On Block Term Tensor Decompositions and Its Applications in Blind Signal Separation

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***Abstract*—We give proofs, phrased in geometric language, of the uniqueness of expressions for block term tensor decompositions introduced by De Lathauwer. And we try to demonstrate our results with some numerical experiments in blind signal separation.**

I. INTRODUCTION

A *tensor decomposition* is the expression of a tensor as a linear combination of other tensors (presumably of lower rank). It arises in numerous application areas (see [1]). In signal processing, the tensor encodes data from received signals, and one wants to decompose the tensor to obtain the transmitted signals. If there is a failure of the uniqueness of decomposition, one cannot definitively determine what the transmitted signals are. In biomedical engineering and medicine, many applications exist in the formation of Brain-Computer Interface for handicapped persons. In communications, similar problems arise: receivers need to eliminate interferences in order to improve on performance. If tensor tools can be used to compute finitely many directions of arrivals of interferences, one can construct notch filters matched to these directions. Therefore, the study of the uniqueness property of this kind of tensor decompositions is of interest and important to mathematicians and also to applied scientists.

Recently, De Lathauwer [2–4] has introduced the concept of *block term tensor decompositions* ($\mathbf{BTD}$), which is natural for certain source separation problems in signal processing and often has better uniqueness properties than the tensor rank decomposition (referred to CANDECOMP or PARAFAC). Block terms are only required to have low "multilinear rank". $\mathbf{BTD}$ allows a fundamentally new approach to signal separation. An analogy is that CANDECOMP or PARAFAC splits data in "atoms" (rank-1 terms) while BTD splits data in molecules (made of atoms). CANDECOMP or PARAFAC owes much of its success to rank-1 terms that capture the essence of components that are in fact more complex and better represented by a block term. That is, it is unlikely that real-life data components (e.g., of natural images) have all columns proportional, all rows proportional, and so on, like a rank-1 term; low (multilinear) rank, as in $\mathbf{BTD}$, is more likely. In addition, block terms may model multidimensional sources, variations around mean activity, mildly nonlinear phenomena, drifts of setting points, frequency shifts, mildly convolutive mixtures, and so on. Efficient variants may become a true alternative to current mainstream algorithms for signal separation. For example, in spread-spectrum systems (see Chapter 13 in [5]) that employ an antenna array at the receiver, the received data are naturally represented by the third-order tensor that shows the signal along the temporal, spectral and spatial axes (see Section 5 Chapter 3 in [5]). For Direct Sequence Code Division Multiple Access (DS-CDMA) systems (see Section 4.5 Chapter 13 in [5]) in simple propagation scenarios that do not cause Inter-Symbol-Interference (ISI) (see Section 5 Chapter 6 in [5]), it is shown that every user contributes a rank-1 term to the received data. When reflections only take place in the far field of the receive array, multiple accesses can be realized through the computation of a decomposition in multilinear rank $(L_1,\ L_1,\ 1), \ldots, (L_R,\ L_R,\ 1)$ terms [6]. In all these applications, the uniqueness of block term tensor decompositions is critically important as it guarantees the unambiguous identification of the signals or information of interest. We refer the reader to [6–11] for the background and applications of block term decomposition in blind source separation.

The results of this paper mainly concern the uniqueness property of $\mathbf{BTD}$ and its applications in blind signal separation. Throughout this paper, for basic definitions, notation and results, we follow [12], which is addressed to both numerical and algebraic geometrical research communities. We hope that the interplay between geometry of tensor and blind signal separation will be fruitful for both sides.

A. *Tensor algebra in geometric language*

Definition I.1. If V is a vector space, let

$$V^* := \{f : V \to \mathbb{R} | f \text{ is linear}\}$$

denote the *dual vector space*. If $\beta \in V^*$ and $b \in W$, one can define a linear map $\beta \otimes b : V \to W$ by $u \mapsto \beta(u)\, b$. Such a linear map has *rank one*. The *rank* of a linear map $f : V \to W$ is the smallest r such that there exist $\beta_1, \ldots, \beta_r \in V^*$ and $b_1, \ldots, b_r \in W$ such that $f = \sum_{i=1}^{r} \beta_i \otimes b_i$.

Definition I.2. Given $\beta_1 \in V_1^*, \beta_2 \in V_2^*, \beta_3 \in V_3^*$, define an element

$$\beta_1 \otimes \beta_2 \otimes \beta_3 \in V_1^* \otimes V_2^* \otimes V_3^*$$

by

$$\beta_1 \otimes \beta_2 \otimes \beta_3 (u_1, u_2, u_3) = \beta_1(u_1)\, \beta_2(u_2)\, \beta_3(u_3)$$

for any $u_i \in V_i$. An element of $V_1^* \otimes V_2^* \otimes V_3^*$ is said to have *rank* one if it is of the form $\beta_1 \otimes \beta_2 \otimes \beta_3$ for some $\beta_i \in V_i^*$. The *rank* of a tensor $\mathcal{T} \in V_1 \otimes V_2 \otimes V_3$, denoted by $R(\mathcal{T})$, is the minimum number r such that $\mathcal{T} = \sum_{u=1}^{r} Z_u$ with each Z_u of rank one.

Definition I.3. Let V_1, V_2, V_3 be vector spaces. A function

$$f : V_1 \times V_2 \times V_3 \to \mathbb{R}$$

is *multilinear* if it is linear in each factor V_l. The space of such multilinear functions is denoted $V_1^* \otimes V_2^* \otimes V_3^*$ and called the *tensor product* of the vector spaces V_1^*, V_2^*, V_3^*. Elements $T \in V_1^* \otimes V_2^* \otimes V_3^*$ are called *tensors*.

Definition I.4. Define the *multilinear rank* (the *duplex rank* or *Tucker rank* of $\mathcal{T} \in V_1 \otimes V_2 \otimes V_3$ to be the 3-tuple of natural numbers

$$\mathrm{R}_{\text{multlin}}(\mathcal{T}) := \left(\dim \mathcal{T}(V_1^*), \dim \mathcal{T}(V_2^*), \dim \mathcal{T}(V_3^*)\right).$$

The number $\dim\left(\mathcal{T}\left(V_j^*\right)\right)$ is sometimes called the *mode j rank* of $\mathcal{T}$.

Definition I.5. (see Definition 11 in [7]) A decomposition of a tensor $\mathcal{T} \in \mathbb{R}^{J \times K \times I}$ in a sum of rank-$(L_r, L_r, 1)$ terms, $1 \le r \le R$, is a decomposition of $\mathcal{T}$ of the form

$$\mathcal{T} = \sum_{r=1}^{R} E_r \otimes a_r,$$

in which the $(J \times K)$ matrix E_r is rank-L_r, $1 \le r \le R$.

It is clear that in $\mathcal{T} = \sum_{r=1}^{R} E_r \otimes a_r$ one can arbitrarily permute the different rank-$(L_r, L_r, 1)$ terms $E_r \otimes a_r$. Also, one can scale E_r, provided that a_r is counter scaled. We call this decomposition to be essentially unique when it is only subject to these trivial indeterminacies.

B. outline of the paper

In Section 2, we give a new proof of De Lathauwer's criterion of uniqueness of block term tensor decompositions (Theorem 2.3 in [8]). Then we apply this criterion to present several conditions for generic uniqueness of tensor decompositions of multilinear rank $(L_r,\ L_r,\ 1)$ terms. In Section 3, we follow L. Chiantini and G. Ottaviani's geometric methods to prove the tangentially weakly defectivity of joins of relevant subspace varieties. In Section 4, we discuss blind signal separation with tensor decomposition model and give several numerical experiments to demonstrate our uniqueness results.

II. Criteria of Uniqueness

As in Chapter 2 in [12], when we need to specify the elements of S and its linear span, we use the notations $\{s_1, s_2, \cdots\}$ and $\langle s_1, s_2, \cdots \rangle$, respectively. Here, we give a new geometric proof of a criterion of uniqueness for block term tensor decomposition, due to De Lathauwer [8].

Theorem II.1. (Theorem 2.3 in [8]) Assume $I \ge R$, $\mathcal{T} = \sum_{r=1}^{R} E_r \otimes a_r$ is essentially unique if and only if for any $E_{j_1},\ \cdots,\ E_{j_s}$, we have

$$\langle E_{j_1},\ \cdots,\ E_{j_s} \rangle \cap \sigma_{L_{j_t}}(\mathbb{R}^{J \times K}) \subset \{E_{j_1},\ \cdots,\ E_{j_s}\}, \quad 1 \le t \le s, \tag{1}$$

where $\sigma_w(\mathbb{R}^{J \times K})$ is the set of matrices in $\mathbb{R}^{J \times K}$ of rank smaller or equal to w.

Proof. $\Leftarrow$

Assume the contrary that $\mathcal{T} = \sum_{r=1}^{R} \tilde{E}_r \otimes a'_r$ is different from $\mathcal{T} = \sum_{r=1}^{R} E_r \otimes a_r$. Since $a_1, \ldots, a_R$ are independent, we have $a'_r = \sum_{j=1}^{R} \alpha_j^r a_j$, where α_j^r are not all zero. From

$$\mathcal{T} = \sum_{r=1}^{R} \tilde{E}_r \otimes a'_r = \sum_{r=1}^{R} (\sum_{j=1}^{R} \alpha_j^r \tilde{E}_j) \otimes a_r.$$

we know that $E_r = \sum_{j=1}^{R} \alpha_j^r \tilde{E}_j$. Taking the inverse of the $R \times R$ matrix $[\alpha_j^r]$, we have $\tilde{E}_r = \sum_{j=1}^{R} \tilde{\alpha}_j^r E_j$. Consequently, there exists $r, j_1, j_2 \in \{1, \ldots, R\}$ such that $j_1 \neq j_1$ and $\tilde{\alpha}_{j_1}^r \cdot \tilde{\alpha}_{j_2}^r \neq 0$. Therefore, we obtain

$$\tilde{E}_r \in \langle E_{j_1}, \ldots, E_{j_s} \rangle \cap \sigma_{L_r}(\mathbb{R}^{J \times K}).$$

But $\tilde{E}_r$ does not belong to $\{E_{j_1}, \ldots, E_{j_s}\}$, contradicts to Eq. (1).

$\Rightarrow$

If there exists $E'_{j_t} \in \langle E_{j_1}, \ldots, E_{j_s} \rangle \cap \sigma_{j_t}\left(\mathbb{R}^{K \times J}\right)$ such that $E'_{j_t} \notin \{E_{j_1}, \ldots, E_{j_s}\}$, without loss of generality, we assume that $E'_{j_t} = E_1 + \chi_2 E_2 + \cdots + \chi_R E_R$. Now

$$\begin{aligned}
&E_1 \otimes a_1 + \cdots + E_R \otimes a_R \\
=&E_1 \otimes a'_{j_t} - E_2 \otimes \chi_2 a_1 - \cdots - E_R \otimes \chi_R a_1 \\
&+ E_2 \otimes a_2 + \cdots + E_R \otimes a_R \\
=&E'_{j_t} \otimes a_1 + E_2 \otimes (a_2 - \chi_2 a_1) + \cdots + E_R \otimes (a_R - \chi_R a_1) \\
=&E'_{j_t} \otimes a_1 + E_2 \otimes a'_2 + \cdots + E_R \otimes a'_R.
\end{aligned}$$

So $\mathcal{T} = \sum_{r=1}^R E_r \otimes a_r$ is not unique. □

Remark III. *Using Kronecker's canonical form (see Chapter 10 in [12] and Chapter IX in [13]), there is a normal form for a general point p of $\sigma_L(\mathbb{R}^{J\times K})$ (L is smaller than J and K), which is*

$$p = b'_1 \otimes c'_1 + \cdots + b'_L \otimes c'_L,$$

and the pencil is

$$\begin{pmatrix} s & & \\ & \ddots & \\ & & s \end{pmatrix}.$$

Theorem III.1. (see Fig. 1 for illustration)

$$\mathcal{T} = \sum_{r=1}^{2} E_r \otimes a_r$$

in definition I.5 is essentially unique if

$$(i):\ I \geq 2,\ J = K > \frac{2L_i + L_j}{2},\ \forall 1 \leq i, j \leq 2;$$

is not unique if

$$(ii):\ I \geq 2,\ J = K = \frac{2L_i + L_j}{2},\ \forall 1 \leq i, j \leq 2.$$

Proof. Case (i):

It is sufficient to prove the case $L_1, L_2 < J = K < \sum_{r=1}^2 L_r$. Let B and C denote vector spaces of dimensions J, K respectively. Split $B = B_1 \oplus B_0 \oplus B_2$ and $C = C_1 \oplus C_0 \oplus C_2$, where B_1, B_0, B_2, C_1, C_0, and C_2 are of dimension $L_1 - l_b$, l_b, $L_2 - l_b$, $L_1 - l_c$, l_c, $L_2 - l_c$, respectively, and $l_b = l_c < \frac{1}{2}L_j$.

Consider

$$\begin{aligned}
E_1 =&b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1-l_b} \otimes c_{1,L_1-l_b} \\
&+ b_{0,1} \otimes c_{1,L_1-l_b+1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\
\in&(B_1 \oplus B_0) \otimes (C_1 \oplus C_0) \\
\cong&\mathbb{R}^{L_1} \otimes \mathbb{R}^{L_1}
\end{aligned}$$

and

$$\begin{aligned}
E_2 =&b_{2,1} \otimes c_{2,1} + \cdots + b_{2,L_1-l_b} \otimes c_{2,L_1-l_b} \\
&+ b_{0,1} \otimes c_{2,L_1-l_b+1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\
\in&(B_2 \oplus B_0) \otimes (C_2 \oplus C_0) \\
\cong&\mathbb{R}^{L_2} \otimes \mathbb{R}^{L_2},
\end{aligned}$$

where $\{b_{0,1}, \ldots, b_{0,l_b}\}$, $\{b_{1,1}, \ldots, b_{1,L_1-l_b}\}$, $\{b_{2,1}, \ldots, b_{2,L_2-l_b}\}$, $\{c_{0,1}, \ldots, c_{0,l_c}\}$, $\{c_{1,1}, \ldots, c_{1,L_1-l_c}\}$, and $\{c_{2,1}, \ldots, c_{2,L_2-l_c}\}$ are bases for B_0, B_1, B_2, C_0, C_1 and C_2 respectively, $J + l_b = L_1 + L_2$, and $K + l_c = L_1 + L_2$.

Let

$$E'_j = b'_1 \otimes c'_1 + \cdots + b'_{L_j} \otimes c'_{L_j}$$

be a general point of $\sigma_{L_j}(\mathbb{R}^{J\times K})$ and set

$$E'_{j_t} = \chi_1 E_1 + \chi_2 E_2.$$

If χ_1, χ_2 are both nonzero, since $l_b = l_c < \frac{1}{2}L_j$, the pencil

$$\begin{pmatrix}
\chi_1 &&&&&&&& \\
& \ddots &&&&&&& \\
&& \chi_1 &&&&&& \\
&&& \chi_1+\chi_2 &&&&& \\
&&&& \ddots &&&& \\
&&&&& \chi_1+\chi_2 &&& \\
&&&&&& \chi_2 && \\
&&&&&&& \ddots & \\
&&&&&&&& \chi_2
\end{pmatrix}$$

has rank bigger than L_j, which implies that E'_j is not a matrix in $\sigma_{L_j}(\mathbb{R}^{J\times K})$. Therefore E'_j is one of $\{E_1, E_2\}$. And the uniqueness follows from Theorem II.1.

Case (ii):

In the second case, we have $l_b = l_c = \frac{1}{2}L_j$, so

$$\begin{aligned}
E_1 =&b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1-l_b} \otimes c_{1,L_1-l_b} \\
&+ b_{0,1} \otimes c_{0,1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\
\in&(B_1 \oplus B_0) \otimes (C_1 \oplus C_0) \\
\cong&\mathbb{R}^{L_1} \otimes \mathbb{R}^{L_1}
\end{aligned}$$

and

$$\begin{aligned}
E_2 =&b_{2,1} \otimes c_{2,1} + \cdots + b_{2,L_2-l_c} \otimes c_{2,L_2-l_c} \\
&+ b_{0,1} \otimes c_{0,1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\
\in&(B_2 \oplus B_0) \otimes (C_2 \oplus C_0) \\
\cong&\mathbb{R}^{L_2} \otimes \mathbb{R}^{L_2}.
\end{aligned}$$

Let E'_j be a general point of $\sigma_{L_j}(\mathbb{R}^{J\times K})$ and set

$$E'_{j_t} = E_1 - E_2,$$

then we have

$$\begin{aligned}
E'_j =&b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1-l_b} \otimes c_{1,L_1-l_b} \\
&- b_{2,1} \otimes c_{2,1} - \cdots - b_{2,L_2-l_c} \otimes c_{2,L_2-l_c}
\end{aligned}$$

has rank equal to L_j, which implies that E'_j is a matrix in $\sigma_{L_j}(\mathbb{R}^{J\times K})$. But E'_j is not in $\{E_1, E_2\}$. The non-uniqueness follows from Theorem II.1.

□

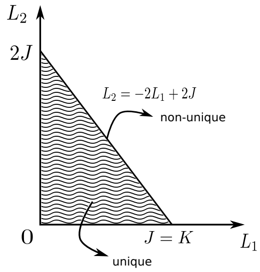

Fig. 1: Illustration for Theorem III.1

Example III.1. For $\mathcal{T} \in \mathbb{R}^{3\times3\times2}$, considering the decomposition in a sum of multilinear rank $(2, 2, 1)$, we have

$$\begin{aligned}\mathcal{T} &= (b_1 \otimes c_1 + b_2 \otimes c_2) \otimes a_1 + (b_2 \otimes c_2 + b_3 \otimes c_3) \otimes a_2 \\ &= (b_1 \otimes c_1 - b_3 \otimes c_3) \otimes a_1 + (b_2 \otimes c_2 + b_3 \otimes c_3) \otimes (a_1 + a_2)\end{aligned}$$

where $\{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}, \{a_1, a_2\}$ are bases for $\mathbb{R}^3, \mathbb{R}^3, \mathbb{R}^2$. So this is not unique.

Theorem III.2. $\mathcal{T} = \sum_{r=1}^{2} E_r \otimes a_r$ in definition I.5 is not unique if

$$I \geq 2,\ \min\{J, K\} = \max\{L_1, L_2\},\ \max\{J, K\} = L_1 + L_2.$$

Proof. It is sufficient to prove the case $L_1 \leq L_2 = K,\ J = L_1 + L_2$. Let B and C denote vector spaces of dimensions J, K respectively. Split $B = B_1 \oplus B_2$ and $C = C_1 \oplus C_2$, where B_1, B_2, C_1, and C_2 are of dimensions L_1, L_2, L_1, $L_2 - L_1$.

Consider

$$\begin{aligned}E_1 &= b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1} \otimes c_{1,L_1} \\ &\in B_1 \otimes C_1 \\ &\cong \mathbb{R}^{L_1} \otimes \mathbb{R}^{L_1}\end{aligned}$$

and

$$\begin{aligned}E_2 =& b_{2,1} \otimes c_{1,1} + \cdots + b_{2,L_1} \otimes c_{1,L_1} \\ &+ b_{2,L_1+1} \otimes c_{1,L_1+1} + \cdots + b_{2,L_2} \otimes c_{2,L_2} \\ \in& B_2 \otimes (C_1 \oplus C_2) \\ \cong& \mathbb{R}^{L_2} \otimes \mathbb{R}^{L_2},\end{aligned}$$

where $\{b_{1,1}, \ldots, b_{1,L_1}\}$, $\{b_{2,1}, \ldots, b_{2,L_2}\}$, $\{c_{1,1}, \ldots, c_{1,L_1}\}$, and $\{c_{2,L_1+1}, \ldots, c_{2,L_2}\}$ are bases for B_1, B_2, C_1 and C_2, respectively.

Let E'_j be a general point of $\sigma_{L_j}(\mathbb{R}^{J\times K})$ and set

$$E'_{j_t} = E_1 + E_2.$$

Then we have

$$\begin{aligned}E'_j =& (b_{1,1} + b_{2,1}) \otimes c_{1,1} + \cdots + (b_{1,1} + b_{2,1}) \otimes c_{1,L_1} \\ &+ b_{2,L_1+1} \otimes c_{2,L_1+1} + \cdots + b_{2,L_2} \otimes c_{2,L_2}.\end{aligned}$$

E'_j has rank equal to L_2, which implies that E'_j is a matrix in $\sigma_{L_2}(\mathbb{R}^{J\times K})$. But E'_j is not in $\{E_1, E_2\}$. The non-uniqueness follows from Theorem II.1.

□

Example III.2. For $\mathcal{T} \in \mathbb{R}^{4\times2\times2}$, considering the decomposition in a sum of multilinear rank $(2, 2, 1)$, we have

$$\begin{aligned}\mathcal{T} =& (b_1 \otimes c_1 + b_2 \otimes c_2) \otimes a_1 \\ &+ (b_3 \otimes c_1 + b_4 \otimes c_2) \otimes a_2 \\ =& ((b_1 + b_3) \otimes c_1 + (b_2 + b_4) \otimes c_2) \otimes a_1 \\ &+ (b_3 \otimes c_1 + b_4 \otimes c_2) \otimes (a_2 - a_1)\end{aligned}$$

where $\{b_1, b_2, b_3, b_4\}, \{c_1, c_2\}, \{a_1, a_2\}$ are basis for $\mathbb{R}^4, \mathbb{R}^2, \mathbb{R}^2$. So this is not unique.

Theorem III.3. $\mathcal{T} = \sum_{r=1}^{R} E_r \otimes a_r$ in definition I.5 is essentially unique if

$$\begin{aligned}&I \geq R,\ K \geq \sum_{r=1}^{R} L_r,\ J \geq 2\max\{L_i\}, \\ &\binom{J}{\max\{L_i\}} \geq R, L_i + L_j > L_k \quad \forall 1 \leq i, j, k \leq R.\end{aligned}$$

Proof. It is sufficient to prove the case $I = R$, $K = \sum_{r=1}^{R} L_r$. Let B and C denote vector spaces of dimensions J, K respectively. Choose the splitting of C as $C = \bigoplus_{1\leq r\leq R} C_r$, and fix a basis $\{b_1, \ldots, b_J\}$ for B.

Without loss of generality, for $1 \leq p \leq R$, we can assume

$$\begin{aligned}E_{j_p} =& b_{j_p,1} \otimes c_{j_p,1} + b_{j_p,2} \otimes c_{j_p,2} \\ &+ \cdots + b_{j_p,L_{j_p}} \otimes c_{j_p,L_{j_p}} \\ \in& B_{j_p} \otimes C_{j_p},\end{aligned}$$

where $\{b_{j_p,1}, \ldots, b_{j_p,L_{j_p}}\} \subset \{b_1, \ldots, b_J\}$ (since $J \geq 2\max\{L_i\}, \binom{J}{\max\{L_i\}} \geq R$), $\{c_{j_p,1}, \ldots, c_{j_p,L_{j_p}}\}$ are bases for B_{j_p}, C_{j_p}, respectively. Further, let

$$E'_{j_t} = b'_1 \otimes c'_1 + \cdots + b'_{L_{j_t}} \otimes c'_{L_{j_t}}$$

be a general point of $\sigma_{L_{j_t}}(\mathbb{R}^{J\times K})$ and set

$$\begin{aligned}E'_{j_t} &= \sum_{1\leq p\leq s} \chi_p E_{j_p} \\ &= \sum_{1\leq p\leq s} \chi_p (b_{j_p,1} \otimes c_{j_p,1} + b_{j_p,2} \otimes c_{j_p,2} \\ &\quad + \cdots + b_{j_p,L_{j_p}} \otimes c_{j_p,L_{j_p}}).\end{aligned}$$

If there exist χ_μ, χ_ν, which are both nonzero, the pencil

$$\begin{pmatrix} x_\mu & & & & & & & & \\ & \ddots & & & & & & & \\ & & x_\mu & & & & & & \\ & & & x_\mu + x_\nu & & & & & \\ & & & & \ddots & & & & \\ & & & & & x_\mu + x_\nu & & & \\ & & & & & & x_\nu & & \\ & & & & & & & \ddots & \\ & & & & & & & & x_\nu \end{pmatrix}$$

has rank at least $L_{j_\mu} + L_{j_\nu}$, which is bigger than L_{j_t}. This implies that E'_{j_t} is not a matrix in $\sigma_{L_{j_t}}(\mathbb{R}^{J\times K})$. Therefore, we prove that $E'_{j_t} \in \{E_{j_1}, \ldots, E_{j_s}\}$. The uniqueness follows from Theorem II.1. □

The following Corollary, which was Theorem 2.1 in [8], can be obtained easily using elementary combinatorics.

Corollary. *(Theorem 2.1 in [8])* $\mathcal{T} = \sum_{r=1}^{R} E_r \otimes a_r$ *in definition I.5 is essentially unique if*

$$I \geq R,\ J,\ K \geq \sum_{r=1}^{R} L_r,\ L_i + L_j > L_k \quad \forall 1 \leq i,j,k \leq R.$$

Remark IV. *The assumption* $I \geq R$ *could be replaced to be* $I \geq 2$*: see condition* C *in [14].*

V. Experiments in Blind Signal Separation

In Chapter 12 in [12], J.M. Landsberg discussed several signal processing applications that are considered as natural tensor decomposition models.

Definition V.1. Blind source(signal) separation (BSS) is the separation of a set of source signals from a set of mixed signals, without the aid of information (or with very little information) about the source signals or the mixing process. This problem is in general highly underdetermined, but useful solutions can be derived under a surprising variety of conditions.

As in section 3 in [8], we consider the following data model. Assume we have R source signals being linearly mixed into K observed signals. For each signal, N samples are available. The following data mode is used in BSS:

$$Y = MS,$$

with $Y \in \mathbb{R}^{K\times N}$ containing the observed data, $S \in \mathbb{R}^{R\times N}$ the R unknown source signals, $M \in \mathbb{R}^{K\times R}$ the unknown mixing matrix. The goal for BSS is to recover the unknown sources in S and the unknown mixing vectors in M, given only the observed data Y. We map each observed signal (each row in Y) to a Hankel matrix. Stacking these Hankel matrices, we obtain a tensor which is assumed to be of low miltilinear rank. By decomposing the tensorized data, one can immediately identify the mixing vectors.

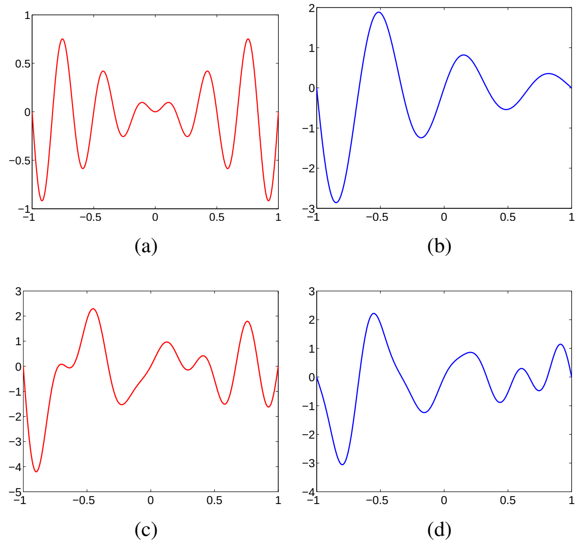

Fig. 2: (a)-(b) show the true signals; (c)-(d) show the observed signals.

BTD can, at least in some interesting cases, be efficiently performed on computer. Some programs can be used for the synthesis of artificial signals. Our further discussion will be on experiments similar to [15].

A. Experiment

In this section, we consider the application of **BTD** to BSS problem. We are given two sources:

$$\begin{aligned} s_1(t) &= \sin(6\pi t)\cdot t, \\ s_2(t) &= e^{-1.25t}\sin(3\pi t), \end{aligned} \tag{2}$$

that are uniformly sampled over $[-1, 1]$ with $N = 501$ samples. The true source is denoted as $S = [s_1, s_2]^T \in \mathcal{R}^{2\times 501}$. The mixing matrix M is given by

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

We obtain the mixed signals $Y = M \cdot S$. The true and observed signals are shown in Fig. 2. We obtain the mixed signals $Y = M \cdot S$. The true and observed signals are shown in Fig. 2. The observations, Y, are mapped to a tensor $\mathcal{T} \in \mathcal{R}^{251\times 251\times 2}$. We compress $\mathcal{T}$ to size $(4\times 4\times 2)$ and $(4\times 2\times 2)$, respectively. The compressed tensors are decomposed in a sum of three rank-(2,2,1) terms. Figs. 3 and 4 show the signals separated from the observed signals by using the compressed tensors of sizes $(4\times 4\times 2)$ and $(4\times 2\times 2)$, correspondingly. We

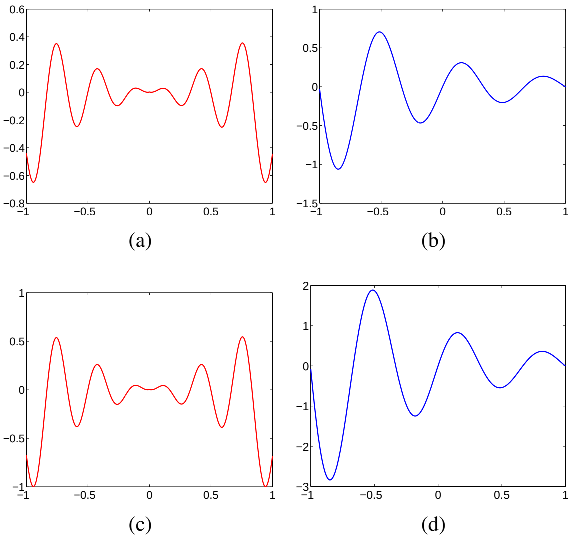

Fig. 3: Separation results using compressed tensor of size $(4 \times 4 \times 2)$: (a)-(b) show the estimated sources; (c)-(d) show the optimally permuted and scaled sources.

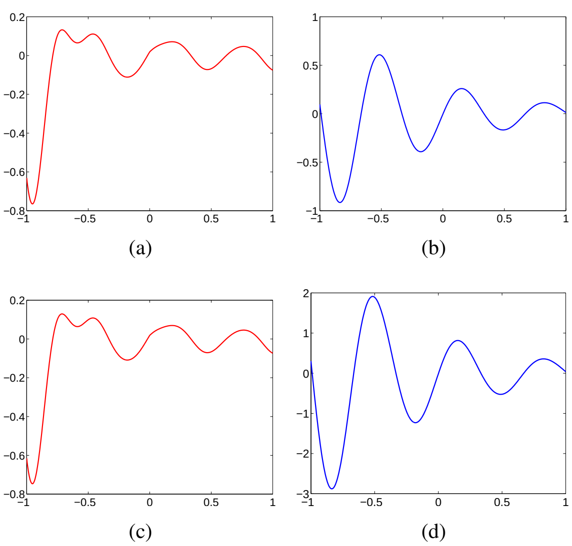

Fig. 4: Separation results using compressed tensor of size $(4 \times 2 \times 2)$: (a)-(b) show the estimated sources; (c)-(d) show the optimally permuted and scaled sources.

observe that with compressed tensor of size $(4 \times 4 \times 2)$, which has a unique decomposition (see Theorem II.1), the true signals can be well separated. However, with compressed tensor of size $(4 \times 2 \times 2)$, which has multiple decompositions (see Theorem III.2), the separated signals are different from the true signals.

Acknowledgment

Qiang Cheng is the corresponding author.

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