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On-diagonal lower estimate of heat kernels on graphs

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ABSTRACT

We establish a new on-diagonal lower estimate of continuous-time heat kernels for large time on graphs. To achieve the goal, we first introduce an upper estimate of heat kernels in natural graph metric, then use the upper estimate and the volume growth condition to show the validity of the on-diagonal lower estimate.

1. Introduction and main result

Over the last decades, there has been remarkable progress in our understanding of upper and lower estimates for heat kernels on Riemannian manifolds [3,4,8,14]. Coulhon and Grigoryan [3] provided an on-diagonal lower estimate for the heat kernel on complete manifolds under polynomial volume growth condition, which reads as follows:

$$p(t,x,x) \geq \frac{1}{4V(x,\sqrt{Ct\log t})}.$$

On the other hand, the celebrated Li–Yau inequality [14] shows that the heat kernel on non-negatively curved manifolds satisfies Gaussian type bounds, that is,

$$\frac{C_l}{V(x,\sqrt{t})}\exp\left(-c_l\frac{d(x,y)^2}{t}\right) \leq p(t,x,y) \leq \frac{C_r}{V(x,\sqrt{t})}\exp\left(-c_r\frac{d(x,y)^2}{t}\right).$$

Similar methods have been used to study heat kernels on graphs. Recently, Bauer et al. [1] established a discrete analogue of the Li–Yau inequality and derived a heat kernel estimate under the curvature condition $CDE(n,0)$. Despite the upper bound in their results is formulated with Gaussian form, the lower bound is not quite Gaussian form and is dependent on the parameter n. Based on this, Horn et al. [11] improved some

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results in [1] and got Gaussian type bounds via introducing the other curvature condition $CDE'(n,0)$. In addition, Lin et al. [15,16] investigated gradient estimates for positive functions and illustrated applications of these results in establishing certain upper bounds and lower bounds of the heat kernel on graphs.

In [5], Davies obtained non-Gaussian upper bounds of the heat kernel on graphs in continuous time setting. The most interesting feature of his results is that they involve certain functions defined as Legendre transform. In [2], Bauer et al. established a sharp version of Davies–Gaffney–Grigoryan Lemma on graphs. As a direct application, it yields Davies's heat kernel estimate. By using Davies's results, Folz [7] established long-range weak Gaussian upper estimates for the heat kernel with respect to adapted metrics.

Comparing with the heat kernel upper bound, it is more difficult to get a heat kernel lower bound. Various techniques for obtaining an on-diagonal heat kernel lower bound were discussed earlier. Especially, the form

$$p(t,x,x) \geq \frac{C_1}{V(x, C_2\sqrt{t})} \tag{1.1}$$

has attracted considerable attention and interest of researchers.

In [6], Delmotte proved that the particular on-diagonal lower bound (1.1) is true on graphs satisfying the continuous-time parabolic Harnack inequality $\mathcal{H}(C_{\mathcal{H}})$. In [11], Horn et al. derived (1.1) under the curvature condition $CDE'(n,0)$. Motivated by the idea of Coulhon and Grigoryan [3], Grigoryan [9], Lust-Piquard [18], in this paper, we shall only use the volume growth condition to obtain an on-diagonal lower estimate of continuous-time heat kernels for large time on graphs. In order to achieve this, we first discuss an upper estimate of heat kernels, which was established by Folz in [7]. Based on this upper estimate, we can establish the on-diagonal lower estimate of continuous-time heat kernels on graphs for large time under the volume growth condition.

Our main result is as follows:

Theorem 1.1. *Let (G, ω, μ) be a weighted graph that satisfies $\mu_0 > 0$ and $D_\mu < \infty$. Assume that, for all $x \in V$ and $r \geq r_0$,*

$$V(x,r) \leq c_0 r^m, \tag{1.2}$$

where r_0, c_0, m are some positive constants. Then, for all large enough t,

$$p(t,x,x) \geq \frac{1}{4V(x, Ct\log t)}, \tag{1.3}$$

where $C > 2e\left(\sqrt{D_\mu} \vee 1\right)$.

Remark 1.1. Although the estimate in Theorem 1.1 may be not as sharp as the Gaussian estimate in [11], its conditions are far weaker than the latter. Actually, Bauer et al. [1] concluded that the curvature conditions $CDE(n,0)$ and $CDE'(n,0)$ imply polynomial volume growth condition. So, in some cases, the estimate in Theorem 1.1 would have broader applications compared to the Gaussian estimate in [11].

Remark 1.2. In fact, the assumption $\mu_0 > 0$ guarantees $\mu(V) = \sum_{x\in V}\mu(x) = \infty$, it follows from the consideration in [13] that $p(t,x,x) \to 0$ as $t \to \infty$, which shows that the lower bound $\frac{1}{4V(x,Ct\log t)}$ in (1.3) has the correct limit as $t \to \infty$.

The remaining parts of this paper are organized as follows. In Section 2, we introduce some concepts and notations used throughout this paper. In Section 3, we review the heat kernel on graphs and show some known results about it. In Section 4, we give the proof of our main result.

2. Preliminaries

In this section, we introduce some concepts and notations which will be used in this paper. For more details, we refer the readers to [1,5,6,11,17,19,20].

Let $G = (V, E)$ be a infinite, connected graph. Here V denotes the vertex set of G and E denotes the edge set of G. One could call the triplet (G, ω, μ), where $G = (V, E)$ is a combinatorial graph, ω are edge weights and μ are measures on vertices, a weighted graph. Let $\omega : V \times V \to [0, \infty)$ be an edge weight function satisfying the following properties: (i) $\omega_{xy} = \omega_{yx}$ for all $x, y \in V$; (ii) $\omega_{xy} > 0$ if and only if x is adjacent to y (also denoted by $x \sim y$); (iii) $\sum_{y\sim x} \omega_{xy} < \infty$ for all $x \in V$. Furthermore, let $\mu : V \to (0, \infty)$ be a positive measure on vertices of G. In this paper, all the graphs in our concern are assumed to satisfy

$$\mu_0 := \inf_{x\in V} \mu(x) > 0$$

and

$$D_\mu := \sup_{x\in V} \frac{m(x)}{\mu(x)} < \infty,$$

where $m(x) := \sum_{y\sim x} \omega_{xy}$.

Let $C(V)$ be the set of real functions on V. For any $1 \le p < \infty$, we denote by

$$\ell^p(V, \mu) = \left\{ f \in C(V) : \sum_{x\in V} \mu(x)|f(x)|^p < \infty \right\}$$

the set of ℓ^p integrable functions on V with respect to the measure μ. For $p = \infty$, let

$$\ell^\infty(V, \mu) = \left\{ f \in C(V) : \sup_{x\in V} |f(x)| < \infty \right\}.$$

The standard inner product is defined by

$$\langle f, g\rangle = \sum_{x\in V} \mu(x)f(x)g(x), \qquad f, g \in \ell^2(V, \mu),$$

which makes $\ell^2(V, \mu)$ a Hilbert space.

Define the μ-Laplacian Δ on the vector space

$$\mathcal{F} := \left\{ f \in C(V) : \sum_{y\sim x} |\omega_{xy} f(y)| < \infty \text{ for all } x \in V \right\}$$

by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy}(f(y) - f(x)), \tag{2.1}$$

where, for each $x \in V$, the sum in the right-hand side of (2.1) exists by the assumption on f. It follows from the finiteness of μ_0 and m that Δ can be restricted on $\ell^p(V, \mu)$ ($p \in [1, \infty]$). Further, it can be checked that $D_\mu < \infty$ is equivalent to the μ-Laplacian Δ being bounded on $\ell^p(V, \mu)$ for all $p \in [1, \infty]$ (see [10]).

A connected graph can be endowed with its natural graph metric $d(x, y)$, i.e., the smallest number of edges of a path between two vertices x and y. For any $r \geq 0$, we define balls $B(x, r) = \{y \in V : d(x, y) \leq r\}$. The volume of a subset U of V is denoted by $V(U)$, where $V(U) = \sum_{x \in U} \mu(x)$. For convenience, we usually abbreviate $V\big(B(x, r)\big)$ as $V(x, r)$. In addition, we say that a graph G has polynomial volume growth of degree $m > 0$, if there is a constant $c > 0$, such that for all $x \in V, r \geq 0$,

$$V(x, r) \leq cr^m.$$

3. The heat kernel on graphs

In this section, we discuss some known results and lemmas about the heat kernel on graphs.

We say that a function $p : (0, +\infty) \times V \times V \to \mathbb{R}$ is a fundamental solution of the heat equation

$$u_t = \Delta u \tag{3.1}$$

on G, if for any bounded initial condition $u_0 : V \to \mathbb{R}$, the function

$$u(t, x) = \sum_{y \in V} \mu(y) p(t, x, y) u_0(y) \qquad (t > 0,\, x \in V)$$

is differentiable in t, satisfies the heat equation (3.1), and for any $x \in V$, $\lim\limits_{t \to 0^+} u(t, x) = u_0(x)$ holds.

It is easy to see that the operator Δ is bounded and self-adjoint under the assumption $D_\mu < \infty$. In this case, there is only one fundamental solution to the heat equation (3.1) for $\ell^2(V, \mu)$-solutions, which can be obtained as follows:

It is well known that $-\Delta$ is a non-negative operator. Using the spectral theorem one defines the heat semigroup P_t associated with $-\Delta$ by

$$P_t := \exp(t\Delta) = \exp(-t(-\Delta)) \qquad (t > 0).$$

Let $p(t, \cdot, \cdot)$ be the integral kernel of P_t on $\ell^2(V, \mu)$, i.e., the function satisfying

$$P_t f(x) = \sum_{y \in V} \mu(y) p(t, x, y) f(y)$$

for all $x \in V$ and $f \in \ell^2(V, \mu)$. Since V is discrete, p is explicitly given by

$$p(t, x, y) = \frac{1}{\mu(x)\mu(y)} \langle P_t \delta_x, \delta_y \rangle. \tag{3.2}$$

Lemma 3.1. *For all $x \in V$, $p(t, x, x)$ is non-increasing for $t \in (0, \infty)$.*

Proof. Let μ_x be the spectral measure of $-\Delta$ at δ_x. It is non-negative and finite. The spectral theorem then implies

$$p(t, x, x) = \frac{1}{\mu(x)^2} \int_0^\infty e^{-t\lambda} d\mu_x.$$

A differentiation under the integral shows

$$\partial_t p(t,x,x) = -\frac{1}{\mu(x)^2}\int_0^\infty \lambda e^{-t\lambda} d\mu_x \le 0,$$

which completes the proof of Lemma 3.1. □

Lemma 3.2 *(see [11,19,20]). For $t, s > 0$ and any $x, y \in V$, we have*

(i) $p(t,x,y) = p(t,y,x)$,
(ii) $p(t,x,y) \ge 0$,
(iii) $\sum_{y\in V} \mu(y)p(t,x,y) \le 1$,
(iv) $\partial_t p(t,x,y) = \Delta_x p(t,x,y) = \Delta_y p(t,x,y)$,
(v) $\sum_{z\in V} \mu(z)p(t,x,z)p(s,z,y) = p(t+s,x,y)$.

Using adapted metrics, Folz [7] established the following long-range, non-Gaussian upper bounds for the heat kernel.

Lemma 3.3 *(see [7]). If $x_1, x_2 \in V$, then for all $t > 0$,*

$$p(t,x_1,x_2) \le \left(\mu(x_1)\mu(x_2)\right)^{-\frac{1}{2}} \exp\left(-\frac{\rho(x_1,x_2)}{2}\log\left(\frac{\rho(x_1,x_2)}{2et}\right) - \Lambda t\right),$$

where $\Lambda \ge 0$ is the bottom of the ℓ^2 spectrum of $-\Delta$ and ρ is a metric on V that satisfies

$$\begin{cases} \frac{1}{\mu(x)}\sum\limits_{y\sim x}\omega_{xy}\rho^2(x,y) \le 1 & \text{for all } x \in V, \\ \rho(x,y) \le 1 & \text{whenever } x,y \in V \text{ and } x \sim y. \end{cases} \tag{3.3}$$

Remark 3.1. Since $D_\mu < \infty$, the metric

$$\rho := \frac{1}{\sqrt{D_\mu} \vee 1} d$$

satisfies the condition (3.3). Therefore, the Lemma 3.3 shows an upper estimate of the heat kernel in natural graph metric $d(\cdot,\cdot)$, that is, for $t > 0$,

$$p(t,x_1,x_2) \le \left(\mu(x_1)\mu(x_2)\right)^{-\frac{1}{2}} \exp\left(-\frac{d(x_1,x_2)}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{d(x_1,x_2)}{2\left(\sqrt{D_\mu}\vee 1\right)et}\right) - \Lambda t\right). \tag{3.4}$$

4. Proof of the main result

In this section we are going to prove our main result. In the process below, some of the inspirations are from Grigoryan [9] and Lust-Piquard [18], in which the authors use the corresponding method to obtain a lower estimate for discrete-time kernel $p_{2n}(x,x)$ in a discrete space.

Proof of Theorem 1.1. Under the assumption $D_\mu < \infty$, the graph is stochastically complete (see [12]), i.e., the heat kernel satisfies

$$\sum_{y\in V} \mu(y)p(t,x,y) = 1.$$

By utilizing properties of the heat kernel and the Cauchy–Schwarz inequality, we get, for any $r > 0$,

$$\begin{aligned} p(2t,x,x) &= \sum_{z\in V} \mu(z)p^2(t,x,z) \\ &\geq \sum_{z\in B(x,r)} \mu(z)p^2(t,x,z) \\ &\geq \frac{1}{V(x,r)} \left(\sum_{z\in B(x,r)} \mu(z)p(t,x,z) \right)^2 \\ &= \frac{1}{V(x,r)} \left(1 - \sum_{z\in B(x,r)^c} \mu(z)p(t,x,z) \right)^2 . \end{aligned} \tag{4.1}$$

From Lemma 3.1 and the inequality (4.1), it follows that

$$p(t,x,x) \geq \frac{1}{V(x,r)} \left(1 - \sum_{z\in B(x,r)^c} \mu(z)p(t,x,z) \right)^2 . \tag{4.2}$$

If we can prove that when $r = r(t) = Ct\log t$, the inequality

$$\sum_{z\in B(x,r)^c} \mu(z)p(t,x,z) \leq \frac{1}{2} \tag{4.3}$$

is valid for large enough t, then the assertion of Theorem 1.1 follows from (4.2) immediately.

Let us now turn to show that the inequality (4.3) holds with $r = Ct\log t$ for large enough t. Using the inequality (3.4), we have the following upper estimate of the heat kernel:

$$\begin{aligned} p(t,x,z) &\leq (\mu(x)\mu(z))^{-\frac{1}{2}} \exp\left(-\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)} \log\left(\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)et} \right) - \Lambda t \right) \\ &\leq \frac{1}{\mu_0} \exp\left(-\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)} \log\left(\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)et} \right) - \Lambda t \right), \end{aligned}$$

where $\mu_0 := \inf_{x\in V} \mu(x) > 0$.

Hence, for any $r > 2\left(\sqrt{D_\mu}\vee 1\right)et$, we have

$$\begin{aligned} &\sum_{z\in B(x,r)^c} \mu(z)p(t,x,z) \\ &\leq \frac{1}{\mu_0} \sum_{z\in B(x,r)^c} \mu(z) \exp\left(-\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)} \log\left(\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)et} \right) - \Lambda t \right) \\ &= \frac{1}{\mu_0} \sum_{k=0}^{\infty} \sum_{z\in B(x,2^{k+1}r)\setminus B(x,2^k r)} \mu(z) \exp\left(-\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)} \log\left(\frac{d(x,z)}{2\left(\sqrt{D_\mu}\vee 1\right)et} \right) - \Lambda t \right) \\ &\leq \frac{1}{\mu_0} \sum_{k=0}^{\infty} \sum_{z\in B(x,2^{k+1}r)\setminus B(x,2^k r)} \mu(z) \exp\left(-\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)} \log\left(\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)et} \right) - \Lambda t \right) \end{aligned}$$

$$\leq \frac{1}{\mu_0}\sum_{k=0}^{\infty}\exp\left(-\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)et}\right)-\Lambda t\right)V(x,2^{k+1}r),$$

where we split the complement of $B(x,r)$ into the union of the annuli $B(x,2^{k+1}r)\backslash B(x,2^k r)$ $(k=0,1,2,\ldots)$ and use the fact that $\exp\left(-\frac{y}{2(\sqrt{D_\mu}\vee 1)}\log\frac{y}{2(\sqrt{D_\mu}\vee 1)et}-\Lambda t\right)$ is decreasing with respect to y when $y > 2\left(\sqrt{D_\mu}\vee 1\right)et$.

Further, by the assumption that $V(x,r)\leq c_0 r^m$ for all $x\in V$ and $r\geq r_0$ in Theorem 1.1, we obtain

$$\sum_{z\in B(x,r)^c}\mu(z)p(t,x,z)\leq \frac{c_0}{\mu_0}\sum_{k=0}^{\infty}(2^{k+1}r)^m\exp\left(-\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)et}\right)-\Lambda t\right)$$

for $r\geq r_0$ and $r>2\left(\sqrt{D_\mu}\vee 1\right)et$.

Set

$$a_k=(2^{k+1}r)^m\exp\left(-\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{2^k r}{2\left(\sqrt{D_\mu}\vee 1\right)et}\right)-\Lambda t\right)\qquad(k=0,1,\cdots).$$

Direct calculation gives

$$\begin{aligned}\frac{a_{k+1}}{a_k}&=2^m\exp\left(-\frac{2^k r}{\sqrt{D_\mu}\vee 1}\log\left(\frac{2^k r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)+\frac{2^{k-1}}{\sqrt{D_\mu}\vee 1}\log\left(\frac{2^{k-1}r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)\right)\\&=2^m\exp\left(-\frac{2^{k-1}r}{\sqrt{D_\mu}\vee 1}\left(\log\left(\frac{2^k r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)^2-\log\left(\frac{2^{k-1}r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)\right)\right)\\&=2^m\exp\left(-\frac{2^{k-1}r}{\sqrt{D_\mu}\vee 1}\log\left(\frac{2^{k+1}r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)\right).\end{aligned}\tag{4.4}$$

Since, for $r>2\left(\sqrt{D_\mu}\vee 1\right)et$,

$$-2^{k-1}r\log\left(\frac{2^{k+1}r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)\leq -2^{-1}r\log\left(\frac{2r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right),$$

we substitute it into (4.4) to obtain

$$\frac{a_{k+1}}{a_k}\leq 2^m\exp\left(-\frac{r}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{2r}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)\right).$$

Furthermore, assuming that $\frac{r}{2(\sqrt{D_\mu}\vee 1)}\log\left(\frac{2r}{(\sqrt{D_\mu}\vee 1)et}\right)\geq m$, we have

$$\frac{a_{k+1}}{a_k}\leq\left(\frac{2}{e}\right)^m<1,$$

thus the sum of $\{a_k\}$ becomes

$$\sum_{k=0}^{\infty}a_k\leq\frac{a_0}{1-\left(\frac{2}{e}\right)^m}.$$

Combining the results discussed above, we obtain

$$\sum_{z\in B(x,r)^c}\mu(z)p(t,x,z)\le Kr^m\exp\left(-\frac{r}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{r}{2\left(\sqrt{D_\mu}\vee 1\right)et}\right)-\Lambda t\right) \tag{4.5}$$

for $r\ge r_0$, $r>2\left(\sqrt{D_\mu}\vee 1\right)et$ and $\frac{r}{2(\sqrt{D_\mu}\vee 1)}\log\left(\frac{2r}{(\sqrt{D_\mu}\vee 1)et}\right)\ge m$, where $K=\frac{2^m c_0}{\mu_0(1-(\frac{2}{e})^m)}$.

Set

$$r=r(t)=Ct\log t,$$

where C is a positive constant satisfying $C>2e\left(\sqrt{D_\mu}\vee 1\right)$. It is not difficult to find that there exists a real number T_1 such that for $t\ge T_1$, $r(t)$ satisfies the following conditions:

$$\text{(i)}\ \ r(t)\ge r_0;\ \ \text{(ii)}\ \ r(t)>2\left(\sqrt{D_\mu}\vee 1\right)et;\ \ \text{(iii)}\ \ \frac{r(t)}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{2r(t)}{\left(\sqrt{D_\mu}\vee 1\right)et}\right)\ge m.$$

In fact, by the monotonicity of $r(t)$, the conditions (i) and (ii) listed above are easy to be achieved, here we assume that t_1 is the minimum of t which satisfies the conditions (i) and (ii). As for the condition (iii), we observe that the left-hand side of (iii) tends to $+\infty$ as $t\to+\infty$, so we can choose a real number t_2 such that the left-hand side of (iii) is equal or larger than m for all $t\ge t_2$. Hence, there exists a real number $T_1\ge\max\{t_1,t_2\}$ such that the conditions (i)–(iii) are satisfied by $r(t)$ in $t\in[T_1,\infty)$.

Substituting $r(t)=Ct\log t$ into (4.5), we obtain

$$\begin{aligned}\sum_{z\in B(x,r)^c}\mu(z)p(t,x,z)&\le K\left(Ct\log t\right)^m\exp\left(-\frac{Ct\log t}{2\left(\sqrt{D_\mu}\vee 1\right)}\log\left(\frac{Ct\log t}{2\left(\sqrt{D_\mu}\vee 1\right)et}\right)-\Lambda t\right)\\&=KC^me^{-\Lambda t}(\log t)^mt^m\left(\frac{C\log t}{2e\left(\sqrt{D_\mu}\vee 1\right)}\right)^{-\frac{Ct\log t}{2(\sqrt{D_\mu}\vee 1)}}\\&=KC^me^{-\Lambda t}t^m(\log t)^{m-\frac{Ct\log t}{2(\sqrt{D_\mu}\vee 1)}}\left(\frac{C}{2e\left(\sqrt{D_\mu}\vee 1\right)}\right)^{-\frac{Ct\log t}{2(\sqrt{D_\mu}\vee 1)}}\end{aligned} \tag{4.6}$$

for $t\ge T_1$.

Under the assumption $C>2e\left(\sqrt{D_\mu}\vee 1\right)$, we conclude that

$$\left(\frac{C}{2e\left(\sqrt{D_\mu}\vee 1\right)}\right)^{-\frac{Ct\log t}{2(\sqrt{D_\mu}\vee 1)}}\to 0\quad\text{as}\quad t\to+\infty.$$

Moreover, it is easy to observe that

$$\begin{aligned}\lim_{t\to+\infty}t^m(\log t)^{m-\log t}&=\lim_{u\to+\infty}e^{mu}u^{m-u}\qquad(u=\log t)\\&=\lim_{u\to+\infty}e^{mu+(m-u)\log u}\\&=\lim_{u\to+\infty}e^{-u(\log u-\frac{m\log u}{u}-m)}\\&=0,\end{aligned}$$

which implies that

$$t^m(\log t)^{m-\frac{Ct\log t}{2(\sqrt{D_\mu}\vee 1)}} \to 0 \quad \text{as} \quad t\to+\infty.$$

We thus conclude that the right-hand side of (4.6) tends to 0 as $t\to+\infty$. So, there exists a real number T ($T\geq T_1$) such that the right-hand side of (4.6) is less than or equal to $\frac{1}{2}$ for all $t\geq T$. This yields the required inequality (4.3).

The proof of Theorem 1.1 is complete. □

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