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Communications in Partial Differential Equations

ISSN: 0360-5302 (Print) 1532-4133 (Online) Journal homepage: http://www.tandfonline.com/loi/lpde20

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To cite this article: Zhi-You Chen & Jann-Long Chern (2016): Topological Multivortex Solutions for the Chern-Simons System with Two Higgs Particles, Communications in Partial Differential Equations, DOI: 10.1080/03605302.2015.1132429

To link to this article: http://dx.doi.org/10.1080/03605302.2015.1132429

Topological Multivortex Solutions for the Chern-Simons System with Two Higgs Particles

Zhi-You Chen[1] and Jann-Long Chern[2]

[1]Department of Mathematics, National Changhua University of Education, Changhua, Taiwan

[2]Department of Mathematics, National Central University, National Center for Theoretical Sciences of Taiwan, Chung-Li, Taiwan

Abstract

In this paper, we prove the uniqueness of topological multivortex solutions to the self-dual abelian Chern-Simons Model if either the Chern-Simons coupling parameter is sufficiently small or sufficiently large. In addition, we also establish the sharp region of the flux for non-topological solutions with a single vortex point.

KEYWORDS: ■.

2010 Mathematics Subject Classfication Primary 35J47, 35J60; Secondary 35A02.

Received December 30, 2014; Accepted December 01 2015.

Zhi-You Chen, Department of Mathematics, National Changhua University of Education, Changhua 500, Taiwan; E-mail: zhiyou@cc.ncue.edu.tw

1. INTRODUCTION

In this paper, we consider the nonlinear elliptic system

$$\begin{cases} \Delta u + \frac{1}{\varepsilon^2} e^v (1 - e^u) = 4\pi \sum_{i=1}^{N_1} \mu_i \delta_{p_i} \text{in } \mathbf{R}^2, \\ \Delta v + \frac{1}{\varepsilon^2} e^u (1 - e^v) = 4\pi \sum_{j=1}^{N_2} v_j \delta_{q_j} \text{in } \mathbf{R}^2, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$, $\varepsilon > 0$, $N_1$and N_2 are two positive integers, $\mu_i \geq 0$ and $v_j \geq 0$ which are called vortex numbers, and δ_p is the Dirac measure at p. System (1.1) arises from a relativistic Abelian Chern-Simons model with two Higgs particles. Let ϕ and χ be two complex scalar fields in $\mathbf{R}^2$ representing two Higgs particles with charges $\mathbf{q}_1$ and $\mathbf{q}_2$. Assume that ϕ and χ have zeros at p_s and q_s with corresponding order μ_s and v_s, respectively, i.e.,

$$|\phi| = |z - p_s|^{\mu_s} \text{for } z \text{near } p_s,\ s = 1, \cdots, N_1,$$

$$|\psi| = |z - q_s|^{v_s} \text{for } z \text{near } q_s,\ s = 1, \cdots, N_2. \tag{1.2}$$

Let $A_r^{(1)}$ and $A_r^{(2)}$, $r = 1, 2$, be two gauge fields and the corresponding covariant derivatives D_r be

$$D_r \phi = \partial_r \phi - i q_I A_r^{(1)} \phi$$

$$D_r \chi = \partial_r \chi - i q_I A_r^{(2)} \chi.$$

Set

$$F_{rs}^{(I)} = \partial_r A_s^{(I)} - \partial_s A_r^{(I)} \tag{1.3}$$

to be the corresponding curvatures associated with D_r on $(2+1)$-dimensional Minkowski space with the metric $(g_{r,s}) = \text{diag}(1, -1, -1)$, where $r, s = 0, 1, 2$. In (10) and (16), the Chern-Simons density takes the form

$$\mathscr{L} = -\frac{\kappa}{4}\varepsilon^{rst}A_r^{(1)}F_{st}^{(2)} - \frac{\kappa}{4}\varepsilon^{rst}A_r^{(2)}F_{st}^{(1)} + \overline{D_r\phi}D^r\phi + \overline{D_r\chi}D^r\chi - V(\phi, \chi),$$

with the Higgs potential density defined by

$$V(\phi, \chi) = \frac{\mathbf{q}_1^2\mathbf{q}_2^2}{\kappa^2}(|\phi|^2(|\chi|^2 - c_2^2)^2 + |\chi|^2(|\phi|^2 - c_1^2)^2),$$

where $\kappa > 0$ is a coupling parameter, c_1 and c_2 are positive constants.

For the static case, by changing of variables, $q_I A_j^{(I)} \to A_j^{(I)}$, $\phi \to c_1\phi$ and $\chi \to c_2\chi$, the self-dual equation for Chern-Simons Lagrangrian $\mathscr{L}$ can be reduced to a system of equations of first order

$$\begin{aligned} &D_1\phi \pm iD_2\phi = 0, \\ &D_1\chi \pm iD_2\chi = 0, \\ &F_{12}^{(1)} \pm \frac{2\mathbf{q}_1\mathbf{q}_2^2}{\kappa^2}|\chi|^2(1-|\phi|^2) = 0 \\ &F_{12}^{(2)} \pm \frac{2\mathbf{q}_1\mathbf{q}_2^2}{\kappa^2}|\phi|^2(1-|\chi|^2) = 0, \end{aligned} \tag{1.4}$$

where

$$D_r\phi = \partial_r\phi - iA_r^{(1)}\phi, \quad D_r\chi = \partial_r\chi - iA_r^{(2)}\chi. \tag{1.5}$$

For the detailed derivation of (1.4) from the Lagrangian $\mathscr{L}$, we refer the readers to (10), (16), (17) and the references therein.

Following the results in (14) by Jaffe and Taubes, (1.4) can be reduced to an elliptic system of second order. Set the operators ∂ and $\overline{\partial}$ by

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2)$$
$$\overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2).$$

From (1.3), we have

$$F_{12}^{(I)} = \partial_1 A_2^{(I)} - \partial_2 A_1^{(I)} = (\partial + \overline{\partial})A_2^{(I)} - i(\partial - \overline{\partial})A_1^{(I)}$$
$$= (-i)[\partial(A_1^{(I)} + iA_2^{(I)}) + \overline{\partial}(-A_1^{(I)} + iA_2^{(I)})]. \tag{1.6}$$

Without loss of generality, we consider the positive sign in (1.4). By (1.5), we have

$$D_1\phi + iD_2\phi = 2\overline{\partial}\phi - iA_1^{(1)}\phi + A_2^{(1)}\phi = 2\overline{\partial}\phi - i(A_1^{(1)} + iA_2^{(1)})\phi$$
$$D_1\chi + iD_2\chi = 2\overline{\partial}\chi - iA_1^{(2)}\chi + A_2^{(2)}\chi = 2\overline{\partial}\chi - i(A_1^{(2)} + iA_2^{(2)})\chi.$$

Hence, from (1.4), we obtain

$$A_1^{(1)} + iA_2^{(1)} = (2i)\frac{\overline{\partial}\phi}{\phi} = (2i)\overline{\partial}\ln\phi$$
$$A_1^{(2)} + iA_2^{(2)} = (2i)\frac{\overline{\partial}\chi}{\chi} = (2i)\overline{\partial}\ln\chi. \tag{1.7}$$

Combining (1.6) and (1.7), we deduce the following relations:

$$F_{12}^{(1)} = 2[\partial\overline{\partial}\ln\phi + \overline{\partial}\partial\ln\overline{\phi}] = \frac{1}{2}\Delta\ln|\phi|^2 \tag{1.8}$$

$$F_{12}^{(2)} = 2[\partial\overline{\partial} \ln \chi + \overline{\partial}\partial \ln \overline{\chi}] = \frac{1}{2}\Delta \ln |\chi|^2.$$

Now, let

$$u = \ln |\phi|^2 \text{and } v = \ln |\chi|^2. \tag{1.9}$$

Then, by (1.4), u and v satisfy (1.1) for $x \notin \mathscr{Z} = \{p_1, \cdots, p_{N_1}, q_1, \cdots, q_{N_2}\}$ and $\varepsilon^2 = \kappa^2/4c_1^2 c_2^2 \mathbf{q}_1 \mathbf{q}_2^2$. It is easy to see that the Dirac measures appearing in (1.1) come from (1.2) and (1.9).

Conversely, suppose that (u, v) is a solution of (1.1). Let $z = x^1 + ix^2$ and

$$\begin{cases} \theta_1(z) = -\sum\limits_{s=1}^{N_1} \arg(z - p_s), \theta_2(z) = -\sum\limits_{s=1}^{N_2} \arg(z - q_s), \\ \phi(z) = e^{\frac{1}{2}u(z)+i\theta_1(z)}, \chi(z) = e^{\frac{1}{2}v(z)+i\theta_2(z)}, \\ A_1^{(1)}(z) = -\mathrm{Re}\{2i\overline{\partial} \ln \phi(z)\}, A_2^{(1)}(z) = -\mathrm{Im}\{2i\overline{\partial} \ln \phi(z)\}, \\ A_1^{(2)}(z) = -\mathrm{Re}\{2i\overline{\partial} \ln \chi(z)\}, A_2^{(2)}(z) = -\mathrm{Im}\{2i\overline{\partial} \ln \chi(z)\}. \end{cases} \tag{1.10}$$

Then $(\phi, \chi, A_r^{(I)}), I = 1, 2, r = 1, 2,$ satisfy (1.4). Therefore we have shown that (1.4) is equivalent to (1.1). In the past decades, the equation derived from Chern-Simons model with one Higgs particle has been intensively studied in, e.g., (4)–(7), (8)–(10), (11)–(16), (18)–(24) and references therein. However, for the system of coupled equations, the study of (1.1) has just begun recently (see, e.g., (2) and (17)).

For (1.1), there are two natural boundary conditions for solutions at infinity, namely,

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0, \tag{1.11}$$

or

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = -\infty. \tag{1.12}$$

We note that if (u, v) is a solution with the boundary condition either (1.11) or (1.12), then, by the maximum principle, we have $u(x) < 0$ and $v(x) < 0$ for all $x \in \mathbf{R}^2 \setminus \mathcal{Z}$. In physics literature, a solution (u, v) satisfying boundary condition (1.11) is called a topological solution and it is easy to see that both $|u|$ and $|v|$ decay exponentially at infinity.

In a recent paper (17), Lin-Ponce-Yang has considered the problem about the existence of the topological solutions of (1.1) for any given set of singularities and the following theorem was proved.

Theorem A.. *For any given points* $p_1, \ldots, p_{N_1}, q_1, \ldots, q_{N_2} \in \mathbf{R}^2$ *and* $\mu_1, \ldots, \mu_{N_1}, \nu_1, \ldots, \nu_{N_2} \in \mathbf{R}^+ \cup \{0\}$, (1.1) *possesses a topological solution.*

In addition to the existence result as stated in Theorem A, it is natural to ask the question about the uniqueness of topological solutions to (1.1). For the case of a single vortex point, the uniqueness result has been proved in (2) for any $\varepsilon > 0$. Moreover, it is not difficult to see that the uniqueness of topological solutions for (1.1) in the case $N_1 = N_2$, $i = j$ and $(p_i, \mu_i) = (q_i, \nu_i)$ is equivalent to the one for topological solutions of

$$\Delta u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{i=1}^{N_1} \mu_i \delta_{p_i} \text{in } \mathbf{R}^2. \tag{1.13}$$

Therefore, according to (1), the uniqueness result for this specific case has also been proved if ε is sufficiently large or small. In this article, we provide a uniqueness result of topological solutions to (1.1) in the general case, stated in the following theorem.

Theorem 1.1. *Let* $p_1, \ldots, p_{N_1}, q_1, \ldots, q_{N_2} \in \mathbf{R}^2$ *and* $\mu_1, \ldots, \mu_{N_1}, \nu_1, \ldots, \nu_{N_2} \in \mathbf{R}^+ \cup \{0\}$ *be given. Then there are two positive constants* ε_0 *and* ε_1*, dependent on* p_i, q_j, μ_i, ν_j*, such that* (1.1) *possesses a unique topological solution for any* $\varepsilon \in (0, \varepsilon_0) \cup (\varepsilon_1, \infty)$.

In (2), we call a solution (u, v) of (1.1) to be non-topological if (u, v) satisfies the boundary condition (1.12), and both $e^u(1 - e^v)$ and $e^v(1 - e^u)$ are in $L^1(\mathbf{R}^2)$. For any entire solution (u, v) of (1.1), we set

$$\beta_{1,\varepsilon}(u, v) = \frac{1}{2\pi\varepsilon^2} \int_{\mathbf{R}^2} e^{v(x)}(1 - e^{u(x)})\, dx \text{ and } \beta_{2,\varepsilon}(u, v) = \frac{1}{2\pi\varepsilon^2} \int_{\mathbf{R}^2} e^{u(x)}(1 - e^{v(x)})\, dx. \quad (1.14)$$

By virtue of (2), the existence of non-topological solutions of (1.1) has been established for the case of none of vortex point or single one. Moreover, for $N_1 = N_2 = N$, (1.1) possesses a non-topological solution (u, u) satisfying $\beta_{1,\varepsilon}(u, u) = \beta_{2,\varepsilon}(u, u) = \Phi$ for any given $\varepsilon > 0$ and $\Phi > 4(N + 1)$ which was obtained in (4). Besides, if $N_1 = N_2 = N$ and $(p_i, \mu_i) = (q_j, \mu_j)$ for all $i, j = 1, \cdots N$, then, based on (5), for any $\varepsilon > 0$ and $\Phi > 4(N + 1)$ with $\Phi \notin \left\{4N\frac{k}{k-1} | k = 2, \cdots, \sum_{j=1}^{M} \mu_j\right\}$, there exists a non-topological solution (u, u) of (1.1) satisfying $\beta_{1,\varepsilon}(u, u) = \beta_{2,\varepsilon}(u, u) = \Phi$. Therefore, there is an intersecting problem about finding the sharp range of flux pair $(\beta_{1,\varepsilon}(u, v), \beta_{2,\varepsilon}(u, v))$ for all non-topological solutions (u, v) of

(1.1). In our second main result, we give an answer to the sharp region of flux pair for non-topological solutions of (1.1) for the case of none of vortex point or single one.

Theorem 1.2. *Let* $(u_\varepsilon(x), v_\varepsilon(x))$ *be a non-topological solution of the equation*

$$\begin{cases} \Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{v_\varepsilon}(1 - e^{u_\varepsilon}) = 4\pi\lambda\delta_p \text{in } \mathbf{R}^2, \\ \Delta v_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon}(1 - e^{v_\varepsilon}) = 4\pi\bar{\lambda}\delta_p \text{in } \mathbf{R}^2 \end{cases} \tag{1.15}$$

for any point $p \in \mathbf{R}^2$ *and constants* $\lambda, \bar{\lambda} \geq 0$. *Then* $\left(\beta_{1,\varepsilon}(u, v), \beta_{2,\varepsilon}(u, v)\right)$ *satisfies*

$$\begin{cases} \beta_{1,\varepsilon}(u, v) \in (2\lambda + 2, \infty), \quad \beta_{2,\varepsilon}(u, v) \in (2\bar{\lambda} + 2, \infty), \\ \left[\beta_{1,\varepsilon}(u, v) - 2(\lambda + 1)\right]\left[\beta_{2,\varepsilon}(u, v) - 2(\bar{\lambda} + 1)\right] > 4(\lambda + 1)(\bar{\lambda} + 1). \end{cases} \tag{1.16}$$

Moreover, (1.15) *possesses a non-topological solution* $(u_\varepsilon(x), v_\varepsilon(x))$ *such that* $\left(\beta_{1,\varepsilon}(u, v), \beta_{2,\varepsilon}(u, v)\right) = (\beta_1, \beta_2)$ *for any pair* (β_1, β_2) *which satisfies* (1.16).

This article is organized as follows. Section 2 is devoted to proving the consequences of Theorem 1.2. In Section 3, we will sketch the proof of Theorem 1.1 about the uniqueness of topological solutions for $\varepsilon > 0$ sufficiently small. In the last section, Section 4, the uniqueness of topological solutions for $\varepsilon > 0$ large enough in Theorem 1.1 will be discussed.

2. SHARP RANGE OF FLUX FOR NON-TOPOLOGICAL SOLUTIONS

Let $(u_\varepsilon(x), v_\varepsilon(x))$ be a non-topological solution of (1.15) and $(u(x), v(x)) = (u_\varepsilon(\varepsilon x + p), v_\varepsilon(\varepsilon x + p))$. Then $(u(x), v(x))$ satisfies the following system

$$\begin{cases} \Delta u + e^v(1 - e^u) = 4\pi\lambda\delta_O \text{in } \mathbf{R}^2, \\ \Delta v + e^u(1 - e^v) = 4\pi\bar{\lambda}\delta_O \text{in } \mathbf{R}^2, \\ \int_{\mathbf{R}^2} e^{v(x)}(1 - e^{u(x)})dx < \infty, \\ \int_{\mathbf{R}^2} e^{u(x)}(1 - e^{v(x)})dx < \infty, \end{cases} \tag{2.1}$$

where $\lambda \geq 0$, $\bar{\lambda} \geq 0$ and $O = (0, 0)$. First,we will refer to the behavior of the non-positive solutions (i.e. $u(x) \leq 0$, $v(x) \leq 0$ in $\mathbf{R}^2 \backslash \{O\}$) near infinity as following lemma.

Lemma 2.1. *If $(u(x), v(x))$ is a non-positive solution of (2.1), then it can be proved that*

$$u(x) = \left[2\lambda - \beta_{1,1}(u, v)\right] \ln|x| + C_1 + o(1) \text{ and } v(x) = \left[2\bar{\lambda} - \beta_{2,1}(u, v)\right] \ln|x| + C_2 + o(1) \text{ near } \infty \tag{2.2}$$

for some constants $C_1, C_2 \in \mathbf{R}$. Moreover, non-positive solution $(u(x), v(x))$ is a topological solution or non-topological solution.

Proof. We set $\widetilde{w}(x) = u(x) - 2\lambda \ln|x|$ and hence $\widetilde{w}(x)$ satisfies

$$\Delta\widetilde{w} + e^v(1 - e^u) = 0 \quad \text{in } \mathbf{R}^2.$$

Define the potential

$$\widetilde{v}(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \ln\left(\frac{|x - y|}{|y|}\right)\Gamma(y)dy$$

where $\Gamma(x) = e^{v(x)}(1 - e^{u(x)})$. Then $\widetilde{v}(x)$ satisfies

$$\Delta\widetilde{v} = e^{v}(1 - e^{u}) \quad \text{in } \mathbf{R}^2$$

and we see that

$$2\pi\widetilde{v}(x) \leq \left(\int_{\mathbf{R}^2} \Gamma(y)dy\right) \ln|x| + C \tag{2.3}$$

where $R_0 \gg 1$, C is a constant independent of x and $x \in \mathbf{R}^2 \setminus B_{R_0}(O)$. Since $\widetilde{w}(x) \leq 0$ in $\mathbf{R}^2 \setminus B_{R_0}(O)$, $\widetilde{w}(x) + \widetilde{v}(x) \leq C(\ln|x| + 1)$ on $\{x : |x| \geq 1\}$ for some constant $C > 0$, which implies $\widetilde{w}(x) + \widetilde{v}(x)$ is a constant in $\mathbf{R}^2$ by Lemma 4.6.1 of (24). It is not difficult to see that

$$\frac{\widetilde{v}(x)}{\ln|x|} \to \frac{1}{2\pi}\int_{\mathbf{R}^2} \Gamma(x)dx \ \text{ uniformly as } \ |x| \to \infty,$$

which implies

$$\frac{u(x)}{\ln|x|} \to 2\lambda - \frac{1}{2\pi}\int_{\mathbf{R}^2} \Gamma(x)dx \ \text{ uniformly as } \ |x| \to \infty$$

and $\beta_{1,1}(u, v) > 2\lambda + 2$ or $\beta_{1,1}(u, v) = 2\lambda$. Similarly for $v(x)$, the result (2.3) is obtained and $\beta_{2,1}(u, v)$ satisfies

$$\begin{cases} \beta_{2,1}(u, v) = 2\bar{\lambda} \ \text{ if } \ \beta_{1,1}(u, v) = 2\lambda, \\ \beta_{2,1}(u, v) > 2\bar{\lambda} + 2 \ \text{ if } \ \beta_{1,1}(u, v) > 2\lambda + 2. \end{cases}$$

Therefore, we we complete this lemma. □

Proposition 2.1. *If* $(u(x), v(x))$ *is a non-topological solution of* (2.1), *then* $\beta_{1,1}(u, v)$ *and* $\beta_{2,1}(u, v)$ *satisfy* (1.16).

Proof. The proof is standard by Pohozaev identity and (2.2). Thus we omit the detail. □

Next, we consider the following ODE system

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) + e^{v(r)}(1 - e^{u(r)}) = 0, \\ v''(r) + \frac{1}{r}v'(r) + e^{u(r)}(1 - e^{v(r)}) = 0, \end{cases} r > 0 \tag{2.4}$$

with the initial value

$$\begin{cases} u(r) = 2\lambda \ln r + \alpha_1 + o(1), \\ v(r) = 2\bar{\lambda} \ln r + \alpha_2 + o(1), \end{cases} \text{as } r \to 0^+ \tag{2.5}$$

where $\alpha_1 \in \mathbf{R}$ and $\alpha_2 \in \mathbf{R}$. For all entire solutions $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ of (2.4)–(2.5), we set their flux pair

$$(\beta_1(\alpha_1, \alpha_2), \beta_2(\alpha_1, \alpha_2)) = \left(\int_0^\infty re^{v(r)}(1 - e^{u(r)})\ dr,\ \int_0^\infty re^{u(r)}(1 - e^{v(r)})\ dr \right). \tag{2.6}$$

According to the flux pair, all entire solutions of (2.4) with boundary condition (1.12) can be classified into the following three types.

Type (I): $\lim_{r\to\infty} u(r) = -\infty,\ \ \lim_{r\to\infty} v(r) = -\infty$ with $\beta_1 < \infty$ and $\beta_2 < \infty$, i.e., (u, v) is a non-topological solution.

Type (II): $\lim_{r\to\infty} u(r) = -\infty,\ \ \lim_{r\to\infty} v(r) = -\infty$ with $2\lambda < \beta_1 \le 2\lambda + 2,\ \ \beta_2 = \infty$.

Type (III): $\lim_{r\to\infty} u(r) = -\infty,\ \ \lim_{r\to\infty} v(r) = -\infty$ with $\beta_1 = \infty,\ \ 2\bar{\lambda} < \beta_2 \le 2\bar{\lambda} + 2$.

Then the structure of solutions sets is described as follows from paper(2).

Theorem 2.1. *There exist strictly increasing functions* $\gamma_u, \gamma_v : (-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ *satisfying* $\gamma_v(\alpha_1) \ge \gamma_u(\alpha_1)$*, and*

$$\lim_{\alpha_1 \to -\infty} \gamma_i(\alpha_1) = -\infty, \quad \lim_{\alpha_1 \to \alpha_1^0} \gamma_i(\alpha_1) = \alpha_2^0, \quad \text{for } i = u, v$$

such that

$$T = \{\alpha = (\alpha_1, \alpha_2) : (\phi_\lambda(r, \alpha), \phi_{\bar{\lambda}}(r, \alpha)) \text{is a topological solution}\} = \{(\alpha_1^0, \alpha_2^0)\},$$

$$\Omega_{NT} = \{\alpha = (\alpha_1, \alpha_2) : (\phi_\lambda(r, \alpha), \phi_{\bar{\lambda}}(r, \alpha)) \text{is a non-topological solution}\}$$

$$= \{(\alpha_1, \alpha_2) : \alpha_1 < \alpha_1^0,\ \gamma_u(\alpha_1) < \alpha_2 < \gamma_v(\alpha_1)\}$$

and

$$\begin{cases} \Gamma_{up} = \{(\alpha_1, \alpha_2) : \alpha_1 < \alpha_1^0,\ \alpha_2 = \gamma_v(\alpha_1)\} \\ \quad\quad = \{\alpha = (\alpha_1, \alpha_2) : (\phi_\lambda(r, \alpha), \phi_{\bar{\lambda}}(r, \alpha)) \text{is a solution of type (III)}\}, \\ \Gamma_{low} = \{(\alpha_1, \alpha_2) : \alpha_1 < \alpha_1^0,\ \alpha_2 = \gamma_u(\alpha_1)\} \\ \quad\quad = \{\alpha = (\alpha_1, \alpha_2) : (\phi_\lambda(r, \alpha), \phi_{\bar{\lambda}}(r, \alpha)) \text{is a solution of type (II)}\}. \end{cases}$$

Moreover, if $(\alpha_1, \alpha_2) \in \Omega_{NT}$*, then* $(\beta_1(\alpha_1, \alpha_2), \beta_2(\alpha_1, \alpha_2))$ *satisfies* (1.16).

Remark 2.1.

(i) From continuity and Theorem 2.1, there exist strictly increasing continuous functions $\tau_1^c : (-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ and $\tau_2^d : (-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ such that

$$\lim_{\alpha_1 \to \alpha_1^0} \tau_1^c(\alpha_1) = \lim_{\alpha_1 \to \alpha_1^0} \tau_2^d(\alpha_1) = \alpha_2^0, \quad \lim_{\alpha_1 \to -\infty} \tau_1^c(\alpha_1) = \lim_{\alpha_1 \to -\infty} \tau_2^d(\alpha_1) = -\infty$$

and $(\alpha_1, \tau_1^c(\alpha_1)), (\alpha_2, \tau_2^d(\alpha_1)) \in \Omega_{NT}$ for all $\alpha_1 < \alpha_1^0$ implies

$$\beta_1(\alpha_1, \tau_1^c(\alpha_1)) = c \text{ and } \beta_2(\alpha_1, \tau_2^d(\alpha_1)) = d \text{ on } (-\infty, \alpha_1^0)$$

for any $(c, d) \in (2\lambda + 2, \infty) \times (2\bar{\lambda} + 2, \infty)$. Moreover, we see that if $c_1 < c_2$(resp.,$d_1 > d_2$), then

$$\tau_1^{c_1}(\alpha_1)) < \tau_1^{c_2}(\alpha_1)(\text{resp., } \tau_2^{d_1}(\alpha_1) < \tau_2^{d_2}(\alpha_1))$$

for any $\alpha_1 \in (-\infty, \alpha_1^0)$ by Lemma 2.3 of (2).

(ii) $\beta_1(\alpha_1, \alpha_2)$ and $\beta_2(\alpha_1, \alpha_2)$ are continuous on Ω_{NT} from (2).

Lemma 2.2. *Suppose $\lambda \geq 0$ and $\bar{\lambda} \geq 0$. Then the following statements are valid.*

(i) *Let $\beta_1 \in (2\lambda + 2, \infty)$. If there exists a sequence $\{(\alpha_1^i, \alpha_2^i)\}_{i\in N} \subseteq \Omega_{NT}$ such that $(\alpha_1^i, \alpha_2^i) \to (\alpha_1^0, \alpha_2^0)$ as $i \to \infty$ and $\beta_1(\alpha_1^i, \alpha_2^i) = \beta_1$ for all $i \in N$, then $\beta_2(\alpha_1^i, \alpha_2^i) \to \infty$ as $i \to \infty$.*

(ii) *Let $\beta_2 \in (2\bar{\lambda} + 2, \infty)$. If there exists a sequence $\{(\alpha_1^i, \alpha_2^i)\}_{i\in N} \subseteq \Omega_{NT}$ such that $(\alpha_1^i, \alpha_2^i) \to (\alpha_1^0, \alpha_2^0)$ as $i \to \infty$ and $\beta_2(\alpha_1^i, \alpha_2^i) = \beta_2$ for all $i \in N$, then $\beta_1(\alpha_1^i, \alpha_2^i) \to \infty$ as $i \to \infty$.*

(iii) *Let $M > 0$, there exists a point $(\alpha_1^M, \alpha_2^M) \in \Omega_{NT}$ such that $\alpha_1^M < \alpha_1^0$ and $\alpha_2^M < \alpha_2^0$ implies*

$$[\beta_1(\alpha_1, \alpha_2) - 2(\lambda + 1)]\left[\beta_2(\alpha_1, \alpha_2) - 2(\bar{\lambda} + 1)\right] > M \quad \text{for any} \quad (\alpha_1, \alpha_2) \in \Omega_{NT} \cap [\alpha_1^M, \alpha_1^0) \times [\alpha_2^M, \alpha_2^0).$$

Proof. By Pohozaev identity, we have

$$\left[\beta_1(\alpha_1^i, \alpha_2^i) - 2(\lambda + 1)\right]\left[\beta_2(\alpha_1^i, \alpha_2^i) - 2(\bar{\lambda} + 1)\right] - 4(\lambda + 1)(\bar{\lambda} + 1) \geq 6\int_0^R re^{u(r;\alpha_1^i,\alpha_2^i)+v(r;\alpha_1^i,\alpha_2^i)}dr$$

for any $R > 0$. It follows that the results (i), (ii) and (iii) hold, and hence we complete this lemma. □

Lemma 2.3. *The following statements are valid.*

(i) *Let* $\beta_1 \in (2\lambda + 2, \infty)$. *If there exists a strictly increasing continuous function* $\tau : (-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ *such that* $\tau(\alpha_1) \to -\infty$ *as* $\alpha_1 \to -\infty$, $(\alpha_1, \tau(\alpha_1)) \subseteq \Omega_{NT}$ *and* $\beta_1(\alpha_1, \tau(\alpha_1)) = \beta_1$ *for all* $\alpha_1 \in (-\infty, \alpha_1^0)$, *then* $\tau(\alpha_1) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1 \to L_{\beta_1}$ *as* $\alpha_1 \to -\infty$ *for some* $L_{\beta_1} \in \mathbf{R}$.

(ii) *Let* $\beta_2 \in (2\bar{\lambda} + 2, \infty)$. *If there exists a strictly increasing continuous function* $\tau : (-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ *such that* $\tau(\alpha_1) \to -\infty$ *as* $\alpha_1 \to -\infty$, $(\alpha_1, \tau(\alpha_1)) \subseteq \Omega_{NT}$ *and* $\beta_2(\alpha_1, \tau(\alpha_1)) = \beta_2$ *for all* $\alpha_1 \in (-\infty, \alpha_1^0)$, *then* $\tau(\alpha_1) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1 \to L_{\beta_2}$ *as* $\alpha_1 \to -\infty$ *for some* $L_{\beta_2} \in \mathbf{R}$.

Proof. The proof of (ii) is similarly to (i), and hence we only prove the result (i). On the contrary. We may assume that there exists a sequence $\{\alpha_1^i\}_{i\in N}$ such that $\lim_{i\to\infty} |\tau(\alpha_1^i) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1^i| = \infty$. Now we set $\widetilde{U}(r; \alpha_1^i) = u(e^{\gamma(\alpha_1^i)}r; \alpha_1^i, \tau(\alpha_1^i)) - 2\lambda \ln e^{\gamma(\alpha_1^i)}r - \alpha_1^i$ and $\widetilde{V}(r; \alpha_1^i) = v(e^{\gamma(\alpha_1^i)}r; \alpha_1^i, \tau(\alpha_1^i)) - 2\bar{\lambda} \ln e^{\gamma(\alpha_1^i)}r - \tau(\alpha_1^i)$, where

$$\gamma(\alpha_1^i) = \begin{cases} -\frac{\tau(\alpha_1^i)}{2+2\bar{\lambda}} & \text{if } \lim_{i\to\infty} \tau(\alpha_1^i) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1^i = \infty, \\ -\frac{\alpha_1^i}{2+2\lambda} & \text{if } \lim_{i\to\infty} \tau(\alpha_1^i) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1^i = -\infty. \end{cases}$$

Then we obtain that $\widetilde{U}(r; \alpha_1^i)$ and $\widetilde{V}(r; \alpha_1^i)$ converge to $\widetilde{U}(r)$ and $\widetilde{V}(r)$ in $C^2([0, R])$ as $i \to \infty$ for any $R > 0$, where

$$\begin{cases} \widetilde{U}''(r) + \frac{1}{r}\widetilde{U}'(r) + r^{2\bar{\lambda}} e^{\widetilde{V}(r)} = 0 \text{ on } [0, \infty), \\ \widetilde{V}''(r) + \frac{1}{r}\widetilde{V}'(r) = 0 \text{ on } [0, \infty), \\ \widetilde{U}(0) = \widetilde{V}(0) = 0, \; \widetilde{U}'(0) = \widetilde{V}'(0) = 0 \end{cases}$$

if $\lim_{i\to\infty} \tau(\alpha_1^i) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1^i = \infty$ and

$$\begin{cases} \widetilde{U}''(r) + \frac{1}{r}\widetilde{U}'(r) = 0 \text{ on } [0, \infty), \\ \widetilde{V}''(r) + \frac{1}{r}\widetilde{V}'(r) + r^{2\lambda} e^{\widetilde{U}(r)} = 0 \text{ on } [0, \infty), \\ \widetilde{U}(0) = \widetilde{V}(0) = 0, \; \widetilde{U}'(0) = \widetilde{V}'(0) = 0 \end{cases}$$

if $\lim_{i\to\infty} \tau(\alpha_1^i) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1^i = -\infty$.

For the case $\lim_{i\to\infty} \tau(\alpha_1^i) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1^i = \infty$, we see that there exist two constants $I, R > 0$ such that

$$e^{\gamma(\alpha_1^i)} R u'(e^{\gamma(\alpha_1^i)} R) < 2\lambda - 2\beta_1 \text{ for all } i \geq I,$$

which contradicts to $\beta_1(\alpha_1^i, \tau(\alpha_1^i)) = \beta_1$ for all $i \in N$. On the other hand, we see that $\widetilde{V}'(r) = -\frac{1}{2+2\bar{\lambda}} r^{1+2\bar{\lambda}}$ for $r \geq 0$, and it follows that

$$\begin{aligned} 0 &= \int_0^\infty \lim_{i\to\infty} r^{1+2\bar{\lambda}} \left\{ e^{2(1+\bar{\lambda})\gamma(\alpha_1^i)+\tau(\alpha_1^i)} e^{\widetilde{V}(r;\alpha_1^i,\tau(\alpha_1^i))} - e^{2(1+\lambda+\bar{\lambda})\gamma(\alpha_1^i)+\alpha_1^i+\tau(\alpha_1^i)} r^{2\lambda} e^{(\widetilde{U}+\widetilde{V})(r;\alpha_1^i,\tau(\alpha_1^i))} \right\} dr \\ &= \lim_{i\to\infty} \int_0^\infty r^{1+2\bar{\lambda}} \left\{ e^{2(1+\bar{\lambda})\gamma(\alpha_1^i)+\tau(\alpha_1^i)} e^{\widetilde{V}(r;\alpha_1^i,\tau(\alpha_1^i))} - e^{2(1+\lambda+\bar{\lambda})\gamma(\alpha_1^i)+\alpha_1^i+\tau(\alpha_1^i)} r^{2\lambda} e^{(\widetilde{U}+\widetilde{V})(r;\alpha_1^i,\tau(\alpha_1^i))} \right\} dr \\ &= \lim_{i\to\infty} \beta_1(\alpha_1^i, \tau(\alpha_1^i)), \end{aligned}$$

which contradicts to $\beta_1(\alpha_1^i, \tau(\alpha_1^i)) = \beta_1$ for all $i \in N$. Thus we complete the proof of (i). □

Lemma 2.4. *For any* $(\beta_1, \beta_2) \in (2\lambda + 2, \infty) \times (2\bar{\lambda} + 2, \infty)$ *and satisfies*

$$[\beta_1 - 2(\lambda + 1)]\left[\beta_2 - 2(\bar{\lambda} + 1)\right] = 4(\lambda + 1)(\bar{\lambda} + 1). \tag{2.7}$$

Then there exists a strictly increasing continuous function $\tau : (-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ *such that*

$$\lim_{\alpha_1 \to -\infty} \tau(\alpha_1) \to -\infty, \quad \lim_{\alpha_1 \to -\infty} \beta_1(\alpha_1, \tau(\alpha_1)) = \beta_1 \quad \text{and} \quad \lim_{\alpha_1 \to -\infty} \beta_2(\alpha_1, \tau(\alpha_1)) = \beta_2.$$

Proof. Let $\beta_1 \in (2\lambda + 2, \infty)$. Then there exists a strictly increasing continuous function $\tau_1^{\beta_1}$: $(-\infty, \alpha_1^0) \to (-\infty, \alpha_2^0)$ such that $\tau_1^{\beta_1}(\alpha_1) \to -\infty$ as $\alpha_1 \to -\infty$, $\lim_{\alpha_1 \to -\infty} \beta_1(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) = \beta_1$ and $\lim_{\alpha_1 \to -\infty} \beta_2(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) \le \infty$. From Lemma 2.3, it follows that $\lim_{\alpha_1 \to -\infty} \tau_1^{\beta_1}(\alpha_1) - \frac{1+\bar{\lambda}}{1+\lambda}\alpha_1 = L_{\beta_1}$ for some $L_{\beta_1} \in \mathbf{R}$. Now we set

$$\begin{cases} \widetilde{U}(r; \alpha_1) = u(e^{-\frac{\alpha_1}{2+2\lambda}} r; \alpha_1, \tau_1^{\beta_1}(\alpha_1)) - 2\lambda \ln e^{-\frac{\alpha_1}{2+2\lambda}} r - \alpha_1, \\ \widetilde{V}(r; \alpha_1) = v(e^{-\frac{\alpha_1}{2+2\lambda}} r; \alpha_1, \tau_1^{\beta_1}(\alpha_1)) - 2\bar{\lambda} \ln e^{-\frac{\alpha_1}{2+2\lambda}} r - \tau_1^{\beta_1}(\alpha_1), \end{cases}$$

and then we obtain that $\widetilde{U}(r; \alpha_1) \le 0, \widetilde{V}(r; \alpha_1) \le 0$ on $[0, \infty)$ and $\left(\widetilde{U}(r; \alpha_1), \widetilde{V}(r; \alpha_1)\right)$ converges to $\left(\widetilde{U}(r), \widetilde{V}(r)\right)$ in $C^2([0, R]) \times C^2([0, R])$ as $\alpha_1 \to -\infty$ for any $R > 0$, where

$$\begin{cases} \widetilde{U}''(r) + \frac{1}{r}\widetilde{U}'(r) + e^{L_{\beta_1}} r^{2\bar{\lambda}} e^{\widetilde{V}(r)} = 0 \text{ on } [0, \infty), \\ \widetilde{V}''(r) + \frac{1}{r}\widetilde{V}'(r) + r^{2\lambda} e^{\widetilde{U}(r)} = 0 \text{ on } [0, \infty), \\ \widetilde{U}(0) = \widetilde{V}(0) = 0, \ \widetilde{U}'(0) = \widetilde{V}'(0) = 0. \end{cases}$$

For the case $\int_0^\infty r^{2\lambda+1} e^{\widetilde{V}(r)} dr = \infty$, it follows that there exist $M, R > 0$ such that

$$e^{-\frac{\alpha_1}{2+2\lambda}} R u'(e^{-\frac{\alpha_1}{2+2\lambda}} R; \alpha_1, \tau_1^{\beta_1}(\alpha_1)) < 2\lambda - 2\beta_1 \text{ for all } \alpha_1 \le -M,$$

which contradicts to $\beta_1(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) = \beta_1$ for all $\alpha_1 < \alpha_1^0$. On the other hand, if $\int_0^\infty r^{2\lambda+1} e^{\widetilde{U}(r)} dr = \infty$, then we see that $\int_0^\infty r^{2\bar{\lambda}+1} e^{\widetilde{V}(r)} dr = 2\lambda + 2$ and $r\widetilde{V}'(r) \to -\infty$ as $r \to \infty$. Hence we have

$$2\lambda + 2 = \int_0^\infty r^{2\bar{\lambda}+1} e^{\widetilde{V}(r)} dr = \lim_{\alpha_1 \to -\infty} \int_0^\infty r e^{v(r;\alpha_1,\tau_1^{\beta_1}(\alpha_1))} \left(1 - r^{2\lambda} e^{\widetilde{U}(r;\alpha_1) + \frac{2\alpha_1}{2+2\lambda}}\right) dr = \beta_1(\tau_1^{\beta_1}(\alpha_1)),$$

which contradicts to $\beta_1(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) = \beta_1$ for all $\alpha_1 < \alpha_1^0$. Thus it follows that $\int_0^\infty r^{2\bar{\lambda}+1} e^{\widetilde{V}(r)} dr = \beta_1$ and $\lim_{\alpha_1 \to -\infty} \beta_2(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) = \int_0^\infty r^{2\lambda+1} e^{\widetilde{U}(r)} dr < \infty$ which satisfy (2.7) because Theorem 1.4 of (3). Therefore, we complete the proof of this result. □

Proof of Theorem 1.2. Let $(\beta_1, \beta_2) \in (2\lambda + 2, \infty) \times (2\bar{\lambda} + 2, \infty)$ satisfies (1.16). From Remark 2.1, we see that $\beta_1(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) = \beta_1$ and $\beta_2(\alpha_1, \tau_1^{\beta_1}(\alpha_1))$ is continuous on $(-\infty, \alpha_1^0)$. Since $\lim_{\alpha_1 \to \alpha_1^0} \beta_2(\alpha_1, \tau_1^{\beta_1}(\alpha_1)) = \infty$ and $\left(\beta_1, \lim_{\alpha_1 \to -\infty} \beta_2(\alpha_1, \tau_1^{\beta_1}(\alpha_1))\right)$ satisfies (2.7) by Lemma 2.3 and Lemma 2.4, the region of $\beta_2(\alpha_1, \tau_1^{\beta_1}(\alpha_1))$ is the interval $\left(2(1 + \bar{\lambda}) + \frac{4(1+\lambda)(1+\bar{\lambda})}{\beta_1 - 2(\lambda+1)}, \infty\right)$ from Lemma 2.2. Therefore, we complete the proof of this theorem by Proposition 2.1. □

3. UNIQUENESS OF TOPOLOGICAL SOLUTIONS FOR ε SMALL

In this section, we will prove Theorem 1.1 for ε sufficiently small from the following two a priori estimates lemmas for any topological solution of (1.1) when ε be small enough.

Lemma 3.1. *Let $\varepsilon > 0$ and $(u_\varepsilon, v_\varepsilon)$ be a topological solution of* (1.1). *Then for each* $0 < d < \frac{1}{4}\min\left\{|p_i - p_j|, |q_k - q_\ell| : \ 1 \le i < j \le N_1,\ 1 \le k < \ell \le N_2\right\}$, *there is a constant* $\varepsilon_0 = \varepsilon_0(d) > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$, *then*

$$\max\{\|u_\varepsilon\|_{C^2(\Omega_d^c)}, \|v_\varepsilon\|_{C^2(\Omega_d^c)}\} \le c_0 \exp[-\frac{c_1}{\varepsilon}], \tag{3.1}$$

for some positive constants c_0, c_1 *depending only on* d *where* $\Omega_d = \bigcup_{i,j}\left\{B_d(p_i) \cup B_d(q_j)\right\}$.

Proof. We divide the proof into two steps.

Step 1. for each compact subset $K \subset \mathbf{R}^2 \setminus \mathscr{Z}$, there are constants $\varepsilon_* > 0$ and $\gamma_0(K) < 0$ such that $\gamma_0(K) \le u_\varepsilon(x),\ v_\varepsilon(x) < 0$ in K for $0 < \varepsilon < \varepsilon_*$ where $\mathscr{Z} = \{p_1, \dots, p_{N_1}, q_1, \dots, q_{N_2}\}$. Let

$$U_\varepsilon(x) = u_\varepsilon(x) - \sum_{i=1}^{N_1} 2\mu_i \ln|x - p_i| \ \text{ and } \ V_\varepsilon(x) = v_\varepsilon(x) - \sum_{j=1}^{N_2} 2v_j \ln|x - q_j|.$$

Thus it suffices to prove that if $\varepsilon > 0$ is sufficiently small, then $\inf_{B_R(O)} U_\varepsilon(x), \inf_{B_R(O)} V_\varepsilon(x) \ge \gamma_0$ for some $\gamma_0 = \gamma_0(R) < 0$ where $R > \max_{i,j}\{|p_i|, |q_j|\}$ and $O = (0, 0)$. On the contrary, without loss of generality, we may assume that there exist a constant $R_0 > \max_{i,j}\{|p_i|, |q_j|\}$ and two sequences $\{\varepsilon_n\}$, $\{x_n\} \subset B_{R_0}(O)$ such that

$$x_n \not\to O, \quad \varepsilon_n \to 0 \ \text{ and } \ U_{\varepsilon_n}(x_n) = \inf_{B_{R_0}(O)} U_\varepsilon(x) \to -\infty \ \text{ as } \ n \to \infty.$$

For simplicity, we let $U_n = U_{\varepsilon_n}$ and $V_n = V_{\varepsilon_n}$. Decompose $U_n = U_{1n} + U_{2n}$ and $V_n = V_{1n} + V_{2n}$ where

$$\begin{cases} \Delta U_{1n} + \frac{1}{\varepsilon_n^2} e^{v_n}(1 - e^{u_n}) = 0 \text{in } B_R(O), \\ \Delta V_{1n} + \frac{1}{\varepsilon_n^2} e^{u_n}(1 - e^{v_n}) = 0 \text{in } B_R(O), \\ U_{1n}(x) = V_{1n}(x) = 0 \text{ on } \partial B_R(O), \end{cases}$$

and

$$\begin{cases} \Delta U_{2n} = 0 \text{in } B_R(O), \\ \Delta V_{2n} = 0 \text{in } B_R(O), \\ U_{2n}(x) = U_n(x), \ V_{1n}(x) = V_n(x) \text{ on } \partial B_R(O) \end{cases}$$

for any $R \geq R_0$. From the Harnack inequality, we see that $U_{2n}(x)$ converges to $-\infty$ uniformly on $B_R(O)$.

On the other hand, following the argument of Lemma 3.1 of (20) and Poincar*é* inequality, we can verify that U_{1n} is bounded in $W_0^{1,q}(B_R(O))$ for each $1 < q < 2$. By passing to a subsequence, we may assume that

$$U_{1n} \rightharpoonup U_\infty \text{ weakly in } W_0^{1,q}(B_R(O)),$$

and strongly in $L^p(B_R(O))$ for $1 \leq p < \frac{2q}{2-q}$. Consequently, $U_n \to -\infty$ and $V_n \to -\infty$ almost everywhere on $B_R(O)$, otherwise, $\lim_{n\to\infty} e^{v_n(x)}(1 - e^{u_n(x)}) = 0$ almost everywhere on $B_R(O)$ from

$$0 \leq \int_{B_R(O)} \lim_{n\to\infty} e^{v_n(x)}(1 - e^{u_n(x)})dx = \lim_{n\to\infty} \int_{B_R(O)} e^{v_n(x)}(1 - e^{u_n(x)})dx < \lim_{n\to\infty} 4\pi \sum_{i=1}^{N_1} \mu_i \varepsilon_n^2 = 0.$$

Consider the equation (1.1) with no vortex and $\varepsilon = 1$. Let $M > 8\left(\sum_{i=1}^{N_1} \mu_i\right)\left(\sum_{j=1}^{N_2} \nu_j\right)$ and by Lemma 2.2(iii), there exists a point $(\alpha_1^*, \alpha_2^*) \in \Omega_{NT}$ such that $\alpha_1^* < \alpha_1^0$ and $\alpha_2^* < \alpha_2^0$ implies

$$[\beta_1(\alpha_1, \alpha_2) - 2]\,[\beta_2(\alpha_1, \alpha_2) - 2] > M \text{ for any } (\alpha_1, \alpha_2) \in \Omega_{NT} \cap [\alpha_1^*, \alpha_1^0) \times [\alpha_2^*, \alpha_2^0). \qquad (3.2)$$

For each n, choose $y_n, y'_n \in \mathbf{R}^2$ such that $u_n(y_n) = \alpha_1^*$ and $v_n(y'_n) = \alpha_2^*$, where

$$|y_n| = \sup\{|x| : u_n(x) = \alpha_1^*\} \text{ and } |y'_n| = \sup\{|x| : v_n(x) = \alpha_2^*\}.$$

Since $u_n, v_n \to -\infty$ almost everywhere on each compact subset and $\lim_{|x|\to\infty}(u_n(x), v_n(x)) \to (0, 0)$, $|y_n|, |y'_n| \to \infty$ as $n \to \infty$, otherwise, there exist a constant $n_* > 0$ and a sequence $\{x_n^u\} \subseteq \mathbf{R}^2$(resp.,$\{x_n^v\} \subseteq \mathbf{R}^2$) such that $|x_n^u| > |y_n|$(resp.,$|x_n^v| > |y'_n|$) and $\Delta u_n(x_n^u) \geq 0$(resp.,$\Delta v_n(x_n^v) \geq 0$)(passing to a subsequence if necessary) for all $n \geq n_*$ if $|y_n|$(resp.,$|y'_n|$) is bounded. Without loss of generality, we let $\hat{u}_n(x) = u_n(\varepsilon_n x + y_n)$, $\hat{v}_n(x) = v_n(\varepsilon_n x + y_n)$ and $|y_n| \geq |y'_n|$ for $n \in \mathbf{N}$, otherwise, we let $\hat{u}_n(x) = u_n(\varepsilon_n x + y'_n)$, $\hat{v}_n(x) = v_n(\varepsilon_n x + y'_n)$ and $|y_n| \leq |y'_n|$ for $n \in \mathbf{N}$. Then $\hat{u}_n$ and $\hat{v}_n$ satisfy

$$\begin{cases} \Delta \hat{u}_n + e^{\hat{v}_n}(1 - e^{\hat{u}_n}) = 0 \text{in } \Omega_n := \left\{|x| < \frac{\min(|y_n|,|y'_n|)}{2\varepsilon_n}\right\}, \\ \Delta \hat{v}_n + e^{\hat{u}_n}(1 - e^{\hat{v}_n}) = 0 \text{in } \Omega_n := \left\{|x| < \frac{\min(|y_n|,|y'_n|)}{2\varepsilon_n}\right\}, \\ \int_{\Omega_n} e^{\hat{v}_n}(1 - e^{\hat{u}_n})dx \leq 4\pi \sum_{i=1}^{N_1} \mu_i, \quad \int_{\Omega_n} e^{\hat{u}_n}(1 - e^{\hat{v}_n})dx \leq 4\pi \sum_{j=1}^{N_2} v_j. \end{cases}$$

To finish this step, we need the fact as follows.

Claim. $\hat{u}_n$ and $\hat{v}_n$ are bounded in $C_{\text{loc}}(\mathbf{R}^2)$.

Proof of the claim. $\hat{u}_n$ is bounded in $C_{\text{loc}}(\mathbf{R}^2)$ from $\hat{u}_n(O) = \alpha_1^*$ and Harnack inequality. Consequently, $\hat{v}_n$ is bounded in $C_{\text{loc}}(\mathbf{R}^2)$ by $\int_{\Omega_n} e^{\hat{u}_n}(1 - e^{\hat{v}_n})dx \leq 4\pi \sum_{j=1}^{N_2} v_j$ for all $n \in \mathbf{N}$. The proof of the claim is completed.

From claim, we see that $(\hat{u}_n, \hat{v}_n)$ converge in $C^2_{\text{loc}}(\Omega_n) \times C^2_{\text{loc}}(\Omega_n)$ to $(\hat{u}_*, \hat{v}_*)$ which is a solution of

$$\begin{cases} \Delta \hat{u}_* + e^{\hat{v}_*}(1 - e^{\hat{u}_*}) = 0 \text{in } \mathbf{R}^2, \\ \Delta \hat{v}_* + e^{\hat{u}_*}(1 - e^{\hat{v}_*}) = 0 \text{in } \mathbf{R}^2, \\ \hat{u}_*(x) \leq 0 \text{ and } \hat{v}_*(x) \leq 0 \text{in } \mathbf{R}^2 \end{cases}$$

with

$$\int_{\mathbf{R}^2} e^{\hat{v}_*(x)}(1 - e^{\hat{u}_*(x)})dx \le 4\pi \sum_{i=1}^{N_1} \mu_i \ \text{ and } \ \int_{\mathbf{R}^2} e^{\hat{u}_*(x)}(1 - e^{\hat{v}_*(x)})dx \le 4\pi \sum_{j=1}^{N_2} \nu_j. \tag{3.3}$$

Due to Proposition 5.1, we get that $\hat{u}_*$ and $\hat{v}_*$ are both radially symmetric with respect to some point $x^* = (x_1^*, x_2^*)$ in $\mathbf{R}^2$. Then $(\hat{u}_*, \hat{v}_*)$ is a topological solution or non-topological solution of (1.15) with $\lambda = \bar{\lambda} = 0$. If $(\hat{u}_*, \hat{v}_*)$ is a topological solution of (1.15) with $\lambda = \bar{\lambda} = 0$, that is, $(\hat{u}_*(x), \hat{v}_*(x)) = (0, 0)$ in $\mathbf{R}^2$, which contradicts to $\hat{u}_*(O) = \alpha_1^* < 0$. Hence $(\hat{u}_*, \hat{v}_*)$ is a non-topological solution of (1.15) with $\lambda = \bar{\lambda} = 0$. Combining $\hat{u}^*(x^*) \ge \alpha_1^*$, $\hat{v}_*(x^*) \ge \alpha_2^*$ and (3.2), we have

$$[\beta_1(\hat{u}^*(x^*), \hat{v}^*(x^*)) - 2]\,[\beta_2(\hat{u}^*(x^*), \hat{v}^*(x^*)) - 2] > 8\left(\sum_{i=1}^{N_1} \mu_i\right)\left(\sum_{j=1}^{N_2} \nu_j\right),$$

which contradicts to (3.3). Thus we finish this step.

Step 2. $(u_\varepsilon, v_\varepsilon) \to (0, 0)$ in $C^0(\Omega_d^c) \times C^0(\Omega_d^c)$ as $\varepsilon \to 0^+$. Moreover, if $\varepsilon > 0$ is sufficiently small, then

$$\max\{\|u_\varepsilon\|_{L^\infty(\Omega_d^c)}, \|v_\varepsilon\|_{L^\infty(\Omega_d^c)}\} \le c_2 \exp[-\frac{c_3}{\varepsilon}] \tag{3.4}$$

for some constants $c_2(d), c_3(d) > 0$. We note that for each $d > 0$, $\|u_\varepsilon\|_{L^\infty(\Omega_d^c)}$ and $\|v_\varepsilon\|_{L^\infty(\Omega_d^c)}$ are attained on $\partial\Omega_d$. Thus it suffices to prove

$$\max\{\|u_\varepsilon\|_{L^\infty(\partial\Omega_d^c)}, \|v_\varepsilon\|_{L^\infty(\partial\Omega_d^c)}\} \le c_2 \exp[-\frac{c_3}{\varepsilon}].$$

From Step 1, there exist two constants $\varepsilon_* > 0$ and $m < 0$ such that $\min\{u_\varepsilon(x), v_\varepsilon(x)\} \geq m$ on $\Omega^c_{\frac{d}{2}}$ for $0 < \varepsilon < \varepsilon_*$. Then, it follows that

$$-\varepsilon^2 \Delta u_\varepsilon + e^{2m} u_\varepsilon \geq 0 \;\text{ on }\; \Omega^c_{\frac{d}{2}}.$$

For each $x_0 \in \partial\Omega^c_d$, we define a comparison function w_ε by

$$w_\varepsilon(x) = (1-m)\exp\left[\frac{e^m}{2d\varepsilon}\left(|x-x_0|^2 - \frac{d^2}{4}\right)\right] \;\text{ for }\; |x-x_0| \leq \frac{d}{2}.$$

It is easy to check that if $\varepsilon > 0$ is sufficiently small

$$-\varepsilon^2 \Delta(u_\varepsilon + w_\varepsilon) + e^{2m}(u_\varepsilon + w_\varepsilon) > 0 \;\text{ on }\; B_{\frac{d}{2}}(x_0).$$

Then the maximum principle implies that

$$(u_\varepsilon + w_\varepsilon)(x) > (u_\varepsilon + w_\varepsilon)|_{|x-x_0|=\frac{d}{2}} > 0 \;\text{ for }\; |x-x_0| \leq \frac{d}{2}.$$

In particular, there is a constant $C > 0$ such that

$$u_\varepsilon(x) > -\exp\left[-\frac{C}{\varepsilon}\right] \;\text{ for }\; |x-x_0| \leq \frac{d}{4}.$$

Similarly for v_ε, and hence (3.4) immediately follows by the constant C is independent of choice of $x_0 \in \partial\Omega_d$. We finish this step.

Now we prove (3.1). From step 2., we have $\min\{\|u_\varepsilon\|_{L^\infty(\Omega^c_d)}, \|v_\varepsilon\|_{L^\infty(\Omega^c_d)}\} \leq c_2 \exp[-\frac{c_3}{\varepsilon}]$ and

$$\begin{cases} \varepsilon^2 \Delta u_\varepsilon = O(1)|u_\varepsilon| \;\text{ in } \Omega^c_{\frac{d}{2}}, \\ \varepsilon^2 \Delta v_\varepsilon = O(1)|v_\varepsilon| \;\text{ in } \Omega^c_{\frac{d}{2}}. \end{cases}$$

Then, Lemma 3.1 is an immediate consequence of the standard elliptic estimate. □

We now investigate the asymptotic behaviors of u_ε and v_ε in each ball $B_d(\xi)$ where $\xi \in \mathscr{Z} = \{p_i, q_j : 1 \le i \le N_1 \text{ and } 1 \le j \le N_2\}$. For each $\xi \in \mathscr{Z}$, let

$$\hat{u}_{\varepsilon,\xi}(x) = u_\varepsilon(\varepsilon x + \xi) \text{ and } \hat{v}_{\varepsilon,\xi}(x) = v_\varepsilon(\varepsilon x + \xi) \text{ for } |x| \le \frac{2d}{\varepsilon}.$$

Then $\hat{u}_{\varepsilon,\xi}(x)$ and $\hat{v}_{\varepsilon,\xi}(x)$ satisfy

$$\begin{cases} \Delta \hat{u}_{\varepsilon,\xi} + e^{\hat{v}_{\varepsilon,\xi}}(1 - e^{\hat{u}_{\varepsilon,\xi}}) = 4\pi\lambda_\xi\delta_0 \text{ for } |x| \le \frac{2d}{\varepsilon}, \\ \Delta \hat{v}_{\varepsilon,\xi} + e^{\hat{u}_{\varepsilon,\xi}}(1 - e^{\hat{v}_{\varepsilon,\xi}}) = 4\pi\bar{\lambda}_\xi\delta_0 \text{ for } |x| \le \frac{2d}{\varepsilon}, \\ |\hat{u}_{\varepsilon,\xi}(x)| = O(e^{-\frac{c}{\varepsilon}}), \quad |\hat{v}_{\varepsilon,\xi}(x)| = O(e^{-\frac{c}{\varepsilon}}) \text{ for } |x| = \frac{2d}{\varepsilon}, \end{cases} \tag{3.5}$$

where $c \in \mathbf{R}$ and

$$(\xi, \lambda_\xi, \bar{\lambda}_\xi) \in \begin{cases} \{(p_i, \mu_i, 0), (q_j, 0, \nu_j) : 1 \le i \le N_1 \text{ and } 1 \le j \le N_2\} \text{ if } p_i \ne q_j, \\ \{(p_i, \mu_i, \nu_j) : 1 \le i \le N_1 \text{ and } 1 \le j \le N_2\} \text{ if } p_i = q_j. \end{cases}$$

Then we have the following property for any topological solution of (1.1) as $\varepsilon \to 0^+$.

Proposition 3.1. *There exist two constants $c > 0$ and $\varepsilon_* > 0$ such that*

$$\int_{|y| \le \frac{2d}{\varepsilon}} (1 - e^{\hat{u}_{\varepsilon,\xi}})(1 - e^{\hat{v}_{\varepsilon,\xi}}) dy = 4\pi\lambda_\xi\bar{\lambda}_\xi + o(e^{-\frac{c}{\varepsilon}})$$

for any $\varepsilon \in (0, \varepsilon_*)$.

Proof. The proof is standard by Pohozaev identity and (3.5), and hence we omit it. □

Lemma 3.2. *Let $(\phi_{\lambda_\xi}(r), \phi_{\bar{\lambda}_\xi}(r))$ be a topological solution of the following equations,*

$$\begin{cases} \Delta \phi_{\lambda_\xi} + e^{\phi_{\bar{\lambda}_\xi}}(1 - e^{\phi_{\lambda_\xi}}) = 4\pi\lambda_\xi\delta_0 \text{ in } \mathbf{R}^2, \\ \Delta \phi_{\bar{\lambda}_\xi} + e^{\phi_{\lambda_\xi}}(1 - e^{\phi_{\bar{\lambda}_\xi}}) = 4\pi\bar{\lambda}_\xi\delta_0 \text{ in } \mathbf{R}^2. \end{cases} \tag{3.6}$$

Then for each $0 < d < \frac{1}{4}\min\{|p_i - p_j|, |q_k - q_\ell| : 1 \le i < j \le N_1, 1 \le k < \ell \le N_2\}$,

$$\max\left\{\|\hat{u}_{\varepsilon,\xi} - \phi_{\lambda_\xi}\|_{C^2(|x|\le\frac{d}{\varepsilon})}, \|\hat{v}_{\varepsilon,\xi} - \phi_{\bar{\lambda}_\xi}\|_{C^2(|x|\le\frac{d}{\varepsilon})}\right\} \to 0 \text{ as } \varepsilon \to 0^+. \tag{3.7}$$

Proof. It follows from Lemma 3.1 that if $\varepsilon > 0$ sufficiently small, then there are positive constants c_0, c_1 such that

$$\max\left\{\|\hat{u}_{\varepsilon,\xi}\|_{L^\infty(\frac{d}{\varepsilon}\le|x|\le\frac{2d}{\varepsilon})}, \|\hat{v}_{\varepsilon,\xi}\|_{L^\infty(\frac{d}{\varepsilon}\le|x|\le\frac{2d}{\varepsilon})}\right\} \le c_0 e^{-\frac{c_1}{\varepsilon}}. \tag{3.8}$$

The proof of Lemma 3.2 will be given in two steps as follows.

Step 1. $\hat{u}_{\varepsilon,\xi} - \phi_{\lambda_\xi} \to 0$ *and* $\hat{v}_{\varepsilon,\xi} - \phi_{\bar{\lambda}_\xi} \to 0$ *in* $C^2_{loc}(\mathbf{R}^2)$ *as* $\varepsilon \to 0^+$. It is easy to see that

$$\begin{cases} \Delta\hat{u}_{\varepsilon,\xi} - e^{\hat{u}+\hat{v}_{\varepsilon,\xi}}\hat{u}_{\varepsilon,\xi} = 4\pi\lambda_\xi\delta_0 \text{ in } \mathbf{R}^2, \\ \Delta\hat{v}_{\varepsilon,\xi} - e^{\hat{v}+\hat{u}_{\varepsilon,\xi}}\hat{v}_{\varepsilon,\xi} = 4\pi\bar{\lambda}_\xi\delta_0 \text{ in } \mathbf{R}^2, \\ \int_{|x|\le\frac{2d}{\varepsilon}} e^{\hat{v}_{\varepsilon,\xi}}(1 - e^{\hat{u}_{\varepsilon,\xi}})dx = 4\pi\lambda_\xi + O(e^{-\frac{c}{\varepsilon}}), \\ \int_{|x|\le\frac{2d}{\varepsilon}} e^{\hat{u}_{\varepsilon,\xi}}(1 - e^{\hat{v}_{\varepsilon,\xi}})dx = 4\pi\bar{\lambda}_\xi + O(e^{-\frac{c}{\varepsilon}}), \end{cases} \tag{3.9}$$

where $\hat{u}(x) \in [\hat{u}_{\varepsilon,\xi}(x), 0]$ and $\hat{v}(x) \in [\hat{v}_{\varepsilon,\xi}(x), 0]$. Indeed, since $\hat{u}_{\varepsilon,\xi}(x) < 0$ and $\hat{v}_{\varepsilon,\xi}(x) < 0$ on $\mathbf{R}^2 \setminus \{(0,0)\}$, it follows from the Harnack inequality that either $\hat{U}_{\varepsilon,\xi}(x)$(resp., $\hat{V}_{\varepsilon,\xi}(x)$) is bounded in $C_{loc}(\mathbf{R}^2)$, or $\hat{U}_{\varepsilon,\xi}(x) \to -\infty$(resp., $\hat{V}_{\varepsilon,\xi}(x) \to -\infty$) uniformly on any compact subset of $\mathbf{R}^2$ where

$$\hat{U}_{\varepsilon,\xi}(x) = \hat{u}_{\varepsilon,\xi}(x) - 2\lambda_\xi \ln|x| \text{ and } \hat{V}_{\varepsilon,\xi}(x) = \hat{v}_{\varepsilon,\xi}(x) - 2\bar{\lambda}_\xi \ln|x|.$$

If $\hat{U}_{\varepsilon,\xi}(x) \to -\infty$ uniformly on any compact subset of $\mathbf{R}^2$, then $\hat{V}_{\varepsilon,\xi}(x)$ also does from (3.9). Hence we have

$$\int_{|y|\le R}(1 - e^{\hat{u}_{\varepsilon,\xi}})(1 - e^{\hat{v}_{\varepsilon,\xi}})dy = \pi R^2 + o(1) \text{ as } \varepsilon \to 0^+$$

for each $R > 0$, which contradicts to Proposition 3.1. Therefore, $\hat{U}_{\varepsilon,\xi}(x)$ and $\hat{V}_{\varepsilon,\xi}(x)$ are both bounded in $C_{loc}(\mathbf{R}^2)$.

By passing to a subsequence, we may assume that $\hat{U}_{\varepsilon,\xi}(x)$ and $\hat{V}_{\varepsilon,\xi}(x)$ converge in $C^2_{loc}(\mathbf{R}^2)$ to functions $U^*(x) \in C^2(\mathbf{R}^2)$ and $V^*(x) \in C^2(\mathbf{R}^2)$, which satisfy $u^*(x) \le 0$, $v^*(x) \le 0$ in $\mathbf{R}^2 \backslash \{O\}$ and

$$\begin{cases} \Delta u^* + e^{v^*}(1 - e^{u^*}) = 4\pi\lambda_\xi \delta_O \ \text{ in } \ \mathbf{R}^2, \\ \Delta v^* + e^{u^*}(1 - e^{v^*}) = 4\pi\bar{\lambda}_\xi \delta_O \ \text{ in } \ \mathbf{R}^2, \\ \int_{\mathbf{R}^2} e^{v^*(x)}(1 - e^{u^*(x)})dx = 4\pi\lambda_\xi, \\ \int_{\mathbf{R}^2} e^{u^*(x)}(1 - e^{v^*(x)})dx = 4\pi\bar{\lambda}_\xi, \end{cases}$$

where $u^*(x) = U^*(x) + 2\lambda_\xi \ln|x|$ and $v^*(x) = V^*(x) + 2\bar{\lambda}_\xi \ln|x|$. Then $(u^*(x), v^*(x))$ is a topological solution of (3.6) by Lemma 2.1. It follows from the uniqueness of topological solution of (3.6) that $(u^*(x), v^*(x)) = (\phi_{\lambda_\xi}(r), \phi_{\bar{\lambda}_\xi}(r))$.

Step 2. $\sup_{|x| \le \frac{2d}{\varepsilon}} \left|\hat{u}_{\varepsilon,\xi}(x) - \phi_{\lambda_\xi}(x)\right| \to 0$ and $\sup_{|x| \le \frac{2d}{\varepsilon}} \left|\hat{v}_{\varepsilon,\xi}(x) - \phi_{\bar{\lambda}_\xi}(x)\right| \to 0$ as $\varepsilon \to 0^+$. For simplicity, we let

$$\theta_{\varepsilon,\xi}(x) = \hat{u}_{\varepsilon,\xi}(x) - \phi_{\lambda_\xi}(x), \quad \eta_{\varepsilon,\xi}(x) = \hat{v}_{\varepsilon,\xi}(x) - \phi_{\bar{\lambda}_\xi}(x).$$

Fix a constant $R_0 > 0$, there exists a constant $\hat{\varepsilon}_* > 0$ such that

$$\min\left\{e^{\hat{u}_{\varepsilon,\xi}(x)}, e^{\hat{v}_{\varepsilon,\xi}(x)}, e^{\phi_{\lambda_\xi}(R_0)}, e^{\phi_{\bar{\lambda}_\xi}(R_0)} : \text{for } |x| \ge R_0\right\} \ge \frac{2}{3} \tag{3.10}$$

for $\varepsilon \in (0, \hat{\varepsilon}_*)$ by step 1 and (3.8). We argue by contradiction. Without loss of generality, we suppose that there exist sequences $\{\varepsilon_n\}$ and $\{x_n\} \subset B_{\frac{2d}{\varepsilon}}(O)$ such that $\varepsilon_n \to 0$ and

$$|\theta_{\varepsilon_n,\xi}(x_n)| = \max_{|x| \le \frac{2d}{\varepsilon_n}} |\theta_{\varepsilon_n,\xi}(x)| \ge \max\left\{\max_{|x| \le \frac{2d}{\varepsilon_n}} |\eta_{\varepsilon_n,\xi}(x)|, \gamma_0\right\}$$

for some constant $\gamma_0 > 0$. Due to step 1, we have $|x_n| \to \infty$ as $n \to \infty$. Thus we may assume that $R_0 < |x_n| < \frac{2d}{\varepsilon_n}$ for large n such that (3.10) holds. If $\theta_{\varepsilon_n,\xi}(x_n) > 0$, then by (3.10), we get

$$0 \geq \Delta\theta_{\varepsilon_n,\xi}(x_n) = e^{\widetilde{u}_n + \hat{v}_{\varepsilon_n,\xi}(x_n)}\theta_{\varepsilon_n,\xi}(x_n) - e^{\widetilde{v}_n}\left[1 - e^{\phi_{\lambda_\xi}(x_n)}\right]\eta_{\varepsilon_n,\xi}(x_n) > 0$$

for large n, where

$$\widetilde{u}_n = t_n\hat{u}_{\varepsilon_n,\xi}(x_n) + (1-t_n)\phi_{\lambda_\xi}(x_n) \text{ and } \widetilde{v}_n = t_n\hat{v}_{\varepsilon_n,\xi}(x_n) + (1-t_n)\phi_{\bar{\lambda}_\xi}(x_n) \text{ for some } t_n \in [0,1].$$

It yields a contradiction and hence we have $\theta_{\varepsilon_n,\xi}(x_n) < \eta_{\varepsilon_n,\xi}(x_n) \leq 0$ for large n. Then

$$\begin{aligned}
0 &\leq \Delta\theta_{\varepsilon_n,\xi}(x_n) \\
&= e^{\phi_{\bar{\lambda}_\xi}(x_n)}\left(1 - e^{\phi_{\lambda_\xi}(x_n)}\right) - e^{\hat{v}_{\varepsilon,\xi}(x_n)}\left(1 - e^{\hat{u}_{\varepsilon,\xi}(x_n)}\right) \\
&< e^{\phi_{\bar{\lambda}_\xi}(x_n)}\left(1 - e^{\phi_{\lambda_\xi}(x_n)}\right) - e^{\hat{u}_{\varepsilon,\xi}(x_n) - \phi_{\lambda_\xi}(x_n) + \phi_{\bar{\lambda}_\xi}(x_n)}\left(1 - e^{\hat{u}_{\varepsilon,\xi}(x_n)}\right) \\
&= e^{\phi_{\bar{\lambda}_\xi}(x_n)}\left[\left(1 - e^{\theta_{\varepsilon_n,\xi}(x_n) + \phi_{\lambda_\xi}(x_n)} - e^{\phi_{\lambda_\xi}(x_n)}\right)\left(1 - e^{\theta_{\varepsilon_n,\xi}(x_n)}\right)\right]
\end{aligned}$$

for large n, which implies

$$e^{\theta_{\varepsilon_n,\xi}(x_n) + \phi_{\lambda_\xi}(x_n)} + e^{\phi_{\lambda_\xi}(x_n)} < 1 \text{ for large n.}$$

Thus for large n, we obtain $\theta_{\varepsilon_n,\xi}(x_n) \leq \ln\left(e^{-\phi_{\lambda_\xi}(x_n)} - 1\right) \leq -|x_n| + C$ for some $C \in \mathbf{R}$. In particular, $\theta_{\varepsilon_n,\xi}(x_n) \to -\infty$ as $n \to \infty$. By Harnack inequality, we see that $\hat{u}_{\varepsilon_n,\xi}(x) =$

$\theta_{\varepsilon_n,\xi}(x) + \phi_{\lambda_\xi}(x) \to -\infty$ uniformly on $B_R(x_n)$ for each $R > 0$, which implies $\hat{v}_{\varepsilon_n,\xi}(x) \to -\infty$ uniformly on $B_R(x_n)$ for each $R > 0$ by (3.9). Consequently,

$$\int_{|x-x_n|<R} (1 - e^{\hat{u}_{\varepsilon_n,\xi}})(1 - e^{\hat{v}_{\varepsilon_n,\xi}})dx = \pi R^2 + o(1) \ \text{ as } \ n \to \infty$$

for each $R > 0$, which contradicts to Proposition 3.1. Hence we finish this step.

Therefore, we complete the proof of this result by step 2 and the following system

$$\begin{cases} \Delta\theta_{\varepsilon,\xi} = e^{\widetilde{u}_{\varepsilon,\xi}+\hat{v}_{\varepsilon,\xi}}\theta_{\varepsilon,\xi} - e^{\widetilde{v}_{\varepsilon,\xi}}\left(1 - e^{\phi_{\lambda_\xi}}\right)\eta_{\varepsilon,\xi} \ \text{ in } \ B_{\frac{2d}{\varepsilon}}(O), \\ \Delta\eta_{\varepsilon,\xi} = e^{\hat{u}_{\varepsilon,\xi}+\widetilde{v}_{\varepsilon,\xi}}\eta_{\varepsilon,\xi} - e^{\widetilde{u}_{\varepsilon,\xi}}\left(1 - e^{\phi_{\bar{\lambda}_\xi}}\right)\theta_{\varepsilon,\xi} \ \text{ in } \ B_{\frac{2d}{\varepsilon}}(O), \end{cases}$$

where

$$\begin{cases} \widetilde{u}_{\varepsilon,\xi}(x) \in \left[\min\left\{\hat{u}_{\varepsilon,\xi}(x), \phi_{\lambda_\xi}(x)\right\}, \max\left\{\hat{u}_{\varepsilon,\xi}(x), \phi_{\lambda_\xi}(x)\right\}\right] \ \text{ for } \ |x| < \frac{2d}{\varepsilon}, \\ \widetilde{v}_{\varepsilon,\xi}(x) \in \left[\min\left\{\hat{v}_{\varepsilon,\xi}(x), \phi_{\bar{\lambda}_\xi}(x)\right\}, \max\left\{\hat{v}_{\varepsilon,\xi}(x), \phi_{\bar{\lambda}_\xi}(x)\right\}\right] \ \text{ for } \ |x| < \frac{2d}{\varepsilon}. \end{cases}$$

□

Theorem 3.1. *Let* $p_1, \ldots, p_{N_1}, q_1, \ldots, q_{N_2} \in \mathbf{R}^2$ *and* $\mu_1, \ldots, \mu_{N_1}, v_1, \ldots, v_{N_2} \in \mathbf{R}^+ \cup \{0\}$ *are given. Then there is a constant* $\varepsilon_0 = \varepsilon_0(p_i, q_j, \mu_i, v_j) > 0$ *such that* (1.1) *possesses a unique topological solution for any* $0 < \varepsilon < \varepsilon_0$.

Proof. On the contrary. We may assume that there exists a sequence $\{\varepsilon_n\}_{n\in N}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ implies that $(u_{1\varepsilon_n}(x), v_{1\varepsilon_n}(x))$ and $(u_{2\varepsilon_n}(x), v_{2\varepsilon_n}(x))$ are two distinct topological solutions of (1.1) for each $n \in N$. Without loss of generality, we let

$$|(u_{1\varepsilon_n} - u_{2\varepsilon_n})(y_n)| = \|u_{1\varepsilon_n} - u_{2\varepsilon_n}\|_{L^\infty(\mathbf{R}^2)} \geq \|v_{1\varepsilon_n} - v_{2\varepsilon_n}\|_{L^\infty(\mathbf{R}^2)}$$

for each $n \in N$. Then we want to show that *there exists a point* $\xi \in \mathcal{Z}$ *such that* $\{y_n\}_{n\in\mathbf{N}} \subseteq B_R(\xi)$(*passing to a subsequence if necessary*) *for some* $0 < R < \hat{R} = \frac{1}{4}\min\big\{|p_i - p_j|, |q_k - q_\ell| :\ 1 \le i < j \le N_1,\ 1 \le k < \ell \le N_2\big\}$. On the contrary. We may assume that $\{y_n\}_{n\in\mathbf{N}} \subseteq \mathbf{R}^2 \setminus \bigcup_{\xi\in\mathcal{Z}} B_{\hat{R}}(\xi)$. Let $\xi \in \mathcal{Z}$ and we set $x_n = \frac{y_n - \xi}{\varepsilon_n}$, $A_n(x) = \frac{(\hat{u}_{1\varepsilon_n,\xi} - \hat{u}_{2\varepsilon_n,\xi})(x)}{\|\hat{u}_{1\varepsilon_n,\xi} - \hat{u}_{2\varepsilon_n,\xi}\|_{L^\infty(\mathbf{R}^2)}}$ and $B_n(x) = \frac{(\hat{v}_{1\varepsilon_n,\xi} - \hat{v}_{2\varepsilon_n,\xi})(x)}{\|\hat{u}_{1\varepsilon_n,\xi} - \hat{u}_{2\varepsilon_n,\xi}\|_{L^\infty(\mathbf{R}^2)}}$ for each $n \in N$. Hence $(A_n(x), B_n(x))$ satisfies

$$\begin{cases} \Delta A_n + e^{\eta_n(x)}(1 - e^{\hat{u}_{\varepsilon_n,\xi}})B_n - e^{\xi_n(x)+\hat{v}_{\varepsilon_n,\xi}}A_n = 0 \text{in } \mathbf{R}^2, \\ \Delta B_n + e^{\xi_n(x)}(1 - e^{\hat{v}_{\varepsilon_n,\xi}})A_n - e^{\eta_n(x)+\hat{u}_{\varepsilon_n,\xi}}B_n = 0 \text{in } \mathbf{R}^2, \\ A_n(x) \to 0,\ \ B_n(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{3.11}$$

where $\xi_n(x) \in \big[\min\{\hat{u}_{1\varepsilon_n,\xi}(x), \hat{u}_{2\varepsilon_n,\xi}(x)\}, \max\{\hat{u}_{1\varepsilon_n,\xi}(x), \hat{u}_{2\varepsilon_n,\xi}(x)\}\big]$ and $\eta_n(x) \in \big[\min\{\hat{v}_{1\varepsilon_n,\xi}(x), \hat{v}_{2\varepsilon_n,\xi}(x)\},\ \max\{\hat{v}_{1\varepsilon_n,\xi}(x), \hat{v}_{2\varepsilon_n,\xi}(x)\}\big]$. To prove this theorem, we need the following fact.

Claim. $\{x_n\}$ *is a bounded sequence.*

Proof of the claim.

Without loss of generality, we may assume that $x_n \to \infty$ as $n \to \infty$. To prove this result, equation (3.11) can be rewritten as

$$\begin{cases} \Delta A_n - A_n = e^{\eta_n(x)}(e^{\hat{u}_{\varepsilon_n,\xi}(x)} - 1)B_n + (e^{\xi_n(x)+\hat{v}_{\varepsilon_n,\xi}(x)} - 1)A_n \text{in } \mathbf{R}^2, \\ \Delta B_n - B_n = e^{\xi_n(x)}(e^{\hat{v}_{\varepsilon_n,\xi}(x)} - 1)A_n + (e^{\eta_n(x)+\hat{u}_{\varepsilon_n,\xi}(x)} - 1)B_n \text{in } \mathbf{R}^2, \\ A_n(x) \to 0,\ \ B_n(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{3.12}$$

and let $K(x, y)$ be the fundamental solution of $\Delta - I$ with zero boundary value on $\mathbf{R}^2$. It is easy to see

$$0 < K(x, y) \le C\frac{e^{-|x-y|}}{\sqrt{|x-y|}} \text{for } |x - y| \ge 1. \tag{3.13}$$

Then

$$
\begin{cases} A_n(x) = \int_{\mathbf{R}^2} K(x,y)[e^{\eta_n(y)}(e^{\hat{u}_{\varepsilon_n,\xi}(y)} - 1)B_n(y) + (e^{\xi_n(y)+\hat{v}_{\varepsilon_n,\xi}(y)} - 1)A_n(y)]dy, \\ B_n(x) = \int_{\mathbf{R}^2} K(x,y)[e^{\xi_n(y)}(e^{\hat{v}_{\varepsilon_n,\xi}(y)} - 1)A_n(y) + (e^{\eta_n(y)+\hat{u}_{\varepsilon_n,\xi}(y)} - 1)B_n(y)]dy. \end{cases} \tag{3.14}
$$

By (3.1), (3.7) and (3.14), we have

$$
|A_n(x_n)| + |B_n(x_n)| \le o(1)(||A_n||_{L^\infty(\mathbf{R}^2)} + ||B_n||_{L^\infty(\mathbf{R}^2)}), \tag{3.15}
$$

for large n, which implies $\{x_n\}$ be bounded. Thus we complete this claim.

Due to claim, it follows that $y_n \in B_{\hat{R}}(\xi)$ for large n, which contradicts to $\{y_n\}_{n\in\mathbf{N}} \subseteq \mathbf{R}^2 \setminus \bigcup_{\xi\in\mathfrak{Z}} B_{\hat{R}}(\xi)$.

Thus there exists a point $\xi_0 \in \mathfrak{Z}$ such that $\{y_n\}_{n\in\mathbf{N}} \subseteq B_R(\xi_0)$(passing to a subsequence if necessary) for some $0 < R < \hat{R} = \frac{1}{4}\min\{|p_i - p_j|, |q_k - q_\ell| : \ 1 \le i < j \le N_1,\ 1 \le k < \ell \le N_2\}$. From claim and Lemma 3.2, we obtain $(\hat{u}_{1\varepsilon_n,\xi_0}(x), \hat{v}_{1\varepsilon_n,\xi_0}(x)) \to (\phi_{\lambda_{\xi_0}}(x), \phi_{\bar{\lambda}_{\xi_0}}(x))$, $(\hat{u}_{2\varepsilon_n,\xi_0}(x), \hat{v}_{2\varepsilon_n,\xi_0}(x)) \to (\phi_{\lambda_{\xi_0}}(x), \phi_{\bar{\lambda}_{\xi_0}}(x))$ and $(A_n(x), B_n(x)) \to (A(x), B(x))$ in $C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ as $n \to \infty$, where $(A(x), B(x)) \in C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ and satisfies

$$
\begin{cases} \Delta A + e^{\phi_{\bar{\lambda}_{\xi_0}}}(1 - e^{\phi_{\lambda_{\xi_0}}})B - e^{\phi_{\lambda_{\xi_0}}+\phi_{\bar{\lambda}_{\xi_0}}}A = 0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta B + e^{\phi_{\lambda_{\xi_0}}}(1 - e^{\phi_{\bar{\lambda}_{\xi_0}}})A - e^{\phi_{\lambda_{\xi_0}}+\phi_{\bar{\lambda}_{\xi_0}}}B = 0 \ \text{ in } \ \mathbf{R}^2. \end{cases}
$$

Now we want to show that $A(x) = A(|x|)$ and $B(x) = B(|x|)$ for all $x \in \mathbf{R}^2$. By contradiction argument, without loss of generality, we may assume that there exist two distinct points $x_1 \in \mathbf{R}^2$ and $x_2 \in \mathbf{R}^2$ such that $|x_1| = |x_2| = R$ and $\left|\frac{A(x_1)}{A(x_2)}\right| =$

$\frac{|\hat{u}_{1\varepsilon_n,\xi_0}(x_1)-\hat{u}_{2\varepsilon_n,\xi_0}(x_1)|}{|\hat{u}_{1\varepsilon_n,\xi_0}(x_2)-\hat{u}_{2\varepsilon_n,\xi_0}(x_2)|} > 1$. From $\phi_{\lambda_{\xi_0}}(R) < 0$, it follows that there exists a constant $\gamma > 1$ such that

$$\frac{|\hat{u}_{1\varepsilon_n,\xi_0}(x_1)-\hat{u}_{2\varepsilon_n,\xi_0}(x_1)|-\phi_{\lambda_{\xi_0}}(R)}{|\hat{u}_{1\varepsilon_n,\xi_0}(x_2)-\hat{u}_{2\varepsilon_n,\xi_0}(x_2)|-\phi_{\lambda_{\xi_0}}(R)} > \gamma > 1 \text{ for large n,}$$

which contradicts to

$$\lim_{n\to\infty}\frac{|\hat{u}_{1\varepsilon_n,\xi_0}(x_1)-\hat{u}_{2\varepsilon_n,\xi_0}(x_1)|-\phi_{\lambda_{\xi_0}}(R)}{|\hat{u}_{1\varepsilon_n,\xi_0}(x_2)-\hat{u}_{2\varepsilon_n,\xi_0}(x_2)|-\phi_{\lambda_{\xi_0}}(R)} = 1.$$

Note that A and B are bounded and not all zero in $\mathbf{R}^2$ because $\{x_n\}$ is bounded. Hence, we have $A \equiv 0$ and $B \equiv 0$ by Lemma 2.4 of (2). Then it yields a contradiction. Therefore, we complete the proof of Theorem 3.1. □

4. UNIQUENESS OF TOPOLOGICAL SOLUTIONS FOR ε LARGE

In this section, we first supply an important estimate in proof of Theorem 1.1 about ε sufficiently large as follows.

Proposition 4.1. *Let* $\varepsilon > 0$, $\bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ and $\bar{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x)$ for $x \in \mathbf{R}^2$ *where* $(u_\varepsilon, v_\varepsilon)$ *is a topological solution of* (1.1). *Then there are two constants* $\varepsilon_* = \varepsilon_*(\{p_i\}, \{q_j\}, \{\mu_i\}, \{\nu_j\}) > 0$ *and* $C = C(\{p_i\}, \{q_j\}, \{\mu_i\}, \{\nu_j\}) > 0$ *such that*

$$\int_{\mathbf{R}^2}(1-e^{\bar{u}_\varepsilon})(1-e^{\bar{v}_\varepsilon})dx \le C \tag{4.1}$$

for all $\varepsilon \ge \varepsilon_*$.

Proof. Let $\varepsilon > 0$ and choose a constant $\delta = \frac{1}{2\varepsilon}\inf\{|\xi_k - \xi_\ell| \,:\, \xi_k, \xi_\ell \in \mathscr{Z} \text{ and } \xi_k \neq \xi_\ell\}$. First, we set $\mathscr{U}_{\varepsilon,\xi}(x) = \bar{u}_\varepsilon(x) - 2\lambda_\xi \ln|x - \frac{1}{\varepsilon}\xi|$ and $\mathscr{V}_{\varepsilon,\xi}(x) = \bar{v}_\varepsilon(x) - 2\bar{\lambda}_\xi \ln|x - \frac{1}{\varepsilon}\xi|$. Then we have

$$
\begin{aligned}
&-\left\{\int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathscr{Z}}B_\delta(\frac{1}{\varepsilon}\xi)} \Delta\bar{u}_\varepsilon(x\cdot\nabla\bar{v}_\varepsilon)dx + \int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathscr{Z}}B_\delta(\frac{1}{\varepsilon}\xi)} \Delta\bar{v}_\varepsilon(x\cdot\nabla\bar{u}_\varepsilon)dx\right\}\\
&= \sum_{\xi\in\mathscr{Z}}\int_{|x-\frac{1}{\varepsilon}\xi|=\delta}\left\{\frac{1}{\delta\varepsilon}\left[\nabla\mathscr{U}_{\varepsilon,\xi}\cdot\left(x-\frac{1}{\varepsilon}\xi\right)\right](\xi\cdot\nabla\mathscr{V}_{\varepsilon,\xi}) + \frac{1}{\delta\varepsilon}\left[\nabla\mathscr{V}_{\varepsilon,\xi}\cdot\left(x-\frac{1}{\varepsilon}\xi\right)\right](\xi\cdot\nabla\mathscr{U}_{\varepsilon,\xi})\right. \qquad (4.2)\\
&+\frac{2\lambda_\xi}{\delta}\left(x\cdot\nabla\mathscr{V}_{\varepsilon,\xi}\right) + \frac{2\bar{\lambda}_\xi}{\delta}\left(x\cdot\nabla\mathscr{U}_{\varepsilon,\xi}\right) + \frac{1}{\delta\varepsilon}\left[\xi\cdot\left(x-\frac{1}{\varepsilon}\xi\right)\right]\left(\nabla\mathscr{U}_{\varepsilon,\xi}\cdot\nabla\mathscr{V}_{\varepsilon,\xi}\right) + \frac{8\lambda_\xi\bar{\lambda}_\xi}{\delta^3\varepsilon}\left[\xi\cdot\left(x-\frac{1}{\varepsilon}\xi\right)\right]\\
&\left.+\frac{4\lambda_\xi\bar{\lambda}_\xi}{\delta} + \delta\left(\nabla\mathscr{U}_{\varepsilon,\xi}\cdot\nabla\mathscr{V}_{\varepsilon,\xi}\right)\right\}d\sigma.
\end{aligned}
$$

Next let $\overline{\mathscr{U}}_\varepsilon(x) = \bar{u}_\varepsilon(x) - 2\sum_{\xi\in\mathscr{Z}}\lambda_\xi \ln\left|x - \frac{1}{\varepsilon}\xi\right|$ and $\{\overline{\mathscr{V}}_\varepsilon(x) = \bar{v}_\varepsilon(x) - 2\sum_{\xi\in\mathscr{Z}}\bar{\lambda}_\xi \ln\left|x - \frac{1}{\varepsilon}\xi\right|$.

It is easy to see that

$$
\begin{cases}
\nabla\mathscr{U}_{\varepsilon,\xi}(x+\frac{1}{\varepsilon}\xi_\ell) = \sum_{\ell\neq\kappa}\frac{2\varepsilon\lambda_{\xi_\kappa}(\varepsilon x+\xi_\ell-\xi_\kappa)}{|\varepsilon x+\xi_\ell-\xi_\kappa|^2} + \nabla\overline{\mathscr{U}}_\varepsilon(x+\frac{1}{\varepsilon}\xi_\ell),\\
\nabla\mathscr{V}_{\varepsilon,\xi}(x+\frac{1}{\varepsilon}\xi_\ell) = \sum_{\ell\neq\kappa}\frac{2\varepsilon\bar{\lambda}_{\xi_\kappa}(\varepsilon x+\xi_\ell-\xi_\kappa)}{|\varepsilon x+\xi_\ell-\xi_\kappa|^2} + \nabla\overline{\mathscr{V}}_\varepsilon(x+\frac{1}{\varepsilon}\xi_\ell),\\
\nabla\overline{\mathscr{U}}_\varepsilon(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{x-y}{|x-y|^2}e^{\bar{v}_\varepsilon(y)}(e^{\bar{u}_\varepsilon(y)}-1)dy, \quad \nabla\overline{\mathscr{V}}_\varepsilon(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{x-y}{|x-y|^2}e^{\bar{u}_\varepsilon(y)}(e^{\bar{v}_\varepsilon(y)}-1)dy
\end{cases} \qquad (4.3)
$$

for $x\in\mathbf{R}^2$. Hence if $|x|\leq 1$, then we obtain that

$$
\begin{cases}
2\pi\left|\nabla\overline{\mathscr{U}}_\varepsilon(x)\right| \leq \int_{|y|\leq 2}\frac{1}{|x-y|}dy + \int_{|y|\geq 2}\frac{2}{|y|}e^{\bar{v}_\varepsilon(y)}(1-e^{\bar{u}_\varepsilon(y)})dy \leq C,\\
2\pi\left|\nabla\overline{\mathscr{V}}_\varepsilon(x)\right| \leq \int_{|y|\leq 2}\frac{1}{|x-y|}dy + \int_{|y|\geq 2}\frac{2}{|y|}e^{\bar{u}_\varepsilon(y)}(1-e^{\bar{v}_\varepsilon(y)})dy \leq C
\end{cases} \qquad (4.4)
$$

for some constant $C = C\big(\{p_i\},\{q_j\},\{\mu_i\},\{\nu_j\}\big) > 0$.

Due to the Pohozaev identity, we see that

$$
\begin{aligned}
&-\left\{\int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathscr{Z}}B_\delta(\frac{1}{\varepsilon}\xi)} \Delta\bar{u}_\varepsilon(x\cdot\nabla\bar{v}_\varepsilon)dx + \int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathscr{Z}}B_\delta(\frac{1}{\varepsilon}\xi)} \Delta\bar{v}_\varepsilon(x\cdot\nabla\bar{u}_\varepsilon)dx\right\}\\
&= 2\int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathscr{Z}}B_\delta(\frac{1}{\varepsilon}\xi)}\left(1-e^{\bar{u}_\varepsilon(x)}\right)\left(1-e^{\bar{v}_\varepsilon(x)}\right)dx
\end{aligned}
$$

$$-\sum_{\xi\in\mathcal{Z}}\int_{|x-\frac{1}{\varepsilon}\xi|=\delta}\frac{1}{\delta}\left[x\cdot\left(x-\frac{1}{\varepsilon}\xi\right)\right]\left(1-e^{\bar{u}_\varepsilon(x)}\right)\left(1-e^{\bar{v}_\varepsilon(x)}\right)d\sigma.$$

Since there exists a constant $\varepsilon_0>0$ such that $|x|<1$ for all $x\in\bigcup_{\xi\in\mathcal{Z}}\partial B_\delta\left(\frac{1}{\varepsilon}\xi\right)$ and $\varepsilon\geq\varepsilon_0$, we get $|\nabla\mathcal{U}_{\varepsilon,\xi}(x)|\leq C$ and $|\nabla\mathcal{V}_{\varepsilon,\xi}(x)|\leq C$ for all $x\in\bigcup_{\xi\in\mathcal{Z}}\partial B_\delta\left(\frac{1}{\varepsilon}\xi\right)$ and $\varepsilon\geq\varepsilon_0$ by (4.3) and (4.4). Then it follows that

$$\begin{aligned}
&\int_{\mathbf{R}^2}\left(1-e^{\bar{u}_\varepsilon(x)}\right)\left(1-e^{\bar{v}_\varepsilon(x)}\right)dx\\
&\leq\int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathcal{Z}}B_\delta(\frac{1}{\varepsilon}\xi)}\left(1-e^{\bar{u}_\varepsilon(x)}\right)\left(1-e^{\bar{v}_\varepsilon(x)}\right)dx+\int_{\cup_{\xi\in\mathcal{Z}}B_\delta(\frac{1}{\varepsilon}\xi)}\left(1-e^{\bar{u}_\varepsilon(x)}\right)\left(1-e^{\bar{v}_\varepsilon(x)}\right)dx\\
&\leq\left|\int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathcal{Z}}B_\delta(\frac{1}{\varepsilon}\xi)}\Delta\bar{u}_\varepsilon(x\cdot\nabla\bar{v}_\varepsilon)dx+\int_{\mathbf{R}^2\setminus\cup_{\xi\in\mathcal{Z}}B_\delta(\frac{1}{\varepsilon}\xi)}\Delta\bar{v}_\varepsilon(x\cdot\nabla\bar{u}_\varepsilon)dx\right|\\
&\quad+\frac{1}{\delta}\sum_{\xi\in\mathcal{Z}}\int_{|x-\frac{1}{\varepsilon}\xi|=\delta}|x|\left|x-\frac{1}{\varepsilon}\xi\right|d\sigma+2\pi\delta(N_1+N_2)\\
&\leq\frac{1}{2}\sum_{\xi\in\mathcal{Z}}\int_{|x-\frac{1}{\varepsilon}\xi|=\delta}\left\{\frac{2}{\delta\varepsilon}\left|\nabla\mathcal{U}_{\varepsilon,\xi}\right|\left|\nabla\mathcal{V}_{\varepsilon,\xi}\right|\left|x-\frac{1}{\varepsilon}\xi\right||\xi|+\frac{2\bar{\lambda}_\xi}{\delta}|x|\left|\nabla\mathcal{U}_{\varepsilon,\xi}\right|+\frac{2\lambda_\xi}{\delta}|x|\left|\nabla\mathcal{V}_{\varepsilon,\xi}\right|\right.\\
&\quad\left.+\frac{1}{\delta\varepsilon}\left|\nabla\mathcal{U}_{\varepsilon,\xi}\right|\left|\nabla\mathcal{V}_{\varepsilon,\xi}\right|\left|x-\frac{1}{\varepsilon}\xi\right||\xi|+\frac{8\lambda_\xi\bar{\lambda}_\xi}{\delta^3\varepsilon}\left|x-\frac{1}{\varepsilon}\xi\right||\xi|+\frac{4\lambda_\xi\bar{\lambda}_\xi}{\delta}+\delta\left|\nabla\mathcal{U}_{\varepsilon,\xi}\right|\left|\nabla\mathcal{V}_{\varepsilon,\xi}\right|\right\}d\sigma\\
&\quad+\frac{1}{\delta}\sum_{\xi\in\mathcal{Z}}\int_{|x-\frac{1}{\varepsilon}\xi|=\delta}|x|\left|x-\frac{1}{\varepsilon}\xi\right|d\sigma+2\pi\delta(N_1+N_2)\\
&\leq C
\end{aligned}$$

for some positive constant $C=C(\{p_i\},\{q_j\},\{\mu_i\},\{v_j\})$. Thus we complete the proof of this result. □

Theorem 4.1. *Let $p_1,\ldots,p_{N_1},q_1,\ldots,q_{N_2}\in\mathbf{R}^2$ and $\mu_1,\ldots,\mu_{N_1},v_1,\ldots,v_{N_2}\in\mathbf{R}^+\cup\{0\}$ are given. Then there is a constant $\varepsilon_1=\varepsilon_1(p_i,q_j,\mu_i,v_j)>0$ such that* (1.1) *possesses a unique topological solution for any $\varepsilon>\varepsilon_1$.*

Proof. Let $\bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x)$ and $\bar{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x)$ for $x \in \mathbf{R}^2$. Then $\bar{u}_\varepsilon(x)$ and $\bar{v}_\varepsilon(x)$ satisfy

$$\begin{cases} \Delta \bar{u}_\varepsilon + e^{\bar{v}_\varepsilon}(1 - e^{\bar{u}_\varepsilon}) = 4\pi \sum_{i=1}^{N_1} \mu_i \delta_{\frac{1}{\varepsilon}p_i} \text{ in } \mathbf{R}^2, \\ \Delta \bar{v}_\varepsilon + e^{\bar{u}_\varepsilon}(1 - e^{\bar{v}_\varepsilon}) = 4\pi \sum_{j=1}^{N_2} v_j \delta_{\frac{1}{\varepsilon}q_j} \text{ in } \mathbf{R}^2. \end{cases}$$

Step 1. For each $R > 0$, there are constants $\varepsilon^*(R) > 0$ and $\gamma_0(R) < 0$ such that $\gamma_0(R) \leq \bar{u}_\varepsilon(x),\ \bar{v}_\varepsilon(x) < 0$ in $\mathbf{R}^2 \setminus B_R(O)$ for $\varepsilon > \varepsilon^*(R)$. Let

$$\bar{U}_\varepsilon(x) = \bar{u}_\varepsilon(x) - \sum_{i=1}^{N_1} 2\mu_i \ln\left|x - \frac{p_i}{\varepsilon}\right| \text{ and } \bar{V}_\varepsilon(x) = \bar{v}_\varepsilon(x) - \sum_{j=1}^{N_2} 2v_j \ln\left|x - \frac{q_j}{\varepsilon}\right|.$$

Thus, by maximum principle, it suffices to prove that if $\varepsilon > 0$ is sufficiently large, then $\inf_{B_R(O)} \bar{U}_\varepsilon(x) \geq \gamma_0$ and $\inf_{B_R(O)} \bar{V}_\varepsilon(x) \geq \gamma_0$ for some $\gamma_0 = \gamma_0(R) < 0$ where $R > \max_{[\varepsilon_0,\infty)} \left\{\left|\frac{p_1}{\varepsilon}\right|, \ldots, \left|\frac{p_{N_1}}{\varepsilon}\right|, \left|\frac{q_1}{\varepsilon}\right|, \ldots, \left|\frac{q_{N_2}}{\varepsilon}\right|\right\}$ for some $\varepsilon_0 > 0$. On the contrary, without loss of generality, we may assume that there are a constant $R_0 > \max_{[\varepsilon_0,\infty)} \left\{\left|\frac{p_1}{\varepsilon}\right|, \ldots, \left|\frac{p_{N_1}}{\varepsilon}\right|, \left|\frac{q_1}{\varepsilon}\right|, \ldots, \left|\frac{q_{N_2}}{\varepsilon}\right|\right\}$ and sequences $\{\varepsilon_n\}$ and $\{x_n\} \subset B_{R_0}(O)$ such that

$$\varepsilon_n \to \infty \text{ and } \bar{U}_{\varepsilon_n}(x_n) = \inf_{B_{R_0}(O)} \bar{U}_{\varepsilon_n}(x) \to -\infty \text{ as } n \to \infty.$$

For simplicity, we let $U_n = \bar{U}_{\varepsilon_n}$ and $V_n = \bar{V}_{\varepsilon_n}$. Decompose $U_n = U_{1n} + U_{2n}$ and $V_n = V_{1n} + V_{2n}$ where

$$\begin{cases} \Delta U_{1n} + e^{\bar{v}_{\varepsilon_n}}(1 - e^{\bar{u}_{\varepsilon_n}}) = 0 \text{in } B_{R'}(O), \\ \Delta V_{1n} + e^{\bar{u}_{\varepsilon_n}}(1 - e^{\bar{v}_{\varepsilon_n}}) = 0 \text{in } B_{R'}(O), \\ U_{1n}(x) = V_{1n}(x) = 0 \text{ on } \partial B_{R'}(O), \end{cases}$$

and

$$\begin{cases} \Delta U_{2n} = 0 \text{in } B_{R'}(O), \\ \Delta V_{2n} = 0 \text{in } B_{R'}(O), \\ U_{2n}(x) = U_n(x),\ V_{2n}(x) = V_n(x) \text{ on } \partial B_{R'}(O) \end{cases}$$

for any $R' \geq R_0$. By the standard elliptic estimate, we can verify that $\{U_{1n}\}_{n\in\mathbf{N}}$ and $\{V_{1n}\}_{n\in\mathbf{N}}$ are bounded in $C\big(B_{R'}\big(\frac{p_i}{\varepsilon_n}\big)\big)$. From the argument of Lemma 3.1 of (20) and Poincaré inequality, we also can verify that

$$U_{1n} \to U_\infty \text{ and } V_{1n} \to V_\infty \text{ strong in } L^2(B_{R'}(O)),$$

and hence $U_n(x) \to -\infty$ as $n \to \infty$ almost everywhere on $B_{R'}(O)$. Since $(\bar{u}_\varepsilon(x), \bar{v}_\varepsilon(x)) \to (0, 0)$ as $|x| \to \infty$ for all $\varepsilon > 0$ and Proposition 4.1, we obtain that there exists a constant $C > 0$ such that

$$\int_{\mathbf{R}^2} e^{\bar{v}_\varepsilon}(1 - e^{\bar{u}_\varepsilon})dx = 4\pi \sum_{i=1}^{N_1} \mu_i, \int_{\mathbf{R}^2} e^{\bar{u}_\varepsilon}(1 - e^{\bar{v}_\varepsilon})dx = 4\pi \sum_{j=1}^{N_2} \nu_j, \text{ and } \int_{\mathbf{R}^2}(1 - e^{\bar{u}_\varepsilon})(1 - e^{\bar{v}_\varepsilon})dx \leq C,$$

which implies

$$\int_{\mathbf{R}^2}(1 - e^{\bar{u}_\varepsilon}) + (1 - e^{\bar{v}_\varepsilon})dx \leq \widetilde{C} \tag{4.5}$$

for some constant $\widetilde{C} > 0$. Thus by $U_n(x) \to -\infty$ as $n \to \infty$ almost everywhere in $\mathbf{R}^2$, we have

$$\widetilde{C} > \lim_{n\to\infty} \int_{\mathbf{R}^2}(1 - e^{\bar{u}_{\varepsilon_n}})dx > \lim_{n\to\infty} \int_{\{|x|\leq R\}}(1 - e^{\bar{u}_{\varepsilon_n}})dx = \int_{\{|x|\leq R\}} \lim_{n\to\infty}(1 - e^{\bar{u}_{\varepsilon_n}})dx = \pi R^2,$$

which yields a contradiction for large R. Therefore, we finish this step.

By step 1, we see that $(\bar{u}_\varepsilon(x), \bar{v}_\varepsilon(x))$ converges to $(\bar{u}(x), \bar{v}(x))$ (passing to a subsequence if necessary) pointwise in $\mathbf{R}^2 \setminus \{O\}$, where

$$\begin{cases} \Delta\bar{u} + e^{\bar{v}}(1 - e^{\bar{u}}) = 4\pi \sum\limits_{i=1}^{N_1} \mu_i \delta_O \ \text{ in } \ \mathbf{R}^2, \\ \Delta\bar{v} + e^{\bar{u}}(1 - e^{\bar{v}}) = 4\pi \sum\limits_{j=1}^{N_2} \nu_i \delta_O \ \text{ in } \ \mathbf{R}^2. \end{cases} \tag{4.6}$$

From (4.5), it follows that

$$\int_{\mathbf{R}^2} (1 - e^{\bar{u}}) + (1 - e^{\bar{v}}) dx \le \liminf_{\varepsilon \to \infty} \int_{\mathbf{R}^2} (1 - e^{\bar{u}_\varepsilon}) + (1 - e^{\bar{v}_\varepsilon}) dx \le \widetilde{C},$$

which implies $(\bar{u}(x), \bar{v}(x)) \to (0, 0)$ as $|x| \to \infty$. Thus $(\bar{u}(x), \bar{v}(x))$ is a topological solution of (4.6), that is, $(\bar{u}(x), \bar{v}(x)) = (\phi_\lambda(|x|), \phi_{\bar{\lambda}}(|x|))$ where $(\lambda, \bar{\lambda}) = \left(\sum_{i=1}^{N_1} \mu_i, \sum_{j=1}^{N_2} \nu_j\right)$.

Step 2. $\overline{U}_\varepsilon(x) = \bar{u}_\varepsilon(x) - \sum_{i=1}^{N_1} 2\mu_i \ln\left|x - \frac{p_i}{\varepsilon}\right|$ and $\overline{V}_\varepsilon(x) = \bar{v}_\varepsilon(x) - \sum_{j=1}^{N_2} 2\nu_j \ln\left|x - \frac{q_j}{\varepsilon}\right|$ converge to $\overline{U}(x)$ and $\overline{V}(x)$ (passing to a subsequence if necessary) in $C^2(\mathbf{R}^2)$, respectively, where $\overline{U}(x) = \phi_\lambda(x) - \sum_{i=1}^{N_1} 2\mu_i \ln|x|$, $\overline{V}(x) = \phi_{\bar{\lambda}}(x) - \sum_{j=1}^{N_2} 2\nu_j \ln|x|$,

$$\begin{cases} \Delta\overline{U}_\varepsilon + e^{\bar{v}_\varepsilon}(1 - e^{\bar{u}_\varepsilon}) = 0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta\overline{V}_\varepsilon + e^{\bar{u}_\varepsilon}(1 - e^{\bar{v}_\varepsilon}) = 0 \ \text{ in } \ \mathbf{R}^2 \end{cases} \quad \text{and} \quad \begin{cases} \Delta\overline{U} + e^{\phi_{\bar{\lambda}}}(1 - e^{\phi_\lambda}) = 0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta\overline{V} + e^{\phi_\lambda}(1 - e^{\phi_{\bar{\lambda}}}) = 0 \ \text{ in } \ \mathbf{R}^2. \end{cases}$$

To prove this step, we need the following fact.

Claim. $\overline{U}_\varepsilon \to \overline{U}$ and $\overline{V}_\varepsilon \to \overline{V}$ as $\varepsilon \to \infty$ (passing to a subsequence if necessary) in $C^2_{loc}(\mathbf{R}^2)$.

Proof of the claim. From Harnack inequality and step 1., we see that $\overline{U}_\varepsilon$ and $\overline{V}_\varepsilon$ are bounded on any compact set of $\mathbf{R}^2 \setminus \mathcal{Z}$. Decompose $\overline{U}_\varepsilon = \overline{U}_{1\varepsilon} + \overline{U}_{2\varepsilon}$ and $\overline{V}_\varepsilon = \overline{V}_{1\varepsilon} + \overline{V}_{2\varepsilon}$ where

$$\begin{cases} \Delta\overline{U}_{1\varepsilon} + e^{\bar{v}_\varepsilon}(1 - e^{\bar{u}_\varepsilon}) = 0 \text{in } B_R(O), \\ \Delta\overline{V}_{1\varepsilon} + e^{\bar{u}_\varepsilon}(1 - e^{\bar{v}_\varepsilon}) = 0 \text{in } B_R(O), \\ \overline{U}_{1\varepsilon}(x) = \overline{V}_{1\varepsilon}(x) = 0 \text{ on } \partial B_R(O), \end{cases}$$

and

$$\begin{cases} \Delta\overline{U}_{2\varepsilon} = 0 \text{in } B_R(O), \\ \Delta\overline{V}_{2\varepsilon} = 0 \text{in } B_R(O), \\ \overline{U}_{2\varepsilon}(x) = \overline{U}_\varepsilon(x),\ \overline{V}_{2\varepsilon}(x) = \overline{V}_\varepsilon(x) \text{ on } \partial B_R(O) \end{cases}$$

for fixed $R > \max_{i,j}\left\{\left|\frac{p_i}{\varepsilon}\right|, \left|\frac{q_j}{\varepsilon}\right|\right\}$. By maximum principle, we get $\|\overline{U}_{2\varepsilon}\|_{C(B_R(O))} \le \|\overline{U}_\varepsilon\|_{C(B_R(O))}$ and $\|\overline{V}_{2\varepsilon}\|_{C(B_R(O))} \le \|\overline{V}_\varepsilon\|_{C(B_R(O))}$, which imply $\overline{U}_{2\varepsilon}(x)$ and $\overline{V}_{2\varepsilon}(x)$ are bounded in $C(B_R(O))$. However, we obtain that $\overline{U}_{1\varepsilon}(x)$ and $\overline{V}_{1\varepsilon}(x)$ are bounded in $C(B_R(O))$ by

$$|\overline{U}_{1\varepsilon}(x)| \le \frac{1}{2\pi}\left|\int_{B_R(O)} \ln|x-y|dy\right| \text{ and } |\overline{V}_{1\varepsilon}(x)| \le \frac{1}{2\pi}\left|\int_{B_R(O)} \ln|x-y|dy\right| \text{ for any } x \in B_R(O).$$

Thus $\overline{U}_\varepsilon$ and $\overline{V}_\varepsilon$ are bounded in $C_{loc}(\mathbf{R}^2)$, and hence we complete this claim.

Now we set

$$\begin{cases} A_\varepsilon(x) = \overline{u}_\varepsilon(x) - \sum_{i=1}^{N_1} 2\mu_i \ln\left|x - \frac{p_i}{\varepsilon}\right| - \phi_\lambda(x) + \sum_{i=1}^{N_1} 2\mu_i \ln|x|, \\ B_\varepsilon(x) = \overline{v}_\varepsilon(x) - \sum_{j=1}^{N_2} 2v_j \ln\left|x - \frac{q_j}{\varepsilon}\right| - \phi_{\bar{\lambda}}(x) + \sum_{j=1}^{N_2} 2v_j \ln|x|. \end{cases}$$

Then $(A_\varepsilon(x), B_\varepsilon(x))$ satisfies

$$\begin{cases} \Delta A_\varepsilon - A_\varepsilon + e^{\bar{v}_\varepsilon}(1 - e^{\bar{u}_\varepsilon}) - e^{\phi_{\bar{\lambda}}}(1 - e^{\phi_\lambda}) = -A_\varepsilon, \\ \Delta B_\varepsilon - B_\varepsilon + e^{\bar{u}_\varepsilon}(1 - e^{\bar{v}_\varepsilon}) - e^{\phi_\lambda}(1 - e^{\phi_{\bar{\lambda}}}) = -B_\varepsilon \end{cases}$$

for $x \in \Lambda_\varepsilon = \bigcup_{i,j} \left\{ B_{1/\varepsilon}(\frac{p_i}{\varepsilon}) \cup B_{1/\varepsilon}(\frac{q_j}{\varepsilon}) \cup B_{1/\varepsilon}(O) \right\}$, and

$$\begin{cases} \Delta A_\varepsilon + e^{\eta_\varepsilon(x)}(1 - e^{\hat{u}_\varepsilon})\left(B_\varepsilon + \sum_{j=1}^{N_2} 2\nu_j \ln \frac{|x - \frac{q_j}{\varepsilon}|}{|x|} \right) - e^{\xi_\varepsilon(x) + \hat{v}_\varepsilon}\left(A_\varepsilon + \sum_{i=1}^{N_1} 2\mu_i \ln \frac{|x - \frac{p_i}{\varepsilon}|}{|x|} \right) = 0, \\ \Delta B_\varepsilon + e^{\xi_\varepsilon(x)}(1 - e^{\hat{v}_\varepsilon})\left(A_\varepsilon + \sum_{i=1}^{N_1} 2\mu_i \ln \frac{|x - \frac{p_i}{\varepsilon}|}{|x|} \right) - e^{\eta_\varepsilon(x) + \hat{u}_\varepsilon}\left(B_\varepsilon + \sum_{j=1}^{N_2} 2\nu_j \ln \frac{|x - \frac{q_j}{\varepsilon}|}{|x|} \right) = 0 \end{cases}$$

for $x \in \mathbf{R}^2 \setminus \bigcup_{i,j} \left\{ B_{1/\varepsilon}(\frac{p_i}{\varepsilon}) \cup B_{1/\varepsilon}(\frac{q_j}{\varepsilon}) \cup B_{1/\varepsilon}(O) \right\}$, where $\xi_n(x) \in \Big[\min\{\bar{u}_\varepsilon(x), \phi_\lambda(x)\}, \max\{\bar{u}_\varepsilon(x), \phi_\lambda(x)\} \Big]$ and $\eta_n(x) \in \Big[\min\{\bar{v}_\varepsilon(x), \phi_{\bar{\lambda}}(x)\}, \max\{\bar{v}_\varepsilon(x), \phi_{\bar{\lambda}}(x)\} \Big]$. Without loss of generality, we may assume

$$\|A_\varepsilon\|_{L^\infty} = \max(\|A_\varepsilon\|_{L^\infty}, \|B_\varepsilon\|_{L^\infty}) \text{ and } |A_\varepsilon(x_\varepsilon)| = \|A_\varepsilon\|_{L^\infty} \text{ for } x_\varepsilon \in \mathbf{R}^2.$$

Since $A_\varepsilon(x)$ and $B_\varepsilon(x)$ converge to zero as $|x| \to \infty$ and step 1., we have

$$|A_\varepsilon(x)| + |B_\varepsilon(x)| \le o(1)(\|A_\varepsilon\|_{L^\infty(\mathbf{R}^2)} + \|B_\varepsilon\|_{L^\infty(\mathbf{R}^2)} + g(\varepsilon)) + f(\varepsilon) \tag{4.7}$$

for some constant $C > 0$ for large $|x|$, where

$$\begin{cases} f(\varepsilon) = \int_{\Lambda_\varepsilon} K(x, y)\Big\{ e^{\phi_{\bar{\lambda}}}(1 - e^{\phi_\lambda}) - e^{\bar{v}_\varepsilon}(1 - e^{\bar{u}_\varepsilon}) + e^{\phi_\lambda}(1 - e^{\phi_{\bar{\lambda}}}) - e^{\bar{u}_\varepsilon}(1 - e^{\bar{v}_\varepsilon}) \Big\} dy, \\ g(\varepsilon) = \sum_{i,j} \left\{ \ln \left\| \frac{|x - \frac{q_j}{\varepsilon}|}{|x|} \right\|_{L^\infty(\mathbf{R}^2 \setminus \Lambda_\varepsilon)} + \ln \left\| \frac{|x - \frac{p_i}{\varepsilon}|}{|x|} \right\|_{L^\infty(\mathbf{R}^2 \setminus \Lambda_\varepsilon)} \right\}. \end{cases}$$

If $\{x_\varepsilon\}$ is a bounded sequence, then $\|A_\varepsilon\|_{L^\infty(\mathbf{R}^2)} \to 0$ and $\|B_\varepsilon\|_{L^\infty(\mathbf{R}^2)} \to 0$ as $\varepsilon \to \infty$ from the claim. On the other hand, if $\{x_\varepsilon\}$ is a unbounded sequence, then by (4.7) and $f(\varepsilon), g(\varepsilon) \to 0$ as $\varepsilon \to \infty$, we also have $\|A_\varepsilon\|_{L^\infty(\mathbf{R}^2)} \to 0$ and $\|B_\varepsilon\|_{L^\infty(\mathbf{R}^2)} \to 0$ as $\varepsilon \to \infty$. Thus $||A_\varepsilon||_{C^2(\mathbf{R}^2)} \to 0$ and $\|B_\varepsilon\|_{C^2(\mathbf{R}^2)} \to 0$ as $\varepsilon \to \infty$. Then we finish this step.

Following the similarly argument as in the proof of Theorem 3.1, we complete the proof of Theorem 4.1. □

Proof of Theorem 1.1.. Combining with Theorem 3.1 and Theorem 4.1, we complete the proof of this theorem. □

5. APPENDIX

In the appendix, we will prove the radial symmetry of the solutions for the following system,

$$\begin{cases} \Delta u + e^v(1-e^u) = 0 \text{ in } \mathbf{R}^2, \\ \Delta v + e^u(1-e^v) = 0 \text{ in } \mathbf{R}^2, \\ \int_{\mathbf{R}^2} e^{v(x)}(1-e^{u(x)})dx < \infty, \\ \int_{\mathbf{R}^2} e^{u(x)}(1-e^{v(x)})dx < \infty. \end{cases} \tag{5.1}$$

Then all non-positive solutions (i.e. $u(x) \le 0$, $v(x) \le 0$ in $\mathbf{R}^2$) of (5.1) are topological solution or non-topological solution from Lemma 2.1. Since the proof of radially symmetric Property of topological solution is standard, we omit it. Hence we will only give a result about the radially symmetric Property of the non-topological solution of (5.1) as follows.

Proposition 5.1. *If* $(u(x), v(x))$ *is a solution of* (5.1) *and satisfying* (1.12), *then* $(u(x), v(x))$ *is radially symmetric at some point in* $\mathbf{R}^2$.

Proof. We will apply the method of moving plane with some modifications to prove this result. It suffices to prove u and v are decreasing when the point $x = (x_1, x_2)$ changes its position along the x_1-axis from the point $(-R, 0)$ to $(-\infty, 0)$. Define the sets $\Sigma_\sigma = \left\{x \in \mathbf{R}^2 : x_1 < \sigma\right\}$, $T_\sigma = \left\{x \in \mathbf{R}^2 : x_1 = \sigma\right\}$, and $u_\sigma(x) = u(x^\sigma), v_\sigma(x) = v(x^\sigma)$ for $x \in \Sigma_\sigma$, where x^σ is the reflection of x with respect to the line $x_1 = \sigma$, i.e., $x^\sigma = (2\sigma - x_1, x_2)$.

Set $w_\sigma(x) = u(x) - u_\sigma(x)$ and $z_\sigma(x) = v(x) - v_\sigma(x)$ for $x \in \Sigma_\sigma$. Then w_σ and z_σ satisfy the following equations respectively.

$$\begin{cases} \Delta w_\sigma - e^{v_\sigma}(e^u - e^{u_\sigma}) = -(e^v - e^{v_\sigma})(1 - e^u) \quad \text{in } \ \Sigma_\sigma, \\ w_\sigma = 0 \text{on } \partial\Sigma_\sigma = T_\sigma, \end{cases} \tag{5.2}$$

and

$$\begin{cases} \Delta z_\sigma - e^{u_\sigma}(e^v - e^{v_\sigma}) = -(e^u - e^{u_\sigma})(1 - e^v) \quad \text{in } \ \Sigma_\sigma, \\ z_\sigma = 0 \text{on } \partial\Sigma_\sigma. \end{cases} \tag{5.3}$$

Now we set $\widetilde{w}_\sigma(x) = \frac{w_\sigma(x)}{g(x)}$ and $\widetilde{z}_\sigma(x) = \frac{z_\sigma(x)}{g(x)}$ for $x \in \Sigma_\sigma$ where $g(x) = (-x_1)^{\frac{1}{2}} + \ln|x|$. Then $\widetilde{w}_\sigma$ and $\widetilde{z}_\sigma$ satisfy the following equations respectively.

$$\begin{cases} \Delta \widetilde{w}_\sigma + \frac{2}{g}\nabla g \cdot \nabla \widetilde{w}_\sigma + \left(C_1(x) + \frac{\Delta g}{g}\right)\widetilde{w}_\sigma = C_2(x)\widetilde{z}_\sigma \quad \text{in } \ \Sigma_\sigma, \\ \widetilde{w}_\sigma = 0 \text{on } \partial\Sigma_\sigma, \end{cases} \tag{5.4}$$

and

$$\begin{cases} \Delta \widetilde{z}_\sigma + \frac{2}{g}\nabla g \cdot \nabla \widetilde{z}_\sigma + \left(C_3(x) + \frac{\Delta g}{g}\right)\widetilde{z}_\sigma = C_4(x)\widetilde{w}_\sigma \quad \text{in } \ \Sigma_\sigma, \\ \widetilde{z}_\sigma = 0 \text{on } \partial\Sigma_\sigma, \end{cases} \tag{5.5}$$

where

$$\begin{cases} \frac{\Delta g(x)}{g(x)} = -\frac{1}{4\left(x_1^2 + (-x_1)^{\frac{3}{2}} \ln|x|\right)}, \\ C_1(x) = \frac{e^{v_\sigma(x)}(e^{u_\sigma(x)} - e^{u(x)})}{u(x) - u_\sigma(x)}, \quad C_2(x) = \frac{(e^{v_\sigma(x)} - e^{v(x)})(1 - e^{u(x)})}{v(x) - v_\sigma(x)}, \\ C_3(x) = \frac{e^{u_\sigma(x)}(e^{v_\sigma(x)} - e^{v(x)})}{v(x) - v_\sigma(x)}, \quad C_4(x) = \frac{(e^{u_\sigma(x)} - e^{u(x)})(1 - e^{v(x)})}{u(x) - u_\sigma(x)}. \end{cases}$$

Define

$$\begin{cases} S_u = \{\rho \in (-\infty, 0) : \widetilde{w}_\sigma < 0 \text{in } \Sigma_\sigma \text{for } \sigma \in (-\infty, \rho)\}, \\ \rho_u = \sup_{\rho \in S_u}\{\rho\}, \end{cases}$$

and

$$\begin{cases} S_v = \{\rho \in (-\infty, 0) : \widetilde{z}_\sigma < 0 \text{in } \Sigma_\sigma \text{for } \sigma \in (-\infty, \rho)\}, \\ \rho_v = \sup_{\rho \in S_v}\{\rho\}. \end{cases}$$

Note that there exists a constant $R_0 > 0$ such that

$$\frac{\partial u}{\partial x_1}(x) > 2, \quad \frac{\partial v}{\partial x_1}(x) > 2, \quad \Big(C_3(x) + \frac{\Delta g}{g}(x)\Big) < 0 \text{ and } \Big(C_1(x) + \frac{\Delta g}{g}(x)\Big) < 0 \text{ for } |x| \geq R_0 \tag{5.6}$$

from Lemma 2.1 and Proposition 2.1. Now we divide the proof into the following steps.

Step 1. $S_u \neq \emptyset$ *and* $S_v \neq \emptyset$*. Moreover, there exists a constant* $\hat{R} \in [-R_0, 0)$ *such that* $\rho_u = \rho_v = \hat{\rho} \geq \hat{R}$*. Furthermore, there exists a constant* $\hat{R}_0 \geq R_0$ *such that if* $\widetilde{w}_\sigma$*(resp.,*$\widetilde{z}_\sigma$*) has a local maximum point* $x_0 \in \Sigma_\sigma$ *and* $\widetilde{w}_\sigma(x_0) > 0$*(resp.,*$\widetilde{z}_\sigma(x_0) > 0$*), then* $|x_0| \leq \hat{R}_0$. On the contrary. Without loss of generality, we may assume that $S_u = \emptyset$, and hence for each $\rho < -R_0$ there exists a point $x_{\sigma(\rho)} \in \Sigma_{\sigma(\rho)}$ such that $|x_{\sigma(\rho)}| \to \infty$ as $\rho \to -\infty$ implies

$$\widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) = \max\{\widetilde{w}_{\sigma(\rho)}(x) : x \in \Sigma_{\sigma(\rho)}\} \geq 0, \ \nabla\widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) = (0,0) \text{ and } \Delta\widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) \leq 0.$$

Then

$$\Big(C_1(x_{\sigma(\rho)}) + \frac{\Delta g}{g}(x_{\sigma(\rho)})\Big)\widetilde{w}_\sigma(x_{\sigma(\rho)}) \geq C_2(x_{\sigma(\rho)})\widetilde{z}_\sigma(x_{\sigma(\rho)}),$$

which implies $\widetilde{z}_\sigma(x_{\sigma(\rho)}) \geq 0$. Consequently, for each $\rho < -R_0$ there exists a $x'_{\sigma(\rho)} \in \Sigma_{\sigma(\rho)}$ such that $|x'_{\sigma(\rho)}| \to \infty$ as $\rho \to -\infty$ implies

$$\widetilde{z}_{\sigma(\rho)}(x'_{\sigma(\rho)}) = \max\{\widetilde{z}_{\sigma(\rho)}(x) : x \in \Sigma_{\sigma(\rho)}\} \geq 0, \quad \nabla\widetilde{z}_{\sigma(\rho)}(x'_{\sigma(\rho)}) = (0,0) \text{ and } \Delta\widetilde{z}_{\sigma(\rho)}(x'_{\sigma(\rho)}) \leq 0. \tag{5.7}$$

Now we want to claim that $\widetilde{w}_\sigma(x'_{\sigma(\rho)}) > 0$. Combining with (5.5) and (5.7), we get

$$\Big(C_3(x'_{\sigma(\rho)}) + \frac{\Delta g}{g}(x'_{\sigma(\rho)})\Big)\widetilde{z}_\sigma(x'_{\sigma(\rho)}) \geq C_4(x'_{\sigma(\rho)})\widetilde{w}_\sigma(x'_{\sigma(\rho)}) \geq C_4(x'_{\sigma(\rho)})\widetilde{w}_\sigma(x_{\sigma(\rho)}),$$

which implies $\widetilde{z}_\sigma(x_{\sigma(\rho)}) = 0$ if $\widetilde{w}_\sigma(x'_{\sigma(\rho)}) = 0$. Thus we see that

$$\widetilde{w}_{\sigma(\rho)}(x) \leq 0 \text{ and } \widetilde{z}_{\sigma(\rho)}(x) \leq 0 \text{ on } \Sigma_{\sigma(\rho)}.$$

Hence, $\widetilde{w}_{\sigma(\rho)}(x) = 0$ and $\widetilde{z}_{\sigma(\rho)}(x) = 0$ on $\Sigma_{\sigma(\rho)} \cup \partial\Sigma_{\sigma(\rho)}$ by (5.4)-(5.5) and maximum principle, which implies $\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x^\sigma)}{\partial 2\sigma(\rho) - x_1} = \frac{\partial w_{\sigma(\rho)}(x)}{\partial x_1} = 0$ on $\Sigma_{\sigma(\rho)}$. It is in contradiction to (5.6). Then $\widetilde{w}_\sigma(x'_{\sigma(\rho)}) > 0$ and hence we complete this claim. Using the similarly argument of proof of claim, we also get $\widetilde{z}_\sigma(x_{\sigma(\rho)}) > 0$. Thus it follows that

$$\Big(C_3(x'_{\sigma(\rho)}) + \frac{\Delta g}{g}(x'_{\sigma(\rho)})\Big)\Big(C_1(x_{\sigma(\rho)}) + \frac{\Delta g}{g}(x_{\sigma(\rho)})\Big)\widetilde{w}_\sigma(x_{\sigma(\rho)}) \leq C_2(x_{\sigma(\rho)})C_4(x'_{\sigma(\rho)})\widetilde{w}_\sigma(x'_{\sigma(\rho)}),$$

that is,

$$\Big(C_3(x'_{\sigma(\rho)}) + \frac{\Delta g}{g}(x'_{\sigma(\rho)})\Big)\Big(C_1(x_{\sigma(\rho)}) + \frac{\Delta g}{g}(x_{\sigma(\rho)})\Big) \leq C_2(x_{\sigma(\rho)})C_4(x'_{\sigma(\rho)}).$$

It is a contradiction to Proposition 2.1, and hence $S_u \neq \emptyset$. Furthermore, there exists a constant $\hat{R}_0 \geq R_0$ such that if $\widetilde{w}_\sigma$(resp.,$\widetilde{z}_\sigma$) has a local maximum point $x_0 \in \Sigma_\sigma$ and $\widetilde{w}_\sigma(x_0) > 0$(resp.,$\widetilde{z}_\sigma(x_0) > 0$), then $|x_0| \leq \hat{R}_0$. However, it is easy to see that $\rho_u = \rho_v = \hat{\rho} \geq \hat{R}$ for some $\hat{R} \in [-R_0, 0)$ by (5.2)-(5.3) and maximum principle. Therefore, we finish this step.

Step 2. $\frac{\partial u}{\partial x_1}(x) > 0$ *and* $\frac{\partial v}{\partial x_1}(x) > 0$ *on* $\Sigma_{\hat{\rho}}$. *Moreover,* $w_{\hat{\rho}} \equiv 0$ *and* $z_{\hat{\rho}} \equiv 0$ *on* $\Sigma_{\hat{\rho}}$. By Hopf lemma, we obtain that

$$\frac{\partial u}{\partial x_1}(x) + \frac{\partial u}{\partial(2\sigma - x_1)}(x^\sigma) = \frac{\partial w_\sigma}{\partial x_1}(x) > 0 \text{ and } \frac{\partial v}{\partial x_1}(x) + \frac{\partial v}{\partial(2\sigma - x_1)}(x^\sigma) = \frac{\partial z_\sigma}{\partial x_1}(x) > 0 \text{ on } T_\sigma$$

for all $\sigma < \hat{\rho}$, which implies $\frac{\partial u}{\partial x_1}(x) > 0$ and $\frac{\partial v}{\partial x_1}(x) > 0$ on $\Sigma_{\hat{\rho}}$. Next, we want show that $w_{\hat{\rho}} \equiv 0$ *and* $z_{\hat{\rho}} \equiv 0$ *on* $\Sigma_{\hat{\rho}}$. Suppose this is not true. Without loss of generality, we may assume $w_{\hat{\rho}} \not\equiv 0$ on $\Sigma_{\hat{\rho}}$, and hence $w_{\hat{\rho}}(x) < 0$ on $\Sigma_{\hat{\rho}}$ by maximum principle, which implies

$$\frac{\partial \widetilde{w}_{\hat{\rho}}}{\partial x_1}(x) > 0 \text{ on } T_{\hat{\rho}} \tag{5.8}$$

from Hopf lemma. According to the definition of $\hat{\rho}$, there exists a positive sequence $\{\varepsilon_k\}_{k\in N}$ such that $\hat{\rho} + \varepsilon_k < 0$ and $(\hat{\rho} + \varepsilon_k) \to \hat{\rho}$ as $k \to \infty$ implies $w_{\hat{\rho}+\varepsilon_k}$ is non-negative somewhere in $\Sigma_{\hat{\rho}+\varepsilon_k}$. By the way, we have $\widetilde{w}_{\hat{\rho}+\varepsilon_k}(x) \to 0$ as $|x| \to \infty$ and $\widetilde{w}_{\hat{\rho}+\varepsilon_k} = 0$ on $T_{\hat{\rho}+\varepsilon_k}$. Hence, for each ε_k there exists $x_k \in \Sigma_{\hat{\rho}+\varepsilon_k}$ such that

$$\widetilde{w}_{\hat{\rho}+\varepsilon_k}(x_k) \geq 0 \text{ and } \nabla \widetilde{w}_{\hat{\rho}+\varepsilon_k}(x_k) = (0, 0). \tag{5.9}$$

Since there exists a constant $\hat{R}_0 \geq R_0$ such that $|x_k| < \hat{R}_0$ for all $k \in N$ due to step 1, there exists a convergent subsequence, we still denote it by x_k, such that $x_k \to x_0$. Due to (5.9), we obtain that $\widetilde{w}_{\hat{\rho}}(x_0) = \lim_{k\to\infty} \widetilde{w}_{\hat{\rho}+\varepsilon_k}(x_k) \geq 0$. Thus we conclude that $x_0 \in T_{\hat{\rho}}$, and by (5.9), it follows that $\frac{\partial \widetilde{w}_{\hat{\rho}}}{\partial x_1}(x_0) = \lim_{k\to\infty} \frac{\partial \widetilde{w}_{\hat{\rho}+\varepsilon_k}}{\partial x_1}(x_k) = 0$, which contradicts to (5.8). Then we finish this step. Therefore, $u(x)$ and $v(x)$ are radially symmetric at some point in $\mathbf{R}^2$. □

ACKNOWLEDGMENT

The authors are grateful to Professor Chang-Shou Lin for all his helpful comments on the Work partially supported by Ministry of Science and Technology of Taiwan under grant no. MOST-103-2115-M-008-011-MY3; subject.

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