^{d+c}\lambda_1(\Lambda) < 2e^{d+c}\rho$, hence $g_t h \Lambda \in V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$.

Case 2. Suppose there exists a nonzero vector X in $\Lambda \cap \overset{\circ}{B}_{d,c}(0, \lambda)$ with $|X|_- < \eta$ (the case $|X|_+ < \eta$ is similar). Call p_{\pm} the projections on the subspaces E_{\pm} and $\|u\|$ the norm of the linear operator u associated with the norm $\|\cdot\|_{d,c}$. The vector $g_t h X$ is in the open ball $\overset{\circ}{B}_{d,c}(0, 2\lambda e^{d+c})$ and we have

$$p_- g_t h X = g_t p_- h p_- X + g_t p_- h p_+ X,$$

hence

$$|g_t h X|_- < e^{d+c} \|p_- h\| \eta + e^{d+c} \|p_- h p_+\| \lambda.$$

We can choose W in order that $\|p_- h p_+\| < \frac{\eta}{\lambda}$ and $\|p_- h\| \leq 2$. Then we have $|g_t h X|_- < 3e^{d+c}\eta$ which implies that $g_t h \Lambda \in V(2e^{d+c}\lambda, 5e^{d+c}\eta, 2e^{d+c}\rho)$.

Case 3. Suppose there exist two distinct nonzero vectors X and X' in $\Lambda \cap \overset{\circ}{B}_{d,c}(0, \lambda)$ such that

$$|X|_-, |X'|_- \geq \eta \text{ and } \frac{1}{1+\eta} < \frac{|X|_-}{|X'|_-} < 1+\eta.$$

The case with $|\cdot|_+$ is similar.
As above,

$$\begin{aligned} |hX|_- &< \|p_-h\| |X|_- + \|p_-hp_+\| \lambda \\ &\leq (\|p_-h\| + \|p_-hp_+\| \frac{\lambda}{\eta}) |X|_-. \end{aligned}$$

We can choose W in order that $\|p_-h\| + \|p_-hp_+\| \frac{\lambda}{\eta} \leq 1+\eta$. We also have

$$|hX'|_- \geq \|p_-hp_-X'\|_{d,c} - \|p_-hp_+\| \|X'\|_{d,c}.$$

We can choose W in order that $\|p_-hp_+\| \frac{\lambda}{\eta} \leq \eta$ and $\|p_- - p_-hp_-\| \leq \eta$. With this choice, we have

$$|hX'|_- \geq |X'|_- - \eta |X'|_- - \eta |X'|_-.$$

It follows that

$$\frac{|g_t hX|_-}{|g_t hX'|_-} = \frac{|hX|_-}{|hX'|_-} \leq \frac{|X|_-}{|X'|_-} \times \frac{1+\eta}{1-2\eta} \leq \frac{(1+\eta)^2}{1-2\eta} \leq 1+5\eta$$

when η is small enough. Inverting the role of X and X' , we get the inequality $\frac{|hX'|_-}{|hX|_-} \leq 1+5\eta$ and we are done. \square

8.5 Proofs of Theorems 1 and 2.1.

We begin by the proof of Theorem 1 which is more difficult. We want to prove that for almost all θ in $M_{d,c}(\mathbb{R})$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\theta) = L_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho(\Lambda) d\mu_S(\Lambda)$$

and that

$$\lim_{n \rightarrow \infty} \frac{-1}{n} \ln r_n(\theta) = \frac{c}{d} L_{d,c}.$$

Let us prove that the convergence almost everywhere of $\frac{1}{n} \ln q_n(\theta)$ to $L_{d,c} = \frac{1}{\mu_S(S)} \int_S \rho(\Lambda) d\mu_S(\Lambda)$, implies the convergence almost everywhere of $\frac{-1}{n} \ln r_n(\theta)$ to $\frac{c}{d} L_{d,c}$. By Dirichlet's theorem we know that for all θ and all integers n , $q_n(\theta)^c r_{n+1}(\theta)^d \leq C$ where C is some universal constant (depending on the norms). Next, by the divergence part of the Khintchin-Groshev theorem (see [4]), we know that for almost all θ , there exist only finitely many $Q \in \mathbb{Z}^c$ such that

$$d(\theta Q, \mathbb{Z}^d) \leq \psi(\|\theta\|_{\mathbb{R}^c})$$

where $\psi(q) = \frac{1}{(q^c \ln q)^{1/d}}$. Therefore, for almost all θ , $\frac{1}{\ln q_n(\theta)} \leq q_n(\theta)^c r_n(\theta)^d \leq C$ for all n large enough. It follows that for almost all θ ,

$$\frac{c}{n} \ln q_n(\theta) + \frac{\ln \ln q_n(\theta)}{n} \geq -\frac{d}{n} \ln r_n(\theta)^d \geq \frac{c}{n} \ln q_n(\theta) - \frac{\ln C}{n}$$

for all large enough n . Since the left hand side of the first inequality and the right hand side of the second inequality both converge to $cL_{d,c}$ when $\lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\theta) = L_{d,c}$, the sequence $(-\frac{d}{n} \ln r_n(\theta)^d)_n$ converges to the same limit for almost all θ . Therefore, the proof of Theorem 1 reduces to prove the first almost everywhere limit.

Let us now prove the first almost everywhere limit. We need two lemmas. The first one is clear.

Lemma 8.9. *For all compact sets K in \mathbb{R}^{d+c} and all $\varepsilon > 0$, there exists $\alpha > 0$ such that for all $g \in B(I_{d+c}, \alpha)$ and all $x \in K$, $d(gx, x) \leq \varepsilon$.*

Lemma 8.10. *Let Λ be in $S' \setminus \mathcal{N}$. Then the return map $R = R_S$ is defined on some neighborhood of Λ and is continuous at Λ .*

Proof. Consider the minimal vectors $X_0 = X_0(\Lambda)$ and $X_1 = X_1(\Lambda)$. By definition of S' the only nonzero vectors $B_{d,c}(0, \lambda_1(\Lambda)) = \mathcal{C}(X_0)$ are $\pm X_0$. Therefore, there exists $\varepsilon > 0$ such that all X in $\Lambda \setminus \{0, \pm X_0\}$ are at a distance $\geq \varepsilon$ from $\mathcal{C}(X_0)$. Since Λ is not in \mathcal{N} , $\pm X_0$ and $\pm X_1$ are the only nonzero vectors of Λ in the cylinder $\mathcal{C}(X_0, X_1)$. Therefore reducing ε if necessary, all X in $\Lambda \setminus \{0, \pm X_0, \pm X_1\}$ are at a distance $\geq \varepsilon$ from $\mathcal{C}(X_0, X_1)$. By the above lemma we can choose $\delta > 0$ such that $\forall g \in B(I_{d+c}, \delta)$, $\forall X \in \mathcal{C}(X_0, X_1) + B_{d,c}(0, 1)$,

$$\max(d(g^{-1}X, X), d(gX, X)) \leq \varepsilon/3.$$

It follows that for all $g \in B(I_{d+c}, \delta)$, $\pm gX_0$ are the only nonzero vectors of $g\Lambda$ in $\mathcal{C}(gX_0)$ and that $\pm gX_0$ and $\pm gX_1$ are the only nonzero vectors of Λ in $\mathcal{C}(gX_0, gX_1)$. It follows that if the lattice $\Gamma = g\Lambda$ is in the set of lattices $B(I_{d+c}, \delta)\Lambda \cap S'$ then $X_0(\Gamma) = gX_0$ and $X_1(\Gamma) = gX_1$. By definition of S , the return times from Λ and Γ are well defined and we have

$$\tau(\Lambda) = \frac{1}{d+1} \ln \frac{|X_1|_-}{|X_0|_+} \quad \text{and} \quad \tau(\Gamma) = \frac{1}{d+1} \ln \frac{|gX_1|_-}{|gX_0|_+},$$

hence $R(\Lambda)$ and $R(\Gamma)$ are defined and

$$\begin{aligned} |\tau(\Lambda) - \tau(\Gamma)| &= \frac{1}{d+1} \left| \ln \frac{|X_1|_- |gX_0|_+}{|X_0|_+ |gX_1|_-} \right| \\ &\leq \frac{1}{d+1} \left(\left| \ln \frac{|X_1|_-}{|gX_1|_-} \right| + \left| \ln \frac{|gX_0|_+}{|X_0|_+} \right| \right), \end{aligned}$$

which goes to zero when δ goes to zero. □

End of proof of Theorem 1. We use Theorem 8.1 with S' and the function $\varphi : S' \rightarrow \mathbb{R}_{\geq 0}$ defined by $\varphi(\Lambda) = \rho \circ R(\Lambda) = \ln \frac{q_1(\Lambda)}{q_0(\Lambda)}$ when $R(\Lambda)$ is defined and by $\varphi(\Lambda) = 0$ otherwise. Since ρ is continuous on S , the above lemma implies that φ is almost everywhere continuous on S' . We

want to find a uniformly continuous function $f : \mathcal{L} \rightarrow \mathbb{R}$ such that $|\varphi| \leq f$. Observe that φ is non negative. By Lemma 3.6, for all lattice $\Lambda \in \mathcal{L}$,

$$q_1(\Lambda)^c r_0(\Lambda)^d \leq C = C_{d,c}$$

where $C_{d,c}$ depends only on c and d . It follows that for all Λ in S' , we have

$$\begin{aligned} \varphi(\Lambda) &= \ln \frac{q_1(\Lambda)}{q_0(\Lambda)} \\ &= \ln \frac{q_1(\Lambda) r_0(\Lambda)^{d/c}}{q_0(\Lambda) r_0(\Lambda)^{d/c}} \\ &\leq \ln C^{d/c} - \ln q_0(\Lambda) r_0(\Lambda)^{d/c}. \end{aligned}$$

Now for Λ in S' , we have $q_0(\Lambda) = r_0(\Lambda) = \lambda_1(\Lambda)$, thus

$$\varphi(\Lambda) \leq \ln C^{d/c} - \frac{d+c}{c} \ln \lambda_1(\Lambda).$$

It is well known (see [13]) that the function $\ln \lambda_1$ is uniformly continuous and it is integrable on \mathcal{L}_{c+d} by Lemma 6.3, consequently we can use Theorem 8.1 with S' and φ . It follows that for almost all θ in $M_{d,c}(\mathbb{R})$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_{S'}^k(\Lambda_\theta) = \frac{1}{\mu_{S'}(S')} \int_{S'} \varphi d\mu_{S'}$$

where $R_{S'}$ is the first return map to S' . Now by Lemmas 3.5 and 4.9, for almost all θ , there is an integer k_0 such that

$$\varphi \circ R_{S'}^k(\Lambda_\theta) = \ln \frac{q_{k+k_0+1}(\theta)}{q_{k+k_0}(\theta)}$$

for all large enough k , where k_0 depends only on θ . It follows that for almost all θ

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ R_{S'}^k(\Lambda_\theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \frac{q_{k+k_0+1}(\theta)}{q_{k+k_0}(\theta)} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln q_n(\theta)$$

So that, the only thing left is the equality

$$\int_S \rho d\mu_S = \int_{S'} \varphi d\mu_{S'}.$$

Now, the image of $\mu_{S'}$ by R is μ_S , hence

$$\int_{S'} \varphi d\mu_{S'} = \int_{S'} \rho \circ R d\mu_{S'} = \int_S \rho d\mu_S.$$

□

Proof of Theorem 2.1. Consider the map $F : S \rightarrow \mathbb{R}$ defined by

$$F(\Lambda) = q_1^c(\Lambda) r_0^d(\Lambda) = |v_1^S(\Lambda)|_-^c |v_0^S(\Lambda)|_+^d$$

and call $\nu = \nu_{d,c}$ the image of the measure $\frac{1}{\mu_S(S)}\mu_S$ by F . Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous and bounded function. We want to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\beta_k(\theta)) = \int_{\mathbb{R}} \varphi(x) d\nu(x)$$

for almost all $\theta \in M_{d,c}(\mathbb{R})$.

Now by Lemmas 3.5 and 4.9, for almost all θ ,

$$F(R^k(\Lambda_\theta)) = q_{k+k_0+1}^c(\theta)r_{k+k_0}^d(\theta) = \beta_{k+k_0}(\theta)$$

for all k large enough. Now the function $\varphi \circ F$ is bounded and continuous, thus by Theorem 8.1 (or Proposition 8.3) we have for almost all θ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ F(R^k(\Lambda_\theta)) &= \frac{1}{\mu_S(S)} \int_S \varphi \circ F(\Lambda) d\mu_S(\Lambda) \\ &= \int_{\mathbb{R}} \varphi(x) d\nu(x) \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\beta_k(\theta)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ F(R^k(\Lambda_\theta)) = \int_{\mathbb{R}} \varphi(x) d\nu(x)$$

and finishes the proof of Theorem 2.1. The proof of Theorem 2.2. is postponed at the end of Section 9. \square

9 On $\liminf q_{n+k}^c r_n^d$

For each θ in $M_{d,c}(\mathbb{R})$, we consider the sequence of best approximation denominators $(Q_n(\theta))_{n \in \mathbb{N}}$, their norms $q_n = \|Q_n(\theta)\|_{\mathbb{R}^c}$, and the sequence $(r_n)_{n \geq 0}$ defined by

$$r_n = d_{\mathbb{R}^d}(\theta Q_n, \mathbb{Z}^d).$$

For a nonnegative integer k , call Bad_k the subset of $M_{d,c}(\mathbb{R})$ defined by

$$Bad_k(d, c) = Bad_k = \left\{ \theta \in M_{d,c}(\mathbb{R}) \setminus M_{d,c}(\mathbb{Q}) : \inf_{n \in \mathbb{N}} q_{n+k}^c r_n^d > 0 \right\}$$

(if $r_n = 0$ for some integer n , θ is not in Bad_k). The sequence of sets $(Bad_k)_{k \geq 0}$ is clearly nondecreasing. The set Bad_0 is the usual set of badly approximable matrices, i.e., the set of matrices $\theta \in M_{d,c}(\mathbb{R})$ such that

$$\inf \left\{ \frac{\|\theta Q - P\|_{\mathbb{R}^d}}{\|Q\|_{\mathbb{R}^c}} : Q \in \mathbb{Z}^c \setminus \{0\}, P \in \mathbb{Z}^d \right\} > 0.$$

When $d = c = 1$, the classical inequality $q_{n+1}r_n \geq \frac{1}{2}$ shows that $Bad_1 = \mathbb{R} \setminus \mathbb{Q}$ while in [12] it is shown that for $c = 1$ and $d \geq 2$, Bad_1 is negligible. Our first goal is to show that $Bad_1 \setminus Bad_0$ is nonempty for $c = 1$ and $d = 2$. Next we will prove that the set

$$\mathcal{B}(d, c) = \mathcal{B} = \cup_{k \geq 0} Bad_k$$

is negligible and does not depend on the choice of the norm.

Proposition 9.1. *If $c = 1$ and $d = 2$ then $\text{Bad}_1 \setminus \text{Bad}_0$ contains uncountably many elements.*

Remark 9.2. *The set $\mathbb{Z}\theta + \mathbb{Z}^2$ is everywhere dense in \mathbb{R}^2 for all in $\theta \in \text{Bad}_1$. Indeed it is known that the first minimum of the lattice*

$$\Lambda_n = \mathbb{Z}^2 + \mathbb{Z}\frac{p_n}{q_n}$$

is $\asymp r_{n-1}$ where p_n is an integer vector such that $r_n = d(q_n\theta, \mathbb{Z}^2)$. This implies that the second minimum of this lattice $\lambda_2(\Lambda_n)$ is $\asymp \frac{1}{q_n r_{n-1}}$. Now, if $q_n r_{n-1}^2 \geq \alpha > 0$ then $\frac{1}{q_n r_{n-1}} \leq \frac{r_{n-1}}{\alpha}$ which goes to zero when $n \rightarrow \infty$. Therefore, $\lambda_2(\Lambda_n)$ goes to zero which implies that $\mathbb{Z}\theta + \mathbb{Z}^2$ is everywhere dense in \mathbb{R}^2 (see [14] or [10]).

Proof. We assume that \mathbb{R}^2 is equipped with the standard Euclidean norm $\|\cdot\| = \|\cdot\|_{\mathbb{R}^2}$. Set $\theta_0 = (0, 0)$ and $\theta_1 = (\frac{1}{5}, \frac{1}{5})$. We construct inductively a sequence $(\theta_n)_{n \geq 0}$ of rational vectors in \mathbb{R}^2 . For each n in \mathbb{N} , let $\Lambda_n = \mathbb{Z}^2 + \theta_n \mathbb{Z}$ be the lattice associated with θ_n . Observe that the least common denominator Q_n of the coordinates of the rational vector θ_n is the inverse of the volume of the lattice Λ_n , $\det \Lambda_n = \frac{1}{Q_n}$ (even for $n = 0$). For $1 \leq i \leq n$, set

$$\begin{aligned} M_{i,n} &= \min\{d(q\theta_n, \mathbb{Z}^2) - d(Q_{i-1}\theta_n, \mathbb{Z}^2) : Q_{i-1} < q < Q_i\}, \\ m_{i,n} &= d(Q_{i-1}\theta_n, \mathbb{Z}^2) - d(Q_i\theta_n, \mathbb{Z}^2) \end{aligned}$$

The sequence $(\theta_n)_{n \geq 0}$ is constructed such that the following properties hold for all $n \geq 1$:

1. $Q_0 = 1 < Q_1 = 5 < Q_2 < \dots < Q_n$ are the best approximations (denominators) of θ_n ,
2. $Q_n > 2nQ_{n-1}$ and given $\theta_0, \theta_1, \dots, \theta_{n-1}$, there are at least two possible choices of θ_n leading to two different values of Q_n (to ensure that we construct an uncountable set),
3. for all $1 \leq i \leq n-1$, $M_{i,n} > 0$ (we need to avoid the situation where $d(q\theta_n, \mathbb{Z}^2) = d(Q_{i-1}\theta_n, \mathbb{Z}^2)$ for some q between Q_{i-1} and Q_i),
4. $\|\theta_n - \theta_{n-1}\| \leq \frac{1}{8Q_{n-1}} \min\{M_{i,j} : 1 \leq i < j \leq n-1\}$,
5. $\|\theta_n - \theta_{n-1}\| \leq \frac{1}{8Q_{n-1}} \min\{m_{i,j} : 1 \leq i \leq j \leq n-1\}$,
6. $\varepsilon_{n-1} = Q_{n-1}(\theta_n - \theta_{n-1})$ is a shortest vector of Λ_n , i.e. $\lambda_1(\Lambda_n) = \|\varepsilon_{n-1}\|$, and $(-1)^{n-1}\varepsilon_{n-1}$ has positive coordinates,
7. $2\lambda_1(\Lambda_n) \leq \lambda_2(\Lambda_n) \leq 30\lambda_1(\Lambda_n)$.

Observe that, with our choices of θ_0 and θ_1 all these conditions hold for $n = 1$ (the conditions 3 and 4 are empty for $n = 1$).

First, let us show that the above conditions imply that the sequence $(\theta_n)_{n \in \mathbb{N}}$ converges to θ in $\text{Bad}_1 \setminus \text{Bad}_0$. By 2 and 4, the sequence $\|\theta_{n-1} - \theta_n\|$ converges to 0 at least at a geometric

rate, hence the sequence $(\theta_n)_{n \geq 1}$ converge to $\theta \in \mathbb{R}^2$. Furthermore, by 4, for all $n \geq 2$,

$$\begin{aligned} \|\theta - \theta_n\| &\leq \sum_{p \geq n+1} \|\theta_p - \theta_{p-1}\| \\ &\leq \sum_{p \geq n+1} \frac{1}{8Q_{p-1}} \min\{M_{i,j} : 1 \leq i < j \leq p-1\} \\ &\leq \frac{1}{4Q_n} \min\{M_{i,n} : 1 \leq i < n\}. \end{aligned}$$

Using 5 instead of 4, we obtain

$$\|\theta - \theta_n\| \leq \frac{1}{4Q_n} \min\{m_{i,n} : 1 \leq i \leq n\}$$

as well. It follows that for all $1 \leq i \leq n-1$ and all $Q_{i-1} < q < Q_i$, we have

$$\begin{aligned} d(q\theta, \mathbb{Z}^2) &\geq d(q\theta_n, \mathbb{Z}^2) - q \|\theta - \theta_n\| \\ &\geq d(Q_{i-1}\theta_n, \mathbb{Z}^2) + M_{i,n} - q \|\theta - \theta_n\| \\ &\geq d(Q_{i-1}\theta, \mathbb{Z}^2) - Q_{i-1} \|\theta - \theta_n\| + M_{i,n} - q \|\theta - \theta_n\| \\ &\geq d(Q_{i-1}\theta, \mathbb{Z}^2) + M_{i,n} - 2Q_i \|\theta - \theta_n\|. \end{aligned}$$

Since $\|\theta - \theta_n\| \leq \frac{1}{4Q_n} M_{i,n}$, $d(q\theta, \mathbb{Z}^2) > d(Q_{i-1}\theta, \mathbb{Z}^2)$. For all $1 \leq i \leq n-1$, we also have

$$\begin{aligned} d(Q_i\theta, \mathbb{Z}^2) &\leq d(Q_i\theta_n, \mathbb{Z}^2) + Q_i \|\theta - \theta_n\| \\ &\leq d(Q_{i-1}\theta_n, \mathbb{Z}^2) - m_{i,n} + \frac{Q_i}{4Q_n} m_{i,n} \\ &\leq d(Q_{i-1}\theta, \mathbb{Z}^2) + Q_{i-1} \|\theta - \theta_n\| - m_{i,n} + \frac{Q_i}{4Q_n} m_{i,n} \\ &\leq d(Q_{i-1}\theta, \mathbb{Z}^2) + \frac{Q_{i-1}}{4Q_n} m_{i,n} - m_{i,n} + \frac{Q_i}{4Q_n} m_{i,n} \\ &< d(Q_{i-1}\theta, \mathbb{Z}^2). \end{aligned}$$

It follows that Q_0, Q_1, \dots, Q_{n-1} are the first n best approximations of θ . Therefore, $(Q_n)_{n \geq 0}$ is the sequence of best approximations of θ and θ_n is the rational approximation of θ associated with Q_n . The standard inequality (see for instance [14])

$$\lambda_1(\Lambda_n) \asymp d(Q_{n-1}\theta, \mathbb{Z}^2)$$

together with 7 imply that $\theta \in \text{Bad}_1 \setminus \text{Bad}_0$.

Let n be integer ≥ 1 . Let us explain the construction θ_{n+1} given that $\theta_0, \dots, \theta_n$ are already constructed. First, choose a primitive point $\alpha_n = k_n\theta_n + (a_n, b_n)$ of Λ_n with $0 \leq k_n < Q_n$ and $(a_n, b_n) \in \mathbb{Z}^2$, in either $\mathbb{R}_{>0}^2$ when n is even or in $\mathbb{R}_{<0}^2$ when n is odd. Just take α_n a point of Λ_n in a square $[x, x+1[\times]0, 1]$ with minimal ordinate when n is even and a point of Λ_n in a square $[x, x+1[\times]-1, 0[$ with maximal ordinate when n is odd. Observe that $\|\alpha_n\|$ can be made arbitrarily large by choosing $|x|$ arbitrarily large.

Call $L_n = \|\alpha_n\|$ the length of the segment $[0, \alpha_n]$. The (Euclidean) distance between two consecutive lines of the set $\mathcal{H}_n = \Lambda_n + \mathbb{R}\alpha_n$ is

$$d_n = \frac{\det \Lambda_n}{L_n} = \frac{1}{Q_n L_n}.$$

We can choose α_n such that

$$L_n^2 \geq n \det \Lambda_n,$$

hence $\frac{L_n}{d_n} = \frac{L_n^2}{\det \Lambda_n} \geq n$. There are at least two integers $p_n \geq 2$ such that

$$10 \frac{L_n}{d_n} \leq p_n \leq 20 \frac{L_n}{d_n}.$$

Suppose p_n is one of these and set

$$\varepsilon_n = \frac{1}{p_n - \frac{k_n}{Q_n}} \alpha_n,$$

$$\theta_{n+1} = \theta_n + \frac{\varepsilon_n}{Q_n},$$

and

$$Q_{n+1} = Q_n p_n - k_n.$$

Since by 1, $Q_0 = 1 < Q_1 = 5 < \dots < Q_j$ are the best approximations of θ_j , $j = 1, \dots, n$, the real number $\min\{m_{i,j} : 1 \leq i \leq j \leq n\}$ is strictly positive. Moreover,

$$\|\varepsilon_n\| \leq \frac{L_n}{p_n - 1} \leq \frac{L_n}{\frac{p_n}{2}} \leq 2 \frac{L_n}{10 \frac{L_n}{d_n}} \leq \frac{d_n}{5}$$

and α_n can be chosen in order that d_n is arbitrarily small, hence we can choose α_n such that $\|\varepsilon_n\| < \|\varepsilon_{n-1}\|$ and

$$\|\theta_{n+1} - \theta_n\| = \frac{1}{Q_n} \|\varepsilon_n\| \leq \frac{1}{8Q_n} \min\{m_{i,j} : 1 \leq i \leq j \leq n\}$$

which is condition 5. Next by 3, $\min\{M_{i,j} : 1 \leq i < j \leq n\}$ is strictly positive. As above, it follows that α_n can be chosen such that

$$\|\theta_{n+1} - \theta_n\| = \frac{1}{Q_n} \|\varepsilon_n\| \leq \frac{1}{8Q_n} \min\{M_{i,j} : 1 \leq i < j \leq n\}$$

which is condition 4. Clearly $Q_{n+1} \geq Q_n(p_n - 1) \geq 2nQ_n$. Notice that the lattice $\Lambda_{n+1} = \mathbb{Z}\theta_{n+1} + \mathbb{Z}^2$ is included in \mathcal{H}_n . Next observe that

$$\begin{aligned} Q_{n+1}\theta_{n+1} &= (Q_n p_n - k_n)\left(\theta_n + \frac{\varepsilon_n}{Q_n}\right) \\ &= (Q_n p_n - k_n)\left(\theta_n + \frac{\alpha_n}{Q_n(p_n - \frac{k_n}{Q_n})}\right) \\ &= Q_n p_n \theta_n - k_n \theta_n + \alpha_n \\ &= p_n Q_n \theta_n + (a_n, b_n) \in \mathbb{Z}^2. \end{aligned}$$

It follows that $Q_{n+1} \det \Lambda_{n+1} = l \in \mathbb{N}$. On the other hand, consider the one dimensional lattice $\Lambda_{n+1} \cap \mathbb{R}\alpha_n$. Because $Q_n \theta_n \in \mathbb{Z}^2$ and $Q_n \theta_{n+1} = Q_n \theta_n + \varepsilon_n$, it contains ε_n and is generated by a vector $v_n = \frac{\varepsilon_n}{m}$ where m is an integer. Next observe that $\theta_n \in \Lambda_{n+1} + \mathbb{R}\alpha_n$, hence $\Lambda_{n+1} + \mathbb{R}\alpha_n = \mathcal{H}_n$. It follows that

$$\begin{aligned} \frac{l}{Q_{n+1}} &= \det \Lambda_{n+1} = \|v_n\| d_n \\ &= \frac{\|\varepsilon_n\|}{m} d_n = \frac{Q_n L_n}{m Q_{n+1}} d_n \\ &= \frac{1}{m Q_{n+1}}, \end{aligned}$$

which implies $m = l = 1$. Therefore, $\det \Lambda_{n+1} = \frac{1}{Q_{n+1}}$ and

$$\Lambda_{n+1} = \{0, \dots, Q_n - 1\} \theta_{n+1} + \mathbb{Z} \varepsilon_n + \mathbb{Z}^2.$$

Since $\|\varepsilon_n\| \leq \frac{d_n}{5}$, and $\Lambda_{n+1} \subset \mathcal{H}_n$, ε_n is the shortest vector of Λ_{n+1} . The choice of the signs for α_n now implies that condition 6 holds. Next

$$\begin{aligned} \lambda_1(\Lambda_{n+1}) &= \|\varepsilon_n\|, \\ 5 \|\varepsilon_n\| &\leq d_n \leq \lambda_2(\Lambda_{n+1}) \leq d_n + \|\varepsilon_n\|. \end{aligned}$$

Since

$$\|\varepsilon_n\| \geq \frac{L_n}{p_n} \geq \frac{L_n}{20 \frac{L_n}{d_n}} = \frac{d_n}{20},$$

we obtain

$$5 \lambda_1(\Lambda_{n+1}) \leq \lambda_2(\Lambda_{n+1}) \leq 21 \|\varepsilon_n\| \leq 30 \lambda_1(\Lambda_{n+1})$$

which implies condition 7. Let us show that Q_0, \dots, Q_{n-1} are the first best approximations of θ_{n+1} .

For all $1 \leq i \leq n-1$ and all $Q_{i-1} < q < Q_i$, we have

$$\begin{aligned} d(q\theta_{n+1}, \mathbb{Z}^2) &\geq d(q\theta_n, \mathbb{Z}^2) - q \|\theta_{n+1} - \theta_n\| \\ &\geq d(Q_{i-1}\theta_n, \mathbb{Z}^2) + M_{i,n} - q \|\theta_{n+1} - \theta_n\| \\ &\geq d(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) - Q_{i-1} \|\theta_{n+1} - \theta_n\| + M_{i,n} - q \|\theta_{n+1} - \theta_n\| \\ &\geq d(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) + M_{i,n} - 2Q_i \|\theta_{n+1} - \theta_n\|. \end{aligned}$$

Since $\|\theta_{n+1} - \theta_n\| \leq \frac{1}{8Q_n} M_{i,n}$, $d(q\theta_{n+1}, \mathbb{Z}^2) > d(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2)$ and hence $M_{i,n+1} > 0$. We also have

$$\begin{aligned} d(Q_i\theta_{n+1}, \mathbb{Z}^2) &\leq d(Q_i\theta_n, \mathbb{Z}^2) + Q_i \|\theta_{n+1} - \theta_n\| \\ &\leq d(Q_{i-1}\theta_n, \mathbb{Z}^2) - m_{i,n} + \frac{Q_i}{8Q_n} m_{i,n} \\ &\leq d(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) + Q_{i-1} \|\theta_{n+1} - \theta_n\| - m_{i,n} + \frac{Q_i}{8Q_n} m_{i,n} \\ &\leq d(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2) + \frac{Q_{i-1}}{8Q_n} m_{i,n} - m_{i,n} + \frac{Q_i}{8Q_n} m_{i,n} \\ &< d(Q_{i-1}\theta_{n+1}, \mathbb{Z}^2). \end{aligned}$$

It follows that Q_0, Q_1, \dots, Q_{n-1} are the first n best approximations of θ_{n+1} . The proof will be done once we will have explained that Q_n and Q_{n+1} are the only best approximations that follow Q_{n-1} and that $M_{n,n+1} > 0$. These are the places where the sign condition 6 plays a role. First observe that ε_n and $-\varepsilon_n$ are the only two shortest vectors of Λ_{n+1} and that

$$\varepsilon_n = Q_n \theta_{n+1} - Q_n \theta_n$$

and

$$\begin{aligned} -\varepsilon_n &= (Q_{n+1} - Q_n) \theta_{n+1} - Q_{n+1} \theta_{n+1} + Q_n \theta_n \\ &= (Q_{n+1} - Q_n) \theta_{n+1} + \text{a vector in } \mathbb{Z}^2. \end{aligned}$$

Together with the inequality $Q_{n+1} - Q_n > Q_n$ this implies that Q_n is a best approximation of θ_{n+1} and that there is no best approximation of θ_{n+1} between Q_n and Q_{n+1} . Next, denoting by \equiv the equivalence mod \mathbb{Z}^2 , we have

$$\begin{aligned} Q_{n-1} \theta_{n+1} &= Q_{n-1} \left(\theta_n + \frac{\varepsilon_n}{Q_n} \right) = Q_{n-1} \left(\theta_{n-1} + \frac{\varepsilon_{n-1}}{Q_{n-1}} + \frac{\varepsilon_n}{Q_n} \right) \\ &\equiv \varepsilon_{n-1} + \frac{Q_{n-1}}{Q_n} \varepsilon_n \end{aligned}$$

and

$$\begin{aligned} (Q_n - Q_{n-1}) \theta_{n+1} &= (Q_n - Q_{n-1}) \left(\theta_n + \frac{\varepsilon_n}{Q_n} \right) \\ &\equiv -Q_{n-1} \theta_n + \left(1 - \frac{Q_{n-1}}{Q_n} \right) \varepsilon_n \\ &= -Q_{n-1} \left(\theta_{n-1} + \frac{\varepsilon_{n-1}}{Q_{n-1}} \right) + \left(1 - \frac{Q_{n-1}}{Q_n} \right) \varepsilon_n \\ &\equiv -\varepsilon_{n-1} + \left(1 - \frac{Q_{n-1}}{Q_n} \right) \varepsilon_n, \end{aligned}$$

by the choice of the signs we obtain that

$$d(Q_{n-1} \theta_{n+1}, \mathbb{Z}^2) < d((Q_n - Q_{n-1}) \theta_{n+1}, \mathbb{Z}^2).$$

If q is an integer $\neq Q_n - Q_{n-1}$ lying in $]Q_{n-1}, Q_n[$, then $q\theta_n$ cannot be $\equiv \pm \varepsilon_{n-1}$ which are the shortest vectors of Λ_n . Hence

$$d(q\theta_n, \mathbb{Z}^2) \geq \min(2 \|\varepsilon_{n-1}\|, \lambda_2(\Lambda_n)) = 2 \|\varepsilon_{n-1}\|.$$

It follows that

$$\begin{aligned} d(q\theta_{n+1}, \mathbb{Z}^2) &\geq d(q\theta_n, \mathbb{Z}^2) - q \|\theta_{n+1} - \theta_n\| \\ &\geq 2 \|\varepsilon_{n-1}\| - \frac{q}{Q_n} \|\varepsilon_n\| \\ &\geq 2 \|\varepsilon_{n-1}\| - \|\varepsilon_n\| > \|\varepsilon_{n-1}\| \\ &\geq \|\varepsilon_{n-1} + \frac{Q_{n-1}}{Q_n} \varepsilon_n\| = d(Q_{n-1} \theta_{n+1}, \mathbb{Z}^2) \end{aligned}$$

which implies both that Q_{n-1} and Q_n are consecutive best approximations of θ_{n+1} and that $M_{n,n+1} > 0$. \square

It is not clear whether the set Bad_1 depends on the norm. However, using an easy result about the relations between best approximation vectors associated with two norms, we can prove:

Proposition 9.3. *The set $\mathcal{B}(d, c)$ does not depend on the norms.*

Proof. We give the proof only in the case $c = 1$. When $c > 1$ one has to extend first the following result about best approximations:

Consider two norms N and N' on \mathbb{R}^d . For $\theta \in \mathbb{R}^d$, call $(q_n)_{n \in \mathbb{N}}$ the sequence of best approximation denominators associated with the norm N and $(q'_n)_{n \in \mathbb{N}}$ the sequence associated with the norm N' . Then (see [14]) there exists an integer k depending only on the norms N and N' such that each interval $]q_n, q_{n+k}]$, contains a best approximation denominator q'_m associated with the norm N' .

It is enough to prove that $\mathbb{R}^d \setminus \text{Bad}_{kp} \subset \mathbb{R}^d \setminus \text{Bad}'_p$ for all p . Let θ be in \mathbb{R}^d and $n \geq k$ be an integer. By the above result, there exist at least one best approximation denominator in each interval $]q_{n+(j-1)k}, q_{n+jk}]$, $j = 0, \dots, p$. Let q'_{n_j} be the largest best approximation denominator in each of these intervals $]q_{n+(j-1)k}, q_{n+jk}]$. For each j , we have

$$r'_{n_j} \leq C r_{n+jk}$$

where C is the constant involved in the norm equivalence. Making use of the above inequality with $j = 0$, we obtain $q'_{n_0+p} r'_{n_0} \leq C^d q'_{n_0+p} r_n^d$. Next $q'_{n_0+p} \leq q_{n+kp}$, hence,

$$q'_{n_0+p} r'_{n_0} \leq C^d q_{n+kp} r_n^d.$$

It follows that $\liminf_{n \rightarrow \infty} q_{n+kp} r_n^d = 0$ implies $\liminf_{n \rightarrow \infty} q'_{n+p} r_n^d = 0$. \square

Theorem 9.4. *The set $\mathcal{B}(d, c) = \cup_{k \geq 0} \text{Bad}_k$ has zero Lebesgue measure.*

By the above proposition, $\mathcal{B}(d, c)$ doesn't depend on the norms and we can suppose that \mathbb{R}^d and \mathbb{R}^c are equipped with the standard Euclidean norms. Let us show that for each k , Bad_k has zero measure. We need two lemmas.

Lemma 9.5. *1. Let $a < b$ be two integers, let $\Lambda = M\mathbb{Z}^{d+c}$ be a lattice in \mathcal{L}_{d+c} and let $(Y_n)_{n=a, \dots, b}$ be a sequence of vectors in \mathbb{Z}^{d+c} . Suppose that for $n = a, \dots, b$,*

- $X_n(\Lambda) = MY_n$,
- *the only nonzero points of Λ in the cylinder $C(X_n(\Lambda), X_{n+1}(\Lambda))$ are $\pm X_n(\Lambda)$ and $\pm X_{n+1}(\Lambda)$.*

Then there exists a open neighborhood W of M in $\text{SL}(d+c, \mathbb{R})$ such that for all lattices $\Lambda' = M'\mathbb{Z}^{d+c} \in \mathcal{L}_{d+c}$ with M' in W , the vectors $Z_n = M'Y_n$ are consecutive minimal vectors of Λ' and

$$\begin{aligned} |Z_n|_+ &\in \left[\frac{1}{2} r_n(\Lambda), 2r_n(\Lambda) \right], \\ |Z_n|_- &\in \left[\frac{1}{2} q_n(\Lambda), 2q_n(\Lambda) \right] \end{aligned}$$

for $n = a, \dots, b$.

2. Suppose furthermore that $a < 0$, $b > 1$ and $\Lambda \in S$. Then for all lattices $\Lambda' = M'\mathbb{Z}^{d+c} \in S$ with M' in W , we have $X_n(\Lambda') = M'Y_n$ for $n = a, \dots, b$.

Proof. 1. Consider a ball $B_{d,c}(0, R)$ that contains all the points MY_n , $n = a, \dots, b$. There is a neighborhood ω of the identity matrix I_{d+c} such that for all A in ω and all X in \mathbb{R}^{d+c} ,

$$\frac{1}{2} \|X\|_{d,c} \leq \|AX\|_{d,c} \leq 2 \|X\|_{d,c},$$

so that

$$\begin{aligned} X \notin B_{d,c}(0, 8R) &\Rightarrow AX \notin B_{d,c}(0, 4R) \\ X \in B_{d,c}(0, R) &\Rightarrow AX \in B_{d,c}(0, 2R). \end{aligned}$$

Two vectors $Z_n = AMY_n$ and $Z_{n+1} = AMY_{n+1}$ are consecutive minimal vectors of $A\Lambda$ as soon as

$$|Z_{n+1}|_- > |Z_n|_-, \quad |Z_{n+1}|_+ < |Z_n|_+$$

and the cylinder $C(Z_n, Z_{n+1})$ contains no other nonzero vector of $A\Lambda$ than $\pm Z_n$ and $\pm Z_{n+1}$. Since, $|X_{n+1}|_- > |X_n|_-$, $|X_{n+1}|_+ < |X_n|_+$, by reducing ω , we can assume $|Z_{n+1}|_- > |Z_n|_-$ and $|Z_{n+1}|_+ < |Z_n|_+$, $n = a, \dots, b$. Since $C(Z_n, Z_{n+1}) = C(AX_n, AX_{n+1}) \subset B_{d,c}(0, 2R)$, the image by A of a vector of Λ that is not in the ball $B_{d,c}(0, 8R)$, cannot enter in the cylinder $C(AX_n, AX_{n+1})$. Therefore, there are only finitely many X in Λ such that AX is in $C(AX_n, AX_{n+1})$. Since by assumption all these vectors X , except $\pm X_n$ and $\pm X_{n+1}$, are at a positive distance from $C(X_n, X_{n+1})$, we obtain that Z_n and Z_{n+1} are consecutive minimal vectors by reducing once again ω . It follows that Z_a, \dots, Z_b are consecutive minimal vectors of the lattice $A\Lambda$. A new reduction of ω ensures that the two inequalities of the lemma hold.

2. We want to see that there is no shift on the indices when $\Lambda \in S$, $a < 0$ and $b > 1$. By the numbering convention (see Section 4.6),

$$|X_0(\Lambda)|_+ = |X_1(\Lambda)|_-, \quad |X_{-1}(\Lambda)|_+ > |X_0(\Lambda)|_-, \quad |X_1(\Lambda)|_+ < |X_2(\Lambda)|_-.$$

By a further reduction of ω , we can assume that the two inequalities hold for the vectors $Z_{-1} = AMY_{-1}$, $Z_0 = AMY_0$, $Z_1 = AMY_1$ and $Z_2 = AMY_2$. Therefore, if AMZ^{d+c} is in S , we must have

$$X_0(A\Lambda) = AMY_0 \text{ and } X_1(A\Lambda) = AMY_1$$

which implies that $X_n(A\Lambda) = Z_n$ for $n = a, \dots, b$. □

Lemma 9.6. *Assume that $d + c \geq 3$. Let Γ be a two dimensional lattice in $\mathcal{L}_2 \setminus \mathcal{N}_{1,1}$ which is in the transversal $S_{1,1} \subset \mathcal{L}_2$ and let k be a non negative integer. Then for all positive real numbers δ , there exists a lattice Γ_δ in $S \setminus \mathcal{N}$ such that*

$$\begin{aligned} r_n(\Gamma_\delta) &\leq 2\delta r_n(\Gamma) \\ q_n(\Gamma_\delta) &\leq 2\delta q_n(\Gamma) \end{aligned}$$

for $n = 0, \dots, k$.

Proof. Let $\Gamma = AZ^2$ be a lattice in $S_{1,1} \setminus \mathcal{N}_{1,1}$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Consider the matrix $M_\delta \in \mathrm{SL}(d+c, \mathbb{R})$ defined by

$$M_\delta = \begin{pmatrix} \delta a_{11} & 0 & 0 & \dots & 0 & \delta a_{12} \\ 0 & \delta^{-\frac{2}{d+c-2}} & 0 & \dots & \dots & 0 \\ 0 & 0 & \delta^{-\frac{2}{d+c-2}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \delta^{-\frac{2}{d+c-2}} & 0 \\ \delta a_{21} & 0 & \dots & \dots & 0 & \delta a_{22} \end{pmatrix}.$$

Let $(U_n = (u_{n,1}, u_{n,2}))_{n \in \mathbb{Z}}$ be the sequence of vectors in \mathbb{Z}^2 such that $(X_n(\Gamma) = AU_n)_{n \in \mathbb{Z}}$ is the sequence of minimal vectors of Γ . For each $n \in \mathbb{Z}$, let Y_n be the element of \mathbb{Z}^{d+c} defined by $y_1 = u_{n,1}$, $y_2 = \dots = y_{d+c-1} = 0$ and $y_{d+c} = u_{n,2}$. If $\delta > 0$ is small enough, then for all $Z \in \mathbb{Z}^{d+c}$ not in the $\mathbb{R}e_1 + \mathbb{R}e_{d+c}$ -plane, we have

$$\|M_\delta Z\|_{d,c} = \max(|M_\delta Z|_+, |M_\delta Z|_-) \geq \delta^{-\frac{2}{d+c-2}} \geq \max(2\delta r_{-1}(\Gamma), 2\delta q_{k+1}(\Gamma)).$$

It follows that none of these vectors $M_\delta Z$ are in one of the cylinders $C(M_\delta Y_n, M_\delta Y_{n+1})$, $n = -1, \dots, k$. Therefore the vectors $X_n = M_\delta Y_n$, $n = -1, \dots, k+1$ are all consecutive minimal vectors of $\Lambda_\delta = M_\delta \mathbb{Z}^{d+1}$ and Λ_δ is in S . With the numbering convention, we have $X_n(\Lambda_\delta) = X_n$ for all $n = -1, \dots, k+1$. Now we fix δ small enough. By the previous lemma applied to Λ_δ , there is a sequence of matrices $(M_p)_p$ in $S \setminus \mathcal{N}$ which converges to M_δ and such that for all p ,

$$X_n(M_p \mathbb{Z}^{d+1}) = M_p Y_n$$

$n = -1, \dots, k$. When p goes to infinity,

$$\begin{aligned} r_n(M_p \mathbb{Z}^{d+1}) &= |M_p Z_n|_+ \rightarrow |M_\delta Z_n|_+ = \delta r_n(\Gamma), \\ q_n(M_p \mathbb{Z}^{d+1}) &= |M_p Z_n|_- \rightarrow |M_\delta Z_n|_- = \delta q_n(\Gamma) \end{aligned}$$

for $n = -1, \dots, k$. So we can take $\Gamma_\delta = M_p \mathbb{Z}^{d+1}$ for some p large enough. \square

End of proof of Theorem 9.4. Let k and $\eta > 0$ be fixed. We want to prove that the set of θ in $M_{d,c}(\mathbb{R})$ such that

$$\liminf_{n \rightarrow \infty} q_{n+k}^c(\theta) r_n^d(\theta) \leq \eta$$

has full measure. By Lemma 3.5, it is enough to show that

$$\liminf_{n \rightarrow \infty} q_{n+k}^c(\Lambda_\theta) r_n^d(\Lambda_\theta) \leq \eta$$

for almost all θ . Fix a two-dimensional lattice Γ in $S_{1,1} \setminus \mathcal{N}_{1,1}$ and let δ be a positive real number with $\delta \leq \frac{1}{4} \left(\frac{\eta}{q_k^c(\Gamma) r_0^d(\Gamma)} \right)^{\frac{1}{d+c}}$. By Lemma 9.6, there exists a lattice Λ_δ in $S \setminus \mathcal{N}$ such that

$$\begin{aligned} r_n(\Lambda_\delta) &\leq 2\delta r_n(\Gamma) \\ q_n(\Lambda_\delta) &\leq 2\delta q_n(\Gamma) \end{aligned}$$

for $n = 0, \dots, k$. Hence,

$$q_k^c(\Lambda_\delta) r_0^d(\Lambda_\delta) \leq \frac{\eta}{2^{d+c}}.$$

By Lemma 9.5, there exists an open neighborhood W of Λ_δ such that for all Λ in W and some integer $m(\Lambda)$, we have both

$$r_{m(\Lambda)}(\Lambda) \leq 4\delta r_0(\Gamma)$$

and

$$q_{m(\Lambda)+k}^c(\Lambda)r_{m(\Lambda)}^d(\Lambda) \leq \eta.$$

Let us show that, if for a given lattice Λ , there exists a sequence $(t_n)_{n \in \mathbb{N}}$ going to infinity such that $g_{t_n}\Lambda \in W$ for all $n \in \mathbb{N}$, then

$$\liminf_{n \rightarrow \infty} q_{n+k}^c(\Lambda)r_n^d(\Lambda) \leq \eta.$$

Indeed, if $g_{t_n}\Lambda \in W$, then for some integer $m(\Lambda, t_n)$, we have

$$(e^{-dt_n}q_{m(\Lambda, t_n)+k}(\Lambda))^c(e^{ct_n}r_{m(\Lambda, t_n)}(\Lambda))^d \leq \eta.$$

So, the only thing to see is that $m(\Lambda, t_n) \rightarrow \infty$ when $n \rightarrow \infty$. Now $e^{dt_n}r_{m(\Lambda, t_n)}(\Lambda) \leq 4\delta r_0(\Gamma)$, hence $r_{m(\Lambda, t_n)}(\Lambda)$ goes to zero when m goes to infinity which implies that $m(\Lambda, t_n)$ goes to infinity.

Making use of the Birkhoff ergodic theorem with the flow g_t , the proof would be already finished if our goal were $\liminf_{n \rightarrow \infty} q_{n+k}^c(\Lambda)r_n^d(\Lambda) \leq \eta$ for almost all lattices. However we want an "almost all" with respect of the Lebesgue measure of $M_{d,c}(\mathbb{R})$.

Let U be a relatively compact nonempty open set in \mathcal{L}_{d+c} such that $\bar{U} \subset W$. One can find a neighborhood V of I_{d+c} in \mathcal{H}_\leq such that for all $\theta \in M_{d,c}(\mathbb{R})$, all $t \geq 0$ and all $h \in V$, we have

$$g_t h \Lambda_\theta = (g_t h g_{-t}) g_t \Lambda_\theta \in U \implies g_t \Lambda_\theta \in W.$$

Call \mathcal{V} the set of θ such that $g_t \Lambda_\theta \notin W$ for all t large enough. By the choices of U and V , for all $h \in V$ and all $\theta \in \mathcal{V}$, $g_t h \Lambda_\theta \notin U$ for all t large enough. If the Lebesgue measure of \mathcal{V} were nonzero then the set of lattices of the form $g_s h \Lambda_\theta$ with $s \in [0, 1]$, $h \in V$ and $\theta \in \mathcal{V}$, would have a nonzero measure. Now, by the Birkhoff ergodic theorem, for almost all lattices Λ , there exists a sequence $t_n \rightarrow \infty$ such that $g_{t_n}\Lambda \in U$ for all n , therefore \mathcal{V} has zero measure. \square

Proof of Theorem 2. 2. By the proof of the first part of Theorem 2, we know that the measure $\nu_{d,c}$ is the image of the measure $\frac{1}{\mu_S(S)}\mu_S$ by the map $F : S \rightarrow \mathbb{R}$ defined by

$$F(\Lambda) = q_1^c(\Lambda)r_0^d(\Lambda) = |v_1^S(\Lambda)|_-^c |v_0^S(\Lambda)|_+^d.$$

We want to prove that the support of the measure $\nu_{d,c}$ contains zero, *i.e.*, that $\nu_{d,c}([0, \eta]) > 0$ for all $\eta > 0$. By the Birkhoff ergodic theorem and by the definition of $\nu_{d,c}$, it is enough to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[0, \eta]}(F(R^i(\Lambda))) > 0$$

for almost all $\Lambda \in S$. By Lemmas 9.5 and 9.6, there exists a non empty open set W in S such that

$$|v_1^S(\Lambda)|_-^c |v_0^S(\Lambda)|_+^d \leq \eta$$

for all $\Lambda \in W$. Hence $1_W \leq 1_{[0,\eta]} \circ F$. By the Birkhoff ergodic theorem, for almost all Λ in S

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1_W \circ R^i(\Lambda) = \frac{1}{\mu_S(S)} \int_S 1_W d\mu_S = a > 0.$$

therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[0,\eta]}(F(R^i(\Lambda))) \geq a > 0.$$

□

10 Miscellaneous Questions

1. In Theorems 1 and 2, we assume that \mathbb{R}^d and \mathbb{R}^c are equipped with standard Euclidean norms. Clearly, Theorems 1 and 2 are valid for any pair of Euclidean norms and by Theorem 6.1, the Levy constant does not depend on the particular choices of the Euclidean norms. However, we do not know what happens for a pair of general norms. Are these theorems valid when \mathbb{R}^d and \mathbb{R}^c are equipped with any norm? If Theorem 1 is valid for any norm, does the Levy's constant depend on the norm?
2. Is the measure $\nu_{d,c}$ in Theorem 2, absolutely continuous with respect to Lebesgue measure? Is the support of $\nu_{d,c}$ an interval? Compute the measure $\nu_{2,1}$.
3. Suppose $c = 1$. Consider a flow $(g_t)_{t \in \mathbb{R}}$ defined by the matrices

$$g_t = \text{diag}(e^{a_1 t}, \dots, e^{a_d t}, e^{-dt}) \in \text{SL}(d+1, \mathbb{R})$$

where the a_i s are positive real numbers with sum d . Best approximation vectors of $\theta \in \mathbb{R}^d$ with respect to the flow $(g_t)_{t \in \mathbb{R}}$ can be defined as follows. A nonzero vector X in \mathbb{Z}^{d+1} is a best approximation vector of θ if there exists $t \geq 0$ such that the interior of the ball $B(0, \|g_t M_\theta X\|_{\mathbb{R}^{d+1}}) \subset \mathbb{R}^d \times \mathbb{R}$ contains no nonzero vector of the lattice $g_t M_\theta \mathbb{Z}^{d+1}$ (equivalently $\|g_t M_\theta X\|_{\mathbb{R}^{d+1}} = \lambda_1(g_t M_\theta \mathbb{Z}^{d+1})$). Arranging the set of best approximation vector according to their heights, we obtain a sequence $(X_n(\theta))_{n \in \mathbb{N}}$ of best approximation vectors associated with θ . Does Theorem 1 hold for these new sequences of best approximation vectors?

4. For a fixed $k \geq 1$, does the set

$$\text{Bad}_k(d, c) = \text{Bad}_k = \{\theta \in M_{d,c}(\mathbb{R}) \setminus M_{d,c}(\mathbb{Q}) : \inf_{n \in \mathbb{N}} q_{n+k}^c(\theta) r_n^d(\theta) > 0\}$$

depend on the norm used to define best approximations vectors?

Observe that by Proposition 9.3, the union $\cup_{k \geq 1} \text{Bad}_k$ does not depend on the choice of the norm.

5. Since $\text{Bad}_1(d, c)$ contains the set of badly approximable matrices $\text{Bad}_0(d, c)$, it has full Hausdorff dimension. But what are the Hausdorff dimensions of $\text{Bad}_1(d, c) \setminus \text{Bad}_0(d, c)$ and of $\cup_{k \geq 0} \text{Bad}_k(d, c) \setminus \text{Bad}_0(d, c)$? Has $\text{Bad}_1(d, c) \setminus \text{Bad}_0(d, c)$ full Hausdorff dimension?

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see also [11]