PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 6, Pages 1755–1762
S 0002-9939(00)05814-7
Article electronically published on October 31, 2000

MENGER CURVATURE AND C^1 REGULARITY OF FRACTALS

YONG LIN AND PERTTI MATTILA

(Communicated by Albert Baernstein II)

ABSTRACT. We show that if E is an s-regular set in $\mathbf{R}^2$ for which the triple integral $\int_E \int_E \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s x\, d\mathcal{H}^s y\, d\mathcal{H}^s z$ of the Menger curvature c is finite and if $0 < s \leq 1/2$, then $\mathcal{H}^s$ almost all of E can be covered with countably many C^1 curves. We give an example to show that this is false for $1/2 < s < 1$.

1. Introduction

The Menger curvature $c(x,y,z)$ of three points x, y and z in the plane $\mathbf{R}^2$ is defined as the reciprocal of the radius of the circle passing through these points. For a historical background, see [K]. In [Me] Melnikov found a remarkable connection between the Menger curvature and the Cauchy kernel $1/z$, $z \in \mathbf{C}$. This led to a rapid development on singular integrals over 1-dimensional subsets of $\mathbf{R}^2$ and on removable sets of bounded analytic functions; see [MV], [MMV], [D], and for a survey [M2].

Another aspect of the Menger curvature is that its integrals can be used to measure smoothness properties of subsets of $\mathbf{R}^n$. Note that $c(x,y,z) = 0$ if and only if the points x, y and z lie on the same line. Let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure. For $\mathcal{H}^s$ measurable sets $E \subset \mathbf{R}^n$ with $0 < \mathcal{H}^s(E) < \infty$ the proper quantity to use is

$$c^{2s}(E) = \int_E \int_E \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s x\, d\mathcal{H}^s y\, d\mathcal{H}^s z.$$

Léger proved in [L] that if $\mathcal{H}^1(E) < \infty$ and $c^2(E) < \infty$, then there are rectifiable curves $\Gamma_1, \Gamma_2, \dots$ such that

$$\mathcal{H}^1\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

Sets with this property are called 1-rectifiable in [M1] and countably $(\mathcal{H}^1, 1)$ rectifiable in [F].

In this paper we study analogous questions for other values of s. It was shown in [Li, Theorem 1.4] that if $E \subset \mathbf{R}^n$ is $\mathcal{H}^s$ measurable and $0 < \mathcal{H}^s(E) < \infty$ for

Received by the editors September 27, 1999.
2000 *Mathematics Subject Classification.* Primary 28A75.
The authors gratefully acknowledge the hospitality of CRM at Universitat Autònoma de Barcelona where part of this work was done. The first author also wants to thank the Academy of Finland for financial support.

some $1 < s \leq n$, then $c^{2s}(E) = \infty$. Hence we restrict to $0 < s < 1$. We also study only subsets of $\mathbf{R}^2$, although with slight modifications the results would extend to $\mathbf{R}^n$. For reasons indicated in Example 2.5 we restrict to the so-called s-regular sets. This means that there is a constant C such that

$$r^s/C \leq \mathcal{H}^s\big(E \cap B(x,r)\big) \leq Cr^s \quad \text{for } x \in E,\ 0 < r < d(E). \tag{1.1}$$

Here $B(x,r)$ is the closed ball with centre x and radius r, and $d(E)$ stands for the diameter of E.

When $0 < s < 1$ and E is compact, (1.1) alone implies that E is contained in one rectifiable curve Γ; see the proof of Theorem 4.1 in [MM]. A rectifiable curve is the same as a Lipschitz image of the interval $[0,1]$. We study here whether Lipschitz images can be replaced by C^1 curves. By a C^1 curve we mean a curve with continuously varying tangent. It is the same as the image of an interval under a regular C^1 map, that is, a C^1 map with non-vanishing derivative. We shall prove in Corollary 2.2 that if E satisfies (1.1), $c^{2s}(E) < \infty$ and $0 < s \leq 1/2$, then there are C^1 curves $\Gamma_1, \Gamma_2, \dots$ such that

$$\mathcal{H}^s\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

We give in 2.4 an example showing that this is false if $1/2 < s < 1$. For $s = 1$ it is again true due to [L], even with the weaker condition $\mathcal{H}^1(E) < \infty$ instead of (1.1), because then covering $\mathcal{H}^1$ almost everything with Lipschitz images or C^1 curves are equivalent as a consequence of Rademacher's theorem; see [F, 3.2.29].

There are also many other characterizations for 1-rectifiable sets. In [MM] analogous conditions for s-dimensional sets were investigated and this paper can be considered as a further contribution to that study.

We would like to thank Immo Hahlomaa for a useful comment.

2. Covering with C^1 curves

We begin with the following result on the existence of tangents. We say that a set $E \subset \mathbf{R}^2$ has a tangent L at x if L is a line through x such that for any $\alpha > 0$, $E \cap B(x,r) \subset C(x,\alpha)$ for all sufficiently small $r > 0$, where $C(x,\alpha)$ is the double-cone with centre x, axis L and angle α.

2.1. Theorem. *Let $0 < s \leq 1/2$ and let $E \subset \mathbf{R}^2$ be $\mathcal{H}^s$ measurable and s-regular. If $x \in E$ and*

$$c^{2s}(E,x) := \int_E \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s y\, d\mathcal{H}^s z < \infty, \tag{2.1}$$

then E has a tangent at x.

Proof. Let $x \in E$. By the s-regularity of E there are positive numbers b and $d < d(E)$ such that for $i = 1, 2, \dots,$

$$\mathcal{H}^s(A_i) \geq b d^{is}, \tag{2.2}$$

where

$$A_i = \big\{y \in E : d^{i+1} < |x-y| \leq d^i\big\}.$$

Set

$$\gamma_i = \int_{A_i} \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s y\, d\mathcal{H}^s z.$$

Then

$$\sum_i \gamma_i \le \int_E \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s y\, d\mathcal{H}^s z < \infty. \tag{2.3}$$

We shall show that for each i there is a line L_i through x such that

$$\mathcal{H}^s\big(A_i \cap L_i(\eta_i\, d^i)\big) \ge bd^{is}/16, \tag{2.4}$$

where

$$\eta_i = (16b^{-1})^{1/s}\, \gamma_i^{1/(2s)}, \tag{2.5}$$

and

$$B(\delta) = \big\{x \in \mathbf{R}^2 : \operatorname{dist}(x,\beta) \le \delta\big\} \quad \text{for } B \subset \mathbf{R}^2,\ \delta > 0.$$

Suppose (2.4) fails for some i. By (2.2) there is a closed quarter-plane Q (a sector with angle $\pi/2$) with vertex at x such that $\mathcal{H}^s(A_i \cap Q) \ge bd^{is}/4$. Further, there is a line L through x such that

$$\mathcal{H}^s(A_i \cap Q \cap H_j) \ge bd^{is}/8 \quad \text{for } j = 1, 2,$$

where H_1 and H_2 are the two closed half-planes whose boundary is L. Since $\mathcal{H}^s\big(A_i \cap L(\eta_i\, d^i)\big) < bd^{is}/16$, we have

$$\mathcal{H}^s\big(A_i \cap Q \cap H_j \setminus L(\eta_i\, d^i)\big) > bd^{is}/16 \quad \text{for } j = 1, 2. \tag{2.6}$$

Let $x_j \in A_i \cap Q \cap H_j \setminus L(\eta_i\, d^i)$ for $j = 1, 2$. We use the following formula, which is an exercise in elementary geometry:

$$c(x, x_1, x_2) = \frac{2 \operatorname{dist}(x_2, L_{x,x_1})}{|x - x_2|\, |x_1 - x_2|}, \tag{2.7}$$

where $L_{y,z}$ denotes the line through two points y and z. This gives

$$c(x, x_1, x_2) \ge \frac{2\eta_i\, d^i}{d^i \cdot d^i} = \frac{2\eta_i}{d^i}.$$

Thus by (2.6) and (2.5)

$$\gamma_i > \left(\frac{\eta_i}{d^i}\right)^{2s} (bd^{is}/16)^2 = (b/16)^2\, \eta_i^{2s} = \gamma_i,$$

which is a contradiction proving (2.4).

Next we show that if

$$\zeta_i = \max\big\{12\eta_i/d, \big(16 \cdot 50^{2s} C b^{-1} d^{-2s} \gamma_i\big)^{1/(3s)}\big\}, \tag{2.8}$$

and if $\zeta_i < d$, then

$$A_i \subset L_i(\zeta_i\, d^i). \tag{2.9}$$

Suppose this fails and let $y_1 \in A_i \setminus L_i(\zeta_i\, d^i)$. Then $\zeta_i < 1$ and $B(y_1, \frac{1}{2}\zeta_i\, d^i) \subset B(x, 2d^i) \setminus L_i(\frac{1}{2}\zeta_i\, d^i)$. Thus for all $y \in B(y_1, \frac{1}{2}\zeta_i\, d^i)$ and $z \in A_i \cap L_i(\frac{1}{12}\zeta_i\, d^{i+1})$ we have by some elementary geometry $d(x, L_{y,z}) \ge \frac{1}{24}\zeta_i\, d^{i+1}$. Hence by (2.7),

$$c(x,y,z) \ge \frac{\zeta_i\, d^{i+1}/12}{d^i \cdot 2d^i} = \frac{\zeta_i\, d}{24 d^i}.$$

Since by (1.1) $\mathcal{H}^s\big(E\cap B(y_1,\frac{1}{2}\zeta_i\,d^i)\big)\geq(\frac{1}{2}\zeta_i\,d^i)^s/C$, and by (2.4) $\mathcal{H}^s\big(A_i\cap L_i(\frac{1}{12}\zeta_i\,d^{i+1})\big)$ $\geq\frac{1}{16}bd^{is}$ (as $L_i(\frac{1}{12}\zeta_i\,d^{i+1})\supset L_i(\eta_i\,d^i)$ by (2.8)), we get

$$\begin{aligned}\gamma_i&\geq\Big(\frac{\zeta_i\,d}{24d^i}\Big)^{2s}C^{-1}\Big(\frac{1}{2}\zeta_i\,d^i\Big)^s\,\frac{1}{16}bd^{is}\\&>\frac{bd^{2s}\,\zeta_i^{3s}}{16\cdot 50^{2s}C}\geq\gamma_i\end{aligned}$$

which proves (2.9).

Let $\alpha_i\in[0,\pi)$ be the angle between the lines L_i and L_{i+1}. We claim that

$$\alpha_i\leq\max\big\{8d^{-1}\eta_i,8d^{-1}\eta_{i+1},8(16/b)^{1/s}d^{-3}(\gamma_i+\gamma_{i+1})^{1/(2s)}\big\}. \tag{2.10}$$

Suppose this is false. Then if $y\in A_i\cap L_i(\eta_i\,d^i)$ and $z\in A_{i+1}\cap L_{i+1}(\eta_{i+1}\,d^{i+1})$, we have by simple elementary geometry $\operatorname{dist}(z,L_{x,y})\geq\frac{1}{4}\alpha_i\,d^{i+2}$. Hence (2.7) gives

$$c(x,y,z)\geq\frac{\frac{1}{2}\alpha_i\,d^{i+2}}{(2d^i)^2}=\frac{\alpha_i\,d^2}{8d^i}.$$

Integrating over $A_i\cap L_i(\eta_i\,d^i)$ and $A_{i+1}\cap L_{i+1}(\eta_{i+1}\,d^{i+1})$ and using (2.4), we obtain

$$\gamma_i+\gamma_{i+1}\geq\Big(\frac{\alpha_i\,d^2}{8d^i}\Big)^{2s}\Big(\frac{1}{16}bd^{(i+1)s}\Big)^2=\frac{d^{6s}\,b^2\,\alpha_i^{2s}}{16^2\cdot 8^{2s}}>\gamma_i+\gamma_{i+1};$$

a contradiction proving (2.10).

By the definition (2.5) of η_i and by (2.10) we have for some $C_1<\infty$ for all i,

$$\alpha_i\leq C_1(\gamma_i+\gamma_{i+1})$$

since $0<s\leq 1/2$. Using (2.3) we find that $\sum_i\alpha_i<\infty$. This means that the lines L_i converge to a line L through x. Applying (2.9) and the fact that $\zeta_i\to 0$, we see that L is a tangent to E at x. This completes the proof.

2.2. Corollary. *If $0<s\leq 1/2$, $E\subset\mathbf{R}^2$ is s-regular, $\mathcal{H}^s$ measurable and $c^{2s}(E)<\infty$, then there are C^1 curves $\Gamma_1,\Gamma_2,\dots$ such that*

$$\mathcal{H}^s\Big(E\setminus\bigcup_{i=1}^{\infty}\Gamma_i\Big)=0.$$

Proof. This follows from Theorem 2.1 and [MM, Theorem 3.9(1)]. The proof is a relatively easy application of Whitney's extension theorem.

2.3. Remark. Even if we would assume that the integral in (2.1) is uniformly bounded for $x\in\mathbf{R}^2$, E is not necessarily contained in a single C^1 curve, that is, the tangent need not vary continuously. For example, let C be a compact s-regular set lying on the unit circle S^1 and let D be an s-regular Cantor set on $\{(x,y):0\leq x\leq 1,y=0\}$ with $0\in D$. Choose a sequence I_j of complementary intervals of D with mid-points x_j and lengths l_j in such a way that $x_j\to 0$, and $l_j/x_j\to 0$ very quickly. Let

$$E=D\cup\bigcup_{j=1}^{\infty}\Big(\frac{1}{4}l_j\,C+x_j\Big).$$

Then E is s-regular. If $l_j/x_j\to 0$ sufficiently quickly, then $c^{2s}(E,\ )$ is uniformly bounded, but E is not contained in any C^1 curve.

We now give an example to show that Theorem 2.1 and Corollary 2.2 fail for $1/2<s<1$.

2.4. Example. Let $1/2 < s < 1$. Then there is a compact s-regular set $E \subset \mathbf{R}^2$ such that $c^{2s}(E,x)$ is uniformly bounded for $x \in E$, but $\mathcal{H}^s(E \cap \Gamma) = 0$ for every C^1 curve Γ.

Proof. We shall construct E with a von Koch-type construction similar to that used in [DS, §20]. Define $r \in (0,1)$ by

$$2r^s = 1.$$

Let $J_{0,1}$ be a closed oriented line-segment of length 1 in $\mathbf{R}^2$. Let $J_{1,1}$ and $J_{1,2}$ be the closed oriented line-segments of length r in $\mathbf{R}^2$ such that the initial point of $J_{1,1}$ is the initial point of $J_{0,1}$, the initial point of $J_{1,2}$ is the mid-point of $J_{0,1}$, and the oriented angle from $J_{0,1}$ to both $J_{1,1}$ and $J_{1,2}$ is 1. Suppose we have constructed the closed oriented line-segments $J_{k,1}, \dots, J_{k,2^k}$ of length r^k. We apply the above operation to each $J_{k,i}$ with the angle 1 replaced by $1/(k+1)$ to obtain the line-segments $J_{k+1,1}, \dots, J_{k+1,2^{k+1}}$ of length r^{k+1}. It is clear that the unions $\bigcup_{i=1}^{2^k} J_{k,i}$ converge as $k \to \infty$ to a compact s-regular set E. For each k and j we denote by $E_{k,j}$ the subset of E generated by $J_{k,j}$ (in the obvious way). Then for all k,

$$E = \bigcup_{j=1}^{2^k} E_{k,j}.$$

Since $\sum_k k^{-1} = \infty$, one sees easily that E has tangent at none of its points. In fact, E approaches all of its points along all directions in the sense that for any $x \in E$ and any line L through x there is a sequence $x_i \in E \setminus \{x\}$ such that $x_i \to x$ and $\operatorname{dist}(x_i, L)/|x_i - x| \to 0$. This together with the s-regularity of E implies that $\mathcal{H}^s\big(E \cap f([0,1])\big) = 0$ for any regular C^1 mapping $f : [0,1] \to \mathbf{R}^2$. This can be checked by using the regularity of f to write $[0,1] = \bigcup_{k=1}^{\infty} A_k$ where each A_k is a Borel set such that for some $e_k \in S^1$

$$\big|(f(x) - f(y))/|f(x) - f(y)| - e_k\big| < 1/2 \quad \text{for } x, y \in A_k.$$

Then $\mathcal{H}^s\big(E \cap f(A_k)\big) = 0$ for all k by the above scatteredness property of E. It remains to show that $c^{2s}(E,\ )$ is uniformly bounded.

Fix $x \in E$. For $y \in E$, $y \neq x$, let $k(y)$ be the largest k such that $x, y \in E_{k-1,j}$ for some j. Here $E_{0,j} = E$. Let $y, z \in E \setminus \{x\}$ with $y \neq z$. Denote $k = k(y)$, $l = k(z)$ and assume that $k \le l$.

Suppose first that $k = l$. Then for some j, $y, z \in E_{k,j}$ whereas $x \notin E_{k,j}$. Let $m = m(y,z)$ be the largest m such that $y, z \in E_{m-1,j}$ for some j. Then $m > k$. It follows from the construction that there is a positive number b, depending only on r, such that

$$\begin{aligned} &|x - z| \ge b^{-1} r^k, \qquad |y - z| \ge b^{-1} r^m, \\ &\operatorname{dist}(z, L_{x,y}) \le b r^m. \end{aligned}$$

If m is not much bigger than k, we can improve the last estimate. Since for $m \le 2k$ the angle between the lines $L_{x,y}$ and $L_{y,z}$ is at most a constant times

$$\sum_{j=k+1}^{m} \frac{1}{j} \approx \log \frac{m}{k} \approx \frac{m-k}{k},$$

we can choose b so that also

$$\operatorname{dist}(z, L_{x,y}) \le b\frac{m-k}{k}\, r^m.$$

Consequently by (2.7) we have both

$$c(x,y,z) \le 2b^3 r^{-k} \qquad \text{and}$$
$$c(x,y,z) \le 2b^3 \frac{m-k}{k}\, r^{-k}.$$

If $k < l$ we get in the same way interchanging x and y in the above argument

$$c(x,y,z) \le 2b^3 r^{-k} \qquad \text{and}$$
$$c(x,y,z) \le 2b^3 \frac{l-k}{k}\, r^{-k}.$$

Set

$$F_k = \{y \in E : k(y) = k\} \qquad \text{and}$$
$$F_m(y) = \{z \in E : k(z) = k(y), k(y,z) = m\}.$$

Then

$$\mathcal{H}^s(F_k) \le C_1\, r^{sk} \qquad \text{and}$$
$$\mathcal{H}^s\big(F_m(y)\big) \le C_1\, r^{sm}$$

where C_1 depends only on r. Therefore, changing $m-k$ to n,

$$\begin{aligned}
c^{2s}(E,x) &= \int_E \int_E c(x,y,z)^{2s} d\mathcal{H}^s z\, d\mathcal{H}^s y \\
&\le 2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \int_{F_k} \int_{F_m(y)} c(x,y,z)^{2s} d\mathcal{H}^s z\, d\mathcal{H}^s y \\
&\quad + 2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \int_{F_k} \int_{F_l} c(x,y,z)^{2s} d\mathcal{H}^s z\, d\mathcal{H}^s y \\
&\le 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{2k} \Big(\frac{m-k}{k}\Big)^{2s} r^{-2sk}\, r^{sk}\, r^{sm} \\
&\quad + 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=2k+1}^{\infty} r^{-2sk}\, r^{sk}\, r^{sm} \\
&= 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{n=1}^{\infty} r^{ns}\, n^{2s} \sum_{k=n}^{\infty} k^{-2s} \\
&\quad + 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} r^{sn} < \infty
\end{aligned}$$

since $2s > 1$. Thus $c^{2s}(E,\ )$ is bounded.

We now show that Theorem 2.1 and Corollary 2.2 fail if we replace the regularity assumption (1.1) by $\mathcal{H}^s(E) < \infty$.

2.5. Example. Given $0 < s < 1$ there exists a compact set $E \subset \mathbf{R}^2$ such that $0 < \mathcal{H}^s(E) < \infty$, $c^{2s}(E,x)$ is uniformly bounded for $x \in \mathbf{R}^2$ and $\mathcal{H}^s(E \cap \Gamma) = 0$ for every C^1 curve Γ.

Proof. Choose an integer $n_1 > 1$, let $J = \{(x,y) \in \mathbf{R}^2 : 0 \le x \le 1, y = 0\}$ and

$$J_i = \{(x,y) \in \mathbf{R}^2 : x = i/n_1, 0 \le y \le r_1\}, \quad i = 1, \dots, n_1,$$

where r_1 is determined by

$$n_1 \, r_1^s = 1.$$

Let μ_1 be the normalized, i.e., $\mu_1\Big(\bigcup_i J_i\Big) = 1$, length measure on $\bigcup_{i=1}^{n_1} J_i$. If we choose n_1 very large, r_1 will be very small and the distance $1/n_1 = r_1^s$ between any J_i and J_{i+1} will be much bigger than r_1. From this we see easily that choosing n_1 large enough, we have

$$c^{2s}(\mu_1, x) = \iint c(x,y,z)^{2s} \, d\mu_1 y \, d\mu_1 z \le C_0$$

for all $x \in \mathbf{R}^2$, where C_0 is an absolute constant.

Next we replace each vertical line segment J_i by n_2 horizontal line segments $J_{i,j}$ of length r_2 such that $n_2 \, r_2^s = r_1^s$ in the same way. Let μ_2 be the normalized length measure on $\bigcup_{i,j} J_{i,j}$. Choosing n_2 sufficiently large we can keep $c^{2s}(\mu_2, x)$ as close to $c^{2s}(\mu_1, x)$, uniformly, as we want. We choose it so that $c^{2s}(\mu_2, x) \le C_0 + \frac{1}{2}$ for $x \in \mathbf{R}^2$. The point here is that for some small $\delta > 0$ looking from any $x \in \mathbf{R}^2$ the part of μ_2 outside $B(x, \delta)$ looks very much like μ_1 and the contribution of μ_2 in $B(x, \delta)$ to $c^{2s}(\mu_2, x)$ is very small.

Continuing this we get unions E_k of line segments $J_{i_1, \dots, i_k}$ of length r_k. Every second time these line segments are horizontal and every second time vertical. We also have uniformly distributed probability measures μ_k on E_k satisfying $c^{2s}(\mu_k, x) \le C_0 + \sum_{i=2}^k 2^{-i}$ for all k and $x \in \mathbf{R}^2$. The sets E_k converge to a compact set E with $0 < \mathcal{H}^s(E) < \infty$ and the measures μ_k converge weakly to a probability measure μ supported on E such that

$$c^{2s}(\mu, x) \le 2 \quad \text{for } x \in \mathbf{R}^2.$$

It is easy to check that μ is comparable with $\mathcal{H}^s$ restricted to E. Thus $c^{2s}(E, x)$ is uniformly bounded.

Finally that $\mathcal{H}^s(E \cap \Gamma) = 0$ for every C^1 curve Γ can be checked with the help of decompositions such as in the proof of Example 2.4.

References

[D] G. David, *Unrectifiable* 1*-sets have vanishing analytic capacity*, Rev. Mat. Iberoamericana **14** (1998), 369–479. MR **99i:**42018

[DS] G. David and S. Semmes, *Singular integrals and rectifiable sets in* $\mathbf{R}^n$*: au-delà des graphes lipschitziens*, Astérisque **193** (1991). MR **92j:**42016

[F] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969. MR **41:**1976

[K] S. Kass, *Karl Menger*, Notices Amer. Math. Soc. **43:5** (1996), 558–561. CMP 96:12

[L] J.-C. Léger, *Menger curvature and rectifiability*, Ann. of Math. **149** (1999), 831–869. CMP 99:17

[Li] Y. Lin, *Menger curvature, singular integrals and analytic capacity*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 111, 1997. MR **98j:**30020

[MM] M. A. Martin and P. Mattila, *k-dimensional regularity classifications for s-fractals*, Trans. Amer. Math. Soc. **305** (1988), 293–315. MR **88j:**28007

[M1] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995. MR **96h:**28006

[M2] P. Mattila, *Rectifiability, analytic capacity, and singular integrals*, Doc. Math. Extra Volume ICM 1998, II, 657–664. MR **99g:**28010

[MMV] P. Mattila, M. S. Melnikov and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. **144** (1996), 127–136. MR **97k:**31004

[Me] M. S. Melnikov, *Analytic capacity: discrete approach and curvature of measure*, Sb. Math. **186** (1995), 57–76. MR **96f:**30020

[MV] M. Melnikov and J. Verdera, *A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs*, Internat. Math. Res. Notices **7** (1995), 325–331. MR **96f:**45011

Department of Mathematics, Renmin University of China, Information School, Beijing, 100872, China

E-mail address: liny9@263.net

Department of Mathematics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland

E-mail address: pmattila@math.jyu.fi