TANGENTIAL WEAK DEFECTIVITY OF JOIN OF SUBSPACE VARIETIES

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Abstract. Using Terracini's Lemma, we study the tangential weak defectivity of join of subspace varieties.

1. Introduction

In [2], L. Chiantini and G. Ottaviani introduced the tangential weak defectivity of join of projective varieties, which was related to uniqueness conditions concerning tensor decompositions (see [1, 3, 4, 10]). In [12], the authors studied the non-tangentially weakly defective join of subspace varieties and used it to obtain some uniqueness results of block term tensor decompositions. The results of this paper mainly concern the tangentially weakly defective join of subspace varieties.

Throughout this paper, for basic definitions, notation and results, we follow [8].

Firstly we recall some basic concepts in algebraic geometry.

1.1. **Notations.** As in [8], for a finite dimensional complex vector space V, $\mathbb{P}V$ denotes the projective space associated to V, π denotes the projection of $V\backslash\{0\}$ onto $\mathbb{P}V$; for a variety $X \subset \mathbb{P}V$, $\hat{X} \subset V$ denotes its inverse image under the projection π, which is the (affine) cone over X in V, and for $x \in X$, $[x]$ denotes $\pi(x)$.

Let S be a subset of $\mathbb{P}V$, then the span $\langle S\rangle$ is by definition the range of π on the usual vector span of $\hat{S}$ in V. The Zariski closure of S in $\mathbb{P}V$ will be denoted by $\bar{S}$.

When we need to specify the elements of S and its linear span, we use the notation $\{s_1, s_2, \cdots\}$ and $\langle s_1, s_2, \cdots\rangle$, respectively.

For $x \in \hat{X}$, $\hat{T}_{[x]}X := \hat{T}_x\hat{X}$ is the affine tangent space to X at $[x]$.

1.2. **Join of subspace varieties.** Let A_j, $1 \le j \le n$, be finite dimensional complex vector spaces.

Definition 1.1. (See Definition 1 in [9]) Let $k_j \le a_j := \dim A_j,\ 1 \le j \le n$ be nonnegative integers. *Subspace varieties*, denoted $Sub_{k_1,\dots,k_n}(A_1 \otimes \ldots \otimes A_n) \in \mathbb{P}(A_1 \otimes \ldots \otimes A_n)$ are defined as

$$\begin{aligned} &Sub_{k_1,\dots,k_n}(A_1 \otimes \ldots \otimes A_n)\\ &:= \overline{\{[T] \in \mathbb{P}(A_1 \otimes \ldots \otimes A_n) \mid \forall j\ \exists A'_j \subset A_j,\ \dim A'_j = k_j,\ T \in A'_1 \otimes \ldots \otimes A'_n\}}. \end{aligned}$$

Definition 1.2. If $X_i,\ i = 1, \dots, k$, $k \le n$ are projective algebraic varieties of $\mathbb{P}^n = \mathbb{P}V, V = \mathbb{C}^{n+1}$, then the *join* of $X_1, \dots, X_k$ is

$$\mathbf{J}(X_1, \dots, X_k) := \overline{\bigcup \{\langle [P_1], \dots, [P_k]\rangle | P_i \in \hat{X}_i,\ 1 \le i \le k\}},$$

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where P_i, $i = 1, \dots, k$, are linearly independent vectors in V. If $X_1 = \cdots = X_k = X$, then we write $\mathbf{J}(X_1, ..., X_k) = \sigma_k(X)$ and we call this the k-th secant variety to X.

The following Terracini's Lemma appears as Proposition 12.11 in [6] and we rephrased it in terms of join of projective varieties.

Theorem 1.3. *Let $P_i \in \hat{X}_i$ be a general point of $\hat{X}_i$ for each $i = 1, \dots, k$, then for $[P] := [P_1 + \ldots + P_k]$,*

$$\hat{T}_{[P]}\mathbf{J}(X_1, \dots, X_k) = \hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k. \tag{1.1}$$

Definition 1.4. (see Definition 2.6 in [2]) Let $P_i \in \hat{X}_i$ be a general point of $\hat{X}_i$, $1 \le i \le k$. When for any $j \in \{1, \dots, k\}$ and $Q_j \in \hat{X}_j$, $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k$ contains $\hat{T}_{[Q_j]}X_j$ only if $[Q_j] \in \{[P_1], \dots, [P_k]\}$, we say $\mathbf{J}(X_1, \dots, X_k)$ is *not tangentially weakly defective*. Otherwise, we say that $\mathbf{J}(X_1, \dots, X_k)$ is *tangentially weakly defective*. $\mathbf{J}(X_1, \dots, X_k)$ is *weakly defective* if the general hyperplane which is tangent to $X_1, \dots, X_k$ at some k general points $[P_1], \dots, [P_k]$, is also tangent at some other point $[Q_j] \neq [P_1], \dots, [P_k]$, $Q_j \in \hat{X}_j$ for some $j \in \{1, \dots, k\}$. Here general means in an open subset of the set of hyperplanes which are tangent to $X_1, \dots, X_k$ at k general points $[P_1], \cdots, [P_k]$ (see [5]).

Remark 1.5. By semicontinuity (see Theorem III.12.8 of [7]), if for one particular set of general points $\{P_1, \dots, P_k\}$, $\hat{T}_{[P_1]}X_1 + \cdots + \hat{T}_{[P_k]}X_k$ contains $\hat{T}_{[Q_j]}X_j$ only if $[Q_j] \in \{[P_1], \dots, [P_k]\}$, then $\mathbf{J}(X_1, \dots, X_k)$ is not tangentially weakly defective.

Remark 1.6. In [12], it is proved that $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \dots, Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is not tangentially weakly defective, if

$$I \ge R,\ J,\ K \ge \sum_{r=1}^{R} L_r,\ L_i + L_j > L_k \quad \forall 1 \le i, j, k \le R.$$

The main results in this paper are the following.

Theorem 1.7. *Assume $I \ge R$, $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \dots, Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly defective, if for some $[X_{j_1}] \in \sigma_{L_{j_1}}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$, $\cdots$, $[X_{j_s}] \in \sigma_{L_{j_s}}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$, $j_1, \dots, j_s \in \{1, \dots, R\}$, there exists $[X'_{jt}] \in \langle [X_{j_1}],\ \cdots,\ [X_{j_s}] \rangle \cap \sigma_{L_{j_t}}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$, but $[X'_{jt}]$ is not in $\{[X_{j_1}],\ \cdots,\ [X_{j_s}]\}$.*

And we use the above theorem to obtain the following tangential weak defectivity results: $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), Sub_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly defective if $I \ge 2,\ J = K = \frac{2L_i + L_j}{2},\ \forall 1 \le i, j \le 2$ ($\frac{2L_i + L_j}{2}$ is not independent from $i,\ j$) or $I \ge 2,\ \min\{J, K\} = \max\{L_1, L_2\}$.

Theorem 1.8. *$J\ \ Sub_{1,L_1,L_1}\ \ \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\ \ , \dots, Sub_{1,L_R,L_R}\ \ \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K$ is weakly defective if $I \ge 2,\ J, K \ge L_1 + \cdots + L_R,\ L_1, \dots, L_R \ge 2$.*

Theorem 1.9. *$J\ \ Sub_{2,L_1,L_1}\ \ \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\ \ , \dots, Sub_{2,L_R,L_R}\ \ \mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K$ is tangentially weakly defective if $I \ge 2,\ J, K \ge L_1 + \cdots + L_R,\ L_1, \dots, L_R \ge 2$.*

2. Proof of Theorem 1.7 and Its Corollaries

Proof. Without loss of generality, we assume that $X'_{j_t} = X_1 + \chi_2 X_2 + \cdots + \chi_R X_R$. Now

$$\begin{aligned} & a_1 \otimes X_1 + \cdots + a_R \otimes X_R \\ & = a_1 \otimes X'_{j_t} - \chi_2 a_1 \otimes X_2 - \cdots - \chi_R a_1 \otimes X_R + a_2 \otimes X_2 + \cdots + a_R \otimes X_R \\ & = a_1 \otimes X'_{j_t} + (a_2 - \chi_2 a_1) \otimes X_2 + \cdots + (a_R - \chi_R a_1) \otimes X_R \\ & = a_1 \otimes X'_{j_t} + a'_2 \otimes X_2 + \cdots + a'_R \otimes X_R. \end{aligned}$$

Let $P_i = a_i \otimes X_i,\ 1 \leq i \leq R$ and $Q = a_1 \otimes X'_{j_t}$. Using Terracini's Lemma 1.3, we have

$$\begin{aligned} & \hat{T}_{[P_1+\cdots+P_R]} \mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \ldots, Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)) \\ & = \hat{T}_{[P_1]} Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K) + \cdots + \hat{T}_{[P_R]} Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K) \end{aligned}$$

contains $\hat{T}_{[Q]} Sub_{1,L_{j_t},L_{j_t}}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K)$. By Remark 1.5, we have $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), \ldots, Sub_{1,L_R,L_R}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly defective.

□

Corollary 2.1. *$\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), Sub_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly defective if $I \geq 2,\ J = K = \frac{2L_i + L_j}{2},\ \forall 1 \leq i, j \leq 2$.*

Proof. It is sufficient to prove the case $L_1, L_2 < J = K < \sum_{r=1}^{2} L_r$. Let A, B and C denote complex vector spaces of dimensions I, J, K respectively. Split $B = B_1 \oplus B_0 \oplus B_2$ and $C = C_1 \oplus C_0 \oplus C_2$, where B_1, B_0, B_2, C_1, C_0, and C_2 are of dimensions $L_1 - l_b$, l_b, $L_2 - l_b$, $L_1 - l_c$, l_c, $L_2 - l_c$, respectively, and $l_b = l_c = \frac{1}{2} L_j$.

Recall that there is a normal form for a general point $[p]$ of $\sigma_L(\mathbb{P}B \times \mathbb{P}C)$ (L is smaller than $\dim B$ and $\dim C$), which is of the form

$$p = b_1 \otimes c_1 + \cdots + b_L \otimes c_L.$$

We may assume that all the $b_i, 1 \leq i \leq L$ are linearly independent in B as well as all the $c_i, 1 \leq i \leq L$ (Otherwise one would have $[p] \in \sigma_{L-1}(\mathbb{P}B \times \mathbb{P}C)$). Then a general element $[\varphi] \in Sub_{1,L,L}(\mathbb{P}A \otimes \mathbb{P}B \times \mathbb{P}C)$ is of the form

$$\varphi = a \otimes (b_1 \otimes c_1 + \cdots + b_L \otimes c_L),$$

where a is a nonzero vector in A.

So we my consider

$$\begin{aligned} X_1 &= b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1 - l_b} \otimes c_{1,L_1 - l_b} + b_{0,1} \otimes c_{0,1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\ & \in (B_1 \oplus B_0) \otimes (C_1 \oplus C_0) \cong \mathbb{C}^{L_1} \otimes \mathbb{C}^{L_1} \end{aligned}$$

and

$$\begin{aligned} X_2 &= b_{2,1} \otimes c_{2,1} + \cdots + b_{2,L_2 - l_c} \otimes c_{2,L_2 - l_c} + b_{0,1} \otimes c_{0,1} + \cdots + b_{0,l_b} \otimes c_{0,l_c} \\ & \in (B_2 \oplus B_0) \otimes (C_2 \oplus C_0) \cong \mathbb{C}^{L_2} \otimes \mathbb{C}^{L_2}. \end{aligned}$$

Let X'_j be a general point of $\sigma_{L_j}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$ and set

$$X'_j = X_1 - X_2,$$

then we have

$$X_j' = b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1-l_b} \otimes c_{1,L_1-l_b} - b_{2,1} \otimes c_{2,1} - \cdots - b_{2,L_2-l_c} \otimes c_{2,L_2-l_c}$$

has rank equal to L_j, which implies that $[X_j']$ is a point in $\sigma_{L_j}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$. But $[X_j']$ is not in $\{[X_1], [X_2]\}$. From Theorem 1.7, we know $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), Sub_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly defective. □

Corollary 2.2. *$\boldsymbol{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), Sub_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly-defective if $I \geq 2$, $\min\{J, K\} = \max\{L_1, L_2\}$.*

Proof. It is sufficient to prove the case $L_1 \leq L_2 = K$. Let B and C denote vector spaces of dimensions J, K respectively. Split $B = B_1 + B_2$ and $C = C_1 \oplus C_2$, where B_1, B_2, C_1, and C_2 are of dimensions L_1, L_2, L_1, $L_2 - L_1$, respectively.

Consider

$$X_1 = b_{1,1} \otimes c_{1,1} + \cdots + b_{1,L_1} \otimes c_{1,L_1} \in B_1 \otimes C_1 \cong \mathbb{C}^{L_1} \otimes \mathbb{C}^{L_1}$$

and

$$\begin{aligned} X_2 &= b_{2,1} \otimes c_{1,1} + \cdots + b_{2,L_1} \otimes c_{1,L_1} + b_{2,L_1+1} \otimes c_{1,L_1+1} + \cdots + b_{2,L_2} \otimes c_{2,L_2} \\ &\in B_2 \otimes (C_1 \oplus C_2) \cong \mathbb{C}^{L_2} \otimes \mathbb{C}^{L_2}, \end{aligned}$$

where $\{b_{1,1}, \ldots, b_{1,L_1}\}$, $\{b_{2,1}, \ldots, b_{2,L_2}\}$, $\{c_{1,1}, \ldots, c_{1,L_1}\}$, and $\{c_{2,L_1+1}, \ldots, c_{2,L_2}\}$ are bases for B_1, B_2, C_1 and C_2, respectively.

Let $[X_j']$ be a general point of $\sigma_{L_j}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$ and set

$$X_j' = X_1 + X_2.$$

Then we have

$$X_j' = (b_{1,1} + b_{2,1}) \otimes c_{1,1} + \cdots + (b_{1,L_1} + b_{2,L_1}) \otimes c_{1,L_1} + b_{2,L_1+1} \otimes c_{2,L_1+1} + \cdots + b_{2,L_2} \otimes c_{2,L_2}.$$

X_j' has rank equal to L_2, which implies that $[X_j']$ is a point in $\sigma_{L_2}(\mathbb{P}^{J-1} \times \mathbb{P}^{K-1})$. But $[X_j']$ is not in $\{[X_1], [X_2]\}$. From Theorem 1.7, we know $\mathbf{J}(Sub_{1,L_1,L_1}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K), Sub_{1,L_2,L_2}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K))$ is tangentially weakly defective. □

3. Proof of Theorem 1.8

Lemma 3.1. *Let A, B, C be complex vector spaces of dimensions I, J, K and $\varphi = a_i \otimes (b_1 \otimes c_1 + \cdots + b_L \otimes c_L)$, where a_i, b_i, c_i are basis for A, B, C, respectively. We have*

$$\hat{T}_\varphi Sub_{1,L,L}(A \otimes B \otimes C) = A \otimes \left\langle \sum_{i=1} b_i \otimes c_i \right\rangle + B \otimes \langle a_i \otimes c_i \rangle + C \otimes \langle a_i \otimes b_i \rangle$$

Proof. Pick a curve $\varphi(t) = a_i(t) \otimes \left(\sum_{i=1}^{L} b_i(t) \otimes c_i(t) \right)$, where $\varphi(0) = \varphi$. Taking derivative with respect to t, we have

$$\varphi'(0) = a_i'(0) \otimes \left(\sum_{i=1} b_i \otimes c_i \right) + a_i \otimes \left(\sum_{i=1} b_i'(0) \otimes c_i \right) + a_i \otimes \left(\sum_{i=1} b_i \otimes c_i'(0) \right).$$

Since $a'(0), b'(0), c'(0)$ are arbitrary vectors in A, B, C, we obtained Lemma 3.1. □

Proof of Theorem 1.8. It is sufficient to prove the case $I = 2, J = K = \sum_{r=1}^{R} L_r$.

Let A, B and C be complex vector spaces of dimensions I, J, K, respectively. Split $B = \bigoplus_{1\le q\le R} B_q$ and $C = \bigoplus_{1\le r\le R} C_r$, where for $1 \le q, r \le R$, B_q and C_r are of dimensions L_q, L_r, respectively.

Choose a general set $\{\varphi_p \in \widehat{Sub}_{1,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K) : 1 \le p \le R\}$. Without loss of generality, we can assume

$$\varphi_p = (a_1 + \lambda^p a_2) \otimes (b_{p,1} \otimes c_{p,1} + b_{p,2} \otimes c_{p,2} + \cdots + b_{p,L_p} \otimes c_{p,L_p}) \in A_p \otimes B_p \otimes C_p,$$

for any $1 \le p \le R$, where $\{a_1 + \lambda^p a_2\}$, $\{b_{p,1}, \ldots, b_{p,L_p}\}$ and $\{c_{p,1}, \ldots, c_{p,L_p}\}$ are bases for A_p, B_p, C_p, respectively. Note that for a general set $\{\varphi_p \in A_p \otimes B_p \otimes C_p : 1 \le p \le R\}$.

Using Terracini's Lemma 1.3 and 3.1, a general hyperplane tangent to

$$J\left(Sub_{1,L_1,L_1}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right), \ldots, Sub_{1,L_R,L_R}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right)\right)$$

at $[\varphi_1 + \cdots + \varphi_R]$ is of the form

$$H = \sum_{1\le p\le R} (\lambda_p a_1^* - a_2^*) \otimes \left(\sum_{1\le j\ne k\le L_p} \mu_{p,j,k} b_{p,j}^* \otimes c_{p,k}^* + \mu'_{p,j,k}\left(b_{p,j}^* \otimes c_{p,j}^* - b_{p,k}^* \otimes c_{p,k}^*\right)\right).$$

It is straightforward to see that H is tangent to $Sub_{1,L_p,L_p}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right)$ at

$$\varsigma_p = (a_1 + \lambda_p a_2) \otimes \sum_{j\ne k} \frac{j-k}{|j-k|} \mu_{p,j,k} b_{p,j} \otimes c_{p,k} + \mu'_{p,j,k}\left(b_{p,j} \otimes c_{p,j} + b_{p,k} \otimes c_{p,k}\right),$$

which is different from $\varphi_1, \cdots, \varphi_R$. So

$$J\left(Sub_{1,L_1,L_1}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right), \ldots, Sub_{1,L_R,L_R}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right)\right)$$

is weakly defective. □

4. Proof of Theorem 1.9

Proof. It is sufficient to prove the case $I = 2, J = K = \sum_{r=1}^{R} L_r$.

Let A, B and C be complex vector spaces of dimensions I, J, K, respectively. Split $B = \bigoplus_{1\le q\le R} B_q$ and $C = \bigoplus_{1\le r\le R} C_r$, where for $1 \le q, r \le R$, B_q and C_r are of dimensions L_q, L_r, respectively.

Choose a general set $\{\varphi_p \in \widehat{Sub}_{2,L_p,L_p}(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K) : 1 \le p \le R\}$. Using Weierstrass normal forms (see Chapter 10 in [8] and Chapter IX in [11]) of tensors of multilinear rank $(2, L_p, L_p)$, we can assume

$$\varphi_1 = \sum_{i=1}^{L_1} (a_1 + \lambda_i a_2) \otimes b_i \otimes c_i,$$

$$\varphi_2 = \sum_{i=L_1+1}^{L_1+L_2} (a_1 + \lambda_i a_2) \otimes b_i \otimes c_i,$$

$$\vdots$$

$$\varphi_R = \sum_{i=L_1+\cdots+L_{R-1}+1}^{L_1+\cdots+L_R} (a_1 + \lambda_i a_2) \otimes b_i \otimes c_i,$$

where $\{a_1, a_2\}, \{b_1, \dots, b_{L_1+\dots+L_R}\}, \{c_1, \dots, c_{L_1+\dots+L_R}\}$ are bases for A, B, C.

Consider

$$\psi_1 = \sum_{i=1}^{L_1-1} (a_1 + \lambda_i a_2) \otimes b_i \otimes c_i + (a_1 + \lambda_{L_1+1} a_2) \otimes b_{L_1+1} \otimes c_{L_1+1}$$

$$\psi_2 = \sum_{i=L_1+2}^{L_1+L_2} (a_1 + \lambda_i a_2) \otimes b_i \otimes c_i + (a_1 + \lambda_{L_1} a_2) \otimes b_{L_1} \otimes c_{L_1}.$$

It is obviously that $\varphi_1 + \dots + \varphi_R = \psi_1 + \psi_2 + \varphi_3 \dots + \varphi_R$. So from Terracini's Lemma 1.3, we know $J\left(Sub_{2,L_1,L_1}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right), \dots, Sub_{2,L_R,L_R}\left(\mathbb{C}^I \otimes \mathbb{C}^J \otimes \mathbb{C}^K\right)\right)$ is tangentially weakly defective. □

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References

[1] E. Ballico and A. Bernardi, *Symmetric tensor rank with a tangent vector: a generic uniqueness theorem*, Proceedings of the American Mathematical Society, 140 (2012), pp. 3377–3384.

[2] L. Chiantini and G. Ottaviani, *On generic identifiability of 3-tensors of small rank*, SIAM Journal on Matrix Analysis and Applications, 33 (2012), pp. 1018–1037.

[3] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven, *An algorithm for generic and low-rank specific identifiability of complex tensors*, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 1265–1287.

[4] ——, *On generic identifiability of symmetric tensors of subgeneric rank*, arXiv preprint arXiv:1504.00547, (2015).

[5] L. Chiantini and D. Sacchi, *Segre functions in multiprojective spaces and tensor analysis*. Trends Hist. Sci. to appear, 2016.

[6] D. Eisenbud and J. Harris, *3264 & all that: Intersection theory in algebraic geometry*, preparation, to appear, (2010).

[7] R. Hartshorne, *Algebraic geometry*, Graduate texts in mathematics, Springer, New York, 1977.

[8] J. M. Landsberg, *Tensors: geometry and applications*, American Mathematical Society, 2012.

[9] J. M. Landsberg and J. Weyman, *On the ideals and singularities of secant varieties of segre varieties*, Bulletin of the London Mathematical Society, (2007).

[10] G. OTTAVIANI, *Identifiability and weak defectivity informal notes for lukecin school september 2-6, 2013*, (2013).

[11] H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices*, Courier Corporation, 2004.

[12] M. Yang, *On partial and generic uniqueness of block term tensor decompositions*, ANNALI DELL'UNIVERSITA'DI FERRARA, 60 (2014), pp. 465–493.

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