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Minimizers of Caffarelli–Kohn–Nirenberg Inequalities with the Singularity on the Boundary

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Abstract

Let Ω be a bounded smooth domain in $\mathbf{R}^N$, $N \geqq 3$, and $D_a^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm:

$$||u||_a^2 = \int_\Omega |x|^{-2a}|\nabla u|^2 \mathrm{d}x.$$

The Caffarelli–Kohn–Nirenberg inequalities state that there is a constant $C > 0$ such that

$$\left(\int_\Omega |x|^{-bq}|u|^q \mathrm{d}x\right)^{\frac{2}{q}} \leqq C \int_\Omega |x|^{-2a}|\nabla u|^2 \mathrm{d}x \tag{0.1}$$

for $u \in D_a^{1,2}(\Omega)$ and

$$-\infty < a < \frac{N-2}{2}, \quad 0 \leqq b-a \leqq 1, \quad q = \frac{2N}{N-2+2(b-a)}.$$

We prove the best constant for (0.1)

$$S(a,b;\Omega) = \inf_{u \in D_a^{1,2}\setminus\{0\}} \frac{\int_\Omega |x|^{-2a}|\nabla u|^2 \mathrm{d}x}{\left(\int_\Omega |x|^{-bq}|u|^q \mathrm{d}x\right)^{\frac{2}{q}}}$$

is always achieved in $D_a^{1,2}(\Omega)$ provided that $0 \in \partial\Omega$ and the mean curvature $H(0) < 0$, where a, b satisfies

$$(i)\ a < b < a+1 \text{ and } N \geqq 3, \text{ or } (ii)\ b = a > 0 \text{ and } N \geqq 4.$$

If $a = 0$ and $1 > b > 0$, then the result was proved by Ghoussoub and Robert [12].

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1. Introduction

The Caffarelli–Kohn–Nirenberg(CKN) inequalities state that there is a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbf{R}^N)$, it holds

$$\left(\int_{\mathbf{R}^N} |x|^{-bq}|u|^q \mathrm{d}x\right)^{\frac{2}{q}} \leqq C \int_{\mathbf{R}^N} |x|^{-2a}|\nabla u|^2 \mathrm{d}x \tag{1.1}$$

for $N \geqq 3$:

$$-\infty < a < \frac{N-2}{2}, \quad 0 \leqq b-a \leqq 1, \quad q = \frac{2N}{N-2+2(b-a)}. \tag{1.2}$$

For a proof of the CKN inequality and its generalization, see [4] and [15]. Let Ω be a domain in $\mathbf{R}^N$, and $D_a^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_a^2 = \int_\Omega |x|^{-2a}|\nabla u|^2 \mathrm{d}x. \tag{1.3}$$

Then inequality (1.1) can be extended to all functions in $D_a^{1,2}(\Omega)$. For a, b and q satisfying (1.2), the best constant of (1.1) is defined by

$$S(a, b; \Omega) = \inf_{u \in D_a^{1,2}(\Omega)\setminus\{0\}} E_{a,b}(u), \tag{1.4}$$

where

$$E_{a,b}(u) = \frac{\int_\Omega |x|^{-2a}|\nabla u|^2 \mathrm{d}x}{(\int_\Omega |x|^{-bq}|u|^q \mathrm{d}x)^{\frac{2}{q}}}.$$

Obviously, the extremal functions for $S(a, b; \Omega)$ are the least-energy solution of the Euler–Lagrangian equation:

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) = |x|^{-bq}u^{q-1}, & u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

In this paper, we want to study the problem whether $S(a, b; \Omega)$ can be attained by some function in $D_a^{1,2}(\Omega)$. When Ω is the whole space $\mathbf{R}^N$, the existence or non-existence of minimizers for (1.4) have been extensively studied for the past 20 years, see [5,6,13,17,20] and references therein. The result can be briefly summarized in the following:

Theorem A. *Suppose a, b and q satisfy the condition* (1.2). Then minimizers exist for the best constant $S(a, b; \mathbf{R}^N)$ if and only if a, b satisfies

$$\text{either } a < b < a+1 \quad \text{or} \quad b = a \geqq 0. \tag{1.6}$$

After Theorem A, it is natural to consider this minimization problem for a domain Ω in $\mathbf{R}^N$. By scaling, for any $\lambda > 0$ the best constant satisfies

$$S(a, b; \lambda\Omega) = S(a, b; \Omega),$$

where $\lambda\Omega = \{\lambda x | x \in \Omega\}$. Thus, if $0 \in \Omega$, then $S(a, b; \Omega) = S(a, b; \mathbf{R}^N)$. Following this equality, we can conclude that $S(a, b; \Omega)$ is never achieved if $0 \in \Omega$ and $\Omega \neq \mathbf{R}^N$. However, if $0 \in \partial\Omega$ and $\overline{\Omega}$ is C^2 near 0, Ghoussoub and Robert [11,12] considered the case $a = 0$ and $b > 0$, and proved the following remarkable result.

Theorem B. *Let* $a = 0,\ 0 < b < 1,$ *and* $0 \in \partial\Omega$. *Suppose the mean curvature* H *of* $\partial\Omega$ *at* 0 *is negative, then minimizers for the best constant* $S(0, b; \Omega)$ *can be attained.*

Throughout the paper, we always assume $0 \in \partial\Omega$ and $\overline{\Omega}$ is C^2 in a neighborhood of 0. Our purpose is to extend the result of Ghoussoub and Robert to cover the whole range of a and b, where a, b satisfy

$$\text{either } (i)\ a < b < a + 1 \quad \text{and } N \geqq 3, \quad \text{or } (ii)\ b = a > 0 \quad \text{and } N \geqq 4. \tag{1.7}$$

Our main result is the following theorem.

Theorem 1.1. *Let* a, b *and* q *satisfy* (1.2) *and* (1.7). *Suppose* $0 \in \partial\Omega$ *and the mean curvature* $H(0) < 0$. *Then the best constant* $S(a, b; \Omega)$ *can be achieved in* $D_a^{1,2}(\Omega)$.

Comparing Theorem 1.1 with Theorem A for the case where $N = 3$ and $a = b > 0$, the minimization problem remains unsolved. We will discuss it in a forthcoming paper. Another remark is that minimizers do not exist in *any* domain $\Omega \neq \mathbf{R}^N$ when $a = b \leqq 0$. See [5,18]. Thus, when $0 \in \partial\Omega$ and $H(0) < 0$, Theorem 1.1 completely solves the existence (or non-existence) of minimizers for CKN inequalities except for $N = 3$ and $a = b > 0$.

Our proof of Theorem 1.1 relies on the transformation of u by letting

$$w(x) = |x|^{-a} u(x) \quad \text{for } x \in \Omega. \tag{1.8}$$

By a straightforward computation, we have for any $u \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\int_\Omega |x|^{-2a} |\nabla u|^2 \, \mathrm{d}x \\ &\quad = \int_\Omega |x|^{-2a} (a^2 |x|^{2a-2} |\nabla w(x)|^2 + 2a |x|^{2a-2} w(x) x \\ &\quad\quad \cdot \nabla w(x) + |x|^{2a} |\nabla w(x)|^2) \mathrm{d}x \\ &\quad = \int_\Omega |\nabla w(x)|^2 \mathrm{d}x - \gamma \int_\Omega \frac{w^2(x)}{|x|^2} \mathrm{d}x, \end{aligned} \tag{1.9}$$

where

$$\gamma = a(N - 2 - a). \tag{1.10}$$

The last equality in (1.9) can be obtained by integration by parts:

$$\begin{aligned}2a\int_{\Omega}|x|^{2a-2}w(x)x\cdot\nabla w(x)\mathrm{d}x \\ = -a(N-2+2a)\int_{\Omega}|x|^{2a-2}w^2(x)\mathrm{d}x.\end{aligned} \tag{1.11}$$

We note that if $a<\frac{N-2}{2}$, then both $\int_{\Omega}|\nabla w|^2\mathrm{d}x$ and $\int_{\Omega}\frac{w^2}{|x|^2}$ are finite, and by the Hardy inequality we have

$$\left(\frac{N-2}{2}\right)^2\int_{\Omega}\frac{w^2}{|x|^2}\,\mathrm{d}x\leqq\int_{\Omega}|\nabla w|^2\ \mathrm{d}x.$$

Therefore, if $a<\frac{N-2}{2}$, then

$$u\in D_a^{1,2}(\Omega)\quad\text{if and only if } w\in H_0^1(\Omega),$$

and furthermore,

$$E_{a,b}(u)=\frac{\int_{\Omega}|\nabla w|^2-\gamma\int_{\Omega}\frac{w^2}{|x|^2}\mathrm{d}x}{(\int_{\Omega}\frac{w^{2^*}}{|x|^s}\mathrm{d}x)^{\frac{2}{2^*}}}, \tag{1.12}$$

where $s=(b-a)q$ and $2^*=\frac{2N}{N-2+2(b-a)}$. It is easy to see s and 2^* satisfy

$$0<s<2\quad\text{and}\quad 2^*=\frac{2(N-s)}{N-2}. \tag{1.13}$$

when $0<b-a<1$, and $s=0,\ 2^*=\frac{2N}{N-2}$ when $b=a$. From now on, we will denote $\frac{2(N-s)}{N-2}$ by $2^*(s)$ for $0\leqq s\leqq 2$. Furthermore, the best constant $S(a,b;\Omega)$ can be expressed in terms of w by

$$S(a,b,\Omega)=\inf_{w\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_{\Omega}|\nabla w|^2-\gamma\int_{\Omega}\frac{w^2}{|x|^2}\mathrm{d}x}{(\int_{\Omega}\frac{w^{2^*(s)}}{|x|^s}\mathrm{d}x)^{\frac{2}{2^*(s)}}}, \tag{1.14}$$

and the condition (1.7) can be translated into:

$$\begin{aligned}&\text{(i) } \gamma<(\tfrac{N-2}{2})^2\quad\text{and}\quad 0<s<2\quad\text{for } N\geqq 3,\quad\text{or}\\ &\text{(ii) } s=0\quad\text{and } 0<\gamma<\left(\tfrac{N-2}{2}\right)^2\quad\text{for } N\geqq 4.\end{aligned} \tag{1.15}$$

Also by (1.8), u is a solution of Equation (1.5) if and only if $w(x)$ is a solution of

$$\begin{cases}\triangle w+\gamma\frac{w}{|x|^2}+\frac{w^{2^*(s)-1}}{|x|^s}=0\quad\text{in}\quad\Omega,\\ w=0\quad\text{on}\quad\partial\Omega.\end{cases} \tag{1.16}$$

Therefore, instead of finding minimizers of (1.4), we will study the minimizing problem of (1.14), and prove the following result which is equivalent to Theorem 1.1.

Theorem 1.2. *Let* $0 \in \partial\Omega$, $H(0) < 0$, *and* γ, s *satisfy* (1.15). *Then there is* $w \in H_0^1(\Omega)$ *which attains the best constant of* (1.14).

To prove Theorem 1.2, we should compare $S(a, b; \Omega)$ with $S(a, b; \mathbf{R}_+^N)$, where $\mathbf{R}_+^N$ is the half-upper space $\{(x', x_N)|x_N > 0\}$. It is easy to see that the inequality,

$$S(a, b; \Omega) \leqq S(a, b; \mathbf{R}_+^N),$$

holds always. However, for the proof of Theorem 1.2 we need the strict inequality

$$S(a, b; \Omega) < S(a, b; \mathbf{R}_+^N). \tag{1.17}$$

In order to derive the strict inequality, we consider a least-energy solution on $\mathbf{R}_+^N$:

$$\begin{cases} \triangle w + \gamma \frac{w}{|x|^2} + \frac{w^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \mathbf{R}_+^N, \\ w = 0 \quad \text{on } \partial\mathbf{R}_+^N. \end{cases} \tag{1.18}$$

Under the condition (1.7), the existence of least-energy solutions was proved by Bartsch et al. [2], Lin and Wang [18].

Let $\alpha_0 \in \left(-\frac{N-2}{2}, 1\right)$ be the root of

$$-\alpha_0(N-2+\alpha_0) + N - 1 = \gamma \tag{1.19}$$

if $\gamma \in (0, (\frac{N-2}{2})^2)$. Note that $\alpha_0(0) = 1$, $\alpha_0(N-1) = 0$ and $\alpha_0(\gamma)$ is decreasing in γ. If $\gamma \leqq 0$, then we simply let $\alpha_0 = 1$. The following theorem is important when (1.17) is concerned. See Section 3.

Theorem 1.3. *Suppose* $w(x) \in H_0^1(\mathbf{R}_+^N)$ *is a positive solution of* (1.18). *Then for any small* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$\begin{aligned} &w(x) \leqq C_\varepsilon |x|^{2-(N+\alpha_0)+\varepsilon}, \quad \textit{and} \\ &|\nabla w(x)| \leqq C_\varepsilon |x|^{1-(N+\alpha_0)+\varepsilon} \end{aligned} \tag{1.20}$$

for $|x| \geqq 1$. *Moreover, the following is true.*

(i) *w is axially symmetric with respect to the* x_N *axis, and there is a positive constant* λ *such that*

$$w(x) = \left(\frac{\lambda}{|x|}\right)^{N-2} w\left(\frac{\lambda^2 x}{|x|^2}\right). \tag{1.21}$$

(ii)

$$2\gamma \int_{\mathbf{R}_+^N} \frac{x_N|x'|^2}{|x|^4} w^2(x)\, dx < \int_{\mathbf{R}^{N-1}} |x'|^2 |\tfrac{\partial w}{\partial x_N}(x', 0)|^2 dx'. \tag{1.22}$$

By (1.17), Theorem 1.2 can be proved through blowup arguments. Although this is a standard procedure, there requires some additional care to go through all details. Usually, blowing up solutions after suitable scaling will converge to a *bounded* entire solution of (1.18). Indeed, the method applied in [7,8] can work out for the case $\gamma < 0$. However, when $\gamma > 0$, positive solutions of (1.18) generally are not bounded near the origin, due to the singular term $\frac{w}{|x|^2}$. Indeed, this is the case when $\gamma \geqq N-1$. Thus, the blowing up method in [7,8] cannot work for the case $\gamma > 0$. Hence we need a non-standard way to re-scale solutions. Fortunately, our method can work out for the case $\gamma > 0$. See Section 5 for the case $s > 0$ and Section 6 for the case $s = 0$. In Section 6, we prove Theorem 1.2 for the case $s = 0$, that is, $b = a$. In this case, $2^* = \frac{2N}{N-2}$ is the Sobolev exponent. Thus, $S(a, a; \Omega)$ has to compare not only with $S(a, a; \mathbf{R}^N_+)$, but also with the Sobolev best constant S_N. Hence as in [3], the dimension plays a crucial role. For $N \geqq 4$, the strict inequality

$$S(a, a; \Omega) < S_N$$

also holds, see Lemma 6.1. In a forthcoming paper, we will study the case $N = 3$.

The blowing up argument for the case $\gamma > 0$ cannot work for the case $\gamma < 0$. The case for $\gamma < 0$ is discussed in Section 4. In Section 2, Theorem 1.3 is proved. The best constant $S(a, b; \Omega)$ is computed in Section 3, where the strict inequality (1.17) is proved.

2. Decay estimate of solutions on $\mathbf{R}^N_+$

In this section, we consider Equation (1.18) on the half space $\mathbf{R}^N_+$:

$$\begin{cases} \triangle w + \gamma \frac{w}{|x|^2} + \frac{w^{2^*(s)-1}}{|x|^s} = 0 & \text{in } \mathbf{R}^N_+, \\ w = 0 & \text{on } \partial \mathbf{R}^N_+, \end{cases} \tag{2.1}$$

where $\gamma < (\frac{N-2}{2})^2$, $0 \leqq s < 2$, and $2^*(s) = \frac{2(N-s)}{N-2}$. It is well-known that if w is locally in $H^1(\mathbf{R}^N_+)$, then w is smooth in $\overline{\mathbf{R}^N_+} \backslash \{0\}$. In the following, we want to estimate w near the origin O.

Lemma 2.1. *Suppose w is a positive solution of (2.1) in $B_1^+(0) = \mathbf{R}^N_+ \bigcap B_1(0)$ and $w \in H^1(B_1^+(0))$. Then for any $\delta > 0$, there exists $C_\delta > 0$ such that*

$$w(x) \leqq C_\delta |x|^{\alpha_0 - \delta} \quad \textit{for} \quad |x| \leqq \frac{1}{2}, \tag{2.2}$$

where α_0 is defined in (1.19).

Proof. By multiplying $\phi^2 w^{2\beta-1}$ on Equation (2.1), we have

$$\begin{aligned} &\frac{2\beta-1}{\beta^2} \int_{\mathbf{R}^N_+} |\nabla(\phi w^\beta)|^2 \, dx - \gamma \int_{\mathbf{R}^N_+} \frac{\phi^2 w^{2\beta}}{|x|^2} \, dx \\ &\leqq \int_{\mathbf{R}^N_+} \frac{\phi^2 w^{2^*(s)+2\beta-2}}{|x|^s} \, dx + O(1), \end{aligned}$$

where ϕ is a cut-off function. Since $\gamma < (\frac{N-2}{2})^2$, by the Sobolev–Hardy inequality, we have $w^\beta \in H^1(B_1^+(0))$ if $\beta > 1$ is close to 1. Therefore $w^{\frac{2N}{N-2}} \in L^q(B_1^+)$ for some $q > 1$ by the Sobolev embedding.

Step 1. We claim there is a $r_0 > 0$ such that

$$\int_{B_{r_0}^+} |\nabla\phi|^2 \,\mathrm{d}x - \gamma \int_{B_{r_0}^+} \frac{\phi^2}{|x|^2} \,\mathrm{d}x - (2^*(s) - 1) \int_{B_{r_0}^+} \frac{w^{2^*(s)-2}\phi^2}{|x|^s} \,\mathrm{d}x \geqq 0 \tag{2.3}$$

for any $\phi \in H_0^1(B_{r_0}^+)$.

By the Hölder inequality,

$$\int_{B_{r_0}^+} \frac{w^{\frac{(2^*(s)-2)N}{2}}}{|x|^{\frac{sN}{2}}} \,\mathrm{d}x \leqq \left(\int_{B_{r_0}^+} w^{\frac{2Nq}{N-2}} \,\mathrm{d}x\right)^{1-\frac{1}{q}} \left(\int_{B_{r_0}^+} |x|^{-\frac{sNq'}{2}} \,\mathrm{d}x\right)^{\frac{1}{q'}},$$

where

$$\frac{1}{q'} = 1 - \frac{(2^*(s)-2)\frac{N}{2} \cdot \frac{N-2}{2N}}{q} = 1 - \frac{4-2s}{2q} < \frac{s}{2}.$$

So, $|x|^{-\frac{sNq'}{2}} \in L^1(B_1^+)$. Thus, for any $\varepsilon > 0$, there is a $r_0 > 0$ such that

$$\left(\int_{B_{r_0}^+} \frac{w^{\frac{(2^*(s)-2)N}{2}}}{|x|^{\frac{sN}{2}}} \,\mathrm{d}x\right)^{\frac{2}{N}} \leqq \varepsilon.$$

Therefore, for any $\phi \in H_0^1(B_{r_0}^+)$,

$$\begin{aligned}\int_{B_{r_0}^+} \frac{w^{2^*(s)-2}\phi^2}{|x|^s} \,\mathrm{d}x &\leqq \left(\int_{B_{r_0}^+} \left(\frac{w^{2^*(s)-2}}{|x|^s}\right)^{\frac{N}{2}} \mathrm{d}x\right)^{\frac{2}{N}} \left(\int_{B_{r_0}^+} \phi^{\frac{2N}{N-2}} \,\mathrm{d}x\right)^{\frac{N-2}{N}} \\ &\leqq \varepsilon ||\nabla\phi||^2_{L^2(B_{r_0}^+)}.\end{aligned}$$

Then by the Hardy inequality we have

$$\begin{aligned}&\textstyle\int_{B_{r_0}^+} |\nabla\phi|^2 \mathrm{d}x - \gamma \int_{B_{r_0}^+} \frac{\phi^2}{|x|^2} \,\mathrm{d}x - (2^*(s)-1) \int_{B_{r_0}^+} \frac{w^{2^*(s)-2}}{|x|^s}\phi^2 \,\mathrm{d}x \\ &\textstyle\geqq (1 - C\varepsilon)||\nabla\phi||^2_{L^2(B_{r_0}^+)} - \gamma \int_{B_{r_0}^+} \frac{\phi^2}{|x|^2} \,\mathrm{d}x \geqq 0,\end{aligned}$$

provided that $\varepsilon > 0$ is small. And (2.3) is proved.

Set

$$\overline{w}(r) = \frac{1}{r^{N-1}} \int_{|x|=r} w \,\mathrm{d}x.$$

Since $w \in L^{\frac{2N}{N-2}}(B_1^+)$, we have

$$\underline{\lim}_{r\to 0} \overline{w}(r) r^{\frac{N-2}{2}} = 0.$$

Then $\forall\, \varepsilon > 0,\ \exists\, r_1 = r_1(\varepsilon) \leqq r_0$ such that

$$\begin{aligned} &\overline{w}(r_1) \leqq \varepsilon r_1^{-\frac{N-2}{2}}, \quad \text{and} \\ &\int_{B_{2r_1}} |\nabla w|^2 \, \mathrm{d}x \leqq \varepsilon. \end{aligned} \tag{2.4}$$

Step 2. We claim

$$w(x) \leqq C\varepsilon |x|^{-\frac{N-2}{2}} \quad \text{for } |x| = r_1, \tag{2.5}$$

holds for some constant $C > 0$.

To see (2.5) holds, we set

$$v_0(y) = w(r_1 y) r_1^{\frac{N-2}{2}}.$$

Then $v_0(y)$ satisfies

$$\begin{cases} \Delta v_0(y) + \gamma \frac{v_0(y)}{|y|^2} + \frac{v_0^{2^*(s)-1}(y)}{|y|^s} = 0, \quad |y| \leqq 2, \\ \int_{|y| \leqq 2} |\nabla v_0(y)|^2 \, \mathrm{d}y = \int_{|x| \leqq 2r_1} |\nabla w|^2(x) \, \mathrm{d}x \leqq \varepsilon. \end{cases}$$

If ε is sufficiently small, then there exists a constant C_1 independent of ε, such that $v_0(y) \leqq C_1$ for $\frac{1}{2} \leqq |y| \leqq \frac{3}{2}$. Then (2.5) follows from the Harnack inequality for linear elliptic PDE. This finishes the proof of Step 2.

Let $z(\theta) = \frac{x_N}{|x|},\ \theta = \frac{x}{|x|} \in S^{N-1}$. Then $z(\theta)$ satisfies

$$\triangle_{S^{N-1}} z(\theta) = -(N-1) z(\theta),$$

where $\triangle_{S^{N-1}}$ is the Laplacian operator on S^{N-1}. Set

$$U(x) = |x|^{-\frac{N-2}{2}} z(\theta). \tag{2.6}$$

Then $U(x)$ satisfies

$$\triangle U(x) + \left[\left(\frac{N-2}{2} \right)^2 + N - 1 \right] \frac{U(x)}{|x|^2} = 0.$$

Thus r_2 can be chosen small such that

$$\triangle U + \gamma \frac{U}{|x|^2} + \frac{U^{2^*(s)-1}}{|x|^s} \leqq 0 \quad \text{for } |x| \leqq r_2. \tag{2.7}$$

Without loss of generality, we may assume $r_1 \leqq r_2$.

Step 3. We claim

$$w(x) \leqq |x|^{-\frac{N-2}{2}} z(\theta) \quad \text{for } |x| \leqq r_1. \tag{2.8}$$

From the proof of (2.5), we have

$$w(x) \leqq C_1\, \varepsilon z(\theta) |x|^{-\frac{N-2}{2}} \quad \text{for } |x| = r_1$$

for some constant C_1 independent of ε and r_1. If ε is small, then

$$w(x) \leqq U(x) \quad \text{for} \ |x| = r_1.$$

Set

$$v(x) = U(x) - w(x) \quad \text{for} \ |x| \leqq r_1.$$

Suppose $\Omega^- = \{x \in B_{r_1} | v(x) < 0\} \neq \emptyset$. Then by (2.7), $v(x)$ satisfies

$$\triangle v(x) + \gamma \frac{v(x)}{|x|^2} + (2^*(s) - 1) \frac{w^{2^*(s)-2}(x)}{|x|^s} v(x) < 0 \quad \text{in} \ \Omega^-,$$

and we have

$$\int_{\Omega^-} |\nabla v(x)|^2 \, dx - \gamma \int_{\Omega^-} \frac{v^2(x)}{|x|^2} \, dx - (2^*(s)-1) \int_{\Omega^-} \frac{w^{2^*(s)-2}(x)}{|x|^2} v^2(x) \, dx < 0,$$

a contradiction to (2.3). Thus Ω^- is an empty set and we have

$$w(x) \leqq |x|^{-\frac{N-2}{2}} z(\theta) \quad \text{for} \ |x| \leqq r_1.$$

Actually, in **Step 2**, we have proved a stronger result:

$$w(x) = o(1)|x|^{-\frac{N-2}{2}} \quad \text{as} \ |x| \to 0. \tag{2.9}$$

By (2.9), we note that

$$\frac{w^{2^*(s)-2}(x)}{|x|^s} = o(1)|x|^{-2} \quad \text{as} \ |x| \to 0.$$

Thus for any $\delta' > 0$ there is $r_3 < r_1$ such that

$$\triangle w(x) + (\gamma + \delta') \frac{w(x)}{|x|^2} \geqq 0 \quad \text{in} \ |x| < r_3. \tag{2.10}$$

For any $\delta > 0$, we set

$$w_\delta(x) = C|x|^{\alpha_0 - \delta} z(\theta). \tag{2.11}$$

Then $w_\delta(x)$ satisfies

$$\triangle w_\delta(x) + (\gamma + \delta') \frac{w_\delta(x)}{|x|^2} = 0,$$

where δ' is chosen so that

$$\gamma + \delta' = -(\alpha_0 - \delta)(N - 2 - \alpha_0 + \delta) + N - 1.$$

Choose $C = C_\delta$ in (2.11) to be large so that

$$w(x) \leqq w_\delta(x) \quad \text{for} \ |x| = r_3.$$

Thus by (2.10) and following the arguments of **Step 2**, we get

$$w(x) \leqq w_\delta(x) \quad \text{for} \ |x| \leqq r_3.$$

Therefore, Lemma 2.1 is proved. □

Now we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\hat{w}(x)$ be the Kelvin transformation of w,

$$\hat{w}(x) = \left(\frac{1}{|x|}\right)^{2-N} w\left(\frac{x}{|x|^2}\right).$$

Then $\hat{w}$ satisfies (2.1) and

$$\int_{\mathbf{R}_+^N} |\nabla \hat{w}|^2 \,\mathrm{d}x = \int_{\mathbf{R}_+^N} |\nabla w|^2 \,\mathrm{d}x,$$

Hence $\hat{w} \in H^1(\mathbf{R}_+^N)$. By Lemma 2.1 for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|\hat{w}(x)| \leqq C_\varepsilon |x|^{\alpha_0 - \varepsilon} \quad \text{for } |x| \leqq 1.$$

Going back to w, we have

$$|w(x)| \leqq C_\varepsilon |x|^{2-(N+\alpha_0)+\varepsilon}. \tag{2.12}$$

After (2.12) is proved, the estimate for $|\nabla w(x)|$ follows from the standard gradient estimate.

To prove the axial symmetry of w, it suffices to prove

$$w(x_1, \dots, -x_{N-1}, x_N) = w(x_1, \dots, x_{N-1}, x_N).$$

The proof will use a variant of the well-known method of moving planes, see [9,10,14,16] and [17]. Let l_θ be the hyperplane $\{(x_1, x_2, \dots, x_{N-1}, x_N) | x_i \in \mathbf{R}, i = 1, \cdots, N-2, \frac{x_N}{x_{N-1}} = \sin\theta\}$, and $\Sigma_\theta = \{x \in \mathbf{R}_+^N | x_N < x_{N-1} \sin\theta\}$. For any $x \in \Sigma_\theta$, we denote x^θ to be the reflection of x with respect to l_θ, and consider

$$v_\theta(x) = w(x^\theta) - w(x), \quad x \in \Sigma_\theta.$$

Then v_θ satisfies

$$\triangle v_\theta + \gamma \frac{v_\theta}{|x|^2} + c_\theta(x) v_\theta = 0, \tag{2.13}$$

where

$$c_\theta(x) = \frac{1}{|x|^s} \frac{w^{2^*(s)-1}(x^\theta) - w^{2^*(s)-1}(x)}{w(x^\theta) - w(x)}. \tag{2.14}$$

By (2.12), there is small $\delta > 0$, such that

$$\frac{w^{2^*(s)-2}}{|x|^s} = O\left(\frac{1}{|x|^{2+\delta}}\right) \quad \text{for large } |x|,$$

and

$$\frac{w^{2^*(s)-2}}{|x|^s} = O\left(\frac{1}{|x|^{2-\delta}}\right) \quad \text{for small } |x|.$$

Choose R large and r_0 small so that for either $|x| < r_0$ or $|x| \geqq R$, the inequality,

$$\frac{\gamma}{|x|^2} + (2^*(s) - 1)\frac{w^{2^*(s)-2}(x)}{|x|^s} < \gamma_0 \frac{1}{|x|^2} \tag{2.15}$$

holds for some $\gamma_0 < (\frac{N-2}{2})^2$. Then by fixing r_0 and R, and θ is chosen to be small such that $v_\theta(x) > 0$ provided that $r_0 \leqq |x| \leqq R$, we claim for such θ,

$$v_\theta(x) > 0 \quad \text{for } x \in \Sigma_\theta. \tag{2.16}$$

To prove (2.16), we let $\Sigma_\theta^- = \{x \in \Sigma_\theta | v_\theta(x) < 0\}$ and suppose $\Sigma_\theta^- \neq \emptyset$(empty set). Then

$$\Sigma_\theta^- \subset \left[\Sigma_\theta \bigcap B_{r_0}(0)\right] \bigcup \left[\Sigma_\theta \bigcap (\mathbf{R}_+^N \backslash B_R(0))\right].$$

If $\Sigma_\theta^- \bigcap B_{r_0}(0) \neq \emptyset$, then $c_\theta \leqq (2^*(s) - 1)w^{2^*(s)-2}(x)|x|^{-s}$. Hence we get

$$\begin{aligned}
\left(\frac{N-2}{2}\right)^2 \int_{\Sigma_\theta^- \cap B_{r_0}} \frac{v_\theta^2}{|x|^2}\,\mathrm{d}x &\leqq \int_{\Sigma_\theta^- \cap B_{r_0}} |\nabla v_\theta|^2\,\mathrm{d}x \\
&\leqq \int_{\Sigma_\theta^- \cap B_{r_0}} \left(\frac{\gamma}{|x|^2} + \frac{(2^* - 1)w^{2^*(s)-2}(x)}{|x|^s}\right) v_\theta^2\,\mathrm{d}x \\
&< \gamma_0 \int_{\Sigma_\theta^- \cap B_{r_0}} \frac{v_\theta^2}{|x|^2}\,\mathrm{d}x,
\end{aligned}$$

a contradiction. Hence $\Sigma_\theta^- \bigcap B_{r_0}(0) = \emptyset$. Similarly, $\Sigma_\theta^- \bigcap (\mathbf{R}_+^N \backslash B_R(0)) = \emptyset$ can be proved. Thus, $\Sigma_\theta^- = \emptyset$ and (2.16) is proved. Let

$$\theta_0 = \sup\left\{\theta \leqq \frac{\pi}{2} | v_{\tilde{\theta}}(x) > 0 \quad \text{in } \Sigma_{\tilde{\theta}}(x) \quad \text{for all } 0 \leqq \tilde{\theta} \leqq \theta\right\}.$$

By applying (2.15) and the process of the method of moving plane, we can show $\theta_0 = \frac{\pi}{2}$. Since the process is well-known, we omit the detail. Since $\theta_0 = \frac{\pi}{2}$, we have

$$w\left(x^{\frac{\pi}{2}}\right) \geqq w(x)\ \forall x \in \Sigma_{\frac{\pi}{2}}.$$

We can do the same process starting from the opposite direction, and show that $w(x^{\frac{\pi}{2}}) \leqq w(x)\ \forall x \in \Sigma_{\frac{\pi}{2}}$, that is

$$w\,(x_1, \ldots, x_{N-2}, -x_{N-1}, x_N) = w(x_1, \ldots, x_{N-2}, x_{N-1}, x_N).$$

To prove (1.21), we let $\hat{w}_\alpha$ be the Kelvin transformation of w with respect to $B_\alpha(0)$, that is,

$$\hat{w}_\alpha(x) = \left(\frac{\alpha}{|x|}\right)^{2-N} w\left(\frac{\alpha^2 x}{|x|^2}\right).$$

Then $\hat{w}_\alpha$ satisfies

$$\begin{cases} \triangle \hat{w}_\alpha + \frac{\gamma}{|x|^2}\hat{w}_\alpha + \frac{\hat{w}_\alpha^{2^*(s)-1}}{|x|^s} = 0 & \text{in}\, \mathbf{R}_+^N \\ \hat{w}_\alpha = 0 & \text{on}\, \partial\mathbf{R}_+^N. \end{cases}$$

Set

$$v_\alpha(x) = \hat{w}_\alpha(x) - w(x)\ \forall x \in B_\alpha^+(0) := \{x | x_N \geqq 0 \text{ and } |x| \leqq \alpha\}.$$

Then $v_\alpha(x)$ satisfies

$$\triangle v_\alpha(x) + \frac{\gamma}{|x|^2} v_\alpha(x) + C_\alpha(x) v_\alpha(x) = 0\ \forall x \in B_\alpha^+(0),$$

where $C_\alpha(x) = \frac{\hat{w}_\alpha^{2^*(s)-1} - w^{2^*(s)-1}}{\hat{w}_\alpha - w}|x|^{-s}$. For α small and $x \in B_\alpha^+(0)$ where $v_\alpha(x) > 0$, we have

$$\begin{aligned} &C_\alpha(x) \leqq \frac{(2^*(s)-1)w^{2^*(s)-2}(x)}{|x|^s} \leqq o(1)|x|^{-2} \quad \text{as } |x| \to 0, \quad \text{and} \\ &\frac{\gamma}{|x|^2} + C_\alpha(x) \leqq \gamma_0 |x|^{-2} \quad \text{for some } \gamma_0 < \left(\frac{N-2}{2}\right)^2. \end{aligned}$$

Then by using the Hardy inequality, we have, if α is small,

$$v_\alpha(x) > 0\ \forall x \in B_\alpha^+(0).$$

The Hopf lemma implies

$$0 > \frac{\partial v_\alpha}{\partial x_N}(0,\dots,0,\alpha) = -\left(2\frac{\partial w}{\partial x_N}(0,\dots,0,\alpha) + \frac{N-2}{\alpha} w(0,\dots,0,\alpha)\right),$$

that is, $x_N^{\frac{N-2}{2}} w(0,\dots,0,x_N)$ is increasing in x_N. Set

$$\alpha_0 = \sup\{\alpha > 0 | v_{\tilde{\alpha}}(x) > 0 \quad \text{in } B_{\tilde{\alpha}}^+(0) \text{ for all } \tilde{\alpha} \leqq \alpha\}.$$

Since $x_N^{\frac{N-2}{2}} w(0,\dots,0,x_N)$ is increasing in x_N if $x_N \leqq \alpha_0$ and

$$\lim_{x_N \to +\infty} x_N^{\frac{N-2}{2}} w(0,\dots,0,x_N) = 0,$$

α_0 should be a finite number. Argued as the method of moving plane, we can show $\hat{w}_{\alpha_0}(x) = w(x)\ \forall x \in B_{\alpha_0}^+(0)$, which is (1.21). The inequality (1.22) will be shown in Corollary 2.2. Thus, Theorem 1.3 is proved. □

Corollary 2.2. *Suppose $w \in H_0^1(\mathbf{R}_+^N)$ is a positive solution of (2.1). Then, letting $x = (x', x_N) \in \mathbf{R}^N$, we have*

$$\begin{aligned} &\left|\frac{\partial w}{\partial x_N}(x',0)\right|^2 |x'|^2 \in L^1(\mathbf{R}^{N-1}), \\ &\frac{x_N |x'|^2}{|x|^4} w^2(x) \quad \textit{and}\ \frac{x_N |x'|^2}{|x|^{s+2}} w^{2^*(s)}(x) \in L^1(\mathbf{R}_+^N). \end{aligned} \tag{2.17}$$

Furthermore, the following inequality holds

$$2\gamma \int_{\mathbf{R}_+^N} \frac{x_N |x'|^2}{|x|^4} w^2(x)\,\mathrm{d}x < \int_{\mathbf{R}^{N-1}} |x'|^2 \left|\frac{\partial w}{\partial x_N}\right|^2 \mathrm{d}x'. \tag{2.18}$$

Proof. By (1.20) of Theorem 1.3, we have

$$\left|\frac{\partial w}{\partial x_N}\right|^2 |x'|^2 \leqq C_\varepsilon |x'|^{4-2(N+\alpha_0)+\varepsilon},$$

where either $\alpha_0 = 1$ if $\gamma \leqq 0$ or α_0 satisfies

$$\alpha_0 > -\frac{N-2}{2} \quad \text{and} -\alpha_0(N-2+\alpha_0)+N-1=\gamma.$$

if $\gamma \in (0, (\frac{N-2}{2})^2)$. Obviously if $\alpha_0 = 1$, we have $4-2(N+1) = -2(N-1) < -(N-1)$ and $|\frac{\partial w}{\partial x_N}|^2 |x'|^2 \in L^1(\mathbf{R}^{N-1})$. In general, we have

$$\begin{aligned} 4-2(N+\alpha_0) &= -2(N-2+\alpha_0) \\ &= -\left[N-2+\sqrt{(N-2)^2-4\gamma+4(N-1)}\right]. \end{aligned}$$

Thus,

$$2(N+\alpha_0)-4 > N-2+\sqrt{4(N-1)} > N-1. \tag{2.19}$$

Therefore, we have $|\frac{\partial w}{\partial x_N}|^2 |x'|^2 \in L^1(\mathbf{R}^{N-1})$. The other two terms of (2.17) can be proved in the same way. Since it is a straightforward computation, we skip the details.

To show (2.18), by scaling, we may assume $w(x)$ satisfies

$$w(x) = |x|^{2-N} w\left(\frac{x}{|x|^2}\right).$$

By direct computation, we have

$$\int_{\mathbf{R}^{N-1}} |x'|^2 \left|\frac{\partial w}{\partial x_N}(x',0)\right|^2 \mathrm{d}x' = \int_{\mathbf{R}^{N-1}} \left|\frac{\partial w}{\partial x_N}\right|^2 \mathrm{d}x', \tag{2.20}$$

By multiplying $\frac{\partial w}{\partial x_N}$ on Equation (2.1) and by integration by parts, we have

$$-\int_{\mathbf{R}^N_+} \frac{\partial w}{\partial x_N} \triangle w \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbf{R}^N_+} \frac{\partial}{\partial x_N}\left(\frac{\partial w}{\partial x_N}\right)^2 \mathrm{d}x = \frac{1}{2}\int_{\mathbf{R}^{N-1}} \left(\frac{\partial w}{\partial x_N}\right)^2 (x',0)\,\mathrm{d}x',$$

and

$$\begin{aligned} \int_{\mathbf{R}^N_+} \frac{\partial w}{\partial x_N}\left(\frac{\gamma w}{|x|^2} + \frac{w^{2^*(s)-1}}{|x|^s}\right) \mathrm{d}x &= \int_{\mathbf{R}^N_+} \left(\frac{\gamma}{2|x|^2}\frac{\partial}{\partial x_N} w^2 + \frac{1}{2^*}\frac{1}{|x|^s}\frac{\partial}{\partial x_N} w^{2^*(s)}\right) \mathrm{d}x \\ &= \int_{\mathbf{R}^N_+} \left(\frac{\gamma x_N}{|x|^4} w^2 + \frac{s}{2^*(s)}\frac{x_N}{|x|^{s+2}} w^{2^*(s)}\right) \mathrm{d}x \\ &> \gamma \int_{\mathbf{R}^N_+} \frac{x_N}{|x|^4} w^2(x)\,\mathrm{d}x. \end{aligned}$$

Hence

$$\int_{\mathbf{R}^{N-1}} \left|\frac{\partial w}{\partial x_N}\right|^2 (x',0)\,\mathrm{d}x' > 2\gamma \int_{\mathbf{R}^N_+} \frac{x_N}{|x|^4} w^2(x)\,\mathrm{d}x.$$

Clearly, (2.18) follows from (2.20) immediately. □

3. Calculation of $S(a, b; \Omega)$

In this section, we will calculate $S(a, b; \Omega)$ if a, b and q satisfy (1.2) and (1.7). Under the assumption (1.7), the best constant $S(a, b; \mathbf{R}^N_+)$ can be achieved. See BARTSCH et al. [2], LIN and WANG [18]. The main result of this section is the following:

Theorem 3.1. *Let Ω be a bounded C^1 domain of $\mathbf{R}^N$ and C^2 at $0 \in \partial\Omega$. Suppose the mean curvature $H(0)$ at 0 is negative. Then*

$$S(a, b; \Omega) < S(a, b; \mathbf{R}^N_+)$$

where a and b satisfy (1.2) *and* (1.7).

Proof. Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_N = \varphi(x')$, $x' = (x_1, \ldots, x_{N-1})$, where $\varphi(0) = 0$, $\nabla'\varphi(0) = \left(\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_{N-1}}\right)(0) = 0$, and the outnormal of $\partial\Omega$ at 0 is $-e_N$.

Let w be a positive solution of

$$\begin{cases} \triangle w + \gamma \frac{w}{|x|^2} + S\frac{w^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in} \quad \mathbf{R}^N_+, \\ w = 0 \quad \text{on}\, \partial\mathbf{R}^N_+, \quad \text{and } \int_{\mathbf{R}^N_+} \frac{w^{2^*(s)}}{|x|^s}\,dx = 1, \end{cases} \tag{3.1}$$

where $S = S(a, b; \mathbf{R}^N_+)$. We note that if (1.7) is satisfied, then $s > 0$, $\gamma < (\frac{N-2}{2})^2$ and $N \geqq 3$, or $s = 0$, $\gamma \in (0, (\frac{N-2}{2})^2)$ and $N \geqq 4$. The existence of w is proved in [2] and [18].

Let U and $\hat{U}$ be a respective neighborhood of 0 such that $\Phi(U) = B_{r_0}(0)$ and $\Phi(\hat{U}) = B_{\frac{r_0}{2}}(0)$, where $x = (x', x_N)$ and

$$\Phi(x) = (x', x_N - \varphi(x')) \quad \text{for } x \in \overline{\Omega} \cap \overline{U}.$$

We define

$$v_\varepsilon(x) := \varepsilon^{-\frac{N-2}{2}} w\left(\frac{\Phi(x)}{\varepsilon}\right) \quad \text{for}\, x \in \Omega \cap U \quad \text{and} \quad \hat{v}_\varepsilon := \eta\, v_\varepsilon \quad \text{in}\ \Omega,$$

where $\eta \in C_0^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\hat{U}$. Then we have

$$S(a, b; \Omega) \leqq \left(\int_\Omega |\nabla \hat{v}_\varepsilon|^2\,dx - \gamma \int_\Omega \frac{\hat{v}_\varepsilon^2}{|x|^2}\,dx\right) \Big/ \left(\int_\Omega \frac{\hat{v}_\varepsilon^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*(s)}}.$$

By a change of variable: first by $z = \Phi(x)$ and second by $y = \frac{z}{\varepsilon}$, we get

$$\begin{aligned}
\int_\Omega |\nabla v_\varepsilon|^2 \mathrm{d}x &= \int_{\Omega\cap U} \eta^2 |\nabla v_\varepsilon|^2 \mathrm{d}x - \int_{\Omega\cap U} \eta(\Delta\eta) v_\varepsilon^2 \mathrm{d}x \\
&= \int_{\mathbf{R}^N_+} |\nabla w(y)|^2 \,\mathrm{d}y \\
&\quad -2\int_{\frac{B^+_{r_0}}{\varepsilon}} \eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \frac{\partial w}{\partial y_N}(y)\,\nabla' w(y)\cdot(\nabla'\varphi)(\varepsilon y')\,\mathrm{d}y \\
&\quad + \int_{\frac{B^+_{r_0}}{\varepsilon}} \eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \left(\frac{\partial w}{\partial y_N}(y)\right)^2 |(\nabla'\varphi)(\varepsilon y')|^2 \,\mathrm{d}y + O(\varepsilon^2) \\
&:= \int_{\mathbf{R}^N_+} |\nabla w(y)|^2 \mathrm{d}y + I_1 + I_2 + O(\varepsilon^2).
\end{aligned}$$

We shall estimate each integral precisely. Since $\partial\Omega$ is C^2 at 0, φ can be expanded by

$$\varphi(y') = \sum_{j=1}^{N-1} \lambda_j y_j^2 + o(1)(|y'|^2). \tag{3.2}$$

By using the decay estimate of $|\nabla w|$ in Theorem 1.3, we see that by (2.19), $2(N+\alpha_0) - 4 > N - 2 + \sqrt{4(N-1)} > N$. Hence $|\frac{\partial w}{\partial x_N}|^2 |y|^2 \in L^1(\mathbf{R}^N_+)$, and then

$$I_2 \leqq C\varepsilon^2 \int_{\mathbf{R}^N_+} \left|\frac{\partial w}{\partial y_N}\right|^2 |y|^2 \mathrm{d}y = O(\varepsilon^2).$$

By integration by parts, the integral I_1 can be computed as follows.

$$\begin{aligned}
I_1 &= -\frac{2}{\varepsilon}\int_{\frac{B^+_{r_0}}{\varepsilon}} \left[\eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \frac{\partial w}{\partial y_N}(y)\,\nabla' w(y)\cdot\nabla'(\varphi(\varepsilon y'))\right] \mathrm{d}y \\
&= \frac{2}{\varepsilon}\int_{\frac{B^+_{r_0}}{\varepsilon}} \left[\eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \nabla'\frac{\partial w}{\partial y_N}(y)\cdot\nabla' w(y)\,\varphi(\varepsilon y')\right] \mathrm{d}y \\
&\quad + \frac{2}{\varepsilon}\int_{\frac{B^+_{r_0}}{\varepsilon}} \left[\eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \frac{\partial w}{\partial y_N}(y) \sum_{i=1}^{N-1} \frac{\partial^2 w}{\partial y_i^2}(y)\,\varphi(\varepsilon y')\right] \mathrm{d}y + O(\varepsilon^2) \\
&:= I_{1,1} + I_{1,2} + O(\varepsilon^2).
\end{aligned}$$

Again, by integration by parts and $\nabla' w \equiv 0$ on $\partial\mathbf{R}^N_+$, we have

$$\begin{aligned}
I_{1,1} &= \frac{2}{\varepsilon}\int_{\frac{B^+_{r_0}}{\varepsilon}} \left[\eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \nabla'\frac{\partial w}{\partial y_N}(y)\cdot\nabla' w(y)\,\varphi(\varepsilon y')\right] \mathrm{d}y \\
&= \frac{1}{\varepsilon}\int_{\frac{B^+_{r_0}}{\varepsilon}} \left[\eta\left(\Phi^{-1}(\varepsilon y)\right)^2 \frac{\partial}{\partial y_N}(|\nabla' w(y)|^2)\varphi(\varepsilon y')\right] \mathrm{d}y \\
&= -\frac{1}{\varepsilon}\int_{\frac{B^+_{r_0}}{\varepsilon}} \left[\frac{\partial}{\partial y_N}(\eta\left(\Phi^{-1}(\varepsilon y)\right)^2)|\nabla' w(y)|^2\varphi(\varepsilon y')\right] \mathrm{d}y = O(\varepsilon^2).
\end{aligned}$$

For the integral $I_{1,2}$, by using equation (3.1) and by integration by parts, we have

$$
\begin{aligned}
I_{1,2} &= \frac{2}{\varepsilon}\int_{\frac{B_{r_0}^+}{\varepsilon}}\left[\eta(\Phi^{-1}(\varepsilon y))^2\frac{\partial w}{\partial y_N}(y)\left(\sum_{i=1}^{N-1}\frac{\partial^2 w}{\partial y_i^2}(y)\right)\varphi(\varepsilon y')\right]\mathrm{d}y\\
&= -\frac{\gamma}{\varepsilon}\int_{\frac{B_{r_0}^+}{\varepsilon}}\left[\eta(\Phi^{-1}(\varepsilon y))^2\left(\frac{\partial}{\partial y_N}w^2(y)\right)\frac{\varphi(\varepsilon y')}{|y|^2}\right]\mathrm{d}y\\
&\quad -\frac{2S}{2^*(s)\varepsilon}\int_{\frac{B_{r_0}^+}{\varepsilon}}\left[\eta\left(\Phi^{-1}(\varepsilon y)\right)^2\left(\frac{\partial}{\partial y_N}w^{2^*(s)}(y)\right)\frac{\varphi(\varepsilon y')}{|y|^s}\right]\mathrm{d}y\\
&\quad -\frac{1}{\varepsilon}\int_{\frac{B_{r_0}^+}{\varepsilon}}\left[\eta(\Phi^{-1}(\varepsilon y))^2\frac{\partial}{\partial y_N}\left(\left(\frac{\partial w}{\partial y_N}(y)\right)^2\right)\varphi(\varepsilon y')\right]\mathrm{d}y\\
&= -\frac{2sS}{2^*(s)\varepsilon}\int_{\mathbf{R}_+^N}\left[\frac{w^{2^*(s)}(y)y_N}{|y|^{s+2}}\varphi(\varepsilon y')\right]\mathrm{d}y-\frac{2\gamma}{\varepsilon}\int_{\mathbf{R}_+^N}\left[\frac{w^2(y)y_N\varphi(\varepsilon y')}{|y|^4}\right]\mathrm{d}y\\
&\quad +\frac{1}{\varepsilon}\int_{\mathbf{R}^{N-1}}\left[\left(\frac{\partial w}{\partial y_N}(y)\right)^2\varphi(\varepsilon y')\right]\mathrm{d}y'+O(\varepsilon^2) := J_1+J_2+J_3+O(\varepsilon^2).
\end{aligned}
$$

Thus by using (3.2) and the axial symmetry of w, we get

$$
\begin{aligned}
J_1+J_2 &= -\frac{2s\varepsilon S}{2^*(s)}\sum_{i=1}^{N-1}\left(\lambda_i\int_{\mathbf{R}_+^N}\frac{w^{2^*(s)}(y)y_i^2y_N}{|y|^{s+2}}\,\mathrm{d}y\right)\\
&\quad -2\gamma\varepsilon\sum_{i=1}^{N-1}\left(\lambda_i\int_{\mathbf{R}_+^N}\frac{w^2(y)y_i^2y_N}{|y|^4}\,\mathrm{d}y\right)+o(1)\varepsilon\\
&= -\frac{2s\varepsilon S}{2^*(s)}H(0)\int_{\mathbf{R}_+^N}\frac{w^{2^*(s)}(y)y_N|y'|^2}{|y|^{s+2}}\,\mathrm{d}y\\
&\quad -2\gamma\varepsilon H(0)\int_{\mathbf{R}_+^N}\frac{w^2(y)y_N|y'|^2}{|y|^4}\,\mathrm{d}y+o(1)\varepsilon\\
&= -C_1H(0)\varepsilon-C_2H(0)\varepsilon+o(1)\varepsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
H(0) &:= \frac{1}{N-1}\sum_{i=1}^{N-1}\lambda_i,\quad C_1 := \frac{2sS}{2^*(s)}\int_{\mathbf{R}_+^N}\frac{w^{2^*(s)}(y)y_N|y'|^2}{|y|^{s+2}}\mathrm{d}y,\\
C_2 &:= 2\gamma\int_{\mathbf{R}_+^N}\frac{w^2(y)y_N|y'|^2}{|y|^4}\mathrm{d}y.
\end{aligned}
\tag{3.3}
$$

Next, we see that

$$
\begin{aligned}
J_3 &= \frac{1}{\varepsilon}\int_{\mathbf{R}^{N-1}}\left(\frac{\partial w}{\partial y_N}(y)\right)^2 \varphi(\varepsilon y')\mathrm{d}S_y \\
&= \varepsilon\sum_{i=1}^{N-1}\lambda_i\int_{\mathbf{R}^{N-1}}\left(\frac{\partial w}{\partial y_N}(y',0)\right)^2 y_i^2\mathrm{d}y' + o(1)\varepsilon \\
&= \frac{\varepsilon}{N-1}\int_{\mathbf{R}^{N-1}}|(\nabla w)(y',0)|^2|y'|^2\mathrm{d}y'\left(\sum_{i=1}^{N-1}\lambda_i\right)+o(1)\varepsilon = C_3H(0)\varepsilon+o(1)\varepsilon,
\end{aligned}
$$

where

$$
C_3 := \int_{\mathbf{R}^{N-1}}|(\nabla w)(y',0)|^2|y'|^2\,\mathrm{d}y'. \tag{3.4}
$$

In summary, we have

$$
\begin{aligned}
&\int_\Omega|\nabla\hat{v}_\varepsilon|^2\,\mathrm{d}x-\gamma\int_\Omega\frac{\hat{v}_\varepsilon^2}{|x|^2}\,\mathrm{d}x \\
&\quad=\int_{\mathbf{R}^N_+}|\nabla w|^2\,\mathrm{d}y-\gamma\int_{\mathbf{R}^N_+}\frac{|w|^2}{|y|^2}\,\mathrm{d}y-(C_1+C_2-C_3)H(0)\varepsilon+o(1)\varepsilon \\
&\quad= S-(C_1+C_2-C_3)H(0)\varepsilon+o(1)\varepsilon.
\end{aligned} \tag{3.5}
$$

Finally, the integral $\int_\Omega\frac{\hat{v}_\varepsilon^{2^*(s)}}{|x|^s}\mathrm{d}x$ can be estimated as follows. By a change of variable $\frac{\Phi(x)}{\varepsilon}=y$, we have

$$
\int_\Omega\frac{\hat{v}_\varepsilon^{2^*(s)}}{|x|^s}\,\mathrm{d}x \geqq \int_{\Omega\cap\tilde{U}}\frac{v_\varepsilon^{2^*(s)}}{|x|^s}\,\mathrm{d}x = \int_{B^+_{\frac{r_0/2}{\varepsilon}}}\frac{w^{2^*(s)}}{\left|\frac{\Phi^{-1}(\varepsilon y)}{\varepsilon}\right|^s}\,\mathrm{d}y. \tag{3.6}
$$

Since $\Phi^{-1}(y)=(y',y_N+\varphi(y'))$, it holds $|\Phi^{-1}(y)|^2=|y|^2+2y_N\varphi(y')+\big(\varphi(y')\big)^2$, and then

$$
\begin{aligned}
\frac{1}{\left|\frac{\Phi^{-1}(\varepsilon y)}{\varepsilon}\right|^s} &= \frac{1}{|y|^s}\cdot\frac{1}{\left(1+\frac{2y_N\varphi(\varepsilon y')}{\varepsilon|y|^2}+\frac{(\varphi(\varepsilon y'))^2}{\varepsilon^2|y|^2}\right)^{\frac{s}{2}}} \\
&= \frac{1}{|y|^s}\left(1-\frac{sy_N\varphi(\varepsilon y')}{\varepsilon|y|^2}-\frac{s\left(\varphi(\varepsilon y')\right)^2}{2\varepsilon^2|y|^2}\right) \\
&\quad + C\frac{1}{|y|^s}\left(\frac{2y_N\varphi(\varepsilon y')}{\varepsilon|y|^2}+\frac{\left(\varphi(\varepsilon y')\right)^2}{\varepsilon^2|y|^2}\right)^2.
\end{aligned} \tag{3.7}
$$

Thus from (3.6) and (3.7), we obtain

$$
\begin{aligned}
\int_\Omega\frac{\hat{v}_\varepsilon^{2^*(s)}}{|x|^s}\,\mathrm{d}x &\geqq \int_{\mathbf{R}^N_+}\frac{w^{2^*(s)}}{|y|^s}\,\mathrm{d}y-\frac{s}{\varepsilon}\int_{B^+_{\frac{r_0/2}{\varepsilon}}}\frac{w(y)^{2^*(s)}y_N\varphi(\varepsilon y')}{|y|^{2+s}}\,\mathrm{d}y+O(\varepsilon^2) \\
&= 1-s\varepsilon H(0)\int_{\mathbf{R}^N_+}\frac{w(y)^{2^*(s)}|y'|^2y_N}{|y|^{2+s}}\,\mathrm{d}y+o(1)\varepsilon \\
&= 1-\frac{2^*(s)C_1}{2S}H(0)\varepsilon+o(1)\varepsilon,
\end{aligned}
$$

where C_1 is the same positive constant as in (3.3), and then

$$\left(\int_\Omega \frac{\hat{v}_\varepsilon^{2^*(s)}}{|x|^s}\,\mathrm{d}x\right)^{-\frac{2}{2^*(s)}} \leqq 1+\frac{C_1}{S}H(0)\varepsilon+o(1)\varepsilon. \tag{3.8}$$

Thus by (3.5) and (3.8), we have

$$\begin{aligned} S(a,b;\Omega) &\leqq (S-(C_1H(0)-C_2H(0)+C_3H(0))\varepsilon+o(1)\varepsilon)\\ &\quad\times\left(1+\frac{C_1}{S}H(0)\varepsilon+O(\varepsilon^2)\right)\\ &= S+(C_3-C_2)H(0)\varepsilon+o(1)\varepsilon, \end{aligned} \tag{3.9}$$

where

$$C_3-C_2=\int_{\mathbf{R}^{N-1}}|\nabla w(y',0)|^2|y'|^2\,\mathrm{d}y'-2\gamma\int_{\mathbf{R}^N_+}\frac{w^2y_N|y'|^2}{|y|^4}\,\mathrm{d}y>0$$

by Corollary 2.2. Since $H(0)<0$, we have for small $\varepsilon>0$, $S(a,b;\Omega)<S(a,b;\mathbf{R}^N_+)$.

4. Proof of Theorem 1.2: the case a $\leqq$ 0 and b > a

Let Ω be a bounded smooth domain. In this section, we consider the case $a\leqq 0$ and $b>a$, or equivalently $\gamma\leqq 0$ and $s>0$ for (1.14). As usual, we first consider the minimizer for subcritical cases:

$$S_i(a,b;\Omega)=\inf\left\{\textstyle\int_\Omega|\nabla w|^2\,\mathrm{d}x-\gamma\int_\Omega\frac{w^2}{|x|^2}\,\mathrm{d}x \mid \int_\Omega\frac{w^{p_i}}{|x|^s}\,\mathrm{d}x=1\right\}, \tag{4.1}$$

where $p_i<2^*(s)$. For each $p_i<2^*(s)$, let $\{\widetilde{w}_k\}$ be a minimizing sequence of (4.1). Since $||\nabla\widetilde{w}_k||_{L^2(\Omega)}\leqq C$, there is a subsequence (still denoted by $\widetilde{w}_k$) such that $\widetilde{w}_k\rightharpoonup w_i$ and $\frac{\widetilde{w}_k}{|x|}\rightharpoonup\frac{w_i}{|x|}$ in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. Then we have

$$\textstyle\int_\Omega|\nabla w_i|^2\,\mathrm{d}x\leqq\lim_{k\to+\infty}\int_\Omega|\nabla\widetilde{w}_k|^2\,\mathrm{d}x,\quad \int_\Omega\frac{w_i^2}{|x|^2}\,\mathrm{d}x\leqq\lim_{k\to+\infty}\int_\Omega\frac{\widetilde{w}_k^2}{|x|^2}\,\mathrm{d}x.$$

Since $\gamma\leqq 0$, we have

$$\begin{aligned}\int_\Omega|\nabla w_i|^2\,\mathrm{d}x-\gamma\int_\Omega\frac{w_i^2}{|x|^2}\,\mathrm{d}x &\leqq \lim_{k\to+\infty}\left[\int_\Omega|\nabla\widetilde{w}_k|^2\,\mathrm{d}x-\gamma\int_\Omega\frac{\widetilde{w}_k^2}{|x|^2}\mathrm{d}x\right]\\ &= S_i(a,b;\Omega),\end{aligned}$$

and

$$\int_\Omega\frac{w_i^{p_i}}{|x|^s}\,\mathrm{d}x=\lim_{k\to+\infty}\int_\Omega\frac{\widetilde{w}_k^{p_i}}{|x|^s}\,\mathrm{d}x=1,$$

Hence w_i is a minimizer of (4.1), and w_i satisfies

$$\begin{cases} \triangle w_i + \gamma \frac{w_i}{|x|^2} + S_i \frac{w_i^{p_i-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ w_i|_{\partial\Omega} = 0, \end{cases} \tag{4.2}$$

where $S_i = S_i(a,b;\Omega)$. Let $p_i \uparrow 2^*(s) = \frac{2(N-s)}{N-2}$. It is easy to see that

$$\lim_{i\to\infty} S_i(a,b;\Omega) = S(a,b;\Omega) \leqq S(a,b;\mathbf{R}^N_+).$$

Since $||\nabla w_i||_{L^2(\Omega)} \leqq C$, there is a subsequence (still denoted by w_i) such that $w_i \rightharpoonup w$ in $H_0^1(\Omega)$. If $w \neq 0$, then we have

$$\begin{aligned} &\int_\Omega \frac{w^{2^*(s)}}{|x|^s}\,\mathrm{d}x \leqq \lim_{i\to+\infty} \int_\Omega \frac{w_i{}^{p_i}}{|x|^s}\,\mathrm{d}x = 1, \\ &\int_\Omega |\nabla w|^2\,\mathrm{d}x - \gamma \int_\Omega \frac{w^2}{|x|^2}\,\mathrm{d}x \\ &\qquad \leqq \lim_{i\to+\infty}\left[\int_\Omega |\nabla w_i|^2\,\mathrm{d}x - \gamma\int_\Omega \frac{w_i^2}{|x|^2}\,\mathrm{d}x\right] = S(a,b;\Omega). \end{aligned}$$

Thus we obtain that

$$\begin{aligned} S(a,b;\Omega) &= \lim_{i\to\infty} S_i(a,b;\Omega) = \lim_{i\to\infty} \frac{\int_\Omega |\nabla w_i|^2\,\mathrm{d}x - \gamma\int_\Omega \frac{w_i^2}{|x|^2}\,\mathrm{d}x}{\left(\int_\Omega \frac{w_i^{p_i}}{|x|^s}\,\mathrm{d}x\right)^{\frac{2}{p_i}}} \\ &= S(a,b;\Omega)\cdot\left(\int_\Omega \frac{w^{2^*(s)}}{|x|^s}\mathrm{d}x\right)^{1-\frac{2}{2^*(s)}} \leqq S(a,b;\Omega). \end{aligned}$$

We can thus deduce that $\int_\Omega \frac{w^{2^*(s)}}{|x|^s}\mathrm{d}x = 1$, w is a minimizer for the best constant S, and w satisfies

$$\begin{cases} \triangle w + \gamma \frac{w}{|x|^2} + S \frac{w^{2^*(s)-1}}{|x|^s} = 0 \quad \text{in } \Omega, \\ w|_{\partial\Omega} = 0. \end{cases}$$

Thus, it remains to prove $w \neq 0$, which is the main result of this section.

Proposition 4.1. *If* $w = 0$, *then* $S(a,b;\Omega) = S(a,b;\mathbf{R}^N_+)$.

Proof. Since $p_j < 2^*(s)$, $\gamma \leqq 0$ and w_j satisfies (4.2), by the assumption $w_i \rightharpoonup 0$ in $H_0^1(\Omega)$, w_j must blow up somewhere in $\overline{\Omega}$. Otherwise, if w_j is uniformly bounded on $\overline{\Omega}$, then $w_j \to 0$ in $L^{2^*(s)}(\frac{\mathrm{d}x}{|x|^s})$, and

$$1 = \lim_{j\to+\infty} \int_\Omega \frac{w_j^{p_j}}{|x|^s}\mathrm{d}x = 0$$

yields a contradiction.

Let

$$w_j(x_j) = \max_{\overline{\Omega}} w_j \to +\infty \quad \text{as} \quad j \to +\infty.$$

Because of $2^*(s) < \frac{2N}{N-2}$, w_j cannot blow up at any point of $\overline{\Omega}\backslash\{0\}$. Thus, we have $x_j \to 0$. Set

$$\varepsilon_j = w_j^{-\frac{2(p_j-2)}{(N-2)(2^*(s)-2)}}(x_j). \tag{4.3}$$

Step 1. We claim $|x_j| = O(\varepsilon_j)$.
Suppose that up to a subsequence,

$$\lim_{j\to+\infty}\frac{|x_j|}{\varepsilon_j} = \infty$$

By scaling, we set

$$v_j(y) = \frac{w_j(x_j+\beta_j y)}{w_j(x_j)} \quad \text{in } \Omega_j, \tag{4.4}$$

where $\Omega_j = \{y \in \mathbf{R}^N | x_j + \beta_j y \in \Omega\}$ and

$$\beta_j = |x_j|^{\frac{s}{2}}\varepsilon_j^{\frac{2-s}{2}} = \left(\frac{|x_j|}{\varepsilon_j}\right)^{\frac{s}{2}}\varepsilon_j.$$

By Equation (4.2), $v_j(y)$ satisfies

$$\begin{cases} \Delta v_j + \gamma\frac{v_j}{|\frac{x_j}{\beta_j}+y|} + S_j\beta_j^2 w_j^{p_j-2}(x_j)\frac{v_j^{p_j-1}}{|x_j+\beta_j y|^s} = 0 \quad \text{in } \Omega_j \\ v_j = 0 \quad \text{on } \partial\Omega_j. \end{cases} \tag{4.5}$$

Note that

$$w_j^{p_j-2}(x_j)\beta_j^2|x_j|^{-s} = 1 \quad \text{and} \quad \frac{|x_j|}{\beta_j} = \left(\frac{|x_j|}{\varepsilon_j}\right)^{1-\frac{s}{2}} \to \infty \quad \text{as } j \to +\infty.$$

Thus for any $R > 0$, $\frac{x_j}{\beta_j} + y \neq 0$ for any $|y| < R$. Recall $v_j(y) \leqq 1$, by the elliptic estimates and by passing to a subsequence if necessary, v_j smoothly converges to v in $\overline{B_R(0)}\bigcap\Omega_j$, and v satisfies either

$$\Delta v + Sv^{2^*(s)-1} = 0 \quad \text{in } \mathbf{R}^N. \tag{4.6}$$

provided $\Omega_j \to \mathbf{R}^N$, or

$$\begin{cases} \Delta v + Sv^{2^*(s)-1} = 0 \quad \text{in } H, \\ v = 0 \quad \text{on } \partial H, \end{cases} \tag{4.7}$$

provided that $\Omega_j \to H$, where H is a half-space of $\mathbf{R}^N$. It is known that both (4.6) and (4.7) has no positive solutions. So, it yields a contradiction. Thus, **Step 1** is established.

Step 2. We claim

$$\lim_{j\to+\infty}\frac{|x_j|}{\varepsilon_j} \neq 0.$$

To prove **Step 2**, we do another scaling:

$$v_j(y) = \frac{w_j(x_j + \varepsilon_j y)}{w_j(x_j)} \quad \text{in } \tilde{\Omega}_j = \{y \in \mathbf{R}^N | x_j + \varepsilon_j y \in \Omega\}. \tag{4.8}$$

Then v_j satisfies

$$\begin{cases} \triangle v_j + \gamma \frac{v_j}{|\frac{x_j}{\varepsilon_j}+y|^2} + S_j \frac{v_j^{p_j-1}}{|\frac{x_j}{\varepsilon_j}+y|^s} = 0 & \text{in } \tilde{\Omega}_j \\ v_j(0) = 1, v_j = 0 & \text{on } \partial\tilde{\Omega}_j. \end{cases} \tag{4.9}$$

Since $\gamma \leqq 0$ and $v_j \leqq 1$, by using the same proof of Lemma 2.1,

$$|v_j(y)| \leqq C|y| \quad \text{for } \overline{B_1(0)} \bigcap \tilde{\Omega}_j.$$

Thus, by the elliptic, for any $\alpha \in (0, 1)$,

$$|v_j|_{C^\alpha(\overline{B_1(0)})} \leqq C$$

for some constant C. Therefore, by passing to a subsequence, v_j uniformly converges to v in any compact subset of $\tilde{\Omega}_j$. Suppose that up to a subsequence, $\frac{x_j}{\varepsilon_j} \to 0$ as $j \to +\infty$. Then $\tilde{\Omega}_j$ converges to $\mathbf{R}_+^N$ after a rotation. Thus v satisfies

$$\begin{cases} \triangle v + \gamma \frac{v}{|y|^2} + S\, \frac{v^{2^*(s)-1}}{|y|^s} = 0 & \text{in } \mathbf{R}_+^N, \\ v = 0 & \text{on } \partial\mathbf{R}_+^N. \end{cases} \tag{4.10}$$

By direct computation, we have

$$\int_{\tilde{\Omega}_j} |\nabla v_j(y)|^2\, \mathrm{d}y = \varepsilon_j^{2+\frac{(N-2)(2^*(s)-2)}{p_j-2}-N} \int_\Omega |\nabla w_j(x)|^2\, \mathrm{d}x \leqq C,$$

because $2 + \frac{(N-2)(2^*(s)-2)}{p_j-2} > 2 + N - 2 = N$, and $\varepsilon_j \to 0$. Thus $v \in H_0^1(\mathbf{R}_+^N)$. By Lemma 2.1, v is continuous at $x = 0$, that is, $v(0) = 0$. But the uniform convergence gives $v(0) = \lim_{j\to+\infty} v_j(0) = 1$, which yields a contradiction. Thus, **Step 2** is proved.

Let v_j be scaled as (4.8). Then by the elliptic estimates again, v_j converges to v in $C^{2,\alpha}$ locally, and after a linear translation, by **Step 2**, we have that v satisfies (4.10) with $S = \lim_{j\to+\infty} S_j = S(a, b; \Omega)$. Note that, by (4.8),

$$\begin{aligned} \int_{\mathbf{R}_+^N} \frac{v^{2^*(s)}}{|y|^s}\, \mathrm{d}y &\leqq \lim_{j\to+\infty} \int_{\tilde{\Omega}_j} \frac{v_j^{p_j}}{|y|^s}\, \mathrm{d}y = w_j^{-p_j}(x_j)\varepsilon_j^{-N+s} \int_{\Omega_j} \frac{w_j^{p_j}}{|x|^s}\, \mathrm{d}x \\ &= w_j^{\frac{N-2}{2-s}\left(p_j - \frac{2(N-s)}{N-2}\right)}(x_j) \leqq 1. \end{aligned}$$

Thus,

$$\begin{aligned} S(a, b; \mathbf{R}_+^N) &\leqq \frac{\int_{\mathbf{R}_+^N} |\nabla v|^2\, \mathrm{d}y - \gamma \int_{\mathbf{R}_+^N} \frac{v^2}{|y|^2}\, \mathrm{d}y}{\left(\int_{\mathbf{R}_+^N} \frac{v^{2^*(s)}}{|y|^s}\, \mathrm{d}y\right)^{\frac{2}{2^*(s)}}} \\ &= S\left(\int_\Omega \frac{v^{2^*(s)}}{|y|^s}\right)^{\frac{2}{2^*(s)}} \leqq S. \end{aligned}$$

Since

$$S(a,b;\Omega) \leqq S(a,b;\mathbf{R}^N_+)$$

holds always, we have

$$S(a,b;\Omega) = S(a,b;\mathbf{R}^N_+) \quad \text{and} \quad \int_{\mathbf{R}^N_+} \frac{v^{2^*(s)}}{|y|^s}\,\mathrm{d}y = 1,$$

which implies v is a minimizer of (1.14). Hence Proposition 4.1 is proved. □

Proof of Theorem 1.2 for the case $\gamma \leqq 0$ and $s > 0$. It is easy to see Theorem 1.2 is the immediate consequence of Theorem 3.1 and Proposition 4.1. □

5. Proof of Theorem 1.1: a > 0 and a < b

In this section, we continue to prove Theorem 1.2, for the case $\gamma \in (0, (\frac{N-2}{2})^2)$ and $s > 0$:

$$S(s,\gamma;\Omega) = \inf_{w\in H^1_0(\Omega)} \left\{ \int_\Omega \left(|\nabla w|^2 - \gamma \frac{w^2}{|x|^2} \right) \mathrm{d}x \,|\, \int_\Omega \frac{w^{2^*(s)}}{|x|^s}\,\mathrm{d}x = 1 \right\}. \tag{5.1}$$

Let $S(a,b;\Omega)$ denote the best constant (1.14). Obviously, we have

$$S(s,\gamma;\Omega) = S(a,b;\Omega),$$

where $s = (b-a)q$, $2^*(s) = q = \frac{2N}{N-2+2(b-a)}$ and $\gamma = a(N-2-a)$. When $\gamma > 0$, minimizers of (5.1) might be unbounded near the origin O. Hence the blowup argument of last section, like (4.5) and (4.9), could not work here, because the limiting function v is a bounded entire solution of (4.10). Therefore, we will use another kind of blowup argument.

To prove Theorem 1.2 for the case $\gamma \geqq 0$, we consider the subcritical case:

$$S_\varepsilon = \inf_{w\in H^1_0(\Omega)} \left\{ \int_\Omega \left(|\nabla w|^2 - \gamma \frac{w^2}{|x|^{2-\varepsilon}} \right) \mathrm{d}x \,|\, \int_\Omega \frac{w^{2^*(s)}}{|x|^{s-\varepsilon}}\mathrm{d}x = 1 \right\}. \tag{5.2}$$

Since

$$2 < 2^*(s) = \frac{2(N-s)}{N-2} < 2^*(s-\varepsilon) = \frac{2(N-s+\varepsilon)}{N-2},$$

the minimizing problem for (5.2) is subcritical for the singular term $|x|^{-(s-\varepsilon)}$. Hence for small $\varepsilon > 0$, there exists a minimizer w_ε of (5.2), and w_ε satisfies

$$\begin{cases} \triangle w_\varepsilon + \gamma \frac{w_\varepsilon}{|x|^{2-\varepsilon}} + S_\varepsilon \frac{w_\varepsilon^{2^*(s)-1}}{|x|^{s-\varepsilon}} = 0 & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

It is easy to see the following holds:

(i) $\lim_{\varepsilon\to 0} S_\varepsilon = S(s,\gamma;\Omega)$,
(ii) There exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leqq ||w_\varepsilon||_{H_0^1(\Omega)} \leqq C_2 \tag{5.4}$$

holds for small $\varepsilon > 0$.

Thus, for any sequence $\varepsilon_j \to 0$, there exists a subsequence $w_j = w_{\varepsilon_j}$ (still denoted by w_j) such that $w_j \rightharpoonup w_0$ weakly in $H_0^1(\Omega)$. If $w_0 \neq 0$, then by using Equation (5.3), w_0 is a minimizer for (5.1). If $w_0 = 0$, we have the following result, which is similar to Proposition 4.1.

Proposition 5.1. *Let w_j be defined in the above. Suppose $w_j \rightharpoonup 0$ in $H_0^1(\Omega)$. Then*

$$S(s,\gamma;\Omega) = S(s,\gamma;\mathbf{R}^N_+).$$

Proof. Let $S_j = S_{\varepsilon_j}$. Then w_j satisfies

$$\begin{cases} \triangle w_j + \gamma \dfrac{w_j}{|x|^{2-\varepsilon_j}} + S_j \dfrac{w_j^{2^*(s)-1}}{|x|^{s-\varepsilon_j}} = 0 \quad \text{in } \Omega, \\ w_j = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Since $w_j \rightharpoonup 0$, w_j must blow up at some point in $\overline{\Omega}$. Because $2 < 2^*(s) < \frac{2N}{N-2}$ is a subcritical exponent to the Sobolev embedding, w_j cannot blow up except at the origin O. Thus for any compact subset $K \subset \overline{\Omega}\backslash\{0\}$,

$$\sup_K |w_j(x)| = o(1), \tag{5.6}$$

where $o(1) \to 0$ as $j \to +\infty$. Set r_j to be

$$\int_{B^+_{r_j}} |\nabla w_j|^2 \,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla w_j|^2 \,\mathrm{d}x, \tag{5.7}$$

where

$$B^+_{r_j} = B_{r_j}(0) \bigcap \overline{\Omega}.$$

By (5.6), $r_j \to 0$ as $j \to +\infty$. Let

$$v_j(y) = w_j(r_j y) r_j^{\frac{N-2}{2}}. \tag{5.8}$$

Then, $v_j(y)$ satisfies

$$\begin{cases} \triangle v_j + \gamma r_j^{\varepsilon_j} \dfrac{v_j(y)}{|y|^{2-\varepsilon_j}} + S_j r_j^{\varepsilon_j} \dfrac{v_j^{2^*(s)-1}}{|y|^{s-\varepsilon_j}} = 0 \quad \text{in } \Omega_j \\ v_j|_{\partial\Omega_j} = 0, \end{cases} \tag{5.9}$$

where $\Omega_j = \{y | r_j y \in \Omega\}$. By the scaling (5.8), we have

$$\int_{B_1^+} |\nabla v_j(y)|^2 \, dy = \int_{B_{r_j}^+} |\nabla w_j|^2 \, dx, \tag{5.10}$$

where $B_1^+ = B_1(0) \bigcap \overline{\Omega}_j$. Since (5.9) is a subcritical elliptic equation in $\Omega_j \setminus \{0\}$, it is not difficult to get the a priori bound:

$$\sup_{\frac{1}{2} \leqq |y| \leqq 2} |v_j(y)| \leqq C.$$

By (5.7) and (5.10), v_j is locally bounded in $H_0^1(\Omega_j)$. By passing to a subsequence (still denoted by v_j), v_j converges weakly to v_0 in any bounded set of $\overline{\mathbf{R}_+^N}$ in $H_0^1(\mathbf{R}_+^N)$. If $v_0 \not\equiv 0$, then v_0 satisfies

$$\begin{cases} \triangle v_0 + \gamma C \frac{v_0}{|y|^2} + SC \frac{v_0^{2^*(s)-1}}{|y|^s} = 0 & \text{in } \mathbf{R}_+^N, \\ v_0 = 0 & \text{on } \partial \mathbf{R}_+^N, \end{cases} \tag{5.11}$$

where

$$C = \lim_{j \to +\infty} r_j^{\varepsilon_j} \leqq 1. \tag{5.12}$$

We note that $C \neq 0$. Otherwise, v_0 is a harmonic function with $\int_{\mathbf{R}_+^N} |\nabla v_0|^2 < +\infty$, which together with the boundary condition $\partial \mathbf{R}_+^N$ yields $v_0 \equiv 0$, a contradiction to the assumption $v_0 \not\equiv 0$. Due to $\gamma > 0$, we have

$$S(s, \gamma; \mathbf{R}_+^N) \leqq S(s, 0; \mathbf{R}_+^N),$$

which implies

$$\frac{(1-C) \int_{\mathbf{R}_+^N} |\nabla v_0|^2 \, dy}{\left(\int_{\mathbf{R}_+^N} \frac{v_0^{2^*(s)}}{|y|^s} \, dy\right)^{\frac{2}{2^*(s)}}} \geqq (1 - C) S(s, \gamma; \mathbf{R}_+^N). \tag{5.13}$$

Note that

$$\int_{\Omega_j} \frac{v_j^{2^*(s)}}{|y|^{s-\varepsilon_j}} \, dy = r_j^{-\varepsilon_j} \int_{\Omega} \frac{w_j^{2^*(s)}}{|y|^{s-\varepsilon_j}} \, dy \leqq r_j^{-\varepsilon_j}.$$

Hence

$$\int_{\mathbf{R}_+^N} \frac{v_0^{2^*(s)}}{|y|^s} \, dy \leqq \lim_{j \to +\infty} r_j^{-\varepsilon_j} = \frac{1}{C}.$$

Thus (5.11) and (5.13) give

$$
\begin{aligned}
S(s,\gamma;\mathbf{R}^N_+) &= CS(s,\gamma;\mathbf{R}^N_+) + (1-C)S(s,\gamma;\mathbf{R}^N_+) \\
&\leqq \frac{C\left(\int_{\mathbf{R}^N_+}|\nabla v_0|^2\,\mathrm{d}y - \gamma\int_{\mathbf{R}^N_+}\frac{v_0^2}{|y|^2}\,\mathrm{d}y\right) + (1-C)\int_{\mathbf{R}^N_+}|\nabla v_0|^2\,\mathrm{d}y}{\left(\int_{\mathbf{R}^N_+}\frac{v_0^{2^*(s)}}{|y|^s}\,\mathrm{d}y\right)^{\frac{2}{2^*(s)}}} \\
&= CS\left(\int_{\mathbf{R}^N_+}\frac{v_0^{2^*(s)}}{|y|^s}\mathrm{d}y\right)^{1-\frac{2}{2^*(s)}} \leqq CSC^{\frac{2}{2^*(s)}-1} = SC^{\frac{2}{2^*(s)}}, \qquad (5.14)
\end{aligned}
$$

which yields

$$
C = 1, \quad S = S(s,\gamma;\mathbf{R}^N_+),
$$

and

$$
\int_{\mathbf{R}^N_+}\frac{v_0^{2^*(s)}}{|y|^s}\,\mathrm{d}y = 1.
$$

Therefore, it suffices for us to exclude the case $v_0 \equiv 0$. Now suppose $v_0 \equiv 0$. Let $\phi \geqq 0$ and

$$
\phi(y) = \begin{cases} 1 & \text{if } |y| \leqq 1 \\ 0 & \text{if } |y| \geqq 2. \end{cases}
$$

Multiplying $-\phi^2 v_j$ on (5.9), applying the Hardy inequality and noting $\gamma r_j^{\varepsilon_j} < (\frac{N-2}{2})^2$, we have

$$
\begin{aligned}
\theta\int_{\Omega_j}|\nabla\phi v_j|^2\,\mathrm{d}y &\leqq \int_{\Omega_j}|\nabla\phi v_j|^2\,\mathrm{d}y - \gamma r_j^{\varepsilon_j}\int_{\Omega_j}\frac{\phi^2 v_j^2}{|y|^{2-\varepsilon_j}}\,\mathrm{d}y + O(\eta_j^2) \\
&= S_j r_j^{\varepsilon_j}\int_{\Omega_j}\frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}}\,\mathrm{d}y + O(\eta_j^{2^*(s)}), \qquad (5.15)
\end{aligned}
$$

where θ is a constant in $(0, 1)$, and η_j is a constant defined by

$$
\eta_j = \sup_{1\leqq|y|\leqq 2}|v_j|.
$$

By our assumption, $\eta_j \to 0$ as $j \to +\infty$. By the scaling (5.8), we have

$$
\int_{\Omega_j}\frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}}\,\mathrm{d}y = r_j^{-\varepsilon_j}\int_{\Omega}\frac{(\phi w_j)^{2^*(s)}}{|x|^{s-\varepsilon_j}}\,\mathrm{d}x \leqq r_j^{-\varepsilon_j}, \qquad (5.16)
$$

also

$$
\left(\int_{\Omega_j}\frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}}\right)^{\frac{2}{2^*}} \geqq \eta_j^2.
$$

Then (5.15) yields

$$\frac{\theta \int_{\Omega_j} |\nabla \phi v_j|^2 \mathrm{d}y}{\left(\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{2-\varepsilon_j}} \,\mathrm{d}y\right)^{\frac{2}{2^*(s)}}} \leqq \frac{\int_{\Omega_j} |\nabla \phi v_j|^2 \,\mathrm{d}y - \gamma r_j^{\varepsilon_j} \int_{\Omega_j} \frac{\phi^2 v_j^2}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y + O(\eta_j^2)}{\left(\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y\right)^{\frac{2}{2^*(s)}}}$$
$$\leqq S_j r_j^{\varepsilon_j \frac{2}{2^*(s)}} + C_3 \leqq C_4$$

for some constants C_3 and $C_4 > 0$. But by (5.10), we have

$$\frac{\theta C_1}{\left(\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y\right)^{\frac{2}{2^*(s)}}} \leqq \frac{\theta \int_{\Omega_j} |\nabla \phi v_j|^2 \,\mathrm{d}y}{\left(\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y\right)^{\frac{2}{2^*(s)}}} \leqq C_4,$$

which yields $\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \mathrm{d}y$ has a positive lower bound.

Since $\operatorname{supp}(\phi v_j) \to B_2^+(O)$ and $S(\gamma, s; B_2^+(O)) = S(\gamma, s; \mathbf{R}_+^N)$, for any $\delta > 0$ there exists j_0 such that if $j \geqq j_0$, then

$$S(\gamma, s; \mathbf{R}_+^N) - \delta \leqq \frac{\int_{\Omega_j} |\nabla \phi v_j|^2 \,\mathrm{d}y - \gamma r_j^{\varepsilon_j} \int_{\Omega_j} \frac{\phi^2 v_j^2}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y}{\left(\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \mathrm{d}y\right)^{\frac{2}{2^*(s)}}}$$
$$\leqq S_j r_j^{\varepsilon_j} \left(\int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y\right)^{1-\frac{2}{2^*(s)}} + o(1)$$
$$\leqq S_j r_j^{\frac{2}{2^*(s)}\varepsilon_j} + o(1) \leqq S_j + o(1).$$

Hence, we have

$$S(\gamma, s; \mathbf{R}_+^N) \leqq \lim_{j \to +\infty} S_j = S(\gamma, s; \Omega).$$

Thus $S(\gamma, s; \mathbf{R}_+^N) = S(\gamma, s; \Omega)$ and

$$\lim_{j \to +\infty} r_j^{\varepsilon_j} = 1 \quad \text{and} \quad \lim_{j \to +\infty} \int_{\Omega_j} \frac{(\phi v_j)^{2^*(s)}}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y = 1. \tag{5.17}$$

This implies

$$\lim_{j \to +\infty} \int_{|y| \geqq \frac{1}{2}} \frac{v_j^{2^*(s)}}{|y|^{s-\varepsilon_j}} \mathrm{d}y = 0 \tag{5.18}$$

because $\int_{\Omega_j} \frac{v_j^{2^*(s)}}{|y|^{s-\varepsilon_j}} \,\mathrm{d}y = r_j^{-\varepsilon_j} \int_{\Omega} \frac{w_j^{2^*(s)}}{|x|^{s-\varepsilon_j}} \,\mathrm{d}x = r_j^{-\varepsilon_j} \to 1$ as $j \to +\infty$.

Let $\hat{v}_j$ be the Kelvin transformation of v_j,

$$\hat{v}_j(y) = |y|^{2-N} v_j\left(\frac{y}{|y|^2}\right). \tag{5.19}$$

Then $\hat{v}_j$ satisfies

$$\triangle \hat{v}_j(y) + \gamma r_j^{\varepsilon_j} \frac{\hat{v}_j(y)}{|y|^{2+\varepsilon_j}} + S_j r_j^{\varepsilon_j} \frac{\hat{v}_j^{2^*(s)-1}(y)}{|y|^{s+\varepsilon_j}} = 0 \quad \text{in } \hat{\Omega}_j,$$

where $\hat{\Omega}_j = \{y | \frac{y}{|y|^2} \in \Omega_j\}$. By noting for $\forall\ y \in \hat{\Omega}_j$,

$$\frac{1}{|y|^{\varepsilon_j}} \leqq \left(\frac{C}{r_j}\right)^{\varepsilon_j} \to 0 \quad \text{as} \quad j \to +\infty,$$

(5.15) still holds:

$$\begin{aligned} \theta \int_{\hat{\Omega}_j} |\nabla \phi \hat{v}_j|^2 \,\mathrm{d}y &\leqq \int_{\hat{\Omega}_j} \left(|\nabla \phi \hat{v}_j|^2 - \gamma r_j^{\varepsilon_j} \frac{\phi^2 \hat{v}_j^2}{|y|^{2+\varepsilon_j}}\right) \mathrm{d}y + O(\eta_j^2) \\ &= S_j r_j^{\varepsilon_j} \int_{\hat{\Omega}_j} \frac{(\phi \hat{v}_j)^{2^*(s)}}{|y|^{s+\varepsilon_j}} \,\mathrm{d}y + O(\eta_j^{2^*(s)}), \end{aligned} \tag{5.20}$$

where ϕ is a cut-off function as defined before. Since by (5.18),

$$\int_{\hat{\Omega}_j} \frac{(\phi \hat{v}_j)^{2^*(s)}}{|y|^{s+\varepsilon_j}} \mathrm{d}y \to 0 \quad \text{as } j \to +\infty,$$

and by (5.6),

$$C_1 \leqq \int_{\Omega \setminus B_1^+} |v_j|^2 \mathrm{d}y \leqq \int_{\hat{\Omega}_j} |\nabla \phi \hat{v}_j|^2 \mathrm{d}y$$

for some constant $C_1 > 0$, we get

$$+\infty \leftarrow \frac{\theta C_1}{\left(\int_{\hat{\Omega}_j} \frac{(\phi \hat{v}_j)^{2^*(s)}}{|y|^{s+\varepsilon_j}}\right)^{\frac{2}{2^*(s)}}} \leqq S_j r_j^{\varepsilon_j} \left(\int_{\hat{\Omega}_j} \frac{(\phi \hat{v}_j)^{2^*(s)}}{|y|^{s+\varepsilon_j}}\right)^{1-\frac{2}{2^*(s)}} + O(1) \leqq C_2,$$

a contradiction. Thus, $v_0 \not\equiv 0$ and then Proposition 5.1 is proved. □

Proof of Theorem 1.2 for the case $\gamma > 0$ and $s > 0$. Clearly, Theorem 1.2 follows from Proposition 5.1 and Theorem 3.1. □

Remark. The blowup argument in this section cannot be applied to the case when $\gamma \leqq 0$, because the constant C of Equation (5.11) cannot be shown to be 1 if $\gamma < 0$.

6. The case: b = a > 0

In this section, the last case $b = a > 0$ for Theorem 1.2 will be proved. Then we will completely finish the proof of Theorem 1.2. In this case, we have $s = 0$ and $2^* = \frac{2N}{N-2}$, the Sobolev exponent. Hence, the best constant is defined by

$$S(a, a; \Omega) = \inf_{H_0^1(\Omega)} \left\{ \int_\Omega \left(\left|\nabla u\right|^2 - \gamma \frac{u^2}{|x|^2} \right) \mathrm{d}x \,\middle|\, \int_\Omega u^{2^*} \mathrm{d}x = 1 \right\}, \tag{6.1}$$

where $0 < \gamma = a(N-2-a) < (\frac{N-2}{2})^2$. Because $s = 0$, the best constant $S(a, a; \Omega)$ is compared not only with $S(a, a; \mathbf{R}_+^N)$, but also with S_N, the best Sobolev constant.

Since $\gamma > 0$, we have

$$S(a, a; \Omega) \leqq \inf_{H_0^1(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \mathrm{d}x \,\middle|\, \int_\Omega u^{\frac{2N}{N-2}} \mathrm{d}x = 1 \right\} = S_N. \tag{6.2}$$

Actually, a strict inequality holds if $N \geqq 4$.

Lemma 6.1. *Let $a = b > 0$ and $N \geqq 4$. Then*

$$S(a, a; \Omega) < S_N. \tag{6.3}$$

Proof. Let $x_0 \in \Omega$, and $\overline{B_{2r_0}(x_0)} \subset \Omega$ for some $r_0 > 0$. We set $\varphi(x)$ to be a cut-off function in $B_{2r_0}(x_0)$:

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad x \in B_{r_0}(x_0) \\ 0 & \text{if} \quad |x - x_0| \geqq 2r_0. \end{cases}$$

Define

$$w_\lambda(x) = \varphi(x) \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{N-2}{2}}.$$

Then by direct computations, we have

$$\begin{aligned} S(a, a; \Omega) &\leqq \frac{\int_\Omega (|\nabla w_\lambda(x)|^2 - \gamma \frac{w_\lambda^2}{|x|^2}) \,\mathrm{d}x}{\left(\int_\Omega w_\lambda^{\frac{2N}{N-2}}(x) \,\mathrm{d}x \right)^{\frac{N-2}{N}}} \\ &= \begin{cases} S_N - C_1 \lambda^{-2} + O(\lambda^{-3}) & \text{if } N \geqq 5, \\ S_N - C_1 \lambda^{-1} \log \lambda + O(\lambda^{-2}) & \text{if } N = 4, \end{cases} \end{aligned}$$

where $\gamma = a(N-2-a) > 0$ and $C_1 > 0$ independent of λ.
Thus $S(a, a; \Omega) < S_N$. □

As before, we consider the subcritical case:

$$S_\varepsilon = \inf_{H_0^1(\Omega)} \left\{ \int_\Omega \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^{2-\varepsilon}} \right) \mathrm{d}x \mid \int_\Omega u^{\frac{2N}{N-2}-\varepsilon}(x)\, \mathrm{d}x = 1 \right\}, \tag{6.4}$$

and it is easy to see $\lim_{\varepsilon\to 0} S_\varepsilon = S(a, a; \Omega)$. For any small $\varepsilon > 0$, (6.4) can be attained by some function $w_\varepsilon \in H_0^1(\Omega)$ and w_ε satisfies

$$\begin{cases} \triangle w_\varepsilon + \gamma \frac{w_\varepsilon}{|x|^{2-\varepsilon}} + S_\varepsilon w_\varepsilon^{\frac{N+2}{N-2}-\varepsilon} = 0 & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

It is not difficult to see

$$0 < C_1 \leqq ||\nabla w_\varepsilon||_{L^2} \leqq C_2, \tag{6.6}$$

where C_j are constants independent of ε. By passing $\varepsilon_j \to 0$, $w_j := w_{\varepsilon_j} \rightharpoonup w$ for some $w \in H_0^1(\Omega)$. If $w \neq 0$, then w is a minimizer of (6.1). Thus, we have to show $w \neq 0$ for Theorem 1.2.

Proof of Theorem 1.2 for the case $b = a > 0$. We suppose $w = 0$. As before, w_j blows up somewhere on $\overline{\Omega}$. Now we want to prove the blowup can not occur at any point except the origin O. We prove it by contradiction. Suppose w_j blows up at $x_0 \neq O$. Without loss of generality, we may assume that there is $r_0 > 0$ such that

$$\begin{cases} w_j(x_j) = \max_{B_{r_0}(x_j)} w_j(x) \to +\infty, & \text{and} \\ x_j \to x_0. \end{cases} \tag{6.7}$$

Since $x_0 \neq O$, the function $|x|^{-2+\varepsilon_j}$ is smooth near x_0. Hence by a Harnack-type inequality, we have

$$\left(\min_{B_{\frac{r_0}{2}}(x_j)} w_j \right) \cdot \left(\max_{B_{\frac{r_0}{4}}(x_j)} w_j \right) \leqq C, \tag{6.8}$$

for some constant C independent of j. This Harnack-type inequality (6.8) has appeared in many papers studying the scalar curvature equation, see [7–9, 19] and reference therein. For example, it can be proved by the method of moving planes as in [9]. Because the nonlinear term satisfies $(\gamma \frac{w}{|x|^{2-\varepsilon}} + S_\varepsilon w^{\frac{N+2}{N-2}-\varepsilon}) w^{-\frac{N+2}{N-2}}$ is decreasing in w, the lower order term $\frac{w(x)}{|x|^{2-\varepsilon_j}}$ in equation (6.5) does not have any effect on the proof in [9]. So, we skip the proof of (6.8).

By scaling, we set

$$\begin{cases} v_j(y) = \frac{w_j(x_j + r_j y)}{w_j(x_j)}, \quad r_j = w_j^{-\frac{2^*-2-\varepsilon}{2}}(x_j) \\ v_j(0) = 1 = \max_{B_{\frac{r_0}{r_j}}(0)} v_j(y), \end{cases} \tag{6.9}$$

and $v_j(y)$ converges to v smoothly in any compact subset of $\mathbf{R}^N$ and v satisfies

$$\triangle v + S v^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbf{R}^N, \tag{6.10}$$

where $S = \lim_{j\to+\infty} S_j = S(a, a; \Omega)$. By (6.9), we have

$$\int_{|y|<\frac{r_0}{r_j}} v_j^{2^*-\varepsilon_j}(y)\,\mathrm{d}y = w_j^{\frac{N-2}{2}\varepsilon_j}(x_j)\int_{B_{r_0}(x_j)} w_j^{2^*-\varepsilon_j}(x)\,\mathrm{d}x. \tag{6.11}$$

We claim

$$\lim_{j\to+\infty} w_j^{\varepsilon_j}(x_j) = 1. \tag{6.12}$$

To prove (6.12), we apply the Pohozaev identity:

$$\begin{aligned}
&\frac{\gamma\varepsilon_j}{2}\int_{B_{\frac{r_0}{2}}(x_0)} \frac{w_j^2}{|x|^{2-\varepsilon_j}} + S_j\left(\frac{N}{2^*-\varepsilon_j} - \frac{N-2}{2}\right)\int_{B_{\frac{r_0}{2}}(x_0)} w_j^{2^*-\varepsilon_j}\,\mathrm{d}x \\
&\quad = -\int_{\partial B_{\frac{r_0}{2}}(x_0)} (x\cdot\nabla w_j)\frac{\partial w_j}{\partial\nu}\,\mathrm{d}\sigma + \frac{1}{2}\int_{\partial B_{\frac{r_0}{2}}(x_0)} |\nabla w_j|^2(x\cdot\nu)\,\mathrm{d}\sigma \\
&\quad + \frac{N-2}{2}\gamma\int_{\partial B_{\frac{r_0}{2}}(x_0)} w_j\frac{\partial w_j}{\partial\nu}\,\mathrm{d}\sigma + \frac{\gamma}{2}\int_{\partial B_{\frac{r_0}{2}}(x_0)} \frac{w_j^2}{|x|^{2-\varepsilon_j}}(x\cdot\nu)\,\mathrm{d}\sigma \\
&\quad + \frac{S_j}{(2^*-\varepsilon_j)}\int_{\partial B_{\frac{r_0}{2}}(x_0)} w_j^{2^*-\varepsilon_j}(x\cdot\nu)\,\mathrm{d}\sigma,
\end{aligned}$$

where $S_j = S_{\varepsilon_j}$. This implies

$$\varepsilon_j\int_{B_{\frac{r_0}{2}}(x_j)} w_j^{2^*-\varepsilon_j}(x)\,\mathrm{d}x \leqq C_1 m_j^2,$$

where $m_j = \min_{\partial B_{\frac{r_0}{2}}} w_j$ and $C_1 > 0$ is a constant. Then by (6.8), we have

$$\varepsilon_j \leqq C\, w_j^{-2}(x_j) \quad \text{and then} \quad \lim_{j\to+\infty} w_j^{\varepsilon_j}(x_j) = 1.$$

By (6.11) and (6.12), we have

$$\int_{\mathbf{R}^N} v^{2^*}(y)\, d \leqq 1.$$

Thus, it yields

$$S_N \leqq \frac{\int|\nabla v|^2}{\int (v^{2^*})^{\frac{2}{2^*}}} = S(a, a; \Omega)\left(\int v^{2^*}\right)^{1-\frac{2}{2^*}} \leqq S(a, a; \Omega),$$

a contradiction to (6.3). Thus, we have proved the origin O is the only blowup point of w_j. Then by applying the argument of Section 5 (that is, the case $\gamma > 0,\ s > 0$) to show that $v_j(y)$ converges to v in $H_0^1(\mathbf{R}_+^N)$, where $v_j(y) = w_j(r_j y) r_j^{\frac{N-2}{2}}$, and v is a positive solution of

$$\begin{cases} \triangle v + \gamma \frac{v}{|y|^2} + S(a, a; \Omega) v^{2^*-1} = 0 & \text{in } \mathbf{R}_+^N \\ v = 0 & \text{on } \partial \mathbf{R}_+^N, \end{cases}$$

with

$$\int_{\mathbf{R}_+^N} v^{2^*}(y)\, \mathrm{d}y = 1.$$

Thus, $S(a, a, \Omega) = S(a, a; \mathbf{R}_+^N)$. But, it yields a contradiction to Theorem 3.1 if $H(0) < 0$. Thus we have completed the proof of Theorem 1.2. □

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References

1. Aubin, T.: Problemes isoperimetriques et espaces de Sobolev. *J. Diff. Geom.* **11**(4), 573–598 (1976)
2. Bartsch, T., Peng, S., Zhang, Z.: Existence and non-existence of solutions to elliptic equations related to the Caffarelli–Kohn–Nirenberg inequalities. *Calc. Var. Partial Differ. Equ.* **30**(1), 113–136 (2007)
3. Brézis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* **36**(4), 437–477 (1983)
4. Caffarelli, L., Kohn, R., Nirenberg, L.: First order interpolation inequalities with weights. *Composit. Math.* **53**(3), 259–275 (1984)
5. Catrina, F., Wang, Z.Q.: On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.* **54**(2), 229–258 (2001)
6. Chou, K.S., Chu, C.W.: On the best constant for a weighted Sobolev–Hardy inequality. *J. Lond. Math. Soc.* (2) **48**(1), 137–151 (1993)
7. Chen, C.C., Lin, C.S.: A spherical Harnack inequality for singular solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30**(3–4), 713–738 (2001, 2002)
8. Chen, C.C., Lin, C.S.: Local behavior of singular positive solutions of semilinear elliptic equations with Sobolev exponent. *Duke Math. J.* **78**(2), 315–334 (1995)
9. Chen, C.C., Lin, C.S.: Estimates of the conformal scalar curvature equation via the method of moving planes. *Comm. Pure Appl. Math.* **50**(10), 971–1017 (1997)
10. Chern, J.L., Lin, C.S.: The symmetry of least-energy solutions for seminliear elliptic equations. *J. Differ. Equ.* **187**, 240–268 (2003)
11. Ghoussoub, N., Robert, F.: Concentration estimates for Emden–Fowler equations with boundary singularities and critical growth. *IMRP Int. Math. Res. Pap.* **21867**, 1–85 (2006)
12. Ghoussoub, N., Robert, F.: The effect of curvature on the best constant in the Hardy–Sobolev inequalities. *Geom. Funct. Anal.* **16**(6), 1201–1245 (2006)

13. LIEB, E.H.: Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. *Ann. Math.* **118**(2), 349–374 (1983)
14. GIDAS, B., NI, W.M., NIRENBERG, L.: Symmetry of positive solutions of nonlinear elliptic equations in R^n. Mathematical Analysis and Applications. Part A, pp. 369–402, Adv. in Math. Suppl. Stud., **7a**, Academic Press, New York London, 1981
15. LIN, C.S.: Interpolation inequalities with weights. *Comm. Partial Differ. Equ.* **11**(14), 1515–1538 (1986)
16. LIN, C.S.: Locating the peaks of solutions via the maximum principle. I. The Neumann problem. *Comm. Pure Appl. Math.* **54**(9), 1065–1095 (2001)
17. LIN, C.-S., WANG, Z.Q.: Symmetry of extremal functions for the Caffarelli–Kohn–Nirenberg inequalities. *Proc. Am. Math. Soc.* **132**(6), 1685–1691 (2004)
18. LIN, C.S., WANG, Z.Q.: Least-energy solutions to a class of anisotropic elliptic equations in cones of $\mathbf{R}^N_+$. Preprint.
19. LI, Y.Y.: Prescribing scalar curvature on S^n and related problems, part I. *J. Differ. Equ.* **120**, 319–410 (1995)
20. TALENTI, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)

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