Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 264 (2018) 550–574

Journal of Differential Equations

www.elsevier.com/locate/jde

Asymptotic behavior of equilibrium states of reaction–diffusion systems with mass conservation

Jann-Long Chern [a,1], Yoshihisa Morita [b,2], Tien-Tsan Shieh [c,d,*]

[a] Department of Mathematics, National Central University, Chung-Li 32001, Taiwan
[b] Department of Applied Mathematics and Informatics, Ryukoku University, Seta Otsu 520-2194, Japan
[c] National Center for Theoretical Science, National Taiwan University, Taipei 106, Taiwan
[d] Department of Mathematics, National Taiwan University, Taipei 106, Taiwan

Received 19 December 2016; revised 5 September 2017
Available online 22 September 2017

Abstract

We deal with a stationary problem of a reaction–diffusion system with a conservation law under the Neumann boundary condition. It is shown that the stationary problem turns to be the Euler–Lagrange equation of an energy functional with a mass constraint. When the domain is the finite interval (0, 1), we investigate the asymptotic profile of a strictly monotone minimizer of the energy as d, the ratio of the diffusion coefficient of the system, tends to zero. In view of a logarithmic function in the leading term of the potential, we get to a scaling parameter κ satisfying the relation $\varepsilon := \sqrt{d} = \sqrt{\log \kappa}/\kappa^2$. The main result shows that a sequence of minimizers converges to a Dirac mass multiplied by the total mass and that by a scaling with κ the asymptotic profile exhibits a parabola in the nonvanishing region. We also prove the existence of an unstable monotone solution when the mass is small.

MSC: 35B35; 35B40; 35K57

* Corresponding author.
E-mail addresses: chern@mx.math.ncu.edu.tw (J.-L. Chern), morita@rins.ryukoku.ac.jp (Y. Morita), ttshieh@ncts.ntu.edu.tw, tshieh@math.ntu.edu.tw (T.-T. Shieh).
[1] Work partially supported by Ministry of Science and Technology of Taiwan under grant MOST-104-2115-M-008-010-MY3.
[2] Work partially supported by JSPS KAKENHI Grant Number, 26287025, 26247013, JST CREST Grant Number JPMJCR14D3, and a grant for overseas research (Kokugai-kenkyu) of Ryukoku University (2016–2017).

http://dx.doi.org/10.1016/j.jde.2017.09.015

Keywords: Reaction–diffusion system; Mass conservation; Equilibrium solution; Asymptotic behavior; Concentration phenomena; Stability

1. Introduction

In the fields of population biology and cell biology concentration phenomena are often observed by aggregation of species and chemical substances respectively. One of the well known models is a Keller–Segel chemotaxis model [21] in which spiky patterns appears by the aggregation of cellular slime mold, though it blows up in a higher dimensional domain (for instance, see [17], [5], [23], [20], [25] and the references therein). In this model the total mass of the slime mold is conserved in a reasonable setting. On the other hand in a study for the cell polarity the authors [19] and [7] proposed simple conceptual models to describe the concentration phenomenon induced by a different mechanism from the chemotaxis model, though the mass conservation property shares in the both models. After their contribution, mathematical studies for the conceptual models are developed in [16], [15], [8], [10] and [9] (see also [13], [14], [11] and [12]). In particular, it is shown in [16], [15] and [8] that the spiky pattern is certainly stable in their model equations.

Motivated by those studies, we are concerned with the following reaction–diffusion system:

$$\begin{cases} u_t = d\Delta u - g(u+\gamma v) + v, \\ v_t = \Delta v + g(u+\gamma v) - v, \end{cases} \qquad x \in \Omega, \tag{1.1}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $0 < d < 1$. We note that the diffusion coefficients of u and v equations are normalized: 1 in the v-equation and d in the u-equation where d stands for the ratio of the two diffusion coefficients. For specific cases $g(u) = au/(u^2+b)$ $(\gamma = 0)$ and $g(w) = w/(w+1)^2$ $(\gamma = 1)$ are provided by [19], where a, b are positive constants.

Here, we deal with the case for $\gamma = 1$ and fix the function $g(w)$ as

$$g(w) = \frac{w}{(w+1)^2}.$$

It is known that there exists a unique nonnegative classical solution satisfying the initial condition

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad u_0, v_0 \in C^0(\overline{\Omega}), \;\; u_0(x) \geq 0, v_0(x) \geq 0 \; (x \in \overline{\Omega})$$

(see [8] and [9]). Under the evolution of the system, the total mass is conserved:

$$\int_\Omega (u(x,t) + v(x,t))dx = \int_\Omega (u_0(x) + v_0(x_0))dx \qquad (t \geq 0).$$

Moreover, the system allows a Lyapunov function (see [8]), that is,

$$\mathcal{L}(u,v)=\int_\Omega \left\{\frac{d}{2}|\nabla(u+v)|^2+(1-d)G(u+v)+\frac{d}{2}(u+v)^2+\frac{1}{2}|\nabla(du+v)|^2\right\}dx,$$

where $G(u):=\int_0^w g(u)du$. Hence, the asymptotic state as $t\to\infty$ belongs to the set of all the equilibrium solutions (see [4]).

In this paper we study the stationary problem of (1.1) and the asymptotic profile of a stationary solution as $d\to 0$, where the problem is given as

$$\begin{cases} d\Delta u-g(u+v)+v=0, \\ \Delta v+g(u+v)-v=0, \end{cases} \qquad x\in\Omega, \tag{1.3}$$

with the functions u and v satisfying the Neumann boundary condition (1.2) and the total mass constraint

$$\int_\Omega (u+v)\,dx=M, \qquad \text{i.e. } \langle u\rangle+\langle v\rangle=M/|\Omega|\equiv m. \tag{1.4}$$

Here, $\langle\cdot\rangle$ is the integral average over Ω:

$$\langle\cdot\rangle:=\frac{1}{|\Omega|}\int_\Omega \cdot\,dx.$$

Set $w=u+v$. By a straightforward computation, the system (1.3) is reduced to the scalar equation for unknown function w and some unknown constant λ:

$$-d\Delta w+(1-d)g(w)+dw=\lambda \quad (x\in\Omega), \tag{1.5}$$

with the Neumann boundary condition $\partial w/\partial\nu=0$ on $\partial\Omega$ and the total mass constraint $\int_\Omega w\,dx=M$. This scalar equation is the Euler–Lagrange equation of the energy functional

$$\mathcal{E}(w)=\int_\Omega \left(\frac{d}{2}|\nabla w|^2+(1-d)G(w)+\frac{d}{2}w^2\right)dx, \tag{1.6}$$

under the total mass constraint (1.4), where the function G is given by

$$G(w)=\int_0^w g(u)\,du=\log|w+1|+\frac{1}{|w+1|}-1.$$

We focus on nonnegative solutions to the system (1.1) minimizing the corresponding energy functional $\mathcal{E}$. Therefore, we shall consider the variational problem

$$\inf_{\mathcal{A}} \mathcal{E}(w) = \inf_{\mathcal{A}} \int_{\Omega} \left(\frac{d}{2} |\nabla w|^2 + (1-d)G(w) + \frac{d}{2} w^2 \right) dx, \tag{1.7}$$

subject to an admissible set

$$\mathcal{A} := \{w \in H^1(\Omega) : \int_{\Omega} w \, dx = M, \ w \geq 0\}.$$

In particular, we analyze the asymptotic behavior of the solution of (1.7) with least energy as $d = \varepsilon^2 \to 0$ ($\varepsilon > 0$). For simplicity, we assume the domain is a one-dimensional interval $\Omega = (0, 1)$. The corresponding Euler–Lagrange equation is given by

$$\begin{aligned} &-\varepsilon^2 w_{xx} + (1-\varepsilon^2) g(w) + \varepsilon^2 w = \lambda \ \ (0 < x < 1), \\ &w_x(0) = w_x(1) = 0, \\ &\int_0^1 w \, dx = M. \end{aligned} \tag{1.8}$$

Invoking of the argument in [3] (see also [24]), we see that the minimizer $w_\varepsilon(x)$ is monotone in x.

We remark that a solution (u^*, v^*) of (1.3) is associate with a solution w^* of (1.5) through the relation

$$(u^*(x), v^*(x)) = \left(M - \langle g(w^*) \rangle + \frac{w^*(x) - M}{1-d}, \ \langle g(w^*) \rangle - \frac{d(w^*(x) - M)}{1-d} \right), \tag{1.9}$$

and it is proved in [8] that for the minimizer w_ε the solution $(u_\varepsilon, v_\varepsilon)$ defined as (1.9) is a stable solution. Moreover, $\mathcal{L}(u^*, v^*) = \mathcal{E}(u^* + v^*)$ holds for any solution (u^*, v^*) of (1.3) since $du^* + v^*$ is constant, the minimizer of $\mathcal{E}$ thereby provides the minimum of $\mathcal{L}(u, v)$.

Let κ_ε be a positive function defined through the relation

$$\varepsilon^2 = \frac{\log \kappa}{\kappa^4}. \tag{1.10}$$

It is seen that for $\epsilon > 0$ small enough the function κ_ϵ is strictly decreasing and $\lim_{\varepsilon \to 0+} \kappa_\varepsilon = +\infty$. Suppose w_ε is a global minimizer of the functional

$$\mathcal{E}_\varepsilon(w) = \int_0^1 \left(\frac{\varepsilon^2}{2} |w_x|^2 + (1-\varepsilon^2) G(w) + \frac{\varepsilon^2}{2} w^2 \right) dx, \tag{1.11}$$

among the admissible set $\mathcal{A}$. Without loss generality, we may assume that w_ε is monotone decreasing.

Theorem 1.1. *Let κ_ε be the solution of (1.10) and the function w_ε be a minimizer of the functional (1.11). Then the following results hold:*

(i) *The sequence* $\{w_\varepsilon\}$ *converges to* $M\delta(x)$ *in the following sense:*

$$\int_0^1 w_\varepsilon(x)dx = M, \qquad \lim_{\varepsilon\to 0}\sup_{\eta\le x\le 1} w_\varepsilon(x) = 0 \quad (\forall \eta \in (0,1)).$$

(ii) *Set* $\mu_\varepsilon := \max_{0\le x\le 1} w_\varepsilon(x) = w_\varepsilon(0)$. *There is a constant* $C_1 > 0$ *and* $\varepsilon_1 > 0$ *such that*

$$\frac{\mu_\varepsilon}{\kappa_\varepsilon} \le C_1 \qquad (0 < \varepsilon < \varepsilon_1).$$

(iii) *Define* $W_\varepsilon(x) := \frac{1}{\kappa_\varepsilon} w_\varepsilon\left(\frac{x}{\kappa_\varepsilon}\right)$. *The sequence* $\{W_\varepsilon\}$ *converges to a function*

$$W_0(x) = \begin{cases} \frac{1}{a} - \frac{ax^2}{4} & \text{for } 0 < x < \frac{\sqrt{2}}{a}, \\ 0 & \text{for } \frac{\sqrt{2}}{a} \le x. \end{cases}$$

in $C^{0,\alpha}_{loc}$ *for* $0 \le \alpha < 1$, *where* $a = \sqrt{\frac{2\sqrt{2}}{3M}}$. *Furthermore,*

$$\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon}{\kappa_\varepsilon} = \frac{1}{a}.$$

By a straightforward estimate, we will obtain the following Corollary. We leave the proof to readers.

Corollary 1.2. *Let* w_ε *be a minimizer of the functional* (1.11) *and* $(u_\varepsilon, v_\varepsilon)$ *be the associative solution to the system* (1.3) *through the relation* (1.9). *Then the sequence* $\{u_\varepsilon\}$ *converges to a Delta function* $M\delta(x)$ *in the same sense describing in* Theorem 1.1 *(i) and the sequence* $\{v_\varepsilon\}$ *converges to* 0 *in* $L^\infty(0, 1)$.

Remark 1.1. Our problem (1.7) is similar to the variational problem arising from the van der Waals/Cahn–Hilliard theory of phase transitions (see [2], [3]),

$$\inf_{\substack{u\in H^1(\Omega)\\ \int_\Omega u\,dx = M}} \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + W(u)\,dx,$$

where $W(u)$ represents a coarse-grain free energy and usually in the form of double-well potential. Unlike the standard one, our nonlinear potential is indeed dependent on the small parameter ε. As $\varepsilon \to 0$, the location of two "wells" is getting away. This is the reason that our solution have a "huge" jump near the boundary which is also different from the classical Maxwell solution. Therefore, instead of getting interface of two different phases, we obtain a Dirac Delta function in its asymptotic limit.

Next we consider the existence of another monotone solution. Let $0 < M < 1$. Then $g'(M) > 0$ and we easily verify that the constant solution $w^* = M$ is a nondegenerate local minimizer of the functional $\mathcal{E}_\varepsilon$ and it cannot be a (global) minimizer for sufficiently small ε. Since w_ε is a minimizer, we suspect that there is another solution, which might be unstable. This leads us to the next theorem.

Theorem 1.3. *Suppose that* $0 < M < 1$. *Then equation* (1.8) *has a strictly monotone decreasing solution* $\tilde{w}_\varepsilon(x)$ *whose energy* $\mathcal{E}_\varepsilon(\tilde{w}_\varepsilon)$ *is bounded away from* 0 *as* $\varepsilon \to 0$. *The reflected* $w = \tilde{w}_\varepsilon(1-x)$ *is a strictly monotone increasing solution. Those solutions are unstable for the gradient flow of* $\mathcal{E}_\varepsilon(w)$.

Corollary 1.4. *Let* $\tilde{w}_\varepsilon$ *be the solution given by* Theorem 1.3 *and let* (u^*, v^*) *be the one defined by* (1.9) *with* $w^* = \tilde{w}_\varepsilon$. *Then* (u^*, v^*) *is unstable in the system* (1.1)–(1.2) *with* $\Omega = (0, 1)$ *and* $d = \varepsilon^2$.

Remark 1.2. As mentioned before the system (1.1) with (1.2) allows the Turing type instability if $g'(M) < 0$, and the instability drives the emergence of the wave patterns in the transient dynamics (see [19]). Thus we were primarily interested in the model equation for this case. On the other hand the above theorem concerns the stable regime for the constant steady state and even this regime the system exhibits a nontrivial structure for the steady state solutions.

Remark 1.3. As seen in §4, our proof of the above theorem is done by a dynamical system argument. Although one might obtain the existence of an unstable solution by utilizing the mountain-pass theorem, we, however, emphasize that the argument here is quite simple.

Remark 1.4. It looks that the variational problem of (1.11) is similar to the stationary problem of the Cahn–Hilliard equation. In fact, the system (1.1) with $\gamma = 1$ can be formally written as a single equation for $w = u + v$ in what follows. First write (1.1) as

$$\begin{aligned} w_t - (1-d)v_t &= d\Delta w - (1-d)g(w) + (1-d)v, \\ w_t &= d\Delta w + (1-d)\Delta v. \end{aligned}$$

Operating $-\Delta$ on the both side of the first equation and differentiating the second one in t yield

$$\begin{aligned} -\Delta[w_t - (1-d)v_t] &= -\Delta[d\Delta w - (1-d)g(w) + (1-d)v], \\ w_{tt} &= d\Delta w_t + (1-d)\Delta v_t. \end{aligned}$$

By eliminating v terms we obtain

$$w_{tt} + w_t = -\Delta[d\Delta w + \tilde{g}(w) - (1+d)w_t], \tag{1.12}$$

where

$$\tilde{g}(w) := -(1-d)g(w) - dw.$$

If w_{tt} is absent from (1.12) and $\tilde{g}$ is a cubic function, it looks close to the viscous Cahn–Hilliard equation

$$(1-\nu)u_t = -\Delta(\Delta u + f(u) - \nu u_t), \qquad (\nu \in (0,1))$$

proposed by [18].

Remark 1.5. As stated in the first part of this section our model equation was presented by a conceptual model for the cell polarity. As a similar model system there is the following one as proposed in [22]

$$\begin{cases} u_t = -\mu[\Gamma(u+v)u - (1-\Gamma(u+v))v], \\ v_t = D\Delta v + \mu[\Gamma(u+v)u - (1-\Gamma(u+v))v], \end{cases} \tag{1.13}$$

where u and v stand for the density of a proliferating population and a migrating population in tumour cells respectively. $\Gamma(u+v)$ is the probability that an immotile cells becomes motile, μ is the exchange rate of phenotype of the cells, and D is the diffusion coefficient for v. This model is called "go-or-rest" model in [22] and numerical simulations are done for the function Γ given by

$$\Gamma(\rho) := \frac{1}{2}(1 + \tanh(\alpha[\rho^* - \rho])),$$

where α and ρ^* is positive parameter. By a biological reason the diffusion of u equation is absent. We notice

$$\Gamma(u+v)u - (1-\Gamma(u+v))v = \Gamma(u+v)(u+v) - v.$$

The system (1.1) with $\gamma = 1$ could be regarded as a singularly perturbed system of (1.13) with $g(w) = \Gamma(w)w$ by the normalization as $D = 1$ and $\mu = 1$. Since our nonlinearity is different from this case, the above results cannot apply to $g(w) = \Gamma(w)w$. Our study, however, would provide a direction of the study to this case.

Remark 1.6. As for the case when $\gamma = 0$ and $g(u) = au/(u^2 + b)$ $(a, b > 0)$, which appears in [7], [19], [15], [16] and [10], similar results to Theorems 1.1 and 1.3 can be obtained by simple modification of the argument in the present paper. We don't state this case in detail to avoid repeating similar arguments.

Remark 1.7. We believe that in a domain Ω with higher dimension the qualitative behaviors of stationary solutions to the equations (1.1)–(1.2) and their structural stability can be understood through our analytical techniques in the paper. Thus, in this case, we conjecture the minimizers of energy functional (1.6) concentrate at points with maximal curvature on the boundary Ω as $d \to 0$. In other words, the limiting function is still a Dirac mass at those points. The detailed discussion will be carried out in the forthcoming paper.

The rest of the paper is organized as follows: in the next section we give the proof for (i) and (ii) of Theorem 1.1 and in §3 we complete the proof of Theorem 1.1 by proving (iii). The proof of Theorem 1.3 is given in §4.

2. Proof of Theorem 1.1 (i) and (ii)

In this paper, we concern with the equilibrium states of the system (1.1) which is corresponding to minimizers of the variational problem (1.7). The existence and regularity results of minimizers are obtained by the standard direct method in the Calculus of Variation and the standard elliptic theory. For fixed ε, the minimizer w_ε satisfies the system (1.8) where λ is the corresponding Lagrange multiplier of the variational problem (1.7).

We define

$$\phi_\varepsilon(x) := \begin{cases} 2M\kappa(1-\kappa x) & (0 \le x \le 1/\kappa), \\ 0 & (1/\kappa \le x \le 1). \end{cases}$$

Lemma 2.1. *There are constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathcal{E}_\varepsilon(\phi_\varepsilon) \le C_0 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon} \quad (0 < \varepsilon \le \varepsilon_0). \tag{2.1}$$

Proof. We simply write κ instead of κ_ε below. Plugging the test function in the functional, we compute the energy term-by-term:

$$\int_0^1 \{(\phi_\varepsilon)_x\}^2 dx = \int_0^{1/\kappa} \left(2M\kappa^2\right)^2 dx = 4M^2\kappa^3,$$

$$\int_0^1 (\phi_\varepsilon)^2 dx = \int_0^{1/\kappa} 4M^2\kappa^2(1-\kappa x)^2 dx = \frac{4}{3}M^2\kappa,$$

$$\int_0^1 \log(\phi_\varepsilon + 1)dx = \left(\frac{1}{\kappa} + \frac{1}{2M\kappa^2}\right)\log(2M\kappa+1) - \frac{1}{\kappa},$$

and

$$\int_0^1 \left(\frac{1}{\phi_\varepsilon + 1} - 1\right) dx = \frac{1}{2M\kappa^2}\log(2M\kappa+1) - \frac{1}{\kappa}.$$

Therefore, we have

$$\begin{aligned} \mathcal{E}_\varepsilon(\phi_\varepsilon) = 2M^2\varepsilon^2\kappa^3 &+ (1-\varepsilon^2)\left[\left(\frac{1}{\kappa} + \frac{1}{2M\kappa^2}\right)\log(2M\kappa+1) - \frac{1}{\kappa}\right] \\ &+ (1-\varepsilon^2)\left[\frac{1}{2M\kappa^2}\log(2M\kappa+1) - \frac{1}{\kappa}\right] + \frac{2}{3}M\varepsilon^2\kappa. \end{aligned}$$

Since the definition of $\kappa = \kappa_\varepsilon$, we see

$$\varepsilon^2\kappa^3 = \frac{\log\kappa}{\kappa},$$

which implies the desired inequality (2.1). □

From the above lemma we see that the minimizer converges to zero almost everywhere as $\varepsilon \to 0$. In addition we have the next lemma.

Lemma 2.2. *Let $\mu_\varepsilon = \max_{0\le x\le L} w_\varepsilon(x)$, where w_ε is the minimizer of (1.11). We have*

$$\lim_{\varepsilon\to+0}\mu_\varepsilon = +\infty.$$

Proof. We will prove by contradiction. Assume that there exists some sequence $\{\varepsilon_j\}$ converging to 0 and some constant $C > 0$ such that $\mu_{\varepsilon_j} \le C$ for all j. Choose a point $w_0 < M$ sufficiently small in an open interval $(0, 1)$ such that the intersection of the tangent line

$$y = G'(w_0)(w - w_0) + G(w_0),$$

and the graph $y = G(w)$ is achieved at the point $w = w_0$ and $w_1 > C$. Note that

$$G'(w_0) = \frac{G(w_1) - G(w_0)}{w_1 - w_0}.$$

Noticing that $y = G(w)$ is convex up to $w = 1$ and concave for $w > 1$, and invoking of the L^∞-boundedness of the sequence $\{w_{\varepsilon_j}\}$, we see

$$G(w_{\varepsilon_j}) \ge G'(w_0)(w_{\varepsilon_j} - w_0) + G(w_0).$$

Therefore,

$$\begin{aligned}
\mathcal{E}_{\varepsilon_j}(w_{\varepsilon_j}) &= \int_0^1 \left(\frac{\varepsilon^2}{2}\left|\frac{dw_{\varepsilon_j}}{dx}\right|^2 + (1-\varepsilon_j^2)G(w_{\varepsilon_j}) + \frac{\varepsilon^2}{2}w_{\varepsilon_j}^2\right)dx \\
&\ge (1-\varepsilon_{\varepsilon_j}^2)\int_0^1 (G'(w_0)(w_{\varepsilon_j} - w_0) + G(w_0))\,dx \\
&> \frac{1}{2}\big(G'(w_0)(M - w_0) + G(w_0)\big) > 0
\end{aligned}$$

for ε_j small enough. The constant lower bound of the energy $\mathcal{E}_{\varepsilon_j}(w_{\varepsilon_j})$ contradict to the result of the previous Lemma 2.1:

$$\mathcal{E}_{\varepsilon_j}(w_{\varepsilon_j}) \le \frac{C_0\log\kappa(\varepsilon_j)}{\kappa(\varepsilon_j)} \to 0.$$

This implies

$$\lim_{\varepsilon\to+0}\mu_\varepsilon=+\infty. \qquad \square$$

By the above lemma we can assert (i) of Theorem 1.1.

Lemma 2.3. *For the same μ_ε in Lemma 2.2*

$$\frac{1}{2\sqrt{2}}\varepsilon\mu_\varepsilon(\log\mu_\varepsilon)^{1/2}\le\mathcal{E}_\varepsilon(w_\varepsilon) \tag{2.2}$$

holds.

Proof. Without loss of generality, we may assume $w_\varepsilon(x)$ is strictly monotone decreasing. Indeed, any stable solution is monotone by [3] but the constant solution $w=M$ of (1.8) cannot be the minimizer because of Lemma 2.1. Thus, we have $\mu_\varepsilon=w_\varepsilon(0)$. Let $\xi=\xi_\varepsilon$ be the point such that

$$\frac{\mu_\varepsilon}{2}=w_\varepsilon(\xi).$$

Put $\varphi_\varepsilon(x):=w_\varepsilon(x)/\mu_\varepsilon$. We have $\varphi(0)=1$, $\varphi(\xi_\varepsilon)=1/2$ and

$$\sigma_\varepsilon:=\mathcal{E}_\varepsilon(w_\varepsilon)=\int_0^1\left(\frac{\varepsilon^2\mu_\varepsilon^2}{2}\{(\varphi_\varepsilon)_x\}^2+(1-\varepsilon^2)G(\mu_\varepsilon\varphi_\varepsilon)+\frac{\varepsilon^2\mu_\varepsilon^2}{2}\varphi_\varepsilon^2\right)dx$$

where

$$G(\mu_\varepsilon\varphi_\varepsilon)=\log(\mu_\varepsilon\varphi_\varepsilon+1)+\frac{1}{\mu_\varepsilon\varphi_\varepsilon+1}-1=\frac{\log\mu_\varepsilon}{2}+\log(\sqrt{\mu_\varepsilon}\varphi_\varepsilon+\frac{1}{\sqrt{\mu_\varepsilon}})+\frac{1}{\mu_\varepsilon\varphi_\varepsilon+1}-1.$$

Because $\lim_{\varepsilon\to0}\mu_\varepsilon=+\infty$ and the definition ξ_ε, we have

$$\log(\sqrt{\mu_\varepsilon}\varphi_\varepsilon+\frac{1}{\sqrt{\mu_\varepsilon}})+\frac{1}{\mu_\varepsilon\varphi_\varepsilon+1}-1\ge\log(\frac{\sqrt{\mu_\varepsilon}}{2}+\frac{1}{\sqrt{\mu_\varepsilon}})+\frac{1}{\mu_\varepsilon+1}-1>0,$$

on the interval $[0,\xi_\varepsilon]$. Thus,

$$G(\mu_\varepsilon\varphi_\varepsilon)\ge\frac{1}{2}\log\mu_\varepsilon \quad (0\le x\le\xi_\varepsilon),$$

for small ε. Utilizing this, we estimate

$$\begin{aligned}\sigma_\varepsilon&\ge\int_0^{\xi_\varepsilon}\frac{\varepsilon^2\mu_\varepsilon^2\{(\varphi_\varepsilon)_x\}^2}{2}+(1-\varepsilon^2)G(\mu_\varepsilon\varphi_\varepsilon)\,dx\\&\ge\int_0^{\xi_\varepsilon}\frac{1}{2}\{\varepsilon\mu_\varepsilon(\varphi_\varepsilon)_x\}^2+\frac{1}{4}\left(\sqrt{\log\mu_\varepsilon}\right)^2\,dx\end{aligned}$$

$$\geq \frac{1}{\sqrt{2}} \int_0^{\xi_\varepsilon} \varepsilon\mu_\varepsilon \sqrt{\log \mu_\varepsilon} |(\varphi_\varepsilon)_x| \, dx \qquad (\text{put } z = \varphi_x(x))$$

$$= \frac{1}{\sqrt{2}} \varepsilon\mu_\varepsilon \sqrt{\log \mu_\varepsilon} \int_1^{1/2} (-1) dz = \frac{1}{2\sqrt{2}} \varepsilon\mu_\varepsilon \sqrt{\log \mu_\varepsilon}. \qquad \square$$

We complete the proof of Theorem 1.1 (ii). Combining Lemmas 2.1 and 2.3, we obtain

$$\frac{\varepsilon\mu_\varepsilon}{2\sqrt{2}} \sqrt{\log \mu_\varepsilon} \leq \mathcal{E}_\varepsilon(u_\varepsilon) \leq \frac{C_0 \log \kappa_\varepsilon}{\kappa_\varepsilon} = C_0 \varepsilon \kappa_\varepsilon \sqrt{\log(\kappa_\varepsilon)}, \tag{2.3}$$

where we used (1.10). Furthermore,

$$\begin{aligned} 2\sqrt{2}C_0\kappa_\varepsilon\sqrt{\log(\kappa_\varepsilon)} &= 2\sqrt{2}C_0\kappa_\varepsilon\sqrt{\log(2\sqrt{2}C_0\kappa_\varepsilon) - \log(2\sqrt{2}C_0)} \\ &\leq 2\sqrt{2}C_0\kappa_\varepsilon\sqrt{\log(2\sqrt{2}C_0\kappa_\varepsilon)}, \end{aligned}$$

for ε small enough. Combining the fact that the function $s(\log s)^{1/2}$ is monotone increasing for large $s > 0$, we conclude that $\mu_\varepsilon \leq 2\sqrt{2}C_0\kappa_\varepsilon$. This proves (ii) of Theorem 1.1. $\square$

3. Proof of Theorem 1.1 (iii)

3.1. Some estimates

In order to prove (iii) of Theorem 1.1 we need elaborate estimates. We rewrite the equation (1.8) as

$$\begin{aligned} &\frac{\varepsilon^2}{2} w_x^2 = (1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2} w^2 - (\lambda_\varepsilon w + A_\varepsilon), \\ &w_x(0) = w_x(1) = 0, \\ &\int_0^1 w \, dx = M. \end{aligned} \tag{3.1}$$

Since we assume w_ε is a monotone decreasing function, the first equation can be written as

$$w_x = -\sqrt{\frac{2}{\varepsilon^2}\left((1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2 - (\lambda_\varepsilon w + A_\varepsilon)\right)},$$

thus

$$-\frac{dw}{\sqrt{\frac{2}{\varepsilon^2}\left((1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2 - (\lambda_\varepsilon w + A_\varepsilon)\right)}} = dx.$$

Set $w_+ = w(0)$ and $w_- = w(1)$. The values $w_+, w_-, \lambda_\varepsilon$ and A_ε satisfy

$$(1-\varepsilon^2)G(w_+) + \frac{\varepsilon^2}{2}w_+^2 = \lambda_\varepsilon w_+ + A_\varepsilon, \tag{3.2}$$

$$(1-\varepsilon^2)G(w_-) + \frac{\varepsilon^2}{2}w_-^2 = \lambda_\varepsilon w_- + A_\varepsilon, \tag{3.3}$$

$$\int_{w_-}^{w_+} \frac{dw}{\sqrt{\frac{2}{\varepsilon^2}\Big((1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2 - (\lambda_\varepsilon w + A_\varepsilon)\Big)}} = 1, \tag{3.4}$$

$$\int_{w_-}^{w_+} \frac{w\,dw}{\sqrt{\frac{2}{\varepsilon^2}\Big((1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2 - (\lambda_\varepsilon w + A_\varepsilon)\Big)}} = M. \tag{3.5}$$

We investigate the behavior of $w_+, w_-, \lambda_\varepsilon$ and A_ε as $\varepsilon \to 0$ in the following discussion. First, invoking of (3.1), we have

$$\begin{aligned}
\mathcal{E}_\varepsilon(w) &= \int_0^1 \left(\frac{\varepsilon^2}{2}|w_x|^2 + (1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2\right) dx \\
&= \int_0^1 \Big(2(1-\varepsilon^2)G(w) + \varepsilon^2 w^2 - (\lambda_\varepsilon w + A_\varepsilon)\Big)\, dx \\
&= \int_0^1 \Big(\varepsilon^2|w_x|^2 + (\lambda_\varepsilon w + A_\varepsilon)\Big)\, dx.
\end{aligned}$$

We claim the following lemma:

Lemma 3.1. *Let $w(x)$ be the monotone decreasing minimizer of (1.11). Then there is an $\varepsilon_c > 0$ such that for $0 < \varepsilon < \varepsilon_c$*

$$w_- = w(1) = \mathcal{O}\left(\sqrt{\frac{\log \kappa_\varepsilon}{\kappa_\varepsilon}}\right), \tag{3.6}$$

$$C_1 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon} \le \lambda_\varepsilon \le C_2 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon}, \tag{3.7}$$

$$A_\varepsilon = \mathcal{O}\left(\frac{\log \kappa_\varepsilon}{\kappa_\varepsilon}\right) \tag{3.8}$$

hold, where C_1 and C_2 are positive constants.

Proof. Because

$$(1-\varepsilon^2)G(w_-) \le \mathcal{E}_\varepsilon(w) \le C\frac{\log(\kappa_\varepsilon)}{\kappa_\varepsilon},$$

and

$$G(w_-) = w_- \int_0^1 G'(tw_-)\,dt = w_-^2 \int_0^1 (1-t)G''(tw_-)\,dt,$$

we have

$$w_- = \mathcal{O}\left(\sqrt{\frac{\log \kappa_\varepsilon}{\kappa_\varepsilon}}\right),$$

which implies (3.6).

Solving the first two equations for λ_ε, we find

$$\lambda_\varepsilon = \frac{(1-\varepsilon^2)(G(w_+)-G(w_-)) + \frac{\varepsilon^2}{2}w_+^2 - \frac{\varepsilon^2}{2}w_-^2}{w_+ - w_-}.$$

Because $w_- \to 0$ and $\varepsilon^2 w_+^2 \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\left(\lambda_\varepsilon - \frac{\log \mu_\varepsilon}{\mu_\varepsilon}\right) = 0.$$

Moreover, by $\mu_\varepsilon \le 2\sqrt{2}C_0\kappa_\varepsilon$,

$$\begin{aligned}\frac{\log \mu_\varepsilon}{\mu_\varepsilon} &= \frac{\kappa_\varepsilon}{\mu_\varepsilon}\frac{1}{\kappa_\varepsilon}(\log \kappa_\varepsilon + \log(\mu_\varepsilon/\kappa_\varepsilon)) \\ &\ge C_1\frac{\log \kappa_\varepsilon}{\kappa_\varepsilon},\end{aligned}$$

therefore,

$$C_1\frac{\log \kappa_\varepsilon}{\kappa_\varepsilon} \le \lambda_\varepsilon.$$

Since the energy $\mathcal{E}_\varepsilon(w_\varepsilon)$ can be written as

$$\mathcal{E}_\varepsilon(w_\varepsilon) = \int_0^1 \left(\varepsilon^2|(w_\varepsilon)_x|^2 + (\lambda_\varepsilon w_\varepsilon + A_\varepsilon)\right)dx,$$

this lead us to the inequalities

$$0 \le \int_0^1 (\lambda_\varepsilon w + A_\varepsilon)\, dx \le C_0 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon},$$

thus

$$0 \le \lambda_\varepsilon M + A_\varepsilon \le C_0 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon}. \tag{3.9}$$

From (3.3)

$$\lambda_\varepsilon w_- = (1-\varepsilon^2) G(w_-) + \frac{\varepsilon^2}{2} w_-^2 - A_\varepsilon > -A_\varepsilon$$

follows. Applying this to (3.9) yields

$$\lambda_\varepsilon M - \lambda_\varepsilon w_- = \lambda_\varepsilon (M - w_-) < \lambda_\varepsilon M + A_\varepsilon \le C_0 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon},$$

thus

$$\lambda_\varepsilon \le C_2 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon}.$$

We have (3.7).

Finally, by

$$-\lambda_\varepsilon M \le A_\varepsilon \le C_0 \frac{\log \kappa_\varepsilon}{\kappa_\varepsilon},$$

we can easily obtain (3.8) and conclude the proof. □

3.2. Rescaling the equation

We complete the proof for (iii) of Theorem 1.1 by applying the result of this subsection. Here, we are going to find the limiting profile of the boundary layer at $x = 0$.

Define rescaled functions $W_\varepsilon : \Omega_\varepsilon := (0, \kappa_\varepsilon) \to \mathbb{R}$ by

$$W_\varepsilon(y) = \frac{1}{\kappa_\varepsilon} w_\varepsilon \left(\frac{y}{\kappa_\varepsilon} \right).$$

The equation (1.8) becomes

$$\begin{aligned} & -W_\varepsilon'' + \frac{(1-\varepsilon^2)}{\log \kappa_\varepsilon} \frac{\kappa_\varepsilon^2 W_\varepsilon}{(\kappa_\varepsilon W_\varepsilon + 1)^2} + \frac{1}{\kappa_\varepsilon^2} W_\varepsilon = \frac{\kappa_\varepsilon \lambda_\varepsilon}{\log \kappa_\varepsilon} \qquad \text{on } \Omega_\varepsilon, \\ & W_\varepsilon'(0) = W_\varepsilon'(\kappa_\varepsilon) = 0, \\ & \int_0^{\kappa_\varepsilon} W(y)\, dy = M. \end{aligned} \tag{3.10}$$

The first of the equation (3.1) is re-written as

$$(W_\varepsilon'(y))^2 = \frac{2}{\log \kappa_\varepsilon}\left[(1-\varepsilon^2)\left(\log(\kappa_\varepsilon W_\varepsilon + 1) + \frac{1}{\kappa_\varepsilon W_\varepsilon + 1} - 1\right) + \frac{\varepsilon^2\kappa_\varepsilon^2}{2}W_\varepsilon^2 - (\lambda_\varepsilon \kappa_\varepsilon W_\varepsilon + A_\varepsilon)\right] \tag{3.11}$$

in the interval Ω_ε.

Because the sequence $\{\mu_\varepsilon/\kappa_\varepsilon\}$ is bounded, this implies $\{W_\varepsilon\}$ is bounded in L^∞-norm. From (3.11), we also know $\{W_\varepsilon\}$ is bounded in $W^{1,\infty}$. Thus, there exists a subsequence $\{\varepsilon_j\}$ such that $\{W_{\varepsilon_j}\}$ converges to a function W_0 in $C^{0,\alpha}(\Omega')$ on any compact subset $\Omega' \subset (0,\infty)$ for $0 \le \alpha < 1$ and the subsequence $\{\mu_{\varepsilon_j}/\kappa_{\varepsilon_j}\}$ converges to some constant.

Lemma 3.2. *Suppose that w_ε is a minimizer for the variational problem*

$$\inf_{w\in\mathcal{A}} \mathcal{E}_\varepsilon(w) = \inf_{w\in\mathcal{A}} \int_0^1 \left(\frac{\varepsilon^2}{2}|w_x|^2 + (1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2\right)dx.$$

Then there exists a subsequence $\{W_{\varepsilon_j}\}$ and a function $W_0 \in C^{0,\alpha}_{loc}$ such that the subsequence W_{ε_j} converges to W_0 in $C^{0,\alpha}_{loc}$ where $\alpha \in [0,1)$.

Lemma 3.3. *Suppose $W_{\varepsilon_j} \to W_0$ in $C^{0,\alpha}_{loc}$. For any $y \in \operatorname{supp}(W_0)$, we have*

$$\lim_{j\to\infty} \frac{\log(\kappa_{\varepsilon_j} W_{\varepsilon_j}(y) + 1)}{\log \kappa_{\varepsilon_j}} = 1.$$

Proof. For $y \in \operatorname{supp}(W_0)$, we have

$$\frac{\log(\kappa_{\varepsilon_j} W_{\varepsilon_j}(y) + 1)}{\log \kappa_{\varepsilon_j}} - \frac{\log(\kappa_{\varepsilon_j} W_0(y) + 1)}{\log \kappa_{\varepsilon_j}} = \frac{\log\left(1 + \frac{W_{\varepsilon_j}(y) - W_0(y)}{W_0(y) + \frac{1}{\kappa_{\varepsilon_j}}}\right)}{\log \kappa_{\varepsilon_j}} \to 0$$

and

$$\lim_{j\to\infty} \frac{\log(\kappa_{\varepsilon_j} W_0(y) + 1)}{\log \kappa_{\varepsilon_j}} = \lim_{j\to\infty} \frac{\log \kappa_{\varepsilon_j} + \log(W_0(y) + \frac{1}{\kappa_{\varepsilon_j}})}{\log \kappa_{\varepsilon_j}} = 1. \qquad \square$$

By (3.11) we have

$$W_\varepsilon'(y) = -\left\{\frac{2}{\log \kappa_\varepsilon}\left[(1-\varepsilon^2)\left(\log(\kappa_\varepsilon W_\varepsilon + 1) + \frac{1}{\kappa_\varepsilon W_\varepsilon + 1} - 1\right) + \frac{\varepsilon^2\kappa_\varepsilon^2}{2}W_\varepsilon^2 - (\lambda_\varepsilon \kappa_\varepsilon W_\varepsilon + A_\varepsilon)\right]\right\}^{1/2}.$$

Using $\{W_\varepsilon\}$ converges to W_0 in $W^{1,p}_{loc}(0,\infty)$ and the above estimates together with (3.7) and (3.8), we find that the limiting function W_0 should satisfies the equation

$$W' = -\left(2\chi_{\mathrm{supp}(W)} - 2a\,W\right)^{1/2} \tag{3.12}$$

on the half real line $(0,\infty)$. The solution of this equation is

$$W_0(y) = \begin{cases} \frac{1}{a} - \frac{ay^2}{2} & \text{for } 0 < y < \frac{\sqrt{2}}{a}, \\ 0 & \text{for } \frac{\sqrt{2}}{a} \le y. \end{cases} \tag{3.13}$$

Here, we set a to be a constant such that $a = \lim_{j\to\infty} \frac{\kappa_{\varepsilon_j}}{\mu_{\varepsilon_j}}$. The exact value of the constant a will be determined later.

3.3. L^1 convergence

Although the mass constraint (3.10) holds for any $\varepsilon > 0$, it is not guaranteed that $M = \int_0^\infty W_0(x)\,dx$ holds. The reason is that the convergence $W_{\varepsilon_j}(x)$ to $W_0(x)$ is only locally uniform in $C^{0,\alpha}$ for $0 \le \alpha < 1$. We compute the limit $\int_0^{\kappa_{\varepsilon_j}} W_{\varepsilon_j}(x)dx$ below. Apply the change of the independent valuable x by $z = w_\varepsilon(x)$, we have

$$M = \int_0^1 w_\varepsilon(x)\,dx = \int_{\gamma_\varepsilon}^{\mu_\varepsilon} \frac{\varepsilon}{\sqrt{2}} \frac{z}{\sqrt{\Phi_\varepsilon(z)}}\,dz,$$

where we put

$$\Phi_\varepsilon(z) := (1-\varepsilon^2)G(z) + \frac{\varepsilon^2}{2}z^2 - \lambda_\varepsilon z - A_\varepsilon, \quad \gamma_\varepsilon := w_\varepsilon(1)$$

(recall $\mu_\varepsilon = w_\varepsilon(0)$). We note

$$\Phi_\varepsilon(\gamma_\varepsilon) = \Phi(\mu_\varepsilon) = 0.$$

Moreover, $G(x)$ is a monotone increasing and

$$G''(z) = g'(z) > 0 \quad (0 \le z < 1), \qquad G''(z) < 0 \quad (1 < z).$$

Hence,

$$\frac{G(z) - G(\gamma_\varepsilon)}{z - \gamma_\varepsilon} \;\; (z \in (\gamma_\varepsilon, 1]), \qquad \frac{G(\mu_\varepsilon) - G(z)}{\mu_\varepsilon - z} \;\; (z \in [1, \mu_\varepsilon))$$

are monotone increasing and decreasing respectively.

We first show

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}^{1} \frac{\varepsilon}{\sqrt{2}}\frac{z}{\sqrt{\Phi_\varepsilon(z)}}dz = 0. \tag{3.14}$$

By $\Phi_\varepsilon(\gamma_\varepsilon) = 0$, we write

$$\begin{aligned}\Phi_\varepsilon(z) &= (1-\varepsilon^2)(G(z)-G(\gamma_\varepsilon)) + \frac{\varepsilon^2}{2}(z^2-\gamma_\varepsilon^2) - \lambda_\varepsilon(z-\gamma_\varepsilon)\\ &= (z-\gamma_\varepsilon)h_\varepsilon(z),\\ h_\varepsilon(z) &:= (1-\varepsilon^2)\frac{G(z)-G(\gamma_\varepsilon)}{z-\gamma_\varepsilon} + \frac{\varepsilon^2}{2}(z+\gamma_\varepsilon) - \lambda_\varepsilon,\end{aligned}$$

where $h_\varepsilon(z)$ can be continuously extended up to $z = \gamma_\varepsilon$ as $h_\varepsilon(\gamma_\varepsilon) = (1-\varepsilon^2)g(\gamma_\varepsilon) + \varepsilon^2\gamma_\varepsilon - \lambda_\varepsilon$. Since $h_\varepsilon(z)$ is monotone increasing in $[\gamma_\varepsilon, 1]$, we have

$$\int_{\gamma_\varepsilon}^{1} \frac{\varepsilon}{\sqrt{2}}\frac{z}{\sqrt{\Phi_\varepsilon(z)}}dz \le \frac{\varepsilon}{\sqrt{2}}\frac{1}{\sqrt{h_\varepsilon(\gamma_\varepsilon)}}\int_{\gamma_\varepsilon}^{1} \frac{z}{\sqrt{z-\gamma_\varepsilon}}dz.$$

Invoking of $\gamma_\varepsilon = O(\sqrt{\log\kappa_\varepsilon/\kappa_\varepsilon})$, we can conclude (3.14).

We next deal with the integral over the range $[1, \mu_\varepsilon]$. Take θ_ε such that

$$\lim_{\varepsilon\to 0}\theta_\varepsilon = 0, \qquad \lim_{\varepsilon\to 0}\kappa_\varepsilon^{\theta_\varepsilon} = \infty,$$

and define

$$\beta_\varepsilon := \kappa_\varepsilon^{1-\theta_\varepsilon}. \tag{3.15}$$

For instance take $\theta_\varepsilon = 1/\sqrt{\log\kappa_\varepsilon}$. We separate the integral as

$$\begin{aligned}&\int_{1}^{\mu_\varepsilon} \frac{\varepsilon}{\sqrt{2}}\frac{z}{\sqrt{\Phi_\varepsilon(z)}}dz = I_\varepsilon + J_\varepsilon,\\ &I_\varepsilon := \int_{1}^{\beta_\varepsilon} \frac{\varepsilon}{\sqrt{2}}\frac{z}{\sqrt{\Phi_\varepsilon(z)}}dz, \quad J_\varepsilon := \int_{\beta_\varepsilon}^{\mu_\varepsilon} \frac{\varepsilon}{\sqrt{2}}\frac{z}{\sqrt{\Phi_\varepsilon(z)}}dz.\end{aligned}$$

We prove $\lim_{\varepsilon\to 0} I_\varepsilon = 0$. By the change of the valuable $z = \kappa_\varepsilon\zeta$ in the integral we obtain

$$I_\varepsilon = \int_{1/\kappa_\varepsilon}^{\beta_\varepsilon/\kappa_\varepsilon} \frac{1}{\sqrt{2}}\frac{\varepsilon\kappa_\varepsilon^2\zeta}{\sqrt{\Phi_\varepsilon(\kappa_\varepsilon\zeta)}}d\zeta = \int_{1/\kappa_\varepsilon}^{1/\kappa_\varepsilon^{\theta_\varepsilon}} \frac{1}{\sqrt{2}}\frac{\zeta}{\sqrt{\Phi_\varepsilon(\kappa_\varepsilon\zeta)/\log\kappa_\varepsilon}}d\zeta,$$

where we used $\varepsilon\kappa_\varepsilon^2 = \sqrt{\log\kappa_\varepsilon}$. Recall $\Phi_\varepsilon(\mu_\varepsilon) = 0$ and put $\alpha_\varepsilon := \mu_\varepsilon/\kappa_\varepsilon$. Then

$$\begin{aligned}
&\Phi_\varepsilon(\kappa_\varepsilon\zeta) = (\alpha_\varepsilon - \zeta)q_\varepsilon(\zeta),\\
&q_\varepsilon(\zeta) := -(1-\varepsilon^2)\frac{G(\kappa_\varepsilon\alpha_\varepsilon) - G(\kappa_\varepsilon\zeta)}{\alpha_\varepsilon - \zeta} - \frac{\varepsilon^2\kappa_\varepsilon^2}{2}(\zeta + \alpha_\varepsilon) + \lambda_\varepsilon\kappa_\varepsilon
\end{aligned}$$

For $\zeta \in [1/\kappa_\varepsilon, \beta_\varepsilon/\kappa_\varepsilon] = [1/\kappa_\varepsilon, 1/\kappa^{\theta_\varepsilon}]$,

$$q_\varepsilon(\zeta) \geq -(1-\varepsilon^2)\frac{G(\kappa_\varepsilon\alpha_\varepsilon) - G(1)}{\alpha_\varepsilon - 1/\kappa_\varepsilon} - \frac{\varepsilon^2\kappa_\varepsilon^2}{2}(1/\kappa^{\theta_\varepsilon} + \alpha_\varepsilon) + \lambda_\varepsilon\kappa_\varepsilon. \tag{3.16}$$

As seen in the proof of Lemma 3.1, we have

$$\lambda_\varepsilon = \frac{(1-\varepsilon^2)(G(\mu_\varepsilon) - G(\gamma_\varepsilon)) + \frac{\varepsilon^2}{2}\mu_\varepsilon^2 - \frac{\varepsilon^2}{2}\gamma_\varepsilon^2}{\mu_\varepsilon - \gamma_\varepsilon}.$$

Applying this equality to the right hand side of (3.16), we assert that there is $c_1 > 0$ such that

$$q_\varepsilon(\zeta)/\log\kappa_\varepsilon \geq c_1/\log\kappa_\varepsilon$$

holds for every small $\varepsilon > 0$. Indeed, we can compute the leading terms of the right hand side of (3.16) as

$$\begin{aligned}
&\kappa_\varepsilon\frac{G(\mu_\varepsilon) - G(\gamma_\varepsilon)}{\mu_\varepsilon - \gamma_\varepsilon} - \frac{G(\kappa_\varepsilon\alpha_\varepsilon) - G(1)}{\alpha_\varepsilon - 1/\kappa_\varepsilon} = \kappa_\varepsilon\left\{\frac{G(\mu_\varepsilon) - G(\gamma_\varepsilon)}{\mu_\varepsilon - \gamma_\varepsilon} - \frac{G(\mu_\varepsilon) - G(1)}{\mu_\varepsilon - 1}\right\}\\
&= \frac{\kappa_\varepsilon}{(\mu_\varepsilon - \gamma_\varepsilon)(\mu_\varepsilon - 1)}\{-G(\mu_\varepsilon)(1-\gamma_\varepsilon) + (G(1) - G(\gamma_\varepsilon))\mu_\varepsilon + G(\gamma_\varepsilon) - G(1)\gamma_\varepsilon\}.
\end{aligned}$$

This term is bounded away from zero as $\varepsilon \to 0$ because of $\kappa_\varepsilon/\mu_\varepsilon = O(1)$, $G(\mu_\varepsilon)/\mu_\varepsilon = O(\log\kappa_\varepsilon/\kappa_\varepsilon)$.

We compute

$$\begin{aligned}
&\int_{1/\kappa_\varepsilon}^{1/\kappa_\varepsilon^{\theta_\varepsilon}} \frac{\zeta}{\sqrt{\alpha_\varepsilon - \zeta}}d\zeta\\
&= \frac{2}{3}\{(\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon})^{3/2} - (\alpha_\varepsilon - 1/\kappa_\varepsilon)^{3/2}\} - 2\alpha_\varepsilon\left(\sqrt{\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}} - \sqrt{\alpha_\varepsilon - 1/\kappa_\varepsilon}\right)\\
&\leq \frac{2\alpha_\varepsilon(1/\kappa_\varepsilon^{\theta_\varepsilon} - 1/\kappa_\varepsilon)}{\sqrt{\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}} + \sqrt{\alpha_\varepsilon - 1/\kappa_\varepsilon}} \leq \frac{\alpha_\varepsilon/\kappa_\varepsilon^{\theta_\varepsilon}}{\sqrt{\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}}}.
\end{aligned}$$

Thus,

$$I_\varepsilon \le \frac{\sqrt{\log \kappa_\varepsilon}}{\sqrt{2c_1}} \int_{1/\kappa_\varepsilon}^{1/\kappa_\varepsilon^{\theta_\varepsilon}} \frac{\zeta}{\sqrt{\alpha_\varepsilon - \zeta}} d\zeta = O(\sqrt{\log \kappa_\varepsilon}/\kappa_\varepsilon^{\theta_\varepsilon}).$$

Putting $\ell_\varepsilon := \sqrt{\log \kappa_\varepsilon}$ and $\theta_\varepsilon = \ell_\varepsilon^{-1}$, we have

$$\frac{\sqrt{\log \kappa_\varepsilon}}{\kappa_\varepsilon^{\theta_\varepsilon}} = \frac{\ell_\varepsilon}{(e^{\ell_\varepsilon^2})^{\ell_\varepsilon^{-1}}} = \frac{\ell_\varepsilon}{e^{\ell_\varepsilon}}.$$

This implies that $I_\varepsilon \to 0$ as $\varepsilon \to 0$.

Finally, by taking a sequence we compute the limit of J_ε. With the same change of variable employed above, we have

$$J_\varepsilon = \int_{1/\kappa_\varepsilon^{\theta_\varepsilon}}^{\alpha_\varepsilon} \frac{1}{\sqrt{2}} \frac{\zeta}{\sqrt{\alpha - \zeta}\sqrt{q_\varepsilon(\zeta)/\log \kappa_\varepsilon}} d\zeta.$$

Taking a subsequence, we yield the limits

$$\lim_{\varepsilon_j \to 0} \frac{\lambda_{\varepsilon_j}\kappa_{\varepsilon_j}}{\log \kappa_{\varepsilon_j}} = a, \qquad \lim_{\varepsilon_j \to 0} \alpha_{\varepsilon_j} =: \tilde{\alpha}.$$

For $\zeta \in [1/\kappa^{\theta_\varepsilon}, \alpha_\varepsilon]$,

$$\kappa_\varepsilon g(\mu_\varepsilon) = \kappa_\varepsilon G'(\kappa_\varepsilon \alpha_\varepsilon) \le \frac{G(\kappa_\varepsilon\alpha_\varepsilon) - G(\kappa_\varepsilon \zeta)}{\alpha_\varepsilon - \zeta} \le \frac{G(\kappa_\varepsilon\alpha_\varepsilon) - G(\kappa^{1-\theta_\varepsilon})}{\alpha_\varepsilon - 1/\kappa^{\theta_\varepsilon}}.$$

We write

$$G(\kappa_\varepsilon\alpha_\varepsilon) - G(\kappa_\varepsilon^{1-\theta_\varepsilon}) = \log \frac{\alpha_\varepsilon\kappa_\varepsilon + 1}{\kappa_\varepsilon^{1-\theta_\varepsilon} + 1} + \frac{\kappa_\varepsilon^{1-\theta_\varepsilon} - \kappa_\varepsilon\alpha_\varepsilon}{(\kappa_\varepsilon\alpha_\varepsilon + 1)(\kappa_\varepsilon^{1-\theta_\varepsilon} + 1)}.$$

By the next estimate

$$\begin{aligned} \frac{1}{\log \kappa_\varepsilon} \log \frac{\alpha_\varepsilon\kappa_\varepsilon + 1}{\kappa_\varepsilon^{1-\theta_\varepsilon} + 1} &= \frac{1}{\log \kappa_\varepsilon} \log \frac{\alpha_\varepsilon \kappa_\varepsilon^{\theta_\varepsilon} + 1/\kappa_\varepsilon^{1-\theta_\varepsilon}}{1 + 1/\kappa_\varepsilon^{1-\theta_\varepsilon}} \\ &\le \frac{1}{\log \kappa_\varepsilon} \log(\alpha_\varepsilon \kappa_\varepsilon^{\theta_\varepsilon} + 1) = \theta_\varepsilon + \frac{1}{\log \kappa_\varepsilon} \log(\alpha_\varepsilon + 1/\kappa_\varepsilon^{\theta_\varepsilon}), \end{aligned}$$

we obtain

$$\lim_{\varepsilon_j \to 0} \frac{1}{\log \kappa_{\varepsilon_j}} q_{\varepsilon_j}(\zeta) = a. \tag{3.17}$$

The interval J_ε is taken on the variable interval $[1/\kappa_\varepsilon^{\theta_\varepsilon}, \alpha_\varepsilon]$. Introducing the new variable $\zeta = 1/\kappa_\varepsilon^{\theta_\varepsilon} + (\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}) \sin^2 \phi$, $(0 \le \phi \le \pi/2)$, we write

$$J_\varepsilon = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{(1/\kappa_\varepsilon^{\theta_\varepsilon} + (\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}) \sin^2\phi) \cdot 2(\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}) \sin\phi \cos\phi \, d\phi}{\sqrt{(\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon})(1-\sin^2\phi)} \sqrt{q_\varepsilon (1/\kappa_\varepsilon^{\theta_\varepsilon} + (\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}) \sin^2\phi)/\log\kappa_\varepsilon}}$$

$$= \sqrt{2(\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon})} \int_0^{\pi/2} \frac{(1/\kappa_\varepsilon^{\theta_\varepsilon} + (\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}) \sin^2\phi) \sin\phi \, d\phi}{\sqrt{q_\varepsilon (1/\kappa_\varepsilon^{\theta_\varepsilon} + (\alpha_\varepsilon - 1/\kappa_\varepsilon^{\theta_\varepsilon}) \sin^2\phi)/\log\kappa_\varepsilon}}.$$

With the aid of (3.17)

$$\lim_{\varepsilon_j \to 0} J_{\varepsilon_j} = \sqrt{2\tilde{\alpha}} \int_0^{\pi/2} \frac{\tilde{\alpha} \sin^3\phi \, d\phi}{\sqrt{a}} = \frac{\sqrt{2}\tilde{\alpha}\sqrt{\tilde{\alpha}}}{\sqrt{a}} \frac{2}{3} = \frac{2\sqrt{2}}{3} \frac{\tilde{\alpha}\sqrt{\tilde{\alpha}}}{\sqrt{a}}.$$

Combining the above computation yields

$$M = \frac{2\sqrt{2}}{3\sqrt{a}} \tilde{\alpha}\sqrt{\tilde{\alpha}}.$$

Using the limiting equation (3.12) and solution (3.13), we know $\tilde{\alpha} = 1/a$. Thus, we obtain

$$M = \frac{2\sqrt{2}}{3a^2}.$$

Since the constant a is independent of the choice of the subsequence $\{\varepsilon_j\}$, the whole sequence $\{W_\varepsilon\}$ converges to W_0 in $C^{0,\alpha}_{loc}$ for $0 \le \alpha < 1$ and $\lim_{\varepsilon\to 0} \frac{\kappa_\varepsilon}{\mu_\varepsilon} = a$. This completes the proof of Theorem 1.1.

3.4. *Boundedness of the support of W_0*

We have proved the convergence of $W_{\varepsilon_j}(y)$ to $W_0(y)$. It, however, has been not discussed that $W_0(y)$ has a bounded support. Here we prove it. For the solution $w_\varepsilon(x)$ we have the expression as

$$x = \int_w^{\mu_\varepsilon} \frac{\varepsilon}{\sqrt{2}} \frac{dz}{\sqrt{\Phi_\varepsilon(z)}},$$

namely the inverse function of the right hand side gives the solution. Taking the change of variables $z = \kappa_\varepsilon \zeta$, $y = \kappa_\varepsilon x$ and $w = \kappa_\varepsilon W$, we obtain

$$x = y/\kappa_\varepsilon = \int_W^{\mu_\varepsilon/\kappa_\varepsilon} \frac{1}{\sqrt{2}} \frac{\varepsilon\kappa_\varepsilon d\zeta}{\sqrt{\Phi_\varepsilon(\kappa\zeta)}},$$

hence,

$$y = \int_W^{\mu_\varepsilon/\kappa_\varepsilon} \frac{1}{\sqrt{2}} \frac{d\zeta}{\sqrt{\Phi_\varepsilon(\kappa_\varepsilon \zeta)/\log \kappa_\varepsilon}}$$

follows and the solution $W_\varepsilon(y)$ is provided by the inverse function of this expression. By choosing β_ε of (3.15), we show

$$x_\varepsilon := \int_{\beta_\varepsilon/\kappa_\varepsilon}^{\mu_\varepsilon/\kappa_\varepsilon} \frac{1}{\sqrt{2}} \frac{d\zeta}{\sqrt{\Phi_\varepsilon(\kappa_\varepsilon \zeta)/\log \kappa_\varepsilon}} \tag{3.18}$$

is uniformly bounded in ε. In view of

$$\lim_{\varepsilon \to 0} \beta_\varepsilon/\kappa_\varepsilon = \lim_{\varepsilon \to 0} 1/\kappa_\varepsilon^{\theta_\varepsilon} = 0,$$

we can assert that $W_{\varepsilon_j}(y) \to 0$ as $\varepsilon_j \to 0$ outside a finite interval.

We utilize the similar argument for J_ε as in the previous section together with (3.17) and show that the right hand side of (3.18) is a bounded for sufficiently small ε. Since the estimate is quite similar, we left the details to the readers.

Consequently, we can conclude the limiting function $W_0(y)$ has a bounded support.

4. Proof of Theorem 1.3

In this section we prove Theorem 1.3 by taking a dynamical system approach. Recall the assumption $M < 1$. We consider the evolution equation

$$\begin{aligned} &w_t = \varepsilon^2 w_{xx} - f_1(w) + \langle f_1(w) \rangle \ \ (0 < x < 1), \\ &w_x = 0 \ \ (x = 0, 1), \end{aligned} \tag{4.1}$$

where

$$f_1(w) := (1 - \varepsilon^2) g(w) + \varepsilon^2 w.$$

For w_0 in

$$Y := \{\phi \in H^2(0,1) : \phi_x(0) = \phi_x(1) = 0, \phi \geq 0, \int_0^1 \phi(x)\, dx = M\},$$

let $w(x, t; w_0)$ be a unique solution of (4.1) with $w(\cdot, 0; w_0) = w_0$. Then the solution generates a smooth semiflow $\{S(t)\}_{t \geq 0}$ defined by $S(t) w_0 := w(\cdot, t; u_0)$ in Y.

First note that the constant solution $\overline{w} = M$ is asymptotically stable equilibrium of this semiflow. The set

$$\mathcal{B} := \{\phi \in Y : \lim_{t \to \infty} S(t)\phi = \overline{w}\},$$

therefore, is positively invariant under the semiflow and open in Y with respect to the $H^2(0,1)$ norm since $0 \notin \mathcal{B}$ and the continuity of the initial data of the solution. Put $\eta := \mathcal{E}_\varepsilon(\overline{w})$ and define

$$\mathcal{W} := \{\phi \in Y : \lim_{t\to\infty} \mathcal{E}_\varepsilon(S(t)\phi) < \eta/2\}.$$

Because of Lemma 2.1, the set $\mathcal{W}$ is non-empty for sufficiently small ε. This set is also positively invariant and open. We define

$$\mathcal{C} := \{\phi \in Y : \phi_x(x) \le 0 \ (0 \le x \le 1)\}.$$

Lemma 4.1. *$\mathcal{C}$ is positively invariant under the follow $S(t)$.*

Proof. Let $U(\cdot,t) = [S(t)\phi](\cdot)$ $(\phi \in \mathcal{C})$. Then $V = U_x(x,t)$ satisfies

$$\begin{aligned} &V_t = \varepsilon^2 V_{xx} - f_1'(U)V \qquad (0 < x < 1), \\ &V(x,t) = 0 \qquad (x = 0, 1), \quad t \ge 0, \end{aligned}$$

and $V(x,0) = \phi_x(x) \le 0$ $(0 \le x \le 1)$. By the maximum principle for the parabolic equations we see

$$V(x,t) \le 0 \quad (0 \le x \le 1), \quad \forall t > 0.$$

This gives the desired assertion. □

Define the sets

$$\mathcal{B}_c := \mathcal{B} \cap \mathcal{C}, \qquad \mathcal{W}_c := \mathcal{W} \cap \mathcal{C}.$$

Then both sets nonempty and open in the relative topology. They are also invariant for $S(t)$. Moreover, nonincreasing of $\mathcal{E}_\varepsilon(S(t)\phi)$ in t implies $\mathcal{B}_c \cap \mathcal{W}_c = \emptyset$. Thus $\mathcal{F} := (Y \cap \mathcal{C}) \setminus (\mathcal{B}_c \cup \mathcal{W}_c)$ is positively invariant, closed and non-empty.

Consider the semiflow restricted on $\mathcal{F}$, denoted by $\tilde{S}(t)$. Since $\tilde{S}(t)$ is a gradient flow and compact, there is a maximal compact invariant set $\mathcal{A}_\mathcal{F}$ in $\mathcal{F}$. We let $E_\mathcal{F}$ be the set of all the equilibrium solutions in $\mathcal{F}$. Then

$$\mathcal{A}_\mathcal{F} := \overline{W^u(E_\mathcal{F})}, \quad W^u(E_\mathcal{F}) := \{\phi \in \mathcal{F} : \lim_{j\to\infty} \tilde{S}(t_j)\phi \in E_\mathcal{F}, \ \exists\{t_j\} \ s.t. \ \lim_{j\to\infty} t_j = -\infty\}$$

(see [4]). We note that any equilibrium cannot be a constant steady state, because there is no constant solution except for $\overline{w} = M$. There are two cases, $\mathcal{A}_\mathcal{F}$ consists of only stable equilibria or it contains at least one unstable equilibria. It suffices to prove that the former case, namely $\mathcal{A}_\mathcal{F} = E_\mathcal{F}$, allows an unstable equilibrium under the flow $S(t)$ though it is stable under $\tilde{S}(t)$. We note that $E_\mathcal{F}$ might be a continuum of equilibria. If an arbitrarily chosen open neighborhood of $E_\mathcal{F}$ in $\mathcal{F}$ intersects $\mathcal{B}_c \cup \mathcal{W}_c$, then it is unstable. Suppose that there is an open neighborhood $\mathcal{U} \subset \mathcal{F}$ of $E_\mathcal{F}$ such that $\mathcal{U} \cap (\mathcal{B}_c \cup \mathcal{W}_c) = \emptyset$. We show a contradiction. Take a point

$$\phi_b \in \overline{\mathcal{B}_c \cup \mathcal{W}_c} \setminus \mathcal{B}_c \cup \mathcal{W}_c.$$

Since $\phi_b \in \mathcal{F}$ and $E_{\mathcal{F}} = \mathcal{A}_{\mathcal{F}}$, there is $\tau_1 > 0$ such that $S(t)\phi_b \in \mathcal{U}$. Taking a sequence $\{\phi_k\}_{k=1,2,...} \subset \mathcal{B}_c \cup \mathcal{W}_c$ such that $\phi_k \to \phi_b$ as $k \to \infty$. Then there is $k = k_1$ such that $S(\tau_1)\phi_{k_1} \in \mathcal{U}$ by the continuity of the flow. This contradicts that any element of $\mathcal{U}$ converges to $\mathcal{A}_{\mathcal{F}}$. In conclusion there is an unstable equilibria in $\mathcal{F}$. □

Remark 4.1. In the above proof we don't exclude the possibility of a stable solution in $\mathcal{F}$. Indeed, in the case that $\mathcal{A}_{\mathcal{F}}$ contains at least one unstable equilibria, there might be a solution which is stable under the flow not only $\tilde{S}(t)$ but also $S(t)$.

Proof of Corollary 1.4. We prove that the unstable equilibrium solution $\tilde{w}_\varepsilon$ of (4.1) leads to an unstable equilibrium solution $(u, v) = (u^*, v^*)$ of the system (1.1)–(1.2) with $d = \varepsilon^2$ and $\Omega = (0, 1)$, where u^*, v^* are defined by $w^* = \tilde{w}_\varepsilon$ in (1.9). We set $\tilde{z}_\varepsilon = du^* + v^*$. If the second variational satisfies

$$\inf_{\varphi\in\mathcal{A}_0} \frac{d}{ds}\mathcal{E}_\varepsilon(\tilde{w}_\varepsilon + s\varphi)_{|s=0} = \inf_{\varphi\in\mathcal{A}_0} \int_0^1 [\varepsilon^2\varphi_x^2 + (1-\varepsilon^2)g'(\tilde{w}_\varepsilon)\varphi^2 + \varepsilon^2\varphi^2]dx < 0,$$

$$\mathcal{A}_0 := \{\varphi \in H^1(0, 1) : \langle\varphi\rangle = 0\},$$

then the spectral comparison in [8] tells that $(u, v) = (u^*, v^*)$ is unstable in (1.1)–(1.2). There is a possibility that the infimum vanishes. Here, we provide another proof which covers both cases. First note that with respect to the variables $w = u + v$, $z = \varepsilon^2 u + v$ we have the Lyapunov function for (1.1)–(1.2) with ε^2 and $\Omega = (0, 1)$, which is written as

$$\mathcal{L}_\varepsilon(w, z) := \int_\Omega [\frac{\varepsilon^2}{2}w_x^2 + (1-\varepsilon^2)G(w) + \frac{\varepsilon^2}{2}w^2 + \frac{1}{2}z_x^2]dx.$$

For any equilibrium solution (u^*, v^*), we see that $z^* = du^* + v^*$ is constant, from which $\mathcal{E}_\varepsilon(w^*) = \mathcal{L}_\varepsilon(w^*, z^*)$ follows. Set

$$e_1 := \mathcal{E}_\varepsilon(\tilde{w}_\varepsilon), \quad e_2 := \sup\{\mathcal{E}_\varepsilon(w^*) : w^* \in E_S, \quad \mathcal{E}_\varepsilon(w^*) < e_1\},$$

where E_S is the set of all the equilibrium solutions for the semiflow $S(t)$ starting from an open ball $B(\tilde{w}_\varepsilon, \sigma)$ at $\tilde{w}_\varepsilon$ with a radius σ which will be determined in the proof of the following claim. Now, we claim that the strict inequality $e_2 < e_1$ holds and its proof will be given later. Since $\tilde{w}_\varepsilon$ is unstable, there is a solution $w(x, t)$ of (4.1) and a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = -\infty$ and

$$\lim_{n\to\infty} w(\cdot, t_n) = \tilde{w}_\varepsilon, \quad \mathcal{E}_\varepsilon(w(\cdot, t_1)) < \mathcal{E}_\varepsilon(w(\cdot, t_2)) < \cdots < \mathcal{E}_\varepsilon(\tilde{w}_\varepsilon) = e_1$$

hold. We denote by $(W(x, t), Z(x, t)) = (u(x, t) + v(x, t), \varepsilon^2 u(x, t) + v(x, t))$ for the solution (1.1)–(1.2) with $d = \varepsilon^2$ and $\Omega = (0, 1)$. Consider the solution $(W_n(x, t), Z_n(x, t))$ satisfying $(W_n(x, t_n), Z_n(x, t_n)) = (w(x, t_n), c_n)$, where $\{c_n\}$ is a sequence of numbers satisfying $\lim_{n\to\infty} c_n = \tilde{z}_\varepsilon$. Then

$$\frac{d}{dt}\mathcal{L}_\varepsilon(W_n(\cdot,t),Z_n(\cdot,t)) = -(1+\varepsilon^2)\int_0^1\left(\frac{\partial W_n}{\partial t}\right)^2 - \int_0^1\left(\frac{\partial Z_n}{\partial x}\right)^2 dx \le 0$$

holds. Recall that the omega-limit set of any solution $(W(x,t), Z(x,t))$ belongs to the set of all the equilibrium solutions. Therefore, by $\mathcal{L}_\varepsilon(w(\cdot,t_n),c_n) = \mathcal{E}_\varepsilon(w(\cdot,t_n))$, we obtain

$$\limsup_{t\to\infty}\mathcal{L}_\varepsilon(W_n(\cdot,t),Z_n(\cdot,t)) \le e_2,$$

from which there is T_n such that $\mathcal{L}_\varepsilon(W_n(\cdot,T_n),Z(\cdot,T_n)) = (\sqrt{e_1}+\sqrt{e_2})^2/4$. By using the triangle inequality and the fact that the derivative of the function $\sqrt{G(\cdot)}$ is bounded on the interval $[0,+\infty)$, there is a positive number r_1 which is independent of n such that

$$\begin{aligned}
\frac{\sqrt{e_1}-\sqrt{e_2}}{2} &= (\mathcal{E}_\varepsilon(\tilde{w}_\varepsilon))^{1/2} - (\mathcal{L}_\varepsilon(W_n(\cdot,T_n),Z_n(\cdot,T_n)))^{1/2} \\
&\le \left\{\int_0^1\left(\frac{\varepsilon^2}{2}|\nabla(\tilde{w}_\varepsilon - W(\cdot,T_n))|^2 + (1-\varepsilon^2)\left((G(\tilde{w}_\varepsilon))^{\frac12} - (G(W(\cdot,T_n)))^{\frac12}\right)^2\right.\right. \\
&\qquad \left.\left. + \frac{\varepsilon^2}{2}(\tilde{w}_\varepsilon - W(\cdot,T_n))^2 + \frac12|\nabla(\tilde{z}_\varepsilon - Z(\cdot,T_n))|^2\right)dx\right\}^{\frac12} \\
&\le r_1\|(\tilde{w}_\varepsilon,\tilde{z}_\varepsilon) - (W(\cdot,T_n),Z(\cdot,T_n))\|_{H^1(0,1)\times H^1(0,1)}.
\end{aligned}$$

This implies that for any n the solution $(W_n(\cdot,t),Z_n(\cdot,t))$ starting from $(w(\cdot,t_n),c_n)$ gets away from a neighborhood of $(\tilde{w}_\varepsilon,\tilde{z}_\varepsilon)$. Therefore we conclude that (u,v) is an unstable equilibrium solution.

Now we prove $e_2 < e_1$. We utilize the Łojasiewicz–Simon type inequality for the nonlocal gradient flow. Let $V := \{\phi \in H^1(0,1) : \int_0^1 \phi\,dx = 0\}$ and V^* be the dual space of V. We denote by $d\mathcal{E}_\varepsilon$ of the Fréchet derivative of $\mathcal{E}_\varepsilon$ on V. Let Ψ be a critical point of $\mathcal{E}_\varepsilon$, that is, $d\mathcal{E}_\varepsilon(\Psi) = 0$. Then there exist constant $\theta \in [1/2,1)$, $C > 0$ and $\sigma > 0$ such that

$$|\mathcal{E}_\varepsilon(w) - \mathcal{E}(\Psi)|^\theta \le C\|d\mathcal{E}_\varepsilon(w)\|_{V^*} \tag{4.2}$$

for all $\|w-\Psi\|_V \le \sigma$ (see [1] and [6]). Apply this inequality to $\Psi = \tilde{w}_\varepsilon$. For any critical point w_c of $\mathcal{E}_\varepsilon$ satisfying $\|w_c - \tilde{w}_\varepsilon\|_V \le \sigma$, the inequality (4.2) leads to $\mathcal{E}_\varepsilon(w_c) = \mathcal{E}(\tilde{w}_\varepsilon)$. This implies that $e_2 < e_1$ must hold. □

Acknowledgments

The authors would like to express their gratitude to the anonymous referees for helpful suggestions and comments to improve the exposition of the article.

References

[1] S. Boussaïd, D. Hilhorst, T.N. Nguyen, Convergence to steady state for the solutions of a nonlocal reaction–diffusion equation, Evol. Equ. and Control Theory 4 (2015) 39–59.

[2] J. Carr, M.E. Gurtin, M. Slemrod, Structured phase transitions on a finite interval, Arch. Ration. Mech. Anal. 86 (1984) 317–351.
[3] M.E. Gurtin, H. Matano, On the structure of equilibrium phase transitions within the gradient theory of fluids, Quart. Appl. Math. 156 (1988) 301–317.
[4] J.K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, American Mathematical Society, Providence, RI, 1988.
[5] M. Herrero, J. Velzquez, Singularity patterns in a chemotaxis model, Math. Ann. 306 (1996) 583–623.
[6] S.-Z. Huang, Gradient Inequalities: With Applications to Asymptotic Behavior and Stability of Gradient-Like Systems, Math. Surveys Monogr., vol. 126, American Mathematical Society, Providence, RI, 2006.
[7] S. Ishihara, M. Otsuji, A. Mochizuki, Transient and steady state of mass-conserved reaction-diffusion systems, Phys. Rev. E 75 (2007), 015203(R).
[8] S. Jimbo, Y. Morita, Lyapunov function and spectrum comparison for a reaction–diffusion system with mass conservation, J. Differential Equations 255 (2013) 1657–1683.
[9] E. Latos, Y. Morita, T. Suzuki, Stability and spectral comparison of a reaction-diffusion system with mass conservation, preprint.
[10] E. Latos, T. Suzuki, Global dynamics of a reaction–diffusion system with mass conservation, J. Math. Anal. Appl. 411 (2014) 107–118.
[11] T. Mori, K. Kuto, T. Tsujikawa, M. Nagayama, S. Yotsutani, Global bifurcation sheet and diagrams of wave-pinning in a reaction–diffusion model for cell polarization, in: Dynamical Systems, Differential Equations and Applications AIMS Proceedings, 2015, pp. 861–877.
[12] T. Mori, K. Kuto, T. Tsujikawa, S. Yotsutani, Exact multiplicity of stationary limiting problem of a cell polarization model, Discrete Contin. Dyn. Syst. Ser. A 36 (2016) 5627–5655.
[13] Y. Mori, Y. Jilkine, L. Edelstein-Keshet, Wave-pinning and cell polarity from bistable reaction–diffusion system, Biophys. J. 94 (2008) 3684–3697.
[14] Y. Mori, Y. Jilkine, L. Edelstein-Keshet, Asymptotic and bifurcation analysis of wave-pinning in a reaction–diffusion model for cell polarization, SIAM J. Appl. Math. 71 (2011) 1401–1427.
[15] Y. Morita, Spectrum comparison for a conserved reaction–diffusion system with a variational property, J. Comput. Anal. Appl. 2 (2012) 57–71.
[16] Y. Morita, T. Ogawa, Stability and bifurcation of nonconstant solutions to a reaction–diffusion system with conservation of mass, Nonlinearity 23 (2010) 1387–1411.
[17] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl. 5 (1995) 581–601.
[18] A. Novick-Cohen, On the viscous Chan–Hilliard equation, in: J.M. Ball (Ed.), Material Instabilities in Continuum Mechanics and Related Mathematical Problems, Clarendon Press, Oxford, 1988, pp. 329–342.
[19] M. Otsuji, S. Ishihara, C. Co, K. Kaibuchi, A. Mochizuki, S. Kuroda, A mass conserved reaction–diffusion system captures properties of cell polarity, PLoS Comput. Biol. 3 (2007) 1040–1054.
[20] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.
[21] E. Keller, L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (1970) 399–415.
[22] K. Pham, A. Chauviere, H. Hatzikirou, X. Li, H.M. Byrne, V. Cristini, J. Lowengrub, Density-dependent quiescence in glioma invasion: instability in a simple reaction–diffusion model for the migration/proliferation dichotomy, J. Biol. Dyn. 6 (2012) 54–71.
[23] T. Senba, T. Suzuki, Parabolic system of chemotaxis: blowup in a finite and the infinite time, Methods Appl. Anal. 8 (2001) 349–367.
[24] T. Suzuki, S. Tasaki, Stationary Fix–Caginalp equation with non-local term, Nonlinear Anal. 71 (2009) 1329–1349.
[25] X. Wang, Q. Xu, Spiky and transition layer steady states of chemotaxis systems via global bifurcation and Helly compactness theorem, J. Math. Biol. 66 (2013) 1241–1266.