A NONLOCAL STOKES SYSTEM WITH VOLUME CONSTRAINTS

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Abstract. In this paper, we introduce a nonlocal model for linear steady Stokes system with physical no-slip boundary condition. We use the idea of volume constraint to enforce the no-slip boundary condition. We prove that the nonlocal model is well-posed and the solution of the nonlocal system converges to the solution of the original Stokes system when the nonlocality vanishes.

1. Introduction. Recently, nonlocal models and corresponding numerical methods attracts lots of attentions and find many successful applications. In solid mechanics, the theory of peridynamics [30] has been shown to be an alternative to conventional models of elasticity and fracture mechanics. Many numerical methods have also been developed to compute peridynamic model based on solid mathematical analysis [8, 23, 31, 10, 9, 34]. Nonlocal methods are also successfully applied in image processing and data analysis [26, 25, 2, 6, 19, 16, 27, 15, 18, 4, 22, 33]. The idea of integral approximation is also applied to derive numerical scheme for solving PDEs on point cloud.

In this paper, we are trying to extend the nonlocal model in Stokes system in fluid mechanics. A nonlocal model was proposed in [11] for Stokes system with periodic boundary condition. In this paper, we consider the no-slip boundary condition. More precisely, for domain $\Omega \subset \mathbb{R}^n$,

$$\left\{ \begin{array}{rcl} \Delta \boldsymbol{u}(\boldsymbol{x}) - \nabla p(\boldsymbol{x}) & = & \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{x}) & = & 0, \quad \boldsymbol{x} \in \Omega. \end{array} \right. \tag{1.1}$$

No-slip boundary condition at the boundary is

$$\boldsymbol{u} = 0, \qquad \text{at } \partial\Omega. \tag{1.2}$$

For the pressure, we impose average zero condition

$$\int_\Omega p(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0. \tag{1.3}$$

Comparing to the periodic boundary condition, the no-slip boundary condition is more natural and more often used in real application. However, the theoretical study with no-slip boundary condition is also much more difficult. The first problem is how to enforce no-slip boundary condition in the nonlocal approach. No-slip boundary condition is basically Dirichlet type boundary condition. Recently, Du et.al. [8] proposed volume constraint to deal with the boundary condition in the nonlocal diffusion problem. They found that in the nonlocal diffusion problem, since the operator is nonlocal, only enforce the boundary condition on the boundary is not enough, it is necessary to extend the boundary condition to a small region adjacent to the boundary. Using this idea, in the nonlocal Stokes system, we extend the no-slip condition to a small layer as shown in Fig. 1. The whole computational domain Ω is decomposed

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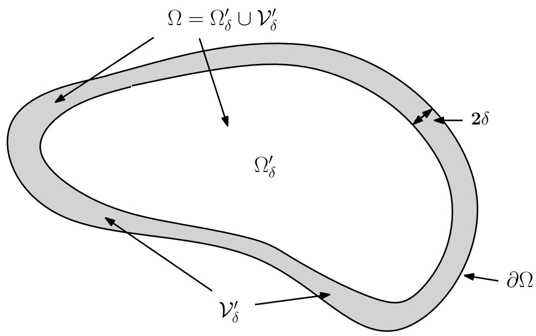

FIG. 1. *Computational domain in non-local Stokes model.*

to two parts $\Omega = \mathcal{V}'_\delta \bigcup \Omega'_\delta$ as shown in Fig. 1 and $\boldsymbol{u}$ is enforced to be zero in V'_δ, i.e.

$$\boldsymbol{u}_\delta(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathcal{V}'_\delta. \tag{1.4}$$

In Ω'_δ, inspired by the point integral method [29], Stokes equation is approximated by an nonlocal approach,

$$\begin{cases} -\mathcal{L}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) + \mathcal{G}_\delta p_\delta(\boldsymbol{x}) &= \displaystyle\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \Omega'_\delta, \\ \mathcal{D}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) - \bar{\mathcal{L}}_\delta p_\delta(\boldsymbol{x}) &= 0, \qquad\qquad\qquad\qquad\qquad \boldsymbol{x} \in \Omega, \end{cases} \tag{1.5}$$

The integral operators in (2.1) are defined as

$$\mathcal{L}_\delta \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}, \tag{1.6}$$

$$\mathcal{G}_\delta p(\boldsymbol{x}) = \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x} - \boldsymbol{y})p(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{1.7}$$

$$\mathcal{D}_\delta \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{1.8}$$

$$\bar{\mathcal{L}}_\delta p(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p(\boldsymbol{x}) - p(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}. \tag{1.9}$$

where

$$R_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta R\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4\delta^2}\right), \quad \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta \bar{R}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4\delta^2}\right) \tag{1.10}$$

C_δ is a normalization factor such that $\int_{\mathbb{R}^n} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} = 1$ and $\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s$. The kernel function $R(r) : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be C^2 smooth and satisfies some mild conditions which are listed in Assumption 1.

Finally, we also need average zero condition for the pressure

$$\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0. \tag{1.11}$$

(1.4), (2.1) and (1.11) form a complete nonlocal formulation for Stokes equation.

Nonlocal integral approximation is actually closely related to many numerical schemes of computational fluid dynamics, such as the smoothed particle hydrodynamics (SPH) [14, 20, 21, 24], vortex methods [1, 7] and others [3, 5, 12, 17, 32]. Analysis to the linear steady Stokes equation in this paper could give some new understanding to the theoretical foundation of these methods.

The Stokes system (2.1) is well-known to be a saddle point problem. To preserve the well-posedness, we have to be very careful in the derivation of the nonlocal approximation. It is demonstrated that the kernel function has to satisfy some additional conditions for the nonlocal Stokes system with periodic boundary by using Fourier analysis [11]. For the problem we consider in this paper, Fourier transform does not apply. To preserve the well-posedness, we add a relaxation term, $\bar{\mathcal{L}}_\delta p_\delta(\boldsymbol{x})$, in the equation of divergence free. The order of this term is $O(\delta)$, so it does not destroy the accuracy of the nonlocal approximation. However, this term is crucial in the analysis of well-posedness.

The rest of the paper is organized as follows. We give the formulating of the nonlocal linear Stokes system in Section 2 and assumptions are also introduced. Then the well-posedness of the nonlocal model is established in Section 3. The vanishing nonlocality limit will be analyzed in Section 4. We will prove the solution of the nonlocal system converges to the solution of the original Stokes system as δ goes to 0. In Section 5, we conclude with a summary and a discussion on future research.

2. Nonlocal Stokes system with volume constraints. Inspired by the volume constraint [8] and the point integral method [29], we consider a nonlocal model of the Stokes system (1.1)-(1.2) as follows:

$$
\left\{
\begin{array}{rcll}
-\mathcal{L}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) + \mathcal{G}_\delta p_\delta(\boldsymbol{x}) & = & \displaystyle\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, & \boldsymbol{x}\in\Omega'_\delta, \\
\mathcal{D}_\delta \boldsymbol{u}_\delta(\boldsymbol{x}) - \bar{\mathcal{L}}_\delta p_\delta(\boldsymbol{x}) & = & 0, & \boldsymbol{x}\in\Omega, \\
\boldsymbol{u}_\delta(\boldsymbol{x}) & = & 0, & \boldsymbol{x}\in\mathcal{V}'_\delta, \\
\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} & = & 0. &
\end{array}
\right.
\tag{2.1}
$$

The integral operators have been defined in (1.6)-(1.9). Here, Ω'_δ and $\mathcal{V}'_\delta$ are subsets of Ω which are defined as

$$
\Omega'_\delta = \left\{\boldsymbol{x}\in\Omega : B\left(\boldsymbol{x},2\delta\right)\cap\partial\Omega = \emptyset\right\}, \quad \mathcal{V}'_\delta = \Omega\backslash\Omega'_\delta. \tag{2.2}
$$

The relation of Ω, $\partial\Omega$, Ω'_δ and $\mathcal{V}'_\delta$ are showed in Fig. 1.

Moreover, we assume that the kernel function $R(r)$ satisfies the conditions in Assumption 1.

Assumption 1.

- *Assumptions on the computational domain: $\Omega, \partial\Omega$ are both compact and C^∞ smooth and Ω satisfies the cone condition.*
- *Assumptions on the kernel function $R(r)$:*
  - *(a) (regularity) $R \in C^1(\mathbb{R}^+)$;*
  - *(b) (positivity and compact support) $R(r) \ge 0$ and $R(r) = 0$ for $\forall r > 1$;*
  - *(c) (nondegeneracy) $\exists \delta_0 > 0$ so that $R(r) \ge \delta_0$ for $0 \le r \le \frac{1}{2}$.*

Based on these assumptions, we could prove the well-posedness and the vanishing nonlocality limit of nonlocal Stokes system (2.1).

Remark 2.1. *If we consider the periodic boundary condition, the well-posedness of the nonlocal system* (2.1) *is easy to prove.*

Applying Fourier transform on (2.1)*, we have a linear system:*

$$A_\delta(\boldsymbol{\xi})\left[\begin{array}{c}\hat{\boldsymbol{u}}_\delta(\boldsymbol{\xi})\\ \hat{p}_\delta(\boldsymbol{\xi})\end{array}\right]=\left[\begin{array}{c}\bar{\lambda}_\delta(\boldsymbol{\xi})\hat{\boldsymbol{f}}(\boldsymbol{\xi})\\ 0\end{array}\right]$$

where

$$A_\delta(\boldsymbol{\xi})=\left[\begin{array}{cc}\lambda_\delta(\boldsymbol{\xi})\boldsymbol{I}_n & i\boldsymbol{b}_\delta(\boldsymbol{\xi})\\ -i\boldsymbol{b}_\delta(\boldsymbol{\xi})^T & c_\delta(\boldsymbol{\xi})\end{array}\right]$$

$i=\sqrt{-1}$, $\boldsymbol{I}_n$ *is an* $n\times n$ *identity matrix and*

$$\begin{aligned}
\lambda_\delta(\boldsymbol{\xi})&=\frac{1}{\delta^2}\int_{\mathbb{R}^n}R_\delta(|\boldsymbol{s}|)(1-\cos(\boldsymbol{\xi}\cdot\boldsymbol{s}))\mathrm{d}\boldsymbol{s},\\
\bar{\lambda}_\delta(\boldsymbol{\xi})&=\int_{\mathbb{R}^n}\bar{R}_\delta(|\boldsymbol{s}|)\cos(\boldsymbol{\xi}\cdot\boldsymbol{s})\mathrm{d}\boldsymbol{s},\\
\boldsymbol{b}_\delta(\boldsymbol{\xi})&=\frac{1}{2\delta^2}\int_{\mathbb{R}^n}\boldsymbol{s}R_\delta(|\boldsymbol{s}|)\sin(\boldsymbol{\xi}\cdot\boldsymbol{s})\mathrm{d}\boldsymbol{s},\\
c_\delta(\boldsymbol{\xi})&=\int_{\mathbb{R}^n}\bar{R}_\delta(|\boldsymbol{s}|)(1-\cos(\boldsymbol{\xi}\cdot\boldsymbol{s}))\mathrm{d}\boldsymbol{s}
\end{aligned}$$

Notice that $\lambda_\delta(\boldsymbol{\xi})>0$ and $c_\delta(\boldsymbol{\xi})>0$ using the assumption that $R_\delta\geq 0$. Then it is easy to verify that the matrix $A_\delta(\boldsymbol{\xi})$ is invertible for any $\boldsymbol{\xi}\neq 0$. The well-posedness follows from the nonsingularity of $A_\delta(\boldsymbol{\xi})$.

3. Well-posedness of the nonlocal Stokes system (2.1)**.** In this section, we will prove the well-posedness of the nonlocal Stokes system (2.1). More precisely, we could prove followng theorem:

THEOREM 3.1. *Suppose Assumption 1 are satisfied. For any* $\boldsymbol{f}\in H^{-1}(\Omega)$*, there exits one and only one pair* $\boldsymbol{u},p$*, such that*

(a) $\boldsymbol{u}\in H^1(\Omega)$, $p\in L^2(\Omega)$*. In addition,*

$$\|\boldsymbol{u}\|_{H^1(\Omega)}+\|p\|_{L^2(\Omega)}\leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)},$$

where $C>0$ *is a constant only depends on* Ω *and kernel function* R*.*

(b) $\boldsymbol{u},p$ *verify the nonlocal Stokes system* (2.1)*.*

In the proof of the well-posedness, we need several technical lemmas.

LEMMA 3.2. *([29]) If* δ *is small enough, for any function* $u\in L^2(\mathcal{M})$*, there exists a constant* $C>0$ *independent on* δ *and* u*, such that*

$$\int_\Omega\int_\Omega R\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{32\delta^2}\right)(u(\boldsymbol{x})-u(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\leq C\int_\Omega\int_\Omega R\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right)(u(\boldsymbol{x})-u(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.$$

LEMMA 3.3. *([28]) For any function* $u\in L_2(\Omega'_\delta)$*, there exists a constant* $C>0$ *only depneds on* Ω*, such that*

$$\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{x})-u(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}+\frac{1}{\delta^2}\int_{\Omega'_\delta}u^2(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\geq C\int_{\Omega'_\delta}|\nabla v|^2\mathrm{d}\boldsymbol{x},$$

where

$$v(\boldsymbol{x})=\frac{1}{w_\delta(\boldsymbol{x})}\int_{\Omega'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y},$$

and $w_\delta(\boldsymbol{x}) = \displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$.

Proof. First, we introduce an extension of u to Ω,

$$\tilde{u}(\boldsymbol{x}) = \begin{cases} u(\boldsymbol{x}), & \boldsymbol{x} \in \Omega'_\delta, \\ 0, & \boldsymbol{x} \in \mathcal{V}'_\delta. \end{cases}$$

It follows that

$$v(\boldsymbol{x}) = \frac{1}{w_\delta(\boldsymbol{x})}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\tilde{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$$

Notice that for any $\boldsymbol{x} \in \Omega'_\delta$, $w_\delta(\boldsymbol{x}) = 1$ and $\displaystyle\int_\Omega \nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} = 0$. Then for any $\boldsymbol{x} \in \Omega'_\delta$, we have

$$\nabla v(\boldsymbol{x}) = \frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega (\nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y}))\tilde{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = -\frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega (\nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y}))(\tilde{u}(\boldsymbol{x}) - \tilde{u}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}$$

This gives that

$$\begin{aligned}
\int_{\Omega'_\delta} |\nabla v(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} &= \frac{1}{\bar{w}_\delta^2}\int_{\Omega'_\delta}\left|\int_\Omega (\nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y}))(\tilde{u}(\boldsymbol{x}) - \tilde{u}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right|^2 \mathrm{d}\boldsymbol{x} \\
&\le \frac{1}{\bar{w}_\delta^2}\int_{\Omega'_\delta}\left(\int_\Omega |\nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}\right)\left(\int_\Omega |\nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y})|\,(\tilde{u}(\boldsymbol{x}) - \tilde{u}(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\le \frac{C}{\delta^2}\int_{\Omega'_\delta}\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\tilde{u}(\boldsymbol{x}) - \tilde{u}(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \frac{C}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{x}) - u(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} + \frac{C}{\delta^2}\int_{\mathcal{V}'_\delta}\left(\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}
\end{aligned}$$

which complete the proof. ☐

Lemma 3.4. *([28]) For any function $u \in L_2(\Omega'_\delta)$, there exists a constant $C > 0$ independent on δ, such that*

$$\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{x}) - u(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} + \int_{\Omega'_\delta} u^2(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \ge C\|u\|^2_{L_2(\Omega'_\delta)},$$

as long as δ small enough.

Now we can prove the main theorem in this section, Theorem 3.1.

Proof. [Proof of Theorem 3.1:]

First, in the nonlocal Stokes system, we replace the condition $\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$ by $\int_{\Omega'_\delta} p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$ and denote the pressure in the original nonlocal Stokes system as $\bar{p}$. It is obvious that

$$\bar{p}_\delta = p_\delta - \frac{1}{|\Omega|}\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{3.1}$$

Multiplying $\boldsymbol{u}_\delta$ on the first equation of (2.1) and multiplying p on the third equation of (2.1) and adding them together, we can get

$$\begin{aligned}
\frac{1}{\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} + \int_\Omega\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
= -2\int_\Omega\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\cdot\boldsymbol{u}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}
\end{aligned} \tag{3.2}$$

Using this equation, we can get the uniqueness of the solution of the nonlocal Stokes equations (2.1) which also imply the existence of the solution in L^2 using standard theory for integral equations.

From (3.2), using Lemma 3.4, there exists $C > 0$, such that

$$\|\boldsymbol{u}_\delta\|_{L^2(\Omega)}^2 \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\boldsymbol{u}_\delta\|_{H^1(\Omega)}. \tag{3.3}$$

In addition, from the first equation of (2.1), $\boldsymbol{u}_\delta$ has following expression, for any $\boldsymbol{x} \in \Omega'_\delta$,

$$\boldsymbol{u}_\delta(\boldsymbol{x}) = \frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - \frac{\delta^2}{w_\delta(\boldsymbol{x})}\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{3.4}$$

Using Lemma 3.3 and (3.2), (3.3), we have

$$\begin{aligned}
&\|\nabla\left(\frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\|_{L^2(\Omega'_\delta)}^2\\
\leq&\frac{C}{\delta^2}\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x})-\boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\boldsymbol{u}_\delta\|_{H^1(\Omega)}
\end{aligned} \tag{3.5}$$

Notice that for any $\boldsymbol{x} \in \Omega'_\delta$, $w_\delta(\boldsymbol{x})$ is a positive constant. Then we have

$$\begin{aligned}
&\|\nabla\left(\frac{1}{2w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\|_{L^2(\Omega'_\delta)}^2\\
\leq&C\int_{\Omega'_\delta}\left|\int_\Omega \nabla_{\boldsymbol{x}}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^2\mathrm{d}\boldsymbol{x} + C\int_{\Omega'_\delta}\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}\\
\leq&\frac{C}{\delta^2}\int_\Omega\left|\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})||\boldsymbol{x}-\boldsymbol{y}|^2p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right|^2\mathrm{d}\boldsymbol{x} + C\int_\Omega\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}\\
\leq&C\int_\Omega\left(\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})|p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x} + C\int_\Omega\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^2\mathrm{d}\boldsymbol{x}\\
\leq&C\|p\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.6}$$

where

$$R'_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta R'\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2}\right), \quad R'(r) = \frac{\mathrm{d}}{\mathrm{d}r}R(r).$$

In addition, direct calculation gives that

$$\|\nabla\left(\frac{\delta^2}{w_\delta(\boldsymbol{x})}\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\|_{L^2(\Omega'_\delta)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \tag{3.7}$$

Putting (3.3)-(3.7) together, we obtain

$$\|\boldsymbol{u}_\delta\|_{H^1(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + C\|p_\delta\|_{L^2(\Omega)} \tag{3.8}$$

Next, we turn to estimate the pressure p. First, considering the problem

$$\nabla\cdot\boldsymbol{v}(\boldsymbol{x}) = p_\delta(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega'_\delta. \tag{3.9}$$

It is well known (e.g. Section 3.3 of [13]) that if Ω'_δ satisfies cone condition, there exists at least one solution of (3.9), denoted by $\boldsymbol{v}$, such that

$$\boldsymbol{v} \in H_0^1(\Omega'_\delta), \quad \|\boldsymbol{v}\|_{H^1(\Omega'_\delta)} \le c\|p\|_{L^2(\Omega'_\delta)} \tag{3.10}$$

Then, we extend $\boldsymbol{v}$ to Ω by assigning the value on $\mathcal{V}'_\delta$ to be 0 and denote the new function also by $\boldsymbol{v}$. Obviously, we have

$$\boldsymbol{v} \in H_0^1(\Omega'_\delta) \cap H_0^1(\Omega), \quad \|\boldsymbol{v}\|_{H^1(\Omega)} \le c\|p_\delta\|_{L^2(\Omega'_\delta)} \tag{3.11}$$

On the other hand, using the second equation of (2.1), $\forall \boldsymbol{x} \in \Omega$

$$\begin{aligned}
\bar{w}_\delta(\boldsymbol{x})p_\delta(\boldsymbol{x}) &= \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&= \int_{\Omega'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\nabla\cdot \boldsymbol{v}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad + \frac{1}{2\delta^2}\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{u}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&= -\frac{1}{2\delta^2}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \bar{\boldsymbol{v}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}
\end{aligned} \tag{3.12}$$

where $\bar{w}_\delta(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ and $\bar{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{u}_\delta$.

Then, it follows that

$$\begin{aligned}
&\frac{1}{2\delta^2}\int_{\Omega'_\delta} \bar{\boldsymbol{v}}(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= -\frac{1}{2\delta^2}\int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\bar{\boldsymbol{v}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega p_\delta^2(\boldsymbol{x})\bar{w}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.13}$$

The first term is positive which is a good term. The second term becomes

$$\begin{aligned}
&-\int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega p_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x})-p_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} - \int_\Omega p_\delta^2(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{3.14}$$

The second term of (3.14) can be controlled by the first term of (3.13). And the first term is bounded by

(3.15)

$$
\begin{aligned}
&\int_{\Omega} p_\delta(\boldsymbol{x}) \left(\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
=&\frac{1}{2}\int_{\mathcal{V}_\delta'}\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + \int_{\Omega_\delta'} p_\delta(\boldsymbol{x}) \left(\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
\geq&\frac{1}{2}\int_{\mathcal{V}_\delta'}\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + \int_{\Omega_\delta'} \left(\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
&- \left| \int_{\Omega_\delta'} \left(\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \right| \\
\geq&\frac{1}{2}\int_{\Omega} \left(\int_{\mathcal{V}_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\mathcal{V}_\delta'} p_\delta^2(\boldsymbol{x}) \left(\int_{\Omega_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Combining (3.13)-(3.15), we get

(3.16)

$$
\frac{1}{2\delta^2}\int_{\Omega_\delta'} \bar{\boldsymbol{v}}(\boldsymbol{x}) \left(\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \geq \quad \int_{\Omega_\delta'} p_\delta^2(\boldsymbol{x}) \left(\int_{\Omega_\delta'} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x}
$$

Now, we are ready to get the estimate of p_δ. Multiplying $\bar{\boldsymbol{v}}$ on both sides of the first equation of (2.1) and integrating over Ω_δ', using the fact that $\bar{\boldsymbol{v}}(\boldsymbol{x}) = 0, \;\; \boldsymbol{x} \in \mathcal{V}_\delta'$, we have

$$
\begin{aligned}
-\frac{1}{2\delta^2}\int_{\Omega}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y}))\cdot(\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{y}))\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\Omega_\delta'} \bar{\boldsymbol{v}}(\boldsymbol{x}) \left(\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})p_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \\
= \int_{\Omega_\delta'} \bar{\boldsymbol{v}}(\boldsymbol{x}) \left(\int_{\Omega} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{f}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \right) \mathrm{d}\boldsymbol{x} \qquad (3.17)
\end{aligned}
$$

Using (3.2), (3.8), (3.11), (3.17) and (3.16), we have

$$
\begin{aligned}
&\frac{1}{2}\|p\|^2_{L^2(\Omega_\delta')} \\
\leq& \left(\frac{1}{2\delta^2}\int_{\Omega}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right)^{1/2} \left(\frac{1}{2\delta^2}\int_{\Omega}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})|\bar{\boldsymbol{v}}(\boldsymbol{x}) - \bar{\boldsymbol{v}}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right)^{1/2} \\
&+ \|\bar{\boldsymbol{v}}\|_{H^1(\Omega_\delta')}\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \\
\leq& \left(\frac{1}{2\delta^2}\int_{\Omega}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right)^{1/2} \left(\left(\frac{1}{2\delta^2}\int_{\Omega}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right)^{1/2} \right. \\
&\left. + \left(\frac{1}{2\delta^2}\int_{\Omega}\int_{\Omega} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}_\delta(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right)^{1/2} \right) + (\|\boldsymbol{v}\|_{H^1(\Omega_\delta')} + \|\boldsymbol{u}_\delta\|_{H^1(\Omega_\delta')})\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \\
\leq& \|\boldsymbol{u}_\delta\|_{H^1(\Omega)}\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \|\boldsymbol{u}_\delta\|^{1/2}_{H^1(\Omega)}\|\boldsymbol{f}\|^{1/2}_{H^{-1}(\Omega)}\|\boldsymbol{v}\|_{H^1(\Omega)} + C(\|p_\delta\|_{L^2(\Omega_\delta')} + \|\boldsymbol{f}\|_{H^{-1}(\Omega)})\|\boldsymbol{f}\|_{H^{-1}(\Omega)} \\
\leq& C(\|p_\delta\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{H^{-1}(\Omega)})\|\boldsymbol{f}\|_{H^{-1}(\Omega)}.
\end{aligned}
$$

(3.18)

On the other hand, using Lemma 3.2, $\|p\|_{L^2(\Omega)}$ can be bounded by $\|p\|_{L^2(\Omega'_\delta)}$. First, notice that using the nondegeneracy assumption in Assumption 1, it is easy to verify that for any $\boldsymbol{x} \in \Omega$,

$$\int_{\Omega'_\delta} \bar{R}_{4\delta}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \geq c_0 > 0. \tag{3.19}$$

where

$$\bar{R}_{4\delta}(\boldsymbol{x}, \boldsymbol{y}) = C_\delta \bar{R}\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|^2}{4(4\delta)^2}\right),$$

C_δ is the normalization factor in (1.10).

$$\begin{aligned}
\|p_\delta\|_{L^2(\Omega)}^2 \leq & C\int_\Omega \left(\int_{\Omega'_\delta} |p_\delta(\boldsymbol{x})|^2 \bar{R}_{4\delta}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right) \mathrm{d}\boldsymbol{x} \\
\leq & C\int_\Omega \left(\int_{\Omega'_\delta} |p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})|^2 \bar{R}_{4\delta}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right) \mathrm{d}\boldsymbol{x} + C\int_\Omega \left(\int_{\Omega'_\delta} |p_\delta(\boldsymbol{y})|^2 \bar{R}_{4\delta}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}\right) \mathrm{d}\boldsymbol{x} \\
\leq & C\int_\Omega \int_\Omega |p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})|^2 \bar{R}_{4\delta}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + C\int_{\Omega'_\delta} |p_\delta(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{x} \\
\leq & C\int_\Omega \int_\Omega |p_\delta(\boldsymbol{x}) - p_\delta(\boldsymbol{y})|^2 \bar{R}_{\delta}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x} + C\|p_\delta\|_{L^2(\Omega'_\delta)}^2
\end{aligned}$$

Using (3.2), (3.8), (3.18) and (3.20), we have

$$\begin{aligned}
\|p_\delta\|_{L^2(\Omega)}^2 \leq & C\|\boldsymbol{u}_\delta\|_{H^1(\Omega)}\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + C\|p_\delta\|_{L^2(\Omega'_\delta)}^2 \\
\leq & C(\|p_\delta\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{H^{-1}(\Omega)})\|\boldsymbol{f}\|_{H^{-1}(\Omega)}
\end{aligned} \tag{3.20}$$

Together with (3.18), we have

$$\|p_\delta\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}. \tag{3.21}$$

This also gives the H^1 estimate of $\boldsymbol{u}_\delta$ using (3.8),

$$\|\boldsymbol{u}_\delta\|_{H^1(\Omega)} \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}. \tag{3.22}$$

and using (3.1)

$$\|\bar{p}_\delta\|_{L^2(\Omega)} \leq \|p_\delta\|_{L^2(\Omega)} + \frac{1}{|\Omega|}\left|\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \leq C\|\boldsymbol{f}\|_{H^{-1}(\Omega)}, \tag{3.23}$$

here we use the fact that

$$\frac{1}{|\Omega|}\left|\int_\Omega p_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \leq \frac{1}{\sqrt{|\Omega|}}\|p_\delta\|_{L^2(\Omega)}.$$

□

4. Vanishing nonlocality. Besides the well-posedness, another problem we are interested is the limit behavior of the nonlocal Stokes system (2.1) as the nonlocality vanish, i.e. $\delta \to 0$. In this section, we will answer this question. Under some assumptions, we can prove that the solution of the nonlocal Stokes system converges to the solution of the Stokes system as $\delta \to 0$. Furthermore, we could give an estimate of the convergence rate. The result is summarized in Theorem 4.2.

Before stating the main theorem, we give several lemmas which will be used in proving the main theorem.

We also need following theorem regarding the nonlocal approximation of the Laplace operator. And the proof can be found in [28].

THEOREM 4.1. *([28]) Let $u \in H^3(\Omega)$ and*

$$r(\boldsymbol{x}) = -\frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{x}) - u(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} - \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta u(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \quad \forall \boldsymbol{x} \in \Omega'_\delta.$$

There exists constants C, T_0 depending only on Ω, so that for any $\delta \le T_0$,

$$\|r(\boldsymbol{x})\|_{L^2(\Omega'_\delta)} \le C\delta \|u\|_{H^3(\Omega)}, \tag{4.1}$$

$$\|\nabla r(\boldsymbol{x})\|_{L^2(\Omega'_\delta)} \le C\|u\|_{H^3(\Omega)}. \tag{4.2}$$

This is the main result in this section regarding the convergence of the nonlocal Stokes system as the nonlocality vanish.

THEOREM 4.2. *Let $\boldsymbol{u}(\boldsymbol{x})$, $p(\boldsymbol{x})$ be solution of Stokes system* (1.1) *and $\boldsymbol{u}_\delta(\boldsymbol{x})$, $p_\delta(\boldsymbol{x})$ be solution of nonlocal Stokes system* (2.1) *with $\boldsymbol{f} \in H^1(\Omega)$. There exists $C > 0$ only depends on Ω and R, such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_{H^1(\Omega'_\delta)} + \|p - p_\delta\|_{L^2(\Omega)} \le C\sqrt{\delta}\|f\|_{H^1(\Omega)}$$

Proof. Let $\boldsymbol{e}_\delta(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_\delta(\boldsymbol{x})$ and $d_\delta = p - p_\delta - \frac{1}{|\Omega'_\delta|}\int_{\Omega'_\delta}(p(\boldsymbol{x}) - p_\delta(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}$, then $\boldsymbol{e}_\delta$, d_δ satisfies

$$\left\{\begin{array}{rcll} -\dfrac{1}{\delta^2}\displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x}) - \boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} + \dfrac{1}{2\delta^2}\displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} &=& \boldsymbol{r_u}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega'_\delta,\\ \boldsymbol{e}_\delta(\boldsymbol{x}) &=& \boldsymbol{u}(\boldsymbol{x}), & \boldsymbol{x} \in \mathcal{V}'_\delta,\\ \dfrac{1}{2\delta^2}\displaystyle\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot \boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - \displaystyle\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x}) - d_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} &=& r_p(\boldsymbol{x}), & \boldsymbol{x} \in \Omega,\\ \displaystyle\int_{\Omega'_\delta} d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &=& 0, & \end{array}\right. \tag{4.3}$$

where

$$\boldsymbol{r_u}(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\Delta \boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}, \qquad \forall \boldsymbol{x} \in \Omega'_\delta \tag{4.4}$$

$$r_p(\boldsymbol{x}) = -\int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p(\boldsymbol{x}) - p(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}, \qquad \forall \boldsymbol{x} \in \Omega. \tag{4.5}$$

First, we focus on the following estimate

(4.6)

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{\Omega'_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{\delta^2}\int_{\Omega'_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}+\frac{1}{\delta^2}\int_{\Omega'_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{2\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}+\frac{1}{\delta^2}\int_{\Omega'_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

The second term of the right hand side of (4.6) can be calculated as

(4.7)

$$
\begin{aligned}
&\frac{1}{\delta^2}\int_{\Omega'_\delta} \boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}-\frac{1}{\delta^2}\int_{\Omega'_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Here we use the definition of $\boldsymbol{e}_\delta$ and the volume constraint condition $\boldsymbol{u}_\delta(\boldsymbol{x})=0,\ \boldsymbol{x}\in\mathcal{V}'_\delta$ to get that $\boldsymbol{e}_\delta(\boldsymbol{x})=u(\boldsymbol{x}),\ \boldsymbol{x}\in\mathcal{V}'_\delta$.

The first term is positive which is good for us. We only need to bound the second term of (4.7) to show that it can be controlled by the first term. First, the second term can be bounded as following

(4.8)

$$
\begin{aligned}
&\frac{1}{\delta^2}\left|\int_{\Omega'_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\le&\frac{1}{\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)^{1/2}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)^{1/2}\mathrm{d}\boldsymbol{x}\\
\le&\frac{1}{\delta^2}\left(\int_{\Omega'_\delta}\frac{1}{2}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+2\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right)\\
\le&\frac{1}{2\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{2}{\delta^2}\int_{\mathcal{V}'_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\left(\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}\right)\mathrm{d}\boldsymbol{y}\\
\le&\frac{1}{2\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{C}{\delta^2}\int_{\mathcal{V}'_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\\
\le&\frac{1}{2\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Here we use Lemma A.1 in Appendix A to get the last inequality.

By substituting (4.8), (4.7) in (4.6), we get

$$\begin{aligned}
&\left|\frac{1}{\delta^2}\int_{\Omega'_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\right| \\
\geq&\ \frac{1}{2\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
&+\frac{1}{2\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}-C\|\boldsymbol{f}\|^2_{H^1(\Omega)}\delta.
\end{aligned}\tag{4.9}$$

This is the key estimate we used to get convergence.

We also need following bound

$$\begin{aligned}
&\left|\frac{1}{\delta^2}\int_{\Omega'_\delta}\boldsymbol{e}_\delta(\boldsymbol{x})\cdot\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{1}{\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right| \\
=&\left|\frac{1}{\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right| \\
\leq&\frac{1}{\delta}\int_\Omega\left(\int_{\mathcal{V}'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{x})||\boldsymbol{u}(\boldsymbol{y})|\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
\leq&\frac{1}{\delta}\left[\int_\Omega\left(\int_{\mathcal{V}'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\int_\Omega\left(\int_{\mathcal{V}'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right]^{1/2} \\
\leq& C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}.
\end{aligned}\tag{4.10}$$

Multiplying $\boldsymbol{e}_\delta(\boldsymbol{x})$, $d_\delta(\boldsymbol{x})$ on both sides of the first and third equation of (4.3) and integrating over Ω'_δ, Ω respectively and adding them together, using (4.9), (4.10), we have

$$\begin{aligned}
&\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}+\frac{1}{2\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&+\int_\Omega\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})|d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
\leq&\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega'_\delta)}\|\boldsymbol{e}_\delta\|_{L^2(\Omega'_\delta)}+\|r_p\|_{L^2(\Omega)}\|d_\delta\|_{L^2(\Omega)}+C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}+C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}.
\end{aligned}\tag{4.11}$$

To simplify the notation, we denote the right hand side of (4.11) as Q^2.

It is well known (e.g. Section 3.3 of [13]) that with the condition that $\int_{\Omega'_\delta}d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=0$, there exists at least one function $\boldsymbol{\psi}$, such that

$$\begin{aligned}
&\nabla\cdot\boldsymbol{\psi}(\boldsymbol{x})=d_\delta(\boldsymbol{x}),\quad \boldsymbol{x}\in\Omega'_\delta,\quad\text{and} \\
&\boldsymbol{\psi}\in H^1_0(\Omega'_\delta),\quad\|\boldsymbol{\psi}\|_{H^1(\Omega'_\delta)}\leq c\|d_\delta\|_{L^2(\Omega'_\delta)}
\end{aligned}\tag{4.12}$$

Then, we extend $\boldsymbol{\psi}$ to Ω by assigning the value on $\mathcal{V}'_\delta$ to be 0 and denote the new function also by $\boldsymbol{\psi}$. Obviously, we have

$$\boldsymbol{\psi}\in H^1_0(\Omega'_\delta)\cap H^1_0(\Omega),\quad\|\boldsymbol{\psi}\|_{H^1(\Omega)}\leq c\|d_\delta\|_{L^2(\Omega'_\delta)}\tag{4.13}$$

Using the third equation of (4.3), we have

$$
\begin{aligned}
\bar{w}_\delta(\boldsymbol{x})d_\delta(\boldsymbol{x}) &= \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - r_p(\boldsymbol{x}) \\
&= \int_{\Omega'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\nabla\cdot\boldsymbol{\psi}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad + \frac{1}{2\delta^2}\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - r_p(\boldsymbol{x}) \\
&= -\frac{1}{2\delta^2}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\bar{\boldsymbol{\psi}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2\delta^2}\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \\
&\quad + \int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - r_p(\boldsymbol{x})
\end{aligned}
\tag{4.14}
$$

where $\bar{w}_\delta(\boldsymbol{x}) = \int_\Omega \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ and $\bar{\boldsymbol{\psi}} = \boldsymbol{\psi} - \boldsymbol{e}_\delta$.

Then, it follows that

$$
\begin{aligned}
&\frac{1}{2\delta^2}\int_{\Omega'_\delta} \bar{\boldsymbol{\psi}}(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= -\frac{1}{2\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\bar{\boldsymbol{\psi}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega d_\delta^2(\boldsymbol{x})\bar{w}_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&\quad - \int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} + \int_\Omega d_\delta(\boldsymbol{x})r_p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}
\end{aligned}
\tag{4.15}
$$

The first term is positive which is a good term. The second term becomes

$$
\begin{aligned}
&-\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} - \frac{1}{2\delta^2}\int_\Omega d_\delta^2(\boldsymbol{x})\left(\int_{\mathcal{V}'_\delta} \bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{4.16}
$$

The second term of (4.16) can be controlled by the first term of (4.15). And the first term is bounded by

(4.17)

$$
\begin{aligned}
&\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
=&\frac{1}{2}\int_{\mathcal{V}_\delta'}\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}+\int_{\Omega_\delta'}d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\geq&\frac{1}{2}\int_{\mathcal{V}_\delta'}\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}+\int_{\Omega_\delta'}\left(\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&-\left|\int_{\Omega_\delta'}\left(\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\geq&\frac{1}{2}\int_\Omega\left(\int_{\mathcal{V}_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y}))^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}-\frac{1}{2}\int_{\mathcal{V}_\delta'}d_\delta^2(\boldsymbol{x})\left(\int_{\Omega_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Combining (4.15)-(4.17), we get

(4.18)

$$
\begin{aligned}
&\frac{1}{2\delta^2}\int_{\Omega_\delta'}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\geq&\int_{\Omega_\delta'}d_\delta^2(\boldsymbol{x})\left(\int_{\Omega_\delta'}\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}-\frac{1}{2\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta'}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega d_\delta(\boldsymbol{x})r_p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
&\left|\frac{1}{2\delta^2}\int_\Omega d_\delta(\boldsymbol{x})\left(\int_{\mathcal{V}_\delta'}R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{u}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\leq&\frac{1}{2\delta}\int_\Omega|d_\delta(\boldsymbol{x})|\left(\int_{\mathcal{V}_\delta'}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\leq&\frac{1}{2\delta}\left[\int_\Omega|d_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}_\delta'}R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\int_\Omega\left(\int_{\mathcal{V}_\delta'}R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right]^{1/2}\\
\leq&C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.19}
$$

and

$$
\begin{aligned}
&\left|\int_\Omega d_\delta(\boldsymbol{x})r_p(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\\
=&\left|\int_\Omega d_\delta(\boldsymbol{x})\left(\int_\Omega\bar{R}_\delta(\boldsymbol{x},\boldsymbol{y})(p(\boldsymbol{x})-p(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
\leq&C\delta\|p\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}\\
\leq&C\delta\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.20}
$$

Multiplying $\bar{\boldsymbol{\psi}}$ on both sides of the first equation of (4.3) and using (4.18), (4.19), (4.20), we have

$$
\begin{aligned}
\|d_\delta\|^2_{L^2(\Omega'_\delta)} \le & \frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\cdot(\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y}))\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \\
& + \frac{1}{\delta^2}\int_{\Omega'_\delta}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x} \\
& + \|\bar{\boldsymbol{\psi}}\|_{L^2(\Omega'_\delta)}\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \qquad (4.21)
\end{aligned}
$$

The first term can be bounded as

$$
\begin{aligned}
& \left|\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\cdot(\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y}))\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right| \\
\le & \left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{1/2}\left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{1/2} \\
\le & \left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{1/2}\left(\left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{\psi}(\boldsymbol{x})-\boldsymbol{\psi}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{1/2}\right. \\
& \left.+\left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right)^{1/2}\right) \\
\le & Q^2 + CQ\|\boldsymbol{\psi}\|_{H^1(\Omega'_\delta)} \le Q^2 + CQ\|d_\delta\|_{L^2(\Omega'_\delta)}.
\end{aligned} \qquad (4.22)
$$

The estimate of the second term of (4.21) is a little involved. First

$$
\begin{aligned}
& \left|\frac{1}{\delta^2}\int_{\Omega'_\delta}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right| \\
\le & \left|\frac{1}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y}))\cdot(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right| \\
& + \left|\frac{1}{\delta^2}\int_{\mathcal{V}'_\delta}\boldsymbol{u}(\boldsymbol{x})\cdot\left(\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right| \\
\le & \left[\left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\bar{\boldsymbol{\psi}}(\boldsymbol{x})-\bar{\boldsymbol{\psi}}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right)^{1/2} + \left(\frac{1}{\delta^2}\int_{\mathcal{V}'_\delta}|\boldsymbol{u}(\boldsymbol{x})|^2\left(\int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right)^{1/2}\right] \\
& \left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right)^{1/2} \\
\le & C\left(\|\boldsymbol{\psi}\|_{H^1(\Omega)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)\left(\frac{1}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right)^{1/2} \\
& + \frac{C}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}.
\end{aligned} \qquad (4.23)
$$

Moreover,

$$
\begin{aligned}
(4.24)\quad & \frac{1}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&\le \frac{2}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{2}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{e}_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&\le \frac{2}{\delta^2}\int_{\Omega'_\delta}|\boldsymbol{e}_\delta(\boldsymbol{x})|^2\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+\frac{2}{\delta^2}\int_{\Omega'_\delta}\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
&\le Q^2+C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Combining (4.23) and (4.24), we get

$$
\begin{aligned}
&\left|\frac{1}{\delta^2}\int_{\Omega'_\delta}\bar{\boldsymbol{\psi}}(\boldsymbol{x})\cdot\left(\int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{e}_\delta(\boldsymbol{x})-\boldsymbol{e}_\delta(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\right|\\
(4.25)\qquad &\le\left(\|d_\delta\|_{L^2(\Omega'_\delta)}+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)(Q+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)})+Q^2+\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}
\end{aligned}
$$

Substituting (4.22) and (4.25) in (4.21),

$$
\begin{aligned}
\|d_\delta\|^2_{L^2(\Omega'_\delta)} \le& Q^2+C\left(\|d_\delta\|_{L^2(\Omega'_\delta)}+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)\left(Q+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)+\|\bar{\boldsymbol{\psi}}\|_{L^2(\Omega'_\delta)}\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega)}\\
&+C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}\\
\le& Q^2+C\left(\|d_\delta\|_{L^2(\Omega'_\delta)}+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)\left(Q+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)\\
(4.26)\qquad &+\left(\|d_\delta\|_{L^2(\Omega'_\delta)}+\|\boldsymbol{e}_\delta\|_{L^2(\Omega'_\delta)}\right)\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega)}+C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)}
\end{aligned}
$$

On the other hand, using Lemma 3.2, $\|d_\delta\|_{L^2(\Omega)}$ can be bounded by $\|d_\delta\|_{L^2(\Omega'_\delta)}$.

(4.27)

$$
\begin{aligned}
\|d_\delta\|^2_{L^2(\Omega)} \le& C\int_\Omega|d_\delta(\boldsymbol{x})|^2\left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\le& C\int_\Omega\left(\int_{\Omega'_\delta}|d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})+d_\delta(\boldsymbol{y})|^2R_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\le& C\int_\Omega\left(\int_{\Omega'_\delta}|d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})|^2R_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}+C\int_\Omega\left(\int_{\Omega'_\delta}|d_\delta(\boldsymbol{y})|^2R_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\right)\mathrm{d}\boldsymbol{x}\\
\le& C\int_\Omega\int_\Omega|d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})|^2R_{4\delta}(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}+C\int_{\Omega'_\delta}|d_\delta(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{x}\\
\le& C\int_\Omega\int_\Omega|d_\delta(\boldsymbol{x})-d_\delta(\boldsymbol{y})|^2R_\delta(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}+C\|d_\delta\|^2_{L^2(\Omega'_\delta)}.
\end{aligned}
$$

Then it follows from (4.11) and above inequality

$$
\begin{aligned}
\|d_\delta\|^2_{L^2(\Omega)} \le& Q^2+C\left(\|d_\delta\|_{L^2(\Omega)}+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)\left(Q+\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)\\
(4.28)\qquad &+\left(\|d_\delta\|_{L^2(\Omega)}+\|\boldsymbol{e}_\delta\|_{L^2(\Omega'_\delta)}\right)\|\boldsymbol{r}_{\boldsymbol{u}}\|_{L^2(\Omega)}
\end{aligned}
$$

Theorem 4.1 gives that

$$\|\boldsymbol{r_u}\|_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|_{H^1(\Omega)}, \quad \|r_p\|_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|_{H^1(\Omega)} \tag{4.29}$$

Following Lemma 3.4 and (4.11), we have

$$\|\boldsymbol{e}_\delta\|^2_{L^2(\Omega'_\delta)} \le Q^2 \le C\delta\|\boldsymbol{f}\|_{H^1(\Omega)}\|\boldsymbol{e}_\delta\|_{L^2(\Omega'_\delta)} + C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \tag{4.30}$$

which implies that

$$\|\boldsymbol{e}_\delta\|^2_{L^2(\Omega'_\delta)} \le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \tag{4.31}$$

Consequently, Q^2 is bounded by

$$Q^2 \le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \tag{4.32}$$

Now, we have the bound of $\|d_\delta\|_{L^2(\Omega)}$ from (4.28) and (4.32),

$$\begin{aligned}\|d_\delta\|^2_{L^2(\Omega)} \le& C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} + \left(\|d_\delta\|_{L^2(\Omega)} + \sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\right)Q \\ \le& C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} + \left(\frac{1}{2}\|d_\delta\|^2_{L^2(\Omega)} + \delta\|\boldsymbol{f}\|^2_{H^1(\Omega)}\right)\end{aligned} \tag{4.33}$$

Therefore

$$\|d_\delta\|^2_{L^2(\Omega)} \le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)} + C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}\|d_\delta\|_{L^2(\Omega)} \tag{4.34}$$

Then the bound of $\|d_\delta\|_{L^2(\Omega)}$ is obtained

$$\|d_\delta\|_{L^2(\Omega)} \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{4.35}$$

The bound of $\|\boldsymbol{e}_\delta\|_{L^2(\Omega'_\delta)}$ follows from (4.31) and (4.35),

$$\|\boldsymbol{e}_\delta\|_{L^2(\Omega'_\delta)} \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}. \tag{4.36}$$

and

$$\|p - p_\delta\|_{L^2(\Omega)} \le \|d_\delta\|_{L^2(\Omega)} + |\bar{d}_\delta| \le C\sqrt{\delta}\|\boldsymbol{f}\|_{H^1(\Omega)}, \tag{4.37}$$

where $\bar{d}_\delta = \frac{1}{|\Omega|}\int_\Omega d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ and we use the fact that

$$|\bar{d}_\delta| = \frac{1}{|\Omega|}\left|\int_\Omega d_\delta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right| \le \frac{1}{\sqrt{|\Omega|}}\|d_\delta\|_{L^2(\Omega)}.$$

Finally, the bound of $\|\boldsymbol{e}_\delta\|_{H^1(\Omega'_\delta)}$ can be drived from

$$\boldsymbol{e}_\delta(\boldsymbol{x}) = \frac{1}{w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{e}_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} + \frac{1}{2w_\delta(\boldsymbol{x})}\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})d_\delta(\boldsymbol{y})\mathrm{d}\boldsymbol{y} - t\boldsymbol{r_u}(\boldsymbol{x}), \tag{4.38}$$

Next, we will estimate the three terms of the right hand side one by one. The third term is easy to bound using Theorem 4.1,

$$\|\delta^2\nabla\boldsymbol{r_u}(\boldsymbol{x})\|_{L^2(\Omega'_\delta)} \le \delta^2\|\boldsymbol{f}\|_{H^1(\Omega)} \tag{4.39}$$

Notice that for any $\boldsymbol{x} \in \Omega'_\delta$, $w_\delta(\boldsymbol{x})$ is a positive constant. Then we have

(4.40)

$$
\begin{aligned}
&\|\nabla \left(\frac{1}{2w_\delta(\boldsymbol{x})} \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \|^2_{L^2(\Omega'_\delta)} \\
\leq & C \int_{\Omega'_\delta} \left| \int_\Omega \nabla_{\boldsymbol{x}} R_\delta(\boldsymbol{x},\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right|^2 \mathrm{d}\boldsymbol{x} + C \int_{\Omega'_\delta} \left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right)^2 \mathrm{d}\boldsymbol{x} \\
\leq & \frac{C}{\delta^2} \int_\Omega \left| \int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})| |\boldsymbol{x}-\boldsymbol{y}|^2 d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right|^2 \mathrm{d}\boldsymbol{x} + C \int_\Omega \left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right)^2 \mathrm{d}\boldsymbol{x} \\
\leq & C \int_\Omega \left(\int_\Omega |R'_\delta(\boldsymbol{x},\boldsymbol{y})| d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right)^2 \mathrm{d}\boldsymbol{x} + C \int_\Omega \left(\int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) d_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right)^2 \mathrm{d}\boldsymbol{x} \\
\leq & C \|d_\delta\|^2_{L^2(\Omega)} \leq C\sqrt{\delta} \|\boldsymbol{f}\|_{H^1(\Omega)}.
\end{aligned}
$$

where

$$
R'_\delta(\boldsymbol{x},\boldsymbol{y}) = C_\delta R'\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4\delta^2} \right), \quad R'(r) = \frac{\mathrm{d}}{\mathrm{d}r} R(r).
$$

The first can be splitted to two terms

(4.41)

$$
\frac{1}{w_\delta(\boldsymbol{x})} \int_\Omega R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{e}_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \frac{1}{w_\delta(\boldsymbol{x})} \int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{e}_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} + \frac{1}{w_\delta(\boldsymbol{x})} \int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{u}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}
$$

Using Lemma A.1,

$$
\|\nabla \left(\frac{1}{w_\delta(\boldsymbol{x})} \int_{\mathcal{V}'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{u}(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \|_{L^2(\Omega'_\delta)} \leq C\sqrt{\delta} \|\boldsymbol{f}\|_{H^1(\Omega)} \tag{4.42}
$$

And it follows from Lemma 3.3 and (4.11),

$$
\|\nabla \left(\frac{1}{w_\delta(\boldsymbol{x})} \int_{\Omega'_\delta} R_\delta(\boldsymbol{x},\boldsymbol{y}) \boldsymbol{e}_\delta(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \right) \|_{L^2(\Omega'_\delta)} \leq C\sqrt{\delta} \|\boldsymbol{f}\|_{H^1(\Omega)} \tag{4.43}
$$

The proof is completed. ☐

5. Discussion and Conclusion. In this paper, we propose a nonlocal model for linear steady Stokes equation with no-slip boundary condition. The main idea is to use volume constraint to enforce the no-slip boundary condition and add a relaxation term in the divergence free condition to maintain the well-posedness of the nonlocal system. We also analyze the vanishing nonlocality limit of the nonlocal system. As the nonlocality scale δ approaches 0, the solution of the nonlocal system converges to the solution of the original Stoke equation.

From the nonlocal system, we can derive a numerical scheme for the original Stokes system on point cloud. Assume we are given a set of sample points P sampling the domain Ω and a subset $S \subset P$ sampling the boundary of Ω. In addition, assume we are given one vector $\mathbf{V} = (V_1, \cdots, V_n)^t$ where V_i is an volume weight of $\boldsymbol{x}_i$ in Ω, so that for any C^1 function f on Ω, $\int_\Omega f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ can be approximated by $\sum_{\boldsymbol{x}_i \in \Omega} f(\boldsymbol{x}_i) V_i$.

Then, the nonlocal Stokes system (2.1) can be discretized as following.

$$\begin{aligned}
-\frac{1}{\delta^2}\sum_{\boldsymbol{x}_j\in\Omega} R_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(\boldsymbol{u}_i-\boldsymbol{u}_j)V_j + \frac{1}{2\delta^2}\sum_{\boldsymbol{x}_j\in\Omega} R_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(\boldsymbol{x}_i-\boldsymbol{x}_j)p_jV_j & \\
= \sum_{\boldsymbol{x}_j\in\Omega} \bar{R}_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)\boldsymbol{f}_jV_j, &\quad \boldsymbol{x}_i\in\Omega'_\delta \\
\frac{1}{2\delta^2}\sum_{\boldsymbol{x}_j\in\Omega} R_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(\boldsymbol{x}_i-\boldsymbol{x}_j)\boldsymbol{u}_jV_j - \sum_{\boldsymbol{x}_j\in\Omega}\bar{R}_\delta(\boldsymbol{x}_i,\boldsymbol{x}_j)(p_i-p_j)V_j =0, &\quad \boldsymbol{x}_i\in\Omega, \\
\boldsymbol{u}_i =0, &\quad \boldsymbol{x}_i\in\mathcal{V}'_\delta \\
\sum_{\boldsymbol{x}_j\in\Omega} p_jV_j =0. &
\end{aligned}$$

This scheme is very simple and easy to implement. However, the accuracy is relatively low. We can show that the error of above scheme is $O\left(\frac{h}{\delta^2}+\delta\right)$, where h is the average distance among the sample points in P. The first term h/δ^2 comes from the error of the numerical integral and the second term δ is from error between nonlocal system and the original Stoke equation. One way to improve the accuracy is to use high order quadrature rule in the computing of the integral transforms. Formally, the error is $O\left(\frac{h^k}{\delta^2}+\delta\right)$, where k is the order of the accuracy of the quadrature rule. To get the optimal error estimate, we should take $\delta = O(h^{k/3})$. If k is large enough ($k>3$), the nonlocal scale δ can be even smaller than h and the numerical scheme is still convergent.

Appendix A. One basic estimates.

LEMMA A.1. *Let $\boldsymbol{u}(\boldsymbol{x})$ be the solution of the Stokes system* (1.1) *and $\boldsymbol{f}\in H^1(\Omega)$, then there is a generic constant $C>0$ and $T_0>0$ only depend on Ω and $\partial\Omega$, for any $\delta<T_0$,*

$$\int_{\mathcal{V}'_\delta} |\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y} \le C\delta^3\|\boldsymbol{f}\|^2_{H^1(\Omega)}.$$

Proof. Both Ω and $\partial\Omega$ are compact and C^∞ smooth. Consequently, it is well known that both Ω and $\partial\Omega$ have positive reaches, which means that there exists $T_0>0$ only depends on Ω and $\partial\Omega$, if $t<T_0$, $\mathcal{V}'_\delta$ can be parametrized as $(\mathbf{z}(\boldsymbol{y}),\tau)\in\partial\Omega\times[0,1]$, where $\boldsymbol{y}=\mathbf{z}(\boldsymbol{y})+\tau(\mathbf{z}'(\boldsymbol{y})-\mathbf{z}(\boldsymbol{y}))$ and $\left|\det\left(\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}(\mathbf{z}(\boldsymbol{y}),\tau)}\right)\right|\le C\delta$ and $C>0$ is a constant only depends on Ω and $\partial\Omega$. Here $\mathbf{z}'(\boldsymbol{y})$ is the intersection point between $\partial\Omega'$ and the line determined by $\mathbf{z}(\boldsymbol{y})$ and $\boldsymbol{y}$. The parametrization is illustrated in Fig.2.

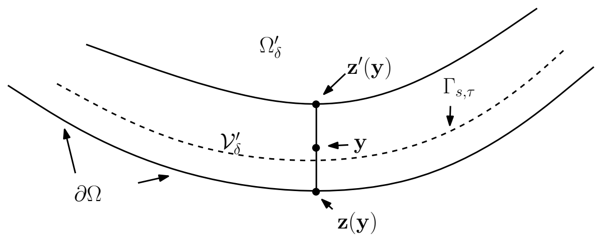

FIG. 2. *Parametrization of* $\mathcal{V}'_\delta$

First, we have

$$\begin{aligned}&\int_{\mathcal{V}'_\delta}|\boldsymbol{u}(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}=\int_{\mathcal{V}'_\delta}|\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}(\mathbf{z}(\boldsymbol{y}))|^2\mathrm{d}\boldsymbol{y}\\ =&\int_{\mathcal{V}'_\delta}\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{u}(\boldsymbol{y}+s(\mathbf{z}(\boldsymbol{y})-\boldsymbol{y}))\mathrm{d}s\right|^2\mathrm{d}\boldsymbol{y}\\ =&\int_{\mathcal{V}'_\delta}\left|\int_0^1(\mathbf{z}(\boldsymbol{y})-\boldsymbol{y})\cdot\nabla\boldsymbol{u}(\boldsymbol{y}+s(\mathbf{z}(\boldsymbol{y})-\boldsymbol{y}))\mathrm{d}s\right|^2\mathrm{d}\boldsymbol{y}\\ \le&\, C\delta^2\int_{\mathcal{V}'_\delta}\int_0^1|\nabla\boldsymbol{u}(\boldsymbol{y}+s(\mathbf{z}(\boldsymbol{y})-\boldsymbol{y}))|^2\,\mathrm{d}s\mathrm{d}\boldsymbol{y}\\ \le&\, C\delta^2\sup_{0\le s\le 1}\int_{\mathcal{V}'_\delta}|\nabla\boldsymbol{u}(\boldsymbol{y}+s(\mathbf{z}(\boldsymbol{y})-\boldsymbol{y}))|^2\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Here, we use the fact that $\|\mathbf{z}(\boldsymbol{y})-\boldsymbol{y}\|_2\le 2\delta$ to get the second last inequality.

Then, the proof can be completed by following estimation.

$$\begin{aligned}&\int_{\mathcal{V}'_\delta}|\nabla\boldsymbol{u}(\boldsymbol{y}+s(\mathbf{z}(\boldsymbol{y})-\boldsymbol{y}))|^2\,\mathrm{d}\boldsymbol{y}\\ \le&\, C\delta\int_0^1\int_{\partial\Omega}|\nabla\boldsymbol{u}(\mathbf{z}(\boldsymbol{y})+(1-s)\tau(\mathbf{z}'(\boldsymbol{y})-\mathbf{z}(\boldsymbol{y})))|^2\,\mathrm{d}\mathbf{z}(\boldsymbol{y})\mathrm{d}\tau\\ \le&\, C\delta\sup_{0\le\tau\le 1}\int_{\partial\Omega}|\nabla\boldsymbol{u}(\mathbf{z}+(1-s)\tau(\mathbf{z}'-\mathbf{z}))|^2\,\mathrm{d}\mathbf{z}\\ \le&\, C\delta\sup_{0\le\tau\le 1}\int_{\Gamma_{s,\tau}}|\nabla\boldsymbol{u}(\tilde{\mathbf{z}})|^2\,\mathrm{d}\tilde{\mathbf{z}}\\ \le&\, C\delta\|\boldsymbol{u}\|^2_{H^2(\Omega)}\le C\delta\|\boldsymbol{f}\|^2_{H^1(\Omega)},\end{aligned}$$

where $\Gamma_{s,\tau}$ is a $k-1$ dimensional manifold given by $\Gamma_{s,\tau}=\{\mathbf{z}+(1-s)\tau(\mathbf{z}'-\mathbf{z}):\mathbf{z}\in\partial\Omega\}$. We use the trace theorem to get the second last inequality and the last inequality is due to that $\boldsymbol{u}$ is the solution of the Stokes system (1.1). □

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