ATIYAH AND TODD CLASSES ARISING FROM INTEGRABLE DISTRIBUTIONS

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ABSTRACT. In this paper, we study the Atiyah class and Todd class of the DG manifold $(F[1], d_F)$ corresponding to an integrable distribution $F \subset T_{\mathbb{K}}M = TM \otimes_{\mathbb{R}} \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We show that these two classes are canonically identical to those of the Lie pair $(T_{\mathbb{K}}M, F)$. As a consequence, the Atiyah class of a complex manifold X is isomorphic to the Atiyah class of the corresponding DG manifold $(T_X^{0,1}[1], \bar{\partial})$. Moreover, if X is a compact Kähler manifold, then the Todd class of X is also isomorphic to the Todd class of the corresponding DG manifold $(T_X^{0,1}[1], \bar{\partial})$.

CONTENTS

1. INTRODUCTION

Atiyah classes were introduced by Atiyah [1] to characterize the obstruction to the existence of holomorphic connections on a holomorphic vector bundle over a complex manifold. In [5], Kapranov showed that the Atiyah class $\alpha_X \in H^1(X, T_X^\vee \otimes \operatorname{End}(T_X))$ of a Kähler manifold X induces an L_∞ algebra structure on $\Omega_X^{0,\bullet-1}(T_X)$ (see [7, 8] for the complex manifold case), which plays an important role in the formalism of Rozansky-Witten theory. In his seminar work [6], Kontsevich indicated a deep link between the Todd genus of complex manifolds and the Duflo element of Lie algebras. See [9] for the formality theorem for smooth DG manifolds, which implies the Kontsevich-Duflo theorem for Lie algebras and Kontsevich's theorem for

complex manifolds [6] under a unified framework. Inspired by the work of Kapranov and Kontsevich, Chen, Stiénon and Xu introduced the Atiyah class of Lie algebroid pairs in [3], which encodes both the Atiyah class of complex manifolds and the Molino class [14] of foliations as special cases. Towards a different direction, Mehta, Stiénon and Xu [13] introduced Atiyah classes and Todd classes of DG manifolds. The Duflo element of a Lie algebra $\mathfrak{g}$ is identified with the Todd class of the associated DG manifold $(\mathfrak{g}[1], d_{\mathrm{CE}})$.

Let $F \subset T_{\mathbb{K}}M = TM \otimes_{\mathbb{R}} \mathbb{K}$ be a subbundle whose space of sections forms an integrable distribution, where $\mathbb{K}$ is either the field $\mathbb{R}$ or $\mathbb{C}$. When $\mathbb{K} = \mathbb{R}$, it corresponds to the tangent bundle of a regular foliation $\mathcal{F}$ in M. We will also say that F is a regular foliation. Then the graded manifold $F[1]$ together with the Chevalley-Eilenberg differential d_F on the Lie algebroid F over $\mathbb{K}$ is a DG manifold. Thus we may consider the Atiyah class and the Todd class [13] of this DG manifold. Meanwhile, $(T_{\mathbb{K}}M, F)$ is a Lie algebroid pair over M. We may also consider the Atiyah class and the Todd class [3] of the Lie pair $(T_{\mathbb{K}}M, F)$.

The main purpose of this paper is to investigate the relation between these two types of Atiyah classes and Todd classes. First of all, we prove the following theorem in Section 2 regarding the cohomology groups of tensor fields of the DG manifold $(F[1], d_F)$.

Theorem A. *For any pair of non-negative integers (n, m), there exists a canonical isomorphism*

$$\Phi_m^n : \ \mathrm{H}^{\bullet}(\mathfrak{T}_m^n(T(F[1])), L_{d_F}) \xrightarrow{\cong} \mathrm{H}_{\mathrm{CE}}^{\bullet}(F, \mathbb{T}_m^n(T_{\mathbb{K}}M/F))$$

from the cohomology of (n, m)-tensor fields $\mathfrak{T}_m^n(T(F[1]))$ of the DG manifold $(F[1], d_F)$ to the Chevelley-Eilenberg cohomology of the F-module $\mathbb{T}_m^n(T_{\mathbb{K}}M/F)$ (see Equation (1.1) for notations). Moreover, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{H}^{\bullet}(\mathfrak{T}_{m_1}^{n_1}(T(F[1])), L_{d_F}) \otimes_{\mathbb{K}} \mathrm{H}^{\bullet}(\mathfrak{T}_{m_2}^{n_2}(T(F[1])), L_{d_F}) & \xrightarrow{\otimes} & \mathrm{H}^{\bullet}(\mathfrak{T}_{m_1+m_2}^{n_1+n_2}(T(F[1])), L_{d_F}) \\
\downarrow{\scriptstyle (\Phi_{m_1}^{n_1}, \Phi_{m_2}^{n_2})} & & \downarrow{\scriptstyle \Phi_{m_1+m_2}^{n_1+n_2}} \\
\mathrm{H}_{\mathrm{CE}}^{\bullet}(F, \mathbb{T}_{m_1}^{n_1}(T_{\mathbb{K}}M/F)) \otimes_{\mathbb{K}} \mathrm{H}_{\mathrm{CE}}^{\bullet}(F, \mathbb{T}_{m_2}^{n_2}(T_{\mathbb{K}}M/F)) & \xrightarrow{\otimes} & \mathrm{H}_{\mathrm{CE}}^{\bullet}(F, \mathbb{T}_{m_1+m_2}^{n_1+n_2}(T_{\mathbb{K}}M/F)).
\end{array}
$$

Moreover, we have a homotopy contraction between $\Gamma(M, \wedge^{\bullet} F^{\vee} \otimes \wedge^{\bullet}(T_{\mathbb{K}}M/F))$ and the space of polyvector fields $\Gamma(F[1], \wedge^{\bullet} T(F[1]))$ of the DG manifold $(F[1], d_F)$. Note that

$$(\Gamma(F[1], \wedge^{\bullet} T(F[1])), L_{d_F}, \wedge, [-, -]_{SN})$$

is a differential Gerstenhaber algebra, thus also a (-1)-shifted derived Poisson algebra [2], where $[-, -]_{SN}$ is the Schouten-Nijenhuis bracket. Op cit, a (-1)-shifted derived Poisson algebra structure is constructed on $\Gamma(M, \wedge^{\bullet} F^{\vee} \otimes \wedge^{\bullet}(T_{\mathbb{K}}M/F))$. We show in Proposition 2.31 that this structure in fact results from the homotopy transfer of the differential Gerstenhaber algebra structure in $\Gamma(F[1], \wedge^{\bullet} T(F[1]))$ via the aforementioned contraction data.

Then we prove the following main theorem in Section 3.

Theorem B. *Under the canonical isomorphism as in Theorem A,*

(1) *The Atiyah class $\alpha_{F[1]}$ of the DG manifold $(F[1], d_F)$ corresponds to the Atiyah class $\alpha_{T_{\mathbb{K}}M/F}$ of the Lie pair $(T_{\mathbb{K}}M, F)$;*

(2) *The scalar Atiyah classes $c_k(F[1])$ $(k = 1, 2, \cdots)$ of the DG manifold $(F[1], d_F)$ correspond to the scalar Atiyah classes $c_k(T_{\mathbb{K}}M/F)$ of the Lie pair $(T_{\mathbb{K}}M, F)$;*

(3) *The Todd class $\mathrm{Td}_{F[1]}$ of the DG manifold $(F[1], d_F)$ corresponds to the Todd class $\mathrm{Td}_{T_{\mathbb{K}}M/F}$ of the Lie pair $(T_{\mathbb{K}}M, F)$.*

As a main application, consider a complex manifold X. Let $T_X^{1,0}$ and $T_X^{0,1}$ be its holomorphic and anti-holomorphic tangent bundle, respectively. By considering the particular integrable distribution $F := T_X^{0,1} \subset$

$T_{\mathbb{C}}X$, we get a Lie pair $(T_{\mathbb{C}}X, T_X^{0,1})$ and a DG manifold $(T_X^{0,1}[1], \bar{\partial})$, where $\bar{\partial} : \Omega_X^{0,\bullet} \to \Omega_X^{0,\bullet+1}$ is the Dolbeault differential operator. Applying Theorems A and B, we have the following

Theorem C. *Let Θ_X and Ω_X be the holomorphic tangent sheaf and holomorphic cotangent sheaf of X, respectively. Then*

(1) There exists a canonical isomorphism

$$\Phi_m^n : \mathrm{H}^\bullet(\mathfrak{T}_m^n(T(T_X^{0,1}[1])), L_{\bar{\partial}}) \xrightarrow{\cong} \mathrm{H}^\bullet(X, (\Omega_X)^{\otimes m} \otimes_{\mathcal{O}_X} (\Theta_X)^{\otimes n}),$$

where $\mathrm{H}^\bullet(X, (\Omega_X)^{\otimes m} \otimes_{\mathcal{O}_X} (\Theta_X)^{\otimes n})$ denotes the sheaf cohomology of $(\Omega_X)^{\otimes m} \otimes_{\mathcal{O}_X} (\Theta_X)^{\otimes n}$.

(2) The canonical isomorphism Φ_2^1 sends the Atiyah class $\alpha_{T_X^{0,1}[1]}$ of the DG manifold $(T_X^{0,1}[1], \bar{\partial})$ to the Atiyah class $\alpha_X \in \mathrm{H}^1(X, \Omega_X^{\otimes 2} \otimes_{\mathcal{O}_X} \Theta_X)$ of the complex manifold X.

(3) The canonical isomorphism $\Phi_\bullet^0$ sends the Todd class $\mathrm{Td}_{T_X^{0,1}[1]}$ of the DG manifold $(T_X^{0,1}[1], \bar{\partial})$ to the Todd class $\mathrm{Td}_{T_X^{1,0}} \in \oplus_k \mathrm{H}^{k,k}(X)$ of the Lie pair $(T_{\mathbb{C}}X, T_X^{0,1})$.

Notations:

In this note, the symbol $\mathbb{K}$ denotes either the field $\mathbb{R}$ or $\mathbb{C}$, and graded means $\mathbb{Z}$-graded. By a graded manifold $\mathcal{M}$ over $\mathbb{K}$, we mean a pair $(M, \mathcal{O}_{\mathcal{M}})$, where M is a smooth manifold, and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of graded commutative $\mathbb{K}$-algebras over M, locally isomorphic to $C^\infty(U, \mathbb{K}) \otimes_{\mathbb{K}} \widehat{S(V^\vee)}$ for a graded vector space V over $\mathbb{K}$, and $U \subset M$ open. We denote by $C^\infty(\mathcal{M}, \mathbb{K})$ the space of $\mathbb{K}$-valued smooth functions on $\mathcal{M}$, i.e., the space of global sections of $\mathcal{O}_{\mathcal{M}}$. A graded vector bundle is a vector bundle in the category of graded manifolds(cf.[12]).

For any graded vector bundle $\mathcal{E}$ over a graded manifold $\mathcal{M}$, denote by

$$\mathbb{T}_p^q(\mathcal{E}) = \mathcal{E}^{\otimes q} \otimes (\mathcal{E}^\vee)^{\otimes p}, \qquad \mathfrak{T}_p^q(\mathcal{E}) = \Gamma(\mathcal{M}; \mathbb{T}_p^q(\mathcal{E})), \tag{1.1}$$

the tensor product of q-copies of $\mathcal{E}$ and p-copies of its dual $\mathcal{E}^\vee$, and the corresponding global section space, respectively. In particular, $\mathfrak{T}_0^1(T\mathcal{M}) = \Gamma(T\mathcal{M})$ is the space of vector fields of $\mathcal{M}$, which will also be denoted by $\mathfrak{X}(\mathcal{M})$.

A DG manifold is a pair $(\mathcal{M}, Q)$, where $\mathcal{M}$ is a graded manifold, and $Q \in \mathfrak{X}(\mathcal{M})$ is a homological vector field on $\mathcal{M}$, i.e., a degree $+1$ derivation of $C^\infty(\mathcal{M})$ such that $[Q, Q] = 0$. A DG vector bundle is a vector bundle in the category of DG manifolds(cf.[12]).

Let $\mathcal{E} \to \mathcal{M}$ be a graded vector bundle and assume that $\mathcal{M}$ is a DG manifold. Then a DG-module structure on $\Gamma(\mathcal{E})$ over $C^\infty(\mathcal{M})$ induces a homological vector field on $\mathcal{E}$ such that $\mathcal{E} \to \mathcal{M}$ is a DG vector bundle. In fact, the category of DG vector bundles and the category of locally free DG-modules are equivalent(cf. [13]).

Let $(\mathcal{M}, Q)$ be a DG manifold. Then for any $p, q \geq 0$, the vector bundle

$$\mathbb{T}_p^q(T\mathcal{M}) = (T\mathcal{M})^{\otimes q} \otimes (T^\vee \mathcal{M})^{\otimes p},$$

together with the Lie derivative L_Q along the homological vector field Q, is a DG vector bundle over $(\mathcal{M}, Q)$. And $(\wedge^q(T\mathcal{M}) \otimes \wedge^p(T^\vee \mathcal{M}), L_Q)$ is a DG subbundle of $(\mathbb{T}_p^q(T\mathcal{M}), L_Q)$.

2. COHOMOLOGY OF DG MANIFOLDS OUT OF INTEGRABLE DISTRIBUTIONS

In this section, we prove Theorem A concerning the cohomology of (n, m)-tensor fields on the DG manifold $(F[1], d_F)$ via constructing a homotopy contraction explicitly.

2.1. **The contraction data.** Let $F \subset T_{\mathbb{K}}M$ be an integrable distribution. For simplicity, we will denote the quotient bundle $T_{\mathbb{K}}M/F$ by B from now on. There is a short exact sequence of vector bundles over M:

$$0 \longrightarrow F \overset{i}{\longrightarrow} T_{\mathbb{K}}M \overset{\mathrm{pr}_B}{\longrightarrow} B = T_{\mathbb{K}}M/F \longrightarrow 0. \tag{2.1}$$

There exists a flat F-connection ∇^{Bott} on B, called Bott representation, which is defined by

$$\nabla^{\mathrm{Bott}}_V Z := \mathrm{pr}_B[V, j(Z)], \quad \forall V \in \Gamma(F), Z \in \Gamma(B),$$

where $j : B \to T_{\mathbb{K}}M$ is any splitting of the short exact sequence (2.1).

Meanwhile, there associates to F a DG manifold $(F[1], d_F)$, whose space of $\mathbb{K}$-valued smooth functions is

$$C^\infty(F[1], \mathbb{K}) = \Gamma(M; \wedge^\bullet F^\vee) :\, = \Omega_F^\bullet.$$

Here $d_F : \Omega_F^\bullet \to \Omega_F^{\bullet+1}$ is the Chevalley-Eilenberg differential of F. Let $\pi : F[1] \to M$ be the bundle projection map. For any vector bundle E over M, we denote the space of smooth sections of the pull-back bundle $\pi^*(E)$ over $F[1]$ by

$$\Gamma(F[1], \pi^*(E)) \cong \Gamma(M; \wedge^\bullet F^\vee \otimes E) := \Omega_F^\bullet(E).$$

In particular, the covariant derivative d_B of the Bott connection ∇^{Bott} on B gives rise to a cochain complex $(\Omega_F^\bullet(B), d_B)$, which is called the Chevalley-Eilenberg complex of F with respect to B. And d_B is also called the F-module structure on B.

The following lemma appeared without explicit proof in [15]. For completeness, we give a proof here.

Lemma 2.2. *For any splitting $j :\, B \to T_{\mathbb{K}}M$ of the short exact sequence* (2.1)*, there exists a contraction data*

$$h_j \overset{\longrightarrow}{\subset} (\mathfrak{X}(F[1]), L_{d_F}) \underset{\psi_j}{\overset{\phi}{\longleftrightarrow}} (\Omega_F^\bullet(B), d_B),$$

satisfying

$$\phi \circ \psi_j = \mathrm{id} :\, (\Omega_F^\bullet(B), d_B) \to (\Omega_F^\bullet(B), d_B), \tag{2.3}$$

$$\mathrm{id} - \psi_j \circ \phi = [L_{d_F}, h_j] :\, (\mathfrak{X}(F[1]), L_{d_F}) \to (\mathfrak{X}(F[1]), L_{d_F}), \tag{2.4}$$

with side conditions

$$h_j \circ \psi_j = 0, \qquad\qquad \phi \circ h_j = 0, \qquad\qquad h_j^2 = 0,$$

where d_B is the Chevalley-Eilenberg differential of F with respect to the F-module B.

Proof. Let us choose a splitting of the short exact sequence (2.1), i.e., a pair of bundle maps $j :\, B \to T_{\mathbb{K}}M$ and $\mathrm{pr}_F :\, T_{\mathbb{K}}M \to F$ such that $\mathrm{pr}_B \circ j = \mathrm{id}_B$, $\mathrm{pr}_F \circ i = \mathrm{id}_F$, and $\mathrm{id}_{T_{\mathbb{K}}M} = j \circ \mathrm{pr}_B + i \circ \mathrm{pr}_F$:

$$0 \longrightarrow F \underset{\mathrm{pr}_F}{\overset{i}{\rightleftarrows}} T_{\mathbb{K}}M \underset{j}{\overset{\mathrm{pr}_B}{\rightleftarrows}} B \longrightarrow 0.$$

By abuse of notations, we will use same notations to denote the corresponding splitting of the short exact sequence:

$$0 \longrightarrow \Omega_F^\bullet(F) \underset{\mathrm{pr}_F}{\overset{i}{\rightleftarrows}} \Omega_F^\bullet(T_{\mathbb{K}}M) \underset{j}{\overset{\mathrm{pr}_B}{\rightleftarrows}} \Omega_F^\bullet(B) \longrightarrow 0.$$

In the sequel, we consider B as a subbundle of $T_{\mathbb{K}}M$ transversal to F via the isomorphism $T_{\mathbb{K}}M \cong F \oplus B$ determined by j. The symbol R denotes the space $C^\infty(M, \mathbb{K})$ of $\mathbb{K}$-valued smooth functions on M. The

space of vector fields on $F[1]$ is canonically isomorphic to the graded Lie algebra of $\mathbb{K}$-linear derivations of $\Omega_F^\bullet$:

$$\mathfrak{X}(F[1]) \cong \operatorname{Der}(\Omega_F^\bullet).$$

First of all, we define the maps ϕ, ψ_j and h_j as follows:

- Let ϕ be the $\Omega_F^\bullet$-linear map

$$\phi := \operatorname{pr}_B \circ \pi_* : \ \mathfrak{X}(F[1]) \to \Omega_F^\bullet(B), \tag{2.5}$$

where $\pi_* : \ \mathfrak{X}(F[1]) \to \Omega_F^\bullet(T_{\mathbb{K}}M)$ is the tangent map of the bundle projection $\pi : \ F[1] \to M$. In fact, $\pi_*(\mathcal{X}) \in \Omega_F^\bullet(T_{\mathbb{K}}M) \cong \Omega_F^\bullet \otimes_R \operatorname{Der}(R)$ is determined by

$$\pi_*(\mathcal{X})(f) = \mathcal{X}(\pi^* f) \in \Omega_F^\bullet, \qquad \forall f \in R.$$

- The map

$$\psi_j : \ \Omega_F^\bullet(B) \to \mathfrak{X}(F[1]) \tag{2.6}$$

is the $\Omega_F^\bullet$-linear extension of the R-linear map

$$\psi_j : \ \Gamma(B) \to \operatorname{Der}(\Omega_F^\bullet) \cong \mathfrak{X}(F[1])$$

specified by the following relations:

$$\psi_j(Z)(\pi^* f) = j(Z)(f), \qquad\qquad \psi_j(Z)(\xi) = \operatorname{pr}_{F^\vee}(L_{j(Z)}\xi) \in \Omega_F^1, \tag{2.7}$$

for all $Z \in \Gamma(B), f \in R$ and $\xi \in \Omega_F^1$. More precisely, we have

$$\iota_V(\psi_j(Z)(\xi)) = j(Z)\langle V, \xi \rangle - \langle \operatorname{pr}_F[j(Z), V], \xi \rangle, \tag{2.8}$$

for all $V \in \Gamma(F)$. It is easy to see that the map ψ_j is well-defined.

- Let $h_j : \ \mathfrak{X}(F[1]) \to \mathfrak{X}(F[1])$ be the $\Omega_F^\bullet$-linear map of degree (-1) determined by

$$h_j(\mathcal{X}) = (-1)^{|\mathcal{X}|} \iota_{\operatorname{pr}_F(\pi_*(\mathcal{X}))} \in \operatorname{Der}(\Omega_F^\bullet) \cong \mathfrak{X}(F[1]) \tag{2.9}$$

for all $\mathcal{X} \in \mathfrak{X}(F[1])$. Here $\operatorname{pr}_F(\pi_*(\mathcal{X})) \in \Omega_F^\bullet(F)$, and ι is the contraction operator

$$\iota_{\alpha \otimes_R V} \omega := \alpha \wedge \iota_V \omega, \ \ \forall \alpha, \omega \in \Omega_F^\bullet, V \in \Gamma(F).$$

In particular, we have, for any $f \in R$,

$$h_j(\mathcal{X})(\pi^* f) = 0, \qquad\qquad h_j(\mathcal{X})(d_F(\pi^* f)) = (-1)^{|\mathcal{X}|} \operatorname{pr}_F(\pi_*(\mathcal{X}))(f). \tag{2.10}$$

The rest of proof is divided into several steps:

- *The map ϕ is a cochain map.*

    It suffices to show that for any $\mathcal{X} \in \mathfrak{X}(F[1])$,

$$\phi([d_F, \mathcal{X}]) = d_B(\phi(\mathcal{X})) \in \Omega_F^\bullet(B).$$

    We prove, for any $\omega \in \Gamma(B^\vee)$, that the identity

$$\langle \omega, \phi([d_F, \mathcal{X}]) \rangle = \langle \omega, d_B(\phi(\mathcal{X})) \rangle \in \Omega_F^\bullet \tag{2.11}$$

    holds. Note that $\operatorname{pr}_B^\vee(\omega) \in \Omega^1(M)$ is locally of the form $\sum_k g_k df_k$, where $f_k, g_k \in R$. It thus follows that

$$\sum g_k d_F(\pi^* f_k) = i^\vee \left(\sum g_k df_k \right) = i^\vee(\operatorname{pr}_B^\vee(\omega)) = 0. \tag{2.12}$$

    Now

$$\text{LHS of (2.11)} = \langle \operatorname{pr}_B^\vee(\omega), \pi_*[d_F, \mathcal{X}] \rangle = \sum g_k \langle df_k, \pi_*[d_F, \mathcal{X}] \rangle = \sum g_k [d_F, \mathcal{X}](\pi^* f_k)$$

$$= \sum g_k d_F(\mathcal{X}(\pi^* f_k)) - \sum (-1)^{|\mathcal{X}|} g_k \mathcal{X}(d_F \pi^* f_k) \quad \text{by Equation (2.12)}$$

$$= d_F \left(\sum g_k \mathcal{X}(\pi^* f_k) \right) - \sum d_F(\pi^* g_k) \mathcal{X}(\pi^* f_k) + (-1)^{|\mathcal{X}|} \sum \mathcal{X}(\pi^* g_k) d_F(\pi^* f_k)$$

$$= d_F\langle\omega,\phi(\mathcal{X})\rangle - \sum d_F(\pi^*g_k)\mathcal{X}(\pi^*f_k) + \sum d_F(\pi^*f_k)\mathcal{X}(\pi^*g_k).$$

Here in the last equality we have used the following identity

$$\langle\omega,\phi(\mathcal{X})\rangle = \langle\mathrm{pr}_B^\vee(\omega),\pi_*(\mathcal{X})\rangle = \sum g_k\langle df_k,\pi_*(\mathcal{X})\rangle = \sum g_k\mathcal{X}(\pi^*f_k).$$

On the other hand, to compute the right-hand side of Equation (2.11), we first note that the Lie algebroid cohomology differential $d_{B^\vee}$ corresponding to the dual F-module structure on $B^\vee$ can be computed by the following commutative diagram

$$
\begin{array}{ccc}
\Gamma(B^\vee) & \xrightarrow{\quad d_{B^\vee}\quad} & \Omega_F^1(B^\vee)\subset\Omega_F^1(T_{\mathbb{K}}M) \\
\Big\downarrow{\scriptstyle\mathrm{pr}_B^\vee} & & \Big\uparrow{\scriptstyle R_F} \\
\Omega^1(M) & \xrightarrow{\quad d_{\mathrm{DR}}\quad} & \Omega^2(M).
\end{array}
$$

Here the right vertical map R_F is defined by

$$R_F(\alpha\wedge\beta) = (i^\vee\otimes\mathrm{id})(\alpha\otimes\beta - \beta\otimes\alpha) = i^\vee(\alpha)\otimes\beta - i^\vee(\beta)\otimes\alpha,$$

for all $\alpha,\beta\in\Omega^1(M)$. And the fact that $\mathrm{Im}(R_F\circ d_{\mathrm{DR}}\circ\mathrm{pr}_B^\vee) = \Omega_F^1(B^\vee)\subset\Omega_F^1(T_{\mathbb{K}}^\vee M)$ follows from the assumption that $F\subset T_{\mathbb{K}}M$ is integrable. Thus,

$$\text{RHS of (2.11)} = d_F\langle\omega,\phi(\mathcal{X})\rangle - \langle d_{B^\vee}(\omega),\phi(\mathcal{X})\rangle$$

$$= d_F\langle\omega,\phi(\mathcal{X})\rangle - \left\langle R_F\left(\sum dg_k\wedge df_k\right),\mathrm{pr}_B(\pi_*(\mathcal{X}))\right\rangle$$

$$= d_F\langle\omega,\phi(\mathcal{X})\rangle - \left\langle\mathrm{pr}_B^\vee\left(\sum d_F(\pi^*g_k)\otimes df_k - d_F(\pi^*f_k)\otimes dg_k\right),\pi_*(\mathcal{X})\right\rangle$$

$$= d_F\langle\omega,\phi(\mathcal{X})\rangle - \sum d_F(\pi^*g_k)\mathcal{X}(\pi^*f_k) + \sum d_F(\pi^*f_k)\mathcal{X}(\pi^*g_k).$$

• *The map ψ_j is a cochain map*, i.e., $\psi_j\circ d_B = L_{d_F}\circ\psi_j$. We have

$$\psi_j(d_B(\alpha\otimes_R Z)) = d_F(\alpha)\otimes_R\psi_j(Z) + (-1)^{|\alpha|}\alpha\otimes_R\psi_j(d_B(Z))$$

and

$$L_{d_F}(\psi_j(\alpha\otimes_R Z)) = d_F(\alpha)\otimes_R\psi_j(Z) + (-1)^{|\alpha|}\alpha\otimes_R L_{d_F}(\psi_j(Z)),$$

for any $\alpha\otimes_R Z\in\Omega_F^\bullet(B)$, where $\alpha\in\Omega_F^\bullet, Z\in\Gamma(B)$. Thus it suffices to prove that

$$\psi_j(d_B(Z)) = L_{d_F}(\psi_j(Z)) = [d_F,\psi_j(Z)]\in\mathfrak{X}(F[1]).$$

Let us examine this relation on the generators $\{\pi^*f, d_F(\pi^*f): f\in R\}$ of the $\mathbb{K}$-algebra $\Omega_F^\bullet$:

On the one hand, for any $V\in\Gamma(F)$, by the defining Equation (2.7) of ψ_j, we have

$$\iota_V(\psi_j(d_B(Z))(\pi^*f)) = \psi_j(\nabla_V^{\mathrm{Bott}}(Z))(\pi^*f) = j(\mathrm{pr}_B([V,j(Z)]))(f)$$

$$= [V,j(Z)](f) - \mathrm{pr}_F([V,j(Z)])(f)$$

$$= V(j(Z)(f)) - (j(Z)(V(f)) - \mathrm{pr}_F([j(Z),V])(f)) \qquad\text{by Equation (2.8)}$$

$$= \iota_V([d_F,\psi_j(Z)](\pi^*f)).$$

Therefore, we have

$$\psi_j(d_B(Z))(\pi^*f) = [d_F,\psi_j(Z)](\pi^*f). \tag{2.13}$$

On the other hand, we have

$$[d_F,\psi_j(Z)](d_F(\pi^*f)) = -d_F([d_F,\psi_j(Z)](\pi^*f)) \qquad\text{by Equation(2.13)}$$

$$= -d_F(\psi_j(d_B(Z))(\pi^*f))$$

$$= -[d_F,\psi_j(d_B(Z))](\pi^*f) + \psi_j(d_B(Z))(d_F(\pi^*f)) \qquad\text{by Equation(2.13)}$$

$$= -\psi_j(d_B^2(Z))(\pi^*f) + \psi_j(d_B(Z))(d_F(\pi^*f))$$

$$= \psi_j(d_B(Z))(d_F(\pi^*f)).$$

Hence, ψ_j is a cochain map.

- *Proof of Equation* (2.3). We first claim that

$$\pi_* \circ \psi_j = j : \quad \Omega_F^\bullet(B) \to \Omega_F^\bullet(T_{\mathbb{K}}M), \tag{2.14}$$

Since both ψ_j and π_* are $\Omega_F^\bullet$-linear, it suffices to show that

$$\pi_*(\psi_j(Z)) = j(Z) \in \Gamma(T_{\mathbb{K}}M) \cong \mathrm{Der}(R), \quad \forall Z \in \Gamma(B).$$

In fact, by the defining formula (2.7) of ψ_j, we have

$$\pi_*(\psi_j(Z))(f) = \psi_j(Z)(\pi^*f) = j(Z)(f), \quad \forall f \in R.$$

This proves Equation (2.14). Hence,

$$\phi \circ \psi_j = \mathrm{pr}_B \circ \pi_* \circ \psi_j = \mathrm{pr}_B \circ j = \mathrm{id}.$$

- *Proof of Equation* (2.4). We prove that

$$\mathcal{X} - \psi_j(\phi(\mathcal{X})) = [L_{d_F}, h_j](\mathcal{X}) = [d_F, h_j(\mathcal{X})] + h_j([d_F, \mathcal{X}]),$$

for all $\mathcal{X} \in \mathfrak{X}(F[1])$. It suffices to show that both sides coincide when acting on elements $\{\pi^*f, d_F(\pi^*f) : f \in R\}$. That is, for any $f \in R$,

$$(\mathcal{X} - \psi_j(\Phi(\mathcal{X})))(\pi^*f) = [d_F, h_j(\mathcal{X})](\pi^*f) + h_j([d_F, \mathcal{X}])(\pi^*f) = [d_F, h_j(\mathcal{X})](\pi^*f), \tag{2.15}$$
$$(\mathcal{X} - \psi_j(\Phi(\mathcal{X})))(d_F(\pi^*f)) = [d_F, h_j(\mathcal{X})](d_F(\pi^*f)) + h_j([d_F, \mathcal{X}])(d_F(\pi^*f)). \tag{2.16}$$

In fact, Equation (2.15) can be verified directly:

$$
\begin{aligned}
[d_F, h_j(\mathcal{X})](\pi^*f) &= (-1)^{|\mathcal{X}|} h_j(\mathcal{X})(d_F(\pi^*f)) && \text{by Equation (2.10)} \\
&= \mathrm{pr}_F(\pi_*(\mathcal{X}))(f) = (\pi_*(X) - \mathrm{pr}_B(\pi_*(\mathcal{X})))(f) && \text{by Equation (2.5)} \\
&= (\pi_*(\mathcal{X}) - (\phi(\mathcal{X})))(f) \\
&= \mathcal{X}(f) - \psi_j(\phi(X))(f).
\end{aligned}
$$

As for Equation (2.16), we compute

$$
\begin{aligned}
&(\mathcal{X} - \psi_j(\phi(\mathcal{X})))(d_F(\pi^*f)) \\
&= (-1)^{|\mathcal{X}|}(d_F((\mathcal{X} - \psi_j(\phi(\mathcal{X})))(\pi^*f)) - [d_F, \mathcal{X} - \psi_j(\phi(\mathcal{X}))](\pi^*f)) && \text{by Equation (2.15)} \\
&= (-1)^{|\mathcal{X}|}(d_F([d_F, h_j(\mathcal{X})](\pi^*f)) - [d_F, \mathcal{X}](\pi^*f) + [d_F, \psi_j(\phi(\mathcal{X}))](\pi^*f)) \\
&= (-1)^{|\mathcal{X}|}(d_F([d_F, h_j(\mathcal{X})](\pi^*f)) - [d_F, \mathcal{X}](\pi^*f) + \psi_j(\phi([d_F, \mathcal{X}]))(\pi^*f)) \\
&= [d_F, h_j(\mathcal{X})](d_F(\pi^*f)) - (-1)^{|\mathcal{X}|}(\pi_*([d_F, \mathcal{X}])(f) - j(\mathrm{pr}_B(\pi_*([d_F, \mathcal{X}]))))(f)) \\
&= [d_F, h_j(\mathcal{X})](d_F(\pi^*f)) + (-1)^{|\mathcal{X}|+1} \mathrm{pr}_F(\pi_*([d_F, \mathcal{X}]))(f) && \text{by Equation (2.10)} \\
&= [d_F, h_j(\mathcal{X})](d_F(\pi^*f)) + h_j([d_F, \mathcal{X}])(d_F(\pi^*f)).
\end{aligned}
$$

- *Side conditions*. Note that by Equation (2.14),

$$(h_j \circ \psi_j)(Z) = \iota_{\mathrm{pr}_F(\pi_*(\psi_j(Z)))} = \iota_{\mathrm{pr}_F(j(Z))} = 0, \quad \forall Z \in \Gamma(B).$$

Since both h_j and ψ_j are $\Omega_F^\bullet$-linear, it follows that $h_j \circ \psi_j = 0$. The verifications of $\phi \circ h_j = 0$ and $h_j^2 = 0$ are similar.

This completes the proof. $\qquad \square$

We generalize this result to the space of tensor fields on $F[1]$ as follows:

Proposition 2.17. *For any splitting* $j : B \to T_{\mathbb{K}}M$ *of the short exact sequence* (2.1), *there exists a contraction data*

$$H_m^n \underset{\Psi_m^n}{\overset{\Phi_m^n}{\rightleftarrows}} (\mathfrak{T}_m^n(T(F[1])), L_{d_F}) \xrightarrow[\Psi_m^n]{\Phi_m^n} (\Omega_F^\bullet(\mathbb{T}_m^n(B)), d_B),$$

satisfying

$$\Phi_m^n \circ \Psi_m^n = \mathrm{id}, \qquad\qquad \Psi_m^n \circ \Phi_m^n + [L_{d_F}, H_m^n] = \mathrm{id},$$

with side conditions

$$H_m^n \circ \Psi_m^n = 0, \qquad\qquad \Phi_m^n \circ H_m^n = 0, \qquad\qquad H_m^n \circ H_m^n = 0.$$

Here for simplicity we denote by d_B the Chevalley-Eilenberg differential of F with respect to the F-module $\mathbb{T}_m^n(B)$ for any pair of non-negative integers (m, n).

Proof. Note that all maps ψ_j, ϕ, h_j of the contraction data are $\Omega_F^\bullet$-linear. Their $\Omega_F^\bullet$-dual maps $\psi_j^\vee, \phi^\vee, h_j^\vee$ give rise to a contraction as well:

$$h_j^\vee \overset{\curvearrowright}{\subset} (\Omega^1(F[1]), L_{d_F}) \overset{\psi_j^\vee}{\underset{\phi^\vee}{\rightleftarrows}} (\Omega_F^\bullet(B^\vee), d_B).$$

By considering the $\Omega_F^\bullet$-dual form of Equations (2.3) and (2.4), it only suffices to show that

$$[L_{d_F}, h_j]^\vee = [L_{d_F}, h_j^\vee] : \Omega^1(F[1]) \to \Omega^1(F[1]).$$

In fact, for all $\alpha \in \Omega^1(F[1]), \mathcal{X} \in \mathfrak{X}(F[1])$, we have

$$\begin{aligned}
\langle [L_{d_F}, h_j^\vee]\alpha, \mathcal{X} \rangle &= \langle L_{d_F}(h_j^\vee(\alpha)), \mathcal{X} \rangle + \langle h_j^\vee(L_{d_F}(\alpha)), \mathcal{X} \rangle \\
&= L_{d_F}\langle h_j^\vee(\alpha), \mathcal{X} \rangle - (-1)^{|\alpha|-1}\langle h_j^\vee(\alpha), L_{d_F}\mathcal{X} \rangle + (-1)^{|\alpha|+1}\langle L_{d_F}\alpha, h_j(\mathcal{X}) \rangle \\
&= (-1)^{|\alpha|}(L_{d_F}\langle \alpha, h_j(\mathcal{X}) \rangle - \langle L_{d_F}\alpha, h_j(\mathcal{X}) \rangle) + \langle \alpha, h_j(L_{d_F}\mathcal{X}) \rangle \\
&= \langle \alpha, L_{d_F}(h_j(\mathcal{X})) + h_j(L_{d_F}\mathcal{X}) \rangle = \langle \alpha, [L_{d_F}, h_j]\mathcal{X} \rangle \\
&= \langle [L_{d_F}, h_j]^\vee(\alpha), \mathcal{X} \rangle.
\end{aligned}$$

Define

$$\Psi_m^n := (\phi^\vee)^{\otimes m} \otimes (\psi_j)^{\otimes n} : (\Omega_F^\bullet(\mathbb{T}_m^n(B)), d_B) \to (\mathfrak{T}_m^n(T(F[1])), L_{d_F}), \tag{2.18}$$

$$\Phi_m^n := (\psi_j^\vee)^{\otimes m} \otimes (\phi)^{\otimes n} : (\mathfrak{T}_m^n(T(F[1])), L_{d_F}) \to (\Omega_F^\bullet(\mathbb{T}_m^n(B)), d_B), \tag{2.19}$$

$$H_m^n = \sum_{i=1}^m (\phi^\vee \circ \psi_j^\vee)^{\otimes(i-1)} \otimes h_j^\vee \otimes (\mathrm{id})^{\otimes(n+m-i)} + \sum_{l=1}^n (\phi^\vee \circ \psi_j^\vee)^{\otimes m} \otimes (\psi_j \circ \phi)^{\otimes(l-1)} \otimes h_j \otimes \mathrm{id}^{\otimes(n-l)}. \tag{2.20}$$

Then according to Lemma 2.2, and the tensor trick in [10, Example 2.6], the maps $\Psi_m^n, \Phi_m^n, H_m^n$ constructed above satisfy the contraction properties. $\square$

Moreover, the quasi-isomorphisms Ψ_m^n and Φ_m^n are, in fact, canonical up to homotopy:

Proposition 2.21. *Assume that $\widehat{j} : B \to T_{\mathbb{K}}M$ is another splitting of the short exact sequence (2.1). Let*

$$\widehat{H}_m^n \overset{\curvearrowright}{\subset} (\mathfrak{T}_m^n(T(F[1])), L_{d_F}) \overset{\widehat{\Phi}_m^n}{\underset{\widehat{\Psi}_m^n}{\rightleftarrows}} (\Omega_F^\bullet(\mathbb{T}_m^n(B)), d_B)$$

be the associated contraction. Then there exist two $\Omega_F^\bullet$-linear maps

$$\Theta_m^n : \Omega_F^\bullet(\mathbb{T}_m^n(B)) \to \mathfrak{T}_m^n(T(F[1])), \qquad\qquad \Xi_m^n : \mathfrak{T}_m^n(T(F[1])) \to \Omega_F^\bullet(\mathbb{T}_m^n(B))$$

such that

$$\widehat{\Psi}_m^n - \Psi_m^n = \Theta_m^n \circ d_B + L_{d_F} \circ \Theta_m^n : \Omega_F^\bullet(\mathbb{T}_m^n(B)) \to \mathfrak{T}_m^n(T(F[1])), \tag{2.22}$$

$$\widehat{\Phi}_m^n - \Phi_m^n = \Xi_m^n \circ L_{d_F} + d_B \circ \Xi_m^n : \mathfrak{T}_m^n(T(F[1])) \to \Omega_F^\bullet(\mathbb{T}_m^n(B)). \tag{2.23}$$

According to the defining formulas (2.18) and (2.19) of Ψ_m^n and Φ_m^n in the proof of Proposition 2.17, the key step is to study how the quasi-isomorphism ψ_j depends on the splitting j:

Lemma 2.24. *Under the same assumptions as in Proposition 2.21, there exists a $\Omega_F^\bullet$-linear homotopy between $\widehat{\Psi}_0^1 = \psi_{\widehat{j}}$ and ψ_j*

$$\Theta : \ \Omega_F^\bullet(B) \to \mathfrak{X}(F[1]),$$

i.e.,

$$\psi_{\widehat{j}} - \psi_j = \Theta \circ d_B + L_{d_F} \circ \Theta : \ \Omega_F^\bullet(B) \to \mathfrak{X}(F[1]). \tag{2.25}$$

Proof. Let $\theta = \widehat{j} - j \in \Gamma(M; \operatorname{Hom}(B, F))$. Define a map

$$\Theta : \ \Omega_F^\bullet(B) \to \operatorname{Der}(\Omega_F^\bullet) \cong \mathfrak{X}(F[1])$$

by

$$\Theta(\alpha \otimes_R Z) = (-1)^{|\alpha|}\alpha \otimes \iota_{\theta(Z)}, \tag{2.26}$$

where $\alpha \in \Omega_F^\bullet, Z \in \Gamma(B)$.

We show that this Θ satisfies Equation (2.25). In fact, both sides of (2.25) are $\Omega_F^\bullet$-linear, since

$$L_{d_F}(\Theta(\alpha \otimes_R Z)) = (-1)^{|\alpha|}[d_F, \alpha \otimes_R \Theta(Z)] = (-1)^{|\alpha|}d_F(\alpha) \otimes_R \Theta(Z) + \alpha \otimes_R L_{d_F}(\Theta(Z)),$$

$$\Theta(d_B(\alpha \otimes_R Z)) = \Theta(d_F(\alpha) \otimes_R Z + (-1)^{|\alpha|}\alpha \otimes_R d_B(Z))$$

$$= (-1)^{|\alpha|+1}d_F(\alpha) \otimes_R \Theta(Z) + \alpha \otimes_R \Theta(d_B(Z)),$$

for all $\alpha \otimes_R Z \in \Omega_F^\bullet(B)$, where $\alpha \in \Omega_F^\bullet, Z \in \Gamma(B)$.

It thus suffices to show that

$$\psi_{\widehat{j}}(Z) - \psi_j(Z) = \Theta(d_B(Z)) + L_{d_F}(\Theta(Z)) \in \mathfrak{X}(F[1]). \tag{2.27}$$

Let us denote by $\operatorname{pr}_F, \widehat{\operatorname{pr}}_F : \ T_\mathbb{K}M \to F$ the projections corresponding to the two splittings j and $\widehat{j}$, respectively. Then

$$\widehat{\operatorname{pr}}_F = \operatorname{pr}_F - \theta \circ \operatorname{pr}_B : \ \Omega_F^\bullet(T_\mathbb{K}M) \to \Omega_F^\bullet(F).$$

For any $f \in R$,

$$(\Theta(d_B(Z)) + L_{d_F}(\Theta(Z)))(\pi^*f) = [d_F, \Theta(Z)](\pi^*f) = [d_F, \iota_{\theta(Z)}](\pi^*f) = \theta(Z)(\pi^*f)$$

$$= (\psi_{\widehat{j}}(Z) - \psi_j(Z))(\pi^*f);$$

For any $\xi \in \Omega_F^1, V \in \Gamma(F)$, we have

$$\iota_V(\psi_{\widehat{j}}(Z)(\xi) - \psi_j(Z)(\xi)) \qquad\qquad \text{by Equation (2.8)}$$

$$= \theta(Z)\langle V, \xi\rangle - \langle \widehat{\operatorname{pr}}_F([\widehat{j}(Z), V]) - \operatorname{pr}_F([j(Z), V]), \xi\rangle$$

$$= \theta(Z)\langle V, \xi\rangle - \langle \operatorname{pr}_F([\widehat{j}(Z), V]) - \theta(\operatorname{pr}_B[\widehat{j}(Z), V]) - \operatorname{pr}_F([j(Z), V]), \xi\rangle$$

$$= \theta(Z)\langle V, \xi\rangle - \langle [\theta(Z), V], \xi\rangle - \langle \theta(\nabla_V^{\operatorname{Bott}}Z), \xi\rangle$$

$$= \langle V, L_{\theta(Z)}(\xi)\rangle - \langle \theta(\iota_V d_B(Z)), \xi\rangle = \iota_V(L_{\theta(Z)}(\xi)) + \iota_V(\Theta(d_B(Z))(\xi))$$

$$= \iota_V(L_{d_F}(\Theta(Z))(\xi) + \Theta(d_B(Z))(\xi)).$$

Since $\Omega_F^\bullet$ is generated by R and Ω_F^1, it follows that Equation (2.27) holds. $\qquad\square$

Proof of Proposition 2.21. Define

$$\Theta_m^n = \sum_{l=1}^n (\phi^\vee)^{\otimes m} \otimes (\psi_{\widehat{j}})^{\otimes(l-1)} \otimes \Theta \otimes (\psi_j)^{\otimes(n-l)} : \ \Omega_F^\bullet(\mathbb{T}_m^n(B)) \to \mathfrak{T}_m^n(T(F[1])),$$

$$\Xi_m^n = \sum_{k=1}^{m} (\psi_{\hat{j}}^\vee)^{\otimes(k-1)} \otimes \Theta^\vee \otimes (\psi_j^\vee)^{\otimes(m-k)} \otimes (\phi)^{\otimes n} : \ \mathfrak{T}_m^n(T(F[1])) \to \Omega_F^\bullet(\mathbb{T}_m^n(B)).$$

Here $\Theta^\vee$ denotes the $\Omega_F^\bullet$-dual of Θ defined in Equation (2.26).

Then by Lemma 2.24, Equations (2.22) and (2.23) follow from straightforward computations. □

2.2. **Proof of Theorem A.** Now we are ready to prove Theorem A:

Proof of Theorem A. Let $j : \ B \to T_\mathbb{K}M$ be a splitting of the short exact sequence (2.1). Applying Proposition 2.17, we obtain an isomorphism

$$\Phi_m^n : \ \mathrm{H}^\bullet(\mathfrak{T}_m^n(T(F[1])), L_{d_F}) \xrightarrow{\cong} \mathrm{H}_{\mathrm{CE}}^\bullet(F, \mathbb{T}_m^n(B)).$$

According to Proposition 2.21, Φ_m^n does not depend on the splitting j, and thus is canonical. Finally, the commutative diagram in Theorem A follows from the commutative diagram on the cochain level

$$
\begin{array}{ccc}
\mathfrak{T}_{m_1}^{n_1}(T(F[1])) \otimes \mathfrak{T}_{m_2}^{n_2}(T(F[1])) & \xrightarrow{\ \otimes\ } & \mathfrak{T}_{m_1+m_2}^{n_1+n_2}(T(F[1])) \\
\Big\downarrow{\scriptstyle \Phi_{m_1}^{n_1} \otimes \Phi_{m_2}^{n_2}} & & \Big\downarrow{\scriptstyle \Phi_{m_1+m_2}^{n_1+n_2}} \\
\Omega_F^\bullet(\mathbb{T}_{m_1}^{n_1}(B)) \otimes \Omega_F^\bullet(\mathbb{T}_{m_2}^{n_2}(B)) & \xrightarrow{\ \otimes\ } & \Omega_F^\bullet(\mathbb{T}_{m_1+m_2}^{n_1+n_2}(B)).
\end{array}
$$

This completes the proof. □

In the sequel, the contraction data $(\Phi_m^n, \Psi_m^n, H_m^n)$ in Proposition 2.17 will simply be denoted by (Φ, Ψ, H):

$$H \, \begin{array}{c}\curvearrowright\end{array} \ (\mathfrak{T}_m^n(T(F[1])), L_{d_F}) \ \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \ (\Omega_F^\bullet(\mathbb{T}_m^n(B)), d_B).$$

It is clear that (Φ, Ψ, H) induces a contraction data on the subcomplexes as well

$$H \, \begin{array}{c}\curvearrowright\end{array} \ (\Gamma(F[1], \wedge^n T(F[1]) \otimes \wedge^m T^\vee(F[1])), L_{d_F}) \ \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \ (\Omega_F^\bullet(\wedge^n B \otimes \wedge^m B^\vee), d_B). \qquad (2.28)$$

The following result is an immediate consequence, which implies that cohomologies of $\Gamma(F[1], \wedge^n T(F[1]) \otimes \wedge^m T^\vee(F[1]))$ are bounded from above:

Corollary 2.29. *The cohomology*

$$\mathrm{H}^r(\Gamma(F[1], \wedge^n T(F[1]) \otimes \wedge^m T^\vee(F[1])), L_{d_F})$$

vanishes under any of the following conditions:

   (1) $m > \mathrm{rank}(B)$*;*
   (2) $n > \mathrm{rank}(B)$*;*
   (3) $r > \mathrm{rank}(F)$*.*

2.3. **Application.** Consider the case $m = 0$. The contraction data reduces to

$$H \, \begin{array}{c}\curvearrowright\end{array} \ (\Gamma(F[1], \wedge^n T(F[1])), L_{d_F}) \ \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \ (\Omega_F^\bullet(\wedge^n B), d_B), \qquad (2.30)$$

where

$$\Psi = \wedge^n \psi_j : \Omega_F^\bullet(\wedge^n B) \to \Gamma(F[1], \wedge^n T(F[1])),$$
$$\Phi = \wedge^n \phi : \Gamma(F[1], \wedge^n T(F[1])) \to \Omega_F^\bullet(\wedge^n B),$$

$$H = \sum_{l=1}^{n} \wedge^{(l-1)}(\psi_j \circ \phi) \wedge h_j \wedge (\mathrm{id})^{\wedge(n-l)} : \Gamma(F[1], \wedge^n T(F[1])) \to \Gamma(F[1], \wedge^n T(F[1])).$$

Note that both $\Omega_F^\bullet(\wedge^\bullet B)$ and $\Gamma(F[1], \wedge^\bullet T(F[1]))$, together with the obvious wedge product, are commutative graded algebras. It is clear that both Ψ and Φ are compatible with the wedge product, thus are morphisms of commutative algebras. However, H in general is not a derivation. In fact, we have

$$H(\mathcal{X}_1 \wedge \mathcal{X}_2) = H(\mathcal{X}_1) \wedge \mathcal{X}_2 + (\Psi \circ \Phi)(\mathcal{X}_1) \wedge H(\mathcal{X}_2), \quad \forall \mathcal{X}_i \in \Gamma(\wedge^\bullet T(F[1])).$$

Note that $(\Gamma(F[1], \wedge^\bullet T(F[1])), L_{d_F})$ together with the wedge product and the Schouten-Nijenhuis bracket $[-, -]_{\mathrm{SN}}$, forms a differential Gerstenhaber algebra. By the standard homotopy transfer technique, we recover the following

Proposition 2.31. *Given a splitting $j : B \to T_\mathbb{K} M$ of the short exact sequence* (2.1), *then*

(1) there exists a (-1)-shifted derived Poisson algebra structure $\{\lambda_k\}_{k \geq 1}$ on $\Omega_F^\bullet(\wedge^\bullet B)$ such that $\lambda_1 = d_B$.

(2) this (-1)-shifted derived Poisson algebra structure coincides with the one out of the Lie pair $(T_\mathbb{K} M, F)$ as in [2, Proposition 4.3].

The following technical lemma is needed:

Lemma 2.32. *For any $\mathcal{X}_1, \mathcal{X}_2 \in \mathfrak{X}(F[1])$ and $Z \in \Gamma(B)$, we have*

$$\phi([h_j(\mathcal{X}_1), h_j(\mathcal{X}_2)]) = 0, \tag{2.33}$$
$$h_j([h_j(\mathcal{X}_1), \psi_j(Z)]) = 0. \tag{2.34}$$

Proof. Note that the space of vertical vector fields

$$\mathfrak{X}(F[1])^{\mathrm{vertical}} = \{\iota_\mathcal{V} \in \mathrm{Der}(\Omega_F^\bullet) : \mathcal{V} \in \Omega_F^\bullet(F)\} \cong \ker(\pi_*)$$

is a Lie subalgebra of $\mathfrak{X}(F[1])$. Since $\mathrm{Im}(h_j) \subset \mathfrak{X}(F[1])^{\mathrm{vertical}}$, it follows that

$$\phi([h_j(\mathcal{X}_1), h_j(\mathcal{X}_2)]) = \mathrm{pr}_B(\pi_*([h_j(\mathcal{X}_1), h_j(\mathcal{X}_2)])) = 0.$$

Meanwhile, Since

$$\pi_*([h_j(\mathcal{X}_1), \psi_j(Z)])(f) = [h_j(\mathcal{X}_1), \psi_j(Z)](\pi^* f) = h_j(\mathcal{X}_1)(j(Z)(f)) = 0,$$

for all $f \in R$, it follows that $\pi_*([h_j(\mathcal{X}_1), \psi_j(Z)]) = 0$. According to Equation (2.9), the latter implies Equation (2.34), as desired. $\square$

Proof of Proposition 2.31. We first use the contraction data (2.30) to prove (1): Note that both $\Omega_F^\bullet(\wedge^\bullet B)$ and $\Gamma(F[1], \wedge^\bullet T(F[1]))$ carry natural graded commutative algebra structures, i.e., the wedge products. Moreover,

$$(\Gamma(F[1], \wedge^\bullet T(F[1])), L_{d_F}, \wedge, [-, -]_{\mathrm{SN}})$$

is a differential Gerstenhaber algebra, and therefore is also a (-1)-shifted derived Poisson algebra. Since both Ψ and Φ are morphisms of the underlying commutative algebras, by the homotopy transfer for derived Poisson algebras [2, Theorem 2.16], there induces a (-1)-shifted derived Poisson algebra structure $\{\lambda_k\}_{k \geq 1}$ on $\Omega_F^\bullet(\wedge^\bullet B)$, whose first bracket λ_1 is d_B and higher brackets are defined iteratively op cit.

To prove (2), let us work out higher brackets $\{\lambda_k\}_{k \geq 2}$ explicitly:

$k = 2$. The binary bracket λ_2 is, by definition

$$\lambda_2(\mathcal{Z}_1, \mathcal{Z}_2) = \Phi([\Psi(\mathcal{Z}_1), \Psi(\mathcal{Z}_2)]_{\mathrm{SN}}), \ \forall \mathcal{Z}_1, \mathcal{Z}_2 \in \Omega_F^\bullet(\wedge^\bullet B).$$

More precisely, we consider the situation on generating elements:

$$\lambda_2(Z_1, Z_2) = \phi([\psi_j(Z_1), \psi_j(Z_2)]_{\mathrm{SN}}) = \mathrm{pr}_B([j(Z_1), j(Z_2)]), \ \forall Z_i \in \Gamma(B), i = 1, 2,$$

$$\lambda_2(Z_1, \xi) = \phi([\psi_j(Z_1), \psi_j(\xi)]_{\mathrm{SN}}) = \psi_j(Z_1)(\xi) = \mathrm{pr}_{F^\vee}(L_{j(Z_1)}\xi), \ \forall Z_1 \in \Gamma(B), \xi \in \Omega_F^1.$$

Similarly, it follows that

$$\lambda_2(Z_1, f) = j(Z_1)(f), \qquad\qquad \lambda_2(\omega_1, \omega_2) = 0,$$

for all $f \in R = C^\infty(M, \mathbb{K})$ and $\omega_i \in \Omega_F^\bullet$.

$k = 3$. The trinary bracket λ_3 is given by

$$\lambda_3(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) = \sum_{\sigma \in \mathrm{sh}(2,1)} \pm \Phi([H([\Psi(\mathcal{Z}_{\sigma(1)}), \Psi(\mathcal{Z}_{\sigma(2)})]_{\mathrm{SN}}), \Psi(\mathcal{Z}_{\sigma(3)})]_{\mathrm{SN}}),$$

for all $\mathcal{Z}_i \in \Omega_F^\bullet(\wedge^\bullet B), 1 \le i \le 3$, where $\mathrm{sh}(2, 1)$ is the set of $(2, 1)$-shuffles, and $\pm$ is some proper Koszul sign. In particular, we have, on the generating elements,

$$\lambda_3(Z_1, Z_2, \xi) = \phi([h_j([\psi_j(Z_1), \psi_j(Z_2)]_{\mathrm{SN}}), \psi_j(\xi)]_{\mathrm{SN}}) = \iota_{\mathrm{pr}_F[j(Z_1), j(Z_2)]}(\xi),$$

for all $Z_1, Z_2 \in \Gamma(B)$ and $\xi \in \Gamma(F^\vee)$, where $\mathrm{pr}_F : \Gamma(T_\mathbb{K}M) \to \Gamma(F)$ is the projection. Furthermore, since

$$h_j([\mathcal{X}, \psi_j(\xi)]_{\mathrm{SN}}) = h_j(\mathcal{X}(\xi)) = 0, \quad \forall \mathcal{X} \in \mathfrak{X}(F[1]), \xi \in \Gamma(F^\vee) \subset \Omega_F^\bullet, \tag{2.35}$$

it follows that λ_3 vanishes if restricted to $\Gamma(B) \times \Gamma(B) \times \Gamma(B), \Gamma(B) \times \Gamma(F^\vee) \times \Gamma(F^\vee)$ and $\Gamma(F^\vee) \times \Gamma(F^\vee) \times \Gamma(F^\vee)$.

$k = 4$. The 4th bracket λ_4 acting on the generating elements is of form

$$\lambda_4(Z_1, Z_2, Z_3, Z_4) = \sum_{\sigma \in \mathrm{sh}(2,2)} \pm \phi([h_j([\psi_j(Z_{\sigma(1)}), \psi_j(Z_{\sigma(2)})]_{\mathrm{SN}}), h_j([\psi_j(Z_{\sigma(1)}), \psi_j(Z_{\sigma(2)})])]_{\mathrm{SN}})$$

$$+ \sum_{\tau \in \mathrm{sh}(2,1,1)} \pm \phi([h_j([h_j([\psi_j(Z_{\tau(1)}), \psi_j(Z_{\tau(2)})]_{\mathrm{SN}}), \psi_j(Z_{\tau(3)})]_{\mathrm{SN}}), \psi_j(Z_{\tau(4)})]_{\mathrm{SN}}),$$

for all $Z_i \in \Gamma(B), 1 \le i \le 4$. It follows from Lemma 2.32 that $\lambda_4(Z_1, Z_2, Z_3, Z_4) = 0$. Moreover, it follows from Equations (2.34) and (2.35) that λ_4 also vanishes with one or more inputs from $\Gamma(F^\vee)$.

$k \ge 5$. The L_∞-brackets $\{\lambda_k\}_{k \ge 5}$ vanish for similar reasons.

It thus follows that the generating relations of the (-1)-shifted derived Poisson algebra structure on $\Omega_F^\bullet(\wedge^\bullet B)$, are exactly those of [2, Proposition 4.3]. This concludes the proof. $\qquad\square$

Remark 2.36. The (-1)-shifted derived Poisson algebra structure on $\Omega_F^\bullet(\wedge^\bullet B)$ induces a strong homotopy Lie-Rinehart algebra structure on the subspace $\Omega_F^\bullet(B)$. This result is due to Vitagliano [15].

According to Theorem 1.1 in [2], this (-1)-shifted derived Poisson algebra is in fact canonical up to isomorphism. On the other hand, $(\mathrm{H}^\bullet(\Gamma(F[1], \wedge^\bullet T(F[1])), L_{d_F}), \wedge, [-, -]_{\mathrm{SN}})$ is canonically a Gerstenhaber algebra. As an immediate consequence of Proposition 2.31 and Theorem A, we have

Corollary 2.37. *There is a canonical isomorphism of Gerstenhaber algebras*

$$\Psi : \ (\mathrm{H}_{\mathrm{CE}}^\bullet(F, \wedge^\bullet B), \wedge, \lambda_2) \xrightarrow{\cong} (\mathrm{H}^\bullet(\Gamma(F[1], \wedge^\bullet T(F[1])), L_{d_F}), \wedge, [-, -]_{\mathrm{SN}}).$$

3. ATIYAH AND TODD CLASSES

3.1. **Atiyah and Todd classes of DG manifolds.** Let $(\mathcal{M}, Q)$ be a DG manifold. The Atiyah class $\alpha_{\mathcal{M}} \in$ $\mathrm{H}^1(\mathfrak{T}_2^1(T\mathcal{M}), L_Q)$ of $(\mathcal{M}, Q)$ is, by definition, the Atiyah class of the tangent DG vector bundle $T\mathcal{M} \to \mathcal{M}$ with respect to the DG Lie algebroid $T\mathcal{M}$ [13]: Choose a $T\mathcal{M}$-connection ∇ on the tangent bundle $T\mathcal{M}$. Then there associates a degree $+1$ L_Q-cocycle $\alpha_{\mathcal{M}}^{\nabla} \in \mathfrak{T}_2^1(T\mathcal{M})$ defined by

$$\alpha_{\mathcal{M}}^{\nabla}(X, Y) = L_Q(\nabla_X Y) - \nabla_{L_Q(X)} Y - (-1)^{|X|} \nabla_X L_Q(Y), \ \forall X, Y \in \Gamma(T\mathcal{M}), \tag{3.1}$$

which will be called the Atiyah cocycle of $\mathcal{M}$ with respect to ∇. The cohomology class

$$\alpha_{\mathcal{M}} = [\alpha_{\mathcal{M}}^{\nabla}] \in \mathrm{H}^1(\mathfrak{T}_2^1(T\mathcal{M}), L_Q),$$

which does not depend on the choice of ∇, is called the Atiyah class of the DG manifold $(\mathcal{M}, Q)$.

Note that

$$\alpha_{\mathcal{M}} \in \mathrm{H}^1(\mathfrak{T}_2^1(T\mathcal{M}), L_Q) \cong \mathrm{H}^1(\Omega^1(\mathcal{M}) \otimes_{C^{\infty}(\mathcal{M}, \mathbb{K})} \Gamma(\mathrm{End}(T\mathcal{M})), L_Q).$$

For any positive integer k, one can form $\alpha_{\mathcal{M}}^k$, the image of $\alpha_{\mathcal{M}}^{\otimes k}$ under the natural map

$$\otimes_{\mathbb{K}}^k \mathrm{H}^1(\Omega^1(\mathcal{M}) \otimes_{C^{\infty}(\mathcal{M}, \mathbb{K})} \Gamma(\mathrm{End}(T\mathcal{M})), L_Q) \to \mathrm{H}^k(\Omega^k(\mathcal{M}) \otimes_{C^{\infty}(\mathcal{M}, \mathbb{K})} \Gamma(\mathrm{End}(T\mathcal{M})), L_Q)$$

induced by the wedge product in $\Omega^{\bullet}(\mathcal{M})$ and the composition in $\mathrm{End}(T\mathcal{M})$.

The scalar Atiyah classes [13] of the DG manifold $(\mathcal{M}, Q)$ are defined by

$$c_k(\mathcal{M}) := \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \mathrm{str}(\alpha_{\mathcal{M}}^k) \in \mathrm{H}^k(\Omega^k(\mathcal{M}), L_Q), \ k = 1, 2, \cdots, \tag{3.2}$$

where $\mathrm{str}: \ \Gamma(\mathrm{End}(T\mathcal{M})) \to C^{\infty}(\mathcal{M}, \mathbb{K})$ is the supertrace map.

The Todd class [13] of the DG manifold $(\mathcal{M}, Q)$ is defined by

$$\mathrm{Td}_{\mathcal{M}} := \mathrm{Ber}\left(\frac{\alpha_{\mathcal{M}}}{1 - e^{-\alpha_{\mathcal{M}}}}\right) \in \prod_{k \geq 0} \mathrm{H}^k(\Omega^k(\mathcal{M}), L_Q), \tag{3.3}$$

where $\mathrm{Ber}: \ \Gamma(\mathrm{End}(T\mathcal{M})) \to C^{\infty}(\mathcal{M}, \mathbb{K})$ is the Berezinian (or the superdeterminant) map [11].

In this paper, we focus on the DG manifold $(F[1], d_F)$ corresponding to an integrable distribution $F \subset T_{\mathbb{K}} M$. The Atiyah, scalar Atiyah and Todd classes of $(F[1], d_F)$, are denoted respectively by:

$$\begin{aligned}
\alpha_{F[1]} &\in \mathrm{H}^1(\mathfrak{T}_2^1(T(F[1])), L_{d_F}), \\
c_k(F[1]) &\in \mathrm{H}^k(\Omega^k(F[1]), L_{d_F}), \quad k = 1, 2, \cdots \\
\mathrm{Td}_{F[1]} &\in \prod_{k \geq 0} \mathrm{H}^k(\Omega^k(F[1]), L_{d_F}).
\end{aligned}$$

3.2. **Atiyah and Todd classes of the Lie pair $(T_{\mathbb{K}} M, F)$.** Given an integrable distribution $F \subset T_{\mathbb{K}} M$, $(T_{\mathbb{K}} M, F)$ is a $\mathbb{K}$-Lie algebroid pair (or Lie pair for short). We briefly recall from [3] the Atiyah and Todd classes of the Lie pair $(T_{\mathbb{K}} M, F)$.

Recall that there is a short exact sequence of vector bundles over M:

$$0 \longrightarrow F \longrightarrow T_{\mathbb{K}} M \longrightarrow T_{\mathbb{K}} M / F \longrightarrow 0.$$

Let us choose a splitting $j: T_{\mathbb{K}} M / F \to T_{\mathbb{K}} M$ of the above short exact sequence and a $T_{\mathbb{K}} M$-connection ∇ on $T_{\mathbb{K}} M / F$ extending the Bott F-connection. The associated Atiyah cocycle

$$R^{\nabla} \in \Omega_F^1(\mathbb{T}_2^1(T_{\mathbb{K}} M / F)) = \Gamma(M; F^{\vee} \otimes F^{\perp} \otimes \mathrm{End}(T_{\mathbb{K}} M / F)),$$

where $F^\perp = (T_\mathbb{K} M / F)^\vee$, is defined by

$$R^\nabla(V, Z)W := \nabla_V \nabla_{j(Z)}(W) - \nabla_{j(Z)} \nabla_V(W) - \nabla_{[V, j(Z)]}(W),$$

for all $V \in \Gamma(F)$ and $Z, W \in \Gamma(T_\mathbb{K} M / F)$. The cohomology class

$$\alpha_{T_\mathbb{K} M / F} = [R^\nabla] \in \mathrm{H}^1_{\mathrm{CE}}(F, \mathbb{T}^1_2(T_\mathbb{K} M / F)) \cong \mathrm{H}^1_{\mathrm{CE}}(F, F^\perp \otimes \mathrm{End}(T_\mathbb{K} M / F))$$

does not depend on the choice of j and ∇, and is called the Atiyah class of the Lie pair $(T_\mathbb{K} M, F)$.

Remark 3.4. On the one hand, when $\mathbb{K} = \mathbb{R}$, this particular class is also known as the Molino class [14] of the foliation F. On the other hand, the Atiyah class of the Lie pair $(T_\mathbb{C} X, T_X^{0,1})$ corresponding to a complex manifold X is identical to Atiyah class α_X [1] of the complex manifold X.

The scalar Atiyah classes of the Lie pair $(T_\mathbb{K} M, F)$ are defined by

$$c_k(T_\mathbb{K} M / F) := \frac{1}{k!} \left(\frac{i}{2\pi} \right)^k \mathrm{tr}(\alpha^k_{T_\mathbb{K} M / F}) \in \mathrm{H}^k_{\mathrm{CE}}(F, \wedge^k F^\perp), \quad k = 1, 2, \cdots. \tag{3.5}$$

Here $\alpha^k_{T_\mathbb{K} M / F}$ is the image of $\alpha^{\otimes k}_{T_\mathbb{K} M / F}$ under the natural map

$$\otimes^k_\mathbb{K}(\mathrm{H}^1_{\mathrm{CE}}(F, F^\perp \otimes \mathrm{End}(T_\mathbb{K} M / F))) \to \mathrm{H}^k_{\mathrm{CE}}(F, \wedge^k F^\perp \otimes \mathrm{End}(T_\mathbb{K} M / F))$$

induced by the composition in $\mathrm{End}(T_\mathbb{K} M / F)$ and the wedge product in $\wedge^\bullet F^\perp$.

The Todd class of the Lie pair $(T_\mathbb{K} M, F)$ is the cohomology class

$$\mathrm{Td}_{T_\mathbb{K} M / F} = \det \left(\frac{\alpha_{T_\mathbb{K} M / F}}{1 - e^{-\alpha_{T_\mathbb{K} M / F}}} \right) \in \bigoplus_{k \geq 0} \mathrm{H}^k_{\mathrm{CE}}(F, \wedge^k F^\perp). \tag{3.6}$$

Remark 3.7. Let X be a compact Kähler manifold. Then the natural inclusion

$$\oplus_k \mathrm{H}^k(X, \Omega_X^k) \hookrightarrow \oplus_k \mathrm{H}^{2k}(X, \mathbb{C})$$

maps the scalar Atiyah classes $c_k(T_X^{1,0})$ and the Todd class $\mathrm{Td}_{T_X^{1,0}}$ of the Lie pair $(T_\mathbb{C} X, T_X^{0,1})$ to the k-th Chern characters $\mathrm{ch}_k(X)$ and the Todd class Td_X of X, respectively.

3.3. Splittings and connections.
From now on, we fix a splitting $j : B \to T_\mathbb{K} M$ of the short exact sequence (2.1) so that we can identify $T_\mathbb{K} M$ with $F \oplus B$.

Let us choose a $T(F[1])$-connection ∇ on $T(F[1])$. It induces an operator

$$\nabla^B : \Omega_F^\bullet(B) \times \Omega_F^\bullet(B) \to \Omega_F^\bullet(B)$$

as follows:

$$\nabla^B_\mathcal{Z} \mathcal{W} = \Phi(\nabla_{\Psi(\mathcal{Z})} \Psi(\mathcal{W})), \quad \forall \mathcal{Z}, \mathcal{W} \in \Omega_F^\bullet(B). \tag{3.8}$$

Here Φ and Ψ are part of the contraction data in the previous section. The following lemma can be easily verified:

Lemma 3.9.   • *The restriction of ∇^B defines a bilinear map*

$$\nabla^B : \Gamma(B) \times \Gamma(B) \to \Gamma(B),$$

which together with the Bott-F-connection on B determines a unique $T_\mathbb{K} M$-connection on B. This particular $T_\mathbb{K} M$-connection on B will also be denoted by ∇^B by abuse of notation.
   • *For all $Z \in \Gamma(B)$ and $V \in \Gamma(F)$, we have*

$$\iota_{\mathrm{pr}_F([Z,V])} = [\nabla^B_Z, \iota_V] = \nabla^B_Z \circ \iota_V - \iota_V \circ \nabla^B_Z : \quad \Omega_F^\bullet(B) \to \Omega_F^{\bullet-1}(B), \tag{3.10}$$

where $\iota_V : \Omega_F^\bullet(B) \to \Omega_F^{\bullet-1}(B)$ is the contraction operator, and $\mathrm{pr}_F : T_\mathbb{K} M \to F$ is the projection determined by the splitting j.

3.4. **Proof of Theorem B.**

3.4.1. *Atiyah classes.* By Theorem A, the first statement of Theorem B follows directly from the following Lemma:

Lemma 3.11. *Under the same hypothesis as in Lemma 3.9, the quasi-isomorphism*

$$\Phi = \Phi_2^1 : \ \mathfrak{T}_2^1(T(F[1])) \to \Omega_F^\bullet(\mathbb{T}_2^1(B))$$

maps the Atiyah cocycle $\alpha_{F[1]}^\nabla$ of the DG manifold $(F[1], d_F)$ with respect to the connection ∇, to the Atiyah cocycle R^{∇^B} of the Lie pair $(T_{\mathbb{K}}M, F)$ with respect to the $T_{\mathbb{K}}M$-connection ∇^B.

Proof. Recall that the Atiyah cocycle $\alpha_{F[1]}^\nabla \in \mathfrak{T}_2^1(T(F[1]))$ is of degree 1. Thus $\Phi(\alpha_{F[1]}^\nabla) \in \Omega_F^1(\mathbb{T}_2^1(B))$. It suffices to show that

$$\Phi(\alpha_{F[1]}^\nabla)(Z, W) = R^{\nabla^B}(-, Z)W \in \Omega_F^1(B),$$

for all $Z, W \in \Gamma(B)$. In fact, by Equation (2.19),

$$\Phi\left(\alpha_{F[1]}^\nabla\right)(Z, W) = \Phi\left(\alpha_{F[1]}^\nabla(\Psi(Z), \Psi(W))\right) \qquad \text{by Equation (3.1)}$$

$$= \Phi\left(L_{d_F}(\nabla_{\Psi(Z)}\Psi(W)) - \nabla_{L_{d_F}(\Psi(Z))}\Psi(W) - \nabla_{\Psi(Z)}L_{d_F}(\Psi(W))\right)$$

$$= d_B\left(\Phi(\nabla_{\Psi(Z)}\Psi(W))\right) - \Phi\left(\nabla_{\Psi(d_B(Z))}\Psi(W)\right) - \Phi\left(\nabla_{\Psi(Z)}(\Psi(d_B(W)))\right) \quad \text{by Equation (3.8)}$$

$$= d_B\left(\nabla_Z^B W\right) - \nabla_{d_B(Z)}^B W - \nabla_Z^B d_B(W).$$

Hence, for any $V \in \Gamma(F)$,

$$\iota_V\left(\Phi\left(\alpha_{F[1]}^\nabla\right)(Z, W)\right) = \iota_V\left(d_B\left(\nabla_Z^B W\right) - \nabla_{d_B(Z)}^B W - \nabla_Z^B d_B(W)\right)$$

$$= \nabla_V^B \nabla_Z^B W - \nabla_{\nabla_V^B Z}^B W - \iota_V(\nabla_Z^B d_B(W)) \qquad \text{by Equation (3.10)}$$

$$= \nabla_V^B \nabla_Z^B W - \nabla_{\nabla_V^B Z}^B W - \nabla_Z^B \nabla_V^B W + \nabla_{\mathrm{pr}_F[Z,V]}^B W$$

$$= \nabla_V^B \nabla_Z^B W - \nabla_Z^B \nabla_V^B W - \nabla_{[V,Z]}^B W = R^{\nabla^B}(V, Z)W.$$

Here we have used the definition of Bott-connection on B, which implies that

$$-\nabla_V^B Z + \mathrm{pr}_F[Z, V] = -\mathrm{pr}_B[V, Z] - \mathrm{pr}_F[V, Z] = -[V, Z].$$

This completes the proof. □

3.4.2. *Scalar Atiyah classes.* In this section, we prove the second statement of Theorem B about scalar Atiyah classes. Consider the supertrace map

$$\mathrm{str} : \ \Gamma(\mathrm{End}(T(F[1]))) \to \Omega_F^\bullet$$

and the trace map

$$\mathrm{tr} : \ \Omega_F^\bullet(\mathrm{End}(B)) \to \Omega_F^\bullet.$$

They are both cochain maps, and $\Omega_F^\bullet$-linear. In an obvious manner, both maps can be extended to

$$\mathrm{str} : \ \Omega^m(F[1]) \otimes_{\Omega_F^\bullet} \Gamma(\mathrm{End}(T(F[1]))) \to \Omega^m(F[1]), \quad \mathrm{tr} : \ \Omega_F^\bullet(\wedge^m B^\vee \otimes \mathrm{End}(B)) \to \Omega_F^\bullet(\wedge^m B^\vee),$$

which are also cochain maps.

Recall that by Proposition 2.17, we have a contraction data

$$H \overset{\curvearrowright}{} \ (\Omega^m(F[1]) \otimes_{\Omega_F^\bullet} \Gamma(\mathrm{End}(T(F[1]))), L_{d_F}) \ \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \ (\Omega_F^\bullet(\wedge^m B^\vee \otimes \mathrm{End}(B)), d_B). \qquad (3.12)$$

Lemma 3.13. *The cochain maps $\Phi \circ \mathrm{str}$ and $\mathrm{tr} \circ \Phi$ are homotopic with $\Phi \circ \mathrm{str} \circ H$ being the homotopy map, i.e.,*

$$\Phi \circ \mathrm{str} - \mathrm{tr} \circ \Phi = d_B \circ (\Phi \circ \mathrm{str} \circ H) + (\Phi \circ \mathrm{str} \circ H) \circ L_{d_F} : \Omega^m(F[1]) \otimes_{\Omega_F^\bullet} \Gamma(\mathrm{End}(T(F[1]))) \to \Omega_F^\bullet(\wedge^m B^\vee).$$

Proof. First of all, we prove the following

$$\Psi \circ \mathrm{tr} = \mathrm{str} \circ \Psi : \Omega_F^\bullet(\wedge^m B^\vee \otimes \mathrm{End}(B)) \to \Omega^m(F[1]). \tag{3.14}$$

When $m = 0$, Equation (3.14) reduces to

$$\mathrm{tr} = \mathrm{str} \circ \Psi_1^1 : \Omega_F^\bullet(\mathrm{End}(B)) \to \Omega_F^\bullet,$$

which is clearly true, since all maps involved are $\Omega_F^\bullet$-linear and elements in $\Gamma(B)$ are purely even. For any $\omega \otimes_{\Omega_F^\bullet} \lambda \in \Omega_F^\bullet(\wedge^m B^\vee \otimes \mathrm{End}(B))$, where $\omega \in \Omega_F^\bullet(\wedge^m B^\vee)$ and $\lambda \in \Omega_F^\bullet(\mathrm{End}(B))$, we have

$$\mathrm{str}(\Psi(\omega \otimes_{\Omega_F^\bullet} \lambda)) = \mathrm{str}((\wedge^m \phi^\vee)(\omega) \otimes_{\Omega_F^\bullet} \Psi_1^1(\lambda)) = (\wedge^m \phi^\vee)(\omega) \otimes_{\Omega_F^\bullet} \mathrm{str}(\Psi_1^1(\lambda))$$
$$= \wedge^m \phi^\vee(\omega) \otimes_{\Omega_F^\bullet} \mathrm{tr}(\lambda) = (\wedge^m \phi^\vee)(\mathrm{tr}(\omega \otimes_{\Omega_F^\bullet} \lambda)) = \Psi(\mathrm{tr}(\omega \otimes_{\Omega_F^\bullet} \lambda)).$$

This proves (3.14).

For any $\Lambda \in \Omega^m(F[1]) \otimes_{\Omega_F^\bullet} \Gamma(\mathrm{End}(T(F[1])))$, by the contraction data (3.12), we have

$$\Lambda = \Psi(\Phi(\Lambda)) + [L_{d_F}, H](\Lambda).$$

Thus it follows that

$$\Phi(\mathrm{str}(\Lambda)) = \Phi(\mathrm{str}(\Psi(\Phi(\Lambda)))) + \Phi(\mathrm{str}([L_{d_F}, H](\Lambda))) \qquad \text{by Equation (3.14)}$$
$$= (\Phi \circ \Psi)(\mathrm{tr}(\Phi(\Lambda))) + (\Phi \circ \mathrm{str})(L_{d_F}(H(\Lambda))) + (\Phi \circ \mathrm{str})(H(L_{d_F}(\Lambda)))$$
$$= \mathrm{tr}(\Phi(\Lambda)) + d_B(\Phi(\mathrm{str}(H(\Lambda)))) + \Phi(\mathrm{str}(H(L_{d_F}(\Lambda)))).$$

Here we used the fact that both Φ and str are cochain maps. $\qquad\square$

As an immediate corollary, we have

Corollary 3.15. *The following diagram*

$$
\begin{array}{ccc}
\mathrm{H}^\bullet(\Omega^m(F[1]) \otimes_{\Omega_F^\bullet} \Gamma(\mathrm{End}(T(F[1]))), L_{d_F}) & \xrightarrow{\ \Phi\ } & \mathrm{H}_{\mathrm{CE}}^\bullet(F, \wedge^m B^\vee \otimes \mathrm{End}(B)) \\
\Big\downarrow{\scriptstyle \mathrm{str}} & & \Big\downarrow{\scriptstyle \mathrm{tr}} \\
\mathrm{H}^\bullet(\Omega^m(F[1]), L_{d_F}) & \xrightarrow{\ \ \ \ \Phi\ \ \ \ } & \mathrm{H}_{\mathrm{CE}}^\bullet(F, \wedge^m B^\vee)
\end{array}
$$

commutes, i.e.,

$$\Phi \circ \mathrm{str} = \mathrm{tr} \circ \Phi : \mathrm{H}^\bullet(\Omega^m(F[1]) \otimes_{\Omega_F^\bullet} \Gamma(\mathrm{End}(T(F[1]))), L_{d_F}) \to \mathrm{H}_{\mathrm{CE}}^\bullet(F, \wedge^m B^\vee).$$

Proof of Theorem B (2). Recall the defining equations (3.2) and (3.5) of the two types of scalar Atiyah classes. Then

$$\Phi(c_k(F[1])) = \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \Phi(\mathrm{str}(\alpha_{F[1]}^k)) \qquad \text{by Corollary 3.15}$$

$$= \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}(\Phi(\alpha_{F[1]}^k)) \quad \text{since } \Phi \text{ is an algebra morphism}$$

$$= \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}((\Phi(\alpha_{F[1]}))^k) \qquad \text{by Theorem B (1)}$$

$$= \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \mathrm{tr}(\alpha_{T_\mathbb{K}M/F}^k) = c_k(T_\mathbb{K}M/F).$$

$\qquad\square$

3.4.3. *Todd classes.* Before we prove the last statement of Theorem B, we recall a general method calculating the determinant and Berezinian of formal power series of cohomology classes. Let

$$P(x) = 1 + \sum_{i \geq 1} b_i x^i,$$

be a formal power series, where $b_i \in \mathbb{K}$. According to [4, Lemma 1.2.2], one can associate to $P(x)$ a sequence of polynomials $K_r(p_1, \cdots, p_r), r = 1, 2, \cdots$, of homogeneous weight r, called m-sequence. Here by saying that K_r is of homogeneous weight r, we mean that K_r is of the form

$$K_r(p_1, \cdots, p_r) = \sum_{n_1 + 2n_2 + \cdots + rn_r = r} c_{n_1 \cdots n_r}(b_i) p_1^{n_1} \cdots p_r^{n_r}.$$

It is well-known that

$$\mathrm{Ber}(P(\mathcal{A})) = 1 + \sum_{r \geq 1} K_r(\mathrm{str}(\mathcal{A}), \cdots, \mathrm{str}(\mathcal{A}^r)),$$

$$\det(P(A)) = 1 + \sum_{r \geq 1} K_r(\mathrm{tr}(A), \cdots, \mathrm{tr}(A^r))$$

for any supermatrix $\mathcal{A}$ in a graded commutative $\mathbb{K}$-algebra, and any matrix A in an ordinary commutative $\mathbb{K}$-algebra.

Let

$$s := \min\{\mathrm{rank}(F), \mathrm{rank}(B)\}.$$

It follows from Corollary 2.29 that for any $\omega \in \mathrm{H}^1(\mathfrak{T}_2^1(T(F[1])), L_{d_F})$, we have $\omega^r = 0$ for all $r > s$. Since K_r is of homogeneous weight r, one has

$$\mathrm{Ber}(P(\omega)) = 1 + \sum_{r=1}^{s} K_r(\mathrm{str}(\omega), \cdots, \mathrm{str}(\omega^r)).$$

Similarly, for any $\lambda \in \mathrm{H}^1_{\mathrm{CE}}(F, B^\vee \otimes \mathrm{End}(B))$, one has

$$\det(P(\lambda)) = 1 + \sum_{r=1}^{s} K_r(\mathrm{tr}(\lambda), \cdots, \mathrm{tr}(\lambda^r)).$$

Lemma 3.16. *With notations as above, we have, for any $\omega \in \mathrm{H}^1(\mathfrak{T}_2^1(T(F[1])), L_{d_F})$*

$$\Phi(\mathrm{Ber}(P(\omega))) = \det(P(\Phi(\omega))) \in \bigoplus_{k \geq 0} \mathrm{H}^k_{\mathrm{CE}}(F, \wedge^k B^\vee).$$

Proof. By Corollary 3.15 and the fact that Φ is an algebra morphism, we have

$$\Phi(\mathrm{str}(\omega^i)) = \mathrm{tr}(\Phi(\omega^i)) = \mathrm{tr}((\Phi(\omega))^i), \quad \forall 1 \leq i \leq s.$$

Therefore,

$$\Phi(\mathrm{Ber}(P(\omega))) = 1 + \sum_{r=1}^{s} K_r(\Phi(\mathrm{str}(\omega)), \cdots, \Phi(\mathrm{str}(\omega^r)))$$

$$= 1 + \sum_{r=1}^{s} K_r(\mathrm{tr}\,\Phi(\omega), \cdots, \mathrm{tr}\,(\Phi(\omega))^r) = \det(P(\Phi(\omega))),$$

as desired. $\square$

Now we prove the last statement of Theorem B:

Proof of Theorem B (3). We take the particular power series

$$P(x) = \frac{x}{1 - e^{-x}}.$$

By the defining formulas (3.3) and (3.6) of Todd classes, we have

$$\mathrm{Td}_{F[1]} = \mathrm{Ber}\, P(\alpha_{F[1]}), \quad \mathrm{Td}_{T_{\mathbb{K}}M/F} = \det P(\alpha_{T_{\mathbb{K}}M/F}).$$

By Theorem B (1), we have

$$\alpha_{T_{\mathbb{K}}M/F} = \Phi(\alpha_{F[1]}).$$

Hence,

$$
\begin{aligned}
\Phi(\mathrm{Td}_{F[1]}) &= \Phi(\mathrm{Ber}\, P(\alpha_{F[1]})) &&\text{by Lemma 3.16} \\
&= \det P(\Phi(\alpha_{F[1]})) &&\text{by Theorem B (1)} \\
&= \det P(\alpha_{T_{\mathbb{K}}M/F}) = \mathrm{Td}_{T_{\mathbb{K}}M/F}.
\end{aligned}
$$

$\square$

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