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Calculus of Variations

Kazdan–Warner equation on graph

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Abstract Let $G = (V, E)$ be a connected finite graph and Δ be the usual graph Laplacian. Using the calculus of variations and a method of upper and lower solutions, we give various conditions such that the Kazdan–Warner equation $\Delta u = c - he^u$ has a solution on V, where c is a constant, and $h : V \to \mathbb{R}$ is a function. We also consider similar equations involving higher order derivatives on graph. Our results can be compared with the original manifold case of Kazdan and Warner (Ann. Math. 99(1):14–47, 1974).

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1 Introduction

A basic problem in Riemannian geometry is that of describing curvatures on a given manifold. Suppose that (Σ, g) is a 2-dimensional compact Riemannian manifold without boundary, and K is the Gaussian curvature on it. Let $\widetilde{g} = e^{2u}g$ be a metric conformal to g, where $u \in C^\infty(\Sigma)$. To find a smooth function $\widetilde{K}$ as the Gaussian curvature of $(\Sigma, \widetilde{g})$, one is led to solving the nonlinear elliptic equation

$$\Delta_g u = K - \widetilde{K}e^{2u}, \tag{1}$$

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where Δ_g denotes the Laplacian operator on (Σ, g). Let v be a solution to $\Delta_g v = K - \overline{K}$. Here and in the sequel, we denote the integral average on Σ by

$$\overline{w} = \frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma w dv_g$$

for any function $w : V \to \mathbb{R}$. Set $\psi = 2(u - v)$. Then ψ satisfies

$$\Delta\psi = 2\overline{K} - (2\widetilde{K}e^{2v})e^{\psi}.$$

If one frees this equation from the geometric situation, then it is a special case of

$$\Delta_g u = c - he^u, \tag{2}$$

where c is a constant, and h is some prescribed function, with neither c nor h depends on geometry of (Σ, g). Clearly one can consider (2) in any dimensional manifold. Now let (Σ, g) be a compact Riemannian manifold of any dimension. Note that the solvability of (2) depends on the sign of c. Let us summarize results of Kazdan and Warner [5]. For this purpose, think of (Σ, g) and $h \in C^\infty(\Sigma)$ as being fixed with $\dim \Sigma \geq 1$.

Case 1 $c < 0$. A necessary condition for a solution is that $\overline{h} < 0$, in which case there is a critical strictly negative constant $c_-(h)$ such that (2) is solvable if $c_-(h) < c < 0$, but not solvable if $c < c_-(h)$.

Case 2 c = 0. When $\dim \Sigma \leq 2$, the Eq. (2) has a solution if and only if both $\overline{h} < 0$ and h is positive somewhere. When $\dim \Sigma \geq 3$, the necessary condition still holds.

Case 3 $c > 0$. When $\dim \Sigma = 1$, so that $\Sigma = S^1$, then (2) has a solution if and only if h is positive somewhere. When $\dim \Sigma = 2$, there is a constant $0 < c_+(h) \leq +\infty$ such that (2) has a solution if h is positive somewhere and if $0 < c < c_+(h)$.

There are tremendous work concerning the Kazdan–Warner problem, among those we refer the reader to Chen and Li [1,2], Ding et al. [3,4], and the references therein.

In this paper, we consider the Kazdan–Warner equation on a finite graph. In our setting, we shall prove the following: In *Case 1*, we have the same conclusion as the manifold case; In *Case 2*, the Eq. (2) has a solution if and only if both $\overline{h} < 0$ and h is positive somewhere; While in *Case 3*, the Eq. (2) has a solution if and only if h is positive somewhere. Following the lines of Kazdan and Warner [5], for results of *Case 2* and *Case 3*, we use the variational method; for results of *Case 1*, we use the principle of upper-lower solutions. It is remarkable that Sobolev spaces on a finite graph are all pre-compact. This leads to a very strong conclusion in *Case 3* compared with the manifold case.

We organized this paper as follows: In Sect. 2, we introduce some notations on graphs and state our main results. In Sect. 3, we give two important lemmas, namely, the Sobolev embedding and the Trudinger–Moser embedding. In Sects. 4–6, we prove Theorems 1–4 respectively. In Sect. 7, we discuss related equations involving higher order derivatives.

2 Settings and main results

Let $G = (V, E)$ be a finite graph, where V denotes the vertex set and E denotes the edge set. Throughout this paper, all graphs are assumed to be *connected*. For any edge $xy \in E$, we assume that its weight $w_{xy} > 0$ and that $w_{xy} = w_{yx}$. Let $\mu : V \to \mathbb{R}^+$ be a finite measure. For any function $u : V \to \mathbb{R}$, the μ-Laplacian (or Laplacian for short) of u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x)), \tag{3}$$

where $y \sim x$ means $xy \in E$. The associated gradient form reads

$$\Gamma(u,v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{4}$$

Write $\Gamma(u) = \Gamma(u,u)$. We denote the length of its gradient by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}. \tag{5}$$

For any function $g : V \to \mathbb{R}$, an integral of g over V is defined by

$$\int_V g d\mu = \sum_{x \in V} \mu(x) g(x), \tag{6}$$

and an integral average of g is denoted by

$$\overline{g} = \frac{1}{\mathrm{Vol(V)}} \int_V g d\mu = \frac{1}{\mathrm{Vol(V)}} \sum_{x \in V} \mu(x) g(x),$$

where $\mathrm{Vol}(V) = \sum_{x \in V} \mu(x)$ stands for the volume of V.

The Kazdan–Warner equation on graph reads

$$\Delta u = c - he^u \quad \text{in} \quad V, \tag{7}$$

where Δ is defined as in (3), $c \in \mathbb{R}$, and $h : V \to \mathbb{R}$ is a function. If $c = 0$, then (7) is reduced to

$$\Delta u = -he^u \quad \text{in} \quad V. \tag{8}$$

Our first result can be stated as following:

Theorem 1 *Let $G = (V, E)$ be a finite graph, and $h(\not\equiv 0)$ be a function on V. Then the Eq. (8) has a solution if and only if h changes sign and $\int_V h d\mu < 0$.*

In cases $c > 0$ and $c < 0$, we have the following:

Theorem 2 *Let $G = (V, E)$ be a finite graph, c be a positive constant, and $h : V \to \mathbb{R}$ be a function. Then the Eq. (7) has a solution if and only if h is positive somewhere.*

Theorem 3 *Let $G = (V, E)$ be a finite graph, c be a negative constant, and $h : V \to \mathbb{R}$ be a function.*

(i) If (7) has a solution, then $\overline{h} < 0$.
(ii) If $\overline{h} < 0$, then there exists a constant $-\infty \le c_-(h) < 0$ depending on h such that (7) has a solution for any $c_-(h) < c < 0$, but has no solution for any $c < c_-(h)$.

Concerning the constant $c_-(h)$ in Theorem 3, we have the following:

Theorem 4 *Let $G = (V, E)$ be a finite graph, c be a negative constant, and $h : V \to \mathbb{R}$ be a function. Suppose that $c_-(h)$ is given as in Theorem 3. If $h(x) \le 0$ for all $x \in V$, but $h \not\equiv 0$, then $c_-(h) = -\infty$.*

3 Preliminaries

Define a Sobolev space and a norm on it by

$$W^{1,2}(V)=\left\{u:V\to\mathbb{R}:\int_V(|\nabla u|^2+u^2)d\mu<+\infty\right\},$$

and

$$\|u\|_{W^{1,2}(V)}=\left(\int_V(|\nabla u|^2+u^2)d\mu\right)^{1/2}$$

respectively. If V is a finite graph, then $W^{1,2}(V)$ is exactly the set of all functions on V, a finite dimensional linear space. This implies the following Sobolev embedding:

Lemma 5 *Let $G=(V,E)$ be a finite graph. The Sobolev space $W^{1,2}(V)$ is pre-compact. Namely, if $\{u_j\}$ is bounded in $W^{1,2}(V)$, then there exists some $u\in W^{1,2}(V)$ such that up to a subsequence, $u_j\to u$ in $W^{1,2}(V)$.*

As a consequence of Lemma 5, we have the following Poincaré inequality:

Lemma 6 *Let $G=(V,E)$ be a finite graph. For all functions $u:V\to\mathbb{R}$ with $\int_V ud\mu=0$, there exists some constant C depending only on G such that*

$$\int_V u^2d\mu\leq C\int_V|\nabla u|^2d\mu.$$

Proof Suppose not. There would exist a sequence of functions $\{u_j\}$ satisfying $\int_V u_jd\mu=0$, $\int_V u_j^2d\mu=1$, but $\int_V|\nabla u_j|^2d\mu\to0$ as $j\to\infty$. Clearly u_j is bounded in $W^{1,2}(V)$. It follows from Lemma 5 that there exists some function u_0 such that up to a subsequence, $u_j\to u_0$ in $W^{1,2}(V)$ as $j\to\infty$. Hence $\int_V|\nabla u_0|^2d\mu=\lim_{j\to\infty}\int_V|\nabla u_j|^2d\mu=0$. This leads to $|\nabla u_0|\equiv0$ and thus $u_0\equiv$ const on V since G is connected. Noting that $\int_V u_0d\mu=\lim_{j\to\infty}\int_V u_jd\mu=0$, we conclude that $u_0\equiv0$ on V, which contradicts $\int_V u_0^2d\mu=\lim_{j\to\infty}\int_V u_j^2d\mu=1$. □

Also we have the following Trudinger–Moser embedding:

Lemma 7 *Let $G=(V,E)$ be a finite graph. For any $\beta\in\mathbb{R}$, there exists a constant C depending only on β and G such that for all functions v with $\int_V|\nabla v|^2d\mu\leq1$ and $\int_V vd\mu=0$, there holds*

$$\int_V e^{\beta v^2}d\mu\leq C.$$

Proof Since the case $\beta\leq0$ is trivial, we assume $\beta>0$. For any function v satisfying $\int_V|\nabla v|^2d\mu\leq1$ and $\int_V vd\mu=0$, we have by Lemma 6 that

$$\int_V v^2d\mu\leq C_0\int_V|\nabla v|^2d\mu\leq C_0,$$

for some constant C_0 depending only on G. Denote $\mu_{\min}=\min_{x\in V}\mu(x)$. In view of (6), the above inequality leads to $\|v\|_{L^\infty(V)}\leq C_0/\mu_{\min}$. Hence

$$\int_V e^{\beta v^2}d\mu\leq e^{\beta C_0^2/\mu_{\min}}\mathrm{Vol}(V).$$

This gives the desired result. □

4 The case $c = 0$

In the case $c = 0$, our approach comes out from that of Kazdan and Warner [5].

Proof of Theorem 1 Necessary condition If (8) has a solution u, then $e^{-u}\Delta u = -h$. Integration by parts gives

$$\begin{aligned}-\int_V hd\mu &= \int_V e^{-u}\Delta u d\mu \\ &= -\int_V \Gamma(e^{-u}, u)d\mu \\ &= -\frac{1}{2}\sum_{x\in V}\sum_{y\sim x} w_{xy}(e^{-u(y)} - e^{-u(x)})(u(y) - u(x)) \\ &> 0,\end{aligned}$$

since $(e^{-u(y)} - e^{-u(x)})(u(y) - u(x)) \leq 0$ for all $x, y \in V$ and u is not a constant. Integrating the Eq. (8), we have

$$\int_V he^u d\mu = -\int_V \Delta u d\mu = 0.$$

This together with $h \not\equiv 0$ implies that h must change sign.

Sufficient condition We use the calculus of variations. Suppose that h changes sign and

$$\int_V hd\mu < 0. \tag{9}$$

Define a set

$$\mathcal{B}_1 = \left\{v \in W^{1,2}(V) : \int_V he^v d\mu = 0,\ \int_V v d\mu = 0\right\}. \tag{10}$$

We *claim* that

$$\mathcal{B}_1 \neq \varnothing. \tag{11}$$

To see this, since h changes sign and (9), we can assume $h(x_1) > 0$ for some $x_1 \in V$. Take a function v_1 satisfying $v_1(x_1) = \ell$ and $v_1(x) = 0$ for all $x \neq x_1$. Hence

$$\begin{aligned}\int_V he^{v_1} d\mu &= \sum_{x\in V}\mu(x)h(x)e^{v_1(x)} \\ &= \mu(x_1)h(x_1)e^{\ell} + \sum_{x\neq x_1}\mu(x)h(x) \\ &= (e^{\ell} - 1)\mu(x_1)h(x_1) + \int_V hd\mu \\ &> 0\end{aligned}$$

for sufficiently large ℓ. Writing $\phi(t) = \int_V he^{tv_1}d\mu$, we have by the above inequality that $\phi(1) > 0$. Obviously $\phi(0) = \int_V hd\mu < 0$. Thus there exists a constant $0 < t_0 < 1$ such that $\phi(t_0) = 0$. Let $v^* = t_0 v_1 - \frac{1}{\mathrm{Vol}(V)}\int_V t_0 v_1 d\mu$, where $\mathrm{Vol}(V) = \sum_{x\in V}\mu(x)$ stands for the volume of V. Then $v^* \in \mathcal{B}_1$. This concludes our claim (11).

We shall minimize the functional $J(v) = \int_V |\nabla v|^2 d\mu$. Let

$$a = \inf_{v\in\mathcal{B}_1} J(v).$$

Take a sequence of functions $\{v_n\} \subset \mathcal{B}_1$ such that $J(v_n) \to a$. Clearly $\int_V |\nabla v_n|^2 d\mu$ is bounded and $\int_V v_n d\mu = 0$. Hence v_n is bounded in $W^{1,2}(V)$. Since V is a finite graph, the Sobolev embedding (Lemma 5) implies that up to a subsequence, $v_n \to v_\infty$ in $W^{1,2}(V)$. Hence $\int_V v_\infty d\mu = 0$, $\int_V he^{v_\infty} d\mu = \lim_{n\to\infty} \int_V he^{v_n} d\mu = 0$, and thus $v_\infty \in \mathcal{B}_1$. Moreover

$$\int_V |\nabla v_\infty|^2 d\mu = \lim_{n\to\infty} \int_V |\nabla v_n|^2 d\mu = a.$$

One can calculate the Euler–Lagrange equation of v_∞ as follows:

$$\Delta v_\infty = -\frac{\lambda}{2} he^{v_\infty} - \frac{\gamma}{2}, \tag{12}$$

where λ and γ are two constants. This is based on the method of Lagrange multipliers. Indeed, for any $\phi \in W^{1,2}(V)$, there holds

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \left\{ \int_V |\nabla(v_\infty + t\phi)|^2 d\mu - \lambda \int_V he^{v_\infty + t\phi} d\mu - \gamma \int_V (v_\infty + t\phi) d\mu \right\} \\ &= 2\int_V \Gamma(v_\infty, \phi) d\mu - \lambda \int_V he^{v_\infty} \phi d\mu - \gamma \int_V \phi d\mu \\ &= -2\int_V (\Delta v_\infty)\phi d\mu - \lambda \int_V he^{v_\infty} \phi d\mu - \gamma \int_V \phi d\mu, \end{aligned} \tag{13}$$

which gives (12) immediately. Integrating the Eq. (12), we have $\gamma = 0$. We *claim* that $\lambda \neq 0$. For otherwise, we conclude from $\Delta v_\infty = 0$ and $\int_V v_\infty d\mu = 0$ that $v_\infty \equiv 0 \notin \mathcal{B}_1$. This is a contradiction. We further *claim* that $\lambda > 0$. This is true because $\int_V h d\mu < 0$ and

$$0 < \int_V e^{-v_\infty} \Delta v_\infty d\mu = -\frac{\lambda}{2} \int_V h d\mu.$$

Thus we can write $\frac{\lambda}{2} = e^{-\vartheta}$ for some constant ϑ. Then $u = v_\infty + \vartheta$ is a desired solution of (8). □

5 The case $c > 0$

Proof of Theorem 2 *Necessary condition* Suppose $c > 0$ and u is a solution to (7). Since $\int_V \Delta u d\mu = 0$, we have

$$\int_V he^u d\mu = c\mathrm{Vol}(V) > 0.$$

Hence h must be positive somewhere on V.

Sufficient condition Suppose $h(x_0) > 0$ for some $x_0 \in V$. Define a set

$$\mathcal{B}_2 = \left\{ v \in W^{1,2}(V) : \int_V he^v d\mu = c\mathrm{Vol}(V) \right\}.$$

We *claim* that $\mathcal{B}_2 \neq \varnothing$. To see this, we set

$$u_\ell(x) = \begin{cases} \ell, & x = x_0 \\ 0, & x \neq x_0. \end{cases}$$

It follows that

$$\int_V he^{u_\ell} d\mu \to +\infty \quad \text{as} \quad \ell \to +\infty.$$

We also set $\widetilde{u}_\ell \equiv -\ell$, which leads to

$$\int_V he^{\widetilde{u}_\ell} d\mu = e^{-\ell} \int_V h d\mu \to 0 \quad \text{as } \ell \to +\infty.$$

Hence there exists a sufficiently large ℓ such that $\int_V he^{u_\ell} d\mu > c\mathrm{Vol}(V)$ and $\int_V he^{\widetilde{u}_\ell} d\mu < c\mathrm{Vol}(V)$. We define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \int_V he^{tu_\ell + (1-t)\widetilde{u}_\ell} d\mu.$$

Then $\phi(0) < c\mathrm{Vol}(V) < \phi(1)$, and thus there exists a $t_0 \in (0, 1)$ such that $\phi(t_0) = c\mathrm{Vol}(V)$. Hence $\mathcal{B}_2 \neq \varnothing$ and our claim follows. We shall solve (7) by minimizing the functional

$$J(u) = \frac{1}{2} \int_V |\nabla u|^2 d\mu + c \int_V u d\mu$$

on $\mathcal{B}_2$. For this purpose, we write $u = v + \overline{u}$, so $\overline{v} = 0$. Then for any $u \in \mathcal{B}_2$, we have

$$\int_V he^v d\mu = c\mathrm{Vol}(V) e^{-\overline{u}} > 0,$$

and thus

$$J(u) = \frac{1}{2} \int_V |\nabla u|^2 d\mu - c\mathrm{Vol}(V) \log \int_V he^v d\mu + c\mathrm{Vol}(V) \log(c\mathrm{Vol}(V)). \tag{14}$$

Let $\widetilde{v} = v / \|\nabla v\|_2$. Then $\int_V \widetilde{v} d\mu = 0$ and $\|\nabla \widetilde{v}\|_2 = 1$. By Lemma 6, $\|\widetilde{v}\|_2 \le C_0$ for some constant C_0 depending only on G. By Lemma 7, for any $\beta \in \mathbb{R}$, one can find a constant C depending only on β and V such that

$$\int_V e^{\beta \widetilde{v}^2} d\mu \le C(\beta, V). \tag{15}$$

This together with an elementary inequality $ab \le \epsilon a^2 + \frac{b^2}{4\epsilon}$ implies that for any $\epsilon > 0$,

$$\begin{aligned} \int_V e^v d\mu &\le \int_V e^{\epsilon \|\nabla v\|_2^2 + \frac{v^2}{4\epsilon \|\nabla v\|_2^2}} d\mu \\ &= e^{\epsilon \|\nabla v\|_2^2} \int_V e^{\frac{v^2}{4\epsilon \|\nabla v\|_2^2}} d\mu \\ &\le C e^{\epsilon \|\nabla v\|_2^2}, \end{aligned}$$

where C is a positive constant depending only on ϵ and G. Hence

$$\int_V he^v d\mu \le C (\max_{x \in V} h(x)) e^{\epsilon \|\nabla v\|_2^2}.$$

In view of (14), the above inequality leads to

$$J(u) \ge \frac{1}{2} \int_V |\nabla u|^2 d\mu - c\mathrm{Vol}(V) \epsilon \|\nabla v\|_2^2 - C_1,$$

where C_1 is some constant depending only on ϵ and G. Choosing $\epsilon = \frac{1}{4c\mathrm{Vol}(V)}$, and noting that $\|\nabla v\|_2 = \|\nabla u\|_2$, we obtain for all $u \in \mathcal{B}_2$,

$$J(u) \ge \frac{1}{4} \int_V |\nabla u|^2 d\mu - C_1. \tag{16}$$

Therefore J has a lower bound on the set $\mathcal{B}_2$. This permits us to consider

$$b = \inf_{u \in \mathcal{B}_2} J(u).$$

Take a sequence of functions $\{u_k\} \subset \mathcal{B}_2$ such that $J(u_k) \to b$. Let $u_k = v_k + \overline{u_k}$. Then $\overline{v_k} = 0$, and it follows from (16) that v_k is bounded in $W^{1,2}(V)$. This together with the equality

$$\int_V u_k d\mu = \frac{1}{c} J(u_k) - \frac{1}{2c} \int_V |\nabla v_k|^2 d\mu$$

implies that $\{\overline{u_k}\}$ is a bounded sequence. Hence $\{u_k\}$ is also bounded in $W^{1,2}(V)$. By the Sobolev embedding (Lemma 5), up to a subsequence, $u_k \to u$ in $W^{1,2}(V)$. It is easy to see that $u \in \mathcal{B}_2$ and $J(u) = b$. Using the same method of (13), we derive the Euler–Lagrange equation of the minimizer u, namely, $\Delta u = c - \lambda h e^u$ for some constant λ. Noting that $\int_V \Delta u d\mu = 0$, we have $\lambda = 1$. Hence u is a solution of the Eq. (7). $\square$

6 The case $c < 0$

In this section, we prove Theorem 3 by using a method of upper and lower solutions. In particular, we show that it suffices to construct an upper solution of the Eq. (7). This is exactly the graph version of the argument of Kazdan and Warner ([5], Sections 9 and 10).

We call a function u_- a lower solution of (7) if for all $x \in V$, there holds

$$\Delta u_-(x) - c + h e^{u_-(x)} \geq 0.$$

Similarly, u_+ is called an upper solution of (7) if for all $x \in V$, it satisfies

$$\Delta u_+(x) - c + h e^{u_+(x)} \leq 0.$$

We begin with the following:

Lemma 8 *Let $c < 0$. If there exist lower and upper solutions, u_- and u_+, of the Eq. (7) with $u_- \leq u_+$, then there exists a solution u of (7) satisfying $u_- \leq u \leq u_+$.*

Proof We follow the lines of Kazdan and Warner ([5], Lemma 9.3). Set $k_1(x) = \max\{1, -h(x)\}$, so that $k_1 \geq 1$ and $k_1 \geq -h$. Let $k(x) = k_1(x) e^{u_+(x)}$. We define $L\varphi \equiv \Delta\varphi - k\varphi$ and $f(x, u) \equiv c - h(x)e^u$. Since $G = (V, E)$ is a finite graph and $\inf_{x \in V} k(x) > 0$, we have that L is a compact operator and $\mathrm{Ker}(L) = \{0\}$. Hence we can define inductively u_{j+1} as the unique solution to

$$L u_{j+1} = f(x, u_j) - k u_j, \tag{17}$$

where $u_0 = u_+$. We *claim* that

$$u_- \leq u_{j+1} \leq u_j \leq \cdots \leq u_+. \tag{18}$$

To see this, we estimate

$$L(u_1 - u_0) = f(x, u_0) - k u_0 - \Delta u_0 + k u_0 \geq 0.$$

Suppose $u_1(x_0) - u_0(x_0) = \max_{x \in V}(u_1(x) - u_0(x)) > 0$. Then $\Delta(u_1 - u_0)(x_0) \leq 0$, and thus $L(u_1 - u_0)(x_0) < 0$. This is a contradiction. Hence $u_1 \leq u_0$ on V. Suppose $u_j \leq u_{j-1}$, we calculate by using the mean value theorem

$$\begin{aligned}L(u_{j+1}-u_j) &= k(u_{j-1}-u_j)+h(e^{u_{j-1}}-e^{u_j})\\ &\geq k_1(e^{u_+}-e^{\xi})(u_{j-1}-u_j)\\ &\geq 0,\end{aligned}$$

where $u_j \leq \xi \leq u_{j-1}$. Similarly as above, we have $u_{j+1} \leq u_j$ on V, and by induction, $u_{j+1} \leq u_j \leq \cdots \leq u_+$ for any j. Noting that

$$L(u_- - u_{j+1}) \geq k(u_j - u_-) + h(e^{u_j} - e^{u_-}),$$

we also have by induction $u_- \leq u_j$ on V for all j. Therefore (18) holds. Since V is finite, it is easy to see that up to a subsequence, $u_j \to u$ uniformly on V. Passing to the limit $j \to +\infty$ in the Eq. (17), one concludes that u is a solution of (7) with $u_- \leq u \leq u_+$. □

Next we show that the Eq. (7) has infinitely many lower solutions. This reduces the proof of Theorem 3 to finding its upper solution.

Lemma 9 *There exists a lower solution u_- of (7) with $c < 0$. Thus (7) has a solution if and only if there exists an upper solution.*

Proof Let $u_- \equiv -A$ for some constant $A > 0$. Since V is finite, we have

$$\Delta u_-(x) - c + h(x)e^{u_-(x)} = -c + h(x)e^{-A} \to -c \quad \text{as } A \to +\infty,$$

uniformly with respect to $x \in V$. Noting that $c < 0$, we can find sufficiently large A such that u_- is a lower solution of (7). □

Proof of Theorem 3 (i) *Necessary condition* If u is a solution of (7), then

$$\begin{aligned}-\int_V h d\mu &= \int_V e^{-u}\Delta u d\mu - c\int_V e^{-u} d\mu\\ &= -\int_V \Gamma(e^{-u}, u) d\mu - c\int_V e^{-u} d\mu\\ &> 0.\end{aligned}$$

(ii) *Sufficient condition* It follows from Lemmas 8 and 9 that (7) has a solution if and only if (7) has an upper solution u_+ satisfying

$$\Delta u_+ \leq c - he^{u_+}.$$

Clearly, if u_+ is an upper solution for a given $c < 0$, then u_+ is also an upper solution for all $\widetilde{c} < 0$ with $c \leq \widetilde{c}$. Therefore, there exists a constant $c_-(h)$ with $-\infty \leq c_-(h) \leq 0$ such that (7) has a solution for any $c > c_-(h)$ but has no solution for any $c < c_-(h)$.

We *claim* that $c_-(h) < 0$ under the assumption $\int_V h d\mu < 0$. To see this, we let v be a solution of $\Delta v = \overline{h} - h$. The existence of v can be seen in the following way. If we consider orthogonality with respect to the standard scalar product, namely $\langle \phi, \psi \rangle = \int_V \phi\psi d\mu$, we have

$$\text{ran}\,\Delta = (\ker \Delta)^{\perp} = \{\text{const}\}^{\perp}.$$

So, since $\overline{h} - h$ is orthogonal to the constant functions such a solution exists by invertibility of Δ on $\{\text{const}\}^{\perp}$ and in the case of constant h a solution can be chosen to be an arbitrary constant since the right hand side satisfies $\overline{h} - h = 0$ in this case.

There exists some constant $a > 0$ such that

$$|e^{av} - 1| \le \frac{-\overline{h}}{2\max_{x\in V}|h(x)|}.$$

Let $e^b = a$. If $c = \frac{a\overline{h}}{2}$ and $u_+ = av + b$, we have

$$\begin{aligned}
\Delta u_+ - c + he^{u_+} &= ah(e^{av} - 1) + \frac{a\overline{h}}{2} \\
&\le a(\max_{x\in V}|h(x)|)|e^{av} - 1| + \frac{a\overline{h}}{2} \\
&\le \frac{a\overline{h}}{2} - \frac{a\overline{h}}{2} \\
&= 0.
\end{aligned}$$

Thus if $c = a\overline{h}/2 < 0$, then the Eq. (7) has an upper solution u_+. Therefore, $\overline{h} < 0$ implies that $c_-(h) \le a\overline{h}/2 < 0$. □

Proof of Theorem 4 We shall show that if $h(x) \le 0$ for all $x \in V$, but $h \not\equiv 0$, then (7) is solvable for all $c < 0$. For this purpose, as in the proof of Theorem 3, we let v be a solution of $\Delta v = \overline{h} - h$. Note that $\overline{h} < 0$. Pick constants a and b such that $a\overline{h} < c$ and $e^{av+b} - a > 0$. Let $u_+ = av + b$. Since $h \le 0$,

$$\begin{aligned}
\Delta u_+ - c + he^{u_+} &= a\Delta v - c + he^{av+b} \\
&= a\overline{h} - ah - c + he^{av+b} \\
&\le h(e^{av+b} - a) \\
&\le 0.
\end{aligned}$$

Hence u_+ is an upper solution. Consequently, $c_-(h) = -\infty$ if $h \le 0$ but $h \not\equiv 0$. □

7 Some extensions

The Eq. (2) involving higher order differential operators was also extensively studied on manifolds, see for examples [6,7] and the references therein. In this section, we shall extend Theorems 1–4 to nonlinear elliptic equations involving higher order derivatives. For this purpose, we define the length of m-order gradient of u by

$$|\nabla^m u| = \begin{cases} |\nabla \Delta^{\frac{m-1}{2}} u|, & \text{when } m \text{ is odd} \\ |\Delta^{\frac{m}{2}} u|, & \text{when } m \text{ is even,} \end{cases} \tag{19}$$

where $|\nabla \Delta^{\frac{m-1}{2}} u|$ is defined as in (5) for the function $\Delta^{\frac{m-1}{2}} u$, and $|\Delta^{\frac{m}{2}} u|$ denotes the usual absolute of the function $\Delta^{\frac{m}{2}} u$. Define a Sobolev space by

$$W^{m,2}(V) = \left\{ v : V \to \mathbb{R} : \int_V (|v|^2 + |\nabla^m v|^2) d\mu < +\infty \right\}$$

and a norm on it by

$$\|v\|_{W^{m,2}(V)} = \left(\int_V (|v|^2 + |\nabla^m v|^2) d\mu \right)^{1/2}.$$

Clearly $W^{m,2}(V)$ is the set of all functions on V since V is finite. Moreover, we have the following Sobolev embedding, the Poincaré inequality and the Trudinger–Moser embedding:

Lemma 10 *Let $G = (V, E)$ be a finite graph. Then for any integer $m > 0$, $W^{m,2}(V)$ is pre-compact.*

Lemma 11 *Let $G = (V, E)$ be a finite graph. For all functions $u : V \to \mathbb{R}$ with $\int_V ud\mu = 0$, there exists some constant C depending only on m and G such that*

$$\int_V u^2 d\mu \le C \int_V |\nabla^m u|^2 d\mu.$$

Proof Similar to the proof of Lemma 6, we suppose the contrary. There exists a sequence of functions $\{u_j\}$ such that $\int_V u_j d\mu = 0$, $\int_V u_j^2 d\mu = 1$ and $\int_V |\nabla^m u_j|^2 d\mu \to 0$ as $j \to \infty$. Noting that G is a finite graph, there would exist a function u^* such that up to a subsequence,

$$\int_V u^{*2} d\mu = \lim_{j\to\infty} \int_V u_j^2 d\mu = 1, \tag{20}$$

$$\int_V u^* d\mu = \lim_{j\to\infty} \int_V u_j d\mu = 0, \tag{21}$$

$$\int_V |\nabla^m u^*|^2 d\mu = \lim_{j\to\infty} \int_V |\nabla^m u_j|^2 d\mu = 0. \tag{22}$$

If m is odd and $m \ge 3$, using the same argument in the proof of Lemma 6, we conclude from (22) that $\Delta^{\frac{m-1}{2}} u^* \equiv 0$ on V since $\int_V \Delta^{\frac{m-1}{2}} u^* d\mu = 0$. While if m is even and $m \ge 4$, (22) leads to $\Delta^{\frac{m}{2}-1} u^* \equiv 0$ since $\int_V \Delta^{\frac{m}{2}-1} u^* d\mu = 0$. In view of (21) and the fact that $\int_V \Delta^k u^* d\mu = 0$ for any $k \ge 1$, after repeating the above procedure finitely many times, we conclude that $u^* \equiv 0$ on V. This contradicts (20). □

Lemma 12 *Let $G = (V, E)$ be a finite graph. Let m be a positive integer. Then for any $\beta \in \mathbb{R}$, there exists a constant C depending only on m, β and G such that for all functions v with $\int_V |\nabla^m v|^2 d\mu \le 1$ and $\int_V v d\mu = 0$, there holds*

$$\int_V e^{\beta v^2} d\mu \le C.$$

Proof The same argument in the proof of Lemma 7. □

We consider an analog of (7), namely

$$\Delta^m u = c - he^u \quad \text{in} \quad V, \tag{23}$$

where m is a positive integer, c is a constant, and $h : V \to \mathbb{R}$ is a function. Obviously (23) is reduced to (7) when $m = 1$. Firstly we have the following:

Theorem 13 *Let $G = (V, E)$ be a finite graph, $h(\not\equiv 0)$ be a function on V, and m be a positive integer. If $c = 0$, h changes sign, and $\int_V hd\mu < 0$, then the Eq. (23) has a solution.*

Proof We give the outline of the proof. Denote

$$\mathcal{B}_3 = \left\{ v \in W^{m,2}(V) : \int_V he^v d\mu = 0, \int_V v d\mu = 0 \right\}.$$

In view of (10), we have that $\mathcal{B}_3 = \mathcal{B}_1$, since V is finite. Hence $\mathcal{B}_3 \ne \varnothing$. Now we minimize the functional $J(v) = \int_V |\nabla^m u|^2 d\mu$ on $\mathcal{B}_3$. The remaining part is completely analogous to that of the proof of Theorem 1, except for replacing Lemma 5 by Lemma 10. We omit the details but leave it to interested readers. □

Secondly, in the case $c > 0$, the same conclusion as Theorem 2 still holds for the Eq. (23) with $m > 1$. Precisely we have the following:

Theorem 14 *Let $G = (V, E)$ be a finite graph, c be a positive constant, $h : V \to \mathbb{R}$ be a function, and m be a positive integer. Then the Eq. (23) has a solution if and only if h is positive somewhere.*

Proof Repeating the arguments of the proof of Theorem 2 except for replacing Lemmas 5 and 7 by Lemmas 10 and 12 respectively, we get the desired result. □

Finally, concerning the case $c < 0$, we obtain a result weaker than Theorem 3.

Theorem 15 *Let $G = (V, E)$ be a finite graph, c be a negative constant, m is a positive integer, and $h : V \to \mathbb{R}$ be a function such that $h(x) < 0$ for all $x \in V$. Then the Eq. (23) has a solution.*

Proof Since the maximum principle is not available for equations involving poly-harmonic operators, we use the calculus of variations instead of the method of upper and lower solutions. Let $c < 0$ be fixed. Consider the functional

$$J(u) = \frac{1}{2}\int_V |\nabla^m u|^2 d\mu + c\int_V u d\mu. \tag{24}$$

Set

$$\mathcal{B}_4 = \left\{u \in W^{m,2}(V) : \int_V he^u d\mu = c\mathrm{Vol}(V)\right\}.$$

Using the same method of proving (11) in the proof of Theorem 2, we have $\mathcal{B}_4 \neq \varnothing$.

We now prove that J has a lower bound on $\mathcal{B}_4$. Let $u \in \mathcal{B}_4$. Write $u = v + \overline{u}$. Then $\overline{v} = 0$ and

$$\int_V he^v d\mu = e^{-\overline{u}} c\mathrm{Vol}(V),$$

which leads to

$$\overline{u} = -\log\left(\frac{1}{c\mathrm{Vol}(V)}\int_V he^v d\mu\right).$$

Hence

$$J(u) = \frac{1}{2}\int_V |\nabla^m u|^2 d\mu - c\mathrm{Vol}(V)\log\left(\frac{1}{c\mathrm{Vol}(V)}\int_V he^v d\mu\right). \tag{25}$$

Since $c < 0$ and $h(x) < 0$ for all $x \in V$, we have $\max_{x\in V} h(x) < 0$, and thus

$$\frac{h}{c\mathrm{Vol}(V)} \geq \delta = \frac{\max_{x\in V} h(x)}{c\mathrm{Vol}(V)} > 0. \tag{26}$$

Inserting (26) into (25), we have

$$J(u) \geq \frac{1}{2}\int_V |\nabla^m u|^2 d\mu - c\mathrm{Vol}(V)\log\delta - c\mathrm{Vol}(V)\log\int_V e^v d\mu. \tag{27}$$

By the Jensen inequality,

$$\frac{1}{\mathrm{Vol}(V)}\int_V e^v d\mu \geq e^{\overline{v}} = 1. \tag{28}$$

Inserting (28) into (27), we obtain

$$J(u) \geq \frac{1}{2}\int_V |\nabla^m u|^2 d\mu - c\mathrm{Vol}(V)\log\delta - c\mathrm{Vol}(V)\log\mathrm{Vol}(V). \tag{29}$$

Therefore J has a lower bound on $\mathcal{B}_4$. Set

$$\tau = \inf_{v\in\mathcal{B}_4} J(v).$$

Take a sequence of functions $\{u_k\} \subset \mathcal{B}_4$ such that $J(u_k) \to \tau$. We have by (29) that

$$\int_V |\nabla^m u_k|^2 d\mu \leq C \tag{30}$$

for some constant C depending only on c, τ, G and h. By (24), we estimate

$$\left|\int_V u_k d\mu\right| \leq \frac{1}{|c|}|J(u_k)| + \frac{1}{2|c|}\int_V |\nabla^m u_k|^2 d\mu. \tag{31}$$

Lemma 11 implies that there exists some constant C depending only on m and G such that

$$\int_V |u_k - \overline{u_k}|^2 d\mu \leq C\int_V |\nabla^m u_k|^2 d\mu \tag{32}$$

Combining (30), (31), and (32), one can see that $\{u_k\}$ is bounded in $W^{m,2}(V)$. Then it follows from Lemma 10 that there exists some function u such that up to a subsequence, $u_k \to u$ in $W^{m,2}(V)$. Clearly $u \in \mathcal{B}_4$ and $J(u) = \lim_{k\to\infty} J(u_k) = \tau$. In other words, u is a minimizer of J on the set $\mathcal{B}_4$. It is not difficult to check that (23) is the Euler–Lagrange equation of u. This completes the proof of the theorem. □

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