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HARNACK AND MEAN VALUE INEQUALITIES ON GRAPHS*

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Abstract We prove a Harnack inequality for positive harmonic functions on graphs which is similar to a classical result of Yau on Riemannian manifolds. Also, we prove a mean value inequality of nonnegative subharmonic functions on graphs.

Key words harmonic function; subharmonic function; Harnack inequality; mean value inequality; graph

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1 Introduction and Main Results

One of the fundamental topics in geometric analysis and partial differential equations is the study of harmonic (subharmonic) functions. In 1975, Yau [1] proved a gradient estimate for positive harmonic functions, which leads to a Harnack type inequality and a Liouville theorem on manifolds with Ricci curvature bounded from below. There is an extensive literature on gradient estimates for various partial differential equations on manifolds, see for examples [2–8]. For the graph case, we refer the reader to [9–11]. Harmonic (subharmonic) functions can also be studied by mean value inequalities. In [12], Li-Schoen obtained an L^p mean value inequality for subharmonic functions on manifold with nonnegative Ricci curvature, which leads to a Liouville theorem for subharmonic L^p functions with $p > 1$.

In [13, 14], Holopainen-Soardi derived several Liouville theorems for p-harmonic functions on graphs. While in [15], Rigoli-Salvatori-Vignati proved that there is no nonnegative subharmonic function belonging to ℓ^p for any $p > 1$. Lipschitz properties of harmonic function on graphs were discovered by Lin-Xi [16]. Hua-Jost [17] proved a graph version of Caccioppoli-type inequality for nonnegative subharmonic functions, and used it to get a Liouville theorem for harmonic or nonnegative subharmonic functions of class ℓ^p for $p > 1$.

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We now fix some notations. Let $G=(X,E)$ be a connected infinite graph, where X denotes the vertex set and E denotes the edge set. We call vertices x and y neighbors, or $x\sim y$, if they are endpoints of the same edge. The degree of x, which is denoted by d_x, is the number of all its neighbors. Throughout this paper we assume that G has bounded degree, namely,

$$d_x\le d<\infty \quad \text{for all} \quad x\in X. \tag{1.1}$$

The Laplace operator Δ on graph reads

$$\Delta u(x)=\frac{1}{d_x}\sum_{y\sim x}[u(y)-u(x)] \tag{1.2}$$

for all functions $u: X\to \mathbb{R}$. A function u is called harmonic (subharmonic) if

$$\Delta u=0\,(\Delta u\ge 0) \quad \text{on} \quad G. \tag{1.3}$$

For $x,y\in X$, the distance $\rho(x,y)$ denotes the minimum number of edges connecting x and y. Let $B(x,r)=\{y\in X:\rho(x,y)\le r\}$ be the ball centered at x with radius r. Given any subset $U\subset X$, the boundary ∂U of U is the set of all vertices $x\in X\backslash U$ having at least one neighbor in U. The volume of U is defined by

$$\mathrm{vol}(U)=N(U),$$

where $N(U)$ denotes the number of vertices on U.

Our first result is the following Harnack inequality.

Theorem 1.1 Let $G=(X,E)$ be a connected infinite graph. Suppose that u is a positive harmonic function on G, x_0 is a point in X. Then there holds for any $R>0$,

$$\sup_{B(x_0,R)} u(x)\le \mathrm{e}^{2R\sqrt{d(d-1)}}\inf_{B(x_0,R)} u(x),$$

where d is a constant given as in (1.1).

Different from [14], our method of proving Theorem 1.1 is to derive a gradient estimate, then use it to obtain a Harnack inequality. Moreover the constant C in [14] did not give any information on how it depends on the radius R, but here we explicitly use R to represent C.

Our second result concerns the mean value inequality, namely,

Theorem 1.2 Let $G=(X,E)$ be a connected infinite graph, x_0 be a point in X and $R>0$. Suppose that v is a nonnegative subharmonic function defined on $B(x_0,R)$. Then for any $\tau\in(0,1/2)$, there is a constant c depending only on R and τ such that

$$\sup_{B(x_0,(1-\tau)R)} v^2\le c\frac{1}{\mathrm{vol}(B(x_0,R/2))}\sum_{B(x_0,R)} v^2.$$

Precisely $c=\mathrm{e}^{2R(1-\tau)\sqrt{d(d-1)}}\left[\frac{64d(d^{R+1}-1)}{\tau^2R(d-1)}+2\right]$, where d is a constant given as in (1.1).

The ℓ^p version of Theorem 1.2 is of its own interest. We stated it as the following.

Theorem 1.3 Let $G=(X,E)$ be a connected infinite graph, x_0 be a point in X, $R>0$ and $0<p\le 2$. Suppose that v is a nonnegative subharmonic function defined on $B(x_0,R)$. Then for any $\tau\in(0,1/2)$, there holds

$$\sup_{B(x_0,(1-\tau)R)} v^p\le \tau^{-2}4^{2/p}C\mathrm{e}^{2R\sqrt{d(d-1)}(1-\tau p/(p+2))}\frac{1}{\mathrm{vol}(B(x_0,R/2))}\sum_{B(x_0,R)} v^p,$$

precisely $C=\frac{64d(d^{R+1}-1)}{R(d-1)}+\frac{1}{2}$, where d is a constant given as in (1.1).

The proof of Theorem 1.1 is based on a gradient estimate. While Theorems 1.2 and 1.3 will be proved by following the lines of [12]. The remaining part of this paper is organized as follows: in Section 2, we give several preliminary lemmas; in Section 3, we prove Theorems 1.1–1.3.

2 Preliminary Lemmas

Let $G = (X, E)$ be a connected infinite graph as in the introduction. Given any function $f : X \to \mathbb{R}$, we say that $f \in \ell^p(X)$ if

$$\sum_{x\in X} | f(x) |^p < +\infty.$$

We define the square of the gradient of f by

$$| \nabla f(x) |^2 = \sum_{y\sim x} | f(y) - f(x) |^2$$

and its Dirichlet integral on $S \subset X$ by

$$I_2(f, S) = \sum_{x\in S} | \nabla f(x) |^2 .$$

Lemma 2.1 Let $S \subset X$ be a finite set. Then u is harmonic on S if and only if it is a minimizer of $I_2(f, S)$ among all functions with the same value on ∂S.

Proof It is the case $p = 2$ of Theorem 3.5 in [14]. □

Lemma 2.2 Let u be harmonic and v be subharmonic in a finite set $S \subset X$ such that $u \geq v$ in ∂S. Then $u \geq v$ in S.

Proof It is a special case of Theorem 3.14 in [14]. □

One way to get the following locally Poincaré inequality is using gradient estimate as in [12]. Here we will give a direct proof.

Lemma 2.3 Let $x_0 \in X$ and $R > 0$. For every function f on $B(x_0, R)$ which vanishes on $\partial B(x_0, R)$, we have the locally poincaré inequality

$$\sum_{B(x_0,R)} | f(x) |^2 \leq c_1 \sum_{B(x_0,R)} | \nabla f(x) |^2,$$

where $c_1 = 2R(d^{R+1} - 1)/(d - 1)$.

Proof Let x_1 denote a vertex with

$$| f(x_1) | = \max_{B(x_0,R)} | f(x) | .$$

Choosing $x_2 \in \partial B(x_0, R)$, then $f(x_2) = 0$. Let P denote a shortest path in $B(x_0, R)$ joining x_1 and x_2. Then by Cauchy-Schwartz inequality we have

$$\begin{aligned}\sum_{B(x_0,R)} | \nabla f(x) |^2 &= \sum_{B(x_0,R)} \sum_{x\sim y} [f(x) - f(y)]^2 \\ &\geq \sum_{(x,y)\in P} [f(x) - f(y)]^2 \\ &\geq \frac{1}{2R} \left[\sum_{(x,y)\in P} (f(x) - f(y)) \right]^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2R} f^2(x_1) \\
&\geq \frac{1}{2R \cdot \operatorname{vol}(B(x_0, R))} \sum_{B(x_0,R)} \mid f(x) \mid^2,
\end{aligned}
$$

where

$$\operatorname{vol}(B(x_0, R)) \leq \sum_{i=0}^{R+1} d^i \leq (d^{R+1} - 1)/(d - 1).$$

The lemma is proved. □

We will also need the following discrete Cacciopoli inequality from [16].

Lemma 2.4 Let $x_0 \in X$ and $R > 0$, v is a nonnegative subharmonic function on G. Then for any $\tau \in (0, 1/2)$, we have

$$\sum_{B(x_0,(1-\tau)R)} \mid \nabla v(x) \mid^2 \leq \frac{4d}{(\tau R)^2} \sum_{B(x_0,R)} v^2(x).$$

3 Proof of Theorems 1.1–1.3

In this section, we will prove a Harnack inequality for nonnegative harmonic functions (Theorem 1.1) and two mean value inequalities for nonnegative subharmonic functions (Theorems 1.2 and 1.3).

3.1 The Proof of Theorem 1.1

Proof Suppose that $\Delta u(x) = -\lambda u(x)$ for all $x \in X$. It is easy to show that λ is a real number, then

$$\frac{1}{d_x} \sum_{y \sim x} [u(y) - u(x)] = -\lambda u(x).$$

It follows that

$$\frac{1}{d_x} \sum_{y \sim x} u(y) = (1 - \lambda) u(x).$$

Since $u(x) > 0$, we calculate

$$
\begin{aligned}
\frac{1}{d_x} \cdot \frac{|\nabla u(x)|^2}{u^2(x)} &= \frac{\frac{1}{d_x} \sum\limits_{y \sim x} \big(u(y) - u(x)\big)^2}{u^2(x)} \\
&= \frac{u^2(x) - 2u(x) \cdot \frac{1}{d_x} \sum\limits_{y \sim x} u(y) + \frac{1}{d_x} \sum\limits_{y \sim x} u^2(y)}{u^2(x)} \\
&\leq 2\lambda - 1 + \frac{\frac{1}{d_x} \cdot \Big(\sum\limits_{y \sim x} u(y) \Big)^2}{u^2(x)} \\
&= 2\lambda - 1 + d_x (1 - \lambda)^2 \\
&\leq 2\lambda - 1 + d - 2d\lambda + d\lambda^2 \\
&= d\lambda^2 - 2(d - 1)\lambda + d - 1.
\end{aligned}
$$

Hence

$$\frac{|\nabla u(x)|^2}{u^2(x)} \le d^2\lambda^2 - 2d(d-1)\lambda + d(d-1)$$
$$= (d\lambda - (d-1))^2 + d - 1.$$

Let $u(x_n) = \sup_{B(x_0,R)} u(x)$, $u(x_1) = \inf_{B(x_0,R)} u(x)$, where $x_1 \sim x_2 \sim \cdots \sim x_n$. Then we have

$$\frac{|u(x_{i+1}) - u(x_i)|}{u(x_i)} \le \frac{\left\{\sum\limits_{y\sim x} (u(y) - u(x_i))^2\right\}^{\frac{1}{2}}}{u(x_i)}$$
$$= \left\{\frac{\sum\limits_{y\sim x} (u(y) - u(x_i))^2}{u^2(x_i)}\right\}^{\frac{1}{2}}$$
$$\le \sqrt{(d\lambda - (d-1))^2 + d - 1}.$$

Therefore

$$\sum_{i=1}^{n-1} \frac{|u(x_{i+1}) - u(x_i)|}{u(x_i)} \le 2R\sqrt{(d\lambda - (d-1))^2 + d - 1}.$$

Then we have

$$\log\frac{u(x_n)}{u(x_1)} = \sum_{i=1}^{n-1} \log\frac{u(x_{i+1})}{u(x_i)}$$
$$= \sum_{i=1}^{n-1} \log\left(1 + \frac{u(x_{i+1}) - u(x_i)}{u(x_i)}\right)$$
$$\le \sum_{i=1}^{n-1} \log\left(1 + \frac{|u(x_{i+1}) - u(x_i)|}{u(x_i)}\right)$$
$$\le \sum_{i=1}^{n-1} \frac{|u(x_{i+1}) - u(x_i)|}{u(x_i)}$$
$$\le 2R\sqrt{(d\lambda - (d-1))^2 + d - 1}.$$

This leads to that

$$\frac{u(x_n)}{u(x_1)} \le \mathrm{e}^{2R\sqrt{(d\lambda-(d-1))^2+d-1}},$$

and that

$$\sup_{B(x_0,R)} u(x) \le \mathrm{e}^{2R\sqrt{(d\lambda-(d-1))^2+d-1}} \inf_{B(x_0,R)} u(x).$$

If $u(x)$ is harmonic, then $\Delta u(x) = 0$, $\lambda = 0$, and we conclude

$$\sup_{B(x_0,R)} u(x) \le \mathrm{e}^{2R\sqrt{d(d-1)}} \inf_{B(x_0,R)} u(x).$$

This ends the proof of the theorem. □

3.2 The Proof of Theorem 1.2

Proof Let h be the harmonic function on $B(x_0, (1-\tau)R)$ which agrees with v on the boundary. Then h is positive in $B(x_0, (1-\tau)R)$ (unless v is identically zero, in which case the theorem is trivial). Since v is subharmonic, we have $v \leq h$ in $B(x_0,(1-\tau)R)$ by Lemma 2.2. It follows from Theorem 1.1 that

$$\begin{aligned}\sup_{B(x_0,(1-\tau)R)} v^2 &\leq \sup_{B(x_0,(1-\tau)R)} h^2 \\ &\leq \mathrm{e}^{2(1-\tau)R\cdot\sqrt{d(d-1)}} \inf_{B(x_0,(1-\tau)R)} h^2 \\ &\leq \mathrm{e}^{2(1-\tau)R\cdot\sqrt{d(d-1)}} \frac{1}{\mathrm{vol}(B(x_0,(1-\tau)R))} \sum_{B(x_0,(1-\tau)R)} h^2. \end{aligned} \tag{3.1}$$

In the following, we will estimate the average value of h^2 by that of v^2. First we note that

$$\sum_{B(x_0,(1-\tau)R)} h^2 \leq 2 \sum_{B(x_0,(1-\tau)R)} (h-v)^2 + 2 \sum_{B(x_0,R)} v^2. \tag{3.2}$$

Since $h-v$ vanishes on $\partial B(x_0,(1-\tau)R)$, we obtain by using Lemma 2.3,

$$\begin{aligned}\sum_{B(x_0,(1-\tau)R)} (h-v)^2 &\leq c_1 \sum_{B(x_0,(1-\tau)R)} |\nabla(h-v)|^2 \\ &\leq 2c_1 \sum_{B(x_0,(1-\tau)R)} (|\nabla h|^2 + |\nabla v|^2),\end{aligned}$$

where $c_1 = 2R(d^{R+1}-1)/(d-1)$ By Lemma 2.1 and the definition of the Dirichlet integral of h, one can easily see that

$$\sum_{B(x_0,(1-\tau)R)} |\nabla h|^2 \leq \sum_{B(x_0,(1-\tau)R)} |\nabla v|^2 .$$

So we get

$$\sum_{B(x_0,(1-\tau)R)} (h-v)^2 \leq 4c_1 \sum_{B(x_0,(1-\tau)R)} |\nabla v|^2 . \tag{3.3}$$

Now we use Lemma 2.4 to estimate the Dirichlet integral of v in terms of the ℓ^2 norm of v. By a straightforward calculation,

$$\sum_{B(x_0,(1-\tau)R)} |\nabla v|^2 \leq 4d(\tau R)^{-2} \sum_{B(x_0,R)} v^2. \tag{3.4}$$

Combining estimates (3.1)–(3.4), we get

$$\begin{aligned}\sup_{B(x_0,(1-\tau)R)} v^2 &\leq \mathrm{e}^{2(1-\tau)R\sqrt{d(d-1)}} \left(8c_1 \cdot \frac{4d}{(\tau R)^2} + 2\right) \frac{1}{\mathrm{vol}\big(B(x_0,(1-\tau)R)\big)} \sum_{B(x_0,R)} v^2 \\ &\leq \mathrm{e}^{2(1-\tau)R\sqrt{d(d-1)}} \left[\frac{64d(d^{R+1}-1)}{\tau^2 R(d-1)} + 2\right] \frac{1}{\mathrm{vol}(B(x_0,R/2))} \sum_{B(x_0,R)} v^2,\end{aligned}$$

here we have used the fact that $\tau \leq 1/2$. The proof of Theorem 1.2 is completed. □

3.3 The Proof of Theorem 1.3

Proof The proof is based on the Moser iteration. For any $\delta \in (0,1/2]$, $\theta \in [1/2, 1-\delta]$, it follows from Theorem 1.2 that

$$\sup_{B(x_0,\theta R)} v^2 \leq \delta^{-2} \mathrm{e}^{c(1-\delta)R} \left[\frac{64d(d^{R+1}-1)}{R(d-1)} + 2\delta^2\right] \frac{1}{\mathrm{vol}(B(x_0,R/2))} \sum_{B(x_0,(\theta+\delta)R)} v^2$$

$$\leq \delta^{-2}\mathrm{e}^{c(1-\delta)R}\left[\frac{64d(d^{R+1}-1)}{R(d-1)}+\frac{1}{2}\right]\frac{1}{\mathrm{vol}(B(x_0,R/2))}\sum_{B(x_0,(\theta+\delta)R)}v^2,$$

where $c=2\sqrt{d(d-1)}$. Since $\theta+\delta\leq 1$, we have

$$\begin{aligned}\sum_{B(x_0,(\theta+\delta)R)}v^2 &\leq \Big(\sup_{B(x_0,(\theta+\delta)R)}v^2\Big)^{1-p/2}\sum_{B(x_0,(\theta+\delta)R)}v^p\\ &\leq \Big(\sup_{B(x_0,(\theta+\delta)R)}v^2\Big)^{1-p/2}\sum_{B(x_0,R)}v^p.\end{aligned}$$

We set

$$M(\theta)=\sup_{B(x_0,\theta R)}v^2, K=\frac{1}{\mathrm{vol}(B(x_0,R/2))}\sum_{B(x_0,R)}v^p.$$

Then we have

$$M(\theta)\leq \delta^{-2}CK\mathrm{e}^{c(1-\delta)R}\big(M(\theta+\delta)\big)^{\lambda},$$

where $\lambda=1-p/2$ and $C=\frac{64d(d^{R+1}-1)}{R(d-1)}+\frac{1}{2}$.

Choosing $\theta_0=1-\tau$ and $\theta_i=\theta_{i-1}+2^{-i}\tau$ for $i=1,2,3\cdots$, then we obtain

$$M(\theta_{i-1})\leq CK4^i\tau^{-2}\mathrm{e}^{c(1-2^{-i}\tau)R}\big(M(\theta_i)\big)^{\lambda}.$$

For any $j\geq 1$, by iteration we have

$$M(\theta_0)\leq C^{\sum_{i=1}^{j}\lambda^{i-1}}K^{\sum_{i=1}^{j}\lambda^{i-1}}4^{\sum_{i=1}^{j}i\lambda^{i-1}}\tau^{-2\sum_{i=1}^{j}\lambda^{i-1}}\mathrm{e}^{[\sum_{i=1}^{j}(1-\tau/2^i)\lambda^{i-1}]cR}\big(M(\theta_j)\big)^{\lambda^j}.$$

Passing to the limit $j\to\infty$, we get

$$\begin{aligned}M(\theta_0)&\leq C^{2/p}K^{2/p}4^{4/p^2}(\tau^{-2})^{2/p}\mathrm{e}^{cR(2/p-2\tau/(p+2))}\\ &\leq 4^{4/p^2}(\tau^{-2})^{2/p}C^{2/p}\mathrm{e}^{cR(2/p-2\tau/(p+2))}\left[\frac{1}{\mathrm{vol}(B(x_0,R/2))}\sum_{B(x_0,R)}v^p\right]^{2/p}.\end{aligned}$$

This implies

$$\sup_{B(x_0,(1-\tau)R)}v^p\leq 4^{2/p}\tau^{-2}C\mathrm{e}^{cR(1-\tau p/(p+2))}\frac{1}{\mathrm{vol}(B(x_0,R/2))}\sum_{B(x_0,R)}v^p.$$

This finishes the proof of Theorem 1.3. □

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