

Witten-Morse functions and Morse inequalities on digraphs

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Abstract. In this paper, we prove that discrete Morse functions on digraphs are flat Witten-Morse functions and Witten complexes of transitive digraphs approach to Morse complexes. We construct a chain complex consisting of the formal linear combinations of paths which are not only critical paths of the transitive closure but also allowed elementary paths of the digraph, and prove that the homology of the new chain complex is isomorphic to the path homology. On the basis of the above results, we give the Morse inequalities on digraphs.

1 Introduction

Digraphs are important topological models in complex networks. A digraph G is determined by a finite set V and a non-empty subset E of $V \times V \setminus \{\text{diag}\}$. V is called the vertex set of G and E is called the directed edge set of G . For vertices $u, v \in V$, the pair $(u, v) \in E$ is denoted as $u \rightarrow v$. G is called *transitive* if for any two directed edges $u \rightarrow v$ and $v \rightarrow w$ of G , there is a directed edge $u \rightarrow w$ of G . The transitive closure of G is the smallest transitive digraph containing G , which is denoted as \bar{G} in this paper if there is no ambiguity.

Let R be the real numbers. For any integer $n \geq 0$, an *elementary n -path* is a sequence $v_0 v_1 \cdots v_n$ of $n + 1$ vertices in V . Let $\Lambda_n(V)$ be the linear space consisting of all the formal linear combinations of the n -paths on V . The i -th face map is defined as the R -linear map

$$d_i : \Lambda_n(V) \longrightarrow \Lambda_{n-1}(V)$$

which sends $v_0 v_1 \cdots v_n$ to $v_0 \cdots \hat{v}_i \cdots v_n$, where \hat{v}_i means omission of the vertex v_i . Let $\partial_n = \sum_{i=0}^n (-1)^i d_i$. Then ∂_n is an R -linear map from $\Lambda_n(V)$ to $\Lambda_{n-1}(V)$ satisfying $\partial_n \partial_{n+1} = 0$ for each $n \geq 0$ (cf. [10, 11, 12, 13, 14, 15, 16]). Hence $\{\Lambda_n(V), \partial_n\}_{n \geq 0}$ is a chain complex.

An *allowed elementary n -path* on G is a n -path $v_0 v_1 \cdots v_n$ on V such that $v_{i-1} \rightarrow v_i$ is a directed edge of G and $v_{i-1} \neq v_i$ for each $1 \leq i \leq n$. Let $P_n(G)$ be the linear space consisting of all the formal linear combinations of allowed elementary n -paths on G . Then $P_n(G)$ is a subspace of $\Lambda_n(V)$, whereas the image of an allowed elementary path under the boundary operator ∂ does not have to be allowed. Consider the space $\Omega_n(G)$ formed by all the linear combinations of the ∂ -invariant n -paths in $P_n(G)$. Obviously, $\Omega_n(G)$ is a subspace of $P_n(G)$.

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The path homology of G is defined as the homology of chain complex $\{\Omega_n(G), \partial_n\}_{n \geq 0}$ and denoted as $H_*(G; R)$. That is,

$$H_m(G; R) = H_m(\{\Omega_n(G), \partial_n\}_{n \geq 0}), \quad m \geq 0.$$

Morse theory can simplify the calculation of homology groups. Using Morse theory, one can determine the cell decomposition of manifolds by studying the negative inertia index of Hessian matrix of Morse functions at the critical points, so as to characterize the homology groups. In 1925, M. Morse first invented the method of Morse theory (cf. [20]). In 1963, J.W. Milnor combed, studied and developed Morse's method, and Morse theory was given in [19]. Since then, there have been numerous researches on Morse theory (cf. [5, 22], etc). In recent years, Morse theory has been applied to cell complexes, simplicial complexes, graphs and other combinatorial objects, and discrete Morse theory has gradually become a hot research topic (cf. [1, 2, 3, 4, 6, 7, 8, 9]).

It is well known that the homology groups of simplicial complexes or cell complexes can be characterized by chain complexes made of the linear combinations of critical simplices. Inspired by this, in this paper, based on [18, 23], we further study the properties of discrete Morse functions on digraphs and critical paths on transitive digraphs, characterize the path homology groups of digraphs with chain complex consisting of the formal linear combinations of paths which are not only critical paths of the transitive closure but also allowed elementary paths of the digraph, prove that Witten complexes of transitive digraphs approach to Morse complexes, which is not necessarily true for general digraphs.

Let G be a digraph and $f : V(G) \rightarrow [0, +\infty)$ a discrete Morse function on G as defined in [23] and Definition 2.1. By [6, Definition 6.1], the (algebraic) discrete gradient vector field of f is defined as an R -linear map $\text{grad} f : P_n(G) \rightarrow P_{n+1}(G)$ such that for any allowed elementary n -path α on G ,

$$(\text{grad} f)(\alpha) = -\langle \partial \gamma, \alpha \rangle \gamma,$$

where $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$. Otherwise $(\text{grad} f)(\alpha) = 0$. Here $\langle \cdot, \cdot \rangle$ is the inner product in $\Lambda_n(V)$ (with respect to which the elementary n -paths are orthonormal).

Let \bar{f} be a discrete Morse function on a transitive digraph and $\bar{V} = \text{grad} \bar{f}$ the discrete gradient vector field on it. By [6, Definition 6.2], the discrete gradient flow is denoted as

$$\bar{\Phi} = \text{Id} + \partial \bar{V} + \bar{V} \partial.$$

Let $\Delta_n(t)$ be the Laplace operator with one-parameter t and $W_n(t)$ the span of the eigenvectors of $\Delta_n(t)$ corresponding to the eigenvalues which tend to 0 as $t \rightarrow \infty$. Denote $\text{Crit}_n(-)$ as the span of all critical n -paths on “-”.

The paper is organized as follows. In Section 2, we review the definition of discrete Morse functions on digraphs and prove that discrete Morse functions on digraphs are discrete flat Witten-Morse functions in Proposition 2.8. Then we prove that Witten complexes

approach to Morse complexes for transitive digraphs in Section 3. Furthermore, we study the path homology of general digraphs in Section 4 which is divided into two subsections. In Subsection 4.1, we give some properties of transitive digraphs. Particularly, we characterize $\overline{\Phi}$ -invariant space with critical paths in Proposition 4.6. Let G be a digraph and \tilde{G} the transitive closure of G . Suppose $\Omega_*(G)$ is \overline{V} -invariant ($\overline{V}(\Omega(G)) \subseteq \Omega(G)$). Then

$$H_m(G; R) \cong H_m(\{R(\alpha + \overline{V}\partial(\alpha)) \cap \Omega_n(G), \partial_n\}_{n \geq 0})$$

where $\alpha \in \text{Crit}(\tilde{G})$. This is proved in Theorem 4.8.

Moreover, in Subsection 4.2, we give a description of path homology of digraphs by homology of a chain complex which is related to critical sets of the transitive closure of G in Corollary 4.11. That is, if $\Omega_*(G)$ is \overline{V} -invariant and $\overline{\Phi}(\alpha) \in \Omega(G)$ for any $\alpha \in \text{Crit}(\tilde{G}) \cap P(G)$, then

$$H_m(\{\text{Crit}_n(\tilde{G}) \cap P_n(G), \tilde{\partial}_n\}_{n \geq 0}) \cong H_m(G; R)$$

where $\tilde{\partial} = (\overline{\Phi}^\infty)^{-1} \circ \partial \circ \overline{\Phi}^\infty$ and $\overline{\Phi}^\infty$ is the stabilization map of $\overline{\Phi}$.

Finally, in Section 5, we give the Morse inequalities on digraphs.

2 Preliminaries

In this section, we mainly review the definition of discrete Morse functions on digraphs and prove that discrete Morse functions on digraphs are flat Witten-Morse functions.

For any allowed elementary paths α and β , if β can be obtained from α by removing some vertices, then we write $\alpha > \beta$ or $\beta < \alpha$.

Definition 2.1. (cf. [23]) A map $f : V(G) \rightarrow [0, +\infty)$ is called a *discrete Morse function* on G , if for any allowed elementary path $\alpha = v_0 v_1 \cdots v_n$ on G , both of the followings hold:

- (i). $\#\{\gamma^{(n+1)} > \alpha^{(n)} \mid f(\gamma) = f(\alpha)\} \leq 1$;
- (ii). $\#\{\beta^{(n-1)} < \alpha^{(n)} \mid f(\beta) = f(\alpha)\} \leq 1$.

where

$$f(\alpha) = f(v_0 v_1 \cdots v_n) = \sum_{i=0}^n f(v_i).$$

For an allowed elementary path α , if in both (i) and (ii), the inequalities hold strictly, then α is called *critical*. Precisely,

Definition 2.2. An allowed elementary n -path $\gamma^{(n)}$ is called *critical*, if both of the followings hold:

- (i)', $\#\{\beta^{(n-1)} < \alpha^{(n)} \mid f(\beta) = f(\alpha)\} = 0$,
- (ii)', $\#\{\gamma^{(n+1)} > \alpha^{(n)} \mid f(\gamma) = f(\alpha)\} = 0$.

It follows from Definition 2.2 that an allowed elementary p -path is not critical if and only if either of the following conditions holds

- (i)" there exists $\beta^{(n-1)} < \alpha^{(n)}$ such that $f(\beta) = f(\alpha)$;
- (ii)" there exists $\gamma^{(n+1)} > \alpha^{(n)}$ such that $f(\gamma) = f(\alpha)$.

A directed loop on G is an allowed elementary path $v_0v_1 \cdots v_nv_0$, $n \geq 1$.

Lemma 2.3. (cf. [18, Lemma 2.4]) *Let G be a digraph and f a discrete Morse function on G . Let $\alpha = v_0v_1 \cdots v_nv_0$ be a directed loop. Then for each $0 \leq i \leq n$, $f(v_i) > 0$.*

Lemma 2.4. (cf. [18, Lemma 2.5]) *Let G be a digraph and f a discrete Morse function on G as defined in Definition 2.1. Then for any allowed elementary path in G , there exists at most one index such that the corresponding vertex is with zero value.*

Lemma 2.5. *Let f be a discrete Morse function on digraph G . Then for any allowed elementary path $\alpha = v_0v_1 \cdots v_n$ in G , (i)" and (ii)" cannot both be true.*

Proof. Suppose to the contrary. By (i)", there exists an allowed elementary $(n-1)$ -path β such that $\beta < \alpha$ and $f(\beta) = f(\alpha)$. Hence, there exists some $0 \leq i \leq n$ such that $f(v_i) = 0$. By (ii)", there exists an allowed elementary $(n+1)$ -path γ such that $f(\gamma) = f(\alpha)$. Hence, there exists a vertex $u \in V(G)$ with $f(u) = 0$ such that $\gamma = v_0 \cdots v_juv_{j+1} \cdots v_n$. We assert that $u \neq v_i$. Suppose to the contrary. Since γ is allowed, it follows that u and v_i are not adjacent. Hence there exists a directed loop in which u is a vertex. This contradicts Lemma 2.3. Therefore, there are two distinct vertices with zero value in γ which contradicts Lemma 2.4.

The lemma follows. □

Definition 2.6. (cf. [9, Definition 0.6]) A function $f : V(G) \rightarrow [0, +\infty)$ is called a *discrete Witten-Morse function* on G if, for any allowed elementary path α ,

- (i) $f(\alpha) < \text{average}\{f(\gamma_1), f(\gamma_2)\}$ where $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$;
- (ii) $f(\alpha) > \text{average}\{f(\beta_1), f(\beta_2)\}$ where $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$.

Note that each Witten-Morse function is, in fact, a Morse function.

Definition 2.7. (cf. [9, Definition 0.7]) A discrete Witten-Morse function is *flat* if for any allowed elementary path α ,

- (i) $f(\alpha) \leq \min\{f(\gamma_1), f(\gamma_2)\}$ where $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$;
- (ii) $f(\alpha) \geq \max\{f(\beta_1), f(\beta_2)\}$ where $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$.

Proposition 2.8. *Let G be a digraph and $f : V(G) \rightarrow [0, +\infty)$ a discrete Morse function on G . Then f is a discrete flat Witten-Morse function.*

Proof. Let α be an arbitrary allowed elementary path on G . Consider the following cases.

CASE 1. α is critical. Then by Definition 2.2, we have that $f(\alpha) < f(\gamma)$ for any $\gamma > \alpha$ and $f(\alpha) > f(\beta)$ for any $\beta < \alpha$. Hence,

$$f(\alpha) < \min\{f(\gamma_1), f(\gamma_2)\}$$

where $\gamma_1 > \alpha$ and $\gamma_2 > \alpha$, and

$$f(\alpha) > \max\{f(\beta_1), f(\beta_2)\}$$

where $\beta_1 < \alpha$ and $\beta_2 < \alpha$.

CASE 2. α is not critical.

SUBCASE 2.1. (i)" holds for α . Then there exists an unique allowed elementary path $\beta < \alpha$ such that $f(\beta) = f(\alpha)$. Let $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$. Then

$$\begin{cases} f(\beta_1) < f(\alpha), f(\beta_2) < f(\alpha), & \text{if } \beta_1 \neq \beta \text{ and } \beta_2 \neq \beta; \\ f(\beta_1) = f(\alpha), f(\beta_2) < f(\alpha), & \text{if } \beta_1 = \beta \text{ and } \beta_2 \neq \beta; \\ f(\beta_1) < f(\alpha), f(\beta_2) = f(\alpha), & \text{if } \beta_1 \neq \beta \text{ and } \beta_2 = \beta. \end{cases}$$

Hence,

$$f(\alpha) \geq \max\{f(\beta_1), f(\beta_2)\}$$

where $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$.

By Lemma 2.3, (ii)" does not hold for α . Then for any $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$, we have that $f(\gamma_1) > f(\alpha)$ and $f(\gamma_2) > f(\alpha)$. Hence,

$$f(\alpha) < \min\{f(\gamma_1), f(\gamma_2)\}$$

where $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$.

SUBCASE 2.2. (ii)" holds for α . Then there exists an unique allowed elementary path $\gamma > \alpha$ such that $f(\gamma) = f(\alpha)$. Let $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$. Similarly,

$$\begin{cases} f(\gamma_1) > f(\alpha), f(\gamma_2) > f(\alpha), & \text{if } \gamma_1 \neq \gamma \text{ and } \gamma_2 \neq \gamma; \\ f(\gamma_1) = f(\alpha), f(\gamma_2) > f(\alpha), & \text{if } \gamma_1 = \gamma \text{ and } \gamma_2 \neq \gamma; \\ f(\gamma_1) > f(\alpha), f(\gamma_2) = f(\alpha), & \text{if } \gamma_1 \neq \gamma \text{ and } \gamma_2 = \gamma. \end{cases}$$

Hence,

$$f(\alpha) \leq \min\{f(\gamma_1), f(\gamma_2)\}$$

where $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $\gamma_1 \neq \gamma_2$.

By Lemma 2.3, (i)" does not hold for α . Then for any $\beta_1 > \alpha$, $\beta_2 > \alpha$ and $\beta_1 \neq \beta_2$, we have that $f(\beta_1) < f(\alpha)$ and $f(\beta_2) < f(\alpha)$. Hence,

$$f(\alpha) > \max\{f(\beta_1), f(\beta_2)\}$$

where $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $\beta_1 \neq \beta_2$.

Combining Case 1 and Case 2, the assertion is proved. \square

Remark 2.9. *The key of Proposition 2.8 is the definition of discrete Morse functions on digraphs. Let f be a discrete Morse function on digraph G . Then $f(\beta) \leq f(\alpha)$ for any $\beta < \alpha$ and $f(\gamma) \geq f(\alpha)$ for any $\gamma > \alpha$. Meanwhile, for any allowed elementary path, (i) and (ii) can not both be true.*

Next, we consider the **equivalent** discrete Morse functions on digraphs.

Definition 2.10. Let G be a digraph and $f : V(G) \rightarrow [0, +\infty)$ a discrete Morse function on G . The set of all vertices $v \in V(G)$ such that $f(v) = 0$ is called *zero-point set* of f , denoted as $S(f)$.

Definition 2.11. (cf. [9, Definition 1.2]) Let f, g be two discrete Morse functions on G . We say f and g are equivalent if for any $n \geq 0$ and every $\alpha^{(n)} < \gamma^{(n+1)}$,

$$f(\alpha) < f(\gamma) \iff g(\alpha) < g(\gamma).$$

Proposition 2.12. *Let f, g be two discrete Morse functions on G such that $S(f) = S(g)$. Let \bar{f}, \bar{g} be the extensions of f, g on transitive closure \bar{G} of G . Then they induce the same Morse complexes.*

Proof. Let $\alpha^{(n)}$ and $\gamma^{(n+1)}$ be allowed elementary paths on \bar{G} such that $\alpha < \gamma$ and $\bar{f}(\alpha) < \bar{f}(\gamma)$. We assert that $\bar{g}(\alpha) < \bar{g}(\gamma)$. Suppose to the contrary, $\bar{g}(\gamma) = \bar{g}(\alpha)$. Then there exists a vertex v in $V(\gamma) \setminus V(\alpha)$ (Consider α and γ as subgraphs of \bar{G}) such that $\bar{g}(v) = 0$. Since $S(f) = S(g)$, it follows that $S(\bar{f}) = S(\bar{g})$. Hence $\bar{f}(v) = 0$ and $\bar{f}(\alpha) = \bar{f}(\gamma)$ which contradict $\bar{f}(\alpha) < \bar{f}(\gamma)$. Thus,

$$\bar{f}(\alpha) < \bar{f}(\gamma) \implies \bar{g}(\alpha) < \bar{g}(\gamma).$$

Similarly, we have that

$$\bar{g}(\alpha) < \bar{g}(\gamma) \implies \bar{f}(\alpha) < \bar{f}(\gamma).$$

Hence, by Definition 2.11, \bar{f} and \bar{g} are equivalent. Therefore, for any allowed elementary path α on \bar{G} ,

$$\begin{aligned} \bar{V}_{\bar{f}}(\alpha) &= \bar{V}_{\bar{g}}(\alpha), \\ \bar{\Phi}_{\bar{f}}(\alpha) &= \bar{\Phi}_{\bar{g}}(\alpha) \end{aligned}$$

and

$$\text{Crit}_{\bar{f}}(\bar{G}) = \text{Crit}_{\bar{g}}(\bar{G})$$

which imply that the induced Morse complexes are the same. \square

3 Witten Complexes of Transitive Digraphs

In this section, we prove that Witten complexes approach to Morse complexes for transitive digraphs.

Let G be a transitive digraph. Similar to [9], consider the chain complex

$$0 \longrightarrow \Omega_n(G) \xrightarrow{\partial} \Omega_{n-1}(G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_0(G) \longrightarrow 0. \quad (3.1)$$

Define a chain homomorphism

$$e^{tf} : \Omega_n(G) \longrightarrow \Omega_n(G)$$

by setting

$$e^{tf}(\alpha) = e^{tf(\alpha)}\alpha \quad (3.2)$$

for any allowed elementary path α on G , and extending linearly to $\Omega(G)$. Replace the boundary operator ∂ with

$$\partial_t = e^{tf}\partial e^{-tf}.$$

Then

$$\begin{aligned} \partial_t(\alpha) &= e^{tf}\partial e^{-tf}(\alpha) \\ &= e^{tf}\partial e^{-tf(\alpha)}\alpha \\ &= e^{-tf(\alpha)}e^{tf}(\partial\alpha) \\ &= \sum_{\beta < \alpha, \beta \in \Omega(G)} e^{t[f(\beta)-f(\alpha)]}\beta. \end{aligned}$$

Hence, $\partial_t(\alpha) \in \Omega(G)$ which implies that

$$0 \longrightarrow \Omega_n(G) \xrightarrow{\partial_t} \Omega_{n-1}(G) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} \Omega_0(G) \longrightarrow 0 \quad (3.3)$$

is still a chain complex. Moreover, by a similar argument to [21, Section 5.1, P.54], we have that

Proposition 3.1. *Let G be a transitive digraph. Then for each $t \in R$, the complexes (3.1) and (3.3) have the same path homology. That is,*

$$H_m(\{\Omega_n(G), \partial_n\}_{n \geq 0}) \cong H_m(\{\Omega_n(G), \partial_t\}_{n \geq 0}).$$

Proof. Note that $\Omega(G) = P(G)$ for transitive digraphs. For any $x \in \text{Ker}\partial$, under the map (3.2), we have that

$$\begin{aligned} \partial_t e^{tf}(x) &= e^{tf}\partial(x) \\ &= e^{tf}(\partial x) \\ &= 0. \end{aligned}$$

That is, $e^{tf}(x) \in \text{Ker}\partial_t$. And if $x = (\partial y) \in \text{Im}\partial$, then

$$\begin{aligned}\partial_t e^{tf}(y) &= e^{tf}\partial(y) \\ &= e^{tf}(x).\end{aligned}$$

Hence, $e^{tf}(x) \in \text{Im}\partial_t$.

Therefore, the invertible map of (4.4) maps ∂ -invariant paths which are closed but not exact in the usual sense to ∂ -invariant paths which are closed but not exact in the sense of ∂_t .

The proposition is proved. □

Let

$$\Delta_n(t) = \partial_t \partial_t^* + \partial_t^* \partial_t$$

be the Laplace operator induced by ∂_t where ∂_t^* is the adjoint of ∂_t with respect to the inner product on the chain spaces $\Lambda_*(V)$ such that all paths are orthonormal. Then by [17, Section 3.1],

$$\text{Ker}(\Delta_n(t)) \cong H_m(\{\Omega_n(G), \partial_t\}_{n \geq 0}).$$

Hence, by Proposition 3.1,

$$\text{Ker}(\Delta_n(t)) \cong H_m(\{\Omega_n(G), \partial_n\}_{n \geq 0}). \quad (3.4)$$

Denote $W_n(t)$ as the span of the eigenvectors of $\Delta_n(t)$ corresponding to the eigenvalues which tend to 0 as $t \rightarrow \infty$. Since $\Delta(t)\partial_t = \partial_t\Delta(t)$, ∂_t preserves the eigenspaces. The Witten complex is defined as

$$0 \longrightarrow W_n(t) \xrightarrow{\partial_t} W_{n-1}(t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0(t) \longrightarrow 0.$$

Let $\text{Crit}_n(G)$ be the span of the critical n -paths on G . We have the following theorem.

Theorem 3.2. *Let G be a transitive digraph and f a discrete Morse function on G . Then*

$$\lim_{t \rightarrow \infty} W_n(t) = \text{Crit}_n(G).$$

Proof. Since G is transitive, $P_n(G) = \Omega_n(G)$ for each $n \geq 0$. By [9, Theorem 2.1] and Proposition 2.8, we have that

$$\Delta_n(t)\alpha = \left[\sum_{\beta < \alpha} \langle \partial\alpha, \beta \rangle^2 e^{2t(f(\beta) - f(\alpha))} + \sum_{\gamma > \alpha} \langle \partial\gamma, \alpha \rangle^2 e^{2t(f(\alpha) - f(\gamma))} \right] \alpha + O(e^{-tc})$$

for some $c > 0$, where γ, α, β are allowed elementary paths on G . Hence, if and only if α is critical, the eigenvalues of $\Delta_n(t)$

$$\langle \Delta_n(t)\alpha, \alpha \rangle = \sum_{\beta < \alpha} \langle \partial\alpha, \beta \rangle^2 e^{2t(f(\beta) - f(\alpha))} + \sum_{\gamma > \alpha} \langle \partial\gamma, \alpha \rangle^2 e^{2t(f(\alpha) - f(\gamma))}$$

tend to 0 as $t \rightarrow \infty$.

The theorem is proved. \square

Corollary 3.3. *Let G be a transitive digraph. Then Witten complex $\{W_n(t), \partial_t\}_{n \geq 0}$ approaches to the complex $\{\text{Crit}_n(G), \tilde{\partial}_n\}_{n \geq 0}$.*

Proof. By [18, Theorem 2.1],

$$H_m(\{\text{Crit}_n(G), \tilde{\partial}_n\}_{n \geq 0}) \cong H_m(\{\Omega_n(G), \partial_n\}_{n \geq 0}).$$

By (3.4), for all t ,

$$H_m(\{W_n(t), \partial_t\}_{n \geq 0}) \cong H_m(\{\Omega_n(G), \partial_n\}_{n \geq 0}).$$

Therefore, by Theorem 3.2, the assertion is followed. \square

Note that for general digraph G , the image of each ∂ -invariant element $x \in \Omega_n(G)$ under ∂_t may be not in $\Omega_{n-1}(G)$. This implies that $\{\Omega(G), \partial_t\}$ is not a chain complex in general. For example,

Example 3.4. *Let G be a square with vertex set $V = \{v_0, v_1, v_2, v_3\}$ and directed edge set $E = \{v_0v_1, v_0v_2, v_1v_3, v_2v_3\}$. Then*

$$\Omega(G) = \{v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_1v_3, v_2v_3, v_0v_1v_3 - v_0v_2v_3\}$$

and

$$\begin{aligned} \partial_t(v_0v_1v_3 - v_0v_2v_3) &= e^{tf} \partial e^{-tf}(v_0v_1v_3 - v_0v_2v_3) \\ &= e^{tf} \partial e^{-tf}(v_0v_1v_3) - e^{tf} \partial e^{-tf}(v_0v_2v_3) \\ &= e^{-tf(v_0v_1v_3)} e^{tf} \partial(v_0v_1v_3) - e^{-tf(v_0v_2v_3)} e^{tf} \partial(v_0v_2v_3) \\ &= [e^{t[f(v_0v_1) - f(v_0v_1v_3)]} v_0v_1 + e^{t[f(v_1v_3) - f(v_0v_1v_3)]} v_1v_3] \\ &\quad - [e^{t[f(v_0v_2) - f(v_0v_2v_3)]} v_0v_2 + e^{t[f(v_2v_3) - f(v_0v_2v_3)]} v_2v_3] \\ &\quad + [e^{t[f(v_0v_3) - f(v_0v_2v_3)]} v_0v_3 - e^{t[f(v_0v_3) - f(v_0v_1v_3)]} v_0v_3]. \end{aligned} \quad (3.5)$$

Since the coefficient of v_0v_3 in (3.5) may **not** be zero, it follows that

$$\partial_t(v_0v_1v_3 - v_0v_2v_3) \notin \Omega_1(G).$$

Moreover,

$$\begin{aligned} \partial^* |_{\Omega(G)}(v_0v_1) &= v_0v_1v_3, \\ (\partial |_{\Omega(G)})^*(v_0v_1) &= v_0v_1v_3 - v_0v_2v_3. \end{aligned}$$

Hence,

$$\partial^* |_{\Omega(G)} \neq (\partial |_{\Omega(G)})^*.$$

Therefore, we will further consider the path homology of general digraphs based on the results of [18] instead of techniques from Hodge theory.

4 Description of Path Homology of Digraphs

In this section, we will characterize the path homology of digraphs by chain complex related to critical sets of transitive closure of digraphs.

4.1 $\bar{\Phi}$ -invariant Module of Transitive Digraph

Firstly, we give some properties of critical paths on transitive digraphs.

Proposition 4.1. *Let G be a digraph. Then G is transitive if and only if for any allowed elementary paths $\gamma^{(n+2)} > \alpha^{(n+1)} > \beta^{(n)}$, there exists an allowed elementary n -path $\alpha'^{(n+1)} \neq \alpha^{(n+1)}$ such that $\gamma > \alpha' > \beta$.*

Proof. Suppose $\gamma^{(n+2)} > \alpha^{(n+1)} > \beta^{(n)}$. Then by [23, Proposition 2.6], there are two cases.

CASE 1. There exists an allowed elementary $(n+1)$ -path $\alpha'^{(n+1)} \neq \alpha^{(n+1)}$ such that $\gamma > \alpha' > \beta$.

CASE 2. β is obtained by removing two subsequent vertices $v_i \rightarrow v_{i+1}$ in γ where $0 \leq i \leq n+1$.

Hence the critical part is to verify that the assertion is followed for Case 2. Let

$$\gamma = v_0 v_1 \cdots v_{n+2}$$

and

$$\beta = v_0 \cdots v_{i-1} v_{i+2} \cdots v_{n+2}.$$

Suppose G is transitive. Then

$$v_0 \cdots v_{i-1} v_{i+1} v_{i+2} \cdots v_{n+2}$$

and

$$v_0 \cdots v_{i-1} v_i v_{i+2} \cdots v_{n+2}$$

are both allowed elementary paths on G which can be denoted as α and α' respectively. Since $v_i \neq v_{i+1}$, it follows that $\alpha \neq \alpha'$.

Hence, summarizing Case 1 and Case 2, we have that if G is transitive, then there exists an allowed elementary n -path $\alpha'^{(n+1)} \neq \alpha^{(n+1)}$ such that $\gamma > \alpha' > \beta$.

On the other hand, suppose for any allowed elementary paths $\gamma^{(n+2)} > \alpha^{(n+1)} > \beta^{(n)}$, there exists an allowed elementary n -path $\alpha'^{(n+1)} \neq \alpha^{(n+1)}$ such that $\gamma > \alpha' > \beta$. Let $u \rightarrow v$ and $v \rightarrow w$ be direct edges of G . Let $\gamma = uvw$, $\alpha = uv$ and $\beta = u$. Then $\alpha' = uw$ must be an allowed elementary path on G . Hence, G is transitive. \square

Lemma 4.2. *Let $\alpha = v_0 \cdots v_n$ ($n > 1$) be a critical path on transitive digraph G . Let f be a discrete Morse function on G . Then there exists at most one $d_j \alpha$ ($0 \leq j \leq n$) such that $d_j \alpha$ is non-critical.*

Proof. Since G is transitive and α is critical, it follows that $d_i\alpha$ is allowed on G and $f(v_i) > 0$ for any $0 \leq i \leq n$. Suppose $\beta = d_j\alpha$ is non-critical for some $0 \leq j \leq n$. Then there exists a unique vertex $u \in V(G)$ with $f(u) = 0$ such that

$$\alpha' = v_0 \cdots v_{j-1} \hat{v}_j v_{j+1} \cdots v_k u v_{k+1} \cdots v_n$$

is an allowed elementary n -path on G , $\alpha' > \beta$ and $f(\alpha') = f(\beta)$.

Step 1. We assert that

$$\alpha' = v_0 \cdots v_{j-1} u v_{j+1} \cdots v_n. \quad (4.1)$$

Suppose to the contrary, either

$$\alpha' = v_0 \cdots v_k u v_{k+1} \cdots v_{j-1} v_{j+1} \cdots v_n$$

or

$$\alpha' = v_0 \cdots v_{j-1} v_{j+1} \cdots v_k u v_{k+1} \cdots v_n.$$

Without loss of generality,

$$\alpha' = v_0 \cdots v_{j-1} v_{j+1} \cdots v_k u v_{k+1} \cdots v_n.$$

Then

$$\gamma = v_0 \cdots v_j \cdots v_k u v_{k+1} \cdots v_n.$$

is an allowed elementary $(n+1)$ -path on G such that $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$. This contradicts that α is critical. Hence the assertion is proved.

Step 2. We will prove that for any $0 \leq i \neq j \leq n$, $d_i\alpha$ is critical. Suppose to the contrary, there exists an allowed elementary path $\beta' = d_i\alpha$ ($i \neq j$) which is non-critical. Then by a discussion similar to the above, there exists a unique vertex $w \in V(G)$ with $f(w) = 0$ and

$$\alpha'' = v_0 \cdots v_{i-1} w v_{i+1} \cdots v_n \quad (4.2)$$

is an allowed elementary n -path on G such that $\alpha'' > \beta'$ and $f(\alpha'') = f(\beta')$. Without loss of generality, $0 \leq i < j \leq n$.

Firstly, we assert that $u \neq w$. Suppose to the contrary, $u = w$. Consider the following two cases.

CASE 1. $j = i + 1$. Then $i = j - 1$. By (4.1) and (4.2), $\gamma = v_0 \cdots v_{i-1} v_i w v_{i+1} \cdots v_n$ is an allowed elementary path on G such that $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$. This contradicts that α is critical.

CASE 2. $j > i + 1$. Then $j - 1 > i$. Hence, $w v_{i+1} \cdots v_{j-1} w$ (or $u v_{i+1} \cdots v_{j-1} u$) is a directed loop with $f(w) = 0$ (or $f(u) = 0$) which contradicts Lemma 2.3.

Combining Case 1 and Case 2, $u \neq w$.

Secondly, according to the value of i , we divide it into two cases to complete the proof of Step 2.

CASE 3. $i \geq 1$. Since G is transitive and $j > i$, $v_{i-1} \rightarrow v_{j-1}$ is a directed edge of G . Thus, $v_{i-1} \rightarrow u$ is also a directed edge of G . Moreover, since $u \neq w$, it follows that $v_{i-1} \rightarrow u$ and $v_{i-1} \rightarrow w$ are two distinct directed edges of G with $f(u) = f(w) = 0$. This contradicts that f is a discrete Morse function on G .

CASE 4. $i = 0$. There are two subcases.

SUBCASE 4.1. $j < n$. Since G is transitive and $j > i$, $v_{i+1} \rightarrow v_{j+1}$ is a directed edge of G . Thus, $w \rightarrow v_{j+1}$ is also a directed edge of G . Moreover, since $u \neq w$, it follows that $u \rightarrow v_{j+1}$ and $w \rightarrow v_{j+1}$ are two distinct directed edges of G with $f(u) = f(w) = 0$. This contradicts that f is a discrete Morse function on G .

SUBCASE 4.2. $j = n$. Then $wv_1 \cdots v_{n-1}u$ is an allowed path on G with $f(u) = f(w) = 0$. Since $u \neq w$, there are two distinct zero-points in the path $wv_{i+1} \cdots v_{n-1}u$ which contradicts Lemma 2.4.

By Step 1 and Step 2, the lemma follows. □

Remark 4.3. Note that in the proof of Lemma 4.2, the condition $n > 1$ ensures that $w \rightarrow v_{i+1}$ and $v_{j-1} \rightarrow u$ are directed edges of G . If $n = 1$, then Lemma 4.2 may not hold. For example, let G is a digraph with $V(G) = \{v_0, v_1, v_2, v_3\}$ and

$$E(G) = \{v_0 \rightarrow v_1, v_2 \rightarrow v_1, v_0 \rightarrow v_3\}.$$

Then G is a transitive digraph. Let f be a function on G with $f(v_2) = f(v_3) = 0$ and $f(v_0) > 0, f(v_1) > 0$. It is easy to verify that f is a discrete Morse function and $\alpha = v_0v_1$ is critical. Let $\beta_1 = v_0 < \alpha$ and $\beta_2 = v_1 < \alpha$. Then $v_0v_3 > \beta_1$ and $f(v_0v_3) = f(\beta_1)$, and $v_2v_1 > \beta_2$ and $f(v_2v_1) = f(\beta_2)$. Hence, β_1 and β_2 are both non-critical.

Denote $P_*^{\bar{\Phi}}(G)$ as the sub-chain complex of $P_*(G)$ consisting of all $\bar{\Phi}$ -invariant chains where G is a transitive digraph.

Lemma 4.4. Let G be a transitive digraph and f a discrete Morse function on G . If $\alpha = uv$ is critical and either $\beta_0 = u$ or $\beta_1 = v$ is not critical. Then $\alpha \notin P_1^{\bar{\Phi}}(G)$ and $(\alpha + \bar{V}\partial(\alpha)) \in P_1^{\bar{\Phi}}(G)$.

Proof. Since α is critical, $f(u) > 0$ and $f(v) > 0$. We divide the proof into the following cases.

CASE 1. Only one of β_0 and β_1 is not critical. Without loss of generality, β_0 is not critical and β_1 is critical. Then there exists a unique vertex $w \in V(G)$ such that $f(w) = 0$ and $f(uw) = f(u)$. Hence

$$\begin{aligned} \alpha + \bar{V}\partial(\alpha) &= uv + \bar{V}(v - u) \\ &= uv - uw \end{aligned}$$

and

$$\begin{aligned}
\overline{\Phi}(\alpha + \overline{V}\partial(\alpha)) &= (\text{Id} + \partial\overline{V} + \overline{V}\partial)(uv - uw) \\
&= (\text{Id} + \partial\overline{V} + \overline{V}\partial)(uv) - (\text{Id} + \partial\overline{V} + \overline{V}\partial)(uw) \\
&= (uv + (\overline{V}\partial)(uv)) - (uw + (\overline{V}\partial)(uw)) \\
&= (uv + \overline{V}(v - u)) - (uw + \overline{V}(w - u)) \\
&= (uv - uw) \in P_1^{\overline{\Phi}}(G).
\end{aligned}$$

CASE 2. Both $\beta_0 = u$ and $\beta_1 = v$ are not critical. Then there exist $\alpha_0 > \beta_0$ and $\alpha_1 > \beta_1$ such that $f(\alpha_0) = f(\beta_0)$ and $f(\alpha_1) = f(\beta_1)$ where α_0 and α_1 are allowed elementary paths on G . By Remark 4.3, we assert that the direction of α_0 and α_1 are not consistent with α . Suppose to the contrary, at least one of α_0 and α_1 is consistent with α . Without loss of generality, $\alpha_0 = uw$, $\alpha_1 = vw'$ and $f(w) = f(w') = 0$. Let $\gamma = uvw'$. Then $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$ which contradicts α is critical. Hence, α_0 and α_1 can be written as uw and $w'v$ respectively with $f(w) = f(w') = 0$.

Moreover, we can prove that $w \neq w'$. Suppose to the contrary, $w = w'$. Let $\gamma = uvw$. Then $\alpha < \gamma$ and $f(\alpha) = f(\gamma)$ which contradicts that α is critical.

Therefore,

$$\begin{aligned}
\overline{\Phi}(\alpha) &= (\text{Id} + \partial\overline{V} + \overline{V}\partial)(\alpha) \\
&= \alpha + \overline{V}\partial(\alpha) \\
&= uv + \overline{V}(v - u) \\
&= uv - w'v - uw
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Phi}(\alpha + \overline{V}\partial(\alpha)) &= \overline{\Phi}(uv - w'v - uw) \\
&= (\text{Id} + \partial\overline{V} + \overline{V}\partial)(uv - w'v - uw) \\
&= (uv - w'v - uw) + \overline{V}\partial(uv - w'v - uw) \\
&= (uv - w'v - uw) + \overline{V}(w' - w) \\
&= uv - w'v - uw \\
&= \alpha + \overline{V}\partial(\alpha).
\end{aligned}$$

The lemma is proved. □

Lemma 4.5. *Let G be a transitive digraph and f a discrete Morse function on G . Then*

$$\overline{\Phi}(\alpha) = 0$$

for any $\alpha \in P(G)$ where α is not critical.

Proof. By Lemma 2.5, there are two cases.

CASE 1. There exists an unique allowed elementary path β' on G such that $\beta' < \alpha$ and $f(\beta') = f(\alpha)$. Then

$$\overline{V}(\beta') = -\langle \partial\alpha, \beta' \rangle \alpha$$

and by Lemma 2.3, $\overline{V}(\alpha) = 0$.

Hence,

$$\begin{aligned} \overline{V}\partial(\alpha) &= \overline{V}\left(\sum_{\beta < \alpha} \langle \partial\alpha, \beta \rangle \beta\right) \\ &= \langle \partial\alpha, \beta' \rangle \overline{V}(\beta') \\ &= -\langle \partial\alpha, \beta' \rangle^2 \alpha \\ &= -\alpha \end{aligned}$$

and

$$\begin{aligned} \overline{\Phi}(\alpha) &= \alpha - \alpha \\ &= 0. \end{aligned}$$

CASE 2. There exists an unique allowed elementary path γ on G such that $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$. Then $\overline{V}(\alpha) = -\langle \partial\gamma, \alpha \rangle \gamma \neq 0$.

Let

$$\alpha = v_0 \cdots v_n, \quad \gamma = v_0 \cdots v_j u v_{j+1} \cdots v_n$$

where $f(u) = 0$. Then

$$\begin{aligned} (\overline{V}\partial + \partial\overline{V})(\alpha) &= \sum_{i=0}^n \overline{V}((-1)^i d_i \alpha) - \langle \partial\gamma, \alpha \rangle \partial\gamma \\ &= \sum_{i=0}^n (-1)^i \overline{V}(d_i \alpha) - (-1)^{j+1} \partial\gamma. \end{aligned} \tag{4.3}$$

Consider the following subcases.

SUBCASE 2.1. $0 \leq i \leq j$. Then

$$\begin{aligned} \overline{V}(d_i \alpha) &= -\langle \partial\alpha', d_i \alpha \rangle \alpha' \\ &= -(-1)^j \alpha' \end{aligned}$$

where

$$\alpha' = v_0 \cdots \hat{v}_i \cdots v_j u v_{j+1} \cdots v_n.$$

Hence, the term containing α' in (4.3) is

$$\begin{aligned} \overline{V}((-1)^i d_i(\alpha)) - (-1)^{j+1} \langle \partial\gamma, \alpha \rangle \alpha' &= -(-1)^{i+j} \alpha' - (-1)^{j+1} (-1)^i \alpha' \\ &= (-1)^{i+j+1} \alpha' - (-1)^{i+j+1} \alpha' \\ &= 0. \end{aligned}$$

SUBCASE 2.2. $j + 1 \leq i \leq n$. Then

$$\begin{aligned}\overline{V}(d_i\alpha) &= -\langle \partial\alpha'', d_i\alpha \rangle \alpha'' \\ &= -(-1)^{j+1}\alpha''\end{aligned}$$

where

$$\alpha'' = v_0 \cdots v_j u v_{j+1} \cdots \hat{v}_i \cdots v_n.$$

Hence, the term containing α'' in (4.3) is

$$\begin{aligned}\overline{V}((-1)^i d_i(\alpha)) - (-1)^{j+1} \langle \partial\gamma, \alpha'' \rangle \alpha'' &= -(-1)^i (-1)^{j+1} \alpha'' - (-1)^{j+1} (-1)^{i+1} \alpha'' \\ &= (-1)^{i+j+2} \alpha'' - (-1)^{i+j+2} \alpha'' \\ &= 0.\end{aligned}$$

Combing Subcase 2.1 and Subcase 2.2, we have that all terms of $\overline{V}\partial(\alpha)$ are cancelled out with terms of $\partial\overline{V}(\alpha)$ and there is only one item left in $\overline{V}\partial(\alpha) + \partial\overline{V}(\alpha)$. Specifically,

$$\begin{aligned}\overline{\Phi}(\alpha) &= \alpha + \overline{V}\partial(\alpha) + \partial\overline{V}(\alpha) \\ &= \alpha + (-(-1)^{j+1}(-1)^{j+1})\alpha \\ &= 0.\end{aligned}$$

Therefore, the assertion follows. □

Next, we give the characterization of the $\overline{\Phi}$ -invariant set of transitive digraphs.

Proposition 4.6. *Let G be a transitive digraph and f a discrete Morse function on G . Then*

$$P_*^{\overline{\Phi}}(G) = R(\alpha + \overline{V}\partial(\alpha))$$

where α is critical in G .

Proof. We divide the proof into the following two steps.

Step 1. We prove that $R(\alpha + \overline{V}\partial(\alpha)) \subseteq P_*^{\overline{\Phi}}(G)$.

Let α is a critical n -path on G . Then $\overline{V}(\alpha) = 0$ and

$$\overline{\Phi}(\alpha) = \alpha + \overline{V}\partial(\alpha).$$

Consider the following cases according to the value of $\overline{V}\partial(\alpha)$.

CASE 1. β is critical for any $\beta < \alpha$. Then $\overline{V}\partial(\alpha) = 0$. Hence,

$$\begin{aligned}\overline{\Phi}(\alpha + \overline{V}\partial(\alpha)) &= \overline{\Phi}(\alpha) \\ &= \alpha + \overline{V}\partial(\alpha)\end{aligned}$$

which implies that $(\alpha + \overline{V}\partial(\alpha)) \in P_*^{\overline{\Phi}}(G)$.

CASE 2. There exists an allowed elementary path $\beta' < \alpha$ such that β' is not critical. Then $n \geq 1$.

Suppose

$$\alpha = v_0 v_1 \cdots v_n$$

and $\beta' = d_i \alpha$. There are two subcases.

SUBCASE 2.1. $n > 1$. Then by Lemma 4.2, β' is unique and $\bar{V}(\beta') = -\langle \partial \alpha', \beta' \rangle \alpha'$ where

$$\alpha' = v_0 \cdots v_{i-1} v'_i v_{i+1} \cdots v_n, \quad v'_i \neq v_i, \quad f(v'_i) = 0.$$

Notice that,

$$\langle \partial \alpha, \beta' \rangle = \langle \partial \alpha', \beta' \rangle.$$

Hence,

$$\begin{aligned} \bar{V} \partial(\alpha') &= \bar{V}(\langle \partial \alpha', \beta' \rangle \beta') \\ &= -\langle \partial \alpha', \beta' \rangle^2 \alpha' \\ &= -\alpha' \end{aligned}$$

and

$$\begin{aligned} \bar{\Phi}(\alpha + \bar{V} \partial(\alpha)) &= \bar{\Phi}(\alpha + \langle \partial \alpha, \beta' \rangle \bar{V}(\beta')) \\ &= \bar{\Phi}(\alpha - \alpha') \\ &= \alpha - \alpha' + \bar{V} \partial(\alpha - \alpha') \\ &= \alpha - \alpha' \\ &= \alpha + \bar{V} \partial(\alpha). \end{aligned}$$

SUBCASE 2.2. $n = 1$. Then by Lemma 4.4, we have that

$$\bar{\Phi}(\alpha + \bar{V} \partial(\alpha)) = \alpha + \bar{V} \partial(\alpha).$$

Combining Case 1 and Case 2, we have that

$$(\alpha + \bar{V} \partial(\alpha)) \in P_*^{\bar{\Phi}}(G)$$

for any $\alpha \in \text{Crit}(G)$.

Step 2. We prove that $P_*^{\bar{\Phi}}(G) \subseteq R(\alpha + \bar{V} \partial(\alpha))$.

Let $x = \sum_{i=1}^m a_i \alpha_i \in P_*^{\bar{\Phi}}(G)$ where $a_i \neq 0$ for $1 \leq i \leq m$ and $\alpha_1, \dots, \alpha_m$ are distinct allowed elementary n -paths on G . Consider the following cases.

CASE 3. Each $\alpha_i, 1 \leq i \leq m$ is not critical. That is, x is a formal linear combination of non-critical paths on G . Then by Lemma 4.5, $\bar{\Phi}(x) = 0$ which contradicts $x \in P_*^{\bar{\Phi}}(G)$. Hence, this case does not hold.

CASE 4. Each $\alpha_i, 1 \leq i \leq m$ is critical. Consider the following two subcases.

SUBCASE 4.1. $\bar{V}\partial(\alpha_i) = 0$ for any α_i . Then

$$x = \sum_{i=1}^m a_i \alpha_i = \sum_{i=1}^m a_i (\alpha_i + \bar{V}\partial(\alpha_i))$$

and by Case 1 of Step 1, we have that $\bar{\Phi}(x) = x$.

SUBCASE 4.2. There exists some α_i such that $\bar{V}\partial(\alpha_i) \neq 0$. Since $\bar{V}\partial(\alpha_i)$ can not be cancelled out by any critical path and $\bar{\Phi}(x) = x$, it follows that there must exist some $\alpha_j, 1 \leq j \neq i \leq m$ such that $\bar{V}\partial(\alpha_j) \neq 0$ and $\bar{V}\partial(\alpha_i)$ is cancelled out by $\bar{V}\partial(\alpha_j)$. Specifically,

- $n > 1$. By Lemma 4.2, there exists an unique allowed elementary path β on G such that $\beta < \alpha_i$ and $\bar{V}(\beta) \neq 0$. Then

$$\begin{aligned} \bar{V}\partial(\alpha_i) &= \bar{V}(\langle \partial\alpha_i, \beta \rangle \beta) \\ &= -\langle \partial\alpha'_i, \beta \rangle \langle \partial\alpha'_i, \beta \rangle \alpha'_i \\ &= -\alpha'_i \end{aligned}$$

where $\alpha'_i > \beta$ and $f(\alpha'_i) = f(\beta)$.

Similarly,

$$\begin{aligned} \bar{V}\partial(\alpha_j) &= \bar{V}(\langle \partial\alpha_j, \beta' \rangle \beta') \\ &= -\langle \partial\alpha_j, \beta' \rangle \langle \partial\alpha'_j, \beta' \rangle \alpha'_j \\ &= -\alpha'_j \end{aligned}$$

where β' is the unique allowed elementary path on G such that $\beta' < \alpha_j$ and $\bar{V}(\beta') \neq 0$, and $\alpha'_j > \beta'$ and $f(\alpha'_j) = f(\beta')$.

Since $\bar{\Phi}(x) = x$, it follows that $a_i \bar{V}\partial(\alpha_i) + a_j \bar{V}\partial(\alpha_j) = 0$. Hence, $\alpha'_i = \alpha'_j$ and $a_i = -a_j$ (In fact, $\beta = \beta'$). Thus,

$$\begin{aligned} a_i(\alpha_i + \bar{V}\partial(\alpha_i)) + a_j(\alpha_j + \bar{V}\partial(\alpha_j)) &= a_i \alpha_i - a_i \alpha'_i + a_j \alpha_j - a_j \alpha'_j \\ &= a_i \alpha_i + a_j \alpha_j. \end{aligned} \tag{4.4}$$

Therefore, by Subcase 4.1 of Step 2 and (4.4),

$$\begin{aligned} x &= \bar{\Phi}(x) \\ &= \bar{\Phi}\left(\sum_{i=1}^m a_i \alpha_i\right) \\ &= \bar{\Phi}\left(\sum_{i=1}^m a_i (\alpha_i + \bar{V}\partial(\alpha_i))\right). \end{aligned}$$

By Step 1,

$$\bar{\Phi}\left(\sum_{i=1}^m a_i (\alpha_i + \bar{V}\partial(\alpha_i))\right) = \sum_{i=1}^m a_i (\alpha_i + \bar{V}\partial(\alpha_i))$$

which implies that x can be written as a formal linear combination of $\alpha + \bar{V}\partial(\alpha)$ where $\alpha \in \text{Crit}(G)$.

- $n=1$. Suppose $\alpha_i = v_0v_1$ and $\beta = v_0$ is not critical. By Lemma 4.4, there exists a unique allowed elementary path $\alpha'_i = v_0v$ such that $\alpha'_i > \beta$ and $f(\alpha'_i) = f(\beta)$. That is,

$$\overline{V}(\beta) = v_0v.$$

Since $\overline{\Phi}(x) = x$, it follows that $\overline{V}\partial(\alpha_i)$ must be cancelled out by some $\overline{V}\partial(\alpha_j)$, $1 \leq j \neq i \leq m$. Let $\alpha_j = u_0u_1$. Then $\beta' = u_0$ is not critical and

$$\overline{V}(\beta') = u_0u$$

where $\alpha'_j = u_0u > \beta'$ and $f(\alpha'_j) = f(\beta')$. Moreover, $v_0 = u_0$, $v = u$ and $a_i = -a_j$. Similarly, if $\beta'' = u_1$ is not critical, then there must exist another critical path α_k ($k \neq i, j$) in x such that β'' is cancelled out by $\overline{V}\partial(\alpha_k)$. Hence, by finite steps, we can have that

$$\sum_{i=1}^m a_i \overline{V}\partial(\alpha_i) = 0.$$

Therefore,

$$\begin{aligned} x &= \sum_{i=1}^m a_i \alpha_i \\ &= \sum_{i=1}^m a_i \alpha_i + \sum_{i=1}^m a_i \overline{V}\partial(\alpha_i) \\ &= \sum_{i=1}^m a_i (\alpha_i + \overline{V}\partial(\alpha_i)). \end{aligned}$$

CASE 5. Some α_i are critical and some are not critical. Without loss of generality, α_1 is not critical. Then $\overline{\Phi}(\alpha_1) = 0$. Since all α_i are distinct, α_1 can not be cancelled out by any critical path and $x \in P_*^{\overline{\Phi}}(G)$, it follows that there must exist some critical path α_i ($2 \leq i \leq m$) such that $\overline{V}\partial(\alpha_i) = \alpha_1$ and $a_1 = a_i$. Thus,

$$a_1 \alpha_1 + a_i \alpha_i = a_i (\alpha_i + \overline{V}\partial(\alpha_i)). \quad (4.5)$$

Moreover, by (4.5) and Step 1,

$$\begin{aligned} \overline{\Phi}(a_1 \alpha_1 + a_i \alpha_i) &= a_i \overline{\Phi}(\alpha_i + \overline{V}\partial(\alpha_i)) \\ &= a_i (\alpha_i + \overline{V}\partial(\alpha_i)) \\ &= a_1 \alpha_1 + a_i \alpha_i. \end{aligned}$$

Hence, any non critical path α_j in x can be written as $\overline{V}\partial(\alpha_i)$ where α_i is critical ($1 \leq i \neq j \leq n$). And after removing such non-critical paths α_j and the corresponding critical paths α_i from x , the rest is a formal linear combination of critical paths which is invariant under $\overline{\Phi}$, denoted as x' . Therefore, by Case 4 of Step 2, x' can be written as a formal linear combination of $\alpha + \overline{V}\partial(\alpha)$ where $\alpha \in \text{Crit}(G)$. So is x .

Combining Case 3, Case 4 and Case 5, we have that for any $x \in P_*^{\bar{\Phi}}(G)$, it can be written as a linear combination of $\alpha + \bar{V}\partial(\alpha)$ where $\alpha \in \text{Crit}(G)$.

Therefore, by Step 1 and Step 2, the proposition is proved. \square

Finally, we give a description of path homology groups of digraphs with critical paths of transitive closure.

Theorem 4.7. (cf. [18, Corollary 2.16]) *Let G be a digraph and f a discrete Morse function on G . Let \bar{f} be the extension of f on \bar{G} and $\bar{V} = \text{grad}\bar{f}$ the discrete gradient vector field on \bar{G} . Suppose $\Omega_*(G)$ is \bar{V} -invariant ($\bar{V}(\Omega_*(G)) \subseteq \Omega_*(G)$). Then*

$$H_m(G; R) \cong H_m(\Omega_*^{\bar{\Phi}}(G)), m \geq 0$$

where

$$\Omega_*^{\bar{\Phi}}(G) = \Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G}).$$

By Proposition 4.6 and Theorem 4.7, we have that

Theorem 4.8. *Let G be a digraph and f a discrete Morse function on G . Let \bar{f} be the extension of f on \bar{G} and $\bar{V} = \text{grad}\bar{f}$ the discrete gradient vector field on \bar{G} . Suppose $\Omega_*(G)$ is \bar{V} -invariant. Then*

$$H_m(G; R) \cong H_m(\{R(\alpha + \bar{V}\partial(\alpha)) \cap \Omega_n(G), \partial_n\}_{n \geq 0}).$$

where $\alpha \in \text{Crit}_n(\bar{G})$.

We give an example to illustrate Theorem 4.8.

Example 4.9. (cf. [18, Example 3.2]) *Let G be a square as follows and \bar{G} the transitive closure of G . Let f be a function on G with*

$$f(v_1) = 0, f(v_i) > 0 \text{ for } i = 0, 2, 3.$$

It is easy to verify that f is a discrete Morse function on G and f can be extended to be a

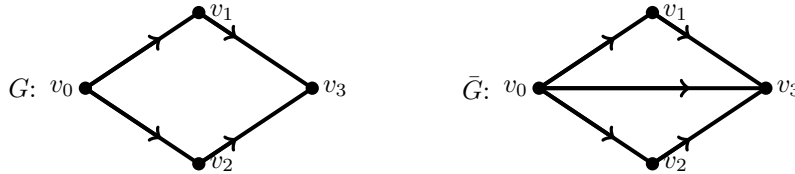


Figure 1: Example 4.9.

discrete Morse function \bar{f} on \bar{G} . Then

$$\begin{aligned} P_*(\bar{G}) &= \{v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_0v_3, v_1v_3, v_2v_3, v_0v_1v_3, v_0v_2v_3\}, \\ \Omega_*(G) &= \{v_0, v_1, v_2, v_3, v_0v_1, v_0v_2, v_1v_3, v_2v_3, v_0v_1v_3 - v_0v_2v_3\}, \\ \text{Crit}_*(\bar{G}) &= \{v_1, v_2, v_0v_2, v_2v_3, v_0v_2v_3\}. \end{aligned}$$

Let $\bar{V} = \text{grad}\bar{f}$ be the discrete gradient vector field on \bar{G} and $\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial$ the discrete gradient flow of \bar{G} . Then

$$\begin{aligned}\bar{V}(v_0) &= v_0v_1, & \bar{V}(v_3) &= -v_1v_3, \\ \bar{V}(v_0v_3) &= v_0v_1v_3, \\ \bar{V}(\alpha) &= 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}.\end{aligned}$$

and

$$\begin{aligned}\bar{\Phi}(v_0) &= v_1, & \bar{\Phi}(v_1) &= v_1, \\ \bar{\Phi}(v_2) &= v_2, & \bar{\Phi}(v_3) &= v_1, \\ \bar{\Phi}(v_0v_1) &= 0, & \bar{\Phi}(v_0v_2) &= v_0v_2 - v_0v_1, \\ \bar{\Phi}(v_1v_3) &= 0, & \bar{\Phi}(v_2v_3) &= v_2v_3 - v_1v_3, \\ \bar{\Phi}(v_0v_3) &= 0, & \bar{\Phi}(v_0v_1v_3) &= 0, \\ \bar{\Phi}(v_0v_2v_3) &= v_0v_2v_3 - v_0v_1v_3.\end{aligned}$$

Hence,

$$\bar{V}(\Omega_n(G)) \subseteq \Omega_{n+1}(G), \quad n \geq 0.$$

and

$$\begin{aligned}R(\{\alpha + \bar{V}\partial(\alpha) \mid \alpha \in \text{Crit}(\bar{G})\}) &= P_*^{\bar{\Phi}}(\bar{G}) \\ &= R(v_1, v_2, v_0v_2 - v_0v_1, v_2v_3 - v_1v_3, v_0v_2v_3 - v_0v_1v_3).\end{aligned}$$

Therefore,

$$\begin{aligned}\partial(v_0v_2 - v_0v_1) &= v_2 - v_1, & \partial(v_2v_3 - v_1v_3) &= v_1 - v_2, \\ \partial(v_0v_2v_3 - v_0v_1v_3) &= (v_0v_2 - v_0v_1) + (v_2v_3 - v_1v_3),\end{aligned}$$

and

$$H_0(\{R(\alpha + \bar{V}\partial(\alpha)) \cap \Omega_n(G), \partial_n\}_{n \geq 0}) = R, \quad (4.6)$$

$$H_m(\{R(\alpha + \bar{V}\partial(\alpha)) \cap \Omega_n(G), \partial_n\}_{n \geq 0}) = 0 \text{ for } m \geq 1, \quad (4.7)$$

where $\alpha \in \text{Crit}_n(\bar{G})$. (4.6) and (4.7) are consistent with the path homology groups $H_m(G; R)$ ($m \geq 0$) given in [10, Proposition 4.7].

4.2 Description of Path Homology of Digraphs

In this section, we prove that path homology of digraph G is isomorphic to the homology of chain complex consisting of the formal linear combinations of paths which are not only critical paths of the transitive closure \bar{G} but also allowed elementary paths of G .

Firstly, we prove an isomorphism of graded R -modules in the following theorem.

Theorem 4.10. *Let G be a digraph and \bar{G} the transitive closure of G . Let \bar{f} be a discrete Morse function on \bar{G} . Suppose $\bar{\Phi}(\alpha) \in \Omega(G)$ for any $\alpha \in \text{Crit}(\bar{G}) \cap P(G)$ where \bar{V} is the discrete gradient vector field on \bar{G} and $\bar{\Phi}$ is the discrete gradient flow of \bar{G} . Then*

$$\bar{\Phi}^\infty|_{\text{Crit}_n(\bar{G}) \cap P_n(G)}: \text{Crit}_n(\bar{G}) \cap P_n(G) \longrightarrow P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G), \quad n \geq 0 \quad (4.8)$$

is a graded R -module isomorphism.

Proof. Let $\alpha \in \text{Crit}_n(\bar{G}) \cap P_n(G)$. By Step 1 of Proposition 4.6, we have that

$$\bar{\Phi}(\alpha) = (\alpha + \bar{V}\partial(\alpha)) \in P_n^{\bar{\Phi}}(\bar{G}).$$

Then

$$\bar{\Phi}^\infty(\alpha) = (\alpha + \bar{V}\partial(\alpha)) \in P_n^{\bar{\Phi}}(\bar{G}). \quad (4.9)$$

Since

$$\bar{\Phi}(\alpha) \in \Omega_n(G),$$

it follows that

$$\bar{\Phi}^\infty(\alpha) = \bar{\Phi}(\alpha) \in \Omega_n(G).$$

Hence,

$$\bar{\Phi}^\infty(\alpha) \in P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G)$$

which implies that (4.8) is well-defined.

Suppose $\alpha', \alpha'' \in \text{Crit}_n(\bar{G}) \cap P_n(G)$ where α', α'' are distinct. Then by (4.9),

$$\begin{aligned} \bar{\Phi}^\infty(\alpha') &= \alpha' + \bar{V}\partial(\alpha') \\ \bar{\Phi}^\infty(\alpha'') &= \alpha'' + \bar{V}\partial(\alpha''). \end{aligned}$$

We assert that

$$\bar{\Phi}^\infty(\alpha') \neq \bar{\Phi}^\infty(\alpha'').$$

Suppose to the contrary,

$$\alpha' + \bar{V}\partial(\alpha') = \alpha'' + \bar{V}\partial(\alpha''). \quad (4.10)$$

Let α be an arbitrary allowed elementary n -path on \bar{G} . Then either $\bar{V}(\alpha) = 0$ or $\bar{V}(\alpha) = \gamma$ where γ is an allowed elementary $(n+1)$ -path on \bar{G} such that $\gamma > \alpha$ and $\bar{f}(\gamma) = \bar{f}(\alpha)$. Hence, both $\bar{V}\partial(\alpha')$ and $\bar{V}\partial(\alpha'')$ are either equal to 0 or formal linear combinations of non-critical paths on \bar{G} . Then by (4.10), $\alpha = \alpha'$ which contradicts that α and α' are distinct. Thus, (4.8) is a monomorphism.

Moreover, by Proposition 4.6, we know that

$$P_n^{\bar{\Phi}}(\bar{G}) = R(\alpha + \bar{V}\partial(\alpha))$$

where $\alpha \in \text{Crit}_n(\bar{G})$. Let $x = \sum_{i=1}^m a_i(\alpha_i + \bar{V}\partial(\alpha_i)) \in P_n^{\bar{\Phi}}(\bar{G})$ where $a_i \in R$ and $\alpha_i \in \text{Crit}(\bar{G})$. For any $1 \leq j \neq k \leq m$, since the pairs

$$\begin{array}{ccc} \alpha_j \text{ and } \alpha_k & \alpha_j \text{ and } \bar{V}\partial(\alpha_j) & \alpha_j \text{ and } \bar{V}\partial(\alpha_k) \\ \alpha_k \text{ and } \bar{V}\partial(\alpha_j) & \alpha_k \text{ and } \bar{V}\partial(\alpha_k) & \end{array}$$

can not cancel out each other in x , it follows that if $x \in P_n(G)$, then $\alpha_i \in P_n(G)$ for each $1 \leq i \leq m$. Hence,

$$P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G) \subseteq \{x = \sum_{i=1}^m a_i(\alpha_i + \bar{V}\partial(\alpha_i)) \mid \alpha_i \in \text{Crit}_n(\bar{G}) \cap P_n(G), \partial x \in P(G)\}.$$

Furthermore, since

$$\bar{\Phi}(\alpha) = (\alpha + \bar{V}\partial(\alpha)) \in \Omega(G)$$

and by Proposition 4.6,

$$\bar{\Phi}(\alpha) \in P_*^{\bar{\Phi}}(\bar{G})$$

where $\alpha \in \text{Crit}(\bar{G}) \cap P(G)$, it follows that

$$P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G) \supseteq \{x = \sum_{i=1}^m a_i(\alpha_i + \bar{V}\partial(\alpha_i)) \mid \alpha_i \in \text{Crit}_n(\bar{G}) \cap P_n(G)\}.$$

Hence,

$$P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G) = \{x = \sum_{i=1}^m a_i(\alpha_i + \bar{V}\partial(\alpha_i)) \mid \alpha_i \in \text{Crit}_n(\bar{G}) \cap P_n(G)\}.$$

That is, $P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G)$ is a R -module of all the formal linear combinations of elements in the form

$$\alpha + \bar{V}\partial(\alpha)$$

where $\alpha \in \text{Crit}_n(\bar{G}) \cap P_n(G)$. This implies that (4.8) is an epimorphism. Therefore, (4.8) is an isomorphism. \square

Next, we give an isomorphism of homology groups.

Corollary 4.11. *Let G be a digraph and \bar{G} the transitive closure of G . Suppose $\Omega_*(G)$ is \bar{V} -invariant and $\bar{\Phi}(\alpha) \in \Omega(G)$ for any $\alpha \in \text{Crit}(\bar{G}) \cap P(G)$ where \bar{V} is the discrete gradient vector field on \bar{G} and $\bar{\Phi}$ is the discrete gradient flow of \bar{G} , respectively. Then*

$$H_m(\{\text{Crit}_n(\bar{G}) \cap P_n(G), \tilde{\partial}_n\}_{n \geq 0}) \cong H_m(G; R) \quad (4.11)$$

where $\tilde{\partial} = (\bar{\Phi}^\infty)^{-1} \circ \partial \circ \bar{\Phi}^\infty$ and $\bar{\Phi}^\infty$ is the stabilization map of $\bar{\Phi}$.

Proof. Let $\alpha \in \text{Crit}_n(\bar{G}) \cap P_n(G)$. Since $\bar{\Phi}(\alpha) \in \Omega(G)$ for any $\alpha \in \text{Crit}(\bar{G}) \cap P(G)$, by the proof of Theorem 4.10, it follows that

$$\bar{\Phi}^\infty(\alpha) = \alpha + \bar{V}\partial(\alpha) \in P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G).$$

By [18, Theorem 2.14 (iii)], $\{P_*^{\bar{\Phi}}(\bar{G}) \cap \Omega_*(G), \partial_*\}$ is a chain complex. Then

$$\partial \bar{\Phi}^\infty(\alpha) \in P_{n-1}^{\bar{\Phi}}(\bar{G}) \cap \Omega_{n-1}(G).$$

By Theorem 4.10, we know that

$$\bar{\Phi}^\infty \big|_{\text{Crit}_*(\bar{G}) \cap P_*(G)}$$

is an isomorphism. Hence

$$\tilde{\partial}(\alpha) = (\bar{\Phi}^\infty)^{-1} \circ \partial \circ \bar{\Phi}^\infty(\alpha) \in \text{Crit}_{n-1}(\bar{G}) \cap P_{n-1}(G).$$

Thus, $\{\text{Crit}_n(\bar{G}) \cap P_n(G), \tilde{\partial}_n\}_{n \geq 0}$ is a chain complex.

Moreover, for each $n \geq 0$, $\bar{\Phi}^\infty \circ \tilde{\partial}_n = \partial_n \circ \bar{\Phi}^\infty$. That is, the following diagram

$$\begin{array}{ccc} \text{Crit}_n(\bar{G}) \cap P_n(G) & \xrightarrow{\tilde{\partial}_n} & \text{Crit}_{n-1}(\bar{G}) \cap P_{n-1}(G) \\ \downarrow \bar{\Phi}^\infty & & \downarrow \bar{\Phi}^\infty \\ P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n(G) & \xrightarrow{\partial_n} & P_{n-1}^{\bar{\Phi}}(\bar{G}) \cap \Omega_{n-1}(G) \end{array}$$

is commutative.

Hence, by Theorem 4.10,

$$H_m(\{\text{Crit}_n(\bar{G}) \cap P_n(G), \tilde{\partial}_n\}_{n \geq 0}) \cong H_m(\{P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n, \partial_n\}_{n \geq 0}). \quad (4.12)$$

By Theorem 4.7,

$$H_m(G; R) \cong H_m(\{P_n^{\bar{\Phi}}(\bar{G}) \cap \Omega_n, \partial_n\}_{n \geq 0}). \quad (4.13)$$

Therefore, by (4.12) and (4.13),

$$H_m(\{\text{Crit}_n(\bar{G}) \cap P_n(G), \tilde{\partial}_n\}_{n \geq 0}) \cong H_m(G; R).$$

□

Finally, we give some examples to illustrate Corollary 4.11.

Example 4.12. (cf. [18, Example 3.5]) Consider the following digraph G and its transitive closure \bar{G} . Let $f : V(G) \rightarrow [0, +\infty)$ be a function on G with $f(v_1) = 0$ and $f(v_i) > 0$, $0 \leq i \leq 5, i \neq 1$. It's easy to verify that f can be extended to be a Morse function \bar{f} on \bar{G} such that $\bar{f}(v) = f(v)$ for all vertices $v \in V(G)$. Then

$$\begin{aligned} P(G) &= \{v_0, v_1, v_2, v_3, v_4, v_5, v_0v_1, v_0v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, \\ &\quad v_5v_3, v_5v_4, v_0v_1v_3, v_0v_1v_4, v_0v_2v_3, v_0v_2v_4\} \\ \Omega(G) &= \{v_0, v_1, v_2, v_3, v_4, v_5, v_0v_1, v_0v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, \\ &\quad v_5v_3, v_5v_4, v_0v_1v_3 - v_0v_2v_3, v_0v_1v_4 - v_0v_2v_4\} \\ \text{Crit}(\bar{G}) &= \{v_1, v_2, v_5, v_0v_2, v_2v_3, v_2v_4, v_5v_3, v_5v_4, v_0v_2v_3, v_0v_2v_4\} \\ \text{Crit}(\bar{G}) \cap P(G) &= \{v_1, v_2, v_5, v_0v_2, v_2v_3, v_2v_4, v_5v_3, v_5v_4, v_0v_2v_3, v_0v_2v_4\}. \end{aligned}$$

Let $\bar{V} = \text{grad} \bar{f}$ be the discrete gradient vector field on \bar{G} . Then

$$\bar{V}(v_0) = v_0 v_1, \quad \bar{V}(v_3) = -v_1 v_3, \quad \bar{V}(v_4) = -v_1 v_4,$$

$$\bar{V}(v_0 v_3) = v_0 v_1 v_3, \quad \bar{V}(v_0 v_4) = v_0 v_1 v_4,$$

$$\bar{V}(\alpha) = 0 \text{ for any other allowed elementary path } \alpha \text{ on } \bar{G}.$$

Hence $\Omega(G)$ is \bar{V} -invariant. Let $\bar{\Phi} = \text{Id} + \partial \bar{V} + \bar{V} \partial$ be the discrete gradient flow of \bar{G} . Then

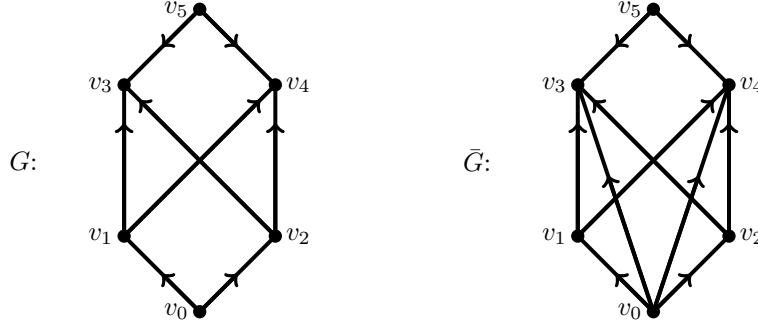


Figure 2: Example 4.12.

$$\bar{\Phi}(v_0) = v_1, \quad \bar{\Phi}(v_1) = v_1,$$

$$\bar{\Phi}(v_2) = v_2, \quad \bar{\Phi}(v_3) = v_1,$$

$$\bar{\Phi}(v_4) = v_1, \quad \bar{\Phi}(v_5) = v_5,$$

$$\bar{\Phi}(v_0 v_1) = 0, \quad \bar{\Phi}(v_0 v_2) = v_0 v_2 - v_0 v_1,$$

$$\bar{\Phi}(v_0 v_3) = 0, \quad \bar{\Phi}(v_0 v_4) = 0,$$

$$\bar{\Phi}(v_1 v_3) = 0, \quad \bar{\Phi}(v_1 v_4) = 0,$$

$$\bar{\Phi}(v_0 v_1 v_3) = 0, \quad \bar{\Phi}(v_2 v_4) = v_2 v_4 - v_1 v_4,$$

$$\bar{\Phi}(v_0 v_1 v_4) = 0, \quad \bar{\Phi}(v_5 v_4) = v_5 v_4 - v_1 v_4,$$

$$\bar{\Phi}(v_2 v_3) = v_2 v_3 - v_1 v_3, \quad \bar{\Phi}(v_0 v_2 v_3) = v_0 v_2 v_3 - v_0 v_1 v_3,$$

$$\bar{\Phi}(v_5 v_3) = v_5 v_3 - v_1 v_3, \quad \bar{\Phi}(v_0 v_2 v_4) = v_0 v_2 v_4 - v_0 v_1 v_4,$$

and for any $\alpha \in \text{Crit}(\bar{G}) \cap P(G)$, $\bar{\Phi}(\alpha) \in \Omega(G)$.

By calculate directly, we have that $\bar{\Phi}^\infty = \bar{\Phi}$. Then

$$\begin{aligned}
\tilde{\partial}(v_0v_2) &= v_2 - v_1, \\
\tilde{\partial}(v_2v_3) &= v_1 - v_2, \\
\tilde{\partial}(v_2v_4) &= v_1 - v_2, \\
\tilde{\partial}(v_5v_3) &= v_1 - v_5, \\
\tilde{\partial}(v_5v_4) &= v_1 - v_5, \\
\tilde{\partial}(v_0v_2v_3) &= v_0v_2 + v_2v_3, \\
\tilde{\partial}(v_0v_2v_4) &= v_0v_2 + v_2v_4.
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_0(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= R \\
H_1(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= R \\
H_m(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= 0 \text{ for } m \geq 2,
\end{aligned}$$

which are consistent with the path homology groups of G .

By [18, Remark 3.6], the condition that $\Omega(G)$ is \bar{V} -invariant in Theorem 4.7 is sufficient but not necessary. The following example will show us that the condition

$$\bar{\Phi}(\text{Crit}(\bar{G}) \cap P(G)) \subseteq \Omega(G) \tag{4.14}$$

in Corollary 4.11 is also sufficient but not necessary.

Example 4.13. (cf. [18, Example 3.5]) We still consider the digraph G in Example 4.12. Different from Example 4.12, we firstly define f with $f(v_0) = 0$ and $f(v_i) > 0$, $0 < i \leq 5$. It is easy to verify that f is a discrete Morse function on G and it can be extended to be a discrete Morse function \bar{f} on \bar{G} . By [18, Example 3.5], we have that

$$\begin{aligned}
\text{Crit}(\bar{G}) &= \{v_0, v_5, v_5v_3, v_5v_4\}, \\
\text{Crit}(\bar{G}) \cap P(G) &= \{v_0, v_5, v_5v_3, v_5v_4\}, \\
\bar{\Phi}(v_5v_3) &= (v_5v_3 - v_0v_3) \notin \Omega(G), \\
\bar{\Phi}(v_5v_4) &= (v_5v_4 - v_0v_4) \notin \Omega(G).
\end{aligned}$$

Hence, (4.14) does not hold.

On the other hand,

$$\begin{aligned}
\tilde{\partial}(v_5v_3) &= v_0 - v_5, \\
\tilde{\partial}(v_5v_4) &= v_0 - v_5.
\end{aligned}$$

Therefore,

$$\begin{aligned} H_0(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= R \\ H_1(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= R \\ H_m(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= 0 \text{ for } m \geq 2, \end{aligned}$$

which are consistent with the path homology groups of G .

Example 4.14. Consider the following digraph G and its transitive closure \bar{G} . Let $f : V(G) \rightarrow [0, +\infty)$ be a function on G with $f(v_0) = 0$ and $f(v_i) > 0, 0 < i \leq 3$. Then f is a discrete Morse function on G which can be extended to \bar{G} . Hence,

$$\begin{aligned} \Omega(G) &= \{v_0, v_1, v_2, v_3, v_0v_1, v_1v_2, v_2v_3, v_0v_3\} \\ \text{Crit}(\bar{G}) &= \{v_0\} \\ \text{Crit}(\bar{G}) \cap P(G) &= \{v_0\} \end{aligned}$$

and

$$\bar{V}(v_2v_3) = -v_0v_2v_3 \notin \Omega(G) \quad \bar{V}(v_1v_2) = -v_0v_1v_2 \notin \Omega(G).$$

However,

$$\begin{aligned} H_0(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= R, \\ H_m(\{\text{Crit}(\bar{G}) \cap P(G), \tilde{\partial}\}) &= 0 \text{ for } m > 0 \end{aligned}$$

which are not consistent with the path homology groups of G (cf. [10, Proposition 4.7]).

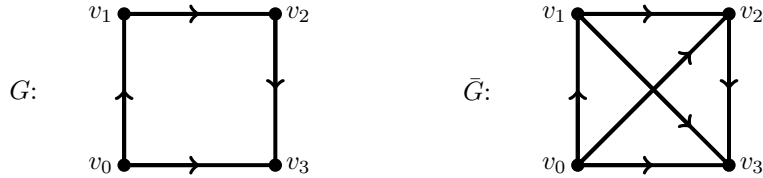


Figure 3: Example 4.14.

Remark 4.15. By Example 4.13 and Example 4.14, we know that if a digraph does not satisfy the conditions in Corollary 4.11, then the isomorphism of homology groups given in (4.11) may or not hold.

Generally speaking, $\text{Crit}_*(\bar{G}) \cap P_*(G)$ in Theorem 4.10 and Corollary 4.11 can not be replaced by $\text{Crit}_*\bar{G} \cap \Omega_*(G)$ which will be illustrated by the following example.

Example 4.16. Consider the digraph given in Example 4.9. Then

$$\text{Crit}_*(\bar{G}) \cap \Omega_*(G) = \{v_1, v_2, v_0v_2, v_2v_3\}.$$

By [18, Example 3.2],

$$\begin{aligned}
\bar{\Phi}(v_0) &= v_1, & \bar{\Phi}(v_1) &= v_1, \\
\bar{\Phi}(v_2) &= v_2, & \bar{\Phi}(v_3) &= v_1, \\
\bar{\Phi}(v_0v_1) &= 0, & \bar{\Phi}(v_0v_2) &= v_0v_2 - v_0v_1, \\
\bar{\Phi}(v_1v_3) &= 0, & \bar{\Phi}(v_2v_3) &= v_2v_3 - v_1v_3, \\
\bar{\Phi}(v_0v_3) &= 0, & \bar{\Phi}(v_0v_1v_3) &= 0, \\
\bar{\Phi}(v_0v_2v_3) &= v_0v_2v_3 - v_0v_1v_3
\end{aligned}$$

where $\bar{\Phi} = \text{Id} + \partial\bar{V} + \bar{V}\partial$ is the discrete gradient flow of \bar{G} . By calculate directly, we have that $\bar{\Phi}^\infty = \bar{\Phi}$. Then

$$P_*^{\bar{\Phi}}(\bar{G}) \cap \Omega_*(G) = \{v_1, v_2, v_0v_2 - v_0v_1, v_2v_3 - v_1v_3, v_0v_2v_3 - v_0v_1v_3\}.$$

Hence, $\text{Crit}_*(\bar{G}) \cap \Omega_*(G)$ and $P_*^{\bar{\Phi}}(\bar{G}) \cap \Omega_*(G)$ can not be isomorphic. Moreover,

$$\begin{aligned}
\tilde{\partial}(v_0v_2) &= v_2 - v_1, \\
\tilde{\partial}(v_2v_3) &= v_1 - v_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
H_0(\{\text{Crit}(\bar{G}) \cap \Omega(G), \tilde{\partial}\}) &= R, \\
H_1(\{\text{Crit}(\bar{G}) \cap \Omega(G), \tilde{\partial}\}) &= R, \\
H_m(\{\text{Crit}(\bar{G}) \cap \Omega(G), \tilde{\partial}\}) &= 0 \text{ for } m \geq 2,
\end{aligned}$$

which are not consistent with the path homology groups of G given in [10, Proposition 4.7].

5 Morse Inequalities

In this section, we will give the Morse inequalities on digraphs by Corollary 4.11.

Given a chain complex

$$0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where C_p is a finite dimensional vector space over R for each $0 \leq p \leq n$. Consider its homology

$$H_p(C_*; R) = \text{Ker}\partial_p / \text{Im}\partial_{p+1}.$$

Then

$$\begin{aligned}
\dim C_p &= \dim \text{Ker}\partial_p + \dim \text{Im}\partial_p, \\
\dim H_p(C_*; R) &= \dim \text{Ker}\partial_p - \dim \text{Im}\partial_{p+1} \\
&= \dim C_p - \dim \text{Im}\partial_p - \dim \text{Im}\partial_{p+1}.
\end{aligned} \tag{5.1}$$

Hence

$$\dim C_p \geq \dim H_p(C_*; R) \quad (5.2)$$

and

$$\begin{aligned} & (\dim C_p - \dim H_p(C_*; R)) - (\dim C_{p-1} - \dim H_{p-1}(C_*; R)) + \cdots \\ & + (-1)^p (\dim C_0 - \dim H_0(C_*; R)) \\ & = (\dim \text{Im} \partial_p + \dim \text{Im} \partial_{p+1}) - (\dim \text{Im} \partial_{p-1} + \dim \text{Im} \partial_p) + \cdots \\ & + (-1)^p (\dim \text{Im} \partial_0 + \dim \text{Im} \partial_1) \\ & = \dim \text{Im} \partial_{p+1} + (-1)^p \dim \text{Im} \partial_0 \\ & = \dim \text{Im} \partial_{p+1} \geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} & \dim C_p - \dim C_{p-1} + \cdots + (-1)^p \dim C_0 \\ & \geq \dim H_p(C_*; R) - \dim H_{p-1}(C_*; R) + \cdots + (-1)^p \dim H_0(C_*; R). \end{aligned} \quad (5.3)$$

Moreover, by (5.1), the Euler characteristic of $\{C_*, \partial_*\}$ is

$$\begin{aligned} \chi &= \sum_{p=0}^n (-1)^p \dim H_p(C_*; R) \\ &= \sum_{p=0}^n (-1)^p \dim C_p. \end{aligned}$$

Hence

$$\begin{aligned} & \dim C_0 - \dim C_1 + \cdots + (-1)^n \dim C_n \\ & = \dim H_0(C_*; R) - \dim H_1(C_*; R) + \cdots + (-1)^n \dim H_n(C_*; R). \end{aligned} \quad (5.4)$$

Let G be a digraph and \bar{G} the transitive closure of G . Denote

$$\begin{aligned} b_m &= \dim H_m(G; R), \\ l_m &= \dim(\text{Crit}_m(\bar{G}) \cap P_m(G)), \\ L_m &= \dim \text{Crit}_m(G). \end{aligned}$$

Then we have the following theorem.

Theorem 5.1. *(Morse Inequalities on Digraphs) Let G be a digraph and \bar{G} the transitive closure of G . Suppose $\Omega_*(G)$ is \bar{V} -invariant and $\bar{\Phi}(\alpha) \in \Omega(G)$ for any $\alpha \in \text{Crit}(\bar{G}) \cap P(G)$ where \bar{V} is the discrete gradient vector field on \bar{G} and $\bar{\Phi}$ is the discrete gradient flow of \bar{G} , respectively. Then*

$$L_m \geq l_m, \quad (5.5)$$

$$l_m - l_{m-1} + \cdots \pm l_0 \geq b_m - b_{m-1} + \cdots \pm b_0 \quad (5.6)$$

and

$$l_0 - l_1 + \cdots \pm l_{\dim G} \geq b_0 - b_1 + \cdots \pm b_{\dim G} \quad (5.7)$$

where $m \geq 0$ and $\dim G = \max\{p \mid \Omega_p(G) \neq 0\}$.

Proof. By Corollary 4.11,

$$H_m(\{\text{Crit}_*(\bar{G}) \cap P_*(G), \bar{\partial}_*\}) \cong H_m(G; R).$$

By substituting C_m and $H_m(C_*; R)$ with $\text{Crit}_m(\bar{G}) \cap P_m(G)$ and $H_m(G; R)$ respectively and by (5.2), we have that

$$l_m \geq b_m.$$

By (5.3) and (5.4), we can obtain (5.6) and (5.7) respectively.

Moreover, since $G \subseteq \bar{G}$, by Definition 2.2, it follows that

$$\text{Crit}_*(\bar{G}) \cap P_*(G) \subseteq \text{Crit}_*(G).$$

Hence (5.5) is proved. \square

We take the digraphs and discrete Morse functions in Example 4.9 and Example 4.12 as examples to illustrate the above results in Theorem 5.1.

Example 5.2. *By Example 4.9, we have that*

$$\begin{aligned} \text{Crit}_*(\bar{G}) \cap P_*(G) &= \{v_1, v_2, v_0v_2, v_2v_3, v_0v_2v_3\}, \\ \text{Crit}_*(G) &= \{v_1, v_2, v_0v_2, v_2v_3, v_0v_1v_3, v_0v_2v_3\}. \end{aligned}$$

and

$$\begin{aligned} \bar{V}(\Omega(G)) &\subseteq \Omega(G) \\ \bar{\Phi}(\text{Crit}_*(\bar{G}) \cap P_*(G)) &\subseteq \Omega_*(G). \end{aligned}$$

Then

$$\begin{aligned} l_0 = 2, \quad l_1 = 2, \quad l_2 = 1, \quad l_i = 0 \text{ for } i > 2, \\ L_0 = 2, \quad L_1 = 2, \quad L_2 = 2, \quad L_i = 0 \text{ for } i > 2. \end{aligned}$$

and

$$b_0 = 1, \quad b_i = 0 \text{ for } i \geq 1.$$

Hence

$$\begin{aligned} L_m &\geq l_m \text{ for any } m \geq 0, \\ 2 &> 1 \text{ for } m = 0, \\ 2 - 2 &> 0 - 1 \text{ for } m = 1, \\ 1 - 2 + 2 &\geq 0 - 0 + 1 \text{ for } m \geq 2, \\ \chi(G) &= 2 - 2 + 1 = 1 - 0 + 0 = 1. \end{aligned}$$

Similarly, by Example 4.12, we have that

$$\begin{aligned}\text{Crit}_*(\bar{G}) \cap P_*(G) &= \{v_1, v_2, v_5, v_0v_2, v_2v_3, v_2v_4, v_5v_3, v_5v_4, v_0v_2v_3, v_0v_2v_4\}, \\ \text{Crit}_*(G) &= \{v_1, v_2, v_5, v_0v_2, v_2v_3, v_2v_4, v_5v_3, v_5v_4, v_0v_1v_3, v_0v_2v_3, v_0v_1v_4, v_0v_2v_4\}, \\ \bar{V}(\Omega(G)) &\subseteq \Omega(G), \\ \bar{\Phi}(\text{Crit}_*(\bar{G}) \cap P_*(G)) &\subseteq \Omega_*(G).\end{aligned}$$

Then

$$\begin{aligned}l_0 &= 3, \quad l_1 = 5, \quad l_2 = 2, \quad l_i = 0 \text{ for } i > 2, \\ L_0 &= 3, \quad L_1 = 5, \quad L_2 = 4, \quad L_i = 0 \text{ for } i > 2.\end{aligned}$$

and

$$b_0 = 1, \quad b_1 = 1, \quad b_i = 0 \text{ for } i > 1.$$

Hence

$$\begin{aligned}L_m &\geq l_m \text{ for any } m \geq 0, \\ 3 &> 1 \text{ for } m = 0, \\ 5 - 3 &> 1 - 1 \text{ for } m = 1, \\ 2 - 5 + 3 &\geq 0 - 1 + 1 \text{ for } m \geq 2, \\ \chi(G) &= 3 - 5 + 2 = 1 - 1 = 0.\end{aligned}$$

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