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Uniqueness of topological multivortex solutions in the Maxwell–Chern–Simons model

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ABSTRACT

In this paper, we prove the uniqueness of topological multivortex solutions for the self-dual Maxwell–Chern–Simons $U(1)$ model if the Chern–Simons coupling parameter is sufficiently large and the charge of electron is sufficiently small or large. On the other hand, we also establish the sharp region of the flux for non-topological solutions and provide the classification of radial solutions of all types in the case of one vortex point.

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1. Introduction

In this article, we study the solutions of the following coupled equations

$$\begin{cases} \Delta u = 2q\big(e^u - 1 - \kappa v\big) + 4\pi \sum_{i=1}^{\ell} n_i \delta_{p_i} \quad \text{in } \mathbf{R}^2, \\ \Delta v = -\kappa q^2\big(e^u - 1 - \kappa v\big) + 2qe^u v \quad \text{in } \mathbf{R}^2, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}$, $n_1, \ldots, n_\ell, q, \kappa$ are positive constants and $\ell \geq 0$ is called the number of vortex points and δ_p is the Dirac measure at p. In particular, if $\ell = 0$, then System (1.1) has no vortices involved. System (1.1) arises from the self-dual Maxwell–Chern–Simons $U(1)$ model on (2+1)-dimensional Minkowski space $\mathbf{R}^{2,1}$ with the metric $\text{diag}(1, -1, -1)$. We refer the readers to [5,16,17,19] and the references therein. Here we briefly introduce the self-dual Maxwell–Chern–Simons $U(1)$ model as follows.

The self-dual Maxwell–Chern–Simons $U(1)$ model is defined by the Lagrangian:

$$\mathcal{L}(A, \phi, N) = \big[(D^q)_\alpha \phi\big]\overline{\big[(D^q)^\alpha \phi\big]} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\mu}{4}\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + \frac{1}{2}\partial_\alpha N \partial^\alpha N - V(|\phi|, N) \tag{1.2}$$

with self-dual potential

$$V(|\phi|, N) = \mathbf{q}^2|\phi|^2 N + \frac{1}{2}\big(\mathbf{q}|\phi|^2 + \mu N - \mathbf{q}\big)^2,$$

where $\mu > 0$ is the Chern–Simons coupling parameter, $\mathbf{q}$ is the charge of electron, $\epsilon^{\varsigma\tau\rho}$ is a totally skew-symmetric tensor with $\epsilon^{012} = 1$, $A = (A_1, A_2)$, $A_\alpha : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ is the gauge field, $\alpha = 0, 1, 2$, $\phi : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{C}$ is the Higgs field, $N : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ is the neutral scalar field, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the field strength, and the covariant derivative D_A is defined as follows:

$$D_A\phi = \Big((D^{\mathbf{q}})_1\phi, (D^{\mathbf{q}})_2\phi\Big) = \nabla\phi - i\mathbf{q}A\phi, \quad i = \sqrt{-1}.$$

We say that (ϕ, A_α) is gauge equivalent to (ψ, B_α), if there exists a function χ such that

$$(\psi, B_\alpha) = (e^{i\chi}\phi, A_\alpha + \partial_\alpha \chi).$$

Then the $\mathcal{L}$ and its Euler–Lagrange equations are invariant under the gauge transformation.

We consider the stationary solutions of the Euler–Lagrange equations for (1.2). The Gauss law constraint obtained from the variation of A_0 is given by

$$\mu F_A = \Delta A_0 - 2\mathbf{q}^2|\phi|^2 A_0 \text{ in } \mathbf{R}^2,$$

where $F_A = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field.

In this article, we study the static configuration for self-dual Maxwell–Chern–Simons $U(1)$ model, and we can write the energy functional as

$$\mathcal{E}_{\mathbf{q},\mu}(A,\phi,N) = \int_{\mathbf{R}^2} \left\{ \left| D_1^{\mathbf{q}}\phi \pm iD_2^{\mathbf{q}}\phi \right|^2 + \mathbf{q}^2|\phi|^2|A_0 \pm N|^2 + \frac{1}{2}|\nabla A_0 \pm \nabla N|^2 \right.$$
$$\left. + \frac{1}{2}\left|F_A \pm \left(\mathbf{q}|\phi|^2 + \mu N - \mathbf{q}\right)\right|^2 \right\} dx \pm \mathbf{q} \int_{\mathbf{R}^2} F_A \, d\mathbf{x}.$$

Thus the field configurations saturating the energy bounded by $\mathcal{E}_{\mathbf{q},\mu}(A,\phi,N) = \pm\mathbf{q}\int_{\mathbf{R}^2} F_A \, d\mathbf{x}$ satisfy the Gauss law constraint and following self-dual equations,

$$\begin{cases} D_1^{\mathbf{q}}\phi \pm iD_2^{\mathbf{q}}\phi = 0, \\ A_0 \pm N = 0, \\ F_A \pm \left(\mathbf{q}|\phi|^2 + \mu N - \mathbf{q}\right) = 0. \end{cases} \tag{1.3}$$

If (A,ϕ,N) is a finite energy solution of (1.3), then $|\phi|$ and N satisfy

$$\left(\mathbf{q}|\phi|^2 + \mu N - \mathbf{q}\right) \to 0 \text{ as } |x| \to \infty.$$

In this paper, we are interested in looking for multivortex solutions so that ϕ vanishes at p_i with corresponding orders n_i, i.e.,

$$|\phi(y)| = |y - p_i|^{n_i} \text{ for } y \text{ near } p_i, \quad i = 1, \cdots, \ell.$$

Without loss of generality, we will only take the upper (plus) sign into account. The flux corresponding to the magnetic field F_A is given by $\Phi = \mathbf{q}\int_{\mathbf{R}^2} F_A \, d\mathbf{x}$, where $F_A = \mathbf{q}|\phi|^2 + \mu N - \mathbf{q}$. With the substitution $(u, v, \kappa, q) = \left(\ln|\phi|^2, -\kappa q^{\frac{3}{2}} N, \frac{\sqrt{\mu}}{\mathbf{q}^2}, \mathbf{q}^2\right)$, or equivalently,

$$\phi(x) = \exp\left[\frac{u}{2} + \mathrm{i}\left(\sum_{i=1}^{\ell} n_i \mathrm{Arg}(x - p_i)\right)\right] \text{ and } N(x) = -\kappa^{-1} q^{-\frac{3}{2}} v(x),$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, (1.3) can be reduced to (1.1). Conversely, once we find a solution (u, v) of (1.1), we may recover A and N from (1.3) by the formula

$$\mu = \kappa^2 q^2, \quad \mathbf{q} = \sqrt{q}, \quad \mathbf{q}A_1 + i\mathbf{q}A_2 = -2i\bar{\partial}\ln\phi \text{ and } N = -\frac{v}{\mathbf{q}\sqrt{\mu}},$$

where $\bar{\partial} = \dfrac{\partial_1 + i\partial_2}{2}$.

For (1.1), there are natural boundary conditions for solutions at infinity, namely,

$$\lim_{|x|\to\infty} u(x) = 0, \quad \lim_{|x|\to\infty} v(x) = 0, \tag{1.4}$$

$$\lim_{|x|\to\infty} u(x) = -\infty, \quad \lim_{|x|\to\infty} v(x) = -\frac{1}{\kappa}. \tag{1.5}$$

We note that if (u, v) is a solution with the boundary condition (1.4), then, by the maximum principle, we have $u(x), v(x) < 0$ on $\mathbf{R}^2\backslash\{p_1, \dots, p_\ell\}$. In physics literature, a solution (u, v) satisfying boundary condition (1.4) is called the topological solution. Correspondingly, a solution (u, v) is called the non-topological solution if it satisfies boundary condition (1.5). Note that one can also consider (1.3) on the 't Hooft type periodic domain, of which solutions are call condensate solutions. There have been several results for condensate solutions obtained in [5,18,19].

From [4], we obtain the existence of topological solutions to (1.1) for all $\kappa, q > 0$. Then it is natural to ask the question about the uniqueness of topological solutions to (1.1). For the case of $\kappa = 0$, $q = \frac{1}{2}$ and $v \equiv 0$, (1.1) reduces to

$$\Delta u = e^u - 1 + 4\pi \sum_{i=1}^{\ell} n_i \delta_{p_i} \quad \text{in } \mathbf{R}^2, \tag{1.6}$$

which is the self-dual equation of the Abelian–Higgs model and the uniqueness of topological solutions of (1.6) was proved in [21]. Similarly, if we set $\mu = \kappa^2 q$, $v = \kappa^{-1}\big(e^u - 1\big)$, $q \to \infty$, then by [5], (1.1) with (1.4) tends formally to

$$\Delta u = \frac{4}{\kappa^2} e^u (e^u - 1) + 4\pi \sum_{i=1}^{\ell} n_i \delta_{p_i} \quad \text{in } \mathbf{R}^2, \tag{1.7}$$

which is the self-dual equation of the Chern–Simons–Higgs model. For (1.7), the uniqueness of topological solutions was obtained in [12] and [8]. Now, we state our main result about the uniqueness of topological solutions of (1.1) as follows.

Theorem 1.1. *Let $p_1, \dots, p_\ell \in \mathbf{R}^2$ and $n_1, \dots, n_\ell \in \mathbf{R}^+$ be given. Then there exist three positive constants $\kappa_0 = \kappa_0(p_i, n_i)$, $q_0 = q_0(p_i, n_i)$ and $q_1 = q_1(p_i, n_i) < q_0$ such that (1.1) possesses one and only one topological solution in $\mathbf{R}^2$ for any $(\kappa, q) \in \big\{(\kappa, q) : 0 < q < q_1 \text{ or } q > q_0 \text{ or } \kappa > \kappa_0\big\}$. Moreover, if $\ell = 0$ or 1, then (1.1) possesses a unique topological solution in $\mathbf{R}^2$ for any $\kappa, q > 0$.*

Moreover, we also discuss the existence of solutions to (1.1) satisfying (1.5). In [15], we get that if $\ell = 0, 1$, then (1.1) possesses a radially symmetric non-topological solution in $\mathbf{R}^2$ for any $\kappa, q > 0$. In addition, for any $(\ell, \kappa, q) \in Z^+ \times (0, \infty) \times (0, \infty)$, we also have that there exist infinitely many non-topological solutions of (1.1) in $\mathbf{R}^2$ and their flux

$\Phi \in \left(4\pi\left(\sum_{i=1}^{\ell} n_i + 1\right), 4\pi\left(\sum_{i=1}^{\ell} n_i + 1\right) + \varepsilon_0\right)$ for some $\varepsilon_0 = \varepsilon_0(n_i, p_i) > 0$, $i = 1, \ldots, \ell$, which is derived in [3]. Hence, there is an intersecting problem about finding the sharp range of flux Φ for all non-topological solutions of (1.1). We note that there have been many well-known results about such a problem for Chern–Simons limit equation (1.7) in, for example, [2,6,13].

To answer the problem about the range of flux for (1.1), we consider the case of $\ell = 0, 1$ and set p_1 to be the origin O. Then (1.1) reduces to the following equations

$$\begin{cases} \Delta u = 2q\big(e^u - 1 - \kappa v\big) + 4\pi \overline{N}\delta_O & \text{in } \mathbf{R}^2, \\ \Delta v = -\kappa q^2\big(e^u - 1 - \kappa v\big) + 2qe^u v & \text{in } \mathbf{R}^2, \end{cases} \tag{1.8}$$

where $\overline{N} \geq 0$. We give a consequence about the range of flux for (1.8) mentioned below.

Theorem 1.2. *If $(u(x), v(x))$ is a non-topological solution of (1.8), then $\Theta_1(u, v) \in (4\overline{N} + 4, \infty)$ and $\Theta_2(u, v) = 0$, where*

$$\begin{cases} \Theta_1(u, v) = \dfrac{q}{\pi} \displaystyle\int_{\mathbf{R}^2} \Big(1 + \kappa v(x) - e^{u(x)}\Big)\, dx, \\ \Theta_2(u, v) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbf{R}^2} \Big\{\kappa q^2 \Big[e^{u(x)} - \kappa v(x) - 1\Big] - 2qe^{u(x)}v(x)\Big\}\, dx. \end{cases}$$

Moreover, (1.8) possesses a non-topological solution $(u(x), v(x))$ such that $\Theta_1(u, v) = \theta$ and $\Theta_2(u, v) = 0$ for any $\theta \in (4\overline{N} + 4, \infty)$.

From [3,4,15], we know that (1.8) possess at least two different types of solutions and hence we want to ask: How many different types of solutions of (1.8) are there? In order to solve this problem, we will investigate the structure of all radial solutions of (1.8) and consider the following ODE system

$$\begin{cases} u''(r) + \dfrac{1}{r}u'(r) = 2q\big(e^u - 1 - \kappa v\big) & \text{for } r > 0, \\ v''(r) + \dfrac{1}{r}v'(r) = -\kappa q^2\big(e^u - 1 - \kappa v\big) + 2qe^u v & \text{for } r > 0 \end{cases} \tag{1.9}$$

with the initial value

$$\begin{cases} u(r) = 2\overline{N}\ln r + \alpha_1 + o(1), \\ v(r) = \alpha_2 + o(1) \end{cases} \quad \text{as } r \to 0^+. \tag{1.10}$$

According to the behaviors at ∞, all solutions of (1.9) defined in $(0, \infty)$ can be classified into the following four types.

Type (I): $\lim_{r\to\infty}(u(r),v(r))=(0,0)$.
Type (II): $\lim_{r\to\infty}(u(r),v(r))=(-\infty,-\frac{1}{\kappa})$.
Type (III): $\lim_{r\to\infty}(u(r),v(r))=(-\infty,\infty)$.
Type (IV): $\lim_{r\to\infty}(u(r),v(r))=(\infty,\infty)$.

Except four types of entire solution stated above, we see that other solutions must blow up at finite $R>0$. Indeed, such solutions admit three types, i.e.,

Type (V): $\lim_{r\to R^-}(u(r),v(r))=(\infty,\infty)$.
Type (VI): $\lim_{r\to R^-}(u(r),v(r))=(\infty,-\infty)$.
Type (VII): $\lim_{r\to R^-}(u(r),v(r))=(\infty,\frac{\kappa q}{2})$.

Let $\alpha=(\alpha_1,\alpha_2)\in\mathbf{R}^2$ and $(u(r,\alpha),v(r,\alpha))$ denote the solution of (1.9)–(1.10). According to the behavior of (u,v), the set of initial data could be classified into the following regions.

$T=\{\alpha|(u(r,\alpha),v(r,\alpha))$ is a Type (I) solution$\}$,
$\Lambda=\{\alpha|(u(r,\alpha),v(r,\alpha))$ is a Type (II) solution$\}$,
$\mathcal{B}_\infty=\{\alpha|(u(r,\alpha),v(r,\alpha))$ is a Type (IV) solution$\}$,
$\mathcal{B}=\{\alpha|(u(r,\alpha),v(r,\alpha))$ is a Type (V) or (VII) solution$\}$,
$\mathcal{B}_v=\{\alpha|(u(r,\alpha),v(r,\alpha))$ is a Type (III) solution$\}$,
$\mathcal{B}_u=\{\alpha|(u(r,\alpha),v(r,\alpha))$ is a Type (VI) solution$\}$.

Then the structure of solutions sets are described as follows.

Theorem 1.3. *The following statements are valid.*

(i) $\mathbf{R}^2=\mathcal{B}_v\bigcup\Lambda\bigcup T\bigcup\mathcal{B}_u\bigcup\mathcal{B}\bigcup\mathcal{B}_\infty$.
(ii) *There exist a point* $(\alpha_1^0,\alpha_2^0)\in\mathbf{R}^2$ *and a strictly increasing function* $\gamma:(-\infty,\infty)\to(-\infty,\infty)$ *such that* $\lim_{\alpha_1\to-\infty}\gamma(\alpha_1)=-\frac{1}{\kappa}$, $\gamma(\alpha_1^0)=\alpha_2^0$ *and*

$$\begin{cases} T=\{(\alpha_1^0,\alpha_2^0)\},\ \ \Lambda=\{(\alpha_1,\alpha_2):\alpha_1<\alpha_1^0,\ \alpha_2=\gamma(\alpha_1)\},\\ \{(\alpha_1,\alpha_2):\alpha_1\in(-\infty,\alpha_1^0),\gamma(\alpha_1)<\alpha_2<\infty\}\supseteq\mathcal{B}_v,\\ \{(\alpha_1,\alpha_2):\alpha_1>-\infty,-\infty<\alpha_2<\gamma(\alpha_1)\}\supseteq\mathcal{B}_u,\\ \{(\alpha_1,\alpha_2):\alpha_1\in(\alpha_1^0,\infty),\ \alpha_2=\gamma(\alpha_1)\}\subseteq\mathcal{B}\bigcup\mathcal{B}_\infty. \end{cases}$$

Now we keep on discussing entire solutions of (1.9)–(1.10) defined for all $r>0$. Let $(u(r),v(r))=(u(r;\alpha_1,\alpha_2),v(r;\alpha_1,\alpha_2))$ be a solution of (1.9)–(1.10). We define

$$\begin{cases} \beta_1(\alpha_1,\alpha_2) = 2q\displaystyle\int_0^\infty r\big(1+\kappa v(r) - e^{u(r)}\big)dr, \\ \beta_2(\alpha_1,\alpha_2) = \displaystyle\int_0^\infty r\Big\{\kappa q^2\big[e^{u(r)} - 1 - \kappa v(r)\big] - 2qe^{u(r)}v(r)\Big\}dr, \end{cases} \tag{1.11}$$

and sometimes denote it by (β_1, β_2) if no confusion arises. Here we call $\beta_1(u, v)$ and $\beta_2(u, v)$ the u-*flux* and v-*flux* with respect to solution (u, v), respectively.

Remark 1.1. Let $(u(r), v(r))$ be an entire (1.9)–(1.10). Then we obtain the following statements from (1.9)–(1.11).

(i) $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ is a solution of Type (I) if and only if $\beta_1(\alpha_1,\alpha_2) = 2\overline{N}$ and $\beta_2(\alpha_1,\alpha_2) = 0$.
(ii) $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ is a solution of Type (II) if and only if $4\overline{N}+4 < \beta_1(\alpha_1,\alpha_2) < \infty$ and $\beta_2(\alpha_1,\alpha_2) = 0$.
(iii) $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ is a solution of Type (III) if and only if $\beta_1(\alpha_1,\alpha_2) = \infty$ and $\beta_2(\alpha_1,\alpha_2) = -\infty$.
(iv) If $\alpha_1 < \alpha_1^0$, then $\lim_{\alpha_1\to\alpha_1^0} \beta_1(\alpha_1,\gamma(\alpha_1)) = \infty$, $\lim_{\alpha_1\to-\infty} \beta_1(\alpha_1,\gamma(\alpha_1)) = 4\overline{N}+4$ and $\beta_2(\alpha_1,\gamma(\alpha_1)) = 0$.

This paper is organized as follows. First we investigate the monotone and non-degeneracy properties of the linearized equations at the solutions of (1.9) in Section 2. We apply the Implicit Function Theorem to prove the uniqueness of the topological solutions for (1.1) with at most one vortex point in Section 3. The existences and classification of radial solutions of all types will be shown in Section 4. In Section 5, we find the sharp range of flux Φ for all non-topological solutions of (1.8). Finally, the uniqueness of the topological multivortex solutions for (1.1) will be derived in Section 6. Roughly speaking, Sections 1–5 are devoted to dealing with (1.1) with at most one vortex point. The situation for multivortex points will be treated by applying the consequences obtained from the case with at most one vortex point.

2. Linearized equations

In this section, we give the proof about the non-degeneracy of linearized equations on the solution of Type (I) to (1.8). Before going to our proof, we need to state some properties concerning solutions. First, we have the following property for solutions with zero boundary values.

Lemma 2.1. *Let $(u(r), v(r))$ be a solution of (1.9)–(1.10) satisfying $u(R_0) = v(R_0) = 0$ for some $R_0 > 0$ (or $R_0 = +\infty$). Then the following are valid.*

(i) $u(r) < 0$, $-\frac{1}{\kappa} \le v(r) < 0$, $u'(r) > 0$ *and* $v'(r) > 0$ *on* $(0, R_0)$. *Furthermore, if* $R_0 = \infty$, *i.e.,* (u, v) *is a topological solution of* (1.8), *then the corresponding* (β_1, β_2) *satisfies* $\beta_1 = 2\overline{N}$ *and* $\beta_2 = 0$.

(ii) $1 + \kappa v(r) - e^{u(r)} \ge 0$ *on* $[0, R_0]$.

Proof. We note that $r = 0$ is not a non-negative local maximum point of $v(r)$ due to $\Delta v(0) = \kappa q^2\big[1 + \kappa v(0)\big]$. To prove it, we divide the proof into two steps.

Step 1. Both $u(r)$ *and* $v(r)$ *do not attain non-negative local maximum on* $(0, R_0)$. *Furthermore, one of* $u(r)$ *and* $v(r)$ *is negative on* $(0, R_0)$. On the contrary, without loss of generality, we may assume that r_u^1 and r_v^1 are both non-negative maximum points of $u(r)$ and $v(r)$ on $(0, R_0)$, respectively. Then we see

$$-\kappa v(r_u^1) - 1 + e^{u(r_u^1)} \le 0 \ \text{ and } \ \kappa q^2\Big[-\kappa v(r_v^1) - 1 + e^{u(r_v^1)}\Big] \ge 2qe^{u(r_v^1)} v(r_v^1) \ge 0,$$

which contradicts $-\kappa v(r_u^1) - 1 + e^{u(r_u^1)} \ge -\kappa v(r_v^1) - 1 + e^{u(r_v^1)}$ when $v(r_v^1) > 0$. If $v(r_v^1) = 0$, then $v(r_u^1) < 0$ and $e^{u(r_u^1)} - 1 < 0$, which contradicts $u(r_u^1) \ge 0$. Therefore, we finish this step.

Step 2. Both $u(r)$ *and* $v(r)$ *are negative on* $(0, R_0)$. *Moreover,* $-\frac{1}{\kappa} \le v(r) < 0$ *on* $[0, R_0)$. On the contrary, we first assume that $u(r)$ possesses a local maximum point $r_u \in (0, R_0)$ such that $u(r_u) \ge 0$ and $-\kappa v(r_u) - 1 + e^{u(r_u)} \le 0$. Then we have $\kappa v(r_u) \ge e^{u(r_u)} - 1 \ge 0$, which contradicts Step 1. If $v(r)$ possesses a local maximum point $r_v \in (0, R_0)$ such that

$$v(r_v) \ge 0 \ \text{ and } \ \kappa q^2\big[e^{u(r_v)} - 1\big] \ge \Big[\kappa^2 q^2 + 2qe^{u(r_v)}\Big] v(r_v),$$

then $u(r_v) \ge 0$. However, it is impossible by Step 1. We complete this step.

Moreover, we can see that one of $u(r)$ and $v(r)$ does not attain non-positive local minimum on $(0, R_0)$ by the similar proofs in Step 1 and Step 2. Thus $u(r)$ and $v(r)$ are both strictly increasing on $(0, R_0)$ and tend to zero as $r \to R_0$. Furthermore, we have $v(0) \ge -\frac{1}{\kappa}$; otherwise, $\Delta v(0) < 0$ and $v(R_0) = 0$, which is in contradiction to $v'(r) > 0$ on $(0, R_0)$. Consequently, $-\frac{1}{\kappa} \le v(r) < 0$ on $[0, R_0)$.

Let $R_0 = \infty$. Combining with (1.9) and Step 2, we get

$$\begin{cases} \Delta u(r) \ge 2qu(r) \ \text{ on } \ [0, \infty), \\ \Delta v(r) \ge (\kappa^2 q + 2)qv(r) \ \text{ on } \ [0, \infty), \\ \lim\limits_{r\to\infty} (u(r), v(r)) = (0, 0). \end{cases}$$

It follows that $|u(r)| = O(e^{-\sqrt{2q}r})$ and $|v(r)| = O(e^{-\sqrt{\kappa^2 q^2 + 2q}r})$ as $r \to \infty$, which imply $1 + \kappa v(r) - e^{u(r)} \in L^1([0, \infty))$ and $e^{u(r)} v(r) \in L^1([0, \infty))$. Then we obtain that $\lim\limits_{r\to\infty} ru'(r) = \lim\limits_{r\to\infty} rv'(r) = 0$, that is, $\beta_1 = 2\overline{N}$ and $\beta_2 = 0$. Hence (i) holds.

Now we want to show that $-\kappa v(r) - 1 + e^{u(r)} \leq 0$ *on* $[0, R_0]$. On the contrary, we may assume that there exist three constants $r_0, r_1, r_2 \in (0, R_0]$ such that $r_0 < r_1 < r_2$ and

$$\begin{cases} -\kappa v(r_0) - 1 + e^{u(r_0)} = -\kappa v(r_2) - 1 + e^{u(r_2)} = 0, \\ -\kappa v(r) - 1 + e^{u(r)} > 0 \ \text{ on } \ (r_0, r_2), \\ -\kappa v'(r_1) + e^{u(r_1)} u'(r_1) = 0 \end{cases} \tag{2.1}$$

from $u(R_0) = v(R_0) = 0$. It is easy to see that

$$\begin{cases} \big(ru'(r)\big)' = 2q\big[-\kappa v(r) - 1 + e^{u(r)}\big] \geq 0 \ \text{ on } \ (r_0, r_2), \\ \big(rv'(r)\big)' = -\kappa q^2\big[-\kappa v(r) - 1 + e^{u(r)}\big] + 2q e^{u(r)} v(r) \leq 0 \ \text{ on } \ (r_0, r_2), \end{cases}$$

which implies

$$-\kappa v'(r) + e^{u(r)} u'(r) \geq \big[-\kappa v'(r_1) + e^{u(r_1)} u'(r_1)\big]\frac{r_1}{r} > 0 \ \text{ on } \ [r_1, r_2]$$

by $\big[e^{u(r)}\big]' > 0$ on $(0, R_0)$. Then it follows that $-\kappa v(r_2) - 1 + e^{u(r_2)} \geq -\kappa v(r_1) - 1 + e^{u(r_1)} > 0$, which contradicts (2.1). Hence the result (ii) is proved. □

Secondly, we have the following useful identity.

Lemma 2.2. *Let* $(u(r), v(r))$ *be a solution of* (1.9)–(1.10) *in* $(0, R]$ *for some* $R > 0$ *or* $R = \infty$. *Then we have the following identity*

$$\frac{\kappa q}{4}\big[ru'(r)\big]^2 + r^2 u'v'(r) + 2qr^2\Big[v(r) + \frac{\kappa v^2(r)}{2} - e^{u(r)} v(r)\Big]$$
$$= \kappa q \overline{N}^2 + 2q \int_0^r t\big[\kappa v(t) + 2 - 2e^{u(t)}\big] v(t)\, dt \tag{2.2}$$

for $r \in (0, R]$.

Proof. By multiplying $\frac{\kappa q}{2} ru'$, rv' and ru' on both sides of the first and second equations of (1.9), and using the initial value (1.10), we can easily obtain (2.2). □

In the following, we will apply the similar method used in [11,9,10] to prove the non-degeneracy properties of the linearized equations. First we investigate the monotone property of solutions of (1.9)–(1.10) and let, for $i = 1, 2$, $\phi_i(r) = \dfrac{\partial U}{\partial \alpha_i}(r)$ and $\psi_i(r) = \dfrac{\partial v}{\partial \alpha_i}(r)$, where $U(r) = u(r) - 2\overline{N} \ln r$. Then (ϕ_i, ψ_i), $i = 1, 2$, satisfy the linearized equations

$$\begin{cases} \Delta\phi_i - 2qe^u\phi_i = -2\kappa q\psi_i \ \text{ on } \ [0, R_0), \\ \Delta\psi_i - \big(\kappa^2q^2 + 2qe^u\big)\psi_i = -qe^u\big(\kappa q - 2v\big)\phi_i \ \text{ on } \ [0, R_0), \\ \phi_1(0) = 1 = \psi_2(0), \phi_2(0) = 0 = \psi_1(0), \phi_i'(0) = 0 = \psi_i'(0), \ i = 1, 2, \end{cases} \tag{2.3}$$

where $u(r)$ and $v(r)$ are both finite on $[0, R_0)$. The monotone property of ϕ_i and ψ_i is as follows.

Lemma 2.3. *If $(u(r), v(r))$ be a solution of (1.9)–(1.10) on $[0, R_v)$, then the corresponding $(\phi_i(r), \psi_i(r))$ satisfies*

$$\begin{cases} \phi_1(r) > 1, \phi_1'(r) > 0, \ \phi_2(r) < 0, \phi_2'(r) < 0, \\ \psi_1(r) < 0, \psi_1'(r) < 0, \ \psi_2(r) > 1, \psi_2'(r) > 0 \end{cases} \tag{2.4}$$

on $(0, R_v)$, where $R_v = \sup\{R : v(r) \le 0 \text{ on } (0, R]\}$. Moreover, if $R_v = \infty$, then $\phi_1(r)$, $\psi_1(r)$, $\phi_2(r)$ and $\psi_2(r)$ are all unbounded on $[0, \infty)$.

Proof. Since $\phi_1(0) = 1$, $\psi_1(0) = 0$ and (2.3), we have

$$\phi_1(r) > 0 \ \text{ and } \ \Delta\psi_1(r) - \big(\kappa^2q^2 + 2qe^{u(r)}\big)\psi_1(r) \le 0 \ \text{ on } \ [0, r_0]$$

for some $r_0 \in (0, R_v)$, and hence $\psi_1(r) < 0$ on $(0, r_0]$ by the maximum principle. From (2.3), we also get

$$r\psi_1'(r) = \int_0^r s\big[\big(\kappa^2q^2 + 2qe^{u(s)}\big)\psi_1(s) - qe^{u(s)}\big(\kappa q - 2v(s)\big)\phi_1(s)\big]ds < 0 \quad \text{on } (0, r_0] \tag{2.5}$$

and

$$\phi_1(r) > 1 \ \text{ and } \ r\phi_1'(r) = 2q\int_0^r s\big(e^{u(s)}\phi_1(s) - \kappa\psi_1(s)\big)\, ds > 0 \quad \text{on } (0, r_0]. \tag{2.6}$$

Since the inequalities in (2.5) and (2.6) hold whenever $\phi_1(r) > 0$ and $\psi_1(r) < 0$, we deduce that $\phi_1(r) > 1$, $\phi_1'(r) > 0$ and $\psi_1(r) < 0$, $\psi_1'(r) < 0$ for all $r \in (0, R_v]$. The situations for $\phi_2(r)$ and $\psi_2(r)$ are similar, and we complete this proof. □

Finally, we state and prove the non-degeneracy property of the linearized equation at a topological solution in the following lemma.

Lemma 2.4. *Let $(u(r), v(r))$ be a solution of (1.9)–(1.10) satisfying $u(R_0) = 0$, $v(R_0) = 0$ for some $R_0 > 0$ (or $R_0 = +\infty$). If $(\phi_i(r), \psi_i(r))$, $i = 1, 2$, is the respective solution pair* *of (2.3), then the corresponding linearized equation (2.7) is non-degenerate, that is, there does not exist a nonzero bounded solution pair* $(\Phi(r), \Psi(r))$ *of*

$$\begin{cases} \Delta\Phi - 2qe^u\Phi + 2\kappa q\Psi = 0 \ \ on \ \ [0, R_0), \\ \Delta\Psi - \big(\kappa^2 q^2 + 2qe^u\big)\Psi + qe^u\big(\kappa q - 2v\big)\Phi = 0 \ \ on \ \ [0, R_0). \end{cases} \tag{2.7}$$

Proof. Let

$$M_\Phi(r) = -\frac{\phi_1(r)}{\phi_2(r)} \text{ and } M_\Psi(r) = -\frac{\psi_1(r)}{\psi_2(r)}.$$

Then we see that if $M_\Phi(r) > M_\Psi(r)$ on $(0, r_0)$ for some $r_0 \le R_0$, then $M'_\Phi(r) < 0$ and $M'_\Psi(r) > 0$ on $(0, r_0)$. To prove this lemma we need the following the fact.

Claim. *There does not exist* $R \in (0, R_0)$ *such that* $M_\Phi(R) = M_\Psi(R)$*.*

Proof of Claim. We prove this claim by a contradiction and hence there exists a smallest $R \in (0, R_0]$ such that $M_\Phi(R) = M_\Psi(R)(\equiv C)$ and $M_\Phi(r) > M_\Psi(r) > 0\ \forall r \in (0, R)$. Let $\Phi_c(r) = \varphi_1(r) + c\varphi_2(r)$ and $\Psi_c(r) = \psi_1(r) + c\psi_2(r)$ for any $c \in \mathbf{R}$. Then $\Phi_c(r)$ and $\Psi_c(r)$ satisfy (2.7) and it follows that

$$\begin{cases} \Phi_C(r) > 0, \Psi_C(r) > 0 \ \forall r \in (0, R), \\ \Phi_C(R) = \Psi_C(R) = 0, \\ \Phi'_C(R) < 0, \Psi'_C(R) < 0 \ \text{ if } R < \infty. \end{cases} \tag{2.8}$$

Taking the differentiation with respect to α_i, $i = 1, 2$, on the both sides of (2.2), then for any $c > 0$ and $r \in (0, R_0]$, we obtain

$$r^2\Big[\frac{\kappa q}{2}\Phi'_c(r)u'(r) + \Phi'_c(r)v'(r) + \Psi'_c(r)u'(r)\Big] + 2qr^2\big[\kappa v(r) + 1 - e^{u(r)}\big]\Psi_c(r)$$
$$= 2qr^2 v(r)e^{u(r)}\Phi_c(r) + 4q\int_0^r t\Big\{\big[\kappa v(t) + 1 - e^{u(t)}\big]\Psi_c(t) - 2e^{u(t)}v(t)\Phi_c(t)\Big\}dt. \tag{2.9}$$

If $R < \infty$ then, by replacing c and r with C and R in (2.9) respectively, we easily have

$$R^2\Big[\frac{\kappa q}{2}\Phi'_c(R)u'(R) + \Phi'_c(R)v'(R) + \Psi'_c(R)u'(R)\Big]$$
$$= 4q\int_0^R t\Big\{\big[\kappa v(t) + 1 - e^{u(t)}\big]\Psi_c(t) - 2e^{u(t)}v(t)\Phi_c(t)\Big\}dt. \tag{2.10}$$

Then, combining with Lemma 2.1, (2.8) and (2.10), we get that

$$0 > R^2\Big[\frac{\kappa q}{2}\Phi_c'(R)u'(R) + \Phi_c'(R)v'(R) + \Psi_c'(R)u'(R)\Big]$$
$$= 4q\int_0^R t\Big\{\big[\kappa v(t) + 1 - e^{u(t)}\big]\Psi_c(t) - 2e^{u(t)}v(t)\Phi_c(t)\Big\}dt > 0,$$

which yields a contradiction. This shows the claim. Therefore, we easily obtain $M_\Phi(r) > M_\Psi(r) > 0$ on $[0, R_0]$ (resp., $[0,\infty)$ if $R_0 = \infty$) and $M_\Phi'(r) < 0, M_\Psi'(r) > 0$ on $(0, R_0)$.

Now we show that *if* $R_0 = \infty$*, then for any* $C > 0$,

$$\textit{either } \Phi_C(r) \textit{ or } \Psi_C(r) \textit{ is unbounded on } [0,\infty). \tag{†}$$

Suppose that (†) is not true. Then $\Phi_C(r)$ and $\Psi_C(r)$ are bounded on $[0,\infty)$ for some $C > 0$. Note that $M_\Phi(r)$ (resp., $M_\Psi(r)$) is strictly decreasing (resp., increasing) to $\lim_{r\to\infty} M_\Phi(r)$ (resp., $\lim_{r\to\infty} M_\Psi(r)$) and $\lim_{r\to\infty} M_\Psi(r) \le \lim_{r\to\infty} M_\Phi(r)$ by maximum principle and Claim, respectively. Hence, we have $C \in \big[\lim_{r\to\infty} M_\Psi(r), \lim_{r\to\infty} M_\Phi(r)\big]$. Indeed, on the contrary, without loss of generality, we may assume that $0 < C < \lim_{r\to\infty} M_\Psi(r)$. Then we obtain that $\Phi_C(r) > 0$ on $[0,\infty)$ and $\Psi_C(r)$ has only one root r_* on $(0,\infty)$, which implies that $\lim_{r\to\infty} \Psi_C(r) = -\infty$ from (2.7). It yields a contradiction.

It follows from $C \in \big[\lim_{r\to\infty} M_\Psi(r), \lim_{r\to\infty} M_\Phi(r)\big]$ that

$$\Phi_C(r) > 0 \ \text{ and } \ \Psi_C(r) > 0 \ \text{ on } \ (0,\infty). \tag{2.11}$$

Since $|u|$ and $|v|$ decay exponentially at ∞, we get that

$$\lim_{r\to\infty} ru'(r) = 0 = \lim_{r\to\infty} rv'(r) \ \text{ and}$$
$$\lim_{r\to\infty} r^2\Phi_C'(r)u'(r) = 0 = \lim_{r\to\infty} r^2\Psi_C'(r)v'(r). \tag{2.12}$$

Moreover,

$$\lim_{r\to\infty} r^2\big[\kappa v(r) + 1 - e^{u(r)}\big] = 0 = \lim_{r\to\infty} r^2 v(r)e^{u(r)}. \tag{2.13}$$

Hence, by virtue of (2.10)–(2.13), we obtain

$$0 = 4q\int_0^\infty t\Big\{\big[\kappa v(t) + 1 - e^{u(t)}\big]\Psi_c(t) - 2e^{u(t)}v(t)\Phi_c(t)\Big\}dt > 0.$$

This contradiction shows that Φ_C or Ψ_C is unbounded, and (†) is proved.

Finally, let (u, v) be a topological solution of (1.8). Then any solution pair $(\Phi(r), \Psi(r))$ of the linearized equations (2.7) can be written as

$$\Phi(r) = c_1\phi_1(r) + c_2\psi_1(r) \quad \text{and} \quad \Psi(r) = c_1\phi_2(r) + c_2\psi_2(r) \text{ for some } c_1, c_2 \in \mathbf{R}.$$

By (†), we easily obtain the non-degeneracy result if $c = c_2/c_1 > 0$. When $c \leq 0$ or $c_1 = 0$, by Lemma 2.3, we also get that both $\Phi_c(r)$ and $\Psi_c(r)$ are unbounded on $[0, \infty)$. The proof of this theorem is completed. □

3. Uniqueness of topological solutions with one vortex point

In this section, we will use Lemma 2.4, Implicit Function Theorem and a continuation argument to establish the existence and uniqueness of topological solutions with one vortex point which, without loss of generality, may be assumed to be the origin point $O = (0, 0)$. First, we will give a uniqueness result for small enough $\kappa > 0$ as follows.

Theorem 3.1. *Let $q > 0$ and $\overline{N} \geq 0$ be fixed. Then there exists a constant $\kappa^* > 0$ such that (1.8) possesses a unique topological solution in $\mathbf{R}^2$ for any $0 < \kappa < \kappa^*$.*

Proof. Let $(u_\kappa^1(x), v_\kappa^1(x))$ and $(u_\kappa^2(x), v_\kappa^2(x))$ be topological solutions of (1.8). Both $(u_\kappa^1(x), v_\kappa^1(x))$ and $(u_\kappa^2(x), v_\kappa^2(x))$ are radially symmetric with respect to O by [14]. It is easy to see that both $(u_\kappa^1(x), v_\kappa^1(x))$ and $(u_\kappa^2(x), v_\kappa^2(x))$ converge to $(u^*(x), 0)$ pointwise as $\kappa \to 0$, where

$$\Delta u^* = 2q\big(e^{u^*} - 1\big) + 4\pi\overline{N}\delta_O \quad \text{in } \mathbf{R}^2. \tag{3.1}$$

Since

$$\int_{\mathbf{R}^2} \big[1 - e^{u^*(x)}\big]dx = \lim_{\kappa\to 0} \int_{\mathbf{R}^2} \Big[1 - e^{u_\kappa^i(x)} - \kappa v_\kappa^i(x)\Big]dx = \frac{2\pi\overline{N}}{q}$$

for $i = 1, 2$ due to $u^*(x) \to 0$ as $|x| \to \infty$, that is, $u^*(x)$ is a unique topological solution of (3.1) and $u^*(x) = u^*(r)$, where $r = |x|$.

Without loss of generality, we may assume $|(u_\kappa^1 - u_\kappa^2)(x_\kappa)| = \|u_\kappa^1 - u_\kappa^2\|_{L^\infty(\mathbf{R}^2)} \geq \|v_\kappa^1 - v_\kappa^2\|_{L^\infty(\mathbf{R}^2)}$ for all κ. Set $\Phi_\kappa = \frac{(u_\kappa^1 - u_\kappa^2)}{\|u_\kappa^1 - u_\kappa^2\|_{L^\infty(\mathbf{R}^2)}}$ and $\Psi_\kappa = \frac{(v_\kappa^1 - v_\kappa^2)}{\|u_\kappa^1 - u_\kappa^2\|_{L^\infty(\mathbf{R}^2)}}$. Then $(\Phi_\kappa, \Psi_\kappa)$ satisfies

$$\begin{cases} \Delta\Phi_\kappa - 2qe^{\xi_\kappa}\Phi_\kappa + 2\kappa q\Psi_\kappa = 0 \text{ in } \mathbf{R}^2, \\ \Delta\Psi_\kappa - \big(\kappa^2 q^2 + 2qe^{u_\kappa^1}\big)\Psi_\kappa + qe^{\xi_\kappa}\big(\kappa q - 2v_\kappa^2\big)\Phi_\kappa = 0 \text{ in } \mathbf{R}^2, \end{cases} \tag{3.2}$$

where

$$\xi_\kappa(|x|) \in \Big[\min\big\{u_\kappa^1(|x|), u_\kappa^2(|x|)\big\}, \max\big\{u_\kappa^1(|x|), u_\kappa^2(|x|)\big\}\Big].$$

First, we claim that $\{x_\kappa\}$ is bounded. Let $\mathcal{P}$ be a 2×2 matrix such that $\mathcal{P}^{-1}\mathcal{M}\mathcal{P} = \Lambda$, where

$$\mathcal{M} = \begin{pmatrix} -2q & 2\kappa q \\ \kappa q^2 & -\kappa^2 q^2 - 2q \end{pmatrix}, \ \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{3.3}$$

and λ_1, λ_2 are distinct negative eigenvalue of $\mathcal{M}$ with $\lambda_1 > \lambda_2$. Introduce a new set variables $\mathcal{A}_\kappa$, $\mathcal{B}_\kappa$ such that $\begin{pmatrix} \mathcal{A}_\kappa \\ \mathcal{B}_\kappa \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} \Phi_\kappa \\ \Psi_\kappa \end{pmatrix}$. From (3.2), we obtain

$$\begin{cases} \Delta \mathcal{A}_\kappa + \lambda_1 \mathcal{A}_\kappa + a_{11}(|x|)\mathcal{A}_\kappa + a_{12}(|x|)\mathcal{B}_\kappa = 0 \ \text{ on } \ [0,\infty), \\ \Delta \mathcal{B}_\kappa + \lambda_2 \mathcal{B}_\kappa + a_{21}(|x|)\mathcal{A}_\kappa + a_{22}(|x|)\mathcal{B}_\kappa = 0 \ \text{ on } \ [0,\infty), \\ \mathcal{A}_\kappa(|x|) \to 0, \ \ \mathcal{B}_\kappa(|x|) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{3.4}$$

where

$$\begin{pmatrix} a_{11}(|x|) & a_{12}(|x|) \\ a_{21}(|x|) & a_{22}(|x|) \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} 2q\big(1 - e^{\xi_\kappa}\big) & 0 \\ q e^{\xi_\kappa}\big(\kappa q - 2v_\kappa^2\big) - \kappa q^2 & 2q\big(1 - e^{u_\kappa^1}\big) \end{pmatrix} \mathcal{P}.$$

Then we have

$$\begin{cases} \mathcal{A}_\kappa(x_\kappa) = -\displaystyle\int_{\mathbf{R}^2} K_1(x_\kappa, y)\Big[a_{11}(|y|)\mathcal{A}_\kappa(y) + a_{12}(|y|)\mathcal{B}_\kappa(y)\Big]dy, \\ \mathcal{B}_\kappa(x_\kappa) = -\displaystyle\int_{\mathbf{R}^2} K_2(x_\kappa, y)\Big[a_{21}(|y|)\mathcal{A}_\kappa(y) + a_{22}(|y|)\mathcal{B}_\kappa(y)\Big]dy, \end{cases} \tag{3.5}$$

where $0 < K_i(x,y) \le Ce^{-\sqrt{-\lambda_i}|x-y|}$ for $|x-y| \ge 1$, $i = 1,2$. Since $u_\kappa^i(|x|) \to u^*(|x|)$, $v_\kappa^i(|x|) \to 0$ pointwise as $\kappa \to 0$ for $i = 1,2$ and $u^*(|x|)$ decay exponentially at ∞, by (3.5), we have

$$|\mathcal{A}_\kappa(x_\kappa)| + |\mathcal{B}_\kappa(x_\kappa)| \le o(1)\Big[\|\mathcal{A}_\kappa\|_{L^\infty} + \|\mathcal{B}_\kappa\|_{L^\infty}\Big] \ \text{ as } \ \kappa \to 0,$$

which contradicts to $\|\Phi_\kappa\|_{L^\infty} = 1$. Thus we complete this claim.

By claim, Φ_k and Ψ_k converges to Φ and Ψ (passing to a subsequence if necessary) in $C^2([0,\infty))$, respectively, where $(\Phi(r), \Psi(r))$ satisfies

$$\begin{cases} \Phi''(r) + \dfrac{1}{r}\Phi'(r) - 2qe^{u^*}\Phi(r) = 0 \ \text{ on } \ [0,\infty), \\ \Psi''(r) + \dfrac{1}{r}\Psi'(r) - 2qe^{u^*}\Psi(r) = 0 \ \text{ on } \ [0,\infty), \\ \Phi(0) = C_1, \ \ \Phi(0) = C_2, \ \ \Phi'(0) = \Psi'(0) = 0 \end{cases}$$

for some $C_1, C_2 \in \mathbf{R}$. Since $\Phi(r)$ and $\Psi(r)$ are bounded on $[0, \infty)$, we have $\Phi \equiv 0$ and $\Psi \equiv 0$. Then the proof of this theorem is completed. □

Lemma 3.1. *Suppose $(u(r), v(r))$ is a solution of Type (I) of (1.9) with respect to $(\kappa, q, \overline{N})$. Then*

$$|U'(r)| \leq \max\left\{2q\ , 2\overline{N}\right\} \ \ \textit{and}\ \ |v'(r)| \leq \max\left\{\kappa q^2 + \frac{2q}{\kappa}\ , \kappa q\overline{N}\right\}$$

for $r \in [0, \infty)$, where $u(r) = U(r) + 2\overline{N} \ln \frac{r^2}{1+r^2}$.

Proof. Without loss of generality, we only prove the case of $U'(r)$ and the case of $v'(r)$ is similarly. Since (u, v) is a topological solution of (1.9) with $(\kappa, q, \overline{N})$, we have

$$\int_0^\infty r\left[e^{u(r)} - 1 - \kappa v(r)\right] dr = -\frac{\overline{N}}{q} \ \text{ and } \ \int_0^\infty r e^{u(r)} v(r) dr = -\frac{\kappa\overline{N}}{2}. \tag{3.6}$$

Combining with Lemma 2.1 and (3.6), we obtain

$$-rU'(r) \leq \begin{cases} 2qr^2 \ \text{ for } r \in [0, 1], \\ 2q \displaystyle\int_0^r t\left[1 + \kappa v(t) - e^{u(t)}\right] dt \ \text{ for } r \in (1, \infty) \end{cases}$$

and

$$-rv'(r) \leq \begin{cases} \left(\kappa q^2 + \frac{2q}{\kappa}\right) r^2 \ \text{ for } r \in [0, 1], \\ \kappa q^2 \displaystyle\int_0^r t\left[e^{u(t)} - 1 - \kappa v(t)\right] dt - 2q \int_0^r t e^{u(t)} v(t) dt \ \text{ for } r \in (1, \infty). \end{cases}$$

Then we complete this result. □

Proposition 3.1. *Suppose $(u(x), v(x))$ is a topological solution of (1.8) with respect to $(\kappa, q, \overline{N})$. Then the linearized operator $L : W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ to $L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$ defined by*

$$L := \begin{pmatrix} \Delta - 2qe^u & 2\kappa q \\ qe^u(\kappa q - 2v) & \Delta - \left(\kappa^2 q^2 + 2qe^u\right) \end{pmatrix} \tag{3.7}$$

is invertible and L^{-1} is a linear bounded operator where $L_r^2(\mathbf{R}^2) = \{z(x) = z(r) | z \in L^2(\mathbf{R}^2)\}$ and $W_r^{2,2}(\mathbf{R}^2) = \{z(x) = z(r) | z, z', z'' \in L^2(\mathbf{R}^2)\}$.

Proof. First, we split the proof into two steps.

Step 1. L is an invertible operator from $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2) \to L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$. To prove it, we only check that the system

$$\begin{cases} \Delta\Phi - 2qe^u\Phi + 2\kappa q\Psi = f \ \text{ in } \ \mathbf{R}^2, \\ \Delta\Psi - \big(\kappa^2 q^2 + 2qe^u\big)\Psi + qe^u(\kappa q - 2v)\Phi = g \ \text{ in } \ \mathbf{R}^2 \end{cases} \tag{3.8}$$

is uniquely solvable in $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ for any pair $(f, g) \in L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$. For any pair (f, g), by Lemma 2.4, there exists one solution $(\Phi, \Psi) \in W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ at most. Hence, it suffices for us to show the existence of solutions. However, for any $R > 0$, the equation

$$\begin{cases} \Delta\Phi - 2qe^u\Phi + 2\kappa q\Psi = f \ \text{ in } \ B_R(O), \\ \Delta\Psi - \big(\kappa^2 q^2 + 2qe^u\big)\Psi + qe^u(\kappa q - 2v)\Phi = g \ \text{ in } \ B_R(O), \\ \Phi = \Psi = 0 \text{ on } \partial B_R(O). \end{cases} \tag{3.9}$$

has a solution (Φ_R, Ψ_R) from the Fredholm alternative theorem. Then by letting $R = R_m \to +\infty$, we want to claim that

$$(\Phi_m, \Psi_m) = \begin{cases} (\Phi_{R_m}, \Psi_{R_m}) & \text{for } x \in B_{R_m}(O), \\ \quad (0, 0) & \text{for } x \in \mathbf{R}^2 \setminus B_{R_m}(O) \end{cases}$$

has a convergent subsequence in $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$.

Proof of Claim. We want to show that

$$||\Phi_m||_{L^\infty(B_{R_m})} + ||\Psi_m||_{L^\infty(B_{R_m})} \le C\{||f||_{L^2(\mathbf{R}^2)} + ||g||_{L^2(\mathbf{R}^2)}\} \tag{3.10}$$

for some positive constant C independent of m. Introduce a new set variables $\mathcal{A}_m$ and $\mathcal{B}_m$ such that $\begin{pmatrix} \mathcal{A}_m \\ \mathcal{B}_m \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} \Phi_m \\ \Psi_m \end{pmatrix}$. From (3.8), we obtain

$$\begin{cases} \Delta\mathcal{A}_m - \lambda_1\mathcal{A}_m + a_{11}(x)\mathcal{A}_m(x) + a_{12}(x)\mathcal{B}_m(x) = \widehat{f} \ \text{ in } \ B_{R_m}(O), \\ \Delta\mathcal{B}_m - \lambda_2\mathcal{B}_m + a_{21}(x)\mathcal{A}_m(x) + a_{22}(x)\mathcal{B}_m(x) = \widehat{g} \ \text{ in } \ B_{R_m}(O), \\ \mathcal{A}_m = \mathcal{B}_m = 0 \text{ on } \partial B_{R_m}(O), \end{cases}$$

where $a_{jk}(x)$ are defined in Theorem 3.1, which satisfies $a_{jk}(x) = 0$ as $|x| = R_m$ for $j, k = 1, 2$ and $\begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix} = \mathcal{P}^{-1} \begin{pmatrix} f \\ g \end{pmatrix}$. To prove above claim, we only show that $\{\mathcal{A}_m\}$ and $\{\mathcal{B}_m\}$ are bounded. On the contrary, without loss of generality, one may assume that

$$\|\mathcal{A}_m\|_{L^\infty(\mathbf{R}^2)} = \max\left(\|\mathcal{A}_m\|_{L^\infty(\mathbf{R}^2)}, \|\mathcal{B}_m\|_{L^\infty(\mathbf{R}^2)}\right) \quad \text{and}$$
$$\|\mathcal{A}_m\|_{L^\infty(\mathbf{R}^2)} \to +\infty \quad \text{as} \quad m \to \infty.$$

Let $x_m \in \mathbf{R}^2$ be such that $|\mathcal{A}_m(x_m)| = \|\mathcal{A}_m\|_{L^\infty(\mathbf{R}^2)}$. Then we see that $\{x_m\}$ be bounded. By letting $\hat{A}_m = \frac{\Phi_m}{\|\Phi_m\|_{L^\infty(\mathbf{R}^2)}}$ and $\hat{B}_m = \frac{\Psi_m}{\|\Phi_m\|_{L^\infty(\mathbf{R}^2)}}$, then $(\hat{A}_m, \hat{B}_m)$ converges to $(\hat{A}, \hat{B})$ (passing to a subsequence if necessary) in $C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$, where $(\hat{A}(x), \hat{B}(x)) = (\hat{A}(|x|), \hat{B}(|x|))$ satisfies

$$\begin{cases} \Delta\hat{A} - 2qe^{u+u_0}\hat{A} + 2\kappa q\hat{B} = 0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta\hat{B} - \left(\kappa^2 q^2 + 2qe^{u+u_0}\right)\hat{B} + qe^{u+u_0}(\kappa q - 2v)\hat{A} = 0 \ \text{ in } \ \mathbf{R}^2, \end{cases}$$

and $\hat{A} \not\equiv 0$. Since $\hat{A}$ and $\hat{B}$ are bounded, by Lemma 2.4, we have $\hat{A} \equiv 0$ and $\hat{B} \equiv 0$, which contradicts $\hat{A} \equiv 0$. Thus (3.10) is established.

Combining with (3.10) and the facts that $|u|$ and $|v|$ decay exponentially at ∞, (Φ_m, Ψ_m) (passing to a subsequence if necessary) satisfies

$$\|\Phi_m\|_{W_r^{2,2}(\mathbf{R}^2)} + \|\Psi_m\|_{W_r^{2,2}(\mathbf{R}^2)} \le C\left[\sum_{i=1}^{2}\sum_{j=1}^{2}\|a_{ij}\|_{L^2(\mathbf{R}^2)} + 1\right]\left(\|f\|_{L^2(\mathbf{R}^2)} + \|g\|_{L^2(\mathbf{R}^2)}\right)$$

for some constant $C > 0$, and hence (Φ_m, Ψ_m) converges to (Φ, Ψ) in $L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$, which implies $\{(\Phi_m, \Psi_m)\}$ is a Cauchy sequence in $L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$. From the above argument, we obtain that

$$\begin{aligned} &\|\Phi_m - \Phi_l\|_{W_r^{2,2}(\mathbf{R}^2)} + \|\Psi_m - \Psi_l\|_{W_r^{2,2}(\mathbf{R}^2)} \\ &\quad \le C\sum_{j=1}^{2}\left[\|a_{1j}\|_{L^2(\mathbf{R}^2)}\|\Phi_m - \Phi_l\|_{L^2(\mathbf{R}^2)} + \|a_{2j}\|_{L^2(\mathbf{R}^2)}\|\Psi_m - \Psi_l\|_{L^2}\right] \end{aligned}$$

for some constant $C > 0$, and then $\{(\Phi_m, \Psi_m)\}$ is a Cauchy sequence in $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$. Hence (Φ_m, Ψ_m) converges to (Φ, Ψ) in $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$, where (Φ, Ψ) satisfies (3.8). Thus we complete this claim. This proves the linearized operator L is 1–1 and onto from $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$ to $L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$, and hence we finish this step.

Step 2. L^{-1} is a linear bounded operator from $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2) \to L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$. By the open mapping theorem, we know the inverse operator of L is bounded from $L_r^2(\mathbf{R}^2) \times L_r^2(\mathbf{R}^2)$ to $W_r^{2,2}(\mathbf{R}^2) \times W_r^{2,2}(\mathbf{R}^2)$.

Therefore, this proposition is proved by Step 1 and Step 2. □

Now we will give a uniqueness result of topological solutions with one vortex point as follows.

Theorem 3.2. *Let $\overline{N} \ge 0$ be given. Then (1.8) possesses a unique topological solution in $\mathbf{R}^2$ for any $(\kappa, q) \in (0, \infty) \times (0, \infty)$.*

Proof. From [14], we only need to consider the radial case. To show it, we define two functions u_0 and $\widetilde{f}$ by

$$u_0(x) = 2\overline{N}\ln\frac{|x|^2}{1+|x|^2} \text{ and } \widetilde{f}(x) = -\frac{4\overline{N}}{(1+|x|^2)^2}. \tag{3.11}$$

Let $(\hat{u}+u_0, v)$ be a topological solution of (1.8). Then $(\hat{u}, v)$ satisfies

$$\begin{cases} \Delta\hat{u} = 2q\Big(e^{\hat{u}+u_0} - \kappa v - 1\Big) - \widetilde{f} \text{ in } \mathbf{R}^2, \\ \Delta\hat{v} = -\kappa q^2\Big(e^{\hat{u}+u_0} - \kappa v - 1\Big) + 2qe^{\hat{u}+u_0}v \text{ in } \mathbf{R}^2, \\ \hat{u}(x) \to 0 \text{ and } \hat{v}(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

Then we split the proof into the following two steps.

Step 1. If (1.8) possesses only one topological solution for each triplet $(\kappa, q, \overline{N}) \in \mathbf{R}_+^2 \times \{\mathbf{R}_+ \cup \{0\}\}$, then there does not exist sequence $\{(\kappa_\tau, q_\tau, N_\tau)\}_{\tau\in\mathbf{N}}$ such that $(\kappa_\tau, q_\tau, N_\tau) \to (\kappa, q, \overline{N})$ as $\tau \to \infty$, and (1.8) possesses at least two topological solutions for any pair $(\kappa_\tau, q_\tau, N_\tau)$. To prove this result, we need the following fact.

Claim. *Suppose (u_τ, v_τ) be a topological solution of (1.8) with respect to $(\kappa_\tau, q_\tau, N_\tau)$ and $(U_\tau(r), v_\tau(r)) = \Big(u_\tau(r) - 2N_\tau \ln\frac{r^2}{1+r^2}, v_\tau(r)\Big)$. Then there exists a subsequence of $\big\{(U_\tau, v_\tau)\big\}$ such that it converges to (U, v) in $C^2\big(\overline{B_R(0)}\big) \times C^2\big(\overline{B_R(0)}\big)$ for any $R > 0$, where $(u(r), v(r)) = (U(r) + u_0(r), v(r))$ is a topological solution of (1.8) with respect to $(\kappa, q, \overline{N})$.*

Proof of Claim. To prove this result, we want to show that $\big\{U_\tau(1)\big\}$ *and* $\big\{v_\tau(1)\big\}$ *are both bounded.* Without loss of generality, we only prove the case $U_\tau(1)$ and another case $v_\tau(1)$ is easy to obtain by $v_\tau(0) \geq -\frac{1}{\kappa}$ and Lemma 2.1. By contradiction argument, we may assume that $U_\tau(1) \to -\infty$ as $\tau \to \infty$. From Lemma 3.1, $|U_\tau'(r)|$ is uniformly bounded in any bounded subinterval of $[0, \infty)$. Since $\big\{U_\tau(r) - U_\tau(1)\big\}$ is uniformly bounded on any bounded subinterval of $[0, \infty)$, we have

$$U_\tau(r) = U_\tau(1) + O(1) \text{ for } 0 \leq r \leq R_0,$$

for any $R_0 > 0$, which implies $U_\tau(r) \to -\infty$ on $[0, \infty)$ pointwise as $\tau \to \infty$. It follows that $v_\tau(r) \to -\frac{1}{\kappa}$ on $[0, \infty)$ pointwise as $\tau \to \infty$ due to $\int_0^\infty t\Big[e^{u_\tau(t)} - \kappa v_\tau(t) - 1\Big]dt = -\frac{N_\tau}{q}$. From (2.2), we get

$$\kappa\overline{N}^2 = \lim_{\tau\to\infty} 2q\int_0^\infty t\Big[2 - \kappa v_\tau(t) + 2e^{u_\tau(t)}v_\tau(t)\Big]v_\tau(t)\, dt = -\frac{6q}{\kappa}\int_0^\infty tdt = -\infty$$

and hence it yields a contradiction. Thus $\big\{U_\tau(1)\big\}$ is bounded.

Then we obtain that $(U_\tau(r), v_\tau(r))$ converges to $(U(r), v(r))$ in $C^2([0,R]) \times C^2([0,R])$ as $\tau \to \infty$ for all $R > 0$ (passing to a subsequence if necessary), where $(u(r), v(r))$ satisfies (1.9)–(1.10) with $(\kappa, q, \overline{N})$ and $u'(r), v'(r) > 0$ on $[0, \infty)$. It follows that

$$\begin{cases} \int_0^\infty r\Big[1+\kappa v(r) - e^{u(r)}\Big]\, dr \le \liminf_{\tau\to\infty} \int_0^\infty r\Big[1+\kappa v_\tau(r) - e^{u_\tau(r)}\Big] dr = \dfrac{\overline{N}}{q}, \\ -\int_0^\infty r e^{u(r)} v(r)\, dr \le \liminf_{k\to\infty} -\int_0^\infty r e^{u_\tau(r)} v_\tau(r) dr = \dfrac{\kappa \overline{N}}{2}, \end{cases}$$

which implies $\lim_{r\to\infty}(u(r), v(r)) = (0,0)$, i.e. (u,v) is a topological solution. In fact, because of the above relations and the monotonicity of $u(r)$ and $v(r)$, the case $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -1/\kappa)$ can be excluded. This completes this claim.

Now we come back to the proof in Step 1. Suppose (u_τ, v_τ) and (u'_τ, v'_τ) are two distinct topological solutions of (1.8) with respect to $(\kappa_\tau, q_\tau, N_\tau)$. By the above claim, (u_τ, v_τ) and (u'_τ, v'_τ) converge to (u,v) as $\tau \to \infty$ (passing to a subsequence if necessary), where (u,v) is a unique topological solution of (1.8) with respect to $(\kappa, q, \overline{N})$. Without loss of generality, we may assume that

$$|(u_\tau - u'_\tau)(x_\tau)| = \|u_\tau - u'_\tau\|_{L^\infty(\mathbf{R}^2)} \ge \|v_\tau - v'_\tau\|_{L^\infty(\mathbf{R}^2)} \ \forall \tau.$$

Set $\Phi_\tau = \dfrac{(u_\tau - u'_\tau)}{\|u_\tau - u'_\tau\|_{L^\infty(\mathbf{R}^2)}}$ and $\Psi_\tau = \dfrac{(v_\tau - v'_\tau)}{\|u_\tau - u'_\tau\|_{L^\infty(\mathbf{R}^2)}}$. Then (Φ_τ, Ψ_τ) satisfies

$$\begin{cases} \Delta\Phi_\tau - 2q_\tau e^{\xi_\tau}\Phi_\tau + 2\kappa q_\tau \Psi_\tau = 0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta\Psi_\tau - \big[\kappa_\tau^2 q_\tau^2 + 2q_\tau e^{u'_\tau}\big]\Psi_\tau + 2q_\tau e^{\xi_\tau}(\kappa_\tau q_\tau - 2v'_\tau)\Phi_\tau = 0 \ \text{ in } \ \mathbf{R}^2, \end{cases}$$

where $\xi_\tau(x) \in \big[\min\{u_\tau(x), u'_\tau(x)\}, \max\{u_\tau(x), u'_\tau(x)\}\big]$. Since (u_τ, v_τ) and (u'_τ, v'_τ) converge to (u,v) as $\tau \to \infty$, we obtain that the sequence of maximum points $\{x_\tau\}$ is bounded from Proposition 3.1. Thus Φ_τ and Ψ_τ converge to Φ and Ψ in $C^2(\mathbf{R}^2)$ (passing to a subsequence if necessary), respectively, where (Φ, Ψ) is radially symmetric with respect to the origin O and satisfies

$$\begin{cases} \Delta\Phi - 2qe^u\Phi + 2\kappa q\Psi = 0 \quad \text{in} \quad \mathbf{R}^2, \\ \Delta\Psi - \big(\kappa^2 q^2 + 2qe^u\big)\Psi + 2qe^u\big(\kappa q - 2v\big)\Phi = 0 \quad \text{in} \quad \mathbf{R}^2. \end{cases}$$

Since Φ and Ψ are bounded and not all zero in $\mathbf{R}^2$, we have $\Phi \equiv 0$ and $\Psi \equiv 0$ by Lemma 2.4, which yields a contradiction. Then we finish the proof of Step 1.

Step 2. Suppose (u_*, v_*) be a topological solution of (1.8) with respect to (κ', q', N'), where $(\kappa', q', N') \in \mathbf{R}_+^2 \times \{\mathbf{R}_+ \cup \{0\}\}$. Let $U_*(r) = u_*(r) - u_0(r)$. Then there exists *a neighborhood B of (κ', q', N') such that for any pair (κ, q, N) in B, there exists a corresponding (U, v) with respect to $(\kappa, q, \overline{N})$, which is close to (U_*, v_*) in $C^2(\overline{B_R(0)}) \times C^2(\overline{B_R(0)})$ for any $R > 0$, where $(u(r), v(r)) = (U(r) + u_0, v(r))$ is a topological solution of (1.8) with respect to $(\kappa, q, \overline{N})$.* Let $F : \{\mathbf{R}^2_+ \times \{\mathbf{R}_+ \cup \{0\}\}\} \times W^{2,2}_r(\mathbf{R}^2) \times W^{2,2}_r(\mathbf{R}^2) \to L^2_r(\mathbf{R}^2) \times L^2_r(\mathbf{R}^2)$ be a continuously differentiable mapping denoted by

$$F(\kappa, q, \overline{N}, \hat{u}, \hat{v}) = \begin{pmatrix} \Delta\hat{u} - 2q\big(-\kappa\hat{v} - 1 + e^{\hat{u}+u_0}\big) + f \\ \Delta\hat{v} + \kappa q^2\big(-\kappa\hat{v} - 1 + e^{\hat{u}+u_0}\big) - 2qe^{\hat{u}+u_0}\hat{v} \end{pmatrix}, \tag{3.12}$$

where f is defined in (3.11). It is clear that $F(\kappa', q', N', U_*, v_*) = (0, 0)^T$. Combining with Lemma 2.4 and Proposition 3.1, we obtain that $F_{(\hat{u},\hat{v})}(\kappa', q', N', U_*, v_*) = L(\hat{u}, \hat{v})$ is invertible and $F^{-1}_{(\hat{u},\hat{v})}(\kappa', q', N', U_*, v_*)$ be a linear bounded operator. Then we finish the proof of this step by Implicit Function Theorem.

Now we define the set $\sum = \{(\kappa, q, \overline{N}) :$ (1.8) possesses a unique topological solution for each pair $(\kappa, q, \overline{N})\}$. By [4], (1.8) possesses a topological solution for each pair $(\kappa, q, \overline{N}) \in \mathbf{R}^2_+ \times \{\mathbf{R}_+ \cup \{0\}\}$, which implies $\sum \subseteq \mathbf{R}^2_+ \times \{\mathbf{R}_+ \cup \{0\}\}$. It is easy to see that the set $\sum$ is nonempty from Theorem 3.1. Due to Step 2, we obtain that either $\partial\sum\bigcap\sum \neq \phi$ or $\sum = \mathbf{R}^2_+ \times \{\mathbf{R}_+ \cup \{0\}\}$. But $\partial\sum\bigcap\sum$ is an empty set by Step 1. Therefore, the proof of Theorem 3.2 was completely established. □

4. Structure of all entire solutions

In this section, we will study the structures of all radial solutions for (1.8). Applying the classifications, we will give the proof of Theorems 1.3.

Remark 4.1. (i) If there exists a constant $r_1 \geq 0$ such that

$$\begin{cases} u'(r_1) \geq 0 \text{ (resp., } \leq 0), \; e^{u(r_1)} - 1 - \kappa v(r_1) > 0 \text{ (resp., } < 0), \\ v'(r_1) \leq 0 \text{ (resp., } \geq 0), \; -\kappa q^2\big[e^{u(r_1)} - 1 - \kappa v(r_1)\big] + 2qe^{u(r_1)}v(r_1) < 0 \text{ (resp., } > 0), \end{cases}$$

then $(u(r), v(r))$ is a solution of type (V) (resp., (III)).

(ii) If $(\alpha_1, \alpha_2) \in \Lambda$, then $\alpha_2 < 0$ because $u(r)$ and $v(r)$ do not both attain to non-negative local maximums on $(0, \infty)$.

(iii) If $(\alpha_1, \alpha_2) \in \mathcal{B}$, then $\alpha_2 > -\frac{1}{\kappa}$ because $u(r)$ and $v(r)$ do not both attain to non-positive local minimums on $(0, R)$, where $\lim_{r\to\mathbf{R}^-}(u(r), (r)) = (\infty, \infty)$.

Lemma 4.1. *Consider the initial value problem (1.9)–(1.10). Then*

(a) *For all $\alpha_1 \in \mathbf{R}$ (resp., $\alpha_2 \in \mathbf{R}$), there exists a constant $\tilde{\alpha}_2 \in \mathbf{R}$ (resp., $\tilde{\alpha}_1 \in \mathbf{R}$) such that $(\alpha_1, \alpha_2) \in \mathcal{B}_v$ (resp., $(\alpha_1, \alpha_2) \in \mathcal{B}_u$) for all $\alpha_2 > \tilde{\alpha}_2$ (resp., $\alpha_1 > \tilde{\alpha}_1$).*
(b) *For all $\alpha_1 \in \mathbf{R}$ (resp., $\alpha_2 \in \mathbf{R}$), there exists a constant $\tilde{\alpha}_2 \in \mathbf{R}$ (resp., $\tilde{\alpha}_1 \in \mathbf{R}$) such that $(\alpha_1, \alpha_2) \in \mathcal{B}_u$ (resp., $(\alpha_1, \alpha_2) \in \mathcal{B}_v$) for all $\alpha_2 < -\frac{1}{\kappa}$ (resp., $\alpha_1 < \tilde{\alpha}_1$).*

(c) *If* $\overline{N} = 0$, *then* $(\alpha_1, \alpha_2) \in \mathcal{B}_v$ *(resp.,* $(\alpha_1, \alpha_2) \in \mathcal{B}_u$*) when* $\alpha_1 < 0$, $\alpha_2 > -\frac{1}{\kappa}$ *(resp.,* $\alpha_1 > 0$, $\alpha_2 < -\frac{1}{\kappa}$*).*

Proof. Let $\alpha_1 \in \mathbf{R}$ and $\hat{\alpha}_2 > 0$ be given. Then there exists a constant $r_0 > 0$ such that $r_0^{2\overline{N}} e^{2\alpha_1} < 1$. It is easy to see that if there exists a constant $\hat{r}(\alpha_2) > 0$ such that $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ satisfies

$$\left[e^{u(r;\alpha_1,\alpha_2)} - 1 - \kappa v(r; \alpha_1, \alpha_2)\right] < 0 \text{ and } v(r; \alpha_1, \alpha_2) > 0 \text{ on } (0, \hat{r}(\alpha_2)) \tag{4.1}$$

for each $\alpha_2 \geq \hat{\alpha}_2$, then we get

$$\begin{cases} rU'(r; \alpha_1, \alpha_2) = 2q \displaystyle\int_0^r t\big(e^{u(t;\alpha_1,\alpha_2)} - 1 - \kappa v(t; \alpha_1, \alpha_2)\big)dt < 0, \\ rv'(r; \alpha_1, \alpha_2) = q \displaystyle\int_0^r t\Big[2e^{u(t;\alpha_1,\alpha_2)}v(t; \alpha_1, \alpha_2) - \kappa q\big(e^{u(t;\alpha_1,\alpha_2)} - 1 - \kappa v(t; \alpha_1, \alpha_2)\big)\Big]dt \\ \qquad > 0 \end{cases} \tag{4.2}$$

for $r \leq \hat{r}(\alpha_2)$, where $U(r; \alpha_1, \alpha_2) = u(r; \alpha_1, \alpha_2) - 2\overline{N} \ln r$. Now we show that $e^{u(r;\alpha_1,\alpha_2)} - 1 - \kappa v(r; \alpha_1, \alpha_2) < 0$ *on* $[0, r_0]$ *for all* $\alpha_2 \geq \hat{\alpha}_2$. On the contrary, we may assume that there exist two positive constants $r_* \in (0, r_0)$ and $\alpha_2^* > \hat{\alpha}_2$ such that

$$e^{u(r_*;\alpha_1,\alpha_2^*)} - 1 - \kappa v(r_*; \alpha_1, \alpha_2^*) = 0 \text{ and } e^{u(r;\alpha_1,\alpha_2^*)} - 1 - \kappa v(r; \alpha_1, \alpha_2^*) < 0 \text{ on } [0, r_*).$$

Combining $r_0^{2\overline{N}} e^{2\alpha_1} < 1$, (4.1) and (4.2), we obtain that $v(r_*; \alpha_1, \alpha_2^*) > \alpha_2^* > 0$ and

$$e^{u(r_*;\alpha_1,\alpha_2^*)} - 1 - \kappa v(r_*; \alpha_1, \alpha_2^*) < r_0^{2\overline{N}} e^{2\alpha_1} - 1 < 0,$$

which yields a contradiction.

Hence, $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ satisfies (4.1) on $[0, r_0]$ for all $\alpha_2 \geq \hat{\alpha}_2$. It follows that there exists a constant $\widetilde{\alpha}_2 \geq \hat{\alpha}_2$ such that

$$\begin{cases} -\kappa v(r_0; \alpha_1, \alpha_2) - 1 + e^{u(r_0;\alpha_1,\alpha_2)} < 0, \\ r_0 u'(r_0; \alpha_1, \alpha_2) < 2\overline{N} + 2q \displaystyle\int_0^{r_0} t\big(-\kappa\widetilde{\alpha}_2 - 1 + t^{2\overline{N}} e^{2\alpha_1}\big)dt < 0 \end{cases}$$

for all $\alpha_2 \geq \widetilde{\alpha}_2$. Therefore, by Remark 4.1(i), $(\alpha_1, \alpha_2) \in \mathcal{B}_v$ for all $\alpha_2 \geq \widetilde{\alpha}_2$, and then (a) is established. The proofs of (b) and (c) are similar, and we omit the details. This lemma is proved. □

Lemma 4.2. *Let (u, v) be a solution of (1.9)–(1.10). Then (u, v) satisfies one of the type (I), type (II), type (III), type (IV), type (V), type (VI) and type (VII). Moreover, $v(r) \in [-1/\kappa, 0]$ on $[0, \infty)$, where $(u(r), v(r))$ is a solution of type (II).*

Proof. Note that both $u(r)$ and $v(r)$ do not attain non-negative local maximum and non-positive local minimum on $(0, \widehat{R})$ in the similar way as in the proof of Lemma 2.1, which implies that both $\lim_{r \to \widehat{R}} u(r) = C_u$ and $\lim_{r \to \widehat{R}} v(r) = C_v$ exist. To show this lemma, we divide the proof into two steps.

Step 1. If $\widehat{R} < \infty$, then $C_u = \infty$. Moreover, $(u(r), v(r))$ satisfies one of the type (V), type (VI) and type (VII). On the contrary, we may assume that $C_u \in [-\infty, \infty)$. If $C_u \in (-\infty, \infty)$, then $|C_v| = \infty$, which implies that $\lim_{r \to \widehat{R}} \left| \int_0^r v(s)ds \right| = \infty$ since $\Delta\left[u + \frac{2}{\kappa q} v\right] = \frac{4}{\kappa} e^u v$. Hence, we obtain that

$$C_u = u(r_0) + r_0 u'(r_0) \ln \frac{\widehat{R}}{r_0} + 2q \lim_{r \to \widehat{R}} \int_{r_0}^{r} s \ln \frac{r}{s} \left[e^{u(s)} - 1 - \kappa v(s)\right] ds = -\infty \text{ (resp., } \infty)$$

when $C_v = \infty$ (resp., $-\infty$). It is in contradiction to $C_u \in (-\infty, \infty)$.

If $C_u = -\infty$, then from (1.9), we have $\lim_{r \to \widehat{R}} \int_0^r v(s)ds = \infty$. It follows that $C_v = \infty$, which contradicts the facts that $\widehat{R} < \infty$ and $\Delta v \le \frac{\kappa^2 q^2}{2} v(r) + \kappa q^2$ on $[r_*, \widehat{R}]$ for some $r_* \in (0, \widehat{R})$. Therefore, we get that $C_u = \infty$. Besides, by the second equation of (1.9), it is easy to see that $C_v = \frac{\kappa q}{2}$ if $|C_v| < \infty$. We finish this step.

Step 2. If $\widehat{R} = \infty$, then $(u(r), v(r))$ satisfies one of the type (I), type (II), type (III) and type (IV). It is easy from (1.9) to see that $(C_u, C_v) = (0, 0)$ when $(C_u, C_v) \in \mathbf{R}^2$. We note that the following equation

$$\begin{cases} w''(r) + \dfrac{1}{r} w'(r) \ge C e^{w(r)} \quad \text{for} \quad r \ge R_0, \\ w(R_0) \ge 0, \ w'(R_0) > 0 \end{cases}$$

does not exist a solution for any positive constants C and R_0. Hence, if $C_u = \infty$, then $C_v = \infty$.

From the second equation of (1.9), we see that if $C_u = -\infty$ and $|C_v| < \infty$, then $C_v = -\frac{1}{\kappa}$. Now we show that if $C_u = -\infty$, then $C_v = \infty$ or $-\frac{1}{\kappa}$. On the contrary, we may assume that $C_v = -\infty$, which implies $\Delta u(r) \ge C$ on $[R, \infty)$ for some positive constants C and R. It is in contradiction to $C_u = -\infty$. Hence, we finish this step.

Finally, the proof of this lemma is complete by Steps 1 and 2. □

Lemma 4.3. *Let* $(\alpha_1^*, \alpha_2^*) \in \Lambda$. *Then* $(\alpha_1^* + \varepsilon, \alpha_2^*) \notin \Lambda$ $\big($*resp.,* $(\alpha_1^*, \alpha_2^* + \varepsilon) \notin \Lambda\big)$ *and* $(\alpha_1^* - \varepsilon, \alpha_2^*) \notin \Lambda$ $\big($*resp.,* $(\alpha_1^*, \alpha_2^* - \varepsilon) \notin \Lambda\big)$ *for all* $|\varepsilon| > 0$.

Proof. Without loss of generality, we consider the case: $\alpha_1^* + \varepsilon$. The proof for the case: $\alpha_2^* + \varepsilon$ can be given in a similar manner. On the contrary. We may assume that $(\alpha_1^*, \alpha_2^*) \in \Lambda$ and $(\alpha_1^* + \widetilde{\varepsilon}, \alpha_2^*) \in \Lambda$ for some $\widetilde{\varepsilon} > 0$. Since $u(r)$ and $v(r)$ do not both attain to non-positive local minimum on $(0, \infty)$, we have $v(r; \alpha_1^* + \widetilde{\varepsilon}, \alpha_2^*) \in [-1/\kappa, 0)$ and $v(r; \alpha_1^*, \alpha_2^*) \in [-1/\kappa, 0)$, which implies

$$(\alpha_1^* + \varepsilon, \alpha_2^*) \in \Lambda, \quad \varphi_1(r; \alpha_1^* + \varepsilon, \alpha_2^*) > 1 \ \text{ and } \ \psi_1(r; \alpha_1^* + \varepsilon, \alpha_2^*) < 0 \ \text{ on } \ (0, \infty)$$

for all $\varepsilon \in [0, \widetilde{\varepsilon}]$ by Lemma 2.3. Consequently, there exist two constants $r_0 > 0$ and $\delta \in (0, \widetilde{\varepsilon})$ such that

$$\psi_1(r; \alpha_1^* + \varepsilon, \alpha_2^*) \leq -1 \ \text{ on } \ (r_0, \infty) \quad \text{for all} \quad \varepsilon \in [0, \delta].$$

Then we have $0 = \lim\limits_{r\to\infty} \big[v(r; \alpha_1^*, \alpha_2^*) - v(r; \alpha_1^* + \delta, \alpha_2^*)\big] = -\delta \lim\limits_{r\to\infty} \psi_1(r; \zeta(r), \alpha_2^*) \geq \delta$, where $\zeta(r) \in [\alpha_1^*, \alpha_1^* + \delta]$. Thus it yields a contradiction and hence we complete the proof of this result. □

Now we prove Theorem 1.3 from Remark 4.1, Lemma 4.1 and Lemma 4.3.

Proof of Theorem 1.3. Combining with continuity and Remark 4.1, we see that $\mathcal{B}_u$ and $\mathcal{B}_v$ are open. Moreover, $\mathcal{B}_u$ and $\mathcal{B}_v$ are both non-empty subsets of $\mathbf{R}^2$ by Lemma 4.1. From Remark 4.1, Lemma 4.1 and Lemma 4.3, there exist a point (α_1^0, α_2^0) and a strictly increasing continuous function $\gamma : (-\infty, \infty) \to (-\infty, \infty)$ such that

$$T = \{(\alpha_1^0, \alpha_2^0)\} = \{(\alpha_1^0, \gamma(\alpha_1^0))\},$$
$$\Lambda = \{(\alpha_1, \gamma(\alpha_1)) : \alpha_1 < \alpha_1^0\} \ \text{ and } \ \{(\alpha_1, \gamma(\alpha_1)) : \alpha_1 > \alpha_1^0\} \subseteq \mathcal{B} \cup \mathcal{B}_\infty.$$

Now we want to show that $\lim\limits_{\alpha_1\to-\infty} \gamma(\alpha_1) = -\frac{1}{\kappa}$. Set $\widetilde{U}(r; \alpha_1) = u(r; \alpha_1, \gamma(\alpha_1)) - 2\overline{N} \ln r - \alpha_1$, and hence we obtain that $\widetilde{U}(r; \alpha_1)$ and $v(r; \alpha_1, \gamma(\alpha_1))$ converge to $\widetilde{U}(r)$ and $v(r)$ in $C^2([0, R])$ as $\alpha_1 \to -\infty$ for any $R > 0$, where

$$\begin{cases} \widetilde{U}''(r) + \dfrac{1}{r}\widetilde{U}'(r) = -2q[\kappa v(r) + 1] \ \text{ for } \ r \geq 0, \\ v''(r) + \dfrac{1}{r}v'(r) = \kappa q^2[\kappa v(r) + 1] \ \text{ for } \ r \geq 0, \\ \widetilde{U}(0) = 0, \ v(0) = \lim\limits_{\alpha_1\to-\infty} \gamma(\alpha_1), \ \widetilde{U}'(0) = v'(0) = 0. \end{cases}$$

By Lemma 4.2, it is easy to see that $\widetilde{U}(r) = 0$ and $v(r) = -\frac{1}{\kappa}$ on $[0, \infty)$, that is, $\lim\limits_{\alpha_1\to-\infty} \gamma(\alpha_1) = -\frac{1}{\kappa}$. Therefore, the proof of Theorem 1.3 is completed. □

5. Sharp range of flux

In this section, we will prove that the interval $(4\overline{N}+4,\infty)$ is a sharp region of the flux for non-topological solutions in (1.8). First, the following proposition will describe the sign of the nonlinear term.

Lemma 5.1. *Let $\mathcal{Z}=\{p_1,\dots,p_\ell\}$ and $(u(x),v(x))$ be a solution of (1.1). Then the following are valid.*

- (i) *If (u,v) is a topological solution of (1.8), then $u(x)<0$ in $\mathbf{R}^2\setminus\mathcal{Z}$ and $-\frac{1}{\kappa}\le v(x)<0$ in $\mathbf{R}^2$.*
- (ii) *If (u,v) is a non-topological solution of (1.8), then $u(x)<0$ in $\mathbf{R}^2\setminus\mathcal{Z}$ and $v(x)<0$ in $\mathbf{R}^2$.*

Proof. We note that all p_i's are not non-negative local maximum points of $v(x)$ due to $\Delta v(p_i)=\kappa q^2\big[1+\kappa v(p_i)\big]$ for $i=1,\dots,\ell$. To prove it, we divide the proof into three steps.

Step 1. $u(x)$ and $v(x)$ do not attain non-negative local maximums in $\mathbf{R}^2\setminus\mathcal{Z}$ and $\mathbf{R}^2$, respectively. Furthermore, one of $u(x)$ and $v(x)$ is negative in $\mathbf{R}^2\setminus\mathcal{Z}$. On the contrary, without loss of generality, we may assume that x_u and x_v are both non-negative maximum points of $u(x)$ and $v(x)$ in $\mathbf{R}^2\setminus\mathcal{Z}$, respectively. Then we see that

$$-\kappa v(x_u)-1+e^{u(x_u)}\le 0 \;\text{ and }\; \kappa q^2\Big[-\kappa v(x_v)-1+e^{u(x_v)}\Big]\ge 2qe^{u(x_v)}v(x_v)\ge 0,$$

which contradicts $-\kappa v(x_u)-1+e^{u(x_u)}\ge -\kappa v(x_v)-1+e^{u(x_v)}$ when $v(x_v)>0$. If $v(x_v)=0$, then $v(x_u)<0$ and $e^{u(x_u)}-1<0$, which contradicts $u(x_u)\ge 0$. Therefore, we finish this step.

Step 2. If $(u(x),v(x))$ is a topological solution or non-topological solution, then $u(x)$ and $v(x)$ are both negative in $\mathbf{R}^2\setminus\mathcal{Z}$ and $\mathbf{R}^2$, respectively. On the contrary, we first assume that $u(x)$ possesses a local maximum point $\hat{x}_u\in\mathbf{R}^2\setminus\mathcal{Z}$ such that $u(\hat{x}_u)\ge 0$ and $-\kappa v(\hat{x}_u)-1+e^{u(\hat{x}_u)}\le 0$. Hence, we have $\kappa v(\hat{x}_u)\ge e^{u(\hat{x}_u)}-1\ge 0$, which contradicts Step 1. If $v(x)$ possesses a local maximum point $\hat{x}_v\in\mathbf{R}^2$ such that

$$v(\hat{x}_v)\ge 0 \;\text{ and }\; \kappa q^2\big[e^{u(\hat{x}_v)}-1\big]\ge\Big[\kappa^2q^2+2qe^{u(\hat{x}_v)}\Big]v(\hat{x}_v),$$

then $u(\hat{x}_v)\ge 0$. However, it is also in contradiction to Step 1. This step is finished.

Step 3. If $(u(x),v(x))$ is a topological solution, then $-\frac{1}{\kappa}\le v(x)<0$ in $\mathbf{R}^2$. It is easy to see that one of $u(x)$ and $v(x)$ does not attain non-positive local minimum in $\mathbf{R}^2\setminus\mathcal{Z}$ from the similar proofs in Step 1 and Step 2. Then by $\lim_{|x|\to\infty}v(x)=0$, $v(x)$ possesses a unique local minimum point x_v^* in $\mathbf{R}^2$ such that $x_v^*\in\mathcal{Z}$ and $\Delta v(x_v^*)\ge 0$. Moreover, since $\Delta v(x_v^*)=\kappa q^2[1+\kappa v(x_v^*)]$, we have $v(x_v^*)\ge -\frac{1}{\kappa}$. Hence, we finish this step.

Consequently, the results (i) and (ii) are proved by Steps 1–3. □

Proposition 5.1. *Let* (u, v) *be a topological solution or non-topological solution of* (1.1). *Then we have*

$$1 + \kappa v(x) - e^{u(x)} > 0 \ \ in \ \ \mathbf{R}^2 \ \ for\ any\ (\kappa, q) \in [0, \infty) \times [0, \infty).$$

Proof. Let $W(x) = 1 + \kappa v(x) - e^{u(x)}$. Then $W(x) \to 0$ as $|x| \to \infty$ from the fact that (u, v) is a topological solution or non-topological solution of (1.1). On the contrary, we may assume that there exists a local minimum point $x_0 \in \mathbf{R}^2$ such that $W(x_0) \leq 0$, $\nabla W(x_0) = 0$ and $\Delta W(x_0) \geq 0$. Then by Lemma 5.1,

$$\Delta W(x_0) = \left[\kappa^2 q^2 + 2qe^{u(x_0)}\right] W(x_0) + 2\kappa q e^{u(x_0)} v(x_0) - \sum_{i=1}^{2} \left[\frac{\partial u}{\partial x_i}(x_0)\right]^2 e^{u(x_0)} < 0,$$

which contradicts $\Delta W(x_0) \geq 0$. Therefore, we complete this result. □

The estimation of the behavior for non-positive solutions (i.e. $u(x) \leq 0$, $v(x) \leq 0$ in $\mathbf{R}^2 \backslash \{O\}$) near infinite will be described as follows.

Lemma 5.2. *If* $(u(x), v(x))$ *is a non-positive solution of* (1.8) *which satisfies* $\Theta_1(u, v) < \infty$, $|\Theta_2(u, v)| < \infty$ *and* $v(x) \geq -\frac{1}{\kappa}$ *for* $x \in \mathbf{R}^2$, *then* $\Theta_2(u, v) = 0$, $\Theta_1(u, v) \in (2\overline{N} + 2, \infty) \cup \{2\overline{N}\}$ *and* $u(x)$ *can be proved that*

$$u(x) = \left[2\overline{N} - \Theta_1(u, v)\right] \ln |x| + C + o(1) \ \ near \ \ \infty \tag{5.1}$$

for some constant $C \in \mathbf{R}$. *Moreover,* $\Theta_2(u, v) = 0$ *and* $\Theta_1(u, v) \in (2\overline{N} + 2, \infty)$ *(resp.,* $\{2\overline{N}\}$*) if and only if* $(u(x), v(x))$ *satisfies* (1.5) *(resp.,* (1.4)*).*

Proof. Using the similar argument in [7] and Proposition 5.1, we get $\Theta_1(u, v) \in (2\overline{N} + 2, \infty) \cup \{2\overline{N}\}$ and

$$\frac{u(x)}{\ln |x|} \to 2\overline{N} - \Theta_1(u, v) \quad \text{as} \ \ |x| \to \infty.$$

Now we want to show that $\Theta_2(u, v) = 0$. Since $v(x) \in \left[-\frac{1}{\kappa}, 0\right]$ in $\mathbf{R}^2$, we have

$$\Theta_1(u, v) + \frac{2}{\kappa q} \Theta_2(u, v) = -\frac{2}{\kappa \pi} \int_{\mathbf{R}^2} e^{u(x)} v(x) dx \in (0, \infty). \tag{5.2}$$

Let $w(x) = u(x) + \frac{2}{\kappa q} v(x)$ and hence $w(x)$ satisfies $w(x) < 0$ in $\mathbf{R}^2 \ \backslash \{O\}$ and

$$\Delta w = \frac{4}{\kappa} e^u v + 4\pi \overline{N} \delta_O \text{ in } \mathbf{R}^2.$$

Then we set $\widetilde{w}(x) = w(x) - 2\overline{N}\ln|x|$ and hence $\widetilde{w}(x)$ satisfies $\Delta\widetilde{w} - \dfrac{4}{\kappa}e^u v = 0$ in $\mathbf{R}^2$. Define the potential

$$\widetilde{v}(x) = \frac{1}{2\pi}\int\limits_{\mathbf{R}^2} \ln\Big(\frac{|x-y|}{|y|}\Big)\Gamma(y)dy,$$

where $\Gamma(x) = -\frac{4}{\kappa}e^{u(x)}v(x)$. Then $\widetilde{v}(x)$ satisfies $\Delta\widetilde{v} + \dfrac{4}{\kappa}e^u v = 0$ in $\mathbf{R}^2$ and we see that

$$2\pi\widetilde{v}(x) \leq \Big(\int\limits_{B_{R_0}(O)} \Gamma(y)dy\Big)\ln|x| + C \tag{5.3}$$

where $R_0 \gg 1$, C is a constant independent of x and $x \in \mathbf{R}^2 \setminus B_{R_0}(O)$. Since $\widetilde{w}(x) \leq 0$ in $\mathbf{R}^2 \setminus B_{R_0}(O)$, $\widetilde{w}(x) + \widetilde{v}(x) \leq C(\ln|x| + 1)$ on $\{x : |x| \geq 1\}$ for some constant $C > 0$, which implies $\widetilde{w}(x) + \widetilde{v}(x)$ is a constant in $\mathbf{R}^2$ by Lemma 4.6.1 of [22]. It is not difficult to see that

$$\frac{\widetilde{v}(x)}{\ln|x|} \to \frac{1}{2\pi}\int\limits_{\mathbf{R}^2}\Gamma(x)dx \ \text{ as } \ |x| \to \infty,$$

which implies

$$\frac{w(x)}{\ln|x|} \to 2\overline{N} - \frac{1}{2\pi}\int\limits_{\mathbf{R}^2}\Gamma(x)dx \quad \text{as} \ \ |x| \to \infty,$$

that is,

$$\frac{u(x)}{\ln|x|} \to 2\overline{N} - \Big[\Theta_1(u,v) + \frac{2}{\kappa q}\Theta_2(u,v)\Big] \quad \text{as} \ \ |x| \to \infty.$$

Thus $\dfrac{v(x)}{\ln|x|} \to \Theta_2(u,v)$ as $|x| \to \infty$, which implies $\Theta_2(u,v) = 0$.

Now we let $(u(x), v(x))$ be a solution of (1.8) satisfying (1.5). Since $v(x) + \frac{1}{\kappa} = O\Big(e^{-\sqrt{\kappa q}|x|}\Big)$ as $|x| \to \infty$, $\displaystyle\int\limits_{|x|=R} \nabla v(x)\cdot\frac{x}{|x|}dx \to 0$ as $R \to \infty$ implies $\Theta_2(u,v) = 0$. By (5.1) and (5.2), we get $\Theta_1(u,v) \in (2\overline{N} + 2, \infty)$. Hence, the proof of this lemma is obtained. □

Proposition 5.2. *Let $(u(x), v(x))$ be a non-topological solution of (1.8). Then $\Theta_1(u,v) \in (4\overline{N} + 4, \infty)$ and $\Theta_2(u,v) = 0$.*

Proof. From Lemma 5.2, we have $\Theta_2(u,v) = 0$. By Pohozaev identity and (5.1), we obtain

$$
\begin{aligned}
&\frac{\kappa}{4}\Theta_1(u,v)\Big[\Theta_1(u,v)-(4\overline{N}+4)\Big] \\
&= -\kappa\Theta_1(u,v)+\frac{q}{\pi}\int\limits_{\mathbf{R}^2} v(x)\big[\kappa v(x)+2-2e^{u(x)}\big]+\frac{1}{\kappa}\,dx \\
&= \frac{q}{\pi}\int\limits_{\mathbf{R}^2}\Big[\kappa v^2(x)+2v(x)+\frac{1}{\kappa}\Big]\,dx \\
&> 0,
\end{aligned}
$$

which implies $\Theta_1(u,v) > 4\overline{N}+4$. Therefore, we complete this proof. □

Lemma 5.3. *If $(u(x), v(x))$ is a non-topological solution of (1.8) with $\overline{N} = 0$, then $(u(x), v(x))$ is radially symmetric at some point in $\mathbf{R}^2$.*

Proof. We will apply the method of moving plane with some modifications to prove this result. It suffices to prove u and v are decreasing when the point $x = (x_1, x_2)$ changes its position along the x_1-axis from the point $(-R, 0)$ to $(-\infty, 0)$. Define the sets $\Sigma_\sigma = \{x \in \mathbf{R}^2 : x_1 < \sigma\}$, $T_\sigma = \{x \in \mathbf{R}^2 : x_1 = \sigma\}$, and $u_\sigma(x) = u(x^\sigma)$, $v_\sigma(x) = v(x^\sigma)$ for $x \in \Sigma_\sigma$, where x^σ is the reflection of x with respect to the line $x_1 = \sigma$, i.e., $x^\sigma = (2\sigma - x_1, x_2)$.

Set $w_\sigma(x) = u(x) - u_\sigma(x)$ and $z_\sigma(x) = v(x) - v_\sigma(x)$ for $x \in \Sigma_\sigma$. Then w_σ and z_σ satisfy the following equations respectively.

$$
\begin{cases}
\Delta w_\sigma - 2q(e^u - e^{u_\sigma}) = \ -2q\kappa z_\sigma \quad \text{in } \ \Sigma_\sigma, \\
w_\sigma = 0 \text{ on } T_\sigma,
\end{cases} \tag{5.4}
$$

and

$$
\begin{cases}
\Delta z_\sigma - \big(2qe^{u_\sigma} + \kappa^2 q^2\big)z_\sigma = \ \big(2qv - \kappa q^2\big)(e^u - e^{u_\sigma}) \quad \text{in } \ \Sigma_\sigma, \\
z_\sigma = 0 \text{ on } T_\sigma.
\end{cases} \tag{5.5}
$$

Now we set $\widetilde{w}_\sigma(x) = \frac{w_\sigma(x)}{g(x)}$ for $x \in \Sigma_\sigma$, where $g(x) = |x_1|^{\frac{1}{2}} + \ln|x|$. Then $\big(\widetilde{w}_\sigma, z_\sigma\big)$ satisfies the following equation

$$
\begin{cases}
\Delta \widetilde{w}_\sigma + \frac{2}{g}\nabla g \cdot \nabla \widetilde{w}_\sigma + \Big(C_1(x) + \frac{\Delta g}{g}\Big)\widetilde{w}_\sigma = C_2(x) z_\sigma \quad \text{in } \ \Sigma_\sigma, \\
\widetilde{w}_\sigma = 0 \text{ on } T_\sigma,
\end{cases} \tag{5.6}
$$

where

$$
C_1(x) = -\frac{2q(e^{u(x)} - e^{u_\sigma(x)})}{u(x) - u_\sigma(x)}, \quad C_2(x) = \frac{-2\kappa q}{g(x)} \ \text{ and } \ \frac{\Delta g(x)}{g(x)} = -\frac{1}{4\big[x_1^2 + |x_1|^{\frac{3}{2}}\ln|x|\big]}.
$$

Define

$$S_u = \{\rho \in (-\infty, 0) : \widetilde{w}_\sigma < 0 \text{ in } \Sigma_\sigma \text{ for } \sigma \in (-\infty, \rho)\}, \quad \rho_u = \sup_{\rho \in S_u} \{\rho\},$$

and

$$S_v = \{\rho \in (-\infty, 0) : z_\sigma < 0 \text{ in } \Sigma_\sigma \text{ for } \sigma \in (-\infty, \rho)\}, \quad \rho_v = \sup_{\rho \in S_v} \{\rho\}.$$

Note that there exists a constant $R_0 > 1$ such that

$$\frac{\partial u}{\partial x_1}(x) > 0, \quad \text{and} \quad \Big(C_1(x) + \frac{\Delta g}{g}(x)\Big) < 0 \text{ for } |x| \geq R_0 \text{ and } x_1 < 0 \tag{5.7}$$

from Lemma 5.2 and Proposition 5.2. Now we divide the proof into the following steps.

Step 1. $S_u \neq \emptyset$ and $S_v \neq \emptyset$. Moreover, there exists a $\hat{R} \in [-R_0, 0)$ such that $\rho_u = \rho_v = \hat{\rho} \geq \hat{R}$. On the contrary, without loss of generality, we may assume that $S_u = \emptyset$, and hence for each $\rho < -R_0$ there exists a point $x_{\sigma(\rho)} \in \Sigma_{\sigma(\rho)}$ such that $|x_{\sigma(\rho)}| \to \infty$ as $\rho \to -\infty$ implies

$$\widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) = \max\{\widetilde{w}_{\sigma(\rho)}(x) : x \in \Sigma_{\sigma(\rho)}\} \geq 0, \ \nabla \widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) = (0,0) \text{ and } \Delta \widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) \leq 0. \tag{5.8}$$

Then $\Big[C_1(x_{\sigma(\rho)}) + \dfrac{\Delta g}{g}(x_{\sigma(\rho)})\Big]\widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}) \geq C_2(x_{\sigma(\rho)}) z_{\sigma(\rho)}(x_{\sigma(\rho)})$, which implies $z_{\sigma(\rho)}(x_{\sigma(\rho)}) \geq 0$. Consequently, for each $\rho < -R_0$ there exists a $x'_{\sigma(\rho)} \in \Sigma_{\sigma(\rho)}$ such that $|x'_{\sigma(\rho)}| \to \infty$ as $\rho \to -\infty$ implies

$$z_{\sigma(\rho)}(x'_{\sigma(\rho)}) = \max\{z_{\sigma(\rho)}(x) : x \in \Sigma_{\sigma(\rho)}\} \geq 0, \ \nabla z_{\sigma(\rho)}(x'_{\sigma(\rho)}) = (0,0) \text{ and } \Delta z_{\sigma(\rho)}(x'_{\sigma(\rho)}) \leq 0.$$

It follows that

$$\begin{aligned}
&-\big(2qe^{u_{\sigma(\rho)}(x'_{\sigma(\rho)})} + \kappa^2 q^2\big) z_{\sigma(\rho)}(x'_{\sigma(\rho)}) \\
&\geq \frac{\big(2qv(x'_{\sigma(\rho)}) - \kappa q^2\big)\big(e^{u(x'_{\sigma(\rho)})} - e^{u_{\sigma(\rho)}(x'_{\sigma(\rho)})}\big)}{\widetilde{w}_{\sigma(\rho)}(x'_{\sigma(\rho)})} \widetilde{w}_{\sigma(\rho)}(x'_{\sigma(\rho)}) \\
&\geq \frac{\big(2qv(x'_{\sigma(\rho)}) - \kappa q^2\big)\big(e^{u(x'_{\sigma(\rho)})} - e^{u_{\sigma(\rho)}(x'_{\sigma(\rho)})}\big)}{\widetilde{w}_{\sigma(\rho)}(x'_{\sigma(\rho)})} \widetilde{w}_{\sigma(\rho)}(x_{\sigma(\rho)}),
\end{aligned}$$

which implies $z_\sigma(x_{\sigma(\rho)}) = 0$ if $\widetilde{w}_\sigma(x'_{\sigma(\rho)}) = 0$ by (5.8). Thus we see that

$$\widetilde{w}_{\sigma(\rho)}(x) \leq 0 \text{ and } z_{\sigma(\rho)}(x) \leq 0 \text{ on } \Sigma_{\sigma(\rho)}.$$

Hence, $\widetilde{w}_{\sigma(\rho)}(x) = 0$ and $z_{\sigma(\rho)}(x) = 0$ on $\Sigma_{\sigma(\rho)} \cup \partial\Sigma_{\sigma(\rho)}$ by (5.5)–(5.6) and the maximum principle, which implies $\frac{\partial u(x)}{\partial x_1} + \frac{\partial u(x^{\sigma(\rho)})}{\partial(2\sigma(\rho)-x_1)} = \frac{\partial w_{\sigma(\rho)}(x)}{\partial x_1} = 0$ on $\Sigma_{\sigma(\rho)}$. It is in contradiction to (5.7). Then $\widetilde{w}_{\sigma(\rho)}(x'_{\sigma(\rho)}) > 0$, and by the similar argument, we also get $z_{\sigma(\rho)}(x_{\sigma(\rho)}) > 0$. Thus it follows that

$$\begin{aligned}&\left(2qe^{u_{\sigma(\rho)}(x'_{\sigma(\rho)})} + \kappa^2 q^2\right)\left[C_1(x_{\sigma(\rho)}) + \frac{\Delta g}{g}(x_{\sigma(\rho)})\right]\\ &\quad \geq C_2(x_{\sigma(\rho)})\left[2qv(x'_{\sigma(\rho)}) - \kappa q^2\right]\left[\frac{e^{u(x'_{\sigma(\rho)})} - e^{u_{\sigma(\rho)}(x'_{\sigma(\rho)})}}{\widetilde{w}_\sigma(x'_{\sigma(\rho)})}\right],\end{aligned}$$

which contradicts (5.7), and hence $S_u \neq \emptyset$. Moreover, it is easy to see that $\rho_u = \rho_v = \hat{\rho} \geq \hat{R}$ for some $\hat{R} \in [-R_0, 0)$ by (5.4)–(5.5) and the maximum principle. Therefore, we finish this step.

Step 2. $\frac{\partial u}{\partial x_1}(x) > 0$ and $\frac{\partial v}{\partial x_1}(x) > 0$ on $\Sigma_{\hat{\rho}}$. Moreover, $w_{\hat{\rho}} \equiv 0$ and $z_{\hat{\rho}} \equiv 0$ on $\Sigma_{\hat{\rho}}$. By Hopf lemma, we obtain that

$$\begin{aligned}\frac{\partial u}{\partial x_1}(x) + \frac{\partial u}{\partial(2\sigma - x_1)}(x^\sigma) &= \frac{\partial w_\sigma}{\partial x_1}(x) > 0 \text{ and}\\ \frac{\partial v}{\partial x_1}(x) + \frac{\partial v}{\partial(2\sigma - x_1)}(x^\sigma) &= \frac{\partial z_\sigma}{\partial x_1}(x) > 0 \text{ on } T_\sigma\end{aligned}$$

for all $\sigma < \hat{\rho}$, which implies $\frac{\partial u}{\partial x_1}(x) > 0$ and $\frac{\partial v}{\partial x_1}(x) > 0$ on $\Sigma_{\hat{\rho}}$. Next, we want show that $w_{\hat{\rho}} \equiv 0$ *and* $z_{\hat{\rho}} \equiv 0$ *on* $\Sigma_{\hat{\rho}}$. Suppose this is not true. Without loss of generality, we may assume $w_{\hat{\rho}} \not\equiv 0$ on $\Sigma_{\hat{\rho}}$, and hence $w_{\hat{\rho}}(x) < 0$ on $\Sigma_{\hat{\rho}}$ by the maximum principle, which implies

$$\frac{\partial \widetilde{w}_{\hat{\rho}}}{\partial x_1}(x) > 0 \text{ on } T_{\hat{\rho}} \tag{5.9}$$

from Hopf lemma. According to the definition of $\hat{\rho}$, there exists a positive sequence $\{\varepsilon_k\}_{k\in N}$ such that $\hat{\rho} + \varepsilon_k < 0$ and $(\hat{\rho} + \varepsilon_k) \to \hat{\rho}$ as $k \to \infty$ implies $\widetilde{w}_{\hat{\rho}+\varepsilon_k}$ is non-negative somewhere in $\Sigma_{\hat{\rho}+\varepsilon_k}$. By the way, we have $\widetilde{w}_{\hat{\rho}+\varepsilon_k}(x) \to 0$ as $|x| \to \infty$ and $\widetilde{w}_{\hat{\rho}+\varepsilon_k} = 0$ on $T_{\hat{\rho}+\varepsilon_k}$. Hence, for each ε_k there exists $x_k \in \Sigma_{\hat{\rho}+\varepsilon_k}$ such that

$$\widetilde{w}_{\hat{\rho}+\varepsilon_k}(x_k) \geq 0 \text{ and } \nabla\widetilde{w}_{\hat{\rho}+\varepsilon_k}(x_k) = (0, 0). \tag{5.10}$$

Since $|x_k| < R_0$ for all $k \in N$ due to Step 1, there exists a convergent subsequence, we still denote it by x_k, such that $x_k \to x_0$. By (5.10) we obtain that $0 \leq \lim_{k\to\infty} \widetilde{w}_{\hat{\rho}+\varepsilon_k}(x_k) = \widetilde{w}_{\hat{\rho}}(x_0)$. Hence, we conclude that $x_0 \in T_{\hat{\rho}}$, and by (5.10), it follows that $0 = \lim_{k\to\infty} \frac{\partial \widetilde{w}_{\hat{\rho}+\varepsilon_k}}{\partial x_1}(x_k) = \frac{\partial \widetilde{w}_{\hat{\rho}}}{\partial x_1}(x_0)$, which contradicts (5.9). Then we finish this step. Therefore, $u(x)$ and $v(x)$ are both radially symmetric with respect to some point in $\mathbf{R}^2$. □

Remark 5.1. The function $\beta_1(\alpha_1, \alpha_2)$ is continuous on the set Λ.

Proof of Theorem 1.2. To prove this theorem, we only check that the function $\beta_1(\alpha_1, \alpha_2)$ is surjective from the set Λ to $(4\overline{N} + 4, \infty)$. Then we divide the proof into following two steps.

Step 1. $\lim_{\alpha_1 \to -\infty} \beta_1(\alpha_1, \gamma(\alpha_1)) = 4\overline{N} + 4$. Set

$$\widetilde{U}(r; \alpha_1) = u(e^{-\frac{\alpha_1}{2\overline{N}+2}} r; \alpha_1, \gamma(\alpha_1)) - 2\overline{N} \ln e^{-\frac{\alpha_1}{2\overline{N}+2}} r - \alpha_1$$

and

$$\widetilde{V}(r; \alpha_1) = v(e^{-\frac{\alpha_1}{2\overline{N}+2}} r; \alpha_1, \gamma(\alpha_1)) - \gamma(\alpha_1).$$

Then we obtain that $\widetilde{U}(r; \alpha_1) + \frac{2}{\kappa q} \widetilde{V}(r; \alpha_1)$ converges to $\widetilde{U}(r)$ in $C^2([0, R])$ as $\alpha_1 \to -\infty$ for any $R > 0$, where

$$\begin{cases} \widetilde{U}''(r) + \dfrac{1}{r} \widetilde{U}'(r) = -\dfrac{4}{\kappa^2} r^{2\overline{N}} e^{\widetilde{U}(r)} \text{ on } [0, \infty), \\ \widetilde{U}(0) = 0, \ \widetilde{U}'(0) = 0. \end{cases}$$

Combining with $\left\{ r\Big[\widetilde{U}(r; \alpha_1) + \frac{2}{\kappa q} \widetilde{V}(r; \alpha_1)\Big]' \right\}' < 0$ on $(0, \infty)$ and $\lim_{r \to \infty} r\widetilde{U}'(r) = 4\overline{N} + 4$, we see that $\lim_{\alpha_1 \to -\infty} \beta_1(\alpha_1, \gamma(\alpha_1)) = \dfrac{4}{\kappa^2} \displaystyle\int_0^\infty r e^{\widetilde{U}(r)} dr = 4\overline{N} + 4$. Thus we finish this step.

Step 2. $\lim_{\alpha_1 \to \alpha_1^0} \beta_1(\alpha_1, \gamma(\alpha_1)) = \infty$. For any $R > 0$, we have

$$\beta_1(\alpha_1, \gamma(\alpha_1)) \Big[\beta_1(\alpha_1, \gamma(\alpha_1)) - (4\overline{N} + 4)\Big] \geq \frac{8q}{\kappa} \int_0^R t \Big[\frac{1}{\sqrt{\kappa}} + \sqrt{\kappa} v(t; \alpha_1, \gamma(\alpha_1))\Big]^2 dt,$$

which implies $\lim_{\alpha_1 \to \alpha_1^0} \beta_1(\alpha_1, \gamma(\alpha_1)) \Big[\beta_1(\alpha_1, \gamma(\alpha_1)) - (4\overline{N} + 4)\Big] \geq \dfrac{8q}{\kappa} \displaystyle\int_0^R t \Big[\frac{1}{\sqrt{\kappa}} + \sqrt{\kappa} v(t; \alpha_1^0, \alpha_2^0)\Big]^2 dt$. Then we obtain that $\lim_{\alpha \to \alpha_1^0} \beta_1(\alpha_1, \gamma(\alpha_1)) \Big[\beta_1(\alpha_1, \gamma(\alpha_1)) - (4\overline{N} + 4)\Big] = \infty$, i.e. $\lim_{\alpha \to \alpha_1^0} \beta_1(\alpha_1, \gamma(\alpha_1)) = \infty$. Hence, we finish this step.

Thus we complete this proof by Steps 1–2, Remark 5.1 and Proposition 5.2. $\square$

6. Uniqueness of topological multivortex solutions

In this section, we will prove Theorem 1.1 via the following three cases with respect to the parameters κ and q for any topological solution of (6.1). Throughout this section, we set $0 < d < \frac{1}{4}\min\left\{|p_i - p_j| : \ 1 \le i < j \le \ell\right\}$ and $(u(x), v(x)) = (\widetilde{u}(x), \kappa\widetilde{v}(x))$, where $(\widetilde{u}(x), \widetilde{v}(x))$ is a topological solution of (1.1). Then $(u(x), v(x))$ satisfies (1.4) and

$$\begin{cases} \Delta u = 2q\big(e^u - 1 - v\big) + 4\pi \sum_{i=1}^{\ell} n_i \delta_{p_i} \quad \text{in } \mathbf{R}^2, \\ \Delta v = -\kappa^2 q^2 \big(e^u - 1 - v\big) + 2qe^u v \quad \text{in } \mathbf{R}^2. \end{cases} \tag{6.1}$$

Remark 6.1. By Lemma 5.1, we see that if (u, v) is a topological solution of (6.1), then $-1 \le v(x) < 0$ and $u(x) < 0$ in $\mathbf{R}^2 \backslash \mathcal{Z}$, where $\mathcal{Z} = \{p_1, \ldots, p_\ell\}$.

First, we consider the case $\kappa^2 q \le \widetilde{L}$ and $q \to \infty$. We give two lemmas deriving priori estimates for any topological solution of (6.1) when q is large.

Lemma 6.1. *Let $(u_\varepsilon^L, v_\varepsilon^L)$ be a topological solution of (6.1), where $\kappa^2 q = L$ and $\varepsilon = \frac{1}{\sqrt{q}}$. For each $\widetilde{L} > 0$, there exists a constant $\varepsilon_0 = \varepsilon_0(d, \widetilde{L}) > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then for each $L \in [0, \widetilde{L}]$,*

$$\max\left\{\|u_\varepsilon^L\|_{C^2(\Omega_d^c)}, \|v_\varepsilon^L\|_{C^2(\Omega_d^c)}\right\} \le \exp[-\frac{c_0}{\varepsilon}] \tag{6.2}$$

for some positive constant c_0 depending on d and $\widetilde{L}$, where $\Omega_d = \cup_{\xi \in \mathcal{Z}} B_d(\xi)$ and $\Omega_d^c = \mathbf{R}^2 \setminus \Omega_d$.

Proof. Let $\widetilde{L} > 0$ be given. We divide the proof into the following three steps.

Step 1. For each compact subset $K \subset \mathbf{R}^2 \setminus \mathcal{Z}$, there are constants $\varepsilon_* > 0$ and $\gamma_0 = \gamma_0(K, \widetilde{L}) < 0$ such that $\gamma_0 \le u_\varepsilon^L(x)$, $v_\varepsilon^L(x) < 0$ in K for $0 < \varepsilon < \varepsilon_*$ and $0 \le L \le \widetilde{L}$, where $\mathcal{Z} = \{p_1, \ldots, p_\ell\}$. Let $U_\varepsilon^L(x) = u_\varepsilon^L(x) - \sum_{i=1}^{\ell} 2n_i \ln|x - p_i|$. Then it suffices to prove that if $\varepsilon > 0$ is sufficiently small, we can deduce that $\inf_{B_R(O)} U_\varepsilon^L(x) \ge \gamma_0$ for some $\gamma_0 = \gamma_0(R, \widetilde{L}) < 0$, where $R > \max_{i=1}^{\ell}\{|p_i|\} + 1$. On the contrary, without loss of generality, we may assume that there are sequences $\{\varepsilon_n\}$, $\{L_n\}$ and $\{x_n\} \subset B_{R_0}(O)$ such that

$$\varepsilon_n \to 0, \ \ L_n \to \hat{L} \ \text{ and } \ U_{\varepsilon_n}^{L_n}(x_n) = \inf_{B_{R_0}(O)} U_{\varepsilon_n}^{L_n}(x) \to -\infty \ \text{ as } \ n \to \infty$$

for some $\hat{L} \in [0, \widetilde{L}]$ and $R_0 > \max_{i=1}^{\ell}\{|p_i|\} + 1$. For simplicity, let $U_n = U_{\varepsilon_n}^{L_n}$ and $V_n = v_{\varepsilon_n}^{L_n}$. We decompose U_n and V_n as $U_n = U_{1n} + U_{2n}$ and $V_n = V_{1n} + V_{2n}$, where

$$\begin{cases} \Delta U_{1n} = \dfrac{2}{\varepsilon_n^2}\big(-v_n - 1 + e^{u_n}\big) \text{ in } B_R(O), \\ \Delta V_{1n} = -\dfrac{1}{\varepsilon_n^2}\Big[L_n\big(-v_n - 1 + e^{u_n}\big) - 2e^{u_n}v_n\Big] \text{ in } B_R(O), \\ U_{1n}(x) = V_{1n}(x) = 0 \ \text{ on } \partial B_R(O), \end{cases}$$

and

$$\begin{cases} \Delta U_{2n} = 0 \text{ in } B_R(O), \\ \Delta V_{2n} = 0 \text{ in } B_R(O), \\ U_{2n}(x) = U_n(x),\ V_{2n}(x) = V_n(x) \ \text{ on } \partial B_R(O) \end{cases}$$

for any $R \geq R_0$. Since $U_n(x_n) \to -\infty$ as $n \to \infty$ and $x_n \in B_R(O)$ for all $n \in N$, there exists a sequence $\{\hat{x}_n\}_{n\in N}$ such that $\hat{x}_n \in \partial B_R(O)$ for all $n \in N$ and $U_n(\hat{x}_n) \to -\infty$ as $n \to \infty$ from Proposition 5.1 and the maximum principle. Then $U_{2n}(x) < 0$ on $\overline{B_R(O)}$ for all $n \in N$ due to $U_n(x) < 0$ on $\partial B_R(O)$ for all $n \in N$ and the maximum principle. Hence, $U_{2n}(x)$ converges to $-\infty$ uniformly on $B_R(O)$ as $n \to \infty$ by Theorem 3 in [1] and the fact that $U_{2n}(\hat{x}_n) = U_n(\hat{x}_n)$.

Moreover, following the argument in the proof of Lemma 3.1 in [20] and the Poincaré inequality, we can verify that $\{U_{1n}\}$ is bounded in $W_0^{1,q}(B_R(O))$ for each $1 < q < 2$. By passing to a subsequence if necessary, we may assume that

$$U_{1n} \rightharpoonup U_\infty \text{ weakly in } W_0^{1,q}(B_R(O)),$$

and strongly in $L^p(B_R(O))$ for $1 \leq p < \frac{2q}{2-q}$. Consequently, $U_n \to -\infty$ almost everywhere on $B_R(O)$. Now, consider (6.1) with $(\varepsilon, \overline{N}) = (1, 0)$. According to the proof of Theorem 1.2, there exists a point $(\alpha_1^*, \alpha_2^*) \in \Lambda$ such that $\alpha_1^* < 0$, $\alpha_2^* < 0$ and

$$\beta_1(\alpha_1, \alpha_2) > 4\sum_{i=1}^{\ell_1} n_i \ \text{ for each } \ (\alpha_1, \alpha_2) \in \Lambda \cap [\alpha_1^*, 0) \times [\alpha_2^*, 0). \tag{6.3}$$

For each n, choose $y_n \in \mathbf{R}^2$ such that $u_n(y_n) = \alpha_1^*$, where $|y_n| = \max\{|x| : u_n(x) = \alpha_1^*\}$. Since $u_n \to -\infty$ almost everywhere on each compact subset of $\mathbf{R}^2$, $|y_n| \to \infty$ as $n \to \infty$. In fact, if $|y_n|$ is bounded, then there exist a constant $n_* > 0$ and a sequence $\{x_n^u\} \subseteq \mathbf{R}^2$ such that $|x_n^u| > |y_n|$ and $\Delta u_n(x_n^u) \geq 0$ (passing to a subsequence if necessary) for all $n \geq n_*$, which is in contradiction to $\Delta u_n(x^u) = \frac{2}{\varepsilon_n^2}\big[-v_n(x^u) - 1 + e^{u_n(x^u)}\big]$ and Proposition 5.1. Without loss of generality, let $\hat{u}_n(x) = u_n(\varepsilon_n x + y_n)$ and $\hat{v}_n(x) = v_n(\varepsilon_n x + y_n)$ for each $n \in N$. Then $\hat{u}_n(x)$ and $\hat{v}_n(x)$ satisfy

$$\begin{cases} \Delta \hat{u}_n = 2\big(-\hat{v}_n - 1 + e^{\hat{u}_n}\big) \text{ in } \Omega_n := \Big\{x : |x| < \dfrac{|y_n|}{2\varepsilon_n}\Big\}, \\ \Delta \hat{v}_n = -\Big[L_n\big(-\hat{v}_n - 1 + e^{\hat{u}_n}\big) - 2e^{\hat{u}_n}\hat{v}_n\Big] \text{ in } \Omega_n := \Big\{x : |x| < \dfrac{|y_n|}{2\varepsilon_n}\Big\}, \\ \displaystyle\int_{\Omega_n} \big(\hat{v}_n + 1 - e^{\hat{u}_n}\big)dx \le 2\pi \sum_{i=1}^{\ell} n_i, \ -\int_{\Omega_n} e^{\hat{u}_n}\hat{v}_n dx \le L_n\pi \sum_{i=1}^{\ell} n_i. \end{cases}$$

To finish this step, we need the following assertion.

Claim. *$\hat{u}_n$ is bounded in $C_{loc}(\mathbf{R}^2)$.*

Proof of Claim. Note that if $\hat{L} = 0$, then we see that $\hat{v}(x) = 0$ in $\mathbf{R}^2$. Since $\big\{\hat{u}_n + \frac{2}{L_n}\hat{v}_n\big\}$ is bounded in $C_{loc}(\mathbf{R}^2)$ by $\hat{u}_n(O) = \alpha_1^*$, Remark 6.1 and Theorem 3 in [1] for $L_n \in (0, \widetilde{L}]$, we get that $\{\hat{u}_n\}$ is bounded in $C_{loc}(\mathbf{R}^2)$ for any $\hat{L} \in (0, \widetilde{L}]$. Moreover, if $\hat{L} = 0$, then $\hat{v}_n \to 0$ in $C_{loc}(\mathbf{R}^2)$. By $\displaystyle\int_{\Omega_n} \big(\hat{v}_n + 1 - e^{\hat{u}_n}\big)dx \le 2\pi \sum_{i=1}^{\ell} n_i$ and $\hat{u}_n(O) = \alpha_1^*$, we obtain that $0 > \hat{u}_n(x) \ge \hat{u}^0(x)$ in Ω_n for large n, where $\hat{u}^0(O) < \alpha_1^*$ and $\Delta\hat{u}^0 = 2\big(e^{\hat{u}^0} - e^{\alpha_1^*}\big)$ in $\mathbf{R}^2$. We complete the proof of this claim.

From the above claim, we see that $(\hat{u}_n, \hat{v}_n)$ converges in $C^2_{loc}(\Omega_n) \times C^2_{loc}(\Omega_n)$ to some $(\hat{u}_*, \hat{v}_*)$, which verifies

$$\begin{cases} \Delta \hat{u}_* = 2\big(-\hat{v}_* - 1 + e^{\hat{u}_*}\big) \ \text{ in } \ \mathbf{R}^2, \\ \Delta \hat{v}_* = -\Big[\hat{L}\big(-\hat{v}_* - 1 + e^{\hat{u}_*}\big) - 2e^{\hat{u}_*}\hat{v}_*\Big] \ \text{ in } \ \mathbf{R}^2, \\ \displaystyle -\int_{\mathbf{R}^2} e^{\hat{u}_*}\hat{v}_* dx \le L^*\pi \sum_{i=1}^{\ell} n_i, \ \int_{\mathbf{R}^2} \big(\hat{v}_* + 1 - e^{\hat{u}_*}\big)dx \le 2\pi \sum_{i=1}^{\ell} n_i. \end{cases} \tag{6.4}$$

Due to $\big(\hat{u}_*(x), \hat{v}_*(x)\big) \in (-\infty, 0] \times \big[-\frac{1}{\kappa}, 0\big]$ in $\mathbf{R}^2$ and Lemma 5.2, $(\hat{u}_*, \hat{v}_*)$ is either a topological solution or non-topological solution of (6.1) with $(\varepsilon, \overline{N}) = (1, 0)$. Since $(0, 0)$ is the unique topological solution of (6.1) with $(\varepsilon, \overline{N}) = (1, 0)$ by Theorem 3.2, $(\hat{u}_*, \hat{v}_*)$ must be a non-topological solution of (6.1) with $(\varepsilon, \overline{N}) = (1, 0)$ from $\hat{u}_*(O) = \alpha_1^* < 0$. It deduces that, by Lemma 5.3, $\hat{u}_*$ and $\hat{v}_*$ are radially symmetric with respect to some point in $\mathbf{R}^2$, which contradicts (6.3). We finish this step.

Step 2. $(u^L_\varepsilon(x), v^L_\varepsilon(x)) \to (0, 0)$ *uniformly on* $\big\{\Omega^c_d \times [0, \widetilde{L}]\big\} \times \big\{\Omega^c_d \times [0, \widetilde{L}]\big\}$ *as* $\varepsilon \to 0^+$. *Moreover, if* $\varepsilon > 0$ *is sufficiently small, then*

$$\max\Big\{\|u^L_\varepsilon\|_{L^\infty(\Omega^c_d)}, \|v^L_\varepsilon\|_{L^\infty(\Omega^c_d)}\Big\} \le c_2 \exp[-\frac{c_3}{\varepsilon}] \tag{6.5}$$

for some constants $c_2(d, L), c_3(d, L) > 0$*, which are bounded on* $[0, \widetilde{L}]$. We note that for each $d > 0$, $\|u_\varepsilon^L\|_{L^\infty(\Omega_d^c)}$ and $\|v_\varepsilon^L\|_{L^\infty(\Omega_d^c)}$ are attained on $\partial\Omega_d$. Then it suffices to prove

$$\max\left\{\|u_\varepsilon^L\|_{L^\infty(\partial\Omega_d^c)}, \|v_\varepsilon^L\|_{L^\infty(\partial\Omega_d^c)}\right\} \leq c_2 \exp[-\frac{c_3}{\varepsilon}].$$

First, we show that

$$\max\left\{\|u_\varepsilon^L\|_{L^\infty(\partial\Omega_d^c)}, \|v_\varepsilon^L\|_{L^\infty(\partial\Omega_d^c)}\right\} \to 0 \ \textit{uniformly on}\ [0, \widetilde{L}] \ \text{as} \ \varepsilon \to 0. \tag{6.6}$$

On the contrary, we may assume that there exist a constant $\hat{L} \in [0, \widetilde{L}]$, sequences $\{x_n\}_{n\in\mathbf{N}} \subseteq \partial\Omega_d^c$ and $\{(\varepsilon_n, L_n)\}_{n\in\mathbf{N}}$ such that $(\varepsilon_n, L_n) \to (0, \hat{L})$ and

$$\left|\left(u_{\varepsilon_n}^{L_n}(x_n), v_{\varepsilon_n}^{L_n}(x_n)\right)\right| \to \delta_0 \ \text{as} \ n \to \infty \tag{6.7}$$

for some $\delta_0 > 0$. Since $\left\{\left(u_{\varepsilon_n}^{L_n}, v_{\varepsilon_n}^{L_n}\right)\right\}_{n\in\mathbf{N}}$ is bounded on $\Omega_{\frac{d}{2}}^c$ from Step 1, $(\hat{u}_n, \hat{v}_n)$ converges to (u^*, v^*) in $C_{loc}^2(\mathbf{R}^2) \times C_{loc}^2(\mathbf{R}^2)$ (passing to a subsequence if necessary) as $n \to \infty$, where $\left(\hat{u}_n(x), \hat{v}_n(x)\right) = \left(u_{\varepsilon_n}^{L_n}(\varepsilon_n x + x_n), \hat{v}_n(x) = v_{\varepsilon_n}^{L_n}(\varepsilon_n x + x_n)\right)$ for $|x| \leq \frac{d}{2\varepsilon_n}$ and (u^*, v^*) satisfies (6.4). Combining Lemma 5.2 and the fact that (u^*, v^*) is non-positive and bounded in $\mathbf{R}^2$, we get $(u^*, v^*) = (0, 0)$, which contradicts (6.7). Hence, we finish this result.

Let $d > 0$ and $\delta > 0$ be given such that the matrix $\mathcal{M}_\delta = \mathcal{M}_\delta(L)$ possesses two negative eigenvalues $\hat{\lambda}_1 = \hat{\lambda}_1(L, \delta)$ and $\hat{\lambda}_2 = \hat{\lambda}_2(L, \delta)$ with $2e^{-\delta} + \hat{\lambda}_1 > 0 > 2e^{-\delta} + \hat{\lambda}_2$ (resp., $2e^{-\delta} + \hat{\lambda}_1 = 0 = 2e^{-\delta} + \hat{\lambda}_2$) for $L > 0$ (resp., $L = 0$), where

$$\mathcal{M}_\delta = \begin{pmatrix} -2e^{-\delta} & 2 \\ L & -2e^{-\delta} - L \end{pmatrix}.$$

From (6.6), there exists a constant $\varepsilon_* > 0$ such that

$$\max\left\{\|u_\varepsilon^L\|_{L^\infty(\Omega_d^c)},\ \|v_\varepsilon^L\|_{L^\infty(\Omega_d^c)}\right\} < \delta \ \text{for} \ 0 < \varepsilon < \varepsilon_* \ \text{and} \ 0 \leq L \leq \widetilde{L}.$$

Then

$$\begin{cases} \varepsilon^2 \Delta u_\varepsilon - 2e^{-\delta} u_\varepsilon + 2v_\varepsilon \leq 0 & \text{on} \quad \Omega_{\frac{d}{2}}^c, \\ \varepsilon^2 \Delta v_\varepsilon + L u_\varepsilon - (2e^{-\delta} + L) v_\varepsilon \leq 0 & \text{on} \quad \Omega_{\frac{d}{2}}^c. \end{cases} \tag{6.8}$$

Let $\mathcal{P}_\delta$ be a 2×2 matrix such that $\mathcal{P}_\delta^{-1} \mathcal{M}_\delta \mathcal{P}_\delta = \Lambda_\delta$ and $\begin{pmatrix} \mathcal{A}_{\varepsilon,L} \\ \mathcal{B}_{\varepsilon,L} \end{pmatrix} = \mathcal{P}_\delta^{-1} \begin{pmatrix} u_\varepsilon^L \\ v_\varepsilon^L \end{pmatrix}$, where

$$\Lambda_\delta = \begin{pmatrix} \hat{\lambda}_1 & 0 \\ 0 & \hat{\lambda}_2 \end{pmatrix}, \ \mathcal{P}_\delta = \begin{pmatrix} 1 & 1 \\ \frac{2e^{-\delta} + \hat{\lambda}_1}{2} & \frac{2e^{-\delta} + \hat{\lambda}_2}{2} \end{pmatrix} \ \text{and} \ \mathcal{P}_\delta^{-1} = \begin{pmatrix} + & + \\ + & - \end{pmatrix}$$

(resp., $\mathcal{P}_\delta^{-1} = \begin{pmatrix} + & 0 \\ + & + \end{pmatrix}$) if $L > 0$ (resp., $L = 0$). Then we obtain that

$$-\varepsilon^2 \Delta \mathcal{A}_{\varepsilon,L} - \hat{\lambda}_1 \mathcal{A}_{\varepsilon,L} \geq 0 \ \text{ on } \ \Omega^c_{d/2}.$$

For each $x_0 \in \partial\Omega^c_d$, we define a comparison function $w_{\varepsilon,L}$ by

$$w_{\varepsilon,L}(x) = (1+m)\exp\left[\frac{\sqrt{-\hat{\lambda}_1}}{2d\varepsilon}\Big(|x-x_0|^2 - \frac{d^2}{4}\Big)\right] \ \text{ on } \ B_{\frac{d}{2}}(x_0),$$

where $m = 2\|\mathcal{P}_\delta^{-1}\|\delta$. It is easy to check that if $\varepsilon > 0$ is sufficiently small, then

$$-\varepsilon^2\Delta(\mathcal{A}_{\varepsilon,L} + w_{\varepsilon,L}) - \hat{\lambda}_1(\mathcal{A}_{\varepsilon,L} + w_{\varepsilon,L}) > 0 \ \text{ on } \ B_{\frac{d}{2}}(x_0).$$

Moreover, the maximum principle implies that

$$(\mathcal{A}_{\varepsilon,L} + w_{\varepsilon,L})(x) > (\mathcal{A}_{\varepsilon,L} + w_{\varepsilon,L})|_{\partial B_{d/2}(x_0)} > 0$$

for $x \in B_{d/2}(x_0)$. In particular, there is a constant $C = C(d, \widetilde{L}) > 0$ such that

$$\max\Big\{|u^L_\varepsilon(x)|, |v^L_\varepsilon(x)|\Big\} < \exp\Big[-\frac{C}{\varepsilon}\Big] \ \text{ for } \ x \in B_{d/4}(x_0) \ \text{ and } \ L \in [0, \widetilde{L}].$$

Since the positive constant C is independent on the choice of $x_0 \in \partial\Omega_d$, (6.5) immediately follows. This step is finished.

Step 3. We now prove (6.2). From Step 2, we have $\max\{\|u^L_\varepsilon\|_{L^\infty(\Omega^c_d)}, \|v^L_\varepsilon\|_{L^\infty(\Omega^c_d)}\} \leq c_2 \exp[-\frac{c_3}{\varepsilon}]$ for all $L \in [0, \widetilde{L}]$ and

$$\begin{cases} \varepsilon^2 \Delta u^L_\varepsilon = O(1)\big(|u^L_\varepsilon|, |v^L_\varepsilon|\big) \ \text{ in } \Omega^c_{\frac{d}{2}}, \\ \varepsilon^2 \Delta v^L_\varepsilon = O(1)\big(|u^L_\varepsilon|, |v^L_\varepsilon|\big) \ \text{ in } \Omega^c_{\frac{d}{2}} \end{cases}$$

for all $L \in [0, \widetilde{L}]$. Then Lemma 6.1 is an immediate consequence by applying the standard elliptic estimates. □

We now investigate the asymptotic behaviors of u^L_ε and v^L_ε in each ball $B_d(\xi)$. For each $\xi \in \mathcal{Z}$, let

$$\hat{u}^L_{\varepsilon,\xi}(x) = u^L_\varepsilon(\varepsilon x + \xi) \ \text{ and } \ \hat{v}^L_{\varepsilon,\xi}(x) = v^L_\varepsilon(\varepsilon x + \xi) \ \text{ for } \ |x| \leq \frac{2d}{\varepsilon}.$$

Then $\hat{u}^L_{\varepsilon,\xi}(x)$ and $\hat{v}^L_{\varepsilon,\xi}(x)$ satisfy

$$\begin{cases} \Delta \hat{u}^L_{\varepsilon,\xi} + 2\big(1 + \hat{v}^L_{\varepsilon,\xi} - e^{\hat{u}^L_{\varepsilon,\xi}}\big) = 4\pi\lambda_\xi\delta_0 \ \text{ for } \ |x| \le \dfrac{2d}{\varepsilon}, \\ \Delta \hat{v}^L_{\varepsilon,\xi} - L\big(1 + \hat{v}^L_{\varepsilon,\xi} - e^{\hat{u}^L_{\varepsilon,\xi}}\big) - 2e^{\hat{u}^L_{\varepsilon,\xi}}\hat{v}^L_{\varepsilon,\xi} = 0 \ \text{ for } \ |x| \le \dfrac{2d}{\varepsilon}, \\ |\hat{u}^L_{\varepsilon,\xi}(x)| = O(e^{-\frac{c}{\varepsilon}}), \ \ |\hat{v}^L_{\varepsilon,\xi}(x)| = O(e^{-\frac{c}{\varepsilon}}) \ \text{ for } \ |x| = \frac{2d}{\varepsilon}, \end{cases}$$

where $c > 0$ and $(\xi, \lambda_\xi) \in \big\{(p_i, n_i) : i = 1, \dots, \ell\big\}$. We have the following property by the Pohozaev identity.

Proposition 6.1. *Let $(u^L_\varepsilon, v^L_\varepsilon)$ be a topological solution of (6.1). Then the following identity holds*

$$\int\limits_{|y| \le \frac{2d}{\varepsilon}} \Big\{2e^{\hat{u}^L_{\varepsilon,\xi}(y)} - 2 - \hat{v}^L_{\varepsilon,\xi}(y)\Big\}\hat{v}^L_{\varepsilon,\xi}(y)dy = 2\pi L\lambda^2_\xi + o(e^{-\frac{c}{\varepsilon}}) \ \text{ as } \ \varepsilon \to 0$$

for some $c > 0$.

Proof. The proof is standard by the Pohozaev identity and we omit the details here. Indeed, the Pohozaev identity for topological multivortex solutions is derived in Proposition 6.2 mentioned later. □

Lemma 6.2. *Let $\big(\phi^L_{\lambda_\xi}(r), \bar{\phi}^L_{\lambda_\xi}(r)\big)$ be a radial topological solution of the following equations with $L \ge 0$*

$$\begin{cases} \Delta \phi^L_{\lambda_\xi} + 2\big(1 + \bar{\phi}^L_{\lambda_\xi} - e^{\phi^L_{\lambda_\xi}}\big) = 4\pi\lambda_\xi\delta_O \ \ in \ \ \mathbf{R}^2, \\ \Delta \bar{\phi}^L_{\lambda_\xi} - L\big(1 + \bar{\phi}^L_{\lambda_\xi} - e^{\phi^L_{\lambda_\xi}}\big) - 2e^{\phi^L_{\lambda_\xi}}\bar{\phi}^L_{\lambda_\xi} = 0 \ \ in \ \ \mathbf{R}^2. \end{cases} \tag{6.9}$$

Then for each pair $(d, \widetilde{L}) \in (0, \infty) \times (0, \infty)$,

$$\max\Big\{\big\|\hat{u}^L_{\varepsilon,\xi} - \phi^L_{\lambda_\xi}\big\|_{C^2(|x| \le \frac{d}{\varepsilon})}, \big\|\hat{v}^L_{\varepsilon,\xi} - \bar{\phi}^L_{\lambda_\xi}\big\|_{C^2(|x| \le \frac{d}{\varepsilon})}\Big\} \to 0$$
$$\textit{uniformly on } [0, \widetilde{L}] \ \textit{ as } \ \varepsilon \to 0^+. \tag{6.10}$$

Proof. From Lemma 6.1, if $\varepsilon > 0$ is sufficiently small, then there are positive constants c_0 and c_1 such that

$$\max_{L \in [0,\widetilde{L}]} \Big\{\|\hat{u}^L_{\varepsilon,\xi}\|_{L^\infty(\frac{d}{\varepsilon} \le |x| \le \frac{2d}{\varepsilon})}, \|\hat{v}^L_{\varepsilon,\xi}\|_{L^\infty(\frac{d}{\varepsilon} \le |x| \le \frac{2d}{\varepsilon})}\Big\} \le c_0 e^{-\frac{c_1}{\varepsilon}}. \tag{6.11}$$

The proof of this lemma will be offered by the following two steps.

Step 1. $\hat{u}^L_{\varepsilon,\xi} - \phi^L_{\lambda_\xi} \to 0$ and $\hat{v}^L_{\varepsilon,\xi} - \bar{\phi}^L_{\lambda_\xi} \to 0$ uniformly on $[0, \widetilde{L}]$ in $C^2_{loc}(\mathbf{R}^2)$ as $\varepsilon \to 0$. It is easy to get

$$\Delta\Big(\hat{u}^L_{\varepsilon,\xi} + \frac{2}{L}\hat{v}^L_{\varepsilon,\xi}\Big) - 2e^{\hat{u}^L_{\varepsilon,\xi}} \frac{\frac{2}{L}\hat{v}^L_{\varepsilon,\xi}}{\hat{u}^L_{\varepsilon,\xi} + \frac{2}{L}\hat{v}^L_{\varepsilon,\xi}} \Big(\hat{u}^L_{\varepsilon,\xi} + \frac{2}{L}\hat{v}^L_{\varepsilon,\xi}\Big) = 0 \ \text{ on } \ B_{\frac{2d}{\varepsilon}}(O) \setminus \{O\}.$$

Indeed, since $\hat{u}^L_{\varepsilon,\xi}(x) < 0$ and $-1 \le \hat{v}^L_{\varepsilon,\xi}(x) < 0$ in $\mathbf{R}^2\backslash\{O\}$, it follows from the Harnack inequality that either $\big\{\hat{U}^L_{\varepsilon,\xi} + \frac{2}{L}\hat{v}^L_{\varepsilon,\xi}\big\}$ is bounded in $C_{loc}\big(\mathbf{R}^2\backslash\{O\}\big)$ or $\hat{U}^L_{\varepsilon,\xi}(x) + \frac{2}{L}\hat{v}^L_{\varepsilon,\xi} \to -\infty$ uniformly on any compact subset of $\mathbf{R}^2\backslash\{O\}$, where $\hat{U}^L_{\varepsilon,\xi}(x) = \hat{u}^L_{\varepsilon,\xi}(x) - 2\lambda_\xi \ln|x|$. Now we want to show that $\big\{\hat{U}^L_{\varepsilon,\xi}\big\}$ *is bounded in* $C_{loc}\big(\mathbf{R}^2\backslash B_2(O)\big)$. On the contrary, we may assume that there exist a constant $\hat{L} \in [0, \widetilde{L}]$ and two sequences $\{\varepsilon_n\}$, $\{L_n\}$ such that $(\varepsilon_n, L_n) \to (0, \hat{L})$ and $\big\{\hat{U}^{L_n}_{\varepsilon_n,\xi}\big\}$ is not bounded in $C_{loc}\big(\mathbf{R}^2\backslash B_2(O)\big)$. First, we consider $\hat{L} \neq 0$ and $\hat{U}^{L_n}_{\varepsilon_n,\xi}(x) + \frac{2}{L_n}\hat{v}^{L_n}_{\varepsilon_n,\xi}(x) \to -\infty$ uniformly on any compact subset of $\mathbf{R}^2\backslash\{O\}$ as $n \to \infty$. By (6.11), we have

$$\begin{aligned}
\int\limits_{|y|\le R} \big(1 + \lim_{n\to\infty} \hat{v}^{L_n}_{\varepsilon_n,\xi}\big)dy &= \lim_{n\to\infty} \int\limits_{|y|\le R} \big(1 + \hat{v}^{L_n}_{\varepsilon_n,\xi} - e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}}\big)dy \\
&\le \frac{2}{\hat{L}} \lim_{n\to\infty} - \int\limits_{|y|\le \frac{d}{\varepsilon_n}} e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}} \hat{v}^{L_n}_{\varepsilon_n,\xi} dy \\
&= 0
\end{aligned}$$

for each $R > 0$, which implies $\hat{v}^{L_n}_{\varepsilon_n,\xi}(x) \to -1$ as $n \to \infty$ in $\mathbf{R}^2$ almost everywhere. It contradicts Proposition 6.1. Hence, $\Big\{\hat{U}^L_{\varepsilon,\xi}\Big\}$ is bounded in $C_{loc}(\mathbf{R}^2\backslash\{O\})$. Secondly, let $\hat{L} = 0$. Since $\hat{U}^{L_n}_{\varepsilon_n,\xi}(x) < -2\lambda_\xi \ln 2$ on $B_{\frac{2d}{\varepsilon_n}}(O)\backslash B_2(O)$ for all $n \in N$ due to $\hat{u}^{L_n}_{\varepsilon_n,\xi}(x) < 0$ in $B_{\frac{2d}{\varepsilon_n}}(O)\backslash\{O\}$, $\Big\{\hat{U}^{L_n}_{\varepsilon_n,\xi}\Big\}$ is either bounded in $C_{loc}\big(\mathbf{R}^2\backslash B_2(O)\big)$ or $\hat{U}^{L_n}_{\varepsilon_n,\xi}(x) \to -\infty$ uniformly on any compact subset of $\mathbf{R}^2\backslash B_2(O)$ as $n \to \infty$ from the Harnack inequality, Proposition 5.1 and $\Delta\hat{U}^{L_n}_{\varepsilon_n,\xi} + 2\left(\dfrac{1 + \hat{v}^{L_n}_{\varepsilon_n,\xi} - e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}}}{\hat{U}^{L_n}_{\varepsilon_n,\xi}}\right)\hat{U}^{L_n}_{\varepsilon_n,\xi} = 0$ in $B_{\frac{2d}{\varepsilon_n}}(O)\backslash B_2(O)$. However, if $\hat{U}^{L_n}_{\varepsilon_n,\xi}(x) \to -\infty$ uniformly on any compact subset of $\mathbf{R}^2\backslash B_2(O)$ as $n \to \infty$, then by Proposition 6.1, we have

$$\begin{aligned}
\infty &= \int\limits_{\mathbf{R}^2} \liminf_{n\to\infty} \big(1 - e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}}\big)\chi_{\{|y|\le \frac{2d}{\varepsilon_n}\}} dy \\
&\le \liminf_{n\to\infty} \int\limits_{|y|\le \frac{2d}{\varepsilon_n}} \big(1 - e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}}\big)dy \\
&= 2\pi\lambda_\xi + \liminf_{n\to\infty} - \int\limits_{|y|\le \frac{2d}{\varepsilon_n}} \hat{v}^{L_n}_{\varepsilon_n,\xi} dy \\
&\le 2\pi\lambda_\xi(1 + \widetilde{L}\lambda_\xi),
\end{aligned}$$

which yields a contradiction.

Consequently, $\big\{\hat{U}^L_{\varepsilon,\xi}\big\}$ is bounded in $C^2_{loc}(\mathbf{R}^2 \setminus B_2(O))$ from (6.1) and the fact that $\big\{\hat{U}^{L_n}_{\varepsilon_n,\xi}\big\}$ is bounded in $C_{loc}(\mathbf{R}^2 \setminus B_2(O))$. Moreover, for each $x \in B_3(O)$, we have

$$\hat{U}^L_{\varepsilon,\xi}(x) = \int\limits_{\partial B_3(O)} \Big[\frac{\partial \Gamma(|x-y|)}{\partial \vec{n}} \hat{U}^L_{\varepsilon,\xi}(y) - \Gamma(|x-y|)\frac{\partial \hat{U}^L_{\varepsilon,\xi}}{\partial \vec{n}}(y)\Big]dy$$
$$- 2\int\limits_{B_3(O)} \Gamma(|x-y|)\Big[1 + \hat{v}^L_{\varepsilon,\xi}(y) - e^{\hat{u}^L_{\varepsilon,\xi}(y)}\Big]dy,$$

where $\Gamma(|x-y|) = \frac{1}{2\pi}\ln|x-y|$ and $\vec{n}$ is the unit outward normal to $\partial B_3(O)$. Then $\{\hat{U}^L_{\varepsilon,\xi}\}$ is bounded in $C(B_3(O))$ and hence both $\{\hat{U}^L_{\varepsilon,\xi}\}$ and $\{\hat{v}^L_{\varepsilon,\xi}\}$ are bounded in $C_{loc}(\mathbf{R}^2)$ from Remark 6.1. By passing to a subsequence if necessary, we may assume that $(\hat{U}^L_{\varepsilon,\xi}, \hat{v}^L_{\varepsilon,\xi})$ converges uniformly to (U^*_L, v^*_L) on $[0, \widetilde{L}]$ in $C^2_{loc}(\mathbf{R}^2) \times C^2_{loc}(\mathbf{R}^2)$ for some $(U^*_L, v^*_L) \in C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ satisfying

$$\begin{cases} \Delta u^*_L + 2\big(1 + v^*_L - e^{u^*_L}\big) = 4\pi\lambda_\xi\delta_0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta v^*_L - L\big(1 + v^*_L - e^{u^*_L}\big) - 2e^{u^*_L}v^*_L = 0 \ \text{ in } \ \mathbf{R}^2, \\ \int\limits_{\mathbf{R}^2} \big(1 + v^*_L(x) - e^{u^*_L(x)}\big)dx = 2\pi\lambda_\xi, \\ \int\limits_{\mathbf{R}^2} L\big(1 + v^*_L(x) - e^{u^*_L(x)}\big) + 2e^{u^*_L(x)}v^*_L(x)dx = 0, \end{cases}$$

where $u^*_L(x) = U^*_L(x) + 2\lambda_\xi \ln|x|$. Then $(u^*_L(x), v^*_L(x))$ is a topological solution of (6.10) and $(u^*_L(x), v^*_L(x)) = (u^*_L(|x|), v^*_L(|x|))$ by the method of moving planes. It turns out that, from the uniqueness of topological solutions for (6.10), $(u^*_L(x), v^*_L(x)) = \big(\phi^L_{\lambda_\xi}(r), \bar{\phi}^L_{\lambda_\xi}(r)\big)$.

Step 2. $\sup_{|x|\le\frac{2d}{\varepsilon}} \big|\hat{u}^L_{\varepsilon,\xi}(x) - \phi^L_{\lambda_\xi}(x)\big| \to 0$ *and* $\sup_{|x|\le\frac{2d}{\varepsilon}} \big|\hat{v}^L_{\varepsilon,\xi}(x) - \bar{\phi}^L_{\lambda_\xi}(x)\big| \to 0$ *uniformly on* $[0, \widetilde{L}]$ *as* $\varepsilon \to 0$. For simplicity, let

$$\theta^L_{\varepsilon,\xi}(x) = \hat{u}^L_{\varepsilon,\xi}(x) - \phi^L_{\lambda_\xi}(x), \ \ \eta^L_{\varepsilon,\xi}(x) = \hat{v}^L_{\varepsilon,\xi}(x) - \bar{\phi}^L_{\lambda_\xi}(x).$$

For $R_0 > 0$ fixed, there exist two constants $\delta = \delta(R_0, L) > 0$ and $\hat{\varepsilon} = \hat{\varepsilon}(L) > 0$ such that

$$e^{\phi^L_{\lambda_\xi}(R_0)} > \max\{1-\delta, 2\delta\}, \ L < \frac{1}{\delta} \ \text{ and}$$
$$\delta + \frac{2}{L}\hat{v}^L_{\varepsilon,\xi}(x) > 0 \quad \text{on } \ \{x : |x| = R_0\} \quad \text{for all } \ \varepsilon \in (0, \hat{\varepsilon}) \tag{6.12}$$

due to Step 1 and the fact that $\phi^L_{\lambda_\xi}(r)$ is strictly increasing and converges to zero as $r \to \infty$. We argue by contradiction. Without loss of generality, we assume that there exist three sequences $\{\varepsilon_n\}$, $\{L_n\}$ and $\{x_n\} \subset B_{\frac{2d}{\varepsilon_n}}(O)$ such that $(\varepsilon_n, L_n) \to (0, \hat{L})$ as $n \to \infty$ and either

$$\big|\theta^{L_n}_{\varepsilon_n,\xi}(x_n)\big| = \max_{|x|\le\frac{2d}{\varepsilon_n}} \big|\theta^{L_n}_{\varepsilon_n,\xi}(x)\big| \ge \gamma_0 \ \text{ or } \ \big|\eta^{L_n}_{\varepsilon_n,\xi}(x'_n)\big| = \max_{|x|\le\frac{2d}{\varepsilon_n}} \big|\eta^{L_n}_{\varepsilon_n,\xi}(x)\big| \ge \gamma_0$$

for some constants $\gamma_0 > 0$ and $\hat{L} \in [0, \widetilde{L}]$. Due to Step 1, we have $|x_n|, |x_n'| \to \infty$ and one may assume that $R_0 < |x_n|, |x_n'| < \frac{2d}{\varepsilon_n}$. Now we want to show that *either* $\theta^{L_n}_{\varepsilon_n,\xi}(x)$ *or* $\eta^{L_n}_{\varepsilon_n,\xi}(x)$ *does not possess positive local maximums on* $\left\{x : R_0 < |x| < \frac{2d}{\varepsilon_n}\right\}$. If $(1-\delta)\theta^{L_n}_{\varepsilon_n,\xi}(x_n) > \max\left\{0, \left|\eta^{L_n}_{\varepsilon_n,\xi}(x_n')\right|\right\}$, then

$$0 \geq \Delta\theta^{L_n}_{\varepsilon_n,\xi}(x_n) = 2e^{\bar{u}_{L_n}}\theta^{L_n}_{\varepsilon_n,\xi}(x_n) - 2\eta^{L_n}_{\varepsilon_n,\xi}(x_n) > 0,$$

which is impossible. Moreover, if $\eta^{L_n}_{\varepsilon_n,\xi}(x_n') \geq (1-\delta)\max\left\{0, \left|\theta^{L_n}_{\varepsilon_n,\xi}(x_n)\right|\right\}$, then

$$\begin{aligned} 0 \geq \Delta\eta^{L_n}_{\varepsilon_n,\xi}(x_n') &= \left(L_n + 2e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}(x_n')}\right)\eta^{L_n}_{\varepsilon_n,\xi}(x_n') \\ &\quad - \left[L_n e^{\hat{u}^{L_n}_{\varepsilon_n,\xi}(x_n')} - 2e^{\tilde{u}_{L_n}}\bar{\phi}^{L_n}_{\lambda_\xi}(x_n')\right]\theta^{L_n}_{\varepsilon_n,\xi}(x_n') > 0, \end{aligned}$$

where $\bar{u}_{L_n} = t\hat{u}^{L_n}_{\varepsilon_n,\xi}(x_n) + (1-t)\phi^{L_n}_{\lambda_\xi}(x_n)$ and $\tilde{u}_{L_n} = t\hat{u}^{L_n}_{\varepsilon_n,\xi}(x_n') + (1-t)\phi^{L_n}_{\lambda_\xi}(x_n')$ for some $t \in [0,1]$. It also yields a contradiction. Hence, either $\theta^{L_n}_{\varepsilon_n,\xi}(x)$ or $\eta^{L_n}_{\varepsilon_n,\xi}(x)$ does not possess positive local maximums on $\left\{x : R_0 < |x| < \frac{2d}{\varepsilon_n}\right\}$. Therefore, $\theta^{L_n}_{\varepsilon_n,\xi}(x)$ and $\eta^{L_n}_{\varepsilon_n,\xi}(x)$ are non-positive on $\left\{x : R_0 < |x| < \frac{2d}{\varepsilon_n}\right\}$ by (6.11). Now, we first consider $\hat{L} \neq 0$. By the similar arguments deriving the claim in the proof of Theorem 3.2, we get that $(\phi^{L_n}_{\lambda_\xi}, \bar{\phi}^{L_n}_{\lambda_\xi}) \to (\phi^{\hat{L}}_{\lambda_\xi}, \bar{\phi}^{\hat{L}}_{\lambda_\xi})$ in $C^2([0,R]) \times C^2([0,R])$ as $n \to \infty$ for all $R > 0$ (passing to a subsequence if necessary). Hence from Step 1 and (6.12), we obtain that there exists a constant $\hat{n}(\hat{L}) > 0$ such that

$$\Delta\left[(1+\delta)\theta_{\varepsilon_n,\xi} + \frac{2}{L_n}\eta_{\varepsilon_n,\xi}\right] = e^{\mu^{L_n}_{\varepsilon_n}}\left(2\delta + \frac{4}{L_n}\hat{v}^{L_n}_{\varepsilon_n,\xi}\right)\theta_{\varepsilon_n,\xi} + \left(e^{\phi^{L_n}_{\lambda_\xi}} - 2\delta\right)\eta_{\varepsilon_n,\xi} \leq 0$$

on $\left\{x : R_0 \leq |x| \leq \frac{2d}{\varepsilon_n}\right\}$ for all $n \geq \hat{n}(\hat{L})$, where $\mu^{L_n}_{\varepsilon_n}(x) \in \left[\hat{u}^{L_n}_{\varepsilon_n,\xi}(x), \phi^{L_n}_{\lambda_\xi}(x)\right]$. It contradicts (6.11) and Step 1 by the maximum principle.

Secondly, for $\hat{L} = 0$, we want to claim that $\sup_{|x| \leq \frac{2d}{\varepsilon_n}} \left|\hat{v}^{L_n}_{\varepsilon_n,\xi}(x)\right| \to 0$ *as* $n \to \infty$. Suppose to the contrary that there exist sequences $\{\varepsilon_n\}$ and $\{x_n'\} \subset B_{\frac{2d}{\varepsilon_n}}(O)$ (passing to a subsequence if necessary) such that $(\varepsilon_n, L_n) \to (0,0)$, $\left\|\hat{v}^{L_n}_{\varepsilon_n,\xi}\right\|_{L^\infty(B_{\frac{2d}{\varepsilon_n}}(O))} \to C_v \in (0,1]$ as $n \to \infty$ and

$$\left|\left[\theta^{L_n}_{\varepsilon_n,\xi} + \frac{2}{L_n}\hat{v}^{L_n}_{\varepsilon_n,\xi}\right](x_n')\right| = \max\left\{\left|\left[\theta^{L_n}_{\varepsilon_n,\xi} + \frac{2}{L_n}\hat{v}^{L_n}_{\varepsilon_n,\xi}\right](x)\right| : x \in B_{\frac{2d}{\varepsilon_n}}(O)\right\} \geq \gamma_1$$

for some $\gamma_1 > 0$. To prove this claim, we need to show that $L_n^{-1}\hat{v}^{L_n}_{\varepsilon_n,\xi}(x_n') \to -\infty$ *as* $n \to \infty$ (*passing to a subsequence if necessary*). On the contrary, we may assume that there exists a constant $C \in [0,\infty)$ such that $L_n^{-1}\hat{v}^{L_n}_{\varepsilon_n,\xi}(x_n') \to -C$ as $n \to \infty$. It is easy to see that $\{\bar{u}_{L_n}\}$ is bounded in $C\left(\{x : R_0 \leq |x| \leq 2d/\varepsilon_n\}\right)$ from Step 1, (6.11) and the maximum principle. Since $\left\|L_n^{-1}\hat{v}_{\varepsilon_n,\xi}^{L_n}\right\|_{L^\infty(B_{\frac{2d}{\varepsilon_n}}(O))} \to \infty$ as $n \to \infty$, we have $\theta_{\varepsilon_n,\xi}^{L_n}(x_n') \to -\infty$ as $n \to \infty$, which implies

$$2e^{\bar{u}_{L_n}(x_n')}\theta_{\varepsilon_n,\xi}^{L_n}(x_n') \to -\infty \quad \text{and} \quad \Delta\theta_{\varepsilon_n,\xi}^{L_n}(x_n') \to \lim_{n\to\infty} 2e^{\bar{u}_{L_n}}\theta_{\varepsilon_n,\xi}^{L_n}(x_n') \quad \text{as} \quad n \to \infty.$$

Consequently,

$$\left|\left[\theta_{\varepsilon_n,\xi}^{L_n} + \frac{2}{L_n}\hat{v}_{\varepsilon_n,\xi}^{L_n}\right](x_n')\right| \neq \max\left\{\left|\left[\theta_{\varepsilon_n,\xi}^{L_n} + \frac{2}{L_n}\hat{v}_{\varepsilon_n,\xi}^{L_n}\right](x)\right| : x \in B_{\frac{2d}{\varepsilon_n}}(O)\right\},$$

which leads to a contradiction. Therefore, we get $L_n^{-1}\hat{v}_{\varepsilon_n,\xi}^{L_n}(x_n') \to -\infty$ as $n \to \infty$.

Combining (6.11) with Step 1 and the maximum principle, we obtain that

$$\begin{aligned} 0 &\le \Delta\left[\theta_{\varepsilon_n,\xi}^{L_n}(x_n') + \frac{2}{L_n}\hat{v}_{\varepsilon_n,\xi}^{L_n}(x_n')\right] \\ &= \frac{4}{L_n}e^{\hat{u}_{\varepsilon_n,\xi}^{L_n}(x_n')}\hat{v}_{\varepsilon_n,\xi}^{L_n}(x_n') + 2 + 2\bar{\phi}_{\lambda_\xi}^{L_n}(x_n') - 2e^{\phi_{\lambda_\xi}^{L_n}(x_n')} \\ &\le \left[\frac{4}{L_n}\hat{v}_{\varepsilon_n,\xi}^{L_n}(x_n') - 2\right]e^{\hat{u}_{\lambda_\xi}^{L_n}(x_n')} + 2 \\ &< 0 \end{aligned}$$

for large n, which yields a contradiction. Hence the claim holds.

Moreover, we have that $\bar{\phi}_{\lambda_\xi}^{L_n} \to \bar{\phi}_{\lambda_\xi}^{\hat{L}}$ in $C^2([0,R])$ as $n \to \infty$ for any $R > 0$, and hence

$$\sup_{|x|\le\frac{2d}{\varepsilon_n}} \left|\eta_{\varepsilon_n,\xi}^{L_n}(x)\right| \to 0$$

as $n \to \infty$ by the above claim. Therefore, for large n, we obtain that

$$0 \le \Delta\theta_{\varepsilon_n,\xi}^{L_n}(x_n) = 2e^{\bar{u}_{L_n}}\theta_{\varepsilon_n,\xi}^{L_n}(x_n) - 2\eta_{\varepsilon_n,\xi}^{L_n}(x_n) < -\gamma_0 < 0,$$

where $\bar{u}_{L_n} = t\hat{u}_{\varepsilon_n,\xi}^{L_n}(x_n) + (1-t)\phi_{\lambda_\xi}^{L_n}(x_n)$ for some $t \in [0,1]$. This contradiction finishes this step. Finally, we complete the proof of this lemma by Steps 1 and 2. □

Theorem 6.1. *Let $\widetilde{L} > 0$, $p_1, \ldots, p_\ell \in \mathbf{R}^2$ and $n_1, \ldots, n_\ell \in \mathbf{R}^+ \cup \{0\}$ be given. Then there exists a constant $q_0 = q_0(p_i, n_i, \widetilde{L}) > 0$ such that (6.1) possesses one and only one topological solution for any $q \ge q_0$ and $0 < \kappa \le \sqrt{\frac{\widetilde{L}}{q}}$.*

Proof. On the contrary, we may assume that there exist sequence $\{(\varepsilon_n, L_n)\}_{n\in N}$ and a constant $\hat{L} \in [0, \widetilde{L}]$ such that $(\varepsilon_n, L_n) \to (0, \hat{L})$ as $n \to \infty$ and that $(u_{1\varepsilon_n}^{L_n}(x), v_{1\varepsilon_n}^{L_n}(x))$ and $(u_{2\varepsilon_n}^{L_n}(x), v_{2\varepsilon_n}^{L_n}(x))$ are two distinct topological solutions of (6.1) for each $n \in N$. Without loss of generality, let

$$\left|(u_{1\varepsilon_n}^{L_n} - u_{2\varepsilon_n}^{L_n})(x_n)\right| = \left\|u_{1\varepsilon_n}^{L_n} - u_{2\varepsilon_n}^{L_n}\right\|_{L^\infty(\mathbf{R}^2)} \ge \left\|v_{1\varepsilon_n}^{L_n} - v_{2\varepsilon_n}^{L_n}\right\|_{L^\infty(\mathbf{R}^2)}$$

for each $n \in N$. Then by (6.2) and the similar technique in the proof of (3.10), we obtain that there exists a point $\xi \in \mathcal{Z}$ such that $\{y_n\} \subseteq B_R(\xi)$ for some $R > 0$. Set $A_n = \dfrac{\hat{u}^{L_n}_{1\varepsilon_n,\xi} - \hat{u}^{L_n}_{2\varepsilon_n,\xi}}{\|\hat{u}^{L_n}_{1\varepsilon_n,\xi} - \hat{u}^{L_n}_{2\varepsilon_n,\xi}\|_{L^\infty(\mathbf{R}^2)}}$ and $B_n = \dfrac{\hat{v}^{L_n}_{1\varepsilon_n,\xi} - \hat{v}^{L_n}_{2\varepsilon_n,\xi}}{\|\hat{u}^{L_n}_{1\varepsilon_n,\xi} - \hat{u}^{L_n}_{2\varepsilon_n,\xi}\|_{L^\infty(\mathbf{R}^2)}}$. Then $(A_n(x), B_n(x))$ satisfies

$$\begin{cases} \Delta A_n + 2B_n - 2e^{\xi_n} A_n = 0 \text{ in } \mathbf{R}^2, \\ \Delta B_n + \big(Ł_n - 2\hat{v}^{L_n}_{2\varepsilon_n,\xi}\big)e^{\xi_n} A_n - \big(L_n + 2e^{\hat{u}^{L_n}_{1\varepsilon_n,\xi}}\big)B_n = 0 \text{ in } \mathbf{R}^2, \\ A_n(x) \to 0, \quad B_n(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{6.13}$$

where $\xi_n(x) \in \big[\min\big\{\hat{u}^{L_n}_{1\varepsilon_n,\xi}(x), \hat{u}^{L_n}_{2\varepsilon_n,\xi}(x)\big\}, \max\big\{\hat{u}^{L_n}_{1\varepsilon_n,\xi}(x), \hat{u}^{L_n}_{2\varepsilon_n,\xi}(x)\big\}\big]$. Applying the similar argument used to derive the claim in the proof of Theorem 3.1, we see that $\{x_n\}$ is a bounded sequence. By the standard elliptic estimates and Lemma 6.2, we obtain $\big(\hat{u}^{L_n}_{1\varepsilon_n,\xi}, \hat{v}^{L_n}_{1\varepsilon_n,\xi}\big) \to \big(\phi^{\hat{L}}_{\lambda,\xi}, \bar{\phi}^{\hat{L}}_{\lambda,\xi}\big)$, $\big(\hat{u}^{L_n}_{2\varepsilon_n,\xi}, \hat{v}^{L_n}_{2\varepsilon_n,\xi}\big) \to \big(\phi^{\hat{L}}_{\lambda,\xi}, \bar{\phi}^{\hat{L}}_{\lambda,\xi}\big)$ and $(A_n, B_n) \to (A, B)$ in $C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ as $n \to \infty$, where $(A(x), B(x)) \in C^2(\mathbf{R}^2) \times C^2(\mathbf{R}^2)$ satisfying

$$\begin{cases} \Delta A + 2B - 2e^{\phi_{\lambda,\xi}} A = 0 \ \text{ in } \ \mathbf{R}^2, \\ \Delta B + (\hat{L} - 2\bar{\phi}_{\lambda,\xi})e^{\phi_{\lambda,\xi}} A - (\hat{L} + 2e^{\phi_{\lambda,\xi}})B = 0 \ \text{ in } \ \mathbf{R}^2. \end{cases}$$

It is clear that $A(x) = A(|x|)$ and $B(x) = B(|x|)$ in $\mathbf{R}^2$ from $\big(\phi^{\hat{L}}_{\lambda,\xi}(x), \bar{\phi}^{\hat{L}}_{\lambda,\xi}(x)\big) = \big(\phi^{\hat{L}}_{\lambda,\xi}(|x|), \bar{\phi}^{\hat{L}}_{\lambda,\xi}(|x|)\big)$.

Note that both A and B are bounded and not equal to zero identically in $\mathbf{R}^2$ because $\{x_n\}$ is bounded. Nevertheless, by Lemma 2.4, it must be the case that $A \equiv 0$ and $B \equiv 0$. This contradiction complete the proof of Theorem 6.1. □

Secondly, we discuss the case $\kappa^2 q = L \le \widetilde{L}$ and $q \to 0$. The following property is very useful for us in proving Theorem 6.2 stated later.

Proposition 6.2. *Let $\varepsilon = \frac{1}{\sqrt{q}}$, $\bar{u}_{\varepsilon,L}(x) = u^L_\varepsilon(\varepsilon x)$ and $\bar{v}_{\varepsilon,L}(x) = v^L_\varepsilon(\varepsilon x)$ for $x \in \mathbf{R}^2$. Then there is a positive constant $C = C(\{p_i\}, \{n_i\}, \widetilde{L})$ such that*

$$0 < \int_{\mathbf{R}^2} \Big[2e^{\bar{u}_{\varepsilon,L}(x)} - 2 - \bar{v}_{\varepsilon,L}(x)\Big]\bar{v}_{\varepsilon,L}(x)dx \le C \tag{6.14}$$

for all $(q, L) \in (0, 1) \times [0, \widetilde{L}]$.

Proof. Fix $L \in [0, \widetilde{L}]$. Let $\varepsilon > 0$ and $\delta = \frac{1}{2\varepsilon}\inf\{|p_k - p_\ell| \ : \ p_k, p_\ell \in \mathcal{Z} \ \text{ and } \ p_k \neq p_\ell\}$. First, we set $\mathcal{U}_{\varepsilon,\xi}(x) = \bar{u}_{\varepsilon,L}(x) - 2\lambda_\xi \ln|x - \frac{1}{\varepsilon}\xi|$ and $\mathcal{V}_\varepsilon(x) = \bar{v}_{\varepsilon,L}(x)$. Then

$$
\begin{aligned}
&- \int\limits_{\mathbf{R}^2 \setminus \cup_{\xi \in \mathcal{Z}} B_\delta(\frac{1}{\varepsilon}\xi)} \left\{ \Delta \bar{u}_{\varepsilon,L}(x \cdot \nabla \bar{v}_{\varepsilon,L}) + \Delta \bar{v}_{\varepsilon,L}(x \cdot \nabla \bar{u}_{\varepsilon,L}) + \frac{\kappa^2 q}{2} \Delta \bar{u}_{\varepsilon,L}(x \cdot \nabla \bar{u}_{\varepsilon,L}) \right\} dx \\
&= \sum_{\xi \in \mathcal{Z}} \int\limits_{|x - \frac{1}{\varepsilon}\xi| = \delta} \left\{ \frac{1}{\delta\varepsilon} \left[\nabla \mathcal{U}_{\varepsilon,\xi} \cdot (x - \frac{1}{\varepsilon}\xi) \right] \Big(\xi \cdot \nabla \mathcal{V}_\varepsilon \Big) + \frac{1}{\delta\varepsilon} \left[\nabla \mathcal{V}_\varepsilon \cdot (x - \frac{1}{\varepsilon}\xi) \right] \Big(\xi \cdot \nabla \mathcal{U}_{\varepsilon,\xi} \Big) \right. \\
&\quad \left. + \tfrac{2\lambda_\xi}{\delta} \Big(x \cdot \nabla \mathcal{V}_\varepsilon \Big) + \tfrac{1}{\delta\varepsilon} \left[\xi \cdot (x - \tfrac{1}{\varepsilon}\xi) \right] (\nabla \mathcal{U}_{\varepsilon,\xi} \cdot \nabla \mathcal{V}_\varepsilon) + \delta (\nabla \mathcal{U}_{\varepsilon,\xi} \cdot \nabla \mathcal{V}_\varepsilon) \right\} d\sigma \\
&\quad + \frac{\kappa^2 q}{2} \sum_{\xi \in \mathcal{Z}} \int\limits_{|x - \frac{1}{\varepsilon}\xi| = \delta} \left\{ \frac{1}{\delta\varepsilon} \left[\nabla \mathcal{U}_{\varepsilon,\xi} \cdot (x - \frac{1}{\varepsilon}\xi) \right] \Big(\xi \cdot \nabla \mathcal{U}_{\varepsilon,\xi} \Big) + \frac{2\lambda_\xi}{\delta} \Big(x \cdot \nabla \mathcal{U}_{\varepsilon,\xi} \Big) \right. \\
&\quad + \tfrac{1}{2\delta\varepsilon} \left[\xi \cdot (x - \tfrac{1}{\varepsilon}\xi) \right] (\nabla \mathcal{U}_{\varepsilon,\xi} \cdot \nabla \mathcal{U}_{\varepsilon,\xi}) + \tfrac{4\lambda_\xi^2}{\delta^3 \varepsilon} \left[\xi \cdot (x - \tfrac{1}{\varepsilon}\xi) \right] + \tfrac{2\lambda_\xi^2 |x|^3}{\delta^4} \\
&\quad \left. + \tfrac{\delta}{2} (\nabla \mathcal{U}_{\varepsilon,\xi} \cdot \nabla \mathcal{U}_{\varepsilon,\xi}) \right\} d\sigma. \qquad (6.15)
\end{aligned}
$$

Let $\overline{\mathcal{U}}_\varepsilon(x) = \bar{u}_{\varepsilon,L}(x) - 2 \sum_{\xi \in \mathcal{Z}} \lambda_\xi \ln \left| x - \frac{1}{\varepsilon}\xi \right|$. Then we have

$$
\begin{cases}
\nabla \mathcal{U}_{\varepsilon,\xi}(x + \frac{1}{\varepsilon}p_\ell) = \sum\limits_{\ell \neq \kappa} \dfrac{2\varepsilon n_\kappa (\varepsilon x + p_\ell - p_\kappa)}{|\varepsilon x + p_\ell - p_\kappa|^2} + \nabla \overline{\mathcal{U}}_\varepsilon(x + \frac{1}{\varepsilon}p_\ell), \\
\nabla \overline{\mathcal{U}}_\varepsilon(x) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{x-y}{|x-y|^2} \left[e^{\bar{u}_{\varepsilon,L}(y)} - 1 - \bar{v}_{\varepsilon,L}(y) \right] dy, \\
\nabla \mathcal{V}_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \frac{x-y}{|x-y|^2} \left\{ L \left[\bar{v}_{\varepsilon,L}(y) + 1 - e^{\bar{u}_{\varepsilon,L}(y)} \right] + 2 e^{\bar{u}_{\varepsilon,L}(y)} \bar{v}_{\varepsilon,L}(y) \right\} dy
\end{cases}
$$

for $x \in \mathbf{R}^2$. Hence, if $|x| \le 1$, we obtain that

$$
\begin{cases}
\pi \left| \nabla \overline{\mathcal{U}}_\varepsilon(x) \right| & \le \int_{|y| \le 2} \frac{1}{|x-y|} dy + \int_{|y| \ge 2} \frac{4}{|y|} \left[1 + \bar{v}_{\varepsilon,L}(y) - e^{\bar{u}_{\varepsilon,L}(y)} \right] dy \\
& \le \int_{|y| \le 2} \frac{1}{|x-y|} dy + 2 \int_{\mathbf{R}^2} \left[1 + \bar{v}_{\varepsilon,L}(y) - e^{\bar{u}_{\varepsilon,L}(y)} \right] dy \le C, \\
2\pi \left| \nabla \mathcal{V}_\varepsilon(x) \right| & \le \int_{|y| \le 2} \frac{1}{|x-y|} dy + \int_{|y| \ge 2} \left\{ \frac{2L}{|y|} \left[1 + \bar{v}_{\varepsilon,L}(y) - e^{\bar{u}_{\varepsilon,L}(y)} \right] \right. \\
& \qquad \left. - \frac{4}{|y|} e^{\bar{u}_{\varepsilon,L}(y)} \bar{v}_{\varepsilon,L}(y) \right\} dy \\
& \le \int_{|y| \le 2} \frac{1}{|x-y|} dy + 4\widetilde{L} \int_{\mathbf{R}^2} \left[1 + \bar{v}_{\varepsilon,L}(y) - e^{\bar{u}_{\varepsilon,L}(y)} \right] dy \le C
\end{cases} \qquad (6.16)
$$

for some positive constant $C = C(\{p_i\}, \{n_i\}, \widetilde{L})$.

Due to the Pohozaev identity, we get that

$$
\begin{aligned}
&- \int\limits_{\mathbf{R}^2 \setminus \cup_{\xi \in \mathcal{Z}} B_\delta(\frac{1}{\varepsilon}\xi)} \left\{ \Delta \bar{u}_{\varepsilon,L}(x \cdot \nabla \bar{v}_{\varepsilon,L}) + \Delta \bar{v}_{\varepsilon,L}(x \cdot \nabla \bar{u}_{\varepsilon,L}) + \frac{\kappa^2 q}{2} \Delta \bar{u}_{\varepsilon,L}(x \cdot \nabla \bar{u}_{\varepsilon,L}) \right\} dx \\
&= 2 \int\limits_{\mathbf{R}^2 \setminus \cup_{\xi \in \mathcal{Z}} B_\delta(\frac{1}{\varepsilon}\xi)} \left[\bar{v}_{\varepsilon,L}(x) + 2 - 2e^{\bar{u}_{\varepsilon,L}(x)} \right] \bar{v}_{\varepsilon,L}(x) dx
\end{aligned}
$$

$$-\sum_{\xi\in\mathcal{Z}}\ \int\limits_{|x-\frac{1}{\varepsilon}\xi|=\delta}\frac{1}{\delta}\left[x\cdot\left(x-\frac{1}{\varepsilon}\xi\right)\right]\left[\bar{v}_{\varepsilon,L}(x)+2-2e^{\bar{u}_{\varepsilon,L}(x)}\right]\bar{v}_{\varepsilon,L}(x)d\sigma,$$

and hence (6.14) holds by (6.15)–(6.16) and Proposition 5.1. $\square$

Theorem 6.2. *Let $\widetilde{L}>0$, $p_1,\dots,p_\ell\in\mathbf{R}^2$ and $n_1,\dots,n_\ell\in\mathbf{R}^+\cup\{0\}$ be given. Then there exists a constant $q_1=q_1(p_i,n_i,\widetilde{L})>0$ such that (6.1) possesses a unique topological solution for any $0<q<q_1$ and $0<\kappa\le\sqrt{\frac{\widetilde{L}}{q}}$.*

Proof. Let $\varepsilon=\frac{1}{\sqrt{q}}$, $\bar{u}_{\varepsilon,L}(x)=u_\varepsilon^L(\varepsilon x)$ and $\bar{v}_{\varepsilon,L}(x)=v_\varepsilon^L(\varepsilon x)$ for $x\in\mathbf{R}^2$, where $(u_\varepsilon^L,v_\varepsilon^L)$ is a topological solution of (6.1). Then $(\bar{u}_{\varepsilon,L},\bar{v}_{\varepsilon,L})$ satisfies

$$\begin{cases}\Delta\bar{u}_{\varepsilon,L}=2\big(-\bar{v}_{\varepsilon,L}-1+e^{\bar{u}_{\varepsilon,L}}\big)+4\pi\sum_{i=1}^{\ell}n_i\delta_{\frac{1}{\varepsilon}p_i} & \text{in } \mathbf{R}^2,\\ \Delta\bar{v}_{\varepsilon,L}=-L\big(-\bar{v}_{\varepsilon,L}-1+e^{\bar{u}_{\varepsilon,L}}\big)+2e^{\bar{u}_{\varepsilon,L}}\bar{v}_{\varepsilon,L} & \text{in } \mathbf{R}^2.\end{cases}$$

Step 1. For each compact subset $K\subset\mathbf{R}^2\backslash\mathcal{Z}$, there are constants $\varepsilon_*>0$ and $\gamma_0(K,\widetilde{L})<0$ such that $\bar{u}_{\varepsilon,L}(x)\ge\gamma_0(K,\widetilde{L})$ in K for $\varepsilon>\varepsilon_*$ and $0\le L\le\widetilde{L}$, where $\mathcal{Z}=\{p_1,\dots,p_\ell\}$.

Let $\overline{U}_{\varepsilon,L}(x)=\bar{u}_{\varepsilon,L}(x)-\sum_{i=1}^{\ell}2n_i\ln\left|x-\frac{p_i}{\varepsilon}\right|$. Then it suffices to prove that for $\varepsilon>0$ small, $\inf_{B_R(O)}\overline{U}_{\varepsilon,L}(x)\ge\gamma_0$ for some $\gamma_0=\gamma_0(R,\widetilde{L})<0$, where $R>\max_{[\varepsilon_0,\infty)}\left\{\left|\frac{p_1}{\varepsilon}\right|,\dots,\left|\frac{p_\ell}{\varepsilon}\right|\right\}+1$ for some $\varepsilon_0>0$. On the contrary, without loss of generality, we may assume that there are a constant $R_0>\max_{[\varepsilon_0,\infty)}\left\{\left|\frac{p_1}{\varepsilon}\right|,\dots,\left|\frac{p_\ell}{\varepsilon}\right|\right\}+1$ and sequences $\{\varepsilon_n\}$, $\{L_n\}$ and $\{x_n\}\subset B_{R_0}(O)$ such that

$$(\varepsilon_n,L_n)\to(\infty,\hat{L})\ \text{ and }\ \overline{U}_{\varepsilon_n,L_n}(x_n)=\inf_{B_{R_0}(O)}\overline{U}_{\varepsilon_n,L_n}(x)\to-\infty\ \text{ as }\ n\to\infty$$

for some $\hat{L}\in[0,\widetilde{L}]$. For simplicity, let $\overline{U}_n=\overline{U}_{\varepsilon_n,L_n}$ and $\bar{v}_n=\bar{v}_{\varepsilon_n,L_n}$. We decompose $\overline{U}_n$ and $\bar{v}_n$ as $\overline{U}_n=U_{1n}+U_{2n}$ and $\bar{v}_n=V_{1n}+V_{2n}$ where

$$\begin{cases}\Delta U_{1n}=2\big(-1-\bar{v}_n+e^{\bar{u}_n}\big)\text{ in } B_{R'}(O),\\ \Delta V_{1n}=-\Big[L_n\big(-\bar{v}_n-1+e^{\bar{u}_n}\big)-2e^{\bar{u}_n}\bar{v}_n\Big]\text{ in } B_{R'}(O),\\ U_{1n}(x)=V_{1n}(x)=0\ \text{ on } \partial B_{R'}(O),\end{cases}$$

and

$$\begin{cases}\Delta U_{2n}=0\text{ in } B_{R'}(O),\\ \Delta V_{2n}=0\text{ in } B_{R'}(O),\\ U_{2n}(x)=\overline{U}_n(x),\ V_{1n}(x)=\bar{v}_n(x)\ \text{ on } \partial B_{R'}(O)\end{cases}$$

for any $R' \geq R_0$. From Theorem 3 in [1] and $e^{\bar{u}_n(x)} - 1 - \bar{v}_n(x) \leq 0$ in $B_{R'}(O)$, we see that $U_{2n}(x)$ converges to $-\infty$ uniformly on $B_{R'}(O)$ as $n \to \infty$. By the standard elliptic estimates, we deduce that $U_{1n}(x)$ is uniformly bounded in $B_{R'}(O)$. Consequently,

$$\overline{U}_n \to -\infty \quad \text{uniformly on } \ B_{R'}(O) \ \text{ as } \ n \to \infty. \tag{6.17}$$

Since $\int\limits_{\mathbf{R}^2} (1 + \bar{v}_n(x) - e^{\bar{u}_n(x)})dx = 2\pi \sum\limits_{i=1}^{\ell} n_i$, we get that

$$\int\limits_{\mathbf{R}^2} e^{\bar{u}_n(x)} \bar{v}_n(x)dx = -L_n \pi \sum_{i=1}^{\ell} n_i. \tag{6.18}$$

Moreover, by (6.14), we have that

$$-\infty < \lim_{n\to\infty} \int\limits_{\mathbf{R}^2} \left[2 + \bar{v}_n(x)\right] \bar{v}_n(x)dx < 2 \lim_{n\to\infty} \int\limits_{\mathbf{R}^2} \bar{v}_n(x)dx,$$

and $\lim\limits_{n\to\infty} \bar{v}_n(x) \to 0$ as $|x| \to \infty$. Then from (6.17), we have

$$\begin{aligned} \infty > 2\pi \sum_{i=1}^{\ell} n_i - \lim_{n\to\infty} \int\limits_{\mathbf{R}^2} \bar{v}_n(x)dx &= \lim_{n\to\infty} \int\limits_{\mathbf{R}^2} \left[1 - e^{\bar{u}_n(x)}\right] dx \\ > \int\limits_{|x|\leq R} \lim_{n\to\infty} \left[1 - e^{\bar{u}_n(x)}\right] dx = \pi R^2, \end{aligned}$$

which leads to a contradiction if R is large. Therefore, we finish this step.

Let $L \in [0, \widetilde{L}]$. Due to Remark 6.1 and Step 1, we see that $(\bar{u}_{\varepsilon,L}, \bar{v}_{\varepsilon,L})$ converges to $(\bar{u}_L, \bar{v}_L)$ (passing to a subsequence if necessary) pointwisely in $\mathbf{R}^2 \setminus \{O\}$ as $\varepsilon \to \infty$, where $(\bar{u}_L, \bar{v}_L)$ verifies

$$\begin{cases} \Delta \bar{u}_L + 2(1 - e^{\bar{u}_L} + \bar{v}_L) = 4\pi \sum\limits_{i=1}^{\ell} n_i \delta_O \ \text{ in } \ \mathbf{R}^2, \\ \Delta \bar{v}_L - L(1 - e^{\bar{u}_L} + \bar{v}_L) - 2e^{\bar{u}_L} \bar{v}_L = 0 \ \text{ in } \ \mathbf{R}^2. \end{cases}$$

Combining (6.14) with (6.18), we obtain that for any $L \in [0, \widetilde{L}]$,

$$0 \geq \int\limits_{\mathbf{R}^2} \bar{v}_L(x)dx \geq \limsup_{\varepsilon\to\infty} \int\limits_{\mathbf{R}^2} \bar{v}_{\varepsilon,L}(x)dx > -\infty,$$

which implies that $\bar{v}_L(x) \to 0$ as $|x| \to \infty$. Then

$$\int_{\mathbf{R}^2} \left(\bar{v}_L(x) + 1 - e^{\bar{u}_L(x)}\right) dx \le \liminf_{\varepsilon\to\infty} \int_{\mathbf{R}^2} \left(\bar{v}_{\varepsilon,L}(x) + 1 - e^{\bar{u}_{\varepsilon,L}(x)}\right) dx = 2\pi \sum_{i=1}^{\ell} n_i,$$

which deduces that $(\bar{u}_L(x), \bar{v}_L(x)) \to \big(0,0\big)$ as $|x| \to \infty$, i.e. $(\bar{u}_L(x), \bar{v}_L(x)) = (\phi_\lambda^L(|x|), \bar{\phi}_\lambda^L(|x|))$, where $\lambda = \sum_{i=1}^{\ell} n_i$.

Step 2. $\overline{U}_{\varepsilon,L}(x) = \bar{u}_{\varepsilon,L}(x) - \sum_{i=1}^{\ell} 2n_i \ln\left|x - \frac{p_i}{\varepsilon}\right|$ *and* $\bar{v}_{\varepsilon,L}(x)$ *converge to* $\overline{U}_L(x)$ *and* $\bar{\phi}_\lambda^L(x)$ *(passing to a subsequence if necessary) in* $C^2(\mathbf{R}^2)$ *uniformly on* $[0, \widetilde{L}]$*, respectively, where* $\overline{U}_L(x) = \phi_\lambda^L(x) - 2\sum_{i=1}^{\ell} n_i \ln|x|$,

$$\begin{cases} \Delta\overline{U}_{\varepsilon,L} = 2\big(-\bar{v}_{\varepsilon,L} - 1 + e^{\bar{u}_{\varepsilon,L}}\big) \ \text{ in } \ \mathbf{R}^2, \\ \Delta\bar{v}_{\varepsilon,L} = -L\big(-\bar{v}_{\varepsilon,L} - 1 + e^{\bar{u}_{\varepsilon,L}}\big) + 2e^{\bar{u}_{\varepsilon,L}}\bar{v}_{\varepsilon,L} \ \text{ in } \ \mathbf{R}^2, \end{cases}$$

and

$$\begin{cases} \Delta\overline{U}_L = 2\big(-\bar{\phi}_\lambda^L - 1 + e^{\phi_\lambda^L}\big) \ \text{ in } \ \mathbf{R}^2, \\ \Delta\bar{\phi}_\lambda^L = -L\big(-\bar{\phi}_\lambda^L - 1 + e^{\phi_\lambda^L}\big) + 2e^{\phi_\lambda^L}\bar{\phi}_\lambda^L \ \text{ in } \ \mathbf{R}^2. \end{cases}$$

To complete this step, we need the following fact.

Claim. $\big(\overline{U}_{\varepsilon,L}, \bar{v}_{\varepsilon,L}\big)$ *converges to* $\big(\overline{U}_L, \bar{\phi}_\lambda^L\big)$ *(passing to a subsequence if necessary) in* $C^2_{loc}(\mathbf{R}^2) \times C^2_{loc}(\mathbf{R}^2)$ *as* $\varepsilon \to \infty$*.*

Proof of Claim. By the maximum principle and Proposition 5.1, $\|\bar{u}_{\varepsilon,L}\|_{L^\infty(\{|x|\ge R\})} = \|\bar{u}_{\varepsilon,L}\|_{L^\infty(\{|x|=R\})}$ for any $R > \max_{[\varepsilon_0,\infty)}\left\{\left|\frac{p_1}{\varepsilon}\right|, \dots, \left|\frac{p_\ell}{\varepsilon}\right|\right\}$. Then $\overline{U}_{\varepsilon,L}(x)$ and $\bar{v}_{\varepsilon,L}(x)$ are uniformly bounded on any compact set of $\mathbf{R}^2 \setminus \{O\}$ from Step 1. Decompose $\overline{U}_{\varepsilon,L}$ as $\overline{U}_{\varepsilon,L} = \overline{U}_{1\varepsilon,L} + \overline{U}_{2\varepsilon,L}$, where

$$\begin{cases} \Delta\overline{U}_{1\varepsilon,L} = 2\big(-\bar{v}_{\varepsilon,L} - 1 + e^{\bar{u}_{\varepsilon,L}}\big) \text{ in } B_R(O), \\ \overline{U}_{1\varepsilon,L}(x) = 0 \ \text{ on } \partial B_R(O) \end{cases} \quad \text{and} \quad \begin{cases} \Delta\overline{U}_{2\varepsilon,L} = 0 \text{ in } B_R(O), \\ \overline{U}_{2\varepsilon,L}(x) = \overline{U}_{\varepsilon,L}(x) \ \text{ on } \partial B_R(O) \end{cases}$$

for $R > \max_{[\varepsilon_0,\infty)}\left\{\left|\frac{p_1}{\varepsilon}\right|, \dots, \left|\frac{p_\ell}{\varepsilon}\right|\right\}$. By the maximum principle, we attain that $\|\overline{U}_{2\varepsilon,L}\|_{C(B_R(O))} \le \|\overline{U}_{\varepsilon,L}\|_{C(B_R(O))}$, which implies that $\{\overline{U}_{2\varepsilon,L}\}$ is bounded in $C(B_R(O))$. Moreover, $\{\overline{U}_{1\varepsilon,L}\}$ is bounded in $C(B_R(O))$ since

$$|\overline{U}_{1\varepsilon,L}(x)| \le \frac{1}{\pi}\left|\int_{B_R(O)} \ln|x-y|dy\right| \quad \text{for any } \ x \in B_R(O).$$

Hence, both $\{\overline{U}_{\varepsilon,L}\}$ and $\{\bar{v}_{\varepsilon,L}\}$ are bounded in $C_{loc}(\mathbf{R}^2)$, and we complete the proof of this claim.

Now, set

$$A_{\varepsilon,L}(x) = \bar{u}_{\varepsilon,L}(x) - \sum_{i=1}^{\ell_1} 2n_i \ln\left|x - \frac{p_i}{\varepsilon}\right| - \phi_\lambda^L(x) + \lambda \ln|x|$$

and

$$B_{\varepsilon,L}(x) = \bar{v}_{\varepsilon,L}(x) - \bar{\phi}_\lambda^L(x),$$

where $\lambda = 2\sum_{i=1}^{\ell} n_i$. Then $(A_{\varepsilon,L}(x), B_{\varepsilon,L}(x))$ satisfies

$$\begin{cases} \Delta A_{\varepsilon,L} + 2B_{\varepsilon,L} - 2e^{\xi_n}\left(A_{\varepsilon,L} - \sum_{i=1}^{\ell} 2n_i \ln \frac{|x - \frac{p_i}{\varepsilon}|}{|x|}\right) = 0, \\ \Delta B_{\varepsilon,L} - \left(L + 2e^{\bar{u}_{\varepsilon,L}}\right)B_{\varepsilon,L} + e^{\xi_{\varepsilon,L}}\left(L - 2\bar{\phi}_\lambda^L\right)\left(A_{\varepsilon,L} - \sum_{i=1}^{\ell} 2n_i \ln \frac{|x - \frac{p_i}{\varepsilon}|}{|x|}\right) = 0, \end{cases}$$

where

$$\xi_{\varepsilon,L}(x) \in \left[\min\{\bar{u}_{\varepsilon,L}(x), \phi_\lambda^L(x)\}, \max\{\bar{u}_{\varepsilon,L}(x), \phi_\lambda^L(x)\}\right]$$

for $x \in \mathbf{R}^2 \backslash \bigcup_{i=1}^{\ell}\left\{B_{1/\varepsilon}\left(\frac{p_i}{\varepsilon}\right) \cup B_{1/\varepsilon}(O)\right\}$ and

$$\begin{cases} \Delta A_{\varepsilon,L} + 2\left(\bar{v}_{\varepsilon,L} + 1 - e^{\bar{u}_{\varepsilon,L}}\right) - 2\left(\bar{\phi}_\lambda^L + 1 - e^{\phi_\lambda^L}\right) = 0, \\ \Delta B_{\varepsilon,L} + L\left(\bar{\phi}_\lambda^L + 1 - e^{\phi_\lambda^L}\right) + 2e^{\phi_\lambda^L}\bar{\phi}_\lambda^L - L\left(\bar{v}_{\varepsilon,L} + 1 - e^{\bar{u}_{\varepsilon,L}}\right) - 2e^{\bar{u}_{\varepsilon,L}}\bar{v}_{\varepsilon,L} = 0 \end{cases}$$

for $x \in \Lambda_\varepsilon = \bigcup_{i=1}^{\ell}\left\{B_{1/\varepsilon}\left(\frac{p_i}{\varepsilon}\right) \cup B_{1/\varepsilon}(O)\right\}$. Let $\mathcal{P}_L$ be a 2×2 matrix such that $\mathcal{P}_L^{-1}\mathcal{M}_L\mathcal{P}_L = \Lambda_L$, where

$$\mathcal{M}_L = \begin{pmatrix} -2 & 2 \\ L & -L-2 \end{pmatrix},$$

$$\Lambda_L = \begin{pmatrix} \lambda_{1,L} & 0 \\ 0 & \lambda_{2,L} \end{pmatrix} = \begin{pmatrix} \frac{-L-4+\sqrt{L^2+8L}}{2} & 0 \\ 0 & \frac{-L-4-\sqrt{L^2+8L}}{2} \end{pmatrix}. \tag{6.19}$$

We introduce new variables $\mathcal{A}_{\varepsilon,L}$ and $\mathcal{B}_{\varepsilon,L}$ defined by $\begin{pmatrix} \mathcal{A}_{\varepsilon,L} \\ \mathcal{B}_{\varepsilon,L} \end{pmatrix} = \mathcal{P}_L^{-1}\begin{pmatrix} \Phi_{\varepsilon,L} \\ \Psi_{\varepsilon,L} \end{pmatrix}$. Then we obtain

$$\begin{cases} \Delta\mathcal{A}_{\varepsilon,L} + \lambda_{1,L}\mathcal{A}_{\varepsilon,L} + a_{11}(x)\mathcal{A}_{\varepsilon,L} + a_{21}(x)\mathcal{B}_{\varepsilon,L} + g_1(x) = 0 \ \text{ on } \ \mathbf{R}^2\backslash\Lambda_\varepsilon, \\ \Delta\mathcal{B}_{\varepsilon,L} + \lambda_{2,L}\mathcal{B}_{\varepsilon,L} + a_{12}(x)\mathcal{A}_{\varepsilon,L} + a_{22}(x)\mathcal{B}_{\varepsilon,L} + g_2(x) = 0 \ \text{ on } \ \mathbf{R}^2\backslash\Lambda_\varepsilon, \\ \Delta\mathcal{A}_{\varepsilon,L} + \lambda_{1,L}\mathcal{A}_{\varepsilon,L} + b_{11}(x) = 0 \ \text{ on } \ \Lambda_\varepsilon, \\ \Delta\mathcal{B}_{\varepsilon,L} + \lambda_{2,L}\mathcal{B}_{\varepsilon,L} + b_{21}(x) = 0 \ \text{ on } \ \Lambda_\varepsilon, \\ \mathcal{A}_{\varepsilon,L}(x) \to 0, \ \ \mathcal{B}_{\varepsilon,L}(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{6.20}$$

where

$$\begin{pmatrix} a_{11}(x) & a_{21}(x) \\ a_{12}(x) & a_{22}(x) \end{pmatrix} = \mathcal{P}_L^{-1} \begin{pmatrix} 2 - 2e^{\xi_{\varepsilon,L}} & 0 \\ e^{\xi_{\varepsilon,L}}\big(L - 2\bar{\phi}_\lambda^L\big) - L & 2 - 2e^{\bar{u}_{\varepsilon,L}} \end{pmatrix} \mathcal{P}_L,$$

$$\begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \mathcal{P}_L^{-1} \begin{pmatrix} 4e^{\xi_{\varepsilon,L}} \displaystyle\sum_{i=1}^{\ell} n_i\Big(\ln\Big|x - \frac{p_i}{\varepsilon}\Big| - \ln|x|\Big) \\ -2e^{\xi_{\varepsilon,L}}\big(L + 2\eta_{\varepsilon,L}\big) \displaystyle\sum_{i=1}^{\ell} n_i\Big(\ln\Big|x - \frac{p_i}{\varepsilon}\Big| - \ln|x|\Big) \end{pmatrix}$$

and

$$\begin{pmatrix} b_{11}(x) \\ b_{12}(x) \end{pmatrix} = \mathcal{P}_L^{-1} \begin{pmatrix} 2A_{\varepsilon,L} + 2e^{\phi_\lambda^L} - 2e^{\bar{u}_{\varepsilon,L}} \\ 2B_{\varepsilon,L} - LA_{\varepsilon,L} + L\big(e^{\bar{u}_{\varepsilon,L}} - e^{\phi_\lambda^L}\big) + 2e^{\phi_\lambda^L}\bar{\phi}_\lambda^L - 2e^{\bar{u}_{\varepsilon,L}}\bar{v}_{\varepsilon,L} \end{pmatrix}.$$

Without loss of generality, we may assume that

$$\|\mathcal{A}_{\varepsilon,L}\|_{L^\infty} = \max(\|\mathcal{A}_{\varepsilon,L}\|_{L^\infty}, \|\mathcal{B}_{\varepsilon,L}\|_{L^\infty}) \ \text{ and } \ |\mathcal{A}_{\varepsilon,L}(x_{\varepsilon,L})| = \|\mathcal{A}_{\varepsilon,L}\|_{L^\infty} \text{ for } x_{\varepsilon,L} \in \mathbf{R}^2.$$

By virtue of $\phi_\lambda^L(x) = O(e^{-\sqrt{-\lambda_{1,L}}|x|})$, $\bar{\phi}_\lambda^L(x) = O(e^{-\sqrt{-\lambda_{1,L}}|x|})$ as $|x| \to +\infty$ and the above claim, we conclude that

$$\begin{aligned} |\mathcal{A}_{\varepsilon,L}(x)| + |\mathcal{B}_{\varepsilon,L}(x)| \le o(1)\Big[\|\mathcal{A}_{\varepsilon,L}\|_{L^\infty(\mathbf{R}^2)} + \|\mathcal{B}_{\varepsilon,L}\|_{L^\infty(\mathbf{R}^2)}\Big] + g(\varepsilon, L) \\ + \|\mathcal{P}_L^{-1}\| f(\varepsilon, L) \end{aligned} \tag{6.21}$$

for some constant $C > 0$ for large $|x|$, where

$$\begin{cases} f(\varepsilon, L) = 2\int_{\Lambda_\varepsilon} K_{1,L}(x,y)\Big[e^{\phi_\lambda(y)} - 4e^{\bar{u}_\varepsilon(y)}\Big]dy \\ \qquad\qquad + \int_{\Lambda_\varepsilon} K_{2,L}(x,y)\Big[L\big(e^{\bar{u}_{\varepsilon,L}(y)} - e^{\phi_\lambda^L(y)}\big) + 2e^{\phi_\lambda^L(y)}\bar{\phi}_\lambda^L(y) - 2e^{\bar{u}_{\varepsilon,L}(y)}\bar{v}_{\varepsilon,L}(y)\Big]dy, \\ g(\varepsilon, L) = \displaystyle\sum_{i=1}^{\ell} \ln\left\|\frac{|x - \frac{p_i}{\varepsilon}|}{|x|}\right\|_{L^\infty(\mathbf{R}^2\backslash\Lambda_\varepsilon)}. \end{cases}$$

If $\{x_{\varepsilon,L}\}$ is a bounded sequence, then $\|\mathcal{A}_{\varepsilon,L}\|_{L^\infty(\mathbf{R}^2)} \to 0$ and $\|\mathcal{B}_{\varepsilon,L}\|_{L^\infty(\mathbf{R}^2)} \to 0$ uniformly on $[0, \widetilde{L}]$ as $\varepsilon \to \infty$ by the above claim. If $\{x_{\varepsilon,L}\}$ is an unbounded sequence, then by (6.21) and the fact that $\big(f(\varepsilon,L),g(\varepsilon,L)\big)\to(0,0)$ uniformly on $[0,\widetilde{L}]\times[0,\widetilde{L}]$ as $\varepsilon\to\infty$, we also have $\|\mathcal{A}_{\varepsilon,L}\|_{L^\infty(\mathbf{R}^2)}\to 0$ and $\|\mathcal{B}_{\varepsilon,L}\|_{L^\infty(\mathbf{R}^2)}\to 0$ uniformly on $[0,\widetilde{L}]$ as $\varepsilon\to\infty$. Hence, $\|A_{\varepsilon,L}\|_{C^2(\mathbf{R}^2)}\to 0$ and $\|B_{\varepsilon,L}\|_{C^2(\mathbf{R}^2)}\to 0$ as $\varepsilon\to\infty$. This step is finished.

Finally, we complete the proof of Theorem 6.2 by Step 2 and the similar argument in the proof of Theorem 6.1. □

Thirdly, we discuss the case $\kappa^2 q\geq L$ for $L>0$ sufficiently large. The lemma mentioned below is very helpful for us to show Theorem 6.3.

Lemma 6.3. *Let $(u_\varepsilon^L,v_\varepsilon^L)$ be a topological solution of (6.1), where $\varepsilon=\frac{1}{\kappa q}=\frac{\kappa}{L}$ and $\kappa^2 q=L$. If $(\xi,\lambda_\xi)\in\{(p_i,n_i): i=1,\dots,\ell\}$ (resp., $\xi=O$), then $\Big(\hat{U}_{\varepsilon,\xi}^L(x),\hat{v}_{\varepsilon,\xi}^L(x)\Big)\to(-\infty,-1)$ uniformly on $\{B_R(O)\backslash\{O\}\}\times B_R(O)$ (resp., $\{B_R(O)\backslash\mathcal{Z}_\varepsilon\}\times B_R(O)$ as $(\varepsilon,L)\to(0,\infty)$ (resp., $(\varepsilon,L)\to(\varepsilon_0,\infty)$ for $\varepsilon_0\in(0,\infty]$) for all $R>0$, where $\mathcal{Z}_\varepsilon=\big\{\frac{p_1}{\varepsilon},\dots,\frac{p_\ell}{\varepsilon}\big\}$,*

$\hat{U}_{\varepsilon,\xi}^L(x)=\hat{u}_{\varepsilon,\xi}^L(x)-2\lambda_\xi\ln|x|$ and $\big(\hat{u}_{\varepsilon,\xi}^L(x),\hat{v}_{\varepsilon,\xi}^L(x)\big)=\big(u_\varepsilon^L(\varepsilon x+\xi),v_\varepsilon^L(\varepsilon x+\xi)\big)$ (resp.,

$$\hat{U}_{\varepsilon,\xi}^L(x)=\hat{u}_{\varepsilon,\xi}^L(x)-2\sum_{i=1}^{\ell}n_i\ln\Big|x-\frac{p_i}{\varepsilon}\Big| \quad \textit{and}$$

$$\big(\hat{u}_{\varepsilon,\xi}^L(x),\hat{v}_{\varepsilon,\xi}^L(x)\big)=\big(u_\varepsilon^L(\varepsilon x+\xi),v_\varepsilon^L(\varepsilon x+\xi)\big)\big).$$

Moreover, if $\xi\in\mathbf{R}^2\backslash\mathcal{Z}$ (resp., there exists a sequence $\{\xi_\varepsilon\}$ such that $\lim\limits_{\varepsilon\to\varepsilon_0}\frac{|p_i-\xi_\varepsilon|}{\varepsilon}=\infty$ for $i\in\{1,\dots,\ell\}$), then either $(\hat{u}_{\varepsilon,\xi}^L(x),\hat{v}_{\varepsilon,\xi}^L(x))\to(-\infty,-1)$ uniformly on any compact subset of $\mathbf{R}^2$ or $\big(\hat{u}_{\varepsilon,\xi}^L,\hat{v}_{\varepsilon,\xi}^L\big)$ converges to (u^*,v^*) in $C^2_{loc}(\mathbf{R}^2)\times C^2_{loc}(\mathbf{R}^2)$ as $(\varepsilon,L)\to(0,\infty)$ (resp., $(\varepsilon,L)\to(\varepsilon_0,\infty)$ for $\varepsilon_0\in(0,\infty]$), where (u^*,v^*) satisfies

$$\begin{cases}\Delta u^*=0 \ \ \textit{in}\ \mathbf{R}^2,\\ \Delta v^*=\big(1+v^*-e^{u^*}\big)\ \ \textit{in}\ \mathbf{R}^2.\end{cases}$$

Proof. First, we consider the part of this lemma as $(\varepsilon,L)\to(0,\infty)$. It is easy to see that $\big(\hat{u}_{\varepsilon,\xi}^L,\hat{v}_{\varepsilon,\xi}^L\big)$ satisfies

$$\begin{cases}\Delta\hat{u}_{\varepsilon,\xi}^L+\dfrac{2}{L}\big(1+\hat{v}_{\varepsilon,\xi}^L-e^{\hat{u}_{\varepsilon,\xi}^L}\big)=4\pi\lambda_\xi\delta_O \ \text{ in } B_{\frac{2d}{\varepsilon}}(O),\\[2mm] \Delta\hat{v}_{\varepsilon,\xi}^L-\big(1+\hat{v}_{\varepsilon,\xi}^L-e^{\hat{u}_{\varepsilon,\xi}^L}\big)-\dfrac{2}{L}e^{\hat{u}_{\varepsilon,\xi}^L}\hat{v}_{\varepsilon,\xi}^L=0 \ \text{ in } B_{\frac{2d}{\varepsilon}}(O).\end{cases}$$

Then

$$\Delta\Big(\hat{u}_{\varepsilon,\xi}^L+\frac{2}{L}\hat{v}_{\varepsilon,\xi}^L\Big)-\frac{4}{L^2}e^{\hat{u}_{\varepsilon,\xi}^L+\frac{2\varepsilon}{\kappa}\hat{v}_{\varepsilon,\xi}^L}e^{-\frac{2\varepsilon}{\kappa}\hat{v}_{\varepsilon,\xi}^L}\hat{v}_{\varepsilon,\xi}^L=0 \ \text{ in } \ B_{\frac{2d}{\varepsilon}}(O)\backslash\{O\}. \tag{6.22}$$

Indeed, since $\hat{u}_{\varepsilon,\xi}^L(x)<0$ and $-1\leq\hat{v}_{\varepsilon,\xi}^L(x)<0$ on $\mathbf{R}^2\backslash\{O\}$, it follows from (6.22) and Theorem 3 in [1] that either $\big\{\hat{u}_{\varepsilon,\xi}^L\big\}$ is bounded in $C_{loc}(\mathbf{R}^2\backslash\{O\})$ or $\hat{u}_{\varepsilon,\xi}^L(x)\to-\infty$

uniformly on any compact subset of $\mathbf{R}^2\backslash\{O\}$ as $(\varepsilon, L) \to (0, \infty)$. If $\{\hat{u}^L_{\varepsilon,\xi}\}$ is bounded in $C_{loc}(\mathbf{R}^2\backslash\{O\})$, then $\{\hat{U}^L_{\varepsilon,\xi}\}$ and $\{\hat{v}^L_{\varepsilon,\xi}\}$ are both bounded in $C_{loc}(\mathbf{R}^2)$ from the standard elliptic estimates and Remark 6.1. By passing to a subsequence if necessary, we may assume that $\hat{U}^L_{\varepsilon,\xi}$ and $\hat{v}^L_{\varepsilon,\xi}$ converge in $C^2_{loc}(\mathbf{R}^2)$ to some functions U^* and $v^* \in C^2(\mathbf{R}^2)$, which satisfy

$$\begin{cases} \Delta u^* = 4\pi\lambda_\xi \delta_O \text{ in } \mathbf{R}^2, \\ \Delta v^* = \left(1 + v^* - e^{u^*}\right) \text{ in } \mathbf{R}^2, \end{cases} \tag{6.23}$$

where $u^*(x) = U^*(x) + 2\lambda_\xi \ln|x|$. However, it contradicts the fact that $\hat{v}^L_{\varepsilon,\xi}(x) \in [-1, 0)$ for all $\varepsilon, L > 0$. Then $\hat{v}^L_{\varepsilon,\xi}$ converges in $C^2_{loc}(\mathbf{R}^2)$ to some function $v^* \in C^2(\mathbf{R}^2)$ as $(\varepsilon, L) \to (0, \infty)$ and $v^*(x)$ satisfies

$$\begin{cases} \Delta v^* = 1 + v^* \text{ in } \mathbf{R}^2, \\ \|v^*\|_{L^\infty(\mathbf{R}^2)} \le 1, \end{cases}$$

which implies that $v^* \equiv -1$.

The conclusion for the case as $(\varepsilon, L) \to (\varepsilon_0, \infty)$ for $\varepsilon_0 \in (0, \infty]$ can also be shown in the same way as mentioned above with some modification:

(i) $B_{\frac{2d}{\varepsilon}}(O)$ and $\{O\}$ replaced by $\mathbf{R}^2$ and $\mathcal{Z}_\varepsilon$, respectively;
(ii) equations in (6.23) replaced by

$$\begin{cases} \Delta u^* = 4\pi \sum_{i=1}^{\ell} n_i \delta_{\frac{p_i}{\varepsilon_0}} \text{ in } \mathbf{R}^2, \\ \Delta v^* = \left(1 + v^* - e^{u^*}\right) \text{ in } \mathbf{R}^2, \end{cases}$$

with $u^*(x) = U^*(x) + 2\sum_{i=1}^{\ell} n_i \ln\left|x - \frac{p_i}{\varepsilon_0}\right|$.

The proof of this lemma is complete. □

Theorem 6.3. *Let $p_1, \ldots, p_\ell \in \mathbf{R}^2$ and $n_1, \ldots, n_\ell \in \mathbf{R}^+ \cup \{0\}$ be given. Then there exists a positive constant $\widetilde{L} = \widetilde{L}(p_i, n_i)$ such that (6.1) possesses a unique topological solution for any κ and q satisfying $\kappa^2 q \ge \widetilde{L}$.*

Proof. Let $(u^L_\varepsilon, v^L_\varepsilon)$ be a topological solution of (6.1). We set $\hat{u}^L_{\varepsilon,\xi}(x) = u^L_\varepsilon(\varepsilon x + \xi)$ and $\hat{v}^L_{\varepsilon,\xi}(x) = \hat{v}^L_\varepsilon(\varepsilon x + \xi)$ in $\mathbf{R}^2$, where $\xi \in \mathbf{R}^2$, $\varepsilon = \frac{1}{\kappa q}$ and $\kappa^2 q = L$. On the contrary, we may assume that there exists a sequence $\{(\varepsilon_n, L_n)\}_{n\in N}$ such that $(\varepsilon_n, L_n) \to (\varepsilon_0, \infty)$ as $n \to \infty$ for some $\varepsilon_0 \in [0, \infty)$ or $\varepsilon_0 = \infty$, which implies that $(u^{L_n}_{1\varepsilon_n}(x), v^{L_n}_{1\varepsilon_n}(x))$ and

$(u^{L_n}_{2\varepsilon_n}(x), v^{L_n}_{2\varepsilon_n}(x))$ are two distinct topological solutions of (6.1) for each $n \in N$. Without loss of generality, let

$$\big(v^{L_n}_{1\varepsilon_n} - v^{L_n}_{2\varepsilon_n}\big)(y_n) = \big\|v^{L_n}_{1\varepsilon_n} - v^{L_n}_{2\varepsilon_n}\big\|_{L^\infty(\mathbf{R}^2)} \geq \big\|u^{L_n}_{1\varepsilon_n} - u^{L_n}_{2\varepsilon_n}\big\|_{L^\infty(\mathbf{R}^2)} \tag{6.24}$$

for each $n \in N$ and $x_{n,\xi} = \frac{y_n - \xi}{\varepsilon_n}$, where $\xi \in \mathcal{Z}$ (resp., $\xi = O$) for $\varepsilon_0 = 0$ (resp., $\varepsilon_0 \in (0, \infty]$). We only consider the situation for $\varepsilon_0 = 0$, the proof of the other case for $\varepsilon_0 \in (0.\infty]$ can be done in the same way.

Suppose that $\{x_{n,\xi}\}$ is bounded for some $\xi \in \mathcal{Z}$. Set

$$(A_n, B_n) = \left(\frac{\hat{u}^{L_n}_{1\varepsilon_n,\xi} - \hat{u}^{L_n}_{2\varepsilon_n,\xi}}{\big\|\hat{v}^{L_n}_{1\varepsilon_n,\xi} - \hat{v}^{L_n}_{2\varepsilon_n,\xi}\big\|_{L^\infty(\mathbf{R}^2)}}, \frac{\hat{v}^{L_n}_{1\varepsilon_n,\xi} - \hat{v}^{L_n}_{2\varepsilon_n,\xi}}{\big\|\hat{v}^{L_n}_{1\varepsilon_n,\xi} - \hat{v}^{L_n}_{2\varepsilon_n,\xi}\big\|_{L^\infty(\mathbf{R}^2)}}\right).$$

Then $(A_n(x), B_n(x))$ satisfies

$$\begin{cases} \Delta A_n + \dfrac{2}{L_n} B_n - \dfrac{2}{L_n} e^{\eta_n} A_n = 0 \text{ in } \mathbf{R}^2, \\ \Delta B_n + \Big(1 - \dfrac{2}{L_n}\hat{v}^{L_n}_{2\varepsilon_n,\xi}\Big) e^{\eta_n} A_n - \Big(1 + \dfrac{2}{L_n} e^{\hat{u}^{L_n}_{1\varepsilon_n,\xi}}\Big) B_n = 0 \text{ in } \mathbf{R}^2, \\ A_n(x) \to 0, \;\; B_n(x) \to 0 \;\text{ as }\; |x| \to \infty, \end{cases} \tag{6.25}$$

where $\eta_n(x) \in \big[\min\big\{\hat{u}^{L_n}_{1\varepsilon_n,\xi}(x), \hat{u}^{L_n}_{2\varepsilon_n,\xi}(x)\big\}, \max\big\{\hat{u}^{L_n}_{1\varepsilon_n,\xi}(x), \hat{u}^{L_n}_{2\varepsilon_n,\xi}(x)\big\}\big]$. By the standard elliptic estimates and Lemma 6.3, we obtain that $\big(\hat{u}^{L_n}_{1\varepsilon_n,\xi}, \hat{v}^{L_n}_{1\varepsilon_n,\xi}\big) \to \big(-\infty, -1\big)$, $\big(\hat{u}^{L_n}_{2\varepsilon_n,\xi}, \hat{v}^{L_n}_{2\varepsilon_n,\xi}\big) \to \big(-\infty, -1\big)$ and $\big(A_n, B_n\big) \to \big(A, B\big)$ in $C^2_{loc}(\mathbf{R}^2) \times C^2_{loc}(\mathbf{R}^2)$ as $n \to \infty$, where $A \in [0, 1]$ and $B(x) \in C^2(\mathbf{R}^2)$ verifying

$$\Delta B = B \;\text{ in }\; \mathbf{R}^2 \;\text{ and }\; \max_{\mathbf{R}^2}\{|B(x)|\} = 1.$$

Now we want to show that $B(x) = B(|x|)$ for all $x \in \mathbf{R}^2$. To prove it by contradiction, without loss of generality, we may assume that there exist two distinct points $x_1 \in \mathbf{R}^2$ and $x_2 \in \mathbf{R}^2$ such that $|x_1| = |x_2| = R$ and $\left|\frac{B(x_1)}{B(x_2)}\right| > 1$. Then there exists a constant $\gamma > 1$ such that

$$\frac{|\hat{v}^{L_n}_{1\varepsilon_n,\xi}(x_1) - \hat{v}^{L_n}_{2\varepsilon_n,\xi}(x_1)| + 1}{|\hat{v}^{L_n}_{1\varepsilon_n,\xi}(x_2) - \hat{v}^{L_n}_{2\varepsilon_n,\xi}(x_2)| + 1} > \gamma > 1 \;\text{ for large } n,$$

which contradicts $\displaystyle\lim_{n\to\infty} \frac{|\hat{v}^{L_n}_{1\varepsilon_n,\xi}(x_1) - \hat{v}^{L_n}_{2\varepsilon_n,\xi}(x_1)| + 1}{|\hat{v}^{L_n}_{1\varepsilon_n,\xi}(x_2) - \hat{v}^{L_n}_{2\varepsilon_n,\xi}(x_2)| + 1} = 1$. Since $B(x) = B(r)$ for $r = |x|$, $B(r)$ satisfies $B'' + \dfrac{1}{r}B' = B$ and $|B(r)| \leq 1$ on $[0, \infty)$. Hence $B(r) = 0$ on $[0, \infty)$, which is impossible because of $B \not\equiv 0$ in $\mathbf{R}^2$.

If $\{x_{n,\xi}\}$ is not a bounded sequence for any $\xi \in \mathcal{Z}$, then we take $\xi = y_n$ for each n. We want to claim that $\eta_n(x) \to 0$ in $C^2_{loc}(\mathbf{R}^2)$ as $n \to \infty$. Furthermore, $A = 1$ and

$\left(u_{1\varepsilon_n}^{L_n}, u_{2\varepsilon_n}^{L_n}\right) \to (0,0)$ in $C^2_{loc}(\mathbf{R}^2) \times C^2_{loc}(\mathbf{R}^2)$ as $n \to \infty$. On the contrary, we may assume that $\eta_n \to -\delta$ (passing to a subsequence if necessary) in $C^2_{loc}(\mathbf{R}^2)$ as $n \to \infty$ for some $\delta \in [0,\infty)$ or $\eta_n \to -\infty$ uniformly on any compact subset of $\mathbf{R}^2$ from (6.24), Lemma 6.3 and $\Delta\left(u_{1\varepsilon_n}^{L_n} - u_{2\varepsilon_n}^{L_n}\right) = \frac{2}{L_n} e^{\eta_n}\left(u_{1\varepsilon_n}^{L_n} - u_{2\varepsilon_n}^{L_n}\right) - \frac{2}{L_n}\left(v_{1\varepsilon_n}^{L_n} - v_{2\varepsilon_n}^{L_n}\right)$. It is easy to see that $(A_n, B_n) \to (A, B)$ in $C^2_{loc}(\mathbf{R}^2) \times C^2_{loc}(\mathbf{R}^2)$ as $n \to \infty$ for some $A \in [0,1]$ and $B(x)$ satisfies $\|B\|_{L^\infty(\mathbf{R}^2)} = 1$,

$$\Delta B = \Big(1 - e^{-\delta}\frac{A}{B}\Big) B \ \text{ in } \mathbf{R}^2 \ \text{ if } \ \eta_n \to -\delta \ \text{ in } \ C^2_{loc}(\mathbf{R}^2) \ \text{ as } \ n \to \infty,$$

and

$$\Delta B = B \ \text{ in } \mathbf{R}^2 \ \text{ if } \ \eta_n \to -\infty \text{ uniformly on any compact subset of as } \ n \to \infty.$$

It yields a contradiction and hence the claim holds.

Now, let $(\varepsilon, \xi) = (1, y_n)$ and $(\mathcal{A}_n, \mathcal{B}_n) = \left(\frac{\hat{u}_{11,\xi}^{L_n} - \hat{u}_{21,\xi}^{L_n}}{\|\hat{v}_{11,\xi}^{L_n} - \hat{v}_{21,\xi}^{L_n}\|_{L^\infty(\mathbf{R}^2)}}, \frac{\hat{v}_{11,\xi}^{L_n} - \hat{v}_{21,\xi}^{L_n}}{\|\hat{v}_{11,\xi}^{L_n} - \hat{v}_{21,\xi}^{L_n}\|_{L^\infty(\mathbf{R}^2)}}\right)$ for all $n \in N$. Then $(\mathcal{A}_n, \mathcal{B}_n)$ satisfies

$$\begin{cases} \Delta \mathcal{A}_n + 2q\mathcal{B}_n - 2qe^{\tau_n}\mathcal{A}_n = 0 \text{ in } \mathbf{R}^2, \\ \Delta \mathcal{B}_n + q\Big(L_n - 2\hat{v}_{21,\xi}^{L_n}\Big) e^{\eta_n}\mathcal{A}_n - q\Big(L_n + 2e^{\hat{u}_{11,\xi}^{L_n}}\Big)\mathcal{B}_n = 0 \text{ in } \mathbf{R}^2, \\ \mathcal{A}_n(x) \to 0, \ \ \mathcal{B}_n(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{6.26}$$

where $\tau_n(x) \in \left[\min\left\{\hat{u}_{11,\xi}^{L_n}(x), \hat{u}_{21,\xi}^{L_n}(x)\right\}, \max\left\{\hat{u}_{11,\xi}^{L_n}(x), \hat{u}_{21,\xi}^{L_n}(x)\right\}\right]$. Due to the above claim, we obtain $\left(\hat{u}_{11,\xi}^{L_n}(O), \hat{u}_{21,\xi}^{L_n}(O), \hat{v}_{11,\xi}^{L_n}(O), \hat{v}_{21,\xi}^{L_n}(O), \mathcal{A}_n(O)\right) \to (0,0,0,0,1)$ as $n \to \infty$. From (6.19) and (6.26), we get that

$$\begin{cases} \Delta \mathcal{B}_n^* + q\lambda_{2,L_n}\mathcal{B}_n^* + 2qa_n\mathcal{A}_n^* + 2qb_n\mathcal{B}_n^* = 0 \text{ in } \mathbf{R}^2, \\ \|\mathcal{B}_n^*\|_{L^\infty(\mathbf{R}^2)} \le 2, \ \ \mathcal{B}_n^*(O) \to \lim_{n\to\infty} \mathcal{A}_n(O) = 1 \ \text{ as } \ n \to \infty, \end{cases}$$

where

$$\begin{pmatrix} \mathcal{A}_n^* \\ \mathcal{B}_n^* \end{pmatrix} = \Big(\frac{\lambda_{1,L_n} - \lambda_{2,L_n}}{2 + \lambda_{1,L_n}}\Big) \mathcal{P}_{L_n}^{-1} \begin{pmatrix} \mathcal{A}_n \\ \mathcal{B}_n \end{pmatrix},$$

$$\mathcal{P}_{L_n}^{-1} = \begin{pmatrix} \dfrac{2 + \lambda_{2,L_n}}{\lambda_{2,L_n} - \lambda_{1,L_n}} & \dfrac{2}{\lambda_{1,L_n} - \lambda_{2,L_n}} \\ \dfrac{2 + \lambda_{1,L_n}}{\lambda_{1,L_n} - \lambda_{2,L_n}} & \dfrac{2}{\lambda_{2,L_n} - \lambda_{1,L_n}} \end{pmatrix}$$

and

$$\begin{pmatrix} a_n(x) \\ b_n(x) \end{pmatrix} = \begin{pmatrix} \frac{(2+\lambda_{1,L_n})(1-e^{\tau_n(x)})}{\lambda_{1,L_n} - \lambda_{2,L_n}} + \frac{\left[L_n - \left(L_n - 2\hat{v}_{21,\xi}^{L_n}(x)\right)e^{\tau_n(x)}\right]}{\lambda_{1,L_n} - \lambda_{2,L_n}} + \frac{(2+\lambda_{1,L_n})(e^{\hat{u}_{11,\xi}^{L_n}(x)} - 1)}{\lambda_{1,L_n} - \lambda_{2,L_n}} \\ \frac{(2+\lambda_{1,L_n})(1-e^{\tau_n(x)})}{\lambda_{1,L_n} - \lambda_{2,L_n}} + \frac{\left[L_n - \left(L_n - 2\hat{v}_{21,\xi}^{L_n}(x)\right)e^{\tau_n(x)}\right]}{\lambda_{1,L_n} - \lambda_{2,L_n}} + \frac{(2+\lambda_{2,L_n})(e^{\hat{u}_{11,\xi}^{L_n}(x)} - 1)}{\lambda_{1,L_n} - \lambda_{2,L_n}} \end{pmatrix}.$$

We note that there exists a positive constant C such that $\max\{\|a_n\|_{L^\infty(\mathbf{R}^2)}, \|b_n\|_{L^\infty(\mathbf{R}^2)}\} \le C$ for large n. It turns out that there exists a constant $n_0 > 0$ such that

$$\Delta \mathcal{B}_n^* \ge \frac{q\lambda_{2,L_n}}{4} \mathcal{B}_n^* \text{ on } B_R(O) \cap \mathcal{E}_n \text{ for all } (n, R) \in [n_0, \infty) \times (0, \infty),$$

where $\mathcal{E}_n = \{x : \mathcal{B}_n^*(x) \ge \frac{1}{2}\}$. Since $\mathcal{B}_n^*(O) \to \lim_{n\to\infty} \mathcal{A}_n(O) = 1$ as $n \to \infty$, there exists a constant $n_1 \ge n_0$ such that

$$B_R(O) \cap \mathcal{E}_n = B_R(O) \text{ for all } (n, R) \in [n_1, \infty) \times (0, \infty),$$

which contradicts $\|\mathcal{B}_n^*\|_{L^\infty(\mathbf{R}^2)} \le 2$ for $n \ge n_1$.

On the other hand, if (6.24) not true and $\|\hat{u}_{1\varepsilon_n}^{L_n} - \hat{u}_{2\varepsilon_n}^{L_n}\|_{L^\infty(\mathbf{R}^2)} \to \infty$ as $n \to \infty$, then by Kelvin transformation of (6.25) and

$$A_n(x) = O(e^{-\sqrt{2/L_n}|x|}), \quad B_n(x) = O(e^{-|x|}) \text{ as } |x| \to \infty$$

for each $n \in \mathbf{N}$, we see that $\frac{\hat{u}_{1\varepsilon_n}^{L_n} - \hat{u}_{2\varepsilon_n}^{L_n}}{\|\hat{u}_{1\varepsilon_n}^{L_n} - \hat{u}_{2\varepsilon_n}^{L_n}\|_{L^\infty(\mathbf{R}^2)}}$ converges to zero uniformly on $B_R^c(O)$ for all $R > 0$. However, $\frac{\hat{u}_{1\varepsilon_n}^{L_n} - \hat{u}_{2\varepsilon_n}^{L_n}}{\|\hat{u}_{1\varepsilon_n}^{L_n} - \hat{u}_{2\varepsilon_n}^{L_n}\|_{L^\infty(\mathbf{R}^2)}}$ converges to 1 uniformly on $B_R(O)$ for all $R > 0$ and hence it yields a contradiction. Therefore, we complete the proof of Theorem 6.3. □

Finally, by combining results in [4], Theorem 3.2 and Theorems 6.1–6.3, we are in a position to demonstrate Theorem 1.1.

Proof of Theorem 1.1. For all $(\kappa, q) \in (0, \infty) \times (0, \infty)$, the existence of topological solutions to (1.1) and the uniqueness of topological solutions to (1.8) have been proved in [4] and by Theorem 3.2, respectively. From Theorem 6.3, there exists a positive constant $L_0 = L_0(p_1, \ldots, p_\ell, n_1, \ldots, n_\ell)$ such that (1.1) possesses a unique topological solution in $\mathbf{R}^2$ for any $(\kappa, q) \in \{(\kappa, q) : \kappa^2 q \ge L_0\}$. Furthermore, Theorem 6.1 and Theorem 6.2 deduce that there exist two positive constants $q_0 = q_0(p_1, \ldots, p_\ell, n_1, \ldots, n_\ell, L_0)$ and $q_1 = q_1(p_1, \ldots, p_\ell, n_1, \ldots, n_\ell, L_0)$ such that (1.1) possesses a unique topological solution in $\{(\kappa, q) : q \ge q_0,\ \kappa^2 q \le L_0\} \cup \{(\kappa, q) : 0 < q \le q_1,\ \kappa^2 q \le L_0\}$. Therefore, this theorem is proved. □

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