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A Gradient Estimate for Positive Functions on Graphs

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Abstract We derive a gradient estimate for positive functions, in particular for positive solutions to the heat equation, on finite or locally finite graphs. Unlike the well known Li-Yau estimate, which is based on the maximum principle, our estimate follows from the graph structure of the gradient form and the Laplacian operator. Though our assumption on graphs is slightly stronger than that of Bauer et al. (J Differ Geom 99:359–405, 2015), our estimate can be easily applied to nonlinear differential equations, as well as differential inequalities. As applications, we estimate the greatest lower bound of Cheng's eigenvalue and an upper bound of the minimal heat kernel, which is recently studied by Bauer et al. (Preprint, 2015) by the Li-Yau estimate. Moreover, generalizing an earlier result of Lin and Yau (Math Res Lett 17:343–356, 2010), we derive a lower bound of nonzero eigenvalues by our gradient estimate.

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1 Introduction

Let $G = (V, E)$ be a finite or locally finite graph, where V denotes the vertex set and E denotes the edge set. For any edge $xy \in E$, we assume its weight $w_{xy} > 0$. The degree of $x \in V$ is defined as $\deg(x) = \sum_{y \sim x} w_{xy}$, here and throughout this paper we write $y \sim x$ if $xy \in E$. Let $\mu : V \to [0, \infty)$ be a measure. Then the μ-Laplacian (or Laplacian for short) on G is defined as

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \big(f(y) - f(x)\big).$$

The associated gradient form reads

$$2\Gamma(f, g)(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \big(f(y) - f(x)\big)\big(g(y) - g(x)\big).$$

Write $\Gamma(f) = \Gamma(f, f)$. Denote

$$D_\mu = \sup_{x \in V} \frac{\deg(x)}{\mu(x)}, \quad d = \sup_{x \in V, \, xy \in E} \frac{\mu(x)}{w_{xy}}. \tag{1}$$

The main result in this paper is the following gradient estimate.

Theorem 1 *Let $G = (V, E)$ be a finite or locally finite graph. Suppose that*

$$D_\mu < +\infty, \quad d < +\infty, \tag{2}$$

where D_μ and d are defined as in (1). Then for any positive function $u : V \to \mathbb{R}$, there holds

$$\frac{\sqrt{2\Gamma(u)}}{u} \le \sqrt{d}\frac{\Delta u}{u} + \sqrt{d}D_\mu + \sqrt{D_\mu}. \tag{3}$$

Several special cases are listed below:
(i) If u is a positive solution to the differential inequality $\Delta u - qu \le 0$ on V, where $q : V \to \mathbb{R}$ is a function, then there holds

$$\frac{\sqrt{2\Gamma(u)}}{u} - \sqrt{d}q \le \sqrt{d}D_\mu + \sqrt{D_\mu}.$$

(ii) If u is a positive solution to the differential inequality $\Delta u - hu^\alpha \le 0$, where $\alpha \in \mathbb{R}$, and $h : V \to \mathbb{R}$ is a function, then there holds

$$\frac{\sqrt{2\Gamma(u)}}{u} - \sqrt{d}hu^{\alpha-1} \le \sqrt{d}D_\mu + \sqrt{D_\mu}.$$

(iii) If u is a positive solution to the differential inequality $\Delta u - \partial_t u \leq qu$, where $q : V \times \mathbb{R} \to \mathbb{R}$ is a function, then there holds

$$\frac{\sqrt{2\Gamma(u)}}{u} - \sqrt{d}\frac{\partial_t u}{u} - \sqrt{d}q \leq \sqrt{d}D_\mu + \sqrt{D_\mu}.$$

(iv) If u is a positive solution to the differential inequality $\Delta u - \partial_t u + au \log u \leq 0$, where $a \in \mathbb{R}$ is a constant, then there holds

$$\frac{\sqrt{2\Gamma(u)}}{u} - \sqrt{d}\frac{\partial_t u}{u} - \sqrt{d}a \log u \leq \sqrt{d}D_\mu + \sqrt{D_\mu}.$$

Remark 2 For the corresponding partial differential equations on complete Riemannian manifolds, (i)–(iv) were extensively studied, see for examples [10–12, 14–17] and the references there in.

At least two points can be seen from Theorem 1: One is that (3) is a global estimate; The other is that (3) can be easily applied to nonlinear elliptic or parabolic equations, as well as differential inequalities. We now analyze the assumption (2), whch is equivalent to

$$\sup_{x \in V} \sharp\{y|y \sim x\} < +\infty, \quad 0 < \inf_{x \in V, \, y \sim x} \frac{\mu(x)}{w_{xy}} \leq \sup_{x \in V, \, y \sim x} \frac{\mu(x)}{w_{xy}} < +\infty, \quad (4)$$

where $\sharp\{y|y \sim x\}$ stands for the number of $y \in V$ which is adjacent to x. In fact, suppose (2) holds. Then $\sharp\{y|y \sim x\} \leq D_\mu d$ and $\frac{1}{D_\mu} \leq \frac{\mu(x)}{w_{xy}} \leq d$ for any $y \sim x$. Hence (4) holds. Conversely, if (4) holds, we have

$$\frac{\deg(x)}{\mu(x)} = \frac{\sum_{y \sim x} w_{xy}}{\mu(x)} \leq \frac{\sharp\{y|y \sim x\}}{\inf_{x \in V, \, y \sim x} \frac{\mu(x)}{w_{xy}}}.$$

Then (2) follows immediately. If we replace (2) by (4) in Theorem 1, then the gradient estimate (3) would be

$$\frac{\sqrt{2\Gamma(u)}}{u} \leq \sqrt{b}\frac{\Delta u}{u} + \sqrt{b}\left(\frac{N}{a} + \sqrt{\frac{N}{ab}}\right), \quad (5)$$

where $N = \sup_{x \in V} \sharp\{y|y \sim x\}$, $a = \inf_{x \in V, \, y \sim x} \frac{\mu(x)}{w_{xy}}$ and $b = \sup_{x \in V, \, y \sim x} \frac{\mu(x)}{w_{xy}}$. Note that the essential assumption of [3] is

$$D_\mu < +\infty, \quad D_w = \sup_{x \in V, \, y \sim x} \frac{\deg(x)}{w_{xy}} < +\infty. \quad (6)$$

It is easy to see that (6) is slightly weaker than (2). All gradient estimates in [3] are about $\sqrt{u}$, where u is a positive solution to a parabolic equation on V. Note that for any positive function u

$$2\Gamma(\sqrt{u})(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \left(\sqrt{u}(y) - \sqrt{u}(x) \right)^2$$

$$\leq \left(\sum_{y \sim x} \frac{w_{xy}}{\mu(x)} \right)^{1/2} \left(\frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \left(\sqrt{u}(y) - \sqrt{u}(x) \right)^4 \right)^{1/2}$$

$$\leq \left(\frac{\deg(x)}{\mu(x)} \right)^{1/2} \left(\frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} \left(u(y) - u(x) \right)^2 \right)^{1/2}$$

$$\leq \sqrt{D_\mu} \sqrt{2\Gamma(u)(x)}. \tag{7}$$

If $u : V \times [0, +\infty) \to \mathbb{R}$ is a positive solution to the parabolic equation $\Delta u - \partial_t u - qu = 0$ on $V \times [0, +\infty)$, we conclude an analog of ([3], Theorem 4.10) by combining (7) with Theorem 1,

$$\frac{\Gamma(\sqrt{u})}{u} - \sqrt{D_\mu d} \frac{\partial_t \sqrt{u}}{\sqrt{u}} - \sqrt{D_\mu d} \frac{q}{2} \leq \frac{D_\mu(\sqrt{D_\mu d} + 1)}{2}.$$

As an application of Case (iii) of Theorem 1, we state the following Harnack inequality.

Theorem 3 *Let $G = (V, E)$ be a finite or locally finite graph satisfying (2). Moreover $\mu_{\max} = \sup_{x \in V} \mu(x) < +\infty$ and $w_{\min} = \inf_{x \in V, y \sim x} w_{xy} > 0$. Assume $u : V \times (-\infty, +\infty) \to \mathbb{R}$ is a positive solution to the heat inequality $\Delta u - \partial_t u \leq qu$, where $q : V \times (-\infty, +\infty) \to \mathbb{R}$ is a function. Then for any (x, T_1) and (y, T_2), $T_1 < T_2$, we have*

$$u(x, T_1) \leq u(y, T_2) \exp \left\{ \left(D_\mu + \sqrt{\frac{D_\mu}{d}} \right) (T_2 - T_1) + \frac{(\mathrm{dist}(x, y))^2}{T_2 - T_1} \sqrt{\frac{d\mu_{\max}}{w_{\min}}} \right.$$

$$\left. + \min \sum_{k=0}^{\ell-1} \left(\int_{t_k}^{t_{k+1}} q(x_k, t) dt + \frac{\ell^2}{(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (q(x_{k+1}, t) - q(x_k, t)) dt \right) \right\},$$

where the minimum takes over all shortest paths $x = x_0, x_1, \cdots, x_\ell = y$ connecting x and y, and $t_k = T_1 + k(T_2 - T_1)/\ell$, $k = 0, 1, \cdots, \ell$. In particular, if there exists some constant C_0 such that $|q(x, t)| \leq C_0$ for all (x, t), then for any $x, y \in V$ and $T_1 < T_2$, there holds

$$u(x, T_1) \leq u(y, T_2) \exp \left\{ \left(D_\mu + \sqrt{\frac{D_\mu}{d}} + \frac{5}{3}C_0 \right) (T_2 - T_1) + \frac{(\mathrm{dist}(x, y))^2}{T_2 - T_1} \sqrt{\frac{d\mu_{\max}}{w_{\min}}} \right\}. \tag{8}$$

In a recent work of Bauer et al. [2], the Li-Yau inequality on graphs, which is due to Bauer et al. [3], is applied to Liouville-type theorems and eigenvalue estimates. Moreover a DGG lemma [6–8] concerning the minimal heat kernel is established on

graphs, and it is used together with the Li-Yau inequality to estimate the upper bound of the minimal heat kernel. Our gradient estimate can be used instead of the Li-Yau estimate in [2]. Using Theorem 1, we can estimate the greatest lower bound of the ℓ^2-spectrum known as Cheng's eigenvalue [4].

Theorem 4 *Let $G = (V, E)$ be a locally finite graph satisfying (2) and λ^* be the greatest lower bound of the ℓ^2-spectrum of the graph Laplacian Δ. Moreover we assume $w_{x,y} = w_{yx}$ for all $y \sim x$ and all $x \in V$. Then we have $\lambda^* \le D_\mu + \sqrt{D_\mu}/\sqrt{d}$.*

While Theorem 3 can be used to get an analog of ([2], Theorem 1.2).

Theorem 5 *Let $G = (V, E)$ be a finite or locally finite graph satisfying (2). Moreover $\mu_{\max} < +\infty$ and $w_{\min} > 0$. Let λ^* be the greatest lower bound of the ℓ^2-spectrum of the graph Laplacian Δ. Moreover we assume $w_{x,y} = w_{yx}$ for all $y \sim x$ and all $x \in V$. Given any $\epsilon > 0$, $0 < \gamma \le 1$, $\beta > 0$. Let $P_t(x, y)$ be the minimum heat kernel of G. Then there exist positive constants $C_1(\beta, \gamma, D_\mu)$ and $C_2(\epsilon, \beta, \gamma, D_\mu, d, \mu_{\max}, w_{\min})$ such that for any $x, y \in V$, and $t \ge \max\{\beta d(x, y), 1\}$,*

$$P_t(x, y) \le \frac{\exp\left(-(1 - \gamma)\lambda^* t\right)}{\sqrt{\mathrm{Vol}(B_x(\sqrt{t}))\mathrm{Vol}(B_y(\sqrt{t}))}} \exp\left\{C_2\sqrt{t} - C_1\frac{(\mathrm{dist}(x, y))^2}{4(1 + 2\epsilon)t}\right\}.$$

Finally we remark that Theorem 1 can also be used to estimate a lower bound of nonzero eigenvalues of the Laplacian on finite connected graphs. Precisely we have an analog of ([13], Theorem 1.8), namely

Theorem 6 *Let $G = (V, E)$ be a finite connected graph, D_μ and d be defined as in (1), and D be its diameter. Moreover we assume $w_{x,y} = w_{yx}$ for all $y \sim x$ and all $x \in V$. Suppose that λ is a nonzero eigenvalue of $-\Delta$. Then there holds*

$$\lambda \ge \frac{1}{Dd\left(\exp\left\{1 + Dd\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\right\} - 1\right)}. \tag{9}$$

Remark 7 If $\mu(x) = \deg(x) = \sum_{y \sim x} w_{xy}$, we have $D_\mu = 1$, and hence (9) becomes

$$\lambda \ge \frac{1}{Dd\left(\exp\left\{1 + Dd\left(1 + \sqrt{\frac{1}{d}}\right)\right\} - 1\right)}.$$

We refer the reader to [1,5] for earlier estimates in terms of the volume of the graph G.

Let us describe the method. The proof of Theorem 1 is based on the positivity of the average of u, i.e., $\frac{1}{\mu(x)}\sum_{y \sim x} w_{xy}u(y)$, and its relation with Δu. To prove Theorem 3, we follow [3] and thereby closely follow [12]. While the proof of Theorems 4, 5, and 6 is adapted from [2] and [13], respectively.

The remaining part of this paper is organized as follows. In Sect. 2, we prove the gradient estimate, Theorem 1. In Sect. 3, we prove the corresponding Harnack inequality, Theorem 3. Finally Theorems 4, 5, and 6 are proved in Sect. 4.

2 Gradient Estimate

In this section, we prove Theorem 1 using a very simple method.

Proof of Theorem 1 Special cases $(i)-(iv)$ are immediate consequences of (3). Hence it suffices to prove (3). Since $u > 0$, we have by definition of $\Gamma(u)$,

$$
2\Gamma(u)(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2
$$

$$
\leq \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} u^2(y) + \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} u^2(x)
$$

$$
= \sum_{y \sim x} \frac{\mu(x)}{w_{xy}} \left(\frac{w_{xy}}{\mu(x)} u(y) \right)^2 + u^2(x) \frac{\deg(x)}{\mu(x)}
$$

$$
\leq d \left(\sum_{y \sim x} \frac{w_{xy}}{\mu(x)} u(y) \right)^2 + D_\mu u^2(x).
$$

Noting that

$$
\sum_{y \sim x} \frac{w_{xy}}{\mu(x)} u(y) = \Delta u(x) + u(x) \frac{\deg(x)}{\mu(x)}
$$

and using an elementary inequality $\sqrt{a^2 + b^2} \leq a + b$, $\forall a, b \geq 0$, we get

$$
\sqrt{2\Gamma(u)(x)} \leq \sqrt{d} \sum_{y \sim x} \frac{w_{xy}}{\mu(x)} u(y) + \sqrt{D_\mu} u(x)
$$

$$
= \sqrt{d} \left(\Delta u(x) + u(x) \frac{\deg(x)}{\mu(x)} \right) + \sqrt{D_\mu} u(x)
$$

$$
\leq \sqrt{d} \Delta u(x) + (\sqrt{D_\mu} + \sqrt{d} D_\mu) u(x).
$$

This leads to (3) and thus ends the proof of the theorem.   □

3 Harnack Inequality

In this section, following the lines of [3,12], we prove a Harnack inequality for positive solution to the parabolic inequality $\Delta u - \partial_t u \leq qu$ using (iii) of Theorem 1.

Proof of Theorem 3 Let u be a positive solution to the inequality $\Delta u - \partial_t u \leq qu$. By (iii) of Theorem 1, we have

$$
-\partial_t \log u \leq D_\mu + \frac{\sqrt{D_\mu}}{\sqrt{d}} + q - \frac{1}{\sqrt{d}} \frac{\sqrt{2\Gamma(u)}}{u}. \tag{10}
$$

We distinguish two cases to proceed.

Case 1 $x \sim y$.

For any $s \in [T_1, T_2]$, we have by (10) that

$$\log u(x, T_1) - \log u(y, T_2) = \log \frac{u(x, T_1)}{u(x, s)} + \log \frac{u(x, s)}{u(y, s)} + \log \frac{u(y, s)}{u(y, T_2)}$$

$$= -\int_{T_1}^{s} \partial_t \log u(x, t) dt + \log \frac{u(x, s)}{u(y, s)} - \int_{s}^{T_2} \partial_t \log u(y, t) dt$$

$$\leq \left(D_\mu + \frac{\sqrt{D_\mu}}{\sqrt{d}} \right) (T_2 - T_1) + \int_{T_1}^{s} q(x, t) dt + \int_{s}^{T_2} q(y, t) dt$$

$$- \frac{1}{\sqrt{d}} \left(\int_{T_1}^{s} \frac{\sqrt{2\Gamma(u)(x, t)}}{u(x, t)} dt + \int_{s}^{T_2} \frac{\sqrt{2\Gamma(u)(y, t)}}{u(y, t)} dt \right)$$

$$+ \log \frac{u(x, s)}{u(y, s)}. \tag{11}$$

We estimate the above terms respectively. Obviously

$$\int_{T_1}^{s} \frac{\sqrt{2\Gamma(u)(x, t)}}{u(x, t)} dt \geq 0. \tag{12}$$

Since

$$2\Gamma(u)(y, t) = \frac{1}{\mu(y)} \sum_{z \sim y} w_{yz} \big(u(z, t) - u(y, t)\big)^2$$

$$\geq \frac{w_{\min}}{\mu_{\max}} \big(u(x, t) - u(y, t)\big)^2,$$

we get

$$- \frac{1}{\sqrt{d}} \int_{s}^{T_2} \frac{\sqrt{2\Gamma(u)(y, t)}}{u(y, t)} dt \leq - \sqrt{\frac{w_{\min}}{d\mu_{\max}}} \int_{s}^{T_2} \left| \frac{u(x, t)}{u(y, t)} - 1 \right| dt. \tag{13}$$

Using an elementary inequality $\log r \leq \sqrt{|r - 1|}, \forall r > 0$, we have

$$\log \frac{u(x, s)}{u(y, s)} \leq \psi(x, y, s), \tag{14}$$

where

$$\psi(x, y, s) = \sqrt{\left| \frac{u(x, s)}{u(y, s)} - 1 \right|}.$$

Inserting (12), (13), (14) into (11), and using ([3], Lemma 5.3), we obtain

$$\log u(x, T_1) - \log u(y, T_2)$$

$$\leq \left(D_\mu + \sqrt{\frac{D_\mu}{d}} \right) (T_2 - T_1) + \psi(x, y, s)$$

$$- \sqrt{\frac{w_{\min}}{d\mu_{\max}}} \int_s^{T_2} \psi^2(x, y, t) dt + \int_{T_1}^s q(x, t) dt + \int_s^{T_2} q(y, t) dt$$

$$\leq \left(D_\mu + \sqrt{\frac{D_\mu}{d}} \right) (T_2 - T_1) + \frac{1}{T_2 - T_1} \sqrt{\frac{d\mu_{\max}}{w_{\min}}}$$

$$+ \int_{T_1}^{T_2} q(x, t) dt + \frac{1}{(T_2 - T_1)^2} \int_{T_1}^{T_2} (t - T_1)^2 (q(y, t) - q(x, t)) dt.$$

Case 2 x is not adjacent to y.

Assume $\text{dist}(x, y) = \ell$. Take a shortest path $x = x_0, x_1, \cdots, x_\ell = y$. Let $T_1 = t_0 < t_1 < \cdots < t_\ell = T_2$, $t_k = t_{k-1} + (T_2 - T_1)/\ell$, $k = 1, \cdots, \ell$. By the result of Case 1, we have

$$\log u(x, T_1) - \log u(y, T_2) = \sum_{k=0}^{\ell-1} (\log u(x_k, t_k) - \log u(x_{k+1}, t_{k+1}))$$

$$\leq \sum_{k=0}^{\ell-1} \left(\left(D_\mu + \sqrt{\frac{D_\mu}{d}} \right) (t_{k+1} - t_k) + \frac{1}{t_{k+1} - t_k} \sqrt{\frac{d\mu_{\max}}{w_{\min}}} \right)$$

$$+ \sum_{k=0}^{\ell-1} \left(\int_{t_k}^{t_{k+1}} q(x_k, t) dt + \frac{1}{(t_{k+1} - t_k)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (q(x_{k+1}, t) - q(x_k, t)) dt \right)$$

$$\leq \left(D_\mu + \sqrt{\frac{D_\mu}{d}} \right) (T_2 - T_1) + \frac{\ell^2}{T_2 - T_1} \sqrt{\frac{d\mu_{\max}}{w_{\min}}}$$

$$+ \sum_{k=0}^{\ell-1} \left(\int_{t_k}^{t_{k+1}} q(x_k, t) dt + \frac{\ell^2}{(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (q(x_{k+1}, t) - q(x_k, t)) dt \right).$$

Therefore we conclude

$$\log u(x, T_1) - \log u(y, T_2) \leq \left(D_\mu + \sqrt{\frac{D_\mu}{d}} \right) (T_2 - T_1) + \frac{(\text{dist}(x, y))^2}{(T_2 - T_1)} \sqrt{\frac{d\mu_{\max}}{w_{\min}}}$$

$$+ \min \mathscr{F}(q)(x, y, T_1, T_2),$$

where

$$\mathscr{F}(q)(x, y, T_1, T_2)$$

$$= \sum_{k=0}^{\ell-1} \left(\int_{t_k}^{t_{k+1}} q(x_k, t) dt + \frac{\ell^2}{(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (q(x_{k+1}, t) - q(x_k, t)) dt \right)$$

and the minimum takes over all shortest paths connecting x and y. Hence the first assertion of the theorem follows immediately.

Moreover, if $|q(x, t)| \leq C_0$ for all (x, t), then we have

$$\sum_{k=0}^{\ell-1} \int_{t_k}^{t_{k+1}} q(x_k, t)dt \leq C_0(T_2 - T_1)$$

and

$$\sum_{k=0}^{\ell-1} \frac{\ell^2}{(T_2 - T_1)^2} \int_{t_k}^{t_{k+1}} (t - t_k)^2 (q(x_{k+1}, t) - q(x_k, t))dt \leq \frac{2C_0}{3}(T_2 - T_1).$$

This gives the desired result and the proof of the theorem is completed. $\qquad\square$

4 Further Applications of the Gradient Estimate

In this section, as applications of Theorem 1, we prove Theorems 4, 5, and 6. For the proof of Theorems 4 and 5, we follow the lines of [2], the essential difference is that we use Theorem 1 instead of the Li-Yau estimate [3]. While the proof of Theorem 6 is an adaptation of [13]. For reader's convenience, we give the details here.

Proof of Theorem 4 Let λ^* be the greatest lower bound of Cheng's eigenvalues. By a result of S. Haeseler and M. Keller ([9], Theorem 3.1), if $\lambda \leq \lambda^*$, then there would be a positive solution u to $\Delta u = -\lambda u$. We conclude from Case (i) of Theorem 1 that

$$\frac{\sqrt{2\Gamma(u)}}{u} + \sqrt{d}\lambda \leq \sqrt{d}D_\mu + \sqrt{D_\mu}.$$

Hence $\lambda \leq D_\mu + \sqrt{D_\mu/d}$. Since λ is arbitrary, we obtain $\lambda^* \leq D_\mu + \sqrt{D_\mu/d}$. $\quad\square$

To prove Theorem 5, we need the following DGG lemma on graphs ([2], Theorem 1.1).

Lemma 1 *Let $P_t(x, y)$ be the minimal heat kernel of the graph $G = (V, E)$. Then for any $\beta > 0$ and $0 < \gamma \leq 1$, there exists a constant C_1 depending only on $\beta, \gamma,$ and D_μ such that for any subsets $B_1, B_2 \subset G, t \geq \max\{\beta \mathrm{dist}(B_1, B_2), 1\}$,*

$$\sum_{x \in B_1} \sum_{y \in B_2} P_t(x, y)\mu(x)\mu(y) \leq e^{-(1-\gamma)\lambda^* t} \sqrt{\mathrm{Vol}(B_1)\mathrm{Vol}(B_2)} \exp\left(-C_1 \frac{(\mathrm{dist}(B_1, B_2))^2}{4t}\right).$$

Proof of Theorem 5 Fix $x, y \in V, \delta > 0, T_1 = t$ and $T_2 = (1 + \delta)t$. Applying the Harnack inequality, Theorem 3, to the minimal heat kernel $P_t(x, y)$,

$$P_t(x, y) \leq P_{(1+\delta)t}(x', y) \exp\left\{\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{(\mathrm{dist}(x, x'))^2}{\delta t}\sqrt{\frac{d\mu_{\max}}{w_{\min}}}\right\}$$

$$\leq P_{(1+\delta)t}(x', y) \exp\left\{\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{1}{\delta}\sqrt{\frac{d\mu_{\max}}{w_{\min}}}\right\}, \quad \forall x' \in B_x(\sqrt{t}).$$

Integrating the above inequality on $B_x(\sqrt{t})$ with respect to x', we have

$$\text{Vol}(B_x(\sqrt{t}))P_t(x, y)$$
$$\leq \exp\left\{\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{1}{\delta}\sqrt{\frac{d\mu_{\max}}{w_{\min}}}\right\} \sum_{x' \in B_x(\sqrt{t})} \mu(x')P_{(1+\delta)t}(x', y). \quad (15)$$

Note that $h(y, s) = \sum_{x' \in B_x(\sqrt{t})} \mu(x')P_s(x', y)$ is also a positive solution to the heat equation. Applying again the Harnack inequality, Theorem 3, to $h(y, s)$ with $T_1 = (1 + \delta)t$ and $T_2 = (1 + 2\delta)t$, we have

$$\text{Vol}(B_y(\sqrt{t}))h(y, (1 + \delta)t)$$
$$\leq \exp\left\{\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{1}{\delta}\sqrt{\frac{d\mu_{\max}}{w_{\min}}}\right\} \sum_{y' \in B_y(\sqrt{t})} \mu(y')h(y', (1 + 2\delta)t).$$

This together with (15) implies that

$$P_t(x, y) \leq \exp\left\{2\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{2}{\delta}\sqrt{\frac{d\mu_{\max}}{w_{\min}}}\right\}$$
$$\frac{1}{\text{Vol}(B_x(\sqrt{t}))\text{Vol}(B_y(\sqrt{t}))} \sum_{x' \in B_x(\sqrt{t})} \sum_{y' \in B_y(\sqrt{t})} \mu(x')\mu(y')P_{(1+2\delta)t}(x', y').$$

Let $t \geq \max\{\beta\text{dist}(x, y), 1\}$. Obviously $t \geq \frac{1}{1+2\delta}\max\{\beta\text{dist}(B_x(\sqrt{t}), B_y(\sqrt{t})), 1\}$. Let $\gamma, 0 < \gamma \leq 1$, be fixed. It follows from Lemma 1 that there exists a constant C_1 depending only on γ, β, and D_μ such that

$$P_t(x, y) \leq \exp\left\{2\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{2}{\delta}\sqrt{\frac{d\mu_{\max}}{w_{\min}}}\right\} \frac{1}{\sqrt{\text{Vol}(B_x(\sqrt{t}))\text{Vol}(B_y(\sqrt{t}))}}$$
$$\exp\left\{-(1 - \gamma)\lambda^*(1 + 2\delta)t - C_1\frac{(\text{dist}(B_x(\sqrt{t}), B_y(\sqrt{t})))^2}{4(1 + 2\delta)t}\right\}. \quad (16)$$

If $\text{dist}(x, y) > 2\sqrt{t}$, then $\text{dist}(B_x(\sqrt{t}), B_y(\sqrt{t})) \geq \text{dist}(x, y) - 2\sqrt{t}$ and thus

$$\frac{(\text{dist}(B_x(\sqrt{t}), B_y(\sqrt{t}))^2}{4(1 + 2\delta)t} \geq \frac{(\text{dist}(x, y))^2}{4(1 + 4\delta)t} - \frac{1}{2\delta}. \quad (17)$$

It is easy to see that (17) still holds if $\mathrm{dist}(x, y) \leq 2\sqrt{t}$. Inserting (17) into (16), we have

$$
P_t(x, y) \leq \frac{1}{\sqrt{\mathrm{Vol}(B_x(\sqrt{t}))\mathrm{Vol}(B_y(\sqrt{t}))}} \exp\left\{2\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\delta t + \frac{2}{\delta}\sqrt{\frac{d\mu_{\max}}{w_{\min}}} + \frac{C_1}{2\delta}\right\}
$$
$$
\exp\left\{-(1-\gamma)\lambda^*(1+2\delta)t - C_1\frac{(\mathrm{dist}(x, y))^2}{4(1+4\delta)t}\right\}. \tag{18}
$$

Note that $t \geq 1$. Choosing $2\delta = \epsilon/\sqrt{t}$ in (18), we obtain for $t \geq \max\{\beta\mathrm{dist}(x, y), 1\}$,

$$
P_t(x, y) \leq \frac{1}{\sqrt{\mathrm{Vol}(B_x(\sqrt{t}))\mathrm{Vol}(B_y(\sqrt{t}))}} \exp\left\{\sqrt{t}\left(D_\mu\epsilon + \sqrt{\frac{D_\mu}{d}}\epsilon + \frac{4}{\epsilon}\sqrt{\frac{d\mu_{\max}}{w_{\min}}} + \frac{C_1}{\epsilon}\right)\right\}
$$
$$
\exp\left\{-(1-\gamma)\lambda^*t - C_1\frac{(\mathrm{dist}(x, y))^2}{4(1+2\epsilon)t}\right\}.
$$

Denoting $C_2 = D_\mu\epsilon + \sqrt{\frac{D_\mu}{d}}\epsilon + \frac{4}{\epsilon}\sqrt{\frac{d\mu_{\max}}{w_{\min}}} + \frac{C_1}{\epsilon}$, we finish the proof of the theorem. $\square$

Proof of Theorem 6 Note that $\int_V \Delta u d\mu = 0$ and that if $-\Delta u = \lambda u$, then $-\Delta(cu) = \lambda cu$ for any constant $c \in \mathbb{R}$. We can assume $-\Delta u = \lambda u$ with $\sup u = 1$ and $\inf u < 0$. Take $x_1, x_\ell \in G$ such that $u(x_1) = \sup u = 1$, $u(x_n) = \inf u < 0$, $x_1x_2\cdots x_\ell$ be the shortest path connecting x_1 and x_ℓ, where $(x_i, x_{i+1}) \in E$. Then $\ell \leq D$. For any $\beta > 1$, note that

$$
\frac{|u(x_i) - u(x_{i+1})|}{\beta - u(x_i)} \leq \frac{\sqrt{\frac{1}{\mu(x_i)}\sum_{x_iy\in E}\frac{\mu(x_i)}{w_{x_iy}}w_{x_iy}\left(u(x_i) - u(y)\right)^2}}{\beta - u(x_i)}
$$
$$
\leq \sqrt{d}\frac{\sqrt{2\Gamma(u)(x_i)}}{\beta - u(x_i)}. \tag{19}
$$

Since $\beta - u > 0$ and $u \leq 1$, we have using Theorem 1,

$$
\frac{\sqrt{2\Gamma(u)}}{\beta - u} = \frac{\sqrt{2\Gamma(\beta - u)}}{\beta - u}
$$
$$
\leq \sqrt{d}\left(\frac{\Delta(\beta - u)}{\beta - u} + D_\mu + \sqrt{\frac{D_\mu}{d}}\right)
$$
$$
= \sqrt{d}\left(\frac{\lambda u}{\beta - u} + D_\mu + \sqrt{\frac{D_\mu}{d}}\right)
$$
$$
\leq \sqrt{d}\left(\frac{1}{\beta - 1}\lambda + D_\mu + \sqrt{\frac{D_\mu}{d}}\right).
$$

This together with (19) implies

$$\sum_{i=1}^{\ell} \frac{|u(x_i) - u(x_{i+1})|}{\beta - u(x_i)} \leq Dd\left(\frac{1}{\beta - 1}\lambda + D_\mu + \sqrt{\frac{D_\mu}{d}}\right). \tag{20}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{\ell} \frac{|u(x_i) - u(x_{i+1})|}{\beta - u(x_i)} &\geq \sum_{i=1}^{\ell} \log\left(1 + \frac{|u(x_i) - u(x_{i+1})|}{\beta - u(x_i)}\right) \\
&\geq \sum_{i=1}^{\ell} \log \frac{\beta - u(x_{i+1})}{\beta - u(x_i)} \\
&= \log \frac{\beta - u(x_\ell)}{\beta - u(x_1)} \\
&\geq \log \frac{\beta}{\beta - 1}. \tag{21}
\end{aligned}$$

Combining (20) and (21), we have

$$\lambda \geq (\beta - 1)\left(\frac{1}{Dd} \log \frac{\beta}{\beta - 1} - D_\mu - \sqrt{\frac{D_\mu}{d}}\right).$$

Choose β such that $\frac{1}{Dd} \log \frac{\beta}{\beta-1} - D_\mu - \sqrt{\frac{D_\mu}{d}} = \frac{1}{Dd}$. We obtain

$$\lambda \geq \frac{1}{Dd\left(\exp\left\{1 + Dd\left(D_\mu + \sqrt{\frac{D_\mu}{d}}\right)\right\} - 1\right)}.$$

This completes the proof of the theorem.   □

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