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Calculus of Variations

On the solutions to a Liouville-type system involving singularity

Zhi-You Chen · Jann-Long Chern · Yong-Li Tang

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Abstract In this paper, we consider a Liouville-type system with singularity in the plane. The existence and uniqueness of solutions to the Dirichlet problem are proved. In addition, the structure of solutions in terms of analogues of the so-called total curvature in geometry or total mass in physics will be offered as well.

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1 Introduction

This paper is concerned with the nonlinear elliptic system

$$\begin{cases} \Delta u + K_1(|x|)e^v = 4\pi m_1 \delta_0, \\ \Delta v + K_2(|x|)e^u = 4\pi m_2 \delta_0, \end{cases} \quad \text{in } \mathbf{R}^2, \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} \partial^2/\partial x_i^2$ is the Laplacian operator in $\mathbf{R}^2$, $m_1, m_2 > 0$, δ_0 is the Dirac measure at the origin, and $K_1(r)$, $K_2(r)$ are positive functions for $r > 0$ satisfying

$$r^{2m_i} K_i \in L^1([0,1)) \cap C^1[(0,\infty)], \quad i = 1, 2, \tag{1.2}$$

$$\lim_{r\to 0} r^{-\beta_i} K_i(r), \ \lim_{r\to\infty} r^{-\gamma_i} K_i(r) \text{ are finite and positive}, \quad i = 1, 2 \tag{1.3}$$

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Z.-Y. Chen (✉)
Department of Mathematics, National Tsing Hua University, Hsin-Chu 30013, Taiwan
e-mail: zhiyou@math.ncu.edu.tw

J.-L. Chern · Y.-L. Tang
Department of Mathematics, National Central University, Chung-Li 32001, Taiwan

J.-L. Chern
e-mail: chern@math.ncu.edu.tw

Y.-L. Tang
e-mail: tangyl@math.ncu.edu.tw

for some $\beta_i, \gamma_i \in \mathbf{R}(i = 1, 2)$ with

$$\beta_1, \gamma_1 > -(2 + 2m_2), \quad \beta_2, \gamma_2 > -(2 + 2m_1). \tag{1.4}$$

We call the system in the form of (1.1) the Liouville-type system, which is a natural extension of the so-called Liouville equation

$$\Delta u + Ke^u = 0 \quad \text{in } \Omega \subset \mathbf{R}^2. \tag{1.5}$$

It is well-known that the Liouville equation is related to many applications in a variety of fields in mathematics and physics. In the aspect of the differential geometry, the Liouville equation stands for the problem of finding a metric whose Gaussian curvature is prescribed [5]. In physics, just to name but a few, it represents the electric potential induced by charge carriers in the theory of electrolytes [24], and the Newtonian potential of a cluster of self-gravitating mass distribution [1,4,25,26]. Moreover, it is also induced by a mean field equation which comes from the spherical Onsager vortex theory, bridging the gap between statistical mechanics of classical vortices and the random surface problem [7,21], and is considered to deal with topics closely related to the abelian model in the Chern–Simons theories [9,10,15,27].

As an extension of the single case, Liouville-type systems have also been used to describe models in the physics of charged particle beams [3,14,20], in the theory of semi-conductors [23], in the theory of chemotaxis [11,19], and other issues in fields of physics, chemistry and ecology. We also remark that another significant extension of the Liouville equation is the Toda system which is closely concerned with the non-abelian Chern–Simons theory. For more details of applications of Liouville-type systems, see for example [2,6,12,13,16–18,22] and references therein.

Throughout this article, we consider the radial case of (1.1), i.e., the following ODE system:

$$\begin{cases} u'' + \dfrac{1}{r}u' + K_1(r)e^v = 0, \\ v'' + \dfrac{1}{r}v' + K_2(r)e^u = 0, \end{cases} \qquad r > 0 \tag{1.6}$$

with $(u(r), v(r))$ to be the specific form:

$$\begin{cases} u(r) = 2m_1 \log r + \alpha_1 + o(1), \\ v(r) = 2m_2 \log r + \alpha_2 + o(1), \end{cases} \qquad \text{as } r \to 0^+, \tag{1.7}$$

where $\alpha_1, \alpha_2 \in \mathbf{R}$. Conventionally, we denote the solution of (1.6)–(1.7) by $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ or simply $(u(r), v(r))$ if there is no confusion. Here we call (α_1, α_2) in (1.7) the *normalized initial data* of solution $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ for (1.6)–(1.7). In fact, if we set

$$U(r) = u(r; \alpha_1, \alpha_2) - 2m_1 \log r, \quad V(r) = v(r; \alpha_1, \alpha_2) - 2m_2 \log r, \tag{1.8}$$

then $(U(r), V(r))$ satisfies

$$\begin{cases} U''(r) + \dfrac{1}{r}U'(r) + r^{2m_2}K_1(r)e^V = 0, \quad r > 0, \\ V''(r) + \dfrac{1}{r}V'(r) + r^{2m_1}K_2(r)e^U = 0, \quad r > 0, \\ U(0) = \alpha_1, \ V(0) = \alpha_2, \\ U'(0) = 0, \ V'(0) = 0. \end{cases} \tag{1.9}$$

Note that every solution $(u(r), v(r))$ of (1.6) is defined for all $r \in (0, \infty)$. Indeed, from (1.9), if $(U(r), V(r))$ is defined on $[0, R)$ and $\lim_{r\to R} U(r) = -\infty$ for some finite $R > 0$, then $\lim_{r\to R} U'(r) = -\infty$ also. Hence,

$$-\infty = \lim_{r\to R} rU'(r) = \lim_{r\to R}\left(-\int_0^r s^{2m_2+1}K_1 e^V\,ds\right) = -\int_0^R s^{2m_2+1}K_1 e^V\,ds$$

which is impossible because $V(r)$ is decreasing in $[0, R)$. For the structure of solutions for the single equation related to (1.6), see for example [7].

In [12], the following system was considered:

$$\begin{cases} \Delta u + V_1(|x|)e^{au+bv} = 0, \\ \Delta v + V_2(|x|)e^{bu+cv} = 0, \end{cases} \quad \text{in } \Omega, \tag{1.10}$$

where Ω is either $B_R(\mathbf{0})$ or $\mathbf{R}^2$, V_1, V_2 are positive functions on Ω, and a, b, c are constants. If $\Omega = B_R(\mathbf{0})$, $0 < k_1 \le V_1, V_2 \le k_2 < \infty$ on $B_R(\mathbf{0})$ and $a, c \ge 0$, then for any $0 < M_1, M_2 < \infty$ satisfying certain assumptions (Theorem 1.1 in [12]), there exists a radially symmetric solution $(u(r), v(r))$ of (1.10) so that $u(R) = v(R) = 0$ and

$$\int_\Omega V_1 e^{au+bv}dx = M_1, \quad \int_\Omega V_2 e^{bu+cv}dx = M_2. \tag{1.11}$$

For $\Omega = \mathbf{R}^2$, $V_1 \equiv V_2 \equiv 1$ and $a, b, c \ge 0$, some sufficient and necessary conditions for existence of an entire radial solution of (1.10) associated with prescribed *finite* "*flux*" (M_1, M_2) (i.e., a solution (u, v) satisfying (1.11)) were established. In this paper, we consider the case of $a = c = 0$ in (1.10) with singularity at the origin. The existence and uniqueness of solutions to the Dirichlet problem will be proved. Furthermore, unbounded flux is possible and really exists, and all solutions can be classified completely in terms of corresponding fluxes.

It is worth mentioning that in the Chern–Simons systems [10], the "*flux*" associated with certain solutions can be unbounded, which is quite different from the situation in single equations. Here, we also investigate such phenomena that occur in the Liouville equation and system.

Now we state the main result on the existence and uniqueness of solutions to the Dirichlet problem of (1.6)–(1.7).

Theorem 1.1 *Suppose that*

$$\frac{rK_i'(r)}{K_i(r)} + 2 \ge 0, \quad r > 0, \ i = 1, 2. \tag{1.12}$$

Then for any $R > 0$, (1.6)–(1.7) possesses one and only one solution $(u(r), v(r))$ satisfying

$$\begin{cases} u(r) < 0, \ v(r) < 0, \quad 0 < r < R, \\ u(R) = v(R) = 0. \end{cases} \tag{1.13}$$

Remark 1.1 In fact, for the existence result, the condition (1.12) can be removed from Theorem 1.1.

Let $(u(r), v(r))$ be a solution of (1.6)–(1.7). We define

$$\Theta_1(u, v) = \int_0^\infty rK_1(r)e^{v(r)}dr, \quad \Theta_2(u, v) = \int_0^\infty rK_2(r)e^{u(r)}dr. \tag{1.14}$$

Clearly, $0 < \Theta_i(u, v) \le \infty$. We sometimes denote it by (Θ_1, Θ_2) if no confusion arises. In the case of $u \equiv v$ in (1.6), $\Theta = \Theta_1(u, v) = \Theta_2(u, v)$ is called, for example, the total curvature coming from the prescribed Gaussian curvature equation or the flux in physics. Here we call $\Theta_1(u, v)$ and $\Theta_2(u, v)$ the *K_1-mass* and *K_2-mass* with respect to solution (u, v) respectively. From standard arguments, $\Theta_1(u, v)$ and $\Theta_2(u, v)$ cannot be infinite simultaneously for any solution $(u(r), v(r))$ of (1.6)–(1.7).

For convenience, we classify a solution $(u(r), v(r))$ of (1.6)–(1.7) in terms of its corresponding K_i-masses pair (Θ_1, Θ_2) as follows:

Type (I): $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$ with $\Theta_1 < \infty$ and $\Theta_2 = \infty$.
Type (II): $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$ with $\Theta_1 = \infty$ and $\Theta_2 < \infty$.
Type (III): $\lim_{r\to\infty}(u(r), v(r)) = (-\infty, -\infty)$ with $\Theta_1 < \infty$ and $\Theta_2 < \infty$.

Our second result shows that the above classification exhausts all possible situations of solutions for (1.6)–(1.7).

Theorem 1.2 *Let $(u(r), v(r))$ be a solution of (1.6)–(1.7). Then $(u(r), v(r))$ must be one of the above three types. Conversely, solutions of all types do exist.*

In the following, we conclude that the corresponding K_i-masses with respect to solutions of types stated above range exactly over certain intervals related to m_i and γ_i.

Theorem 1.3 *Consider (1.6)–(1.7). Then*

(a) *For any $\theta_1 \in (0, 2+2m_1+\gamma_2]$, there exist infinitely many solutions $(u(r), v(r))$ of Type (I) such that*

$$(\Theta_1(u, v), \Theta_2(u, v)) = (\theta_1, \infty).$$

Furthermore, if $(u(r), v(r))$ is a solution of Type (I), then $\Theta_1(u, v) \le 2+2m_1+\gamma_2$ and $\Theta_2(u, v) = \infty$.

(b) *For any $\theta_2 \in (0, 2+2m_2+\gamma_1]$, there exist infinitely many solutions $(u(r), v(r))$ of Type (II) such that*

$$(\Theta_1(u, v), \Theta_2(u, v)) = (\infty, \theta_2).$$

Furthermore, if $(u(r), v(r))$ is a solution of Type (II), then $\Theta_1(u, v) = \infty$ and $\Theta_2(u, v) \le 2+2m_2+\gamma_1$.

For the special case of $K_1 \equiv K_2 \equiv 1$ in (1.6), i.e.,

$$\begin{cases} u'' + \frac{1}{r}u' + e^v = 0, \\ v'' + \frac{1}{r}v' + e^u = 0, \end{cases} \qquad r > 0, \tag{1.15}$$

we have some further consequences. Before stating our final result, we first introduce the *linearized system* of (1.15)–(1.7) with respect to solution $(u(r), v(r))$:

$$\begin{cases} A'' + \frac{1}{r}A' + r^{2m_2}e^{V(r)}B = 0, \\ B'' + \frac{1}{r}B' + r^{2m_1}e^{U(r)}A = 0, \end{cases} \qquad r > 0 \tag{1.16}$$

where $(U(r), V(r))$ is defined in (1.8). The linearized system (1.16) is *degenerate* if it possesses a nonconstant bounded solution $(A(r), B(r))$ on $(0, \infty)$.

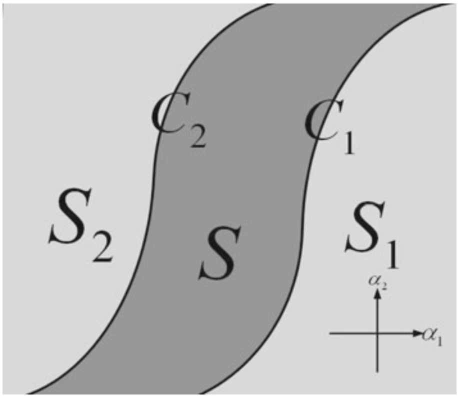

Fig. 1 Structure of solutions of (1.15)–(1.7)

Theorem 1.4 *Consider* (1.15)–(1.7). *Then*

(a) *All conclusions in Theorems 1.2 and 1.3 hold ($\gamma_1 = \gamma_2 = 0$ in this case).*
(b) *For any $\theta_1 > 2 + 2m_1$ and $\theta_2 > 2 + 2m_2$ satisfying*

$$\theta_1\theta_2 - (2m_2 + 2)\theta_1 - (2m_1 + 2)\theta_2 = 0, \tag{1.17}$$

there exist infinitely many solutions $(u(r), v(r))$ of Type (III) such that

$$(\Theta_1(u, v), \Theta_2(u, v)) = (\theta_1, \theta_2).$$

Furthermore, if $(u(r), v(r))$ is a solution of Type (III), then $(\theta_1, \theta_2) = (\Theta_1(u, v), \Theta_2(u, v))$ satisfies (1.17)*, and its corresponding linearized system* (1.16) *is degenerate.*

Example 1.1 In the specific case of $K_1 \equiv K_2 \equiv 1$, solutions of (1.15)–(1.7), to the Dirichlet problem and related to associated K_i-masses, are all clarified by Theorem 1.1 and Theorem 1.4. By adopting the notations introduced in (2.2), Sect. 2, we illustrate the structure of solutions of (1.15)–(1.7) below.

We organize this article as follows. In Sect. 2, some results concerning with K_i-masses will be made. Theorems 1.2 and 1.3 will be proved in Sect. 3. We give a complete verification of Theorem 1.4 in Sect. 4. Finally, Sect. 5 is devoted to the proof of Theorem 1.1.

2 K_i-masses associated with solutions

First of all, we note that for any solution $(u(r), v(r))$ of (1.6)–(1.7) and from (1.14),

$$\Theta_1(u, v) = \int_0^\infty r^{2m_2+1} K_1 e^{V(r)} dr, \quad \Theta_2(u, v) = \int_0^\infty r^{2m_1+1} K_2 e^{U(r)} dr, \tag{2.1}$$

where $(U(r), V(r))$ is defined in (1.8) and satisfies (1.9). Occasionally, we denote $\Theta_i(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ by $\Theta_i(\alpha_1, \alpha_2)$ simply to indicate the dependence of (α_1, α_2).

Our first lemma in this section gives the fact that if one of K_i-masses is finite, then the other one is bounded from below by a positive constant.

Lemma 2.1 *Let* $(u(r), v(r)) = (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ *be a solution of* (1.6)–(1.7). *If* $\Theta_1(u, v) < \infty$*(resp.,* $\Theta_2(u, v) < \infty$*), then* $\Theta_2(u, v) > 2 + 2m_2 + \gamma_1$ *(resp.,* $\Theta_1(u, v) > 2 + 2m_1 + \gamma_2$*).*

Proof Let $\Theta_1(u,v)<\infty$ and $(U(r),V(r))$ be defined as in (1.8). Without loss of generality, we assume $\Theta_2(u,v)<\infty$. Then by (1.9) and (2.1), we have that $rV'(r)>-\Theta_2$ for $r\geq 0$. Hence

$$V(r)>-\Theta_2\log r+c_1,\quad r\geq R$$

and, by (1.3),

$$K_1(r)\geq c_2r^{\gamma_1},\quad r\geq R$$

for some $R>0$ and $c_1,c_2>0$. Therefore, by combining the results above, we obtain that

$$\begin{aligned}\Theta_1&>\int_R^\infty r^{2m_2+1}K_1(r)e^{V(r)}dr\\&\geq c_3\int_R^\infty s^{2m_2+1+\gamma_1-\Theta_2}\,ds\end{aligned}$$

for some $c_3>0$, which implies $\Theta_2>2+2m_2+\gamma_1$ because of the finiteness of Θ_1. We complete this proof. □

From the above lemma, any solution $(u(r;\alpha_1,\alpha_2),v(r;\alpha_1,\alpha_2))$ of (1.6)–(1.7) can be classified into the following regions in terms of normalized initial data (α_1,α_2) depending on its corresponding K_i-masses (Θ_1,Θ_2):

$$\begin{cases}\mathcal{S}=\{(\alpha_1,\alpha_2):\Theta_1(u,v)<\infty,\Theta_2(u,v)<\infty\},\\\mathcal{S}_1=\{(\alpha_1,\alpha_2):0<\Theta_1(u,v)<\kappa_1,\Theta_2(u,v)=\infty\},\\\mathcal{S}_2=\{(\alpha_1,\alpha_2):0<\Theta_2(u,v)<\kappa_2,\Theta_1(u,v)=\infty\},\\\mathcal{C}_1=\{(\alpha_1,\alpha_2):\Theta_1(u,v)=\kappa_1,\Theta_2(u,v)=\infty\},\\\mathcal{C}_2=\{(\alpha_1,\alpha_2):\Theta_2(u,v)=\kappa_2,\Theta_1(u,v)=\infty\},\end{cases}\tag{2.2}$$

where

$$\kappa_1=2+2m_1+\gamma_2,\quad \kappa_2=2+2m_2+\gamma_1.\tag{2.3}$$

It is clear that $\mathbf{R}^2=\mathcal{S}\cup\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{C}_1\cup\mathcal{C}_2$.

Let $(u(r),v(r))$ be a solution of (1.6)–(1.7), and $(U(r),V(r))$ be defined as in (1.8). Then from (1.9), it is easy to see that

$$\begin{cases}(rU')'+r^{2m_2+1}K_1(r)e^V=0,\quad r>0,\\(rV')'+r^{2m_1+1}K_2(r)e^U=0,\quad r>0.\end{cases}$$

Multiplying $rV'(r)$ and $rU'(r)$ on the first and second equation of the above relation respectively, adding them together, and then integrating from 0 to r, we have

$$\int_0^r\left[(sU')(sV')\right]'ds+\int_0^r\left(s^{2m_2+2}K_1e^VV'+s^{2m_1+2}K_2e^UU'\right)ds=0\tag{2.4}$$

and hence obtain the Pohozaev-type identity as follows:

$$\begin{aligned}&r^2U'(r)V'(r)+r^{2m_2+2}K_1(r)e^V+r^{2m_1+2}K_2(r)e^U\\&\quad=\int_0^r\left\{s^{2m_2+1}\left[(2m_2+2)K_1+sK_1'\right]e^V+s^{2m_1+1}\left[(2m_1+2)K_2+sK_2'\right]e^U\right\}ds.\end{aligned}\tag{2.5}$$

For finite K_i-masses, they satisfy a relation as shown in the following.

Lemma 2.2 *If* $(\alpha_1, \alpha_2) \in \mathcal{S}$*, then* $(\Theta_1, \Theta_2) = (\Theta_1(\alpha_1, \alpha_2), \Theta_2(\alpha_1, \alpha_2))$ *satisfies*

$$\Theta_1\Theta_2 - (2m_2+2)\Theta_1 - (2m_1+2)\Theta_2 = \int_0^\infty \left(r^{2m_2+2}K_1'e^V + r^{2m_1+2}K_2'e^U\right) dr.$$

Proof Let $(U(r), V(r))$ be defined in (1.8) associated with the solution $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1, \alpha_2))$. Since $\Theta_1 + \Theta_2$ is finite, then by (2.1), we have that

$$r_k^{2m_2+2}K_1(r_k)e^{V(r_k)} + r_k^{2m_1+2}K_2(r_k)e^{U(r_k)} \to 0 \quad \text{as } k \to \infty$$

for some sequence $\{r_k\}$ with $r_k \to 0$ as $k \to \infty$. Hence by taking $r = r_k$ on both sides of (2.5) and then letting $k \to \infty$, we complete the proof of this lemma. □

Remark 2.1 (i) In the case of $K_1 \equiv K_2 \equiv 1$, Lemma 2.2 implies

$$\Theta_1\Theta_2 - (2m_2+2)\Theta_1 - (2m_1+2)\Theta_2 = 0$$

for any solution $(u(r), v(r))$ of (1.15)–(1.7) with both $\Theta_1(u, v)$ and $\Theta_2(u, v)$ finite, i.e., solution of Type (III).

(ii) For any $(\alpha_2, \alpha_2) \in \mathcal{S}$, we conclude that

$$rU'(r) < -(2+2m_1+\gamma_2), \quad rV'(r) < -(2+2m_2+\gamma_1) \quad \text{for large } r,$$

where $(U(r), V(r))$ is defined in (1.8) associated with $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$. This is due to Lemma 2.1 and the facts that $rU'(r)$ and $rV'(r)$ decrease to $-\Theta_1(\alpha_1,\alpha_2)$ and $-\Theta_2(\alpha_1,\alpha_2)$ respectively as $r \to \infty$ by (1.9).

Actually, $(U(r), V(r))$ behave logarithmically at infinity, related to its corresponding K_i-masses if both are finite.

Lemma 2.3 *Let* $(\alpha_1, \alpha_2) \in \mathcal{S}$ *and* $(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ *be the solution of* (1.6)–(1.7)*. Then*

$$\begin{cases} U(r) = -\Theta_1(\alpha_1,\alpha_2)\log r + O(1) & \text{at } r = \infty, \\ V(r) = -\Theta_2(\alpha_1,\alpha_2)\log r + O(1) & \text{at } r = \infty, \end{cases} \tag{2.6}$$

where $(U(r), V(r))$ *is defined in* (1.8)*.*

Proof From (1.9), we see that

$$rU'(r) = -\int_0^r s^{2m_2+1}K_1e^V ds, \quad rV'(r) = -\int_0^r s^{2m_1+1}K_2e^U ds, \quad r \ge 0.$$

Then

$$\lim_{r\to\infty} rU'(r) = -\Theta_1(\alpha_1,\alpha_2), \quad \lim_{r\to\infty} rV'(r) = -\Theta_2(\alpha_1,\alpha_2).$$

By Remark 2.1(ii), we may choose some $p > 0$ such that $rV'(r) + p < -(2+2m_2+\gamma_1)$ for $r \ge R$ if R is large. Then $V(r) < -(p+2+2m_2+\gamma_1)\log r + c_1$ for $r \ge R$ and some $c_1 > 0$, which implies that

$$e^{V(r)} < c_2 r^{-(p+2+2m_2+\gamma_1)}, \quad r \ge R,$$

where $c_2 = e^{c_1}$.

Let $W(r) = U(r) + \Theta_1(\alpha_1, \alpha_2)\log r$ for $r > 0$. Then by the above result, (1.3) and (2.1), we obtain that for $r \geq R$,

$$\begin{aligned} rW'(r) &= rU'(r) + \Theta_1(\alpha_1, \alpha_2) \\ &= -\int_0^r s^{2m_2+1} K_1 e^{V(s)} ds + \Theta_1(\alpha_1, \alpha_2) \\ &= \int_r^\infty s^{2m_2+1} K_1 e^{V(s)} ds \\ &< c_3 \int_r^\infty s^{2m_2+1} \cdot s^{\gamma_1} \cdot s^{-(p+2+2m_2+\gamma_1)} ds \\ &= c_3 \int_r^\infty s^{-(p+1)} ds \\ &= c_4 r^{-p} \end{aligned}$$

for some $c_3, c_4 > 0$. Therefore, we assure that $W(r)$ is bounded on $[R, \infty)$, and hence the expression of $U(r)$ in (2.6) holds. The situation for $V(r)$ is similar, and then this lemma is proved. □

3 Existence of solutions of Types (I)–(III)

In this section, we prove the existence of solutions of all types stated in Sect. 1. Additionally, the structure of solutions in terms of normalized initial data depending on corresponding K_i-masses will be established as well.

We first show that System (1.6)–(1.7) possesses solutions with one of associated K_i-masses being infinite, i.e., solutions of Types (I) and (II).

Proposition 3.1 *Consider* (1.6)–(1.7) *and let*

$$\lambda_1 = \max\{-(2m_2 + \beta_1), -(2m_1 + \gamma_2)\}, \quad \lambda_2 = \max\{-(2m_1 + \beta_2), -(2m_2 + \gamma_1)\}.$$

Then

(a) *For any $\xi \in [\lambda_1, 2)$, there exist two constants $\bar{\alpha}_1(\xi) > 0$ and $\bar{\alpha}_2(\xi) < 0$ such that*

$$\Theta_1(\alpha_1, \alpha_2) < 2 - \xi, \quad \Theta_2(\alpha_1, \alpha_2) = \infty$$

for all $\alpha_1 \geq \bar{\alpha}_1(\xi)$ and $\alpha_2 \leq \bar{\alpha}_2(\xi)$.

(b) *For any $\zeta \in [\lambda_2, 2)$, there exist two constants $\hat{\alpha}_2(\zeta) > 0$ and $\hat{\alpha}_1(\zeta) < 0$ such that*

$$\Theta_1(\alpha_1, \alpha_2) = \infty, \quad \Theta_2(\alpha_1, \alpha_2) < 2 - \zeta$$

for all $\alpha_1 \leq \hat{\alpha}_1(\zeta)$ and $\alpha_2 \geq \hat{\alpha}_2(\zeta)$.

Proof Let $(u(r), v(r)) = (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ be a solution of (1.6)–(1.7) and $(U(r), V(r)) = (U(r; \alpha_1, \alpha_2), V(r; \alpha_1, \alpha_2))$ be defined as in (1.8) with respect to the solution $(u(r), v(r))$. We note that $\lambda_1, \lambda_2 < 2$ by the assumptions in (1.4).

(a) We split this proof into the following steps.

Step 1. For $\xi \in [\lambda_1, 2)$ and $\eta < -(2m_1 + \beta_2)$, we define

$$\begin{cases} w(r) = w(r;\alpha_1,\alpha_2) = U(r) + \log(1 + r^{2-\xi}), & r \ge 0, \\ z(r) = z(r;\alpha_1,\alpha_2) = V(r) + \log(1 + r^{2-\eta}), & r \ge 0. \end{cases}$$

Then $w(r)$ and $z(r)$ satisfy

$$\begin{cases} (rw')'(r) = \dfrac{r^{1-\xi}}{(1+r^{2-\xi})^2}\left[(2-\xi)^2 - \dfrac{r^{\xi+2m_2}(1+r^{2-\xi})^2}{1+r^{2-\eta}}K_1 e^z\right], & r > 0, \\ (rz')'(r) = \dfrac{r^{1-\eta}}{(1+r^{2-\eta})^2}\left[(2-\eta)^2 - \dfrac{r^{\eta+2m_1}(1+r^{2-\eta})^2}{1+r^{2-\xi}}K_2 e^w\right], & r > 0, \\ w(0) = \alpha_1, z(0) = \alpha_2;\ \lim\limits_{r\to 0} rw'(r) = 0,\ \lim\limits_{r\to 0} rz'(r) = 0. \end{cases} \tag{3.1}$$

Since $2m_1 + \xi + \gamma_2 \ge 0$ and $2 - \eta > 0$, there exist $\alpha_1^* > 0$ and $R_0 > 1$ such that for $r \ge R_0$ and $\alpha_1 \ge \alpha_1^*$, we have

$$\frac{r^{1-\eta}}{(1+r^{2-\eta})^2}\left[(2-\eta)^2 - \frac{r^{\eta+2m_1}(1+r^{2-\eta})^2}{1+r^{2-\xi}}K_2(r)e^{\alpha_1}\right] \le -\frac{k_2 e^{\alpha_1}}{r} \tag{3.2}$$

where $k_2 = \lim_{r\to\infty} r^{-\gamma_2} K_2(r) > 0$. In addition, from the behavior of $K_1(r)$ at infinity and (3.2), we get

$$r^{\xi+2m_2-(k_2 e^{\alpha_1^*}\log R_1)} \cdot \frac{(1+r^{2-\xi})^2}{1+r^{2-\eta}}K_1(r) < \frac{(2-\xi)^2}{2}, \quad r \ge R_1 \tag{3.3}$$

if $R_1 \ge R_0$ is sufficiently large. Moreover, by (3.1) and (3.2), we also get $z(r) \le -(k_2 e^{\alpha_1^*}\log R_2)\log r$ for $\alpha_1 \ge \alpha_1^*$ and $r \ge R_3$ for some large $R_3 \ge R_2 \ge R_1$. Hence

$$e^{z(r)} \le r^{-k_2 e^{\alpha_1^*}\log R_2}, \quad r \ge R_3, \alpha_1 \ge \alpha_1^*. \tag{3.4}$$

By combining (3.1)–(3.4), we obtain

$$\begin{cases} (2-\xi)^2 - \dfrac{r^{\xi+2m_2}(1+r^{2-\xi})^2}{1+r^{2-\eta}}K_1(r)e^{z(r)} > 0, \\ (2-\eta)^2 - \dfrac{r^{\eta+2m_1}(1+r^{2-\eta})^2}{1+r^{2-\xi}}K_2(r)e^{w(r)} < 0, \end{cases} \quad r \ge R_3,\ \alpha_1 \ge \alpha_1^*. \tag{3.5}$$

On the other hand, because of $\xi + 2m_2 + \beta_1 \ge 0$ and $\eta + 2m_1 + \beta_2 \le 0$, we have that (3.5) holds for $r \in (0, R_3]$, $\alpha_1 \ge \bar{\alpha}_1$ and $\alpha_2 \le \bar{\alpha}_2$ if we choose $\bar{\alpha}_1 \ge \alpha_1^*$ sufficiently large and $\bar{\alpha}_2 < 0$ sufficiently small. Therefore, from (3.1) and the above results, we conclude that

$$(rw')'(r) > 0, \quad (rz')'(r) < 0, \quad r > 0,\ \alpha_1 \ge \bar{\alpha}_1,\ \alpha_2 \le \bar{\alpha}_2, \tag{3.6}$$

and hence $w(r;\alpha_1,\alpha_2)$ is positive for all $r \ge 0$, $\alpha_1 \ge \bar{\alpha}_1$ and $\alpha_2 \le \bar{\alpha}_2$.

Step 2. From Step 1, we see that for $\alpha_1 \ge \bar{\alpha}_1$ and $\alpha_2 \le \bar{\alpha}_2$, $U(r) = U(r;\alpha_1,\alpha_2) > -\log(1 + r^{2-\xi})$ and then $e^{U(r)} > 1/(1 + r^{2-\xi})$ for $r \ge 0$. Hence

$$\Theta_2(\alpha_1,\alpha_2) = \int_0^\infty r^{2m_1+1}K_2(r)e^{U(r)}dr \ge C\int_R^\infty r^{2m_1+1}\cdot r^{\gamma_2}\cdot r^{-(2-\xi)}dr$$
$$= C\int_R^\infty r^{2m_1+\gamma_2+\xi-1}dr = \infty$$

for some $C > 0$ and large R since $2m_1 + \gamma_2 + \xi - 1 \ge -1$.

Step 3. By (3.6), we obtain that

$$0 < \lim_{r\to\infty} rw'(r) = \lim_{r\to\infty}\left[rU'(r) + \frac{(2-\xi)r^{2-\xi}}{1+r^{2-\xi}}\right] = -\Theta_1(\alpha_1,\alpha_2) + (2-\xi),$$

and then $\Theta_1(\alpha_1,\alpha_2) < 2-\xi$. Hence (a) is proved.

(b) For $\zeta \in [\lambda_2, 2)$ and $\eta < -(2m_2+\beta_1)$, we define

$$\begin{cases} w(r) = w(r;\alpha_1,\alpha_2) = U(r) + \log(1+r^{2-\eta}), & r \ge 0, \\ z(r) = z(r;\alpha_1,\alpha_2) = V(r) + \log(1+r^{2-\zeta}), & r \ge 0. \end{cases}$$

Then by the similar arguments as in the proof of (a), we omit the details and hence (b) is proved. □

The next result gives us the existence of solutions of Type (III) to (1.6)–(1.7).

Proposition 3.2 *The region $\mathcal{S}$ is nonempty, i.e., System* (1.6)–(1.7) *possesses solutions of Type (III).*

Proof We prove this proposition by contradiction. Suppose that, without loss of generality, there exist $(\alpha_1,\alpha_2) \in \mathcal{S}_1 \cup \mathcal{C}_1$ and a sequence $\{(\alpha_1^i,\alpha_2^i)\}$ in $\mathcal{S}_2 \cup \mathcal{C}_2$ such that $(\alpha_1^i,\alpha_2^i) \to (\alpha_1,\alpha_2)$ as $i\to\infty$. Since $(\alpha_1,\alpha_2) \in \mathcal{S}_1 \cup \mathcal{C}_1$, $\Theta_2(\alpha_1,\alpha_2) = \infty$ and hence

$$\int_0^R r^{2m_1+1} K_2(r) e^{U(r;\alpha_1,\alpha_2)} dr > 2 + 2m_2 + \gamma_1$$

if R is large, where $U(r;\alpha_1,\alpha_2)$ is defined in (1.8) with respect to (α_1,α_2). By the continuity of solutions with respect to initial data and applying the bounded convergence theorem, we obtain that

$$\begin{aligned} 2 + 2m_2 + \gamma_1 &< \int_0^R r^{2m_1+1} K_2(r) e^{U(r;\alpha_1,\alpha_2)} dr \\ &= \lim_{i\to\infty} \int_0^R r^{2m_1+1} K_2(r) e^{U(r;\alpha_1^i,\alpha_2^i)} dr \\ &< \lim_{i\to\infty} \Theta_2\left(\alpha_1^i,\alpha_2^i\right) \\ &\le 2 + 2m_2 + \gamma_1 \end{aligned}$$

since $(\alpha_1^i,\alpha_2^i) \in \mathcal{S}_2 \cup \mathcal{C}_2$, which leads to a contradiction. Hence this proof is complete. □

Remark 3.1 From the proof of Proposition 3.2, we know that $\overline{\mathcal{S}_1 \cup \mathcal{C}_1}$ and $\overline{\mathcal{S}_2 \cup \mathcal{C}_2}$ are disjoint.

Proof of Theorem 1.2 We recall that for any solution of (1.6)–(1.7), the corresponding K_i-masses cannot be infinite simultaneously. For existence parts, Propositions 3.1 and 3.2 fulfill these requirements. Hence we complete the proof of Theorem 1.2. □

To obtain further geometric properties of the regions defined in (2.2), the following concept related to linearized systems needs to be introduced. Let $(u(r), v(r)) =$

$(u(r;\alpha_1,\alpha_2), v(r;\alpha_1,\alpha_2))$ be a solution of (1.6)–(1.7) and $(U(r;\alpha_1,\alpha_2), V(r;\alpha_1,\alpha_2))$ be defined in (1.8). We define

$$\begin{cases} \phi_i(r) = \phi_i(r;\alpha_1,\alpha_2) = \dfrac{\partial U(r;\alpha_1,\alpha_2)}{\partial \alpha_i}, \\ \psi_i(r) = \psi_i(r;\alpha_1,\alpha_2) = \dfrac{\partial V(r;\alpha_1,\alpha_2)}{\partial \alpha_i}, \end{cases} \quad i = 1, 2. \tag{3.7}$$

Then $\phi_i(r)$ and $\psi_i(r)$ $(i = 1, 2)$ satisfy the linearized systems of (1.9)

$$\begin{cases} \phi_i''(r) + \dfrac{1}{r}\phi_i'(r) + r^{2m_2}K_1(r)e^V\psi_i = 0, \quad r > 0, \\ \psi_i''(r) + \dfrac{1}{r}\psi_i'(r) + r^{2m_1}K_2(r)e^U\phi_i = 0, \quad r > 0, \\ \phi_1(0) = 1, \ \phi_2(0) = 0, \ \psi_1(0) = 0, \ \psi_2(0) = 1, \\ \phi_1'(0) = \phi_2'(0) = \psi_1'(0) = \psi_2'(0) = 0. \end{cases} \tag{3.8}$$

The linearized systems play a significant role in deriving the structure of solutions of (1.6)–(1.7). We first present the monotone properties of ϕ_i and ψ_i in the following lemma.

Lemma 3.1 *Let $(u(r), v(r))$ be a solution of (1.6)–(1.7), and $\phi_i(r), \psi_i(r)$ $(i = 1, 2)$ be defined as in (3.7). Then for any $r > 0$,*

$$\begin{cases} \phi_1(r) > 0, \ \phi_1'(r) > 0; \quad \psi_1(r) < 0, \ \psi_1'(r) < 0; \\ \phi_2(r) < 0, \ \phi_2'(r) < 0; \quad \psi_2(r) > 0, \ \psi_2'(r) > 0. \end{cases} \tag{3.9}$$

Proof We refer the reader to [8,9] for the proof of this lemma. In fact, from (3.8), it is easy to see that $\psi_1(r)$ decreases strictly and hence $\phi_1(r)$ increases strictly near $r = 0$. We omit the details here. □

In the following, some geometric properties of the regions defined in (2.2) are offered.

Proposition 3.3 *The regions $\mathcal{S}, \mathcal{S}_1$ and $\mathcal{S}_2$ are nonempty simply connected open subsets of $\mathbf{R}^2$. Furthermore, the following properties hold.*

(a) *$\Theta_1(\alpha_1,\alpha_2)$ is continuous for $(\alpha_1,\alpha_2) \in \mathcal{S}_1 \cup \mathcal{C}_1 \cup \mathcal{S}$, and $\Theta_2(\alpha_1,\alpha_2)$ is continuous for $(\alpha_1,\alpha_2) \in \mathcal{S}_2 \cup \mathcal{C}_2 \cup \mathcal{S}$.*
(b) *If $(\alpha_1,\alpha_{21}), (\alpha_1,\alpha_{22}) \in \mathcal{S}$ (resp., $\mathcal{S}_1, \mathcal{S}_2$) with $\alpha_{21} < \alpha_{22}$, then $(\alpha_1,\alpha_2) \in \mathcal{S}$ (resp., $\mathcal{S}_1, \mathcal{S}_2$) for $\alpha_2 \in (\alpha_{21},\alpha_{22})$. Similarly, if $(\alpha_{11},\alpha_2), (\alpha_{12},\alpha_2) \in \mathcal{S}$ (resp., $\mathcal{S}_1, \mathcal{S}_2$) with $\alpha_{11} < \alpha_{12}$, then $(\alpha_1,\alpha_2) \in \mathcal{S}$ (resp., $\mathcal{S}_1, \mathcal{S}_2$) for $\alpha_1 \in (\alpha_{11},\alpha_{12})$.*
(c) *$\mathcal{C}_1$ and $\mathcal{C}_2$ are curves in $\mathbf{R}^2$.*

Proof (a) Let $(\alpha_{10},\alpha_{20}) \in \mathcal{S}_1 \cup \mathcal{C}_1 \cup \mathcal{S}$, i.e., $\Theta_1(\alpha_{10},\alpha_{20}) < \infty$ and $\Theta_2(\alpha_{10},\alpha_{20}) > 2+2m_2+\gamma_1$. Then $\Theta_2(\alpha_{10},\alpha_{20}) > 2+2m_2+\gamma_1+\varepsilon$ for some $\varepsilon > 0$. Since $rV'(r;\alpha_{10},\alpha_{20})$ decreases to $-\Theta_2(\alpha_{10},\alpha_{20})$ as $r \to \infty$, we may select $r_0 > 0$ sufficiently large to assure that

$$rV'(r;\alpha_{10},\alpha_{20}) < -(2+2m_2+\gamma_1+\varepsilon), \quad r \geq r_0.$$

Moreover, by the continuity of solutions with respect to initial data, there exists $\delta > 0$ such that for $(\alpha_1,\alpha_2) \in B_\delta((\alpha_{10},\alpha_{20}))$, we have $rV'(r_0;\alpha_1,\alpha_2) \leq -(2+2m_2+\gamma_1+\varepsilon)$. Then

$$rV'(r;\alpha_1,\alpha_2) \leq -(2+2m_2+\gamma_1+\varepsilon), \quad r \geq r_0$$

for $(\alpha_1,\alpha_2) \in B_\delta((\alpha_{10},\alpha_{20}))$, which implies that

$$V(r;\alpha_1,\alpha_2) \leq c_1 - (2+2m_2+\gamma_1+\varepsilon)\log r, \quad r \geq r_0$$

for $(\alpha_1, \alpha_2) \in B_\delta((\alpha_{10}, \alpha_{20}))$ with some constant c_1. Hence,

$$e^{V(r;\alpha_1,\alpha_2)} \le c_2 r^{-(2+2m_2+\gamma_1+\varepsilon)}, \quad r \ge r_0 \tag{3.10}$$

for $(\alpha_1, \alpha_2) \in B_\delta((\alpha_{10}, \alpha_{20}))$, where $c_2 = e^{c_1}$. Due to the above results, we obtain that

$$\Theta_1(\alpha_1, \alpha_2) < \infty, \quad (\alpha_1, \alpha_2) \in B_\delta((\alpha_{10}, \alpha_{20})),$$

which implies that the set $\mathcal{S}_1 \cup \mathcal{C}_1 \cup \mathcal{S}$ is open. In addition, by (3.10) and applying the Lebesgue dominated convergence theorem, we conclude that $\Theta_1(\alpha_1, \alpha_2)$ is continuous at $(\alpha_{10}, \alpha_{20})$. The situation for Θ_2 is similar, and then we complete the proof of (a).

(b) Let $(\alpha_1, \alpha_{21}), (\alpha_1, \alpha_{22}) \in \mathcal{S}$ and $\alpha_{21} < \alpha_{22}$. Then $\Theta_1(\alpha_1, \alpha_{2i})$ and $\Theta_2(\alpha_1, \alpha_{2i})$, $i = 1, 2$, are all finite. By Lemma 3.1, we see that $\psi_2(r) > 0$ and $\phi_2(r) < 0$. This means that for any fixed $r > 0$, $V(r; \alpha_1, \alpha_2)$ and $U(r; \alpha_1, \alpha_2)$ are strictly increasing and decreasing with respect to α_2 respectively. Hence for any $\alpha_2 \in (\alpha_{21}, \alpha_{22})$, we have that both $\Theta_1(\alpha_1, \alpha_2)$ and $\Theta_2(\alpha_1, \alpha_2)$ are also finite, and then $(\alpha_1, \alpha_2) \in \mathcal{S}$. The proofs of other cases are similar, and we omit the details. We complete the proof of (b).

Define

$$\rho_1(\alpha_1) = \sup\{\alpha_2 : (\alpha_1, \alpha_2) \in \mathcal{S}_1\}, \quad \rho_2(\alpha_1) = \inf\{\alpha_2 : (\alpha_1, \alpha_2) \in \mathcal{S}_2\} \tag{3.11}$$

and

$$\sigma_1(\alpha_1) = \inf\{\alpha_2 : (\alpha_1, \alpha_2) \in \mathcal{S}\}, \quad \sigma_2(\alpha_1) = \sup\{\alpha_2 : (\alpha_1, \alpha_2) \in \mathcal{S}\}. \tag{3.12}$$

Then

$$\mathcal{S}_1 = \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathbf{R}, \alpha_2 < \rho_1(\alpha_1)\}, \quad \mathcal{S}_2 = \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathbf{R}, \alpha_2 > \rho_2(\alpha_1)\}$$

and

$$\mathcal{S} = \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathbf{R}, \sigma_1(\alpha_1) < \alpha_2 < \sigma_2(\alpha_1)\}.$$

It is easy to see that $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ are nonempty simply connected open subsets of $\mathbf{R}^2$.

(c) First we have

$$\mathcal{C}_1 = \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathbf{R}, \rho_1(\alpha_1) \le \alpha_2 \le \sigma_1(\alpha_1)\},$$

and

$$\mathcal{C}_2 = \{(\alpha_1, \alpha_2) : \alpha_1 \in \mathbf{R}, \sigma_2(\alpha_1) \le \alpha_2 \le \rho_2(\alpha_1)\}.$$

To claim that $\mathcal{C}_1$ is a curve, i.e., $\rho_1 = \sigma_1$, it suffices to show that for any $\varepsilon > 0$, $(\alpha_1 + \varepsilon, \alpha_2), (\alpha_1, \alpha_2 + \varepsilon) \notin \mathcal{C}_1$ whenever $(\alpha_1, \alpha_2) \in \mathcal{C}_1$. If $(\alpha_1, \alpha_2) \in \mathcal{C}_1$, then $\Theta_1(\alpha_1, \alpha_2) = \kappa_1$, where κ_1 is defined in (2.3). For any $\varepsilon > 0$, we have $V(r; \alpha_1+\varepsilon, \alpha_2) < V(r; \alpha_1, \alpha_2)$ and then $e^{V(r;\alpha_1+\varepsilon,\alpha_2)} < e^{V(r;\alpha_1,\alpha_2)}$ for all $r > 0$ by Lemma 3.1. Hence $\Theta_1(\alpha_1+\varepsilon, \alpha_2) < \Theta_1(\alpha_1, \alpha_2)$ and then $(\alpha_1 + \varepsilon, \alpha_2) \notin \mathcal{C}_1$ (in fact, $(\alpha_1 + \varepsilon, \alpha_2) \in \mathcal{S}_1$). Similarly, from Lemma 3.1 again, we also have $V(r; \alpha_1, \alpha_2 + \varepsilon) > V(r; \alpha_1, \alpha_2)$ which implies $\Theta_1(\alpha_1, \alpha_2 + \varepsilon) > \Theta_1(\alpha_1, \alpha_2)$. Therefore $(\alpha_1, \alpha_2 + \varepsilon) \notin \mathcal{C}_1$ (in fact, $(\alpha_1, \alpha_2 + \varepsilon) \in \mathcal{S} \cup \mathcal{S}_2$). The situation for $\mathcal{C}_2$ can be done in the same way, and hence (c) is proved. □

Now we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3 To prove (a), it is enough to show that the range of $\Theta_1(\alpha_1, \alpha_2)$ over $\mathcal{S}_1 \cup \mathcal{C}_1$ is exactly the interval $(0, 2 + 2m_1 + \gamma_2]$. Let $\theta_1 \in (0, 2 + 2m_1 + \gamma_2)$. Then we can choose some $\xi \in [\lambda_1, 2)$, where λ_1 is defined as in Proposition 3.1, so that $2 - \xi < \theta_1$. For such ξ, Proposition 3.1(a) assures that $\Theta_1(\alpha_1^*, \alpha_2^*) < 2-\xi < \theta_1$ for some $(\alpha_1^*, \alpha_2^*) \in \mathcal{S}_1$. In addition, we also know that $\Theta_1(\alpha_1^*, \rho_1(\alpha_1^*)) = 2+2m_1+\gamma_2 > \theta_1$, where ρ_1 is defined in (3.11). Therefore, by virtue of Proposition 3.3(a) and (b), we obtain that there exists $\alpha_2 \in (\alpha_2^*, \rho_1(\alpha_1^*))$ satisfying $(\alpha_1^*, \alpha_2) \in \mathcal{S}_1$ and $\Theta_1(\alpha_1^*, \alpha_2) = \theta_1$. Then (a) is proved.

The proof of (b) is similar, and we omit the details. Hence the proof of Theorem 1.3 is complete. □

4 The case of $K_1 \equiv K_2 \equiv 1$

Throughout this section, we consider the case of $K_1 \equiv K_2 \equiv 1$ in (1.6). In this case, $\beta_i = \gamma_i = 0 (i = 1, 2)$ in (1.3). The logarithmic behaviors of ϕ_i and ψ_i at infinity are proved below, and the differentiation properties of K_i-masses with respect to normalized initial data follow.

Lemma 4.1 *Let* $(\alpha_1, \alpha_2) \in \mathcal{S}$ *and* $(\phi_i(r), \psi_i(r)) = (\phi_i(r; \alpha_1, \alpha_2), \psi_i(r; \alpha_1, \alpha_2)), i = 1, 2,$ *be defined as in (3.7). Then*

(a) $\phi_i(r) = C_\phi^i \log r + \mu_i + o(1)$ *at* $r = \infty$ *for some* $C_\phi^1 > 0$, $C_\phi^2 < 0$ *and* $\mu_i \in \mathbf{R}$, $i = 1, 2$. *Furthermore,* $C_\phi^i = C_\phi^i(\alpha_1, \alpha_2)$ *is continuous with respect to* (α_1, α_2) *on* $\mathcal{S}$, $i = 1, 2$.

(b) $\psi_i(r) = C_\psi^i \log r + \nu_i + o(1)$ *at* $r = \infty$ *for some* $C_\psi^2 > 0$, $C_\psi^1 < 0$ *and* $\nu_i \in \mathbf{R}$, $i = 1, 2$. *Furthermore,* $C_\psi^i = C_\psi^i(\alpha_1, \alpha_2)$ *is continuous with respect to* (α_1, α_2) *on* $\mathcal{S}$, $i = 1, 2$.

Proof We only prove the results involving ϕ_1 and ψ_1. The others involving ϕ_2 and ψ_2 are similar, and we omit the details. We divide the proof into the following steps.

Step 1. We first show that

$$\begin{cases} \phi_1(r) = C_\phi^1 \log r + \mu_1 + o(1), \\ \psi_1(r) = C_\psi^1 \log r + \nu_1 + o(1), \end{cases} \quad \text{at } r = \infty$$

for some $C_\phi^1 = C_\phi^1(\alpha_1, \alpha_2) > 0$, $C_\psi^1 = C_\psi^1(\alpha_1, \alpha_2) < 0$ and $\mu_1, \nu_1 \in \mathbf{R}$. Let $(U(r), V(r))$ be defined in (1.8) associated with the solution $(u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ of (1.15)–(1.7). Since $(\alpha_1, \alpha_2) \in \mathcal{S}$ and from Remark 2.1(ii), then for R_0 large,

$$\eta \equiv \min_{r \geq R_0} \{-rU'(r) - (2+2m_1), -rV'(r) - (2+2m_2)\} > 0.$$

Choose $0 < \varepsilon < \eta$ and $C_0 > 0$. Then for $r \geq R_0$, we have

$$\begin{aligned} \Delta[C_0 r^{1+\varepsilon} - (\phi_1 - \psi_1)] &= (1+\varepsilon)^2 C_0 r^{\varepsilon-1} + r^{2m_2} e^V \psi_1 - r^{2m_1} e^U \phi_1 \\ &\geq (1+\varepsilon)^2 C_0 r^{\varepsilon-1} - r^{-2-\varepsilon}(\phi_1 - \psi_1) \\ &\geq (1+\varepsilon)^2 C_0 r^{\varepsilon-1} - (1+\varepsilon)^2 r^{-2}(\phi_1 - \psi_1) \\ &= (1+\varepsilon)^2 r^{-2}[C_0 r^{1+\varepsilon} - (\phi_1 - \psi_1)], \quad r \geq R_1 \end{aligned}$$

for some $R_1 \geq R_0$ since $\phi_1 > 0$, $\psi_1 < 0$ and the definition of η. Hence

$$-\psi_1(r) < \phi_1(r) - \psi_1(r) \leq C_0 r^{1+\varepsilon}, \quad r \geq R_1$$

if C_0 is large.

Now, from (3.8) and the above results, we have

$$\Delta \phi_1 = -r^{2m_2} e^V \psi_1(r) \leq \hat{C}_0 r^{-1}, \quad r \geq R_1$$

for some $\hat{C}_0 > 0$, which implies

$$\phi_1(r) \le C_1 r, \quad r \ge R_1$$

for some $C_1 > 0$. Moreover,

$$\Delta\psi_1 = -r^{2m_1} e^U \phi_1(r) \ge -\hat{C}_1 r^{-1-\varepsilon}, \quad r \ge R_1,$$

for some $\hat{C}_1 > 0$, and hence we also obtain

$$\psi_1(r) \ge -C_2 r^{1-\varepsilon}, \quad r \ge R_1$$

for some $C_2 > 0$. By repeating the process as above if necessary, we conclude

$$\lim_{r\to\infty} r\phi_1' = -\int_0^\infty r^{2m_2+1} e^V \psi_1 dr = C_\phi^1, \quad \lim_{r\to\infty} r\psi_1' = -\int_0^\infty r^{2m_1+1} e^U \phi_1 dr = C_\psi^1$$

for some finite $C_\phi^1 > 0$ and $C_\psi^1 < 0$.

Finally, set $w(r) = \phi_1(r) - C_\phi^1 \log r$. Then $w(r)$ satisfies

$$\Delta w = -r^{2m_2} e^V \psi_1, \quad \lim_{r\to\infty} r w'(r) = 0,$$

which implies

$$r w'(r) = \int_r^\infty s^{2m_2+1} e^V \psi_1 ds < 0, \quad r > 0.$$

By means of the choice of ε and the behavior of $\psi_1(r)$ at infinity, we get

$$r w'(r) \ge -C r^{-\varepsilon/2} \quad \text{for large } r$$

for some $C > 0$, and then $w(r)$ is bounded from below. Therefore $\lim_{r\to\infty} w(r)$ exists and finite since $w'(r) < 0$. The situation for $\psi_1(r)$ is similar, and then we finish this step.

Step 2. Next we prove that $C_\phi^1(\alpha_1, \alpha_2)$ and $C_\psi^1(\alpha_1, \alpha_2)$ are continuous with respect to (α_1, α_2) on $\mathcal{S}$. Let $(\hat{\alpha}_1, \hat{\alpha}_2) \in \mathcal{S}$. Then by Remark 2.1(ii), there exist constants $\varepsilon, \delta > 0$ and $r_0 > 1$ such that

$$rU'(r; \alpha_1, \alpha_2) < -(2 + 2m_1) - \varepsilon, rV'(r; \alpha_1, \alpha_2) < -(2 + 2m_2) - \varepsilon \tag{4.1}$$

for $r \ge r_0$ and $(\alpha_1, \alpha_2) \in B_\delta\left((\hat{\alpha}_1, \hat{\alpha}_2)\right)$. Define

$$X(r; \alpha_1, \alpha_2) = \frac{|\phi_1(r; \alpha_1, \alpha_2) - \phi_1(r; \hat{\alpha}_1, \hat{\alpha}_2)|}{\log r}$$

and

$$Y(r; \alpha_1, \alpha_2) = \frac{|\psi_1(r; \alpha_1, \alpha_2) - \psi_1(r; \hat{\alpha}_1, \hat{\alpha}_2)|}{\log r}.$$

Then from (3.8), we get

$$\phi_1(r) = \phi_1(r_0) + r_0\phi_1'(r_0)(\log r - \log r_0) - \int_{r_0}^r (\log r - \log s) s^{2m_2+1} e^V \psi_1 ds$$

and

$$\psi_1(r) = \psi_1(r_0) + r_0\psi_1'(r_0)(\log r - \log r_0) - \int_{r_0}^{r} (\log r - \log s)s^{2m_1+1}e^U \phi_1 ds.$$

By subtracting with respect to (α_1, α_2) and $(\hat{\alpha}_1, \hat{\alpha}_2)$, divided by $\log r$, and using (4.1), we obtain that for $(\alpha_1, \alpha_2) \in B_\delta((\hat{\alpha}_1, \hat{\alpha}_2))$,

$$\begin{aligned} X(r; \alpha_1, \alpha_2) \le\ & X(r_0; \alpha_1, \alpha_2) + r_0|\phi_1'(r_0; \alpha_1, \alpha_2) - \phi_1'(r_0; \hat{\alpha}_1, \hat{\alpha}_2)| \\ & + C\int_{r_0}^{r} s^{-1-\varepsilon/2} Y(s; \alpha_1, \alpha_2) ds, \quad r \ge r_0, \end{aligned}$$

and

$$\begin{aligned} Y(r; \alpha_1, \alpha_2) \le\ & Y(r_0; \alpha_1, \alpha_2) + r_0|\psi_1'(r_0; \alpha_1, \alpha_2) - \psi_1'(r_0; \hat{\alpha}_1, \hat{\alpha}_2)| \\ & + C\int_{r_0}^{r} s^{-1-\varepsilon/2} X(s; \alpha_1, \alpha_2)\, ds, \quad r \ge r_0 \end{aligned}$$

for some $C > 0$. Hence

$$(X+Y)(r; \alpha_1, \alpha_2) \le I(\alpha_1, \alpha_2) + C\int_{r_0}^{r} s^{-1-\varepsilon/2}[(X+Y)(s; \alpha_1, \alpha_2)]\, ds, \quad r \ge r_0$$

for $(\alpha_1, \alpha_2) \in B_\delta((\hat{\alpha}_1, \hat{\alpha}_2))$, where

$$\begin{aligned} I(\alpha_1, \alpha_2) =\ & (X+Y)(r_0; \alpha_1, \alpha_2) + r_0|\phi_1'(r_0; \alpha_1, \alpha_2) - \phi_1'(r_0; \hat{\alpha}_1, \hat{\alpha}_2)| \\ & + r_0|\psi_1'(r_0; \alpha_1, \alpha_2) - \psi_1'(r_0; \hat{\alpha}_1, \hat{\alpha}_2)|. \end{aligned}$$

Therefore, by the Gronwall inequality, we have

$$(X+Y)(r; \alpha_1, \alpha_2) \le I(\alpha_1, \alpha_2) \exp\left\{\left(\frac{2C}{\varepsilon}\right) r_0^{-\varepsilon/2}\right\}, \quad r \ge r_0$$

for $(\alpha_1, \alpha_2) \in B_\delta((\hat{\alpha}_1, \hat{\alpha}_2))$. Since $I(\alpha_1, \alpha_2) \to 0$ as $(\alpha_1, \alpha_2) \to (\hat{\alpha}_1, \hat{\alpha}_2)$, we deduce that $X(r; \alpha_1, \alpha_2)$ and $Y(r; \alpha_1, \alpha_2)$ converge uniformly to 0 on $[r_0, \infty)$ as $(\alpha_1, \alpha_2) \to (\hat{\alpha}_1, \hat{\alpha}_2)$. Then Step 2 and hence this lemma is proved. □

Lemma 4.2 *Both $\Theta_1(\alpha_1, \alpha_2)$ and $\Theta_2(\alpha_1, \alpha_2)$ are continuously differentiable on $\mathcal{S}$. Furthermore,*

$$\begin{cases} \dfrac{\partial\Theta_1(\alpha_1, \alpha_2)}{\partial\alpha_i} = \displaystyle\int_0^\infty r^{2m_2+1}e^V \psi_i(r; \alpha_1, \alpha_2)dr = -\lim_{r\to\infty} r\phi_i'(r; \alpha_1, \alpha_2), \\ \dfrac{\partial\Theta_2(\alpha_1, \alpha_2)}{\partial\alpha_i} = \displaystyle\int_0^\infty r^{2m_1+1}e^U \phi_i(r; \alpha_1, \alpha_2)dr = -\lim_{r\to\infty} r\psi_i'(r; \alpha_1, \alpha_2) \end{cases} \tag{4.2}$$

for $(\alpha_1, \alpha_2) \in \mathcal{S}$ and $i = 1, 2$.

Proof From Proposition 3.3, we see that $\mathcal{S}$ is open and $\Theta_1(\alpha_1,\alpha_2)$, $\Theta_2(\alpha_1,\alpha_2)$ are continuous on $\mathcal{S}$. Let $(\hat\alpha_1,\hat\alpha_2)\in\mathcal{S}$. We prove that Θ_1, Θ_2 are differentiable at $(\hat\alpha_1,\hat\alpha_2)$. By Remark 2.1(ii), (1.15) and the continuity with respect to initial data, there exist positive constants R, δ and ε such that

$$rU'(r;\alpha_1,\alpha_2) < -(2+2m_1)-\varepsilon,\, rV'(r;\alpha_1,\alpha_2) < -(2+2m_2)-\varepsilon \tag{4.3}$$

for $r\geq R$ and $(\alpha_1,\alpha_2)\in B_\delta\left((\hat{\alpha_1},\hat{\alpha_2})\right)$. Set

$$\begin{cases} g_h(r) = r^{2m_2+1}\left[\dfrac{e^{V(r;\hat\alpha_1+h,\hat\alpha_2)}-e^{V(r;\hat\alpha_1,\hat\alpha_2)}}{h}\right], \\ f_h(r) = r^{2m_1+1}\left[\dfrac{e^{U(r;\hat\alpha_1+h,\hat\alpha_2)}-e^{U(r;\hat\alpha_1,\hat\alpha_2)}}{h}\right], \end{cases}$$

and

$$\begin{cases} g(r) = r^{2m_2+1}e^{V(r;\hat\alpha_1,\hat\alpha_2)}\psi_1(r;\hat\alpha_1,\hat\alpha_2), \\ f(r) = r^{2m_1+1}e^{U(r;\hat\alpha_1,\hat\alpha_2)}\phi_1(r;\hat\alpha_1,\hat\alpha_2). \end{cases}$$

Then

$$\lim_{h\to 0} g_h(r) = g(r), \quad \lim_{h\to 0} f_h(r) = f(r)$$

and

$$\begin{cases} |g_h(r)| \leq r^{2m_2+1}e^{V(r;\alpha_1^1,\hat\alpha_2)}\left|\psi_1(r;\alpha_1^2,\hat\alpha_2)\right|, \\ |f_h(r)| \leq r^{2m_1+1}e^{U(r;\alpha_1^3,\hat\alpha_2)}\left|\phi_1(r;\alpha_1^4,\hat\alpha_2)\right| \end{cases} \tag{4.4}$$

for some α_1^i, $i=1,2,3,4$, between $\hat\alpha_1+h$ and $\hat\alpha_1$.

Moreover, by Lemma 4.1, we have

$$C_\phi^1(\hat\alpha_1,\hat\alpha_2) - \frac{\varepsilon}{2}\phi_1(r_0;\hat\alpha_1,\hat\alpha_2) < -2C_\phi^1(\hat\alpha_1,\hat\alpha_2),$$

and

$$C_\psi^1(\hat\alpha_1,\hat\alpha_2) - \frac{\varepsilon}{2}\psi_1(r_0;\hat\alpha_1,\hat\alpha_2) > -2C_\psi^1(\hat\alpha_1,\hat\alpha_2)$$

for some large $r_0\geq R$. Then by Lemma 4.1 again, (3.8) and the above inequalities, there exists $\delta_1\in(0,\delta)$ such that

$$r\phi_1'(r;\alpha_1,\alpha_2) - \frac{\varepsilon}{2}\phi_1(r;\alpha_1,\alpha_2) < -C_\phi^1(\hat\alpha_1,\hat\alpha_2) < 0,$$

and

$$r\psi_1'(r;\alpha_1,\alpha_2) - \frac{\varepsilon}{2}\psi_1(r;\alpha_1,\alpha_2) > -C_\psi^1(\hat\alpha_1,\hat\alpha_2) > 0$$

for $r\geq r_0$ and $(\alpha_1,\alpha_2)\in B_{\delta_1}((\hat\alpha_1,\hat\alpha_2))$. Hence

$$\left[r^{-\varepsilon/2}\phi_1(r;\alpha_1,\alpha_2)\right]' < 0, \quad \left[r^{-\varepsilon/2}\psi_1(r;\alpha_1,\alpha_2)\right]' > 0, \quad r\geq r_0 \tag{4.5}$$

for $(\alpha_1,\alpha_2)\in B_{\delta_1}((\hat\alpha_1,\hat\alpha_2))$. By combining (4.3), (4.4) and (4.5), we get

$$\max\{|g_h(r)|,|f_h(r)|\} \leq Mr^{-1-\varepsilon/2}, \quad r\geq r_0,\ |h|<\delta_1$$

for some $M > 0$. Since $g_h(r)$ and $f_h(r)$ are bounded on $[0, r_0]$ and $|h| < \delta_1$, the Lebesgue dominated convergence theorem implies

$$\lim_{h\to 0}\int_0^\infty f_h(r)dr = \int_0^\infty f(r)dr, \quad \lim_{h\to 0}\int_0^\infty g_h(r)dr = \int_0^\infty g(r)dr.$$

Finally, $\partial\Theta_1/\partial\alpha_1$ and $\partial\Theta_2/\partial\alpha_1$ are continuous on $\mathcal{S}$ by (4.3), (4.5) and the Lebesgue dominated convergence theorem again. The proofs of other parts are similar, and we omit the details. Hence this lemma is proved. □

In addition, the following gives us the fact that all K_1-masses equal identically along a unique curve passing a given normalized initial data in $\mathbf{R}^2$, and so do K_2-masses.

Proposition 4.1 (a) *For any $\boldsymbol{\alpha}^* = (\alpha_1^*, \alpha_2^*) \in \mathbf{R}^2$, there exists a unique smooth curve $\Gamma(\boldsymbol{\alpha}^*)$ in $\mathbf{R}^2$ such that $\boldsymbol{\alpha}^* \in \Gamma(\boldsymbol{\alpha}^*)$ and*

$$\Theta_i(\boldsymbol{\alpha}^*) = \Theta_i(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \Gamma(\boldsymbol{\alpha}^*), \ i = 1, 2. \tag{4.6}$$

Moreover, $\Theta_1(\alpha_1, \alpha_2)$ (resp., $\Theta_2(\alpha_1, \alpha_2)$) is increasing (resp., decreasing) as α_2 increasing and $(\alpha_1, \alpha_2) \in \mathcal{S}$; $\Theta_1(\alpha_1, \alpha_2)$ (resp., $\Theta_2(\alpha_1, \alpha_2)$) is decreasing (resp., increasing) as α_1 increasing and $(\alpha_1, \alpha_2) \in \mathcal{S}$.

(b) *The linearized system (1.16) associated with $(\alpha_1, \alpha_2) \in \mathcal{S}$ is degenerate.*

Proof (a) Let $(u(r), v(r)) = (u(r; \alpha_1^*, \alpha_2^*), v(r; \alpha_1^*, \alpha_2^*))$ be the solution of (1.15)–(1.7), and $(U(r), V(r))$ be defined in (1.8). We define

$$\Gamma(\boldsymbol{\alpha}^*) = \{(\Gamma_1(t), \Gamma_2(t)) : t > 0\}$$

by

$$\Gamma_1(t) = \alpha_1^* + (2m_1 + 2)\log t, \quad \Gamma_2(t) = \alpha_2^* + (2m_2 + 2)\log t, \ t > 0. \tag{4.7}$$

Then for any $t > 0$,

$$(U_t(r), V_t(r)) \equiv (U(tr) + (2m_1 + 2)\log t, V(tr) + (2m_2 + 2)\log t)$$

is the unique solution of (1.9) with initial data $(\Gamma_1(t), \Gamma_2(t))$. Moreover, we also have

$$\int_0^\infty r^{2m_2+1}e^{V_t(r)}dr = \int_0^\infty r^{2m_2+1}e^{V(r)}dr,$$

and

$$\int_0^\infty r^{2m_1+1}e^{U_t(r)}dr = \int_0^\infty r^{2m_1+1}e^{U(r)}dr.$$

Hence $\Theta_i(\alpha_1^*, \alpha_2^*) = \Theta_i(\Gamma_1(t), \Gamma_2(t))$ for all $t > 0, i = 1, 2$. By Proposition 3.3, we see that such curve $\Gamma(\boldsymbol{\alpha}^*)$ is unique for any $\boldsymbol{\alpha}^* \in \mathbf{R}^2$ fixed. The remaining parts in (a) can be proved by Lemmas 4.1 and 4.2.

(b) Let $(\alpha_1^*, \alpha_2^*) \in \mathcal{S}$ and $M = (2m_2 + 2)/(2m_1 + 2)$. Then we have

$$\frac{\partial\Theta_i}{\partial\alpha_1}(\alpha_1^*, \alpha_2^*) + M\frac{\partial\Theta_i}{\partial\alpha_2}(\alpha_1^*, \alpha_2^*) = 0, \quad i = 1, 2.$$

In fact, from the proof of (a), we know that $(d\Theta_i/dt)(\Gamma_1(t), \Gamma_2(t)) = 0$ and then

$$\frac{\partial\Theta_i}{\partial\alpha_1}(\Gamma_1(t), \Gamma_2(t)) + M\frac{\partial\Theta_i}{\partial\alpha_2}(\Gamma_1(t), \Gamma_2(t)) = 0, \quad t > 0, i = 1, 2$$

and $\Gamma_1(1) = \alpha_1^*$, $\Gamma_2(1) = \alpha_2^*$, where $\Gamma_1(t)$, $\Gamma_2(t)$ are defined in (4.7). By Lemma 4.2, we have

$$\lim_{r\to\infty} r[\phi_1'(r;\alpha_1^*,\alpha_2^*) + M\phi_2'(r;\alpha_1^*,\alpha_2^*)] = 0,$$

and

$$\lim_{r\to\infty} r[\psi_1'(r;\alpha_1^*,\alpha_2^*) + M\psi_2'(r;\alpha_1^*,\alpha_2^*)] = 0.$$

Hence $\phi_1(r;\alpha_1^*,\alpha_2^*) + M\phi_2(r;\alpha_1^*,\alpha_2^*)$ and $\psi_1(r;\alpha_1^*,\alpha_2^*) + M\psi_2(r;\alpha_1^*,\alpha_2^*)$, which satisfy (1.16) with respect to (α_1^*, α_2^*), are both bounded on $[0,\infty)$ by Lemma 4.1. The proof of (b) is complete. □

Before finishing this section, a verification of Theorem 1.4 is given.

Proof of Theorem 1.4 By Proposition 3.3 and Lemma 4.2, we have that the ranges of $\Theta_1(\alpha_1, \alpha_2)$ and $\Theta_2(\alpha_1, \alpha_2)$ over the region $\mathcal{S}$ are $(2 + 2m_1, \infty)$ and $(2 + 2m_2, \infty)$ respectively. The remaining parts of Theorem 1.4 can be obtained by virtue of Remark 2.1(i) and Proposition 4.1(b). Hence we complete the proof of Theorem 1.4. □

Remark 4.1 In Theorem 1.4, we consider (1.15)–(1.7) with $m_1, m_2 > 0$. In fact, by applying similar arguments used in this section, the conclusions of Theorem 1.4 also hold for (1.15)–(1.7) with $m_1, m_2 \in (-1, 0]$.

5 Dirichlet problem

We consider (1.9) throughout this section, i.e.,

$$\begin{cases} U''(r) + \frac{1}{r}U'(r) + r^{2m_2}K_1(r)e^V = 0, & r > 0, \\ V''(r) + \frac{1}{r}V'(r) + r^{2m_1}K_2(r)e^U = 0, & r > 0, \\ U(0) = \alpha_1,\ V(0) = \alpha_2, \\ U'(0) = 0,\ V'(0) = 0. \end{cases} \tag{5.1}$$

where $\alpha_1, \alpha_2 > 0$.

Remark 5.1 If $(U(r), V(r))$ is a solution of (5.1), then it is easy to see that $U(r)$ and $V(r)$ must vanish on $(0,\infty)$.

Due to Remark 5.1, any solution of (5.1) can be categorized into the following types: a solution $(U(r), V(r))$ is a *Dirichlet-type* solution if $U(r)$ and $V(r)$ are both positive before some finite point but vanish at that point (*vanishing point*); it is a *U-crossing* (resp., *V-crossing*) solution if $U(r)$ (resp., $V(r)$) vanishes first at some finite point where $V(r)$ (resp., $U(r)$) is still positive.

For convenience, we use the following notations for the regions of initial data corresponding to various types of solutions of (5.1):

$$\begin{cases} \mathcal{D} = \{(\alpha_1,\alpha_2) : (U(r;\alpha_1,\alpha_2), V(r;\alpha_1,\alpha_2)) \text{ is Dirichlet-type}\}, \\ \mathcal{D}_1 = \{(\alpha_1,\alpha_2) : (U(r;\alpha_1,\alpha_2), V(r;\alpha_1,\alpha_2)) \text{ is } U\text{-crossing}\}, \\ \mathcal{D}_2 = \{(\alpha_1,\alpha_2) : (U(r;\alpha_1,\alpha_2), V(r;\alpha_1,\alpha_2)) \text{ is } V\text{-crossing}\}. \end{cases} \tag{5.2}$$

Lemma 5.1 *Consider the initial value problem (5.1). Then*

(a) *For all* $\alpha_1 > 0$ *(resp.,* $\alpha_2 > 0$*), there exists* $\tilde{\alpha}_2 > 0$ *(resp.,* $\tilde{\alpha}_1 > 0$*) such that* $(\alpha_1, \alpha_2) \in \mathcal{D}_1$ *(resp.,* $(\alpha_1, \alpha_2) \in \mathcal{D}_2$*) for all* $\alpha_2 > \tilde{\alpha}_2$ *(resp.,* $\alpha_1 > \tilde{\alpha}_1$*).*

(b) *For all* $\alpha_1 > 0$ *(resp.,* $\alpha_2 > 0$*), there exists* $\tilde{\alpha}_2 > 0$ *(resp.,* $\tilde{\alpha}_1 > 0$*) such that* $(\alpha_1, \alpha_2) \in \mathcal{D}_2$ *(resp.,* $(\alpha_1, \alpha_2) \in \mathcal{D}_1$*) for all* $\alpha_2 < \tilde{\alpha}_2$ *(resp.,* $\alpha_1 < \tilde{\alpha}_1$*).*

Proof First from (5.1), we obtain that

$$\begin{cases} U(r;\alpha_1,\alpha_2) = \alpha_1 - \int_0^r (\log r - \log s) s^{2m_2+1} K_1(s) e^V \, ds, \\ V(r;\alpha_1,\alpha_2) = \alpha_2 - \int_0^r (\log r - \log s) s^{2m_1+1} K_2(s) e^U \, ds, \end{cases} \tag{5.3}$$

where $(U(r;\alpha_1,\alpha_2), V(r;\alpha_1,\alpha_2))$ is the solution of (5.1). To prove (a), let $\alpha_1 > 0$ be fixed and define

$$A = \max\{r^{-\beta_2} K_2(r) : r \in (0,1]\} > 0, \quad B = \min\{r^{-\beta_1} K_1(r) : r \in (0,1]\} > 0.$$

Then for $r \in (0, 1]$, we get that

$$\begin{aligned} \int_0^r (\log r - \log s) s^{2m_1+1+\beta_2} ds &\le \int_0^1 (-\log s) s^{2m_1+1+\beta_2} ds \\ &= \frac{1}{2m_1+2+\beta_2} \int_0^1 s^{2m_1+1+\beta_2} ds \\ &= \frac{1}{(2m_1+2+\beta_2)^2}, \end{aligned}$$

and hence by (5.3),

$$\begin{aligned} V(r;\alpha_1,\alpha_2) &\ge \alpha_2 - Ae^{\alpha_1} \int_0^r (\log r - \log s) s^{2m_1+1+\beta_2} ds \\ &\ge \alpha_2 - \frac{Ae^{\alpha_1}}{(2m_1+2+\beta_2)^2} \\ &\ge \frac{\alpha_2}{2}, \quad r \in (0,1] \end{aligned}$$

if $\alpha_2 > 2Ae^{\alpha_1}/(2m_1+2+\beta_2)^2$.

Additionally, we also have

$$U(1;\alpha_1,\alpha_2) \le \alpha_1 - Be^{\alpha_2/2} \int_0^1 (-\log s) s^{2m_2+1+\beta_1} \, ds.$$

Therefore, if we choose α_2 sufficiently large, then results above deduce that $V(r;\alpha_1,\alpha_2) > 0$ on $[0, 1]$ but $U(1;\alpha_1,\alpha_2) < 0$. This means that $(\alpha_1, \alpha_2) \in \mathcal{D}_1$ for large α_2, and hence (a) is established. The proof of (b) is similar, and we omit the details. Then this lemma is proved. □

The following results consist of the geometric structure of solutions to (5.1) in terms of initial data.

Lemma 5.2 *Consider the initial value problem* (5.1) *and notations defined in* (5.2)*. Then*

(a) *$\mathcal{D}_1$ and $\mathcal{D}_2$ are simply connected open sets and*

$$\mathcal{D} \cup \mathcal{D}_1 \cup \mathcal{D}_2 = (0, \infty) \times (0, \infty).$$

(b) *There exists a strictly increasing function $\tau : (0, \infty) \to (0, \infty)$ satisfying $\lim_{\alpha_1 \to 0^+} \tau(\alpha_1) = 0$, $\lim_{\alpha_1 \to \infty} \tau(\alpha_1) = \infty$ such that*

$$\mathcal{D} = \{(\alpha_1, \tau(\alpha_1)) : \alpha_1 > 0\}.$$

Proof It is not difficult to see that (a) holds by Lemma 5.1 and Remark 5.1. To prove (b), it suffices to show that $(\alpha_1^* + \varepsilon, \alpha_2^*)$, $(\alpha_1^*, \alpha_2^* + \varepsilon) \notin \mathcal{D}$ for any $\varepsilon > 0$ whenever $(\alpha_1^*, \alpha_2^*) \in \mathcal{D}$. In fact, let $R > 0$ be the point so that $U(r; \alpha_1^*, \alpha_2^*)$, $V(r; \alpha_1^*, \alpha_2^*) > 0$ for $r \in [0, R)$ and $U(R; \alpha_1^*, \alpha_2^*) = V(R; \alpha_1^*, \alpha_2^*) = 0$. Then from Lemma 3.1, we have that

$$U(r; \alpha_1^* + \varepsilon, \alpha_2^*) - U(r; \alpha_1^*, \alpha_2^*) = \varepsilon \phi_1(r; \hat{\alpha}_1(r), \alpha_2^*) > 0, \quad r \in (0, R),$$

and

$$V(R; \alpha_1^* + \varepsilon, \alpha_2^*) - V(R; \alpha_1^*, \alpha_2^*) = \varepsilon \psi_1(R; \tilde{\alpha}_1, \alpha_2^*) < 0,$$

for some $\hat{\alpha}_1(r), \tilde{\alpha}_1 \in (\alpha_1^*, \alpha_1^* + \varepsilon)$. This deduces that $(\alpha_1^* + \varepsilon, \alpha_2^*) \in \mathcal{D}_2$. Similarly, we also have $(\alpha_1^*, \alpha_2^* + \varepsilon) \in \mathcal{D}_1$ for any $\varepsilon > 0$, and hence (b) is proved. □

We now give the existence of solutions to the Dirichlet problem of (5.1) below.

Proposition 5.1 *For any $R > 0$, (5.1) possesses a Dirichlet-type solution which is positive on $[0, R)$ while vanishes at R.*

Proof Let τ be defined as in Lemma 5.2(b). We introduce the function $\mathcal{R} : (0, \infty) \to (0, \infty)$ to be the point so that the Dirichlet-type solution $U(r; \alpha_1, \tau(\alpha_1))$, $V(r; \alpha_1, \tau(\alpha_1)) > 0$ for $r \in [0, \mathcal{R}(\alpha_1))$ and $U(\mathcal{R}(\alpha_1); \alpha_1, \tau(\alpha_1)) = V(\mathcal{R}(\alpha_1); \alpha_1, \tau(\alpha_1)) = 0$. We will claim that $\mathcal{R}(\alpha_1)$ is a continuous and onto function satisfying

$$\lim_{\alpha_1 \to 0^+} \mathcal{R}(\alpha_1) = 0, \quad \lim_{\alpha_1 \to \infty} \mathcal{R}(\alpha_1) = \infty. \tag{5.4}$$

By the continuity of solutions with respect to initial data, it is not difficult to get that $\mathcal{R}(\alpha_1)$ is continuous and $\lim_{\alpha_1 \to 0^+} \mathcal{R}(\alpha_1) = 0$. Hence it remains to prove that $\lim_{\alpha_1 \to \infty} \mathcal{R}(\alpha_1) = \infty$.

We apply the scaling arguments to achieve our goal. Let $d > 0$ be given and consider initial data (α_1, α_2) in the form (s, ds), $s > 0$ specifically. Define

$$\begin{cases} \hat{U}_s(r) = U\left(e^{-\frac{(cd)s}{2}} r; s, ds\right) - s, \\ \hat{V}_s(r) = V\left(e^{-\frac{(cd)s}{2}} r; s, ds\right) - ds, \end{cases}$$

where

$$c > \max\left\{\frac{2}{(2 + 2m_2 + \beta_1)d}, \frac{2}{(2 + 2m_1 + \beta_2)d}\right\} > 0.$$

Then $(\hat{U}_s(r), \hat{V}_s(r))$ satisfies

$$\begin{cases} \Delta \hat{U}_s + e^{\left[1-cd\left(1+\frac{2m_2+\beta_1}{2}\right)\right]s} r^{2m_2} e^{\frac{(cd)\beta_1 s}{2}} K_1\left(e^{-\frac{(cd)s}{2}} r\right) e^{\hat{V}_s} = 0, \quad r > 0, \\ \Delta \hat{V}_s + e^{\left[1-cd\left(1+\frac{2m_1+\beta_2}{2}\right)\right]s} r^{2m_1} e^{\frac{(cd)\beta_2 s}{2}} K_2\left(e^{-\frac{(cd)s}{2}} r\right) e^{\hat{U}_s} = 0, \quad r > 0, \\ \hat{U}_s(0) = \hat{V}_s(0) = 0, \lim_{r\to 0} r\hat{U}_s'(r) = \lim_{r\to 0} r\hat{V}_s'(r) = 0. \end{cases} \tag{5.5}$$

Note that $\hat{U}_s(r), \hat{V}_s(r)$ are negative for all $r > 0$. Let $\{(s_j, ds_j)\}$ be a sequence in $\mathbf{R}_+^2$ satisfying $s_j \to \infty$ as $j \to \infty$. Set $(\hat{U}_j, \hat{V}_j) = (\hat{U}_{s_j}, \hat{V}_{s_j})$. Since $e^{\hat{U}_j}, e^{\hat{V}_j} \le 1$ and $r^{-\beta_1} K_1(r)$, $r^{-\beta_2} K_2(r)$ are bounded on $[0, R]$ for any $R > 0$, we have that $r^{-(1+2m_2+\beta_1)}|\hat{U}_j'(r)|, r^{-(1+2m_1+\beta_2)}|\hat{V}_j'(r)|$ and hence $|\hat{U}_j(r)|, |\hat{V}_j(r)|$ are all bounded on $[0, R]$ for any $R > 0$. By using standard elliptic estimates, we obtain that $(\hat{U}_j, \hat{V}_j)$ converges to some $(\hat{U}, \hat{V})$ (passing to a subsequence if necessary) in $C^2([0, R]) \times C^2([0, R])$ for any $R > 0$. Then $(\hat{U}_j, \hat{V}_j)$ converges to $(\hat{U}, \hat{V})$ pointwisely on $[0, \infty)$ and $(\hat{U}, \hat{V})$ satisfies

$$\begin{cases} \Delta \hat{U}(r) = 0, \quad \Delta \hat{V}(r) = 0, \\ \hat{U}(0) = \hat{V}(0) = 0, \lim_{r\to 0} r\hat{U}'(r) = \lim_{r\to 0} r\hat{V}'(r) = 0, \end{cases}$$

which implies that $(\hat{U}(r), \hat{V}(r)) \equiv (0, 0)$.

Assume that $\lim_{j\to\infty} \mathcal{R}(s_j) = R^*$. Then $R^* > 0$, and from (5.5), we obtain that for $r \ge 0$,

$$|\hat{U}_j(r)| = \int_0^r (\log r - \log t) e^{\left[1-cd\left(1+\frac{2m_2+\beta_1}{2}\right)\right]s_j} t^{1+2m_2} e^{\frac{(cd)\beta_1 s_j}{2}} K_1\left(e^{-\frac{(cd)s_j}{2}} t\right) e^{\hat{V}_j}\, dt,$$

$$|\hat{V}_j(r)| = \int_0^r (\log r - \log t) e^{\left[1-cd\left(1+\frac{2m_1+\beta_2}{2}\right)\right]s_j} t^{1+2m_1} e^{\frac{(cd)\beta_2 s_j}{2}} K_2\left(e^{-\frac{(cd)s_j}{2}} t\right) e^{\hat{U}_j}\, dt.$$

Note that

$$1 - cd\left(1 + \frac{2m_2 + \beta_1}{2}\right) < 0, \quad 1 - cd\left(1 + \frac{2m_1 + \beta_2}{2}\right) < 0$$

due to the choice of c, and

$$e^{\frac{(cd)\beta_i s_j}{2}} K_i\left(e^{-\frac{(cd)s_j}{2}} t\right) \quad \text{is bounded,} \quad i = 1, 2,\ j \in \mathbf{N},$$

for t on any compact subset of $[0, \infty)$ by the behaviors of $K_1(r)$, $K_2(r)$ at the origin. If R^* is finite, then by the pointwise convergence of $(\hat{U}_j, \hat{V}_j)$ on $[0, \infty)$ and applying Fatou's lemma, we get

$$\limsup_{j\to\infty} \left|\hat{U}_j\left(e^{\frac{cds_j}{2}} r\right)\right| = \limsup_{j\to\infty} \left|\hat{V}_j\left(e^{\frac{cds_j}{2}} r\right)\right| = 0 \quad \text{for any fixed } r > 0.$$

Therefore,

$$V(r; s_j, ds_j) \to \infty \quad \text{as } j \to \infty \text{ for any } d > 0 \text{ and } r > 0.$$

However, by Lemma 5.2(b), without loss of generality, we may choose d to be small enough so that $ds_j \le \tau(s_j)$ for large j, then a contradiction occurs since for any $\varepsilon > 0$, $\mathcal{R}(s_j) < R^* + \varepsilon$

for large j and thus $V(R^*+\varepsilon; s_j, ds_j) < 0$ for large j by Lemma 3.1. Consequently, (5.4) holds and we complete the proof of Proposition 5.1. □

To attain our uniqueness result of the Dirichlet problem, we introduce the following auxiliary functions:

$$\begin{cases} \Phi(r;\alpha_1,\alpha_2,C) = \phi_1(r;\alpha_1,\alpha_2) + C\phi_2(r;\alpha_1,\alpha_2), & r>0, \\ \Psi(r;\alpha_1,\alpha_2,C) = \psi_1(r;\alpha_1,\alpha_2) + C\psi_2(r;\alpha_1,\alpha_2), & r>0, \end{cases} \tag{5.6}$$

and

$$\begin{cases} C_\Phi(r;\alpha_1,\alpha_2) = -\dfrac{\phi_1}{\phi_2}(r;\alpha_1,\alpha_2), & r>0, \\ C_\Psi(r;\alpha_1,\alpha_2) = -\dfrac{\psi_1}{\psi_2}(r;\alpha_1,\alpha_2), & r>0, \end{cases} \tag{5.7}$$

where ϕ_i and $\psi_i (i = 1, 2)$ are defined in (3.7) with respect to the solution $(U(r;\alpha_1,\alpha_2), V(r;\alpha_1,\alpha_2))$ of (5.1) and $C \in \mathbf{R}$. For simplification, we leave out the symbol of initial data (α_1, α_2) in the functions defined by (5.6) and (5.7) if no confusion arises. Then $\Phi(r; C)$ and $\Psi(r; C)$ satisfy

$$\begin{cases} \Phi''(r;C) + \dfrac{1}{r}\Phi'(r;C) + r^{2m_2}K_1(r)e^{V(r)}\Psi(r;C) = 0, & r>0, \\ \Psi''(r;C) + \dfrac{1}{r}\Psi'(r;C) + r^{2m_1}K_2(r)e^{U(r)}\Phi(r;C) = 0, & r>0. \\ \Phi(0;C) = 1,\ \Psi(0;C) = C, \\ \Phi'(0;C) = \Psi'(0;C) = 0, \end{cases} \tag{5.8}$$

Remark 5.2 (i) By Lemma 3.1, it is easy to see that $C_\Phi(r) \to +\infty$ and $C_\Psi(r) \to 0$ as $r \to 0$.

(ii) $C_\Phi(r)$ and $C_\Psi(r)$ cannot be constant on an interval. Indeed, if $C_\Phi(r) \equiv K$ for $r \in [a,b]$, then $\Phi(r;K) = 0$ for $r \in [a,b]$ which is impossible by (5.8).

The following assertions are crucial to proving the uniqueness of Dirichlet-type solutions of (5.1).

Lemma 5.3 *Let $(u(r), v(r))$ be a Dirichlet-type solution of (1.6)–(1.7) with vanishing point $R > 0$. Then $C_\Phi(r)$ is strictly decreasing and $C_\Psi(r)$ is strictly increasing on $(0, R]$, where $C_\Phi(r), C_\Psi(r)$ are defined by $(U(r), V(r))$ associated with $(u(r), v(r))$. Furthermore, $C_\Phi(R) > C_\Psi(R)$.*

Proof First of all, by Remark 5.2(i), we have $C_\Phi(r) > C_\Psi(r)$ for $r \in (0, r_0)$ for some $0 < r_0 \le R$.

Claim $C'_\Phi(r) < 0$ *and* $C'_\Psi(r) > 0$ *for* $r \in (0, r_0)$.

Proof of Claim. Suppose that $C_\Phi(r)$ is not strictly decreasing on $(0, r_0)$. Then by Remark 5.2(ii), there exist $0 < r_1 < r_2 \le r_0$ such that

$$C'_\Phi(r_1) < 0,\ C'_\Phi(r_2) > 0, \quad C_\Phi(r_1) = C_\Phi(r_2) \equiv C_0$$

and

$$0 < C_\Psi(r) < C_\Phi(r) < C_0, \quad r \in (r_1, r_2).$$

By combining (5.6), (5.7) and Lemma 3.1, we obtain

$$\begin{cases} \Phi(r;C_0) < 0 < \Psi(r;C_0), & r \in (r_1, r_2), \\ \Phi(r_1;C_0) = \Phi(r_2;C_0) = 0, \end{cases} \tag{5.9}$$

which implies that $\Phi(r; C_0)$ has a local minimum at some $\bar{r} \in (r_1, r_2)$ and $\Phi''(\bar{r}; C_0) \ge 0$. However, from (5.8) and (5.9), we have

$$\Phi''(\bar{r}; C_0) = -\bar{r}^{2m_2} K_1(\bar{r}) e^{V(\bar{r})} \Psi(\bar{r}; C_0) < 0.$$

This is a contradiction. The proof for $C_\Psi(r)$ is similar and we complete the proof of this claim.

Now, suppose there exists $R_0 \in (0, R]$ such that $C_\Phi(R_0) = C_\Psi(R_0) \equiv C$ and $C_\Phi(r) > C_\Psi(r) > 0$ for $r \in (0, R_0)$. Then from the claim above, we obtain

$$\begin{cases} \Phi(r;C) > 0, \Psi(r;C) > 0, & r \in (0, R_0), \\ \Phi(R_0;C) = \Psi(R_0;C) = 0, \\ \Phi'(R_0;C) < 0, \Psi'(R_0;C) < 0. \end{cases} \tag{5.10}$$

By taking the differentiation with respect to α_i, $i = 1, 2$, on both sides of (2.5) and definitions of $\Phi(r; C)$ and $\Psi(r; C)$, we get

$$\begin{aligned} & r^2 V'(r)\Phi'(r;C) + r^2 U'(r)\Psi'(r;C) + r^2 K_1 e^v \Psi(r;C) + r^2 K_2 e^u \Phi(r;C) \\ & = \int_0^r \left\{ s\left[(2m_2+2)K_1 + sK_1'\right] e^v \Psi(s;C) + s\left[(2m_1+2)K_2 + sK_2'\right] e^u \Phi(s;C) \right\} ds, \end{aligned}$$

and hence

$$\begin{aligned} & r^2 v'(r)\Phi'(r;C) + r^2 u'(r)\Psi'(r;C) + r^2 K_1 e^v \Psi(r;C) + r^2 K_2 e^u \Phi(r;C) \\ & = \int_0^r \left\{ s\left[2K_1 + sK_1'\right] e^v \Psi(s;C) + s\left[2K_2 + sK_2'\right] e^u \Phi(s;C) \right\} ds. \end{aligned}$$

Consequently, we deduce

$$\begin{aligned} 0 & \ge R_0^2 v'(R_0)\Phi'(R_0;C) + R_0^2 u'(R_0)\Psi'(R_0;C) \\ & = \int_0^{R_0} \left\{ s\left[2K_1 + sK_1'\right] e^v \Psi(s;C) + s\left[2K_2 + sK_2'\right] e^u \Phi(s;C) \right\} ds > 0 \end{aligned}$$

by (1.4), (1.12) and (5.10), which is a contradiction. Therefore the graphs of C_Φ and C_Ψ do not intersect on $[0, R]$. The proof of this lemma is complete. □

Based on Lemma 5.3, it is easy to obtain the following consequences.

Lemma 5.4 *If $(u(r), v(r))$ is a Dirichlet-type solution of (1.6)–(1.7) and define*

$$C^* = -\frac{\phi_1}{\phi_2}(R), \quad C_* = -\frac{\psi_1}{\psi_2}(R),$$

then $\Phi(r; C)$ and $\Psi(r; C)$ satisfy the following properties.

(a) *If $C > C^*$, then $\Psi(r; C) > 0$ on $[0, R]$ and $\Phi(R; C) < 0$.*
(b) *If $C = C^*$, then $\Phi(r; C), \Psi(r; C) > 0$ on $[0, R)$, $\Phi(R; C) = 0$ and $\Psi(R; C) > 0$.*
(c) *If $C_* < C < C^*$, then $\Phi(r; C), \Psi(r; C) > 0$ on $[0, R]$.*

(d) *If* $C = C_*$, *then* $\Phi(r; C), \Psi(r; C) > 0$ *on* $[0, R)$, $\Phi(R; C) > 0$ *and* $\Psi(R; C) = 0$.
(e) *If* $0 < C < C_*$, *then* $\Phi(r; C) > 0$ *on* $[0, R]$ *and* $\Psi(R; C) < 0$.
(f) *If* $C \le 0$, *then* $\Phi(r; C) > 0$ *and* $\Psi(r; C) < 0$ *on* $[0, R]$.

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1 Let $R > 0$ be given and $(u(r), v(r)) = (u(r; \alpha_1, \alpha_2), v(r; \alpha_1, \alpha_2))$ be a solution of (1.6)–(1.7) satisfying $u(R) = v(R) = 0$. If we define $(\tilde{U}(r), \tilde{V}(r)) = (U(r)+2m_1 \log R, V(r)+2m_2 \log R)$, where $(U(r), V(r))$ is defined in (1.8) associated with $(u(r), v(r))$. Then $(\tilde{U}(r), \tilde{V}(r))$ satisfies (5.1) replacing $K_1(r)$ and $K_2(r)$ by $R^{-2m_2} K_1(r)$ and $R^{-2m_1} K_2(r)$, respectively. By Proposition 5.1 combining the monotone property (Lemma 3.1) and the continuity of solutions with respect to normalized initial data, there exists a strictly increasing function $\tilde{\tau} : (-\infty, \infty) \to (-\infty, \infty)$ satisfying $\lim_{\alpha_1 \to -\infty} \tilde{\tau}(\alpha_1) = -\infty, \lim_{\alpha_1 \to \infty} \tilde{\tau}(\alpha_1) = \infty$ such that $u(r; \alpha_1, \tilde{\tau}(\alpha_1)), v(r; \alpha_1, \tilde{\tau}(\alpha_1)) < 0$ for $r \in (0, \mathcal{R}(\alpha_1)), u(\mathcal{R}(\alpha_1); \alpha_1, \tilde{\tau}(\alpha_1)) = v(\mathcal{R}(\alpha_1); \alpha_1, \tilde{\tau}(\alpha_1)) = 0$, and $\lim_{\alpha_1 \to -\infty} \mathcal{R}(\alpha_1) = \infty, \lim_{\alpha_1 \to \infty} \mathcal{R}(\alpha_1) = 0$. Then the existence result is proved. To prove the uniqueness result, it suffices to show that the function $\mathcal{R}(\alpha_1)$ is strictly decreasing on $(0, \infty)$. We prove this by contradiction arguments. Without loss of generality, suppose there exists an $\hat{\alpha}_1 \in \mathbf{R}$ such that $\mathcal{R}(\hat{\alpha}_1)$ is a local maximum value, and two sequences $\{\alpha_1^i\}$ and $\{\bar{\alpha}_1^i\}$ such that for $i \in N$,

$$\alpha_1^i < \bar{\alpha}_1^i, \quad \mathcal{R}(\alpha_1^i) = \mathcal{R}(\bar{\alpha}_1^i) \quad \text{and} \quad \lim_{i \to \infty} \alpha_1^i = \lim_{i \to \infty} \bar{\alpha}_1^i = \hat{\alpha}_1.$$

Let $(u_i(r), v_i(r)), (\bar{u}_i(r), \bar{v}_i(r))$ be solutions of (1.6)–(1.7) associated with normalized initial data $(\alpha_1^i, \tilde{\tau}(\alpha_1^i)), (\bar{\alpha}_1^i, \tilde{\tau}(\bar{\alpha}_1^i))$ respectively. Without loss of generality, we may assume for some $\varepsilon > 0$ fixed,

$$||u_i - \bar{u}_i||_{L^\infty([0, \mathcal{R}(\hat{\alpha}_1)+\varepsilon])} \ge ||v_i - \bar{v}_i||_{L^\infty([0, \mathcal{R}(\hat{\alpha}_1)+\varepsilon])} \quad \text{for all } i.$$

Set

$$\Lambda_i(r) = \frac{(u_i - \bar{u}_i)(r)}{||u_i - \bar{u}_i||_{L^\infty([0, \mathcal{R}(\hat{\alpha}_1)+\varepsilon])}}, \quad \Omega_i(r) = \frac{(v_i - \bar{v}_i)(r)}{||v_i - \bar{v}_i||_{L^\infty([0, \mathcal{R}(\hat{\alpha}_1)+\varepsilon])}}.$$

Then $(\Lambda_i(r), \Omega_i(r))$ satisfies

$$\begin{cases} \Delta \Lambda_i + K_1(r) e^{\eta_i(r)} \Omega_i = 0, \\ \Delta \Omega_i + K_2(r) e^{\xi_i(r)} \Lambda_i = 0, \\ \Lambda_i(\mathcal{R}(\alpha_1^i)) = \Omega_i(\mathcal{R}(\alpha_1^i)) = 0, \end{cases}$$

where $\xi_i(r) \in (u_i(r), \bar{u}_i(r))$ and $\eta_i(r) \in (v_i(r), \bar{v}_i(r))$ for $r \in [0, \mathcal{R}(\hat{\alpha}_1) + \varepsilon]$. By applying standard elliptic estimates, we obtain $(\Lambda_i, \Omega_i) \to (\Lambda, \Omega)$ (passing to a subsequence if necessary) in $C^2([0, \mathcal{R}(\hat{\alpha}_1) + \varepsilon]) \times C^2([0, \mathcal{R}(\hat{\alpha}_1) + \varepsilon])$ and (Λ, Ω) satisfies

$$\begin{cases} \Delta \Lambda + K_1(r) e^{v(r; \hat{\alpha}_1, \tilde{\tau}(\hat{\alpha}_1))} \Omega = 0, \\ \Delta \Omega + K_2(r) e^{u(r; \hat{\alpha}_1, \tilde{\tau}(\hat{\alpha}_1))} \Lambda = 0, \\ \Lambda(\mathcal{R}(\hat{\alpha}_1)) = \Omega(\mathcal{R}(\hat{\alpha}_1)) = 0, \\ \|\Lambda\|_{L^\infty([0, \mathcal{R}(\hat{\alpha}_1)+\varepsilon])} = \|\Omega\|_{L^\infty([0, \mathcal{R}(\hat{\alpha}_1)+\varepsilon])} = 1. \end{cases}$$

However, by Lemma 5.4, we know that either $\Lambda(\mathcal{R}(\hat{\alpha}_1)) \neq 0$ or $\Omega(\mathcal{R}(\hat{\alpha}_1)) \neq 0$, which leads to a contradiction. Therefore, the function $\mathcal{R}(\alpha_1)$ is strictly increasing on $(0, \infty)$ and we finish this proof. □

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