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Existence of Solutions to Mean Field Equations on Graphs

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Abstract: In this paper, we prove two existence results of solutions to mean field equations

$$\Delta u + e^u = \rho\delta_0$$

and

$$\Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}$$

on an arbitrary connected finite graph, where $\rho > 0$ and $\lambda > 0$ are constants, M is a positive integer, and $p_1, \ldots, p_M$ are arbitrarily chosen distinct vertices on the graph.

1. Introduction

The mean field equation

$$\Delta u + e^u = \rho\delta_0 \tag{1.1}$$

has its origin in the prescribed curvature problem in geometry, where constant $\rho > 0$ and δ_0 is the Dirac delta mass at the zero point. Closely related is the Kazdan–Warner equation [9]

$$\Delta u + he^u = c. \tag{1.2}$$

The name of the Eq. (1.1) comes from statistical physics as the mean field limits of the Euler flow [1]. It has also been shown to be related to the Chern–Simons–Higgs model. The existence of solutions to Eq. (1.1) has been studied in [3,4,10,11] on Euclidean spaces and on the two dimensional flat tori. For example, on the two dimensional flat tori, when $\rho \neq 8m\pi$ for any $m \in \mathbf{Z}$, Eq. (1.1) always has solutions, see [3,4]. When $\rho = 8\pi$, it was shown in [10] that Eq. (1.1) has solutions if and only if the Green's function on the two dimensional flat tori has critical points other than the three half period points.

In [5], Grigorigan, Lin and Yang have obtained a few sufficient conditions when Eq. (1.2) has a solution on a finite graph. There are several further results regarding the solutions of (1.2) on graphs in [6–8].

In this paper, we study Eq. (1.1) and also the following mean field equation on graphs:

$$\Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}, \tag{1.3}$$

where $\lambda > 0$, M is any fixed positive integer, and $p_1, \ldots, p_M$ are arbitrarily chosen distinct vertices on the graph and δ_{p_j} is the Dirac delta mass at the vertex p_j.

Caffarelli and Yang in [2] proved an existence result of solutions to Eq. (1.3) on doubly periodic regions in $\mathbf{R}^2$ (the 2-tori), depending on the value of the parameter λ.

Let $G = (V, E)$ be a connected finite graph, where V is the set of vertices and E is the set of edges. For a vertex $x \in V$, we denote d_x as the degree of x, i.e. the number of vertices in V connected to x. For any function $f : V \to \mathbb{R}$, we use the notation $\int_V f(x) d\mu(x) = \sum_{x \in V} f(x)\mu(x)$ to denote the integral of function f on vertex set V. We use the notation $V(G) = \sum_{x \in V} \mu(x)$ to denote the volume of graph G.

In this paper, we show that Eq. (1.1) always has a solution on any connected finite graph (Theorem 2.1), in contrast to the continuous case. We shall also prove an existence result for Eq. (1.3) on a connected finite graph (Theorem 2.2), depending on the value of the parameter λ, which is in line with the result of Caffarelli and Yang on the 2-tori.

We obtain these results by a mostly straightforward adaption of existing treatments from the continuous case [1,5,9]. Once we have the setup, some analysis tend to simplify on finite graphs since there is only a finite number of degrees of freedom. Theorem 2.1 on the other hand shows that the existence of solutions for (1.1) on the discrete two dimensional tori graph given as the quotient of the two dimensional lattice infinite graph by a rank 2 sublattice of finite index, differs from that on the continuous limit– the two dimensional flat tori, when the parameter ρ takes on certain special values such as 8π.

Remark 1. As a side remark, it appears interesting to study the Green's function on the 2-tori by studying the corresponding discrete Green's function on the 2-tori graph stated above. For example, when the torus parameter $\tau = \frac{1}{2} + i$, there exist two additional critical points of the Green's function besides the half periods by [10]. A computer study aided by this discrete Green's function indicates that the slope of the line through these two additional critical points of the Green's function is equal to $\frac{25}{64}$.

2. Settings and Main Results

Let $G = (V, E)$ be a connected finite graph. Denote $N = |V|$. We assume positive symmetric weights $w_{xy} = \omega_{yx}$ on edges $xy \in E$. Let $\mu : V \to \mathbb{R}^+$ be a finite measure. For any function $u : V \to \mathbb{R}$, the Laplace operator acting on u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)),$$

where $y \sim x$ means $xy \in E$. The gradient form of u is by definition

$$\Gamma(u) = \frac{1}{2} \int_V |\nabla u|^2 := \sum_{x \in V} \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2.$$

As in [5], we define a Sobolev Space and a norm by

$$W^{1,2}(V) = \{u : V \to \mathbb{R} : \int_V (|\nabla u|^2 + u^2) d\mu < +\infty\}$$

and

$$||u||_{W^{1,2}(V)} = (\int_V (|\nabla u|^2 + u^2) d\mu)^{1/2}$$

respectively. Since V is a finite graph, $W^{1,2}(V)$ is $V^{\mathbb{R}}$, the finite dimensional vector space of all real functions on V. We have the following Sobolev embedding (Lemma 5 in [5]):

Lemma 2.1. *Let $G = (V, E)$ be a finite graph. The Sobolev Space $W^{1,2}(V)$ is precompact. Namely, if $\{u_j\}$ is bounded in $W^{1,2}(V)$, then there exits some $u \in W^{1,2}(V)$ such that there is a subsequence u_{n_i}, $u_{n_i} \to u$ in $W^{1,2}(V)$.*

Remark 2. For finite graphs, Lemma 2.1 can be avoided for the purpose of the present paper. But we include it for potential generalizations to infinite graphs.

By using the variational principle (see the similar approach in [9] and [5]), we prove the following

Theorem 2.1. *Equation (1.1) has a solution on G.*

Using an iteration method, we next prove the following

Theorem 2.2. *There is a critical value λ_c depending on G satisfying*

$$\lambda_c \geq \frac{16\pi M}{|V|},$$

such that when $\lambda > \lambda_c$, the Eq. (1.3) has a solution on G, and when $\lambda < \lambda_c$, the Eq. (1.3) has no solution.

3. The Proof of Theorem 2.1

In this section, we fix the vertex $x_0 \in V$ and denote the δ_0 as the Dirac delta mass at the vertex x_0.

Proof. For $u \in W^{1,2}(V)$, we consider the functional

$$J(u) = \frac{1}{2} \int_V |\nabla u|^2 + \int_V \rho \cdot \delta_0 \cdot u.$$

Let the set

$$B = \{u \in W^{1,2}(V) : \int_V e^u = \int_V \rho \cdot \delta_0 = \rho\}.$$

We can choose that $u(x) \equiv \log \frac{\rho}{V(G)}$, then $\int_V e^u = \rho$, therefore $B \neq \emptyset$.

For any $u \in B$, $\int_V e^u = \rho$, choose $x_D \in V$ such that

$$e^{u(x_D)} = \min_{x \in V} \{e^{u(x)}\},$$

then

$$Ne^{u(x_D)} \leq \rho,$$
$$u(x_D) \leq \log \frac{\rho}{N}.$$

Choose a shortest path on G from x_0 to x_D (therefore non-backtracking): $x_0 \sim x_1 \sim \cdots x_{D-1} \sim x_D$,

$$\begin{aligned}
\frac{1}{2}\int_V |\bigtriangledown u|^2 &= \sum_{x\in V}\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))^2 \\
&\geq D_{eg}[(u(x_1)-u(x_0))^2+(u(x_2)-u(x_1))^2+\cdots+(u(x_D)-u(x_{D-1}))^2] \\
&\geq D_{eg}\frac{(u(x_0)-u(x_D))^2}{D} \\
&\geq \frac{D_{eg}}{D}\cdot(u(x_0)-\log\frac{\rho}{N})^2 \\
&\geq \frac{D_{eg}}{2D}\cdot u^2(x_0)-\frac{2D_{eg}}{D}\cdot\log^2\frac{\rho}{N}.
\end{aligned}$$

where $D_{eg} = \min_{x\in V, y\sim x}\frac{\omega_{xy}}{2\mu(x)}$, and we use Cauchy-Schwartz inequality in the proof of second inequality.

So there exists $c = \max\{\log^2\frac{\rho}{N}, \frac{4D\rho}{D_{eg}}+4\} > 0$, such that when $|u(x_0)| \geq c$,

$$\frac{1}{4}\int_V |\bigtriangledown u|^2 \geq \rho\cdot|u(x_0)|.$$

Therefore we have in this case

$$J(u) \geq \frac{1}{4}\int_V |\bigtriangledown u|^2. \tag{3.1}$$

When $|u(x_0)| < c$,

$$J(u) > \frac{1}{2}\int_V |\bigtriangledown u|^2 - \rho c. \tag{3.2}$$

Therefore $J(u)$ has a lower bound on B. So we can choose

$$u_k(x) \in B,\ J(u_k(x)) \to b \quad (k\to\infty),$$

where $b = \inf_{u\in B} J(u)$.

From (3.1) and (3.2), for all k,

$$\int_V |\bigtriangledown u_k|^2 \leq c_1$$

for some constant c_1, since $|J(u_k)| \leq c_2$ for some constant c_2. As

$$J(u_k) = \frac{1}{2}\int_V |\bigtriangledown u_k|^2 + \rho\cdot u_k(x_0),$$

there exists a constant c', such that $|u_k(x_0)| \le c'$ for all k. For any $x \in V$, choose a shortest path on G from x_0 to x:

$$x_0 \sim x_1 \sim \ldots x_{L-1} \sim x,$$

where L is at most the diameter of G which is bounded by N,

$$\begin{aligned}|u_k(x)| &\le |u_k(x) - u_k(x_{L-1})| + |u_k(x_{L-1}) - u_k(x_{L-2})| + \cdots \\ &\quad + |u_k(x_1) - u_k(x_0)| + |u_k(x_0)| \\ &\le L \cdot [|u_k(x) - u_k(x_{L-1})|^2 + \cdots + |u_k(x_1) - u_k(x_0)|^2]^{1/2} + c' \\ &\le \frac{L}{D_{eg}} \int_V |\bigtriangledown u_k|^2 + c'.\end{aligned}$$

As $L \le N$, the L^∞ norm of $u_k(x)$ is uniformly bounded, and therefore $u_k(x)$ are uniformly bounded in $W^{1,2}(V)$. From the Sobolev embedding (Lemma 2.1), there exits a subsequence $u_{k1}(x) \to u_\infty(x) \in W^{1,2}(V)$ in $W^{1,2}(V)$, and

$$\int_V e^{u_\infty} = \lim_{k_1 \to \infty} \int_V e^{u_{k1}} = \rho.$$

Finally we prove that u_∞ is the solution of Eq. (1.1). This is based on the method of Lagrange maltiplies. Let

$$L(t, \lambda) = \frac{1}{2} \int_V |\bigtriangledown (u_\infty + t\varphi)|^2 + \int_V \rho \cdot \delta_0 (u_\infty + t\varphi) + \lambda(-\int_V e^{u_\infty + t\varphi} + \rho),$$

where $\varphi \in W^{1,2}(V)$. So we have

$$\frac{\partial L}{\partial \lambda} |_{t=0} = -\int_V e^{u_\infty} + \rho = 0,$$

since $u_\infty \in B$. And

$$0 = \frac{\partial L}{\partial t}|_{t=0} = -\int \Delta u_\infty \cdot \varphi + \int \rho \cdot \delta_0 \cdot \varphi - \lambda \int e^{u_\infty} \cdot \varphi = 0.$$

Therefore by the variational principle,

$$-\Delta u_\infty + \rho \cdot \delta_0 - \lambda \cdot e^{u_\infty} = 0.$$

Since $\int_V \Delta u_\infty = 0$, we have

$$\lambda \int_V e^{u_\infty} = \int_V \rho \cdot \delta_0 = \rho.$$

So $\lambda = 1$, and

$$\Delta u_\infty + e^{u_\infty} = \rho \cdot \delta_0.$$

This finishes the proof of Theorem 2.1. □

4. The Proof of Theorem 2.2

We use the method of upper and lower solutions to prove Theorem 2.2, adapting methods from [1,9] and [5] to the graph setting.

Lemma 4.1 (Maximum principle). *Let $G = (V, E)$, where V is a finite set, and $K \geq 0$ is a constant. Suppose a real function $u(x) : V \to \mathbb{R}$ satisfies*

$$(\Delta - K)u(x) \geq 0 \textit{ for all } x \in V,$$

then $u(x) \leq 0$ for all $x \in V$.

Proof. Let $u(x_0) = \max_{x \in V} \{u(x)\}$, we only need to show that $u(x_0) \leq 0$. Suppose this is not the case. Since

$$(\Delta - K)u(x_0) \geq 0,$$

we have

$$\sum_{y \sim x_0} u(y) \geq (d_{x_0} + K)u(x_0) \geq d_{x_0} u(x_0),$$

where we have used the assumption that $u(x_0) > 0$, and that $K \geq 0$ in the last inequality. This implies that for any $y \sim x_0, u(y) \geq u(x_0)$. Since G is a connected graph, by induction, for any $xy \in E, u(y) = u(x_0)$. From

$$K \int_V u(x) \leq \int \Delta u(x) = 0$$

and $K \geq 0$ we get that $u(x_0) \leq 0$. This is a contradiction. □

Let u_0 be a solution of the Poisson equation

$$\Delta u_0 = -\frac{4\pi M}{|V|} + 4\pi \sum_{j=1}^{M} \delta_{p_j}. \tag{4.1}$$

The solution of (4.1) always exists, as the integral of the right side is equal to 0. Inserting $u = u_0 + v$ into Eq. (1.3), we get

$$\Delta v = \lambda e^{u_0+v}(e^{u_0+v} - 1) + \frac{4\pi M}{|V|}. \tag{4.2}$$

Sum the two sides of the about equation, we get

$$\lambda(e^{u_0+v} - \frac{1}{2})^2 = \frac{\lambda}{4} - \frac{4\pi M}{|V|},$$

which implies that

$$\lambda \geq \frac{16\pi M}{|V|}. \tag{4.3}$$

We call a function v_+ an upper solution of (4.2) if for any $x \in V$, it satisfies

$$\Delta v_+(x) \geq \lambda e^{u_0(x)+v_+(x)}(e^{u_0(x)+v_+(x)} - 1) + \frac{4\pi M}{|V|}. \tag{4.4}$$

Let $v_0 = -u_0$, we define a sequence $\{v_n\}$ by iterating for a constant $K \geq 2\lambda$,

$$(\Delta - K)v_n = \lambda e^{u_0+v_{n-1}}(e^{u_0+v_{n-1}} - 1) - Kv_{n-1} + \frac{4\pi M}{|V|}. \tag{4.5}$$

We next prove that $\{v_n\}$ is a monotone sequence and it converges to a solution of Eq. (4.2).

Lemma 4.2. *Let $\{v_n\}$ be a sequence defined by (4.5). Then*

$$v_0 \geq v_1 \geq v_2 \geq \cdots \geq v_n \cdots \geq v_+$$

for any upper solution v_+ of (4.2).

Proof. We prove the Lemma by induction. As $v_0 = -u_0$, for v_1 we have by (4.5),

$$(\Delta - K)v_1 = Ku_0 + \frac{4\pi M}{|V|}.$$

Together with (4.1), we obtain

$$(\Delta - K)(v_1 - v_0)(x) = 4\pi \sum_{j=1}^{M} \delta_{p_j}(x) \geq 0$$

for any $x \in V$, and

$$K \int_V (v_1 - v_0) = -4\pi M < 0.$$

Therefore $v_1 - v_0 \leq 0$ by Lemma 4.1. Suppose that $v_0 \geq v_1 \geq \cdots \geq v_k$ for $k \geq 1$. From (4.5) and $K \geq 2\lambda$, we get

$$\begin{aligned}
(\Delta - K)(v_{k+1} - v_k) &= \lambda e^{2u_0+2v_k} - \lambda e^{u_0+v_k} - Kv_k - \lambda e^{2u_0+2v_{k-1}} + \lambda e^{u_0+v_{k-1}} + Kv_k \\
&= \lambda e^{2u_0}(e^{2v_k} - e^{2v_{k-1}}) - \lambda e^{u_0}(e^{v_k} - e^{v_{k-1}}) - K(v_k - v_{k-1}) \\
&\geq \lambda e^{2u_0}(e^{2v_k} - e^{2v_{k-1}}) - K(v_k - v_{k-1}) \\
&= 2\lambda e^{2u_0+2v^*}(v_k - v_{k-1}) - K(v_k - v_{k-1}) \\
&\geq K(e^{2u_0+2v_0} - 1)(v_k - v_{k-1}) \\
&\geq 0.
\end{aligned}$$

Where $v_k \leq v^* \leq v_{k-1} \leq v_0$. Lemma 4.1 then implies that $v_{k+1} - v_k \leq 0$ on V.

Next we prove that $v_k \geq v_+$ for any k.

First consider the case $k = 0$. From (4.1) and (4.4) ,

$$\begin{aligned}
\Delta(v_+ - v_0) &\geq \lambda e^{u_0+v_+}(e^{u_0+v_+} - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j} \\
&\geq \lambda e^{u_0+v_+}(e^{u_0+v_+} - 1) \\
&= \lambda e^{v_+-v_0}(e^{v_+-v_0} - 1).
\end{aligned} \tag{4.6}$$

Let $v_+(x_0) - v_0(x_0) = \max_{x\in V}\{v_+(x_0) - v_0(x_0)\}$. We only need to prove that $v_+(x_0) - v_0(x_0) \le 0$. Suppose not, then from (4.6) we have

$$\Delta(v_+ - v_0)(x_0) > 0$$

which contradicts with the assumption that x_0 is a point where $v_+ - v_0$ attains maximum in V. Hence $v_+ - v_0 \le 0$ in V. Now suppose that $v_+ \le u_k$ for $k \ge 0$. From (4.4) and (4.5), we have

$$\begin{aligned}(\Delta - K)(v_+ - v_{k+1}) &= \lambda e^{2u_0}(e^{2v_+} - e^{2v_k}) - K(v_+ - v_k) - \lambda e^{u_0}(e^{v_+} - e^{v_k})\\ &\ge \lambda e^{2u_0}(e^{2v_+} - e^{2v_k}) - K(v_+ - v_k)\\ &= 2\lambda e^{2u_0+2v^*}(v_+ - v_k) - K(v_+ - v_k)\\ &\ge K(e^{2u_0+2v_0} - 1)(v_+ - v_k)\\ &= 0,\end{aligned}$$

where $v_+ \le v^* \le v_k \le v_0$. So Lemma 4.1 implies that $v_{k+1} \ge v_+$.

This finishes the proof of Lemma 4.2. □

Lemma 4.3. *The Eq.* (1.3) *has a solution on G, when* λ *is sufficiently big.*

Proof. We only need to prove that Eq. (4.2) has an upper solution v_+. Suppose u_0 is a solution of (4.1). Choose $v_+ = -c'' < 0$ to be a constant function, where $-c''$ is sufficiently small such that $u_0 + v_+ < 0$ in V. Then $e^{u_0+v_+} - 1 < 0$. So we can choose $\lambda > 0$ big enough such that

$$\lambda e^{u_0+v_+}(e^{u_0+v_+} - 1) + \frac{4\pi M}{|V|} < 0.$$

Therefore

$$0 = \Delta v_+ > \lambda e^{u_0+v_+}(e^{u_0+v_+} - 1) + \frac{4\pi M}{|V|}.$$

So $v_+ \equiv -c$ is an upper solution of (4.2). □

Lemma 4.4. *If u is a solution of Eq.* (1.3) *on G, then* $u < 0$ *on G.*

Proof. Let $u(x_0) = \max_{x\in V}\{u(x)\}$, we only need to show that $u(x_0) < 0$. Suppose $u(x_0) \ge 0$. Then $e^{u(x_0)} - 1 \ge 0$. From Eq. (1.3) we get that

$$\Delta u(x_0) \ge 0,$$

that is

$$\sum_{y\sim x} u(y) \ge d_{x_0} u(x_0).$$

This implies that for any

$$y \sim x, u(y) \ge u(x_0).$$

Since G is a connected finite graph, by iterating the above process, we get that for any

$$y \in V, u(y) = u(x_0).$$

So the left side of Eq. (1.3) is 0 and the right side is positive on $p_j \in V$, which is a contradiction. □

Now we prove Theorem 2.2, which is similar to the proof of Lemma 4 in [2].

Proof. Denote

$$\Lambda = \{\lambda > 0 | \lambda \text{ is such that equation (1.3) has a solution}\}.$$

We will show that Λ is an interval. Suppose that $\lambda' \in \Lambda$. We need to prove that

$$[\lambda', +\infty) \in \Lambda.$$

In fact, let $u' = u_0 + v'$ is the solution of Eq. (1.3) at $\lambda = \lambda'$, where v' is the corresponding solution of Eq. (4.2). Since

$$u' = u_0 + v' < 0,$$

we see that v' is an upper solution of Eq. (4.2) for any $\lambda \geq \lambda'$. By Lemma 4.2, we obtain that $\lambda \in \Lambda$ as desired.

Set $\lambda_c = \inf\{\lambda | \lambda \in \Lambda\}$. Then $\lambda \geq \frac{16\pi M}{|V|}$ for any $\lambda > \lambda_c$ by (4.3) and that Λ is an interval. Taking the limit, we get that

$$\lambda_c \geq \frac{16\pi N}{|V|}.$$

□

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