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Evaluating solutions on an elliptic problem in a gravitational gauge field theory ☆

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Abstract

An elliptic equation arising from the study of static solutions with prescribed zeros and poles of the Einstein equations coupled with the classical sigma model and an Abelian gauge field, is considered. We classify the solutions and establish the uniqueness of radially symmetric solutions. We also complete a classification of symmetric solutions of an elliptic equation on the sphere.

Keywords: Nonlinear elliptic equation; A gravitational gauge field theory; Uniqueness and classification of symmetric solutions

1. Introduction

In this paper we study the equation

$$\Delta u = 2e^{\eta}\left(\frac{e^u - 1}{e^u + 1}\right) + 4\pi \sum_{j=1}^{L} \delta_{p_j} - 4\pi \sum_{j=1}^{M} \delta_{q_j} \quad \text{in } \mathbb{R}^2, \tag{1}$$

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where δ_p is the Dirac distribution concentrated at $p \in \mathbb{R}^2$ and η is the function given by

$$e^{\eta(x)} = g_0\left[\frac{e^{u(x)}}{(e^{u(x)}+1)^2}\prod_{j=1}^{L}|x-p_j|^{-2}\prod_{j=1}^{M}|x-q_j|^{-2}\right]^{8\pi G}. \tag{2}$$

Here $g_0 > 0$ is an arbitrary constant, G is Newton's gravitational constant, $p_1, p_2, \dots, p_L$ and $q_1, q_2, \dots, q_M$ are the locations of vortices and antivortices of unit charges. This model is initiated by Yang [8,10] on the coexistence of cosmic strings and antistrings in an Abelian gauge field theory originating with Schroers [6,7] in which the classical sigma model is coupled with a gauge field.

In the theory, with a Minkowski spacetime of signature $(+---)$, the action density of the gauged sigma model is given by

$$\mathcal{L} = -\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} + \frac{1}{2}g^{\mu\nu}(D_\mu\phi)\cdot(D_\nu\phi) - \frac{1}{2}(\mathbf{n}\cdot\phi)^2,$$

where the field configuration $\phi = (\phi_1, \phi_2, \phi_3)$ is a spin vector which maps $\mathbb{R}^2$ into the unit 2-sphere S^2, $\mathbf{n}$ is the north pole of S^2, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the electromagnetic field induced from the Abelian gauge field A_μ ($\mu, \nu = 0, 1, 2, 3$ with $t = x_0$), $D_\mu\phi = \partial_\mu\phi + A_\mu(\mathbf{n}\times\phi)$ is the gauge-covariant derivative, and the gauge group is the subgroup of $O(3)$ that leaves $\mathbf{n}$ invariant. Through a stereographic projection from the south pole $\mathbf{n}$ of S^2 into the complex plane, we set

$$\psi = \psi_1 + \mathrm{i}\psi_2, \quad \psi_1 = \frac{\phi_1}{1+\phi_3}, \; \psi_2 = \frac{\phi_2}{1+\phi_3}.$$

So that with the substitution $A_\mu \mapsto -A_\mu$ and the new gauge-covariant derivative given by $D_\mu\psi = \partial_\mu\psi + \mathrm{i}A_\mu\psi$, we arrive at an Abelian gauge theory with the action density

$$\mathcal{L} = -\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} + \frac{2}{(1+|\psi|^2)^2}g^{\mu\nu}(D_\mu\psi)(\overline{D_\nu\psi}) - \frac{1}{2}\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)^2. \tag{3}$$

With the coupling of gravity, the equations of motion of the field-theoretic model (3) are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{4}$$

$$\frac{1}{\sqrt{|g|}}D_\mu\left(\frac{\sqrt{|g|}g^{\mu\nu}}{(1+|\psi|^2)^2}D_\nu\psi\right) = f, \tag{5}$$

$$\frac{1}{\sqrt{|g|}}\partial_{\mu'}\left(g^{\mu\nu}g^{\mu'\nu'}\sqrt{|g|}F_{\nu\nu'}\right) = j^\mu, \tag{6}$$

where $G_{\mu\nu}$ is the Einstein tensor, G is Newton's gravitational constant, Λ is the cosmological constant, $T_{\mu\nu}$ is the energy-momentum tensor defined by

$$T_{\mu\nu} = -g^{\mu'\nu'}F_{\mu\mu'}F_{\nu\nu'} + \frac{2}{(1+|\psi|^2)^2}\left(D_\mu\psi\overline{D_\nu\psi} + \overline{D_\mu\psi}D_\nu\psi\right) - g_{\mu\nu}\mathcal{L}, \tag{7}$$

and f, j^μ are the force and current density terms respectively given by

$$f = \frac{(1-|\psi|^2 - 2g^{\mu\nu}D_\mu\psi\overline{D_\nu\psi})}{(1+|\psi|^2)^3}\psi,$$

$$j^\mu = \frac{2\mathrm{i}}{(1+|\psi|^2)^2}g^{\mu\nu}(\overline{\psi}D_\nu\psi - \psi\overline{D_\nu\psi}).$$

Let x_0 be the temporal component and (x_i), $i = 1, 2, 3$, the spatial coordinates. To looking for static the field configurations which depend only on spatial variable x_1, x_2, we assume $A_0 = A_3 = 0$ and the metric tensor is of the form

$$g_{\mu\nu} = \operatorname{diag}\big(1, -e^\eta, -e^\eta, -1\big), \tag{8}$$

with η being a function of x_1, x_2 only. In this manner, the energy density $\mathcal{H} = T_{00}$ has the representation

$$\mathcal{H} = e^{-\eta}(F_{12} + J_{12}) + \frac{1}{2}\left(e^{-\eta}F_{12} - \frac{1-|\psi|^2}{1+|\psi|^2}\right)^2 + \frac{2e^{-\eta}}{(1+|\psi|^2)^2}|D_1\psi + \mathrm{i}D_2\psi|^2, \tag{9}$$

in which $J_{jk} = \partial_j J_k - \partial_k J_j$ with a new current density

$$J_k = \frac{\mathrm{i}}{1+|\psi|^2}(\psi\overline{D_k\psi} - \overline{\psi}D_k\psi), \quad k = 1, 2.$$

As a consequence of (9), one can obtain the following Bogomol'nyi equations

$$D_1\psi + \mathrm{i}D_2\psi = 0, \tag{10}$$

$$F_{12} = e^\eta\frac{1-|\psi|^2}{1+|\psi|^2}. \tag{11}$$

Note that the metric (8) implies the Einstein tensor $G_{\mu\nu}$ assumes the form

$$G_{00} = -G_{33} = -K_g, \qquad G_{\mu\nu} = 0 \quad \text{for other } (\mu, \nu) \text{ pairs}, \tag{12}$$

where K_g is the Gauss curvature of the 2-surface $(\mathbb{R}^2, e^\eta\delta_{jk})$ which can be given by

$$K_g = -\frac{1}{2}e^{-\eta}\Delta\eta$$

with $\Delta = \partial_1^2 + \partial_2^2$ being the Laplace operator on $\mathbb{R}^2$. From (12) we see that the cosmological constant $\Lambda = 0$ and the Einstein equations (4) are boiled down to a single equation

$$K_g = 8\pi G\mathcal{H}, \tag{13}$$

which relates the Gauss curvature to the energy density. The sets $\{p_j\}_{j=1}^L, \{q_j\}_{j=1}^M$ of zeros and poles of ψ are the sites for strings and antistrings to reside. Using the substitution $u = \log|\psi|^2$, Eqs. (10) and (11) is then reduced to Eq. (1). A solution pair (ψ, A) of (10) and (11) can be constructed from u via the standard procedure

$$\psi(z)=\exp\left(\frac{1}{2}u(z)+\mathrm{i}\theta(z)\right),\qquad \theta(z)=\sum_{j=1}^{L}\arg(z-p_j)-\sum_{j=1}^{M}\arg(z-q_j),$$

$$A_1(z)=-\mathrm{Re}\left\{2\mathrm{i}\bar\partial\log\psi(z)\right\},\qquad A_2(z)=-\mathrm{Im}\left\{2\mathrm{i}\bar\partial\log\psi(z)\right\}.$$

Hence the energy density $\mathcal{H}$ can be rewritten by

$$e^{\eta}\mathcal{H}=\Delta\left(\log\left(1+e^{u}\right)-\frac{1}{2}u\right)+2\pi\sum_{j=1}^{L}\delta_{p_j}+2\pi\sum_{j=1}^{M}\delta_{q_j}\quad \text{in } \mathbb{R}^2.$$

It follows from (13) that the factor e^{η} is explicitly determined by the expression (2).

The solutions of (1) can be evaluated by a functional β that associates each solution u with the value

$$\beta(u)=-\frac{1}{\pi}\int_{\mathbb{R}^2}e^{\eta}\left(\frac{e^{u}-1}{e^{u}+1}\right)dx.$$

We remark that β describes the value of magnetic flux in the model corresponding to a solution u, and on the other hand, it characterizes the asymptotic behavior of u near infinity, that is (via a potential analysis [2]),

$$\beta=2(L-M)-\lim_{|x|\to\infty}\frac{u(x)}{\log|x|}. \tag{14}$$

Here arise the following interesting problem. For a prescribed value $\lambda\in\mathbb{R}$, we inquire whether there exists any solution u such that $\beta(u)=\lambda$. And, if such solution exists, we wish to seek for its uniqueness. In this paper, we concentrate our study on radially symmetric solutions and on the cases either (i) $L>0$, $M=0$ or (ii) $L=0$, $M>0$. In either cases, we assume that the locations $\{p_j\}_{i=1}^{L}$ (or $\{q_j\}_{j=1}^{M}$) of vortices (or antivortices) cluster at the origin. Specifically, letting $L=N>0$ and $M=0$, Eq. (1) is reduced to

$$\Delta u=-|x|^{-2aN}h\left(e^{u}\right)+4\pi N\delta_0\quad \text{in } \mathbb{R}^2, \tag{15}$$

where

$$h(t)=-2g_0\left[\frac{t}{(t+1)^2}\right]^{a}\left(\frac{t-1}{t+1}\right),\qquad a=8\pi G.$$

The functional β, depending on $u(x)=u(r)$ with $r=|x|$, is given by

$$\beta(u)=\int_0^{\infty}r^{-2aN+1}h\left(e^{u(r)}\right)dr. \tag{16}$$

Here are our consequences.

Theorem 1.1. *Let u be a radially symmetric solution of Eq. (15). Then the following statements are true.*

(a) *If $aN < 1$, then $\beta(u) \in E := (-\infty, \frac{8(aN-1)}{a}) \cup \{2N\} \cup (\frac{4}{a}, \infty)$. Conversely, for any prescribed value $\lambda \in E$, there exists a unique (radially symmetric) solution u_λ such that $\beta(u_\lambda) = \lambda$. In addition, u_λ has the asymptotic behavior*

$$u_\lambda(r) = (2N - \lambda)\log r + O(1)$$

as $r \to \infty$; especially, when $\lambda = 2N$, $u_\lambda(r) \to 0$ as $r \to \infty$.

(b) *If $aN = 1$, then $\beta(u) \equiv \text{constant} \in \{0, 2N, 4N\}$.*

(c) *If $aN \geqslant 2$, then every solution u satisfies that $u(r) \to \infty$ as $r \to \infty$. Moreover,*

$$0 < \beta(u) < \frac{4}{a}.$$

Note. When $L = 0$ and $M = N > 0$ in Eq. (1), the problem is read as the following:

$$\Delta u = -|x|^{-2aN} h\big(e^u\big) - 4\pi N\delta_0 \quad \text{in } \mathbb{R}^2. \tag{17}$$

Note that $h(t^{-1}) = -h(t)$. Taking $u \mapsto -u$ interchanges Eq. (17) with (15). Clearly, $\beta(-u) = -\beta(u)$. So we have mirror consequences of Theorem 1.1 for this case.

As an interesting aspect of this theory, the model can be performed an Abelian gauge field over a compact Riemann surface [10], in which there arises a nonlinear elliptic equation on the unit sphere S^2 about locating vortices and antivortices at the north and south poles of S^2. In fact, denoting the north and south poles respectively by $\mathbf{n}$ and $\mathbf{s}$, here is the equation:

$$\begin{cases} \Delta_g u = 2e^\eta\left(\dfrac{e^u - 1}{e^u + 1}\right) + 4\pi N\delta_{\mathbf{n}} + 4\pi P\delta_{\mathbf{s}}, \\ \Delta_g\left(\dfrac{\eta}{16\pi G} + \log\big(1 + e^u\big) - \dfrac{1}{2}u\right) = \dfrac{K_0}{8\pi G} - 2\pi|N|\delta_{\mathbf{n}} - 2\pi|P|\delta_{\mathbf{s}}, \end{cases} \tag{18}$$

where Δ_g is the Laplace–Beltrami operator on (S^2, g) with is the standard metric g, K_0 is the associated (constant) Gauss curvature and N, P are integers for which positive or negative integer represents the corresponding number of prescribed zeros or poles of u respectively. We remark that, as a result in [10], a necessary condition for the existence of a solution u solving Eq. (18) is that the Newton gravitational constant G satisfies the quantization condition

$$4\pi G\big(|N| + |P|\big) = 1. \tag{19}$$

To classify the symmetric solution u of (18) with respect to $\mathbf{n}$ and $\mathbf{s}$, here are the known consequences:

Theorem A. *(See [10].) Consider symmetric solutions of (18).*

(i) *Existence: If $|N| = |P| > 0$, then there is a (symmetric) solution.*

(ii) *Nonexistence*: *If* $|N| > 0$, $P = 0$ *or* $N = 0$, $|P| > 0$, *then there is no solution*; *if* $|N| \neq |P|$ *with* $NP > 0$, *then there is no solution.*

There remains an unsolved case in Theorem A, namely $|N| \neq |P|$ with $NP < 0$. To look for the answer, we note that the symmetric case of (18) can be equivalently put into the framework of Eq. (15) associated with a prescribed asymptotic behavior at infinity. In fact, since (18) is invariant under the transformation $(\eta, u) \mapsto (\eta, -u)$, it suffices to consider the case $N > 0$, $P < 0$ and $|N| < |P|$. Note that, through a stereographic projection from the south pole of S^2, we are looking for a radially symmetric solution $u = u(r)$, $r = |x|$, satisfying

$$\begin{cases} \Delta u = 2e^{\eta}\left(\dfrac{e^u - 1}{e^u + 1}\right) + 4\pi N\delta_0 \quad \text{in } \mathbb{R}^2, \\ \lim\limits_{|x|\to\infty} \dfrac{u(r)}{\log r} = -2P, \end{cases} \tag{20}$$

with

$$e^{\eta} = \lambda r^{-2aN}\left(\frac{e^u}{(e^u + 1)^2}\right)^a, \quad a = 8\pi G,$$

where $\lambda > 0$ is a constant. In view of (19), $a(|N| + |P|) = 2$, we have $aN < 1$. So that, if there is a solution u solving (20), then by Theorem 1.1, it follows that

$$-2P \in \left(-\infty, 2N - \frac{4}{a}\right) \cup \{0\} \cup \left(-6N + \frac{8}{a}, \infty\right).$$

Since $2N - (4/a) < 0$, it must hold that $-2P > -6N + (8/a)$, or equivalently,

$$\frac{4}{a} = 2\big(|N| + |P|\big) > -4N + \frac{8}{a} > \frac{4}{a},$$

yielding a contradiction. Therefore, we complete the classification in Theorem A with the following corollary:

Corollary 1. *If* $|N| \neq |P|$ *with* $NP < 0$, *then Eq.* (18) *admits no symmetric solution.*

Therefore, *A symmetric solution of Eq.* (18) *exists only when the zeros or poles located at the north and south poles of* S^2 *are balanced, in the sense that there are equal numbers of zeros or poles clustered at* **n** *and* **s**.

Remark. In the case $aN = 1$ in Theorem 1.1, the constant associated with the value of β is dependent (only) on the given of the parameter g_0 in the expression (2). For example, as one may see in [9, Theorem 11.3.7] with the choice $g_0 = 4^a N$, we conclude that $\beta = 2N$. On the other hand, in view of the context of (20) with $P = \pm N$, the existence results in Theorem A indicate that both $\beta = 0$ and $\beta = 4N$ are also possible provided suitable g_0 is selected.

The paper is organized as follows. We divided the proof of Theorem 1.1 into several aspects. In Section 2 we include a preliminary classification of solutions. In Section 3 we establish the uniqueness of solutions in the case $aN < 1$. In Section 4 we identify the range of $\beta(u)$ and carry out the main theorem.

2. Preliminaries

To make sense of the solution classification, we associate the radially symmetric solutions of Eq. (15) with real numbers in the sense that $u(r) = u(r;s)$ with $s \in \mathbb{R}$ satisfies

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) = -r^{-2aN}h\bigl(e^{u(r)}\bigr), & r = |x| > 0, \\ u(r) = 2N\log r + s + o(1), \qquad u'(r) = \frac{2N}{r} + o(1) & \text{as } r \to 0. \end{cases} \tag{21}$$

Since $h : \mathbb{R}^+ \to \mathbb{R}$ is a bounded function, u cannot blow up at a finite r. Let Λ be the solution set of (21). We divide Λ into three classes that

$$\mathfrak{S}^+ = \Bigl\{u \in \Lambda \colon \lim_{r\to\infty} u(r) = \infty\Bigr\};$$
$$\mathfrak{S}^* = \Bigl\{u \in \Lambda \colon \lim_{r\to\infty} u(r) = 0\Bigr\};$$
$$\mathfrak{S}^- = \Bigl\{u \in \Lambda \colon \lim_{r\to\infty} u(r) = -\infty\Bigr\}.$$

Then the set of real numbers is split up into the components:

$$J^+ = \bigl\{s \in \mathbb{R} \colon u(r;s) \text{ is of the class } \mathfrak{S}^+\bigr\};$$
$$J^* = \bigl\{s \in \mathbb{R} \colon u(r;s) \text{ is of the class } \mathfrak{S}^*\bigr\};$$
$$J^- = \bigl\{s \in \mathbb{R} \colon u(r;s) \text{ is of the class } \mathfrak{S}^-\bigr\}.$$

Note that $J^+ \cup J^* \cup J^- = \mathbb{R}$. In the following we conclude some elementary facts associated with these settings.

(P1) By the continuous dependence of u on s, J^+ and J^- are open sets whose boundary satisfies $\partial J^+ \cup \partial J^- \subset J^*$.

(P2) By the maximum principle, we see that
- (i) if $u \in \mathfrak{S}^* \cup \mathfrak{S}^-$, then $u < 0$;
- (ii) if $u \in \mathfrak{S}^* \cup \mathfrak{S}^+$, then $u' > 0$.

The $\beta(u)$ defined in (16) is treated as a function of s which is given by

$$\beta(s) = \beta\bigl(u(\cdot, s)\bigr) = \int_0^\infty r^{-2aN+1}h\bigl(e^{u(r;s)}\bigr)\,dr.$$

(P3) According to Eq. (21), we have

$$ru'(r) = 2N - \int_0^r \tau^{-2aN+1} h\big(e^{u(\tau)}\big)\,d\tau, \tag{22}$$

from which it is not hard to see that $|\beta| < \infty$, $\lim_{r\to\infty} ru'(r) = 2N - \beta$ and

$$u(r) = (2N - \beta)\log r + O(1), \quad r \to \infty. \tag{23}$$

Clearly, $\beta(u) = 2N$ provided $u \in \mathfrak{S}^*$. Assume $s \in J^-$. Since $e^{u(r;s)}$ vanishes as $r \to \infty$, we extract constants $\varrho, r_0 > 0$ such that $e^{au} \leqslant \varrho h(e^u)$ for all $r \geqslant r_0$. By (23),

$$\int_{r_0}^{\infty} r^{-2aN+1} e^{au(r)}\,dr \leqslant \varrho \int_{r_0}^{\infty} r^{-2aN+1} h\big(e^{u(r)}\big)\,dr < \infty.$$

So that

$$\beta(s) > \frac{2}{a}, \quad s \in J^-. \tag{24}$$

Similarly, if $s \in J^+$, $e^{-u(r;s)}$ vanishes as $r \to \infty$. Hence applying the fact $h(t^{-1}) = -h(t)$ we conclude that

$$\beta(s) < 4N - \frac{2}{a}, \quad s \in J^+. \tag{25}$$

A Pohozaev-type identity associated with Eq. (21) is established as follows. By multiplying (21) by the factor $\tau u'(\tau)$ and taking integration both sides over $(0, r)$,

$$\begin{aligned}
\big[ru'(r)\big]^2 - 4N^2 &= -2\int_0^r \tau^{-2aN+2} h\big(e^{u(\tau)}\big)u'(\tau)\,d\tau \\
&= -2\int_0^r \tau^{-2aN+2} \frac{d}{d\tau} H\big(e^{u(\tau)}\big)\,d\tau \\
&= -2\big[\tau^{-2aN+2} H\big(e^{u(\tau)}\big)\big]_{\tau=0}^{\tau=r} \\
&\quad + 4(1 - aN)\int_0^r \tau^{-2aN+1} H\big(e^{u(\tau)}\big)\,d\tau,
\end{aligned} \tag{26}$$

in which $H : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$H(t) = H_\sigma(t) = \int_\sigma^t \frac{h(\zeta)}{\zeta}\,d\zeta,$$

where σ may denote either 0 or ∞ regarding the following context. Denote

$$I_r = \int_0^r \tau^{-2aN+1} h\big(e^{u(\tau)}\big)\, d\tau.$$

From (22), $I_r = 2N - ru'(r)$. Using I_r in substitution for $ru'(r)$ in (26), we get

$$I_r\left(I_r - \frac{4}{a}\right) = -2\big[\tau^{-2aN+2} H\big(e^{u(\tau)}\big)\big]_{\tau=0}^{\tau=r} + 4(1-aN)\int_0^r \tau^{-2aN+1} G\big(u(\tau)\big)\, d\tau, \tag{27}$$

where $G(\zeta) = H(e^\zeta) - (1/a)h(e^\zeta)$. Note that

$$G'(\zeta) = \frac{-2e^{(a+1)\zeta}}{(e^\zeta+1)^{2a+2}}\big(e^\zeta - e^{\mu_0}\big), \quad e^{\mu_0} = 1 + \frac{1}{a}. \tag{28}$$

Therefore, G is increasing in the interval $(-\infty, \mu_0)$, decreasing in (μ_0, ∞) and attains its maximum at $\zeta = \mu_0 > 0$.

Theorem 2.1. *Assume $aN < 1$. Let $u = u(r; s)$ be the solution of (21) with $s \in \mathbb{R}$. Then the following statements are true.*

(i) *If $s \in J^-$, then $\beta(s) > \frac{4}{a}$. Moreover, $\beta(s_j) \to \infty$ whenever $\{s_j\} \subset J^-$ and $s_j \to \partial J^-$.*
(ii) *If $s \in J^+$, then $\beta(s) < 0$. Moreover, $\beta(s_j) \to -\infty$ whenever $\{s_j\} \subset J^+$ and $s_j \to \partial J^+$.*

Proof. Assume $u \in \mathfrak{S}^-$. Recall that $u < 0$ and $u(r) \to -\infty$ as $r \to \infty$. Consider

$$G(\zeta) = H_0\big(e^\zeta\big) - (1/a)h\big(e^\zeta\big) = \int_0^{\exp(\zeta)} \frac{h(t)}{t}\, dt - (1/a)h\big(e^\zeta\big). \tag{29}$$

Clearly, $G(-\infty) = 0$. Thus, by (28), $G(u)$ is positive. On the other hand, applying L'Hospital's Rule,

$$\lim_{r\to\infty} r^{-2aN+2} H_0\big(e^{u(r)}\big) = \frac{1}{2(aN-1)} \lim_{r\to\infty} r^{-2aN+2} h\big(e^u\big)\big(ru'\big) = 0,$$

in which the estimates (23) and (24) are taken into account. Letting $r \to \infty$ in (27), we get

$$\beta\left(\beta - \frac{4}{a}\right) = 4(1-aN)\int_0^\infty r^{-2aN+1} G\big(u(r)\big)\, dr > 0. \tag{30}$$

So that $\beta > 4/a$, as desired. Similarly, in the case $u \in \mathfrak{S}^+$ where $u(r) \to \infty$ as $r \to \infty$, we consider

$$G(\zeta) = H_\infty\left(e^\zeta\right) - (1/a)h\left(e^\zeta\right) = \int_\infty^{\exp(\zeta)} \frac{h(t)}{t}\,dt - (1/a)h\left(e^\zeta\right).$$

Note that $G(\infty) = 0$. Since $h(\zeta^{-1}) = -h(\zeta)$,

$$G(-\infty) = -\left(\int_0^1 + \int_1^\infty\right)\frac{h(\zeta)}{\zeta}\,d\zeta = -\left(\int_0^1 - \int_0^1\right)\frac{h(\zeta)}{\zeta}\,d\zeta = 0.$$

From (28) again, $G(u)$ is positive. By (23), (25) and L'Hospital's Rule,

$$\lim_{r\to\infty} r^{-2aN+2} H_\infty\left(e^{u(r)}\right) = \frac{1}{2(1-aN)} \lim_{r\to\infty} r^{-2aN+2} h\left(e^{-u}\right)\left(ru'\right) = 0.$$

In view of (25) and (30), $\beta < 0$. To complete the first conclusion, we assume, contrarily, that there exist a sequence $s_j \in J^-$ and $C > 0$ such that $s_j \to s^* \in \partial J^-$ and $|\beta(s_j)| < C$. Let

$$u_j(r) = u(r; s_j), \qquad u^*(r) = u\left(r; s^*\right).$$

Note that $s^* \in J^*$. For $r > 0$, we have $\|u_j - u^*\|_{L^\infty(0,r)} \to 0$ as $s_j \to s^*$. Since I_r has a uniform bound with various r by means of (27), it is possible to extract $\bar{C} > 0$ such that

$$0 < \int_0^r \tau^{-2aN+1} H_0\left(e^{u^*(\tau)}\right) d\tau < \bar{C}, \tag{31}$$

regardless of the selection of r. This indicates $r^{-2aN+1} H_0(e^{u^*(r)})$ is integrable over $(0, \infty)$; however, it is impossible because $r^{-2aN+1} = r^{-1+\epsilon}$ for some $\epsilon > 0$ and

$$\lim_{r\to\infty} H_0\left(e^{u^*(r)}\right) = \int_0^1 \frac{h(\zeta)}{\zeta}\,d\zeta > 0.$$

Therefore, $\beta(s_j) \to \infty$ whenever $s_j \to \partial J^-$ and $s_j \in J^-$. The second conclusion for the asymptotic behavior of $\beta(s)$ comes from the same argument by using H_∞ in place of H_0 in (31). We omit the details here. □

Let $u(r) = u(r; s)$ solve (21). Denote

$$\varphi(r) = \varphi(r; s) = \frac{\partial u}{\partial s}(r; s).$$

Clearly, φ satisfies

$$\begin{cases} \varphi''(r) + \frac{1}{r}\varphi'(r) = Q(r)\varphi(r), & r > 0, \\ \varphi(0) = 1, \quad \varphi'(0) = 0, \end{cases} \tag{32}$$

where $Q(r) = -r^{-2aN} h'(e^{u(r)})e^{u(r)}$. Especially when $aN = 1$, the function $w(r) = ru'(r)$ satisfies (32) with $w(0) = 2N$. Thus, $w = 2N\varphi$. This leads to the following theorem.

Theorem 2.2. *If $aN = 1$, then $\beta(s) \equiv \text{constant} \in \{0, 2N, 4N\}$.*

Proof. By definition,

$$2N\varphi'(r) = w'(r) = ru''(r) + u'(r) = -r^{-2aN+1} h\big(e^{u(r)}\big), \quad u(r) = u(r; s). \tag{33}$$

From property (P2) before, we have (i) if $s \in J^- \cup J^*$, then $\varphi'(\cdot, s) < 0$; (ii) if $s \in J^+$, then $\varphi'(r; s) < 0$ for $r \in (0, r_0)$ and $\varphi'(r; s) > 0$ for $r \in (r_0, \infty)$ where $u(r_0) = 0$. On the other hand, in the light of (23) and (26), it is not hard to see that

$$\beta(s) = 0 \quad \text{for } s \in J^+; \qquad \beta(s) = 2N \quad \text{for } s \in J^*; \qquad \beta(s) = 4N \quad \text{for } s \in J^-.$$

Note that $w(r; s) = ru'(r; s) \to 2N - \beta(s)$ as $r \to \infty$. Hence

$$\lim_{r\to\infty} \varphi(r; s) = \lim_{r\to\infty} w(r; s)/2N = 1, 0, -1,$$

for $s \in J^+, J^*, J^-$ respectively. So $|\varphi(r; s)| \leqslant 1$ for all (r, s). Let $s_1, s_2 \in \mathbb{R}$. Then

$$\big|u(r; s_1) - u(r; s_2)\big| \leqslant \left|\frac{\partial u}{\partial s}(r, \cdot)\right| |s_1 - s_2| \leqslant |s_1 - s_2|, \quad r > 0,$$

indicating that the function $u(\cdot, s_1) - u(\cdot, s_2)$ must be bounded. In view of the asymptotic behavior characterized in (23), it follows that $\beta(s_1) = \beta(s_2)$. Therefore the theorem is proved. □

Theorem 2.3. *Let $aN \geqslant 2$. Assume u is a solution of Eq. (21). Then u satisfies that $u(r) \to \infty$ as $r \to \infty$. Moreover,*

$$0 < \beta(u) < \frac{4}{a}.$$

Proof. From (27), we have

$$\beta\left(\beta - \frac{4}{a}\right) = 4(1 - aN) \int_0^\infty r^{-2aN+1} G\big(u(r)\big)\, dr < 0. \tag{34}$$

Obviously, (21) admits no $\mathfrak{S}^-$ type solution because of (23) and (34). Assume there exists a solution $u^* \in \mathfrak{S}^*$. Note that $\beta(u^*) = 2N$ and $ru^*(r) \to 0$ as $r \to \infty$. From (26),

$$\frac{N^2}{aN-1} = \int_0^\infty \tau^{-2aN+1} H\left(e^{u(\tau)}\right) d\tau$$
$$> \frac{1}{a} \int_0^\infty \tau^{-2aN+1} h\left(e^{u(\tau)}\right) d\tau = \frac{2N}{a},$$

indicating $aN < 2$. This contradicts our assumption. So there is no solution of $\mathfrak{S}^*$ type. Therefore, nothing but $\mathfrak{S}^*$ type solutions can exist in this case; furthermore, the range of β follows readily from (34). □

3. Uniqueness of the evaluation with $aN < 1$

The following theorem shows the monotonicity of $\beta(s)$, by which the sets J^-, J^* and J^+ are identified.

Theorem 3.1. *If J^* is not empty, then there is $s^* \in \mathbb{R}$ such that $J^- = (-\infty, s^*)$, $J^* = \{s^*\}$ and $J^+ = (s^*, \infty)$. Moreover, $\beta'(s) > 0$ for $s \in \mathbb{R} - \{s^*\}$ and*

$$\lim_{s \to s^{*-}} \beta(s) = \infty, \qquad \lim_{s \to s^{*+}} \beta(s) = -\infty.$$

We now anticipate the required pieces in advance of assembling Theorem 3.1. We apply the techniques introduced in [1]. Recall the linear equation (32) we mention before. Let us rewrite it as follows:

$$\begin{cases} \varphi''(r) + \dfrac{1}{r}\varphi'(r) = f\left(e^{u(r)}\right)q(r)\varphi(r), & r > 0, \\ \varphi(0) = 1, \qquad \varphi'(0) = 0, \end{cases} \tag{35}$$

where $f(t) = -at^2 + (2a+2)t - a$ and

$$q(r) = 2g_0 r^{-2aN}\left[\frac{e^u}{(e^u+1)^2}\right]^a \left(\frac{1}{(e^u+1)^2}\right) > 0.$$

Obviously, the quadratic polynomial f has two positive zeros at $t = T_1, T_2$ given by

$$T_1 = \frac{a+1-\sqrt{2a+1}}{a}, \qquad T_2 = \frac{a+1+\sqrt{2a+1}}{a};$$

f is increasing in $(-\infty, a+1)$ and decreasing in $(a+1, \infty)$. To figure out the behavior of φ, we compare it with the function $w_c(r) = ru'(r) + c$, where c is a constant yet to be determined. From (21) and (35), w_c satisfies the equation

$$w_c''(r) + \frac{1}{r}w_c'(r) = f\left(e^{u(r)}\right)q(r)w_c(r) + q(r)p_c\left(e^{u(r)}\right), \quad r > 0, \tag{36}$$

where p_c is the quadratic polynomial given by

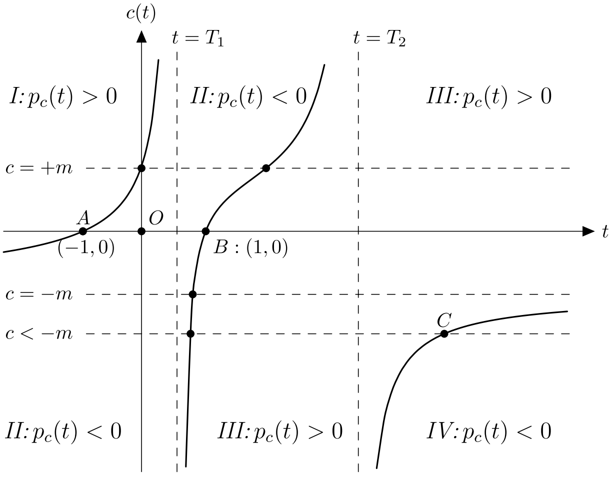

Fig. 1. c is a function of t with $p_c(t) = p(c,t) = 0$.

$$\begin{aligned} p_c(t) &= 2(1-aN)(t+1)(t-1) - cf(t) \\ &= a(c+m)t^2 - 2c(a+1)t + a(c-m), \quad m = 2(1-aN)/a > 0. \end{aligned} \tag{37}$$

If $c = -m$, the function $p_c(t)$ has only one zero at $t = a/(a+1)$. If $c \in \mathbb{R}\backslash\{-m\}$, then $p_c(t)$ admits two distinct zeros. As shown in Fig. 1, $p_c(t) = p(c,t)$ can be treated as a function of $(c,t) \in \mathbb{R}^2$ and the level set $\{(c,t)\colon\ p_c(t) = 0\}$ consists of three disjoint curves containing the points A, B and C in the diagram respectively. They divide the plane into four regions denoted by I, II, III and IV, in which $p_c(t)$ is positive in regions I, III and is negative in II, IV. From (37), the level set $p_c(t) = 0$ defines c as a function of t, namely

$$c(t) = \frac{am(t+1)(t-1)}{f(t)}$$

for $t \in \mathbb{R} - \{T_1, T_2\}$. Note that

$$\lim_{t\to\xi^-} c(t) = \infty, \qquad \lim_{t\to\xi^+} c(t) = -\infty, \qquad \lim_{t\to-\infty} c(t) = \lim_{t\to\infty} c(t) = -m,$$

where ξ denotes T_1, T_2. Now we combine Eqs. (35) and (36) as follows. Since

$$\frac{d}{dr}\big[r\big(\varphi w_c' - w_c\varphi'\big)\big] = \varphi\big(rw_c'\big)' - w_c\big(r\varphi'\big)',$$

by taking integration on both sides of the formula, we get the identity

$$\int_{r_1}^{r_2} rP_c\varphi\,dr = r_2\big[\varphi(r_2)w_c'(r_2) - w_c(r_2)\varphi'(r_2)\big] - r_1\big[\varphi(r_1)w_c'(r_1) - w_c(r_1)\varphi'(r_1)\big], \tag{38}$$

where $P_c(r) = q(r)p_c(e^{u(r)})$.

Lemma 3.1. *Let φ be the solution of (35) with $u(r) = u(r; s)$.*

(a) *If $s \in J^*$, then φ is positive.*
(b) *If $s \in J^+$, then φ changes sign at most twice.*
(c) *If $s \in J^-$, then φ changes sign either once or twice.*

Proof. Let $u(r) = u(r; s)$. We perform the proof case by case with s in the following.

Case 1. For $s \in J^*$, we assume contrarily that $z_1 > 0$ is the first zero of φ for which $\varphi(z_1) = 0$ and $\varphi(r) > 0$ in $(0, z_1)$. We consider $c = 0$ in (38). Note that $w(r) = w_0(r) = ru'(r) > 0$ for all r. Since $e^u < 1$ and $p_0(t) < 0$ in $(0, 1)$, we have $P_0(r)$ negative in $(0, z_1)$. So that

$$0 > \int_0^{z_1} r P_0 \varphi \, dr = -z_1 w(z_1) \varphi'(z_1) > 0, \tag{39}$$

which is a contradiction. Therefore, $\varphi(r; s)$ does not change sign provided $s \in J^*$.

Case 2. Let $s \in J^+$. Assume φ changes sign at least three times. Let z be the second zero of φ. Pick r_1, r_2 with $0 < r_1 < z < r_2$, such that

$$\begin{cases} \varphi(r) < 0 & \text{for } r \in (r_1, z), \\ \varphi(r) > 0 & \text{for } r \in (z, r_2), \end{cases} \tag{40}$$

and $\phi(r)$ has local minimum and maximum at $r = r_1, r_2$ respectively. Let $\sigma_1 = e^{u(r_1)}$. From (35), we have

$$f(\sigma_1) q(r_1) \varphi(r_1) = \varphi''(r_1) \geqslant 0,$$

indicating $f(\sigma_1) \leqslant 0$. Thus either $\sigma_1 \leqslant T_1$ or $\sigma_1 \geqslant T_2$, where recall that T_1, T_2 are the zeros of f with $T_1 < T_2$. Now we show, in the following, that either of these situations for σ_1 cannot occur.

(2-1) Assume $\sigma_1 \leqslant T_1$. Let z_1 be the first zero of φ. Clearly, $z_1 < r_1$. Since $T_1 < 1$ and $u(r)$ decreases as r decreases, it follows that $e^{u(r)} < 1$ for all $r \in (0, z_1)$. Consider (38) and let $w = w_{c=0}$. $P_0(r)$ is negative for all $r \in (0, z_1)$ because $p_0(t) < 0$ for $t \in (0, 1)$. Therefore, from the same expression as (39), we obtain a contradiction.

(2-2) Assume $\sigma_1 \geqslant T_2$. Let $\zeta = e^{u(z)}$ and $\sigma_2 = e^{u(r_2)}$. Clearly,

$$1 < T_2 \leqslant \sigma_1 < \zeta < \sigma_2.$$

By virtue of the behavior of $c(t)$, as shown in Fig. 1, we let $c < -m$ be the constant such that $p_c(\zeta) = 0$, $p_c(t) > 0$ for $t \in (\sigma_1, \zeta)$ and $p_c(t) < 0$ for $t \in (\zeta, \sigma_2)$. Hence

$$\begin{cases} P_c(r) > 0 & \text{for } r \in (r_1, z), \\ P_c(r) < 0 & \text{for } r \in (z, r_2). \end{cases} \tag{41}$$

With our selection of r_1, r_2, c and in view of (40) and (41), the left-hand side (LHS) of (38) is negative, whereas the right-hand side (RHS) is positive. That is a contradiction. Here we also make use of (33), concluding $w_c' > 0$ provided $e^u > 1$. The second assertion of this lemma is concluded.

Case 3. Suppose $s \in J^-$. We have to exclude the situations that $\varphi > 0$ as well as that φ changes sign more than twice. Recall that $e^u < 1$ and $w_c' < 0$ for any $\mathfrak{S}^-$ solution u. We divide the argument into the following subcases.

(3-1) Assume φ is positive all the time. Then either φ is bounded or $\varphi(r) \to \infty$ as $r \to \infty$. Consider the identity (38) with $c = 0$. Let $w = ru'$. From (21), (24) and (35), $w(r)$ approaches to $2N - \beta < 0$ and $\varphi(r) = O(\log r)$ as $r \to \infty$. Moreover, from (23), (24) and (33), we have $rw' = O(r^{-\ell})$ at infinity for some $\ell > 0$. Because $p_0(t) < 0$ for $t \in (0, 1)$,

$$0 > \int_0^\infty r P_0 \varphi \, dr = \lim_{r\to\infty} r\big[\varphi(r)w'(r) - w(r)\varphi'(r)\big]$$
$$= (\beta - 2N) \lim_{r\to\infty} r\varphi'(r) \geqslant 0,$$

which is impossible.

(3-2) Assume φ changes sign more than twice. Let z be the second zero of φ. Pick $0 < r_1 < z < r_2$ such that φ satisfies (40) and has a local minimum and a local maximum at $r = r_1, r_2$ respectively. Let z_1 be the first zero of φ. Clearly, $z_1 < r_1$. Let $w = ru'$. Since $\varphi'(z_1) < 0$, from (38),

$$0 > \int_0^{z_1} r P_0 \varphi \, dr = -z_1 w(z_1)\varphi'(z_1),$$

implying that

$$u'(z_1) < 0. \tag{42}$$

Hence $u'(r) < 0$ for all $r \geqslant z_1$ by the property of $\mathfrak{S}^-$ solutions. Furthermore, from (35),

$$f\big(e^{u(r_1)}\big)q(r_1)\varphi(r_1) = \varphi''(r_1) \geqslant 0,$$

which reveals that $e^{u(r_1)} \leqslant T_1$. The numbers $\sigma_1 := e^{u(r_1)}$, $\zeta := e^{u(z)}$ and $\sigma_2 := e^{u(r_2)}$ thus satisfy

$$0 < \sigma_2 < \zeta < \sigma_1 \leqslant T_1.$$

From the c–t diagram shown in Fig. 1, there exists a (unique) constant c ($> m$) such that $p_c(\zeta) = 0$, $p_c(t) > 0$ for $t \in (\sigma_2, \zeta)$ and $p_c(t) < 0$ for $t \in (\zeta, \sigma_1)$. Hence

$$\begin{cases} P_c(r) < 0 & \text{for } r \in (r_1, z), \\ P_c(r) > 0 & \text{for } r \in (z, r_2). \end{cases} \tag{43}$$

By (40) and (43) with such choice of r_1, r_2, c, the LHS of (38) is positive while the RHS is negative; it is impossible. The proof is completed. □

Lemma 3.2. $\lim_{r\to\infty} r\varphi'(r;s) \neq 0$ *for any* $s \in \mathbb{R}$.

Proof. Let $\varphi(r) = \varphi(r;s)$ be the solution of (35) with $u(r) = u(r;s)$ dependent on $s \in \mathbb{R}$. The conclusion for $s \in J^*$ follows readily from (35), Lemma 3.1 and the fact that $e^{u(r)} \to 1$ as $r \to \infty$. In fact, pick a sufficiently large $r_0 > 0$, we have

$$\begin{aligned} r\varphi'(r) &= r_0\varphi(r_0) - \int_{r_0}^{r} q(\tau)\varphi(\tau)\big(e^{u(\tau)} - T_1\big)\big(e^{u(\tau)} - T_2\big)\tau\, d\tau \\ &\geqslant r_0\varphi(r_0) := C \end{aligned} \tag{44}$$

for $r > r_0$, here $T_1 < 1 < T_2$. We complete the proof for other cases by contradiction. Assume

$$\lim_{r\to\infty} r\varphi'(r) = 0.$$

From the assumption, it is necessary that φ is bounded. Recall that for any $\mathfrak{S}^-$ or $\mathfrak{S}^+$ solution u, the function w_c given in (36) is also a bounded function; it holds that $rw_c'(r) \to 0$ as $r \to 0$. Now we consider the cases $s \in J^-$ and $s \in J^+$ in the following.

Case 1. Let $s \in J^-$. Note that $w_c' < 0$ for $\mathfrak{S}^-$ solution. According to Lemma 3.1, there are two possible situations as follows.

(1-1) If φ changes sign twice, the proof is the same as the subcase (3-2) of Lemma 3.1, just with the modification $r_2 = \infty$. The details are omitted here.

(1-2) Assume φ changes sign only once. Let z_1 be the zero of φ. Let c_1 be the number such that $w_{c_1}(z_1) = 0$. Observe that for $c = m$, p_m has zeros at 0 and at some number $\sigma > 1$; especially, p_m is negative in $(0, 1)$; see Fig. 1 for illustration. So by (38) we have

$$0 > \int_0^{z_1} rP_m\varphi\, dr = -z_1 w_m(z_1)\varphi'(z_1),$$

which implies that $w_m(z_1) < 0$ and hence $c_1 > m$. As a result, p_{c_1} has two positive zeros θ_1, θ_2 with $\theta_1 < 1 < \theta_2$. In particular,

$$\begin{cases} p_{c_1} > 0 & \text{in } (0, \theta_1), \\ p_{c_1} < 0 & \text{in } (\theta_1, 1). \end{cases} \tag{45}$$

Now we claim that

$$P_{c_1}(z_1) < 0. \tag{46}$$

Indeed, if $P_{c_1}(z_1) \geqslant 0$ or equivalently $p_{c_1}(e^{u(z_1)}) \geqslant 0$, then $e^{u(z_1)} \leqslant \theta_1$ and thus by (42), $e^{u(r)} < \theta_1$ for all $r > z_1$. Hence $P_{c_1}(r)$ is positive for all $r > z_1$. So we have

$$0 < \int_{z_1}^{\infty} r P_{c_1} \varphi \, dr = \lim_{r \to \infty} r\big[\varphi(r) w_{c_1}'(r) - w_{c_1}(r)\varphi'(r)\big] = 0,$$

which is a contradiction. On the other hand, since $e^{u(r)} \to 0$ as $r \to 0, \infty$, we have $P_{c_1} > 0$ near the origin and infinity. This together with (46) indicates that there are r_1, r_2 with $r_1 < r_2$ such that $z_1 \in (r_1, r_2)$ and

$$\begin{cases} P_{c_1}(r) < 0 & \text{for } r \in (r_1, r_2), \\ P_{c_1}(r) > 0 & \text{for } r \in (0, r_1) \cup (r_2, \infty). \end{cases} \tag{47}$$

Let $w = w_{c=0} = ru'$. Note that

$$r\big[\varphi(r) w'(r) - w(r)\varphi'(r)\big] = \int_0^r \tau P_0(\tau)\varphi(\tau)\, d\tau := \Theta(r). \tag{48}$$

Clearly, $\Theta'(r) < 0$ in $(0, z_1)$ and > 0 in (z_1, ∞), and in addition, $\lim_{r\to\infty} \Theta(r) = 0$. So that Θ is negative all the time. Hence for any $r \in \mathbb{R} - \{z_1\}$,

$$\left(\frac{w}{\varphi}\right)'(r) < 0. \tag{49}$$

Set

$$\frac{w(r_1)}{\varphi(r_1)} = K_1, \qquad \frac{w(r_2)}{\varphi(r_2)} = K_2.$$

From (49) we conclude that

$$\begin{cases} K_1\varphi(r) < w(r) & \text{for } r \in (0, r_1), \\ K_1\varphi(r) > w(r) & \text{for } r \in (r_1, z_1); \end{cases} \qquad \begin{cases} K_2\varphi(r) > w(r) & \text{for } r \in (z_1, r_2), \\ K_2\varphi(r) < w(r) & \text{for } r \in (r_2, \infty). \end{cases} \tag{50}$$

Recall that $z_1 \in (r_1, r_2)$. So that from (47), letting $\chi(r) = r^{2aN} q(r)$,

$$\begin{aligned} &\big(r^{2-2aN} - r_1^{2-2aN}\big)\chi(r) p_{c_1}\big(e^{u(r)}\big) < 0 \quad \text{for } r \in (0, z_1), \\ &\big(r^{2-2aN} - r_2^{2-2aN}\big)\chi(r) p_{c_1}\big(e^{u(r)}\big) > 0 \quad \text{for } r \in (z_1, \infty). \end{aligned} \tag{51}$$

Since $w(z_1) < 0$ via (42), we have $w(r_2) < 0$ and thus $K_2 > 0$. On the other hand, letting $b > 0$ be the site at which u attains its maximum, from (45)–(46), we have $P_{c_1}(b) < 0$, because $e^{u(b)} > e^{u(z_1)} > \theta_1$. Hence $b \in (r_1, z_1)$. Note that u' is positive in $(0, b)$ and negative in (b, ∞). In particular, $w(b) = 0$. So that $K_1 > 0$. Recall that $\varphi(z_1) = w_{c_1}(z_1) = 0$ by definition. So that, applying (38),

$$\left(\int_0^b + \int_b^{z_1}\right) r P_{c_1} \varphi \, dr = \int_0^{z_1} r P_{c_1} \varphi \, dr = 0,$$

where we note that $P_{c_1}\varphi < 0$ in the interval (b, z_1). Combining (47), (50) and (51),

$$0 < K_1 \int_0^b r P_{c_1}\varphi\, dr < \int_0^b r^2 P_{c_1} u'\, dr < r_1^{2-2aN} \int_0^b \chi p_{c_1}\big(e^u\big) u'\, dr,$$

$$0 = K_2 \int_{z_1}^\infty r P_{c_1}\varphi\, dr < \int_{z_1}^\infty r^2 P_{c_1} u'\, dr < r_2^{2-2aN} \int_{z_1}^\infty \chi p_{c_1}\big(e^u\big) u'\, dr.$$

Since $P_{c_1} u' > 0$ in (b, z_1), we improve the first formula above to be

$$\int_0^{z_1} \chi p_{c_1}\big(e^u\big) u'\, dr = \left(\int_0^b + \int_b^{z_1}\right) \chi p_{c_1}\big(e^u\big) u'\, dr > 0.$$

Therefore, since $u \to -\infty$ as $r \to 0, \infty$,

$$0 < \int_0^\infty \chi p_{c_1}\big(e^u\big) u'\, dr = 2g_0 \int_0^\infty \frac{e^{au}}{(e^u + 1)^{2a+2}} p_{c_1}\big(e^u\big) u'\, dr$$

$$= 2g_0 \int_{e^{u(0)}}^{e^{u(\infty)}} \frac{\tau^{a-1}}{(\tau + 1)^{2a+2}} p_{c_1}(\tau)\, d\tau = 0,$$

which is a contradiction. So we complete the proof for the case $u \in \mathfrak{S}^-$.

Case 2. Assume $s \in J^+$. In this case, from (33), we have $w_c' < 0$ in $(0, \eta)$ and > 0 in (η, ∞), here η denotes the point at which $u(\eta) = 0$. Especially, w_c is positive for any $c \geqslant 0$. According to Lemma 3.1, there are three possible situations to be considered, namely, φ changing sign either once, twice or φ being positive all the time.

(2-1) If φ changes sign twice, the proof is the same as *Case* 2 of Lemma 3.1 by letting $r_2 = \infty$ therein. So we omit the details.

(2-2) Assume φ changes sign only once. Let z_1 be the zero of φ. Letting $w = w_{c=0}$ and testing (38) with $c = 0$, we have

$$0 > \int_0^\eta r P_0 \varphi\, dr = -\eta w(\eta) \varphi'(\eta), \tag{52}$$

and thus $\varphi'(\eta) > 0$. So that we may extract $r_0 \in (\eta, z_1)$ such that φ has a local maximum at r_0. Let $\sigma_0 = e^{u(r_0)}$. From (35),

$$f(\sigma_0) q(r_0) \varphi(r_0) = \varphi''(r_0) \leqslant 0,$$

implying that $\sigma_0 \geqslant T_2$. Note that $\sigma_1 := e^{u(z_1)} > \sigma_0 > T_2$, because $u(r)$ increases as r increases. So we can pick the constant c such that $p_c(\sigma_1) = 0$, $p_c > 0$ in (σ_0, σ_1) and $p_c < 0$ in (σ_1, ∞). Hence

$$\begin{cases} P_c(r) > 0 & \text{for } r \in (r_0, z_1), \\ P_c(r) < 0 & \text{for } r \in (z_1, \infty). \end{cases}$$

Testing (38) again,

$$0 < \int_{r_0}^{\infty} r P_c \varphi \, dr = -r_0 \varphi(r_0) w_c'(r_0) < 0,$$

a contradiction.

(2-3) Assume φ does not change sign, namely, $\varphi(r) > 0$ for all $r > 0$. Recall that η is the point at which $u(\eta) = 0$. Let $w = w_{c=0}$. According to (48), since $\Theta'(r)$ is positive in $(0, 1)$ and negative in $(1, \infty)$ such that $\lim_{r\to\infty} \Theta(r) = 0$, it follows that $\Theta(r) < 0$ for all $r > 0$ and thus (49) is also valid here. Let $w(\eta)/\varphi(\eta) = K$. Then,

$$K\varphi(r) > w(r) \quad \text{for } r \in (\eta, \infty).$$

Note that $(r^{2-2aN} - \eta^{2-2aN})\chi(r) p_0(e^{u(r)}) > 0$ for $r \in (0, \infty)$ where $\chi(r) = r^{2aN} q(r)$. Thus,

$$0 = K \int_{\eta}^{\infty} r P_0 \varphi \, dr > \int_{\eta}^{\infty} r^2 P_0 u' \, dr > \eta^{2-2aN} \int_{\eta}^{\infty} \chi p_0\bigl(e^u\bigr) u' \, dr.$$

Therefore,

$$\begin{aligned} 0 > \int_{\eta}^{\infty} \chi p_0\bigl(e^u\bigr) u' \, dr &= 2g_0 \int_{\eta}^{\infty} \frac{e^{au}}{(e^u + 1)^{2a+2}} p_0\bigl(e^u\bigr) u' \, dr \\ &= 2g_0 \int_{1}^{\infty} \frac{\tau^{a-1}}{(\tau + 1)^{2a+2}} p_0(\tau) \, d\tau > 0, \end{aligned}$$

a contradiction. The lemma is concluded. □

Proof of Theorem 3.1. In view of (35), it follows from Theorem 2.1 and Lemma 3.2, that the derivative of $\beta(s)$ given by

$$\beta'(s) = -\lim_{r\to\infty} r\varphi'(r; s),$$

never vanishes for either $s \in J^-$ or $s \in J^+$; furthermore, both J^- and J^+ are unbounded intervals. As a result, J^* consists of a single point, say, s^*. To conclude the theorem, we show that $J^- \cap (s^*, \infty)$ is empty. In fact, if J^- contains a point $s > s^*$, then $(s^*, \infty) \subset J^-$ by the monotonicity of $\beta(s)$. Especially, $\beta'(s) < 0$ and thus

$$\varphi(r; s) \text{ changes sign exactly twice for } s \in \left(s^*, \infty\right), \tag{53}$$

according to Lemma 3.1. By (44) and the continuous dependence of $\varphi(r; s)$ on s, we select $r_0 > 0$ sufficiently large and pick $s_0 > s^*$, sufficiently close to s^*, such that

$$\varphi(r; s_0) > 0 \quad \text{for } r \in (0, r_0), \qquad \varphi'(r_0; s_0) > 0.$$

Let $\varphi(r) = \varphi(r; s_0)$. From (53), there exist $r_1, z_1, r_2 > 0$ satisfying $r_0 < r_1 < z_1 < r_2$, such that $\varphi(z_1) = 0$ and

$$\begin{cases} \varphi(r) > 0 & \text{for } r \in (r_1, z_1), \\ \varphi(r) < 0 & \text{for } r \in (z_1, r_2); \end{cases} \tag{54}$$

in addition, φ has local maximum and minimum at r_1, r_2 respectively. From (35) and (42),

$$e^{u(r_2)} < e^{u(z_1)} < e^{u(r_1)} \leqslant T_1.$$

Hence we can choose a constant c such that

$$\begin{cases} P_c(r) < 0 & \text{for } r \in (r_1, z_1), \\ P_c(r) > 0 & \text{for } r \in (z_1, r_2). \end{cases} \tag{55}$$

By (54) and (55) with such choice of r_1, r_2, c, the LHS of (38) is negative while the RHS is positive, yielding a contradiction. Therefore, J^- does not contain any point in (s^*, ∞). So we identify that $J^- = (-\infty, s^*)$ and $J^+ = (s^*, \infty)$. Applying Theorem 2.1 thus concludes the theorem. □

4. Concluding the main theorem

Let us make up the final episode. First of all, we are going to apply a shooting technique developed in [3–5] to detect the limit of $\beta(s)$ as $s \to \pm\infty$. Consider the problem

$$\begin{cases} u''(r) + \dfrac{1}{r}u'(r) + K(r)e^{au(r)} = 0, & r \in (0, 1], \\ u(r) = \lambda \log r + s + o(1), \qquad u'(r) = \dfrac{\lambda}{r} + o(1) & \text{as } r \to 0, \end{cases} \tag{56}$$

where λ and s are constants. Through a straightforward adaptation of Lemma 2.2 in [4], we have the following lemma.

Lemma 4.1. *Suppose $K(r) = K_0 r^p + O(r^{p+k})$ near $r = 0$ where $K_0 > 0$, $k > 0$ and p are constants. Assume $a\lambda > -p - 2$. Then for each real constant s, (56) possesses a unique solution $u = u(r; s, \lambda)$ which is a continuous function of s and λ. Furthermore, $u(r; s, \lambda)$ satisfies*

$$\begin{cases} u(1; s, \lambda) = s + O\left(e^{as}\right), \\ u'(1; s, \lambda) = \lambda + O\left(e^{as}\right) \end{cases} \tag{57}$$

as $s \to -\infty$;

$$\begin{cases} u(1; s, \lambda) = -s + C + O\left(e^{-\mu as}\right), \\ u'(1; s, \lambda) = -\left(\lambda + \frac{4}{a} + \frac{2p}{a}\right) + O\left(e^{-\mu as}\right) \end{cases} \tag{58}$$

as $s \to \infty$. *Here* C *is a constant independent of* s *and* μ *is a constant satisfying* $0 < \mu \leqslant 1$.

In order to bring the problem (21) into the framework of Lemma 4.1 and describe the behavior of the solution u at the site $r = 1$ in response to a large s, we decompose the solution $u(r; s)$ of (21) into two components:

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) + K_1(r)e^{au(r)} = 0, \quad r \in (0, 1], \\ u(r) = 2N \log r + s + o(1) \quad \text{as } r \to 0, \end{cases} \tag{59}$$

$$\begin{cases} u''(r) + \frac{1}{r}u'(r) + K_1(r)e^{au(r)} = 0, \quad r \in [1, \infty), \\ u(r) = (2N - \beta) \log r + \sigma + o(1) \quad \text{as } r \to \infty, \end{cases} \tag{60}$$

where the last equation blends with (23) for a suitable constant $\sigma = \sigma(s)$ and

$$\begin{aligned} K_1(r) &= 2g_0 r^{-2aN} \frac{1 - e^{u(r)}}{(1 + e^{u(r)})^{2a+1}} \\ &= 2g_0 r^{-2aN} + O\left(r^{-2aN+2N}\right), \quad r \to 0. \end{aligned}$$

With the substitution $\lambda = 2N$ and $p = -2aN$, from Lemma 4.1 the solution $u = u(r; s)$ of (59) satisfies

$$\begin{cases} u(1; s) = s + O\left(e^{as}\right) \\ u'(1; s) = 2N + O\left(e^{as}\right) \end{cases} \quad \text{as } s \to -\infty; \tag{61}$$

$$\begin{cases} u(1; s) = -s + C + O\left(e^{-\mu as}\right) \\ u'(1; s) = -2N - \frac{4}{a} + 4N + O\left(e^{-\mu as}\right) \end{cases} \quad \text{as } s \to \infty. \tag{62}$$

On the other hand, we set

$$v(r) = u\left(\frac{1}{r}\right) - \frac{4}{a} \log r,$$

and Eq. (60) is transformed into the following problem for $v(r) = v(r; \sigma, \beta)$:

$$\begin{cases} v''(r) + \frac{1}{r}v'(r) + K_2(r)e^{av(r)} = 0, & r \in (0, 1], \\ v(r) = \left(\beta - 2N - \frac{4}{a}\right)\log r + \sigma + o(1) & \text{as } r \to 0, \end{cases} \tag{63}$$

in which

$$\begin{aligned} K_2(r) = K_1\left(\frac{1}{r}\right) &= 2g_0 r^{2aN} \frac{1 - e^{u(1/r)}}{(1 + e^{u(1/r)})^{2a+1}} \\ &= 2g_0 r^{2aN} + O\left(r^{2aN+\beta-2N}\right), \quad r \to 0. \end{aligned}$$

Clearly, at the conjunction site $r = 1$ we have

$$v(1; \sigma, \beta) = u(1; s) \quad \text{and} \quad v'(1; \sigma, \beta) = -u'(1; s) - \frac{4}{a}. \tag{64}$$

Here β, σ are dependent on s. Theorem 3.1 can be reinforced by the following theorem.

Theorem 4.1. *Let $aN < 1$. Then*

$$\lim_{s \to -\infty} \beta(s) = \frac{4}{a}, \qquad \lim_{s \to \infty} \beta(s) = \frac{8}{a}(aN - 1). \tag{65}$$

Proof. By Theorem 3.1, $J^- = (-\infty, s^*)$, $J^* = \{s^*\}$ and $J^+ = (s^*, \infty)$ for some $s^* \in \mathbb{R}$. Suppose s is sufficiently closed to $-\infty$. Applying Lemma 4.1 to problem (63), $v(r; \sigma)$ satisfies

$$\begin{cases} v(1; \sigma) = \sigma + O\left(e^{a\sigma}\right) \\ v'(1; \sigma) = \beta - 2N - \frac{4}{a} + O\left(e^{a\sigma}\right) \end{cases} \quad \text{as } \sigma \to -\infty; \tag{66}$$

$$\begin{cases} v(1; \sigma) = -\sigma + C + O\left(e^{-\mu a\sigma}\right) \\ v'(1; \sigma) = -2N - \beta + O\left(e^{-\mu a\sigma}\right) \end{cases} \quad \text{as } \sigma \to \infty. \tag{67}$$

So there are two possible situations as follows:

(i) Assume $\lim_{s \to -\infty} \sigma(s) = -\infty$. Then, by (61), (64) and (66), $\lim_{s \to -\infty} \beta(s) = 0$.
(ii) Assume $\lim_{s \to -\infty} \sigma(s) = \infty$. Then, by (61), (64) and (67), $\lim_{s \to -\infty} \beta(s) = \frac{4}{a}$.

Case (i) cannot happen because of (24). Therefore,

$$\lim_{s \to -\infty} \beta(s) = \frac{4}{a}.$$

To conclude the theorem, we consider $s > 0$ sufficiently large. Rewrite the problem (63) by letting $\psi = -v$; in fact, ψ satisfies

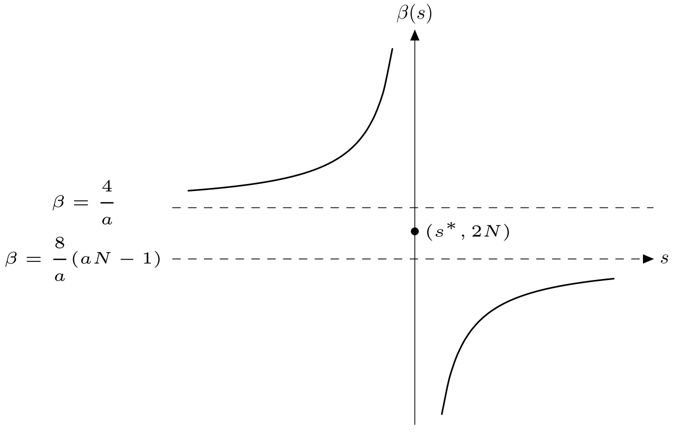

Fig. 2. $\beta(s)$ is increasing in $\mathbb{R} - \{s^*\}$.

$$\begin{cases} \psi''(r) + \dfrac{1}{r}\psi'(r) + K_3(r)e^{a\psi(r)} = 0, & r \in (0,1], \\ \psi(r) = \left(2N + \dfrac{4}{a} - \beta\right)\log r + \xi + o(1) & \text{as } r \to 0, \end{cases} \tag{68}$$

where

$$\begin{aligned} K_3(r) &= -r^{-8}e^{2au(1/r)}K_2(r) = 2g_0 r^{2aN-8}e^{2au(1/r)}\frac{e^{u(1/r)}-1}{(e^{u(1/r)}+1)^{2a+1}} \\ &= 2g_0 r^{2aN-8} + O\left(r^{2aN-8+2N-\beta}\right), \quad r \to 0. \end{aligned}$$

Note that $2N - \beta(s) > 2(1-aN)/a > 0$ whenever $s > s^*$ by (25). By Lemma 4.1 again, $\psi(r;\xi)$ satisfies

$$\begin{cases} \psi(1;\xi) = \xi + O\left(e^{a\xi}\right) \\ \psi'(1;\xi) = 2N + \dfrac{4}{a} - \beta + O\left(e^{a\xi}\right) \end{cases} \quad \text{as } \xi \to -\infty; \tag{69}$$

$$\begin{cases} \psi(1;\xi) = -\xi + C + O\left(e^{-\mu a\xi}\right) \\ \psi'(1;\xi) = \beta - 6N + \dfrac{8}{a} + O\left(e^{-\mu a\xi}\right) \end{cases} \quad \text{as } \xi \to \infty. \tag{70}$$

Here $\xi = \xi(s)$. By (62), (64) and (69)–(70), it follows that $\lim_{s\to\infty}\xi(s) = \pm\infty$; furthermore, one of the following assertions is valid:

(i′) If $\lim_{s\to\infty}\xi(s) = -\infty$, then $\lim_{s\to\infty}\beta(s) = \frac{4}{a} > 0$.
(ii′) If $\lim_{s\to\infty}\xi(s) = \infty$, then $\lim_{s\to\infty}\beta(s) = \frac{8}{a}(aN-1) < 0$.

Obviously, case (i′) is not true because it contradicts the fact that $\beta < 0$ in Theorem 2.1. Therefore, $\beta(s) \to 8(aN-1)/a$ as $s \to \infty$. $\square$

Concluding Theorem 1.1. The first conclusion of Theorem 1.1 for $aN < 1$ follows from Theorems 2.1, 3.1 and 4.1. The behavior of $\beta(s)$ is illustrated in Fig. 2. Moreover, Theorems 2.2 and 2.3 account for the conclusions for the cases $aN = 1$ and $aN \geqslant 2$ respectively. □

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