The solution structure of the O(3) sigma model in a Maxwell-Chern-Simons theory

Sze-Guang Yang, Zhi-You Chen, and Jann-Long Chern

Citation: Journal of Mathematical Physics **58**, 071503 (2017); doi: 10.1063/1.4994060
View online: http://dx.doi.org/10.1063/1.4994060
View Table of Contents: http://aip.scitation.org/toc/jmp/58/7
Published by the American Institute of Physics

JOURNAL OF MATHEMATICAL PHYSICS **58**, 071503 (2017)

The solution structure of the O(3) sigma model in a Maxwell-Chern-Simons theory

Sze-Guang Yang,[1,a)] Zhi-You Chen,[2,b)] and Jann-Long Chern[1,c)]
[1]*Department of Mathematics, National Central University, Chung-Li 32001, Taiwan*
[2]*Department of Mathematics, National Changhua University of Education, Changhua 500, Taiwan*

(Received 24 December 2015; accepted 2 July 2017; published online 14 July 2017)

In this paper, a system of semilinear elliptic equations arising from a relativistic self-dual Maxwell-Chern-Simons $O(3)$ sigma model is considered. We reveal the uniqueness aspect of the topological solutions for the model. The uniqueness result is associated with a clear solution structure of the equations of the radially symmetric case. We locate each solution set denoted by a planar diagram. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4994060]

I. INTRODUCTION

In the paper, we consider the following system of semilinear elliptic equations which comes from the self-dual equations arising in the Maxwell-Chern-Simons gauged $O(3)$ sigma model:

$$\begin{cases} \Delta U = 2q\Big(-N + s - \dfrac{1-e^U}{1+e^U}\Big) + 4\pi d\,\delta_0, \\ \Delta N = -\kappa^2 q^2\Big(-N + s - \dfrac{1-e^U}{1+e^U}\Big) + 4q\dfrac{e^U}{(1+e^U)^2}N, \end{cases} \tag{1}$$

where U, N are unknown functions, δ_0 denotes the Dirac distribution concentrated at the origin, d is a positive integer, κ, q are positive constants, and s is a parameter with $0 \le s < 1$.

The Maxwell-Chern-Simons gauged $O(3)$ sigma model is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4q}F_{\alpha\beta}F^{\alpha\beta} + \frac{\kappa}{2}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma + \frac{1}{2}|D_\alpha\phi|^2 + \frac{1}{2q}(\partial_\alpha N)^2 - \mathcal{V}(\phi, N), \tag{2}$$

where $A_\alpha : \mathbb{R}^{1,2} \to \mathbb{R}^3$ $(\alpha = 0, 1, 2)$ is the gauge field, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the corresponding curvature tensor, $D_\alpha\phi = \partial_\alpha\phi + A_\alpha\mathbf{n}\times\phi$ is the gauge covariant derivative, $\mathbf{n} = (0, 0, 1)$, $\phi = (\phi_1\phi_2, \phi_3) : \mathbb{R}^{1,2} \to \mathbb{S}^2$ is a vector field, $N : \mathbb{R}^{1,2} \to \mathbb{R}$ is a scalar field, $q, \kappa > 0$ are parameters, $\varepsilon_{\alpha\beta\gamma}, \alpha, \beta, \gamma = 0, 1, 2$, is the totally skew-symmetric tensor with $\varepsilon_{012} = 1$, and $\mathcal{V}(\phi, N)$ is a potential given by

$$\mathcal{V}(\phi, N) = \frac{q}{2}(\kappa N + s - \mathbf{n}\cdot\phi)^2 + \frac{1}{2}N^2(\mathbf{n}\times\phi)^2,$$

in which $s \in \mathbb{R}$ is a constant. The Gauss equation (as a variational equation for A_0) of (2) is given by

$$\frac{1}{q}\partial_\alpha F_{0\alpha} + \kappa F_{12} + \mathbf{n}\cdot\phi\times D_0\phi = 0. \tag{3}$$

By virtue of (3), the static energy corresponding to (2) can be written as

a)E-mail: sgyang@math.ncu.edu.tw.
b)E-mail: zhiyou@cc.ncue.edu.tw.
c)Author to whom correspondence should be addressed: chern@math.ncu.edu.tw

$$
\begin{aligned}
E &= \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2q}(|\partial_j A_0|^2 + |F_{12}|^2) + \frac{1}{2}(|A_0|^2|\mathbf{n}\times\phi|^2 + |D_j\phi|^2) \right. \\
&\quad \left. + \frac{1}{2q}|\partial_j N|^2 + \mathcal{V}(\phi, N) \right\} \\
&= \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2q}|\partial_j A_0 \mp \partial N|^2 + \frac{1}{2q}|F_{12} \mp q(\kappa N + s - \mathbf{n}\cdot\phi)|^2 \right. \\
&\quad + \frac{1}{2}|\mathbf{n}\times\phi|^2|A_0 \mp N|^2 \Big\} + \frac{1}{2}|D_1\phi \pm \phi\times D_2\phi|^2 \\
&\quad \left. \pm \phi\cdot D_1\phi\times D_2\phi + (s - \mathbf{n}\cdot\phi)F_{12} \right\}.
\end{aligned}
$$

We choose the upper sign and thus obtain the lower energy bound

$$
E \geq \int_{\mathbb{R}^2} d^2x \left\{ \phi\cdot D_1\phi\times D_2\phi + (s - \mathbf{n}\cdot\phi)F_{12} \right\}.
$$

When the field configurations saturate the energy bound, we have the Bogomol'nyi equations

$$
\begin{cases}
A_0 = N, \\
F_{12} = q(\kappa N + s - \mathbf{n}\cdot\phi), \\
D_1\phi = -\phi\times D_2\phi.
\end{cases} \tag{4}
$$

Finite energy condition indicates the following boundary conditions:

$$
N \to 0, \qquad s - \mathbf{n}\cdot\phi \to 0, \qquad \text{as } |x| \to \infty, \tag{5}
$$

$$
|\mathbf{n}\times\phi| \to 0, \qquad \kappa N + s - \mathbf{n}\cdot\phi \to 0, \qquad \text{as } |x| \to \infty. \tag{6}
$$

Let $u = u_1 + \mathrm{i}u_2$ be the projection of ϕ onto the complex plane through the south pole $-\mathbf{n}$, which is given by

$$
u_1 = \frac{\phi_1}{1+\phi_3}, \qquad u_2 = \frac{\phi_2}{1+\phi_3}.
$$

Equation (4) is rewritten as

$$
\begin{cases}
A_0 = N, \\
F_{12} = q(\kappa N + s - \mathbf{n}\cdot\phi), \\
\overline{\partial} u = -\mathrm{i}\overline{\alpha} u,
\end{cases} \tag{7}
$$

where $\overline{\partial} = (\partial_1 + \mathrm{i}\,\partial_2)/2$ and $\alpha = (A_1 - \mathrm{i}A_2)/2$. Set $U = \log\phi$ and $V = -\kappa N$. Equation (7) is transformed into

$$
\begin{cases}
\Delta U = 2q\Big(-V + s - \dfrac{1-e^U}{1+e^U}\Big) + 4\pi\displaystyle\sum_{j=1}^{m}\delta_{p_j} - 4\pi\sum_{j=1}^{n}\delta_{q_j}, \\
\Delta V = -\kappa^2 q^2\Big(-V + s - \dfrac{1-e^U}{1+e^U}\Big) + 4q\dfrac{e^U}{(1+e^U)^2}V,
\end{cases} \tag{8}
$$

where δ_p is the Dirac distribution concentrated at $p \in \mathbb{R}^2$ and p_j, q_j are the locations of vortices and antivortices. Conditions (5) and (6) indicate three kinds of boundary conditions for solutions of (8) as follows:

(a) Topological: $U(x) \to \log\frac{1-s}{1+s}$ and $N(x) \to 0$, as $|x| \to \infty$.
(b) Non-topological I: $U(x) \to -\infty$ and $N(x) \to s - 1$, as $|x| \to \infty$.
(c) Non-topological II: $U(x) \to +\infty$ and $N(x) \to s + 1$, as $|x| \to \infty$.

If $m = d \geq 0, n = 0$ with $p_1 = \cdots = p_d = 0$, Eq. (1) is a special case of Eq. (8). For the details of the theory, we refer the readers to Refs. 11 and 13, for example.

Existence and asymptotic structure of those solutions of (8) which satisfy the topological boundary condition[2,3,10,15] have been studied in Refs. 9 and 11. Recently, there have been papers concerning the solution structure of the elliptic equations coming from the CS (Chern-Simons) gauge theory. For example, one may refer to Refs. 7 and 8 for the CS gauged $O(3)$ sigma model,[5] for the gravitational sigma model,[4,6] for the self-dual $U(1) \times U(1)$ two-particle system, and[12] for a perturbation of the $SU(3)$ Toda system. Classification of radially symmetric solutions is associated with the uniqueness of the topological solutions of the related models and can probably be an approach to realize more complicated solutions.

In this paper, we prove the uniqueness of the topological solutions and identify the solution sets of all solution types for the radially symmetric case of (1). We portray our main theorems as follows.

Theorem 1.1. *Equation (1) possesses exactly one solution satisfying the topological boundary condition for each* $d \geq 0$. *Moreover, the topological solutions continuously deform with respect to* d *in the sense that for any* $d_0 \geq 0$, *there exist a number* $\varepsilon > 0$ *and a continuous mapping* $\mathcal{P}: \rho \mapsto (U_\rho, N_\rho) \in C^0_{\mathrm{loc}}(\mathbb{R}^2) \times C^0_{\mathrm{loc}}(\mathbb{R}^2)$ *such that* U_ρ, N_ρ *solve Eq. (1) with* $d = \rho$ *and satisfy the topological boundary condition whenever* $\rho \in (d_0 - \varepsilon, d_0 + \varepsilon) \cap (0, \infty)$.

The radially symmetric case of Eq. (1) is characterized by

$$\begin{cases} U'' + r^{-1}U' = 2q\Big(-N + s - \dfrac{1 - e^U}{1 + e^U}\Big), \quad r > 0, \\ N'' + r^{-1}N' = -\kappa^2 q^2\Big(-N + s - \dfrac{1 - e^U}{1 + e^U}\Big) + 4q\dfrac{e^U}{(1 + e^U)^2}N, \quad r > 0, \\ U(r; c_1, c_2) = c_1 + 2d\log r + o(1), \quad \text{as } r \to 0, \\ N(r; c_1, c_2) = c_2 + o(1) \quad \text{as } r \to 0 \end{cases} \tag{9}$$

for $c_1, c_2 \in \mathbb{R}$. Define

$$\begin{aligned} E_I &= \{(c_1, c_2) : \quad U(r; c_1, c_2) \to \log\frac{1 - s}{1 + s}, \quad N(r; c_1, c_2) \to 0, \quad \text{as } r \to \infty\}, \\ E_{II} &= \{(c_1, c_2) : \quad U(r; c_1, c_2) \to -\infty, \quad N(r; c_1, c_2) \to s - 1, \quad \text{as } r \to \infty\}, \\ E_{III} &= \{(c_1, c_2) : \quad U(r; c_1, c_2) \to +\infty, \quad N(r; c_1, c_2) \to s + 1, \quad \text{as } r \to \infty\}, \\ E_{IV} &= \{(c_1, c_2) : \quad U(r; c_1, c_2) \to -\infty, \quad N(r; c_1, c_2) \to +\infty, \quad \text{as } r \to \infty\}, \\ E_V &= \{(c_1, c_2) : \quad U(r; c_1, c_2) \to +\infty, \quad N(r; c_1, c_2) \to -\infty, \quad \text{as } r \to \infty\}. \end{aligned}$$

Theorem 1.2. *Assume* $\kappa^2 q \geq 4$.*The sets* $E_I, E_{II}, E_{III}, E_{IV}, E_V$ *can be identified as follows.*

(i) $E_I \cup E_{II} \cup E_{III} \cup E_{IV} \cup E_V = \mathbb{R}^2$.
(ii) *There exist* $c^* \in \mathbb{R}$ *and a continuous function* $\Gamma : \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{aligned} E_I &= \{(c, \Gamma(c)) : c = c^*\}, \\ E_{II} &= \{(c, \Gamma(c)) : -\infty < c < c^*\}, \\ E_{III} &= \{(c, \Gamma(c)) : c^* < c < \infty\}, \\ E_{IV} &= \{(c_1, c_2) : c_1 \in \mathbb{R}, \ c_2 < \Gamma(c_1)\}, \\ E_V &= \{(c_1, c_2) : c_1 \in \mathbb{R}, \ c_2 > \Gamma(c_1)\}. \end{aligned}$$

Moreover, $\Gamma(c) \to s + 1$ *as* $c \to \infty$ *and* $\Gamma(c) \to s - 1$ *as* $c \to -\infty$ (please see Fig. 1 for an illustration).

Preliminaries. With the substitution

$$s = \frac{a - 1}{a + 1}, \qquad u = U + \log a, \qquad v = -\frac{2}{\kappa^2 q}N,$$

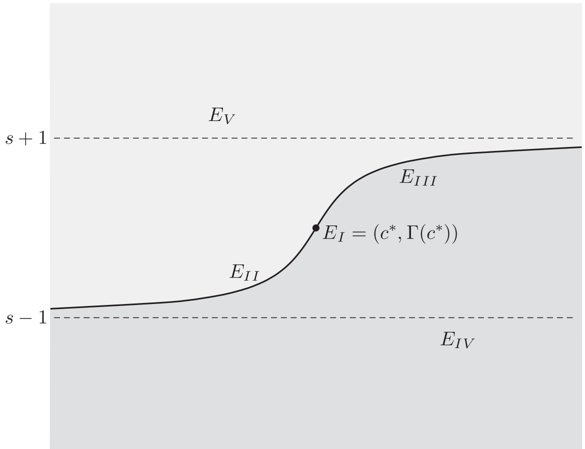

FIG. 1. The location of each solution set.

Equation (9) can be transformed into the system

$$\begin{cases} u'' + r^{-1}u' = 2q\Big[\lambda v + \dfrac{2a(e^u - 1)}{(a+1)(e^u + a)}\Big], & r > 0,\\ v'' + r^{-1}v' = 2q\Big[\lambda v + \dfrac{2a(e^u - 1)}{(a+1)(e^u + a)}\Big] + 4q\dfrac{ae^u}{(e^u + a)^2}v, & r > 0,\\ u(r) = \alpha_1 + 2d\log r + o(1), \quad \text{as } r \to 0, & \\ v(r) = \alpha_2 + o(1) \quad \text{as } r \to 0, \quad \alpha_1, \alpha_2 \in \mathbb{R}, & \end{cases} \tag{10}$$

where $a \geq 1$ (by $0 \leq s < 1$) and $\lambda = \kappa^2 q/2$. The topological and non-topological solutions boundary conditions associated with Eq. (10) are given by

$$\text{(Topological)} \qquad u(x) \to 0, v(x) \to 0 \quad \text{as } |x| \to \infty, \tag{11}$$

$$\text{(Non-topological I)} \qquad u(x) \to -\infty, v(x) \to \frac{2}{\lambda(a+1)} \quad \text{as } |x| \to \infty, \tag{12}$$

$$\text{(Non-topological II)} \qquad u(x) \to +\infty, v(x) \to \frac{-2a}{\lambda(a+1)} \quad \text{as } |x| \to \infty. \tag{13}$$

By the maximum principle, it is clear that the solutions satisfying either (11) or (12) have the following property:

$$u(x) < 0, \qquad v(x) > 0, \qquad \text{for all } x \in \mathbb{R}^2, \tag{14}$$

and thus, letting $w = \lambda v + \frac{2a(e^u-1)}{(a+1)(e^u+a)}$, we have

$$\Delta w - 2q\Big[\lambda + \frac{2ae^u}{(e^u + a)^3}\Big]w = 4q\lambda\frac{ae^u}{(e^u + a)^2}v + \frac{2ae^u(a - e^u)}{(e^u + a)^3}|\nabla u|^2 > 0.$$

So that

$$\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)} < 0 \tag{15}$$

for topological and non-topological I solutions. From (10), $\Delta v = 2q[f(e^u)v + g(e^u)]$, where

$$f(t) = \lambda + \frac{2at}{(t+a)^2}, \qquad g(t) = \Big(\frac{2a}{a+1}\Big)\Big(\frac{t-1}{t+a}\Big)$$

are bounded functions. Thus, the solutions of (10) are all entire solutions. Furthermore, by Lemma 2.3., u, v cannot oscillate infinitely many times. Especially, since $g(0) = -2/(a+1)$, $g(\infty) = 2a/(a+1)$, and $g' > 0$, it follows that

$$\begin{aligned} &v'(r) > 0, \quad \text{for all } r > 0, \text{provided } v(0) > 2/\lambda(a+1), \\ &v'(r) < 0, \quad \text{for all } r > 0, \text{provided } v(0) < -2a/\lambda(a+1). \end{aligned} \tag{16}$$

By the way, from Lemma 2.1. and Lemma 2.2., we have

$$-2a/\lambda(a+1) \le v \le 2/\lambda(a+1) \tag{17}$$

whenever u, v satisfy one of the boundary conditions (11)–(13). It is not difficult to observe that all the possible solution types are included as follows:

$$\begin{array}{lllll} \text{Type I:} & u(r) \to 0 & \text{and} & v(r) \to 0 & \text{as } r \to \infty, \\ \text{type II:} & u(r) \to -\infty & \text{and} & v(r) \to 2/\lambda(a+1) & \text{as } r \to \infty, \\ \text{type III:} & u(r) \to +\infty & \text{and} & v(r) \to -2a/\lambda(a+1) & \text{as } r \to \infty, \\ \text{type IV:} & u(r) \to -\infty & \text{and} & v(r) \to -\infty & \text{as } r \to \infty, \\ \text{type V:} & u(r) \to +\infty & \text{and} & v(r) \to +\infty & \text{as } r \to \infty. \end{array}$$

Let $\beta = (1/2\pi) \int_{\mathbb{R}^2} F_{12}$, where F_{12} is given in (7). We have

$$\beta = \beta(u, v) = \int_0^\infty -2q\Big[\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)}\Big] r\, dr.$$

Assume $|\beta| < \infty$. Clearly, $u(r) = (2d - \beta)\log r + O(1)$ as $r \to \infty$. Since $v(r) \to$ constant, we have $rv'(r) \to 0$ as $r \to \infty$; thus, β has an alternative expression

$$\beta = \int_0^\infty 4q \frac{ae^u}{(e^u + a)^2} v\, r\, dr. \tag{18}$$

If u, v are of type I, then $\beta = 2d$. If u, v are of type II, we have

$$\int_0^\infty r^{2d-\beta+1}\, dr < c \int_0^\infty \frac{e^u}{a+1} r\, dr < c' \int_0^\infty \frac{e^u}{(e^u + a)^2} v\, r\, dr < \infty$$

and thus $\beta > 2d + 2$. Let u, v belong to type III, in which $v(r) < 0$ in (R, ∞) for some large $R > 0$. Then

$$\int_R^\infty r^{\beta-2d+1}\, dr < c \int_R^\infty e^{-u} r\, dr < c' \int_R^\infty \frac{e^u}{(e^u + a)^2}(-v)\, r\, dr < \infty.$$

So $\beta < 2d - 2$. Besides, type IV and V solutions are characterized by $\beta = +\infty$ and $-\infty$, respectively. We summarize the above remark as follows:

$$\begin{cases} \beta = 2d, & \text{for type I solutions,} \\ \beta > 2d + 2, & \text{for type II solutions,} \\ \beta < 2d - 2, & \text{for type III solutions,} \\ \beta = +\infty, & \text{for type IV solutions,} \\ \beta = -\infty, & \text{for type V solutions.} \end{cases} \tag{19}$$

The paper is organized as follows. In Sec. II, we establish the non-degeneracy for the linearized equation of Eq. (10) and show the uniqueness of the topological solution. In Sec. III, we locate the regions of the initial data which account for each solution type we introduce as above and carry out a classification of the radially symmetric solutions of Eq. (9).

II. NON-DEGENERACY AND UNIQUENESS

In this section, we will establish the non-degeneracy property for the linearized equations of (10). By introducing a functional deformation method and the non-degeneracy property, we can prove the uniqueness of the topological solutions.
Consider the linearized equation of (10)

$$\begin{cases} \phi'' + r^{-1}\phi' = F_1(\phi, \psi), \\ \psi'' + r^{-1}\psi' = F_1(\phi, \psi) + F_2(\phi, \psi), \end{cases} \tag{20}$$

where

$$F_1(\phi,\psi) = 4q\frac{ae^u}{(e^u + a)^2}\phi + 2q\lambda\psi,$$

$$F_2(\phi,\psi) = 4q\frac{ae^u(a - e^u)}{(e^u + a)^3}v\phi + 4q\frac{ae^u}{(e^u + a)^2}\psi.$$

Then the non-degeneracy property of (20) is in the following:

Theorem 2.1. *If* (u, v) *is a solution of* (10) and (11), *then* $(\phi,\psi) \equiv (0,0)$ *is the only bounded solution of* (20).

Remark 2.1. If (u, v) is a solution of (10), then, by $\Delta v(0) = 2q\left[\lambda v(0) - \frac{2a}{a(a+1)}\right]$, $r = 0$ cannot be a non-positive local minimum point of $v(r)$.

In the following, we first prove the monotone property of the solution of type I.

Lemma 2.1. Let $(u(r), v(r))$ *be a solution of* (10). *Then the following conditions are valid.*

(i) *Both* $u(r)$ *and* $-v(r)$ *cannot attain any non-negative local maximum on* $(0,\infty)$.
(ii) *Both* $u(r)$ *and* $-v(r)$ *cannot attain any non-positive local minimum on* $(0,\infty)$.
(iii) *If* $(u(r), v(r))$ *is a solution of type I, then* $u'(r) > 0$ *and* $v'(r) < 0$ *on* $(0,\infty)$.

Proof. Let $(u(r), v(r))$ be a solution of (10). First, we show the result of part (i). On the contrary, we assume that r_u^* and r_v^* are the respective first non-negative local maximum points of $u(r)$ and $-v(r)$ on $[0,\infty)$. By Remark 2.1, we have $r_v^* \neq 0$. From (10) and $\left[\frac{2a(e^x-1)}{(a+1)(e^x+a)}\right]' > 0\ \forall x \in \mathbb{R}$, we easily obtain that $u(r)$ [respectively, $-v(r)$] does not possess a non-negative local minimum on $(0, r_v^*]$ (respectively, $(0, r_u^*]$), and the following equations hold:

$$\begin{cases} \lambda v(r_u^*) + \dfrac{2a(e^{u(r_u^*)} - 1)}{(a + 1)(e^{u(r_u^*)} + a)} \leq 0, \\ \lambda v(r_v^*) + \dfrac{2a(e^{u(r_v^*)} - 1)}{(a + 1)(e^{u(r_v^*)} + a)} \geq -\dfrac{2ae^{u(r_v^*)}}{(a + 1)(e^{u(r_v^*)} + a)^2}v(r_v^*) \geq 0. \end{cases}$$

By the first inequality and $\left[\frac{2a(e^x-1)}{(a+1)(e^x+a)}\right]' > 0\ \forall x \in \mathbb{R}$, we easily get

$$\lambda v(r_v^*) + \frac{2a(e^{u(r_v^*)} - 1)}{(a+1)(e^{u(r_v^*)}+a)} < \lambda v(r_u^*) + \frac{2a(e^{u(r_u^*)} - 1)}{(a+1)(e^{u(r_u^*)}+a)} \leq 0$$

which contradicts with the second inequality. This completes the proof of (i). Similarly, we can show the result of part (ii).

Next, in order to show the result of part (iii), we need the following claim.

Claim. If $(u(r), v(r))$ *is a solution of type I, then* $u(r) < 0$ *and* $-v(r) < 0\ \forall r \in (0,\infty)$.

Proof of Claim. On the contrary, since $(u(r), v(r)) \to (0,0)$ as $r \to \infty$ and by the boundary conditions of (10), we easily obtain that $u(r)$ and $-v(r)$ possess a respective non-negative local maximum point which contradicts to the result of part (i).

This proves our **Claim.** Now, suppose the result of part (iii) is not true. Then by above **Claim**, we obtain that $u(r)$ and $-v(r)$ possess a respective non-positive local minimum point which contradicts to the result of part (ii).

This shows part (iii), and the proof of Lemma 2.1. is complete. □

Next, we establish the range of $v(r)$ for solutions of type II and III as follows.

Lemma 2.2. Let $(u(r), v(r))$ *be a solution of* (10). *Then the following conditions are valid.*

(i) *If* $(u(r), v(r))$ *is a solution of type II, then*

$$u(r) < 0 \text{ and } 0 \le v(r) \le 2/\lambda(a+1)\ \forall r \in (0, \infty).$$

(ii) *If* $(u(r), v(r))$ *is a solution of type III, then* $-2a/\lambda(a+1) \le v(r) \le 2/\lambda(a+1)\ \forall r \in [0, \infty)$.

Proof. (i) Let (u, v) be a solution of type II for Eq. (10). Then $u(r) \to -\infty$ and $v(r) \to 2/\lambda(a+1)$ as $r \to \infty$. By (i) and (ii) of Lemma 2.1. and the boundary conditions of (10), we easily obtain $u(r) < 0$ and $v(r) \ge 0\ \forall r \in (0, \infty)$. From (10), we have

(*) $\Delta v = 2q[f(e^u)v + g(e^u)]$,
where $f(t) = \lambda + \frac{2at}{(t+a)^2}$ and $g(t) = \left(\frac{2a}{a+1}\right)\left(\frac{t-1}{t+a}\right)$ are bounded functions.

If $v(0) > \frac{2}{\lambda(a+1)}$ then, since $g'(t) > 0$, $\lim_{t\to 0} g(t) = -\frac{2}{a+1}$ and $v(r) \to 2/\lambda(a+1)$ as $r \to \infty$, by (10) and (*), we obtain that $v(r)$ possesses a non-negative local maximum which contradicts with (ii) of Lemma 2.1. Hence, we have $0 < v(0) \le \frac{2}{\lambda(a+1)}$ and $v(r) \to \frac{2}{\lambda(a+1)}$ as $r \to \infty$. By using (ii) of Lemma 2.1. again, we finally obtain $0 \le v(r) \le 2/\lambda(a+1)$ on $(0, \infty)$. This proves part (i) of this lemma.

(ii) Let (u, v) be a solution of type III for Eq. (10). Then, we have $u(r) \to +\infty$ and $v(r) \to -2a/\lambda(a+1)$ as $r \to \infty$. Similar to the proof of above part (i), by above (*), $g'(t) > 0$, $\lim_{t\to 0} g(t) = -\frac{2}{a+1}$, $\lim_{t\to\infty} g(t) = \frac{2a}{a+1}$, the boundary conditions of (10) and (i) and (ii) of Lemma 2.1., we obtain that $-2a/\lambda(a+1) \le v(0) \le 2/\lambda(a+1)$, and $v(r)$ does not possess a local minimum (respectively, local maximum) at $r_v \in (0, \infty)$ such that $v(r_v) \ge \frac{2}{\lambda(a+1)}$ (respectively, $v(r_v) \le -\frac{2a}{\lambda(a+1)}$). These imply that

$$-2a/\lambda(a+1) \le v(r) \le 2/\lambda(a+1)\ \forall r \in [0, \infty).$$

Thus, we get the result of part (ii). The proof of Lemma 2.2. is complete. □

Now for each solution of (10), we have the following non-oscillation result.

Lemma 2.3. Let $(u(r), v(r))$ *be a solution of* (10). *Then* u, v *cannot oscillate infinitely many times.*

Proof. By the results of (i) and (ii) in Lemma 2.1., we easily obtain the conclusion. □

Let $\varphi_{11}, \varphi_{12}$ and $\varphi_{21}, \varphi_{22}$ solve (20), respectively, with the initial data

$$\begin{cases} \varphi_{11}(0) = 0,\ \varphi_{11}'(0) = 0, \\ \varphi_{21}(0) = 1,\ \varphi_{21}'(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \varphi_{12}(0) = 1,\ \varphi_{12}'(0) = 0, \\ \varphi_{22}(0) = 0,\ \varphi_{22}'(0) = 0. \end{cases} \tag{21}$$

Lemma 2.4. $\varphi_{ij} > 0$ *and* $\varphi_{ij}' > 0$ *for all* $i, j = 1, 2$.

Proof. Since $F_1(\phi, \psi), F_2(\phi, \psi) > 0$ whenever $\phi, \psi > 0$, it follows readily that each φ_{ij} is an increasing function. □

We introduce the auxiliary functions $A = ru'$ and $B(r) = rv'$. Then

$$A'' + r^{-1}A' = F_1(A, B) + G_1,$$
$$B'' + r^{-1}B' = F_1(A, B) + F_2(A, B) + G_2,$$

in which G_1, G_2 are given by

$$G_1 = 4q\left[\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)}\right],$$
$$G_2 = 4q\left[\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)}\right] + 8q\frac{ae^u}{(e^u + a)^2}v.$$

Lemma 2.5. Let ϕ, ψ *solve* (20). *Set* $\Lambda = (\phi - \psi)A' - A(\phi' - \psi') - (\phi B' - B\phi')$. *Then*

$$\Lambda(\tau) = \frac{1}{\tau}\int_0^\tau r\left[(G_1 - G_2)\phi - G_1\psi\right] dr, \qquad \tau > 0. \tag{22}$$

Proof. Let $\xi = \phi - \psi$. From (20), $(r\xi')' = -rF_2(\phi,\psi)$. Making use of the expression

$$f(rg')' - g(rf')' = \frac{d}{dr}[r(fg' - gf')],$$

we have

$$\begin{aligned}
\tau[\xi A' - A\xi'](\tau) &= \int_0^\tau \xi(rA')' - A(r\xi')'\,dr\\
&= \int_0^\tau r\Big[(\phi-\psi)F_1(A,B) + AF_2(\phi,\psi) + (\phi-\psi)G_1\Big]\,dr\\
&= \int_0^\tau r\Big[F_1(A\phi - A\psi, B\phi - B\psi) + F_2(A\phi, A\psi) + G_1(\phi-\psi)\Big]\,dr,\\
\tau[\phi B' - B\phi'](\tau) &= \int_0^\tau \phi(rB')' - B(r\phi')'\,dr\\
&= \int_0^\tau r\Big[\phi(F_1+F_2)(A,B) - BF_1(\phi,\psi) + G_2\phi\Big]\,dr\\
&= \int_0^\tau r\Big[F_1(A\phi - B\phi, B\phi - B\psi) + F_2(A\phi, B\phi) + G_2\phi\Big]\,dr.
\end{aligned}$$

Hence

$$\begin{aligned}
\tau\Lambda(\tau) &= \int_0^\tau r\Big[F_1(B\phi - A\psi, 0) + F_2(0, A\psi - B\phi) + G_1(\phi-\psi) - G_2\phi\Big]\,dr\\
&= \int_0^\tau r\Big[(G_1 - G_2)\phi - G_1\psi\Big]\,dr.
\end{aligned}$$

□

We use the notation Q_0, Q_1 to denote the following functions:

$$Q_0 = \frac{\varphi_{11}}{\varphi_{12}} \qquad \text{and} \qquad Q_1 = \frac{\varphi_{21}}{\varphi_{22}}.$$

Clearly, $Q_0(0) = 0$, $Q_1(0) = 1$, and $Q_0, Q_1 > 0$ (Lemma 2.4.).

Lemma 2.6. Let u, v *satisfy* (10) and (11). *Then* $Q_0'(r) > 0$, $Q_1'(r) < 0$, *and* $Q_0(r) < Q_1(r)$ *for all* $r \in (0,\infty)$.

Proof. We construct the proof by two steps.
(**A1**) *If there is* $M > 0$ *such that* $Q_0(r) < Q_1(r)$ *for* $r \in (0, M)$, *then* $Q_0'(r) > 0$ *and* $Q_1'(r) < 0$ *whenever* $r \in (0, M)$.

Assume there is $r_1 \in (0, M)$ such that $Q_0'(r_1) = 0$ and $Q_0(r) < Q_1(r)$ for all $r \in (0, r_1)$. Without loss of generality, we may assume

$$Q_0(r) < Q_0(r_1), \qquad r \in (0, r_1). \tag{23}$$

Set $Q_0(r_1) = c$. Consider the solution ϕ, ψ of (20) given by

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\varphi_{11}\\ \varphi_{21}\end{pmatrix} - c\begin{pmatrix}\varphi_{12}\\ \varphi_{22}\end{pmatrix}. \tag{24}$$

Note that

$$\phi(r_1) = 0, \quad \phi'(r_1) = 0, \quad \text{and} \quad \psi(r_1) > 0.$$

So $\phi''(r_1) > 0$ by virtue of (20). This contradicts (23). Therefore, $Q_0' > 0$.

Since $\lim_{r\to 0^+} Q_1(r) = \infty$, $Q_1'(r) < 0$ for a sufficiently small $r > 0$. Hence, we have $Q_1' < 0$ according to the similar argument of the proof as in the case $Q_0' > 0$.

(**A2**) *Q_1 and Q_0 cannot meet.*

Suppose (A2) fails. Select $M > 0$ such that $Q_0(M) = Q_1(M) = c^*$ and $Q_1(r) > c^* > Q_0(r)$ whenever $r \in (0, M)$. Consider the solution ϕ, ψ given by (24) with $c = c^*$. Note that

$$\begin{cases} \phi(r) < 0 \quad \text{and} \quad \psi(r) > 0, & r \in (0, M), \\ \phi(M) = \psi(M) = 0. \end{cases} \tag{25}$$

Applying (15) and (22),

$$0 > -M\Big[-A(\phi' - \psi') + B\phi'\Big](M) = \int_0^M r\Big[(G_1 - G_2)\phi - G_1\psi\Big]\,dr > 0,$$

which is a contradiction. The proof is completed. □

Proof of Theorem 2.1. Let ϕ, ψ be a nontrivial solution of (20) which is characterized by (24) with some $c > 0$. If ϕ, ψ are bounded, then $r\phi'(r), r\psi'(r) \to 0$ as $r \to \infty$. Since $\varphi_{12}, \varphi_{22} \to \infty$ as $r \to \infty$, it follows that

$$Q_0 - c = \phi/\varphi_{12} \to 0, \qquad r \to \infty,$$
$$Q_1 - c = \psi/\varphi_{22} \to 0, \qquad r \to \infty.$$

By Lemma 2.6., $Q_0(r) \nearrow c$ and $Q_1(r) \searrow c$ as $r \to \infty$. In particular, $\phi < 0$ and $\psi > 0$. On the other hand, since A, B are bounded, we have $rA', rB' \to 0$ as $r \to \infty$. Therefore,

$$\begin{aligned} 0 &= \lim_{\tau \to 0} \tau[(\phi - \psi)A' - A(\phi' - \psi') - (\phi B' - B\phi')] \\ &= \int_0^\infty r\Big[(G_1 - G_2)\phi - G_1\psi\Big]\,dr > 0. \end{aligned}$$

This yields a contradiction. So Eq. (24) cannot possess any non-trivial bounded solution. □

Lemma 2.7. Assume $d = 0$. Let u, v satisfy (10) and (11). *Then $u = v \equiv 0$.*

Proof. $\Delta u \le 0$ in $\mathbb{R}^2$ and $u(x) \to 0$ as $|x| \to \infty$. So $u, v \equiv 0$ by the maximum principle. □

Function Spaces. Define the scalar products $\langle , \rangle_{X_\alpha}$ and $\langle , \rangle_{Y_\alpha}$, $0 < \alpha < 1$, for functions in the spaces $L^2_{loc}(\mathbb{R}^2)$ and $W^{2,2}_{loc}(\mathbb{R}^2)$,

$$\langle u, v\rangle_{X_\alpha} = \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})\, uv\, dx, \qquad u, v \in L^2_{loc}(\mathbb{R}^2),$$
$$\langle u, v\rangle_{Y_\alpha} = \langle \Delta u, \Delta v\rangle_{X_\alpha} + \int_{\mathbb{R}^2} \frac{uv}{1 + |x|^{2+\alpha}}\, dx, \qquad u, v \in W^{2,2}_{loc}(\mathbb{R}^2).$$

Let X_α, Y_α be the Hilbert spaces given by

$$X_\alpha = \Big\{u \in L^2_{loc}(\mathbb{R}^2) : \langle u, u\rangle_{X_\alpha} < +\infty\Big\}, \qquad Y_\alpha = \Big\{u \in W^{2,2}_{loc}(\mathbb{R}^2) : \langle u, u\rangle_{Y_\alpha} < +\infty\Big\}.$$

The X_α-norm and Y_α-norm are induced by

$$\|u\|_{X_\alpha} = \sqrt{\langle u, u\rangle_{X_\alpha}}, \qquad \|u\|_{Y_\alpha} = \sqrt{\langle u, u\rangle_{Y_\alpha}}.$$

We observe that $X_\alpha \hookrightarrow L^1(\mathbb{R}^2)$ and $Y_\alpha \subset C^0_{loc}(\mathbb{R}^2)$ by Hölder's inequality and the local regularity of the Laplace operator, respectively. Let the product spaces $X_\alpha \times X_\alpha$ and $Y_\alpha \times Y_\alpha$ be equipped with inner products $\langle u, v\rangle_{X_\alpha \times X_\alpha}$ and $\langle u, v\rangle_{Y_\alpha \times Y_\alpha}$ which are given by

$$\langle f, g\rangle_{X_\alpha \times X_\alpha} = \langle f_1, g_1\rangle_{X_\alpha} + \langle f_2, g_2\rangle_{X_\alpha}, \quad \langle u, v\rangle_{Y_\alpha \times Y_\alpha} = \langle u_1, v_1\rangle_{Y_\alpha} + \langle u_2, v_2\rangle_{Y_\alpha},$$

where $f = (f_1, f_2), g = (g_1, g_2) \in X_\alpha \times X_\alpha$ and $u = (u_1, u_2), v = (v_1, v_2) \in Y_\alpha \times Y_\alpha$. In the following content, we denote the subspaces consisting of radially symmetric functions by X^r_α and Y^r_α.

We introduce the mapping $\mathcal{P}(d,\xi,\eta)=(\mathcal{P}_1,\mathcal{P}_2)(d,\xi,\eta)$ for $d\geq 0$, $\xi,\eta\in Y_\alpha^r$,

$$\mathcal{P}_1=\rho^{-1}\Big\{\Delta\xi-2q\Big[\lambda\eta+\frac{2a(|x|^{2d}e^{\xi}-1)}{(a+1)(|x|^{2d}e^{\xi}+a)}\Big]\Big\},$$
$$\mathcal{P}_2=\rho^{-1}\Big\{\Delta\eta-2q\Big[\lambda\eta+\frac{2a(|x|^{2d}e^{\xi}-1)}{(a+1)(|x|^{2d}e^{\xi}+a)}\Big]-4q\frac{a|x|^{2d}e^{\xi}}{(|x|^{2d}e^{\xi}+a)^2}\eta\Big\},$$

where $\rho=1+|x|^{4d+4}$. The triplet $(d,\xi+2d\log r,\eta)$ solving (10) is equivalent to $\mathcal{P}(d,\xi,\eta)=0$. According to Ref. 1, Lemma 1.1, there exists $C>0$ such that

$$|u(x)|\leq C\,\|u\|_{Y_\alpha}(\log^+|x|+1),\qquad x\in\mathbb{R}^2, \tag{26}$$

for all $u\in Y_\alpha$, where $\log^+|x|=\max\{0,\ \log|x|\}$. Then

$$\mathcal{P}(d,\xi,\eta)\in X_\alpha^r\times X_\alpha^r$$

whenever $d\geq 0$ and $\xi,\eta\in Y_\alpha^r$. Consider the partial derivative $L=\partial\mathcal{P}(d,\xi,\eta)/\partial(\xi,\eta)$ at a fixed $d>0$ and $(\xi,\eta)\in Y_\alpha^r\times Y_\alpha^r$,

$$L:\begin{pmatrix}\phi\\ \psi\end{pmatrix}\mapsto\rho^{-1}\begin{pmatrix}\Delta\phi-L_1(\phi,\psi)\\ \Delta\psi-L_1(\phi,\psi)-L_2(\phi,\psi)\end{pmatrix},\qquad \phi,\psi\in Y_\alpha^r, \tag{27}$$

in which

$$L_1(\phi,\psi)=4q\frac{a|x|^{2d}e^{\xi}}{(|x|^{2d}e^{\xi}+a)^2}\phi+2q\lambda\psi,$$
$$L_2(\phi,\psi)=4q\frac{a|x|^{2d}e^{\xi}(a-|x|^{2d}e^{\xi})}{(|x|^{2d}e^{\xi}+a)^3}\eta\phi+4q\frac{a|x|^{2d}e^{\xi}}{(|x|^{2d}e^{\xi}+a)^2}\psi.$$

Lemma 2.8. Let $d>0$ and $\xi,\eta\in Y_\alpha^r$. If $(\xi,\eta)=(u-2d\log r,v)$, where $(u,\ v)$ solves (10) and (11) *with d, then L is a surjective mapping from $Y_\alpha^r\times Y_\alpha^r$ to $X_\alpha^r\times X_\alpha^r$.*

Proof. Let $\mathrm{I}mL=\{g\in X_\alpha^r\times X_\alpha^r: Lf=g,\ \exists f\in Y_\alpha^r\times Y_\alpha^r\}$. We claim that $\mathrm{I}mL=X_\alpha^r\times X_\alpha^r$. Suppose the assertion fails. Let $\zeta=(\zeta_1,\zeta_2)$ be a non-zero element of $X_\alpha^r\times X_\alpha^r$ and $\zeta\notin\mathrm{I}mL$. Since $\mathrm{I}mL$ is closed (via Ref. 1, Proposition 2.1), we may decompose $X_\alpha^r\times X_\alpha^r=\mathrm{I}mL\oplus(\mathrm{I}mL)^\perp$. Assume (without loss of generality) that $\zeta\perp\mathrm{I}mL$. We have

$$0=\langle L(\phi,\psi),\zeta\rangle=\int_{\mathbb{R}^2}\Big\{[\Delta\phi-L_1(\phi,\psi)]f+[\Delta\psi-L_1(\phi,\psi)-L_2(\phi,\psi)]g\Big\}\,dx \tag{28}$$

whenever $\phi,\psi\in Y_\alpha^r$, in which $f=\rho^{-1}(1+|x|^{2+\alpha})\zeta_1\geq 0$ and $g=\rho^{-1}(1+|x|^{2+\alpha})\zeta_2\geq 0$. Since $C_0^\infty(\mathbb{R}^2)\subset Y_\alpha$, by elliptic regularity, f, g are C^2-functions. For $\phi,\psi\in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned}0&=\int_{\mathbb{R}^2}\phi\Delta f+\psi\Delta g-L_1(f\phi+g\phi,f\psi+g\psi)-L_2(g\phi,g\psi)\,dx\\ &=\int_{\mathbb{R}^2}\Big[\Delta f-L_1(f+g,0)-L_2(g,0)\Big]\phi+\Big[\Delta g-L_1(0,f+g)-L_2(0,g)\Big]\psi\,dx.\end{aligned}$$

So f, g solve

$$\begin{cases}\Delta f-L_1(f+g,0)-L_2(g,0)=0,\\ \Delta g-L_1(0,f+g)-L_2(0,g)=0.\end{cases} \tag{29}$$

Let $\theta=f+g$. Note that $\theta\to 0$ as $|x|\to\infty$. Since $X_\alpha\subset L^1(\mathbb{R}^2)$, we get $f,g\to 0$ as $|x|\to\infty$. By (29), (g,θ) satisfies (20). Thus by Theorem 2.1., we have $g\equiv 0$ and $\theta\equiv 0$; hence $f=g=0$, that is, $\zeta_1=\zeta_2=0$, which contradicts our assumption. Therefore, $\mathrm{I}mL=X_\alpha^r\times X_\alpha^r$. □

Assume u_0, v_0 solve (10) and (11) with $d=d_0>0$. Let $u_0=\xi_0+2d\log r$ and $v_0=\eta_0$. Clearly, ξ_0,η_0 satisfy the assumption of Lemma 2.8. and $\mathcal{P}(d_0,\xi_0,\eta_0)=0$. By applying a specific version of the implicit function theorem ([Ref. 14, Theorem 2.7.5]) to $\mathcal{P}$, we find a continuous mapping

$$h:\ d\mapsto(\xi(d),\eta(d))$$

from a neighborhood $\mathcal{N}(d_0)$ of d_0 to $Y^r_\alpha \times Y^r_\alpha$ such that $\xi(d_0) = \xi_0$, $\eta(d_0) = \eta_0$, and

$$\mathcal{P}(d, \xi(d), \eta(d)) = 0 \qquad \text{for all} \quad d \in \mathcal{N}. \tag{30}$$

This establishes the existence of a family of solutions u_d, v_d to (10) which are given by

$$u_d = \xi(d) + 2d \log r, \qquad v_d = \eta(d), \qquad d \in \mathcal{N}(d_0). \tag{31}$$

Lemma 2.9. Assume u_0, v_0 solve (10) and (11) *with $d = d_0 > 0$. Then there exists $\delta > 0$ such that the solutions u_d, v_d given by* (31) *satisfy the boundary condition* (11) *whenever $|d - d_0| < \delta$.*

Proof. Since $X_\alpha \hookrightarrow L^1(\mathbb{R}^2)$, by the continuity of h, we have

$$2\pi\left|\frac{\beta(d)}{2} - d_0\right| \le \|\Delta(\xi(d) - \xi_0)\|_{L^1(\mathbb{R}^2)} \le C\|\Delta(\xi(d) - \xi_0)\|_{X_\alpha} \le C\|\xi(d) - \xi_0\|_{Y_\alpha} \to 0$$

as $d \to d_0$ for some constant $C > 0$. From (19), u_d, v_d must belong to type I solutions as d is sufficiently close to d_0. □

Proof of Theorem 1.1. If $d = 0$, the only type I solution of (10) is the trivial solution by Lemma 2.7. Assume that there is $d_1 \in (0, \infty)$ such that Eq. (10) with $d = d_1$ possesses at least two solutions satisfying the boundary condition (11). Let

$$\mathcal{J} = \{\sigma \ge 0 : \ \forall\, d \in [0, \sigma],\ \exists \text{ a unique } (u, v) \text{ satisfying (10) and (11)}\}.$$

Set $d^* = \sup \mathcal{J}$. Then $0 \le d^* \le d_1$. We claim that $d^* \in \mathcal{J}$. In fact, if $d^* \notin \mathcal{J}$, there exist two (different) solutions $(u, v), (\bar{u}, \bar{v})$ of (10) and (11) with $d = d^*$. By virtue of Lemma 2.9., there are two sequences (u_d, v_d) and $(\bar{u}_d, \bar{v}_d)$ of type I solutions which converge to $(u, v), (\bar{u}, \bar{v})$ as $d \to d^*$, which indicates $\sup \mathcal{J} < d^*$, a contradiction. Let (u, v) be the unique solution of (10) and (11) with $d = d^*$. We select two sequences (d_j, u_j, v_j) and $(d_j, \bar{u}_j, \bar{v}_j)$ such that

(i) $d_j > d^*$ and $d_j \to d^*$ as $j \to \infty$;
(ii) for each j, (u_j, v_j) and $(\bar{u}_j, \bar{v}_j)$ are two different solutions which solve (10) and (11) with $d = d_j$.

Let

$$\phi_j = (u_j - \bar{u}_j)/m_j \qquad \text{and} \qquad \psi_j = (v_j - \bar{v}_j)/m_j,$$

where $m_j = \max\{\|u_j - \bar{u}_j\|_{L^\infty}, \|v_j - \bar{v}_j\|_{L^\infty}\}$. Then

$$1 < \|\phi_j\|_{L^\infty} + \|\psi_j\|_{L^\infty} < 2 \tag{32}$$

and ϕ_j, ψ_j satisfy

$$\begin{aligned}
\Delta\phi_j &= 2q\Big[\lambda\psi_j + \frac{2ae^{a_j}}{(e^{a_j} + a)^2}\phi_j\Big], \\
\Delta\psi_j &= 2q\Big[\lambda\psi_j + \frac{2ae^{a_j}}{(e^{a_j} + a)^2}\phi_j\Big] \\
&\quad + 4q\Big[\frac{ae^{u_j}}{(e^{u_j} + a)^2}\psi_j + \frac{ae^{b_j}(a - e^{b_j})}{(e^{b_j} + a)^3}\bar{v}_j\phi_j\Big],
\end{aligned}$$

where a_j, b_j are located between u_j and $\bar{u}_j$. Let x_j, y_j be the maximum points given by

$$(u_j - \bar{u}_j)(x_j) = \|u_j - \bar{u}_j\|_{L^\infty} \qquad \text{and} \qquad (v_j - \bar{v}_j)(y_j) = \|v_j - \bar{v}_j\|_{L^\infty}.$$

By Ref. 4, Lemma 3.2, x_j, y_j are bounded sequences. So ϕ_j, ψ_j converge. Let $\phi_j \to \phi, \psi_j \to \psi$. Note that ϕ, ψ are bounded functions which solve (20). Hence $\phi = \psi = 0$ by Theorem 2.1. This contradicts (32). So $d^* = \infty$. Moreover, the continuous deformation follows readily from (31) and Lemma 2.9. Theorem 1.1 is concluded. □

III. LOCATE THE SOLUTION SETS

Using the uniqueness result shown in Sec. II, we are going to identify each solution set in the present section.

Let $u(\cdot, \alpha_1, \alpha_2), v(\cdot, \alpha_1, \alpha_2)$ satisfy Eq. (10) with the initial data $\alpha_1, \alpha_2 \in \mathbb{R}$. Consider the functions $\phi = \partial u/\partial\alpha_2, \psi = \partial v/\partial\alpha_2$. (ϕ, ψ) solve Eq. (20) with $(\phi, \psi)(0) = (0, 1)$. From Lemma 2.4., $\phi, \psi > 0$ and thus, we have

$$u(\cdot, \alpha_1, \alpha_2) < u(\cdot, \alpha_1, \alpha_2'), \qquad v(\cdot, \alpha_1, \alpha_2) < v(\cdot, \alpha_1, \alpha_2') \tag{33}$$

whenever $\alpha_2 < \alpha_2'$. Let

$$E_i = \{(\alpha_1, \alpha_2) : u(\cdot, \alpha_1, \alpha_2), v(\cdot, \alpha_1, \alpha_2) \text{ solve (10) and satisfy type } i \text{ boundary condition}\},$$

for $i = I, II, III, IV, V$ as given in Sec. I. Note that $\bigcup_i E_i = \mathbb{R}^2$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.

Theorem 3.1. *For each $\alpha_1 \in \mathbb{R}$, there exists exactly one $\alpha_2 \in \mathbb{R}$ such that*

$$(\alpha_1, \alpha_2) \in E_I \cup E_{II} \cup E_{III}.$$

Lemma 3.1. $E_{IV} \cap \overline{E}_V = \emptyset$ and $E_V \cap \overline{E}_{IV} = \emptyset$, where $\overline{E}$ denotes the closure of E.

Proof. Assume $(\alpha_1, \alpha_2) \in E_{IV} \cap \overline{E}_V$. Note that $v(r, \alpha_1, \alpha_2) \to -\infty$ as $r \to \infty$. In particular, there is $R > 0$ so large that $v(R, \alpha_1, \alpha_2) < -2a/\lambda(a+1)$. By continuity, we may extract $(\alpha_1', \alpha_2') \in E_V$ sufficiently close to (α_1, α_2) such that $v(R, \alpha_1', \alpha_2') < -2a/\lambda(a + 1)$. However, $v(r, \alpha_1', \alpha_2') \to \infty$ as $r \to \infty$ and $v(0, \alpha_1', \alpha_2') \geq -2a/\lambda(a+1)$ [by (16)], which indicates that $v(\cdot, \alpha_1', \alpha_2')$ has a negative minimum which is less than $-2a/\lambda(a + 1)$ at some $r = R'$. Thus,

$$0 \leq v''(R', \alpha_1', \alpha_2') = 2q\Big[\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)} + \frac{2ae^u}{(e^u + a)^2}v\Big](R', \alpha_1', \alpha_2') < 0.$$

That is a contradiction. So $E_{IV} \cap \overline{E}_V$ must be empty. Similarly, $E_V \cap \overline{E}_{IV} = \emptyset$. We omit the details. □

Lemma 3.2. $E_{II} \cap \overline{E}_{III} = \emptyset$ and $E_{III} \cap \overline{E}_{II} = \emptyset$.

Proof. $E_{III} \cap \overline{E}_{II} = \emptyset$ is evident because $v > 0$ [by (14)] for type II solutions but $v(r) < 0$ as $r \to \infty$ from type III boundary condition. To show that $E_{II} \cap \overline{E}_{III} = \emptyset$, we let $(\alpha_1, \alpha_2) \in E_{II} \cap \overline{E}_{III}$. Note that $u(r; \alpha_1, \alpha_2) \to -\infty$ as $r \to 0, \infty$ and $u(r; \alpha_1, \alpha_2) < 0, v(r; \alpha_1, \alpha_2) > 0$ for all $r > 0$ from (14). By continuity, we may pick an interval $[r_1, r_2] \subset (0, \infty)$ and $(\alpha_1', \alpha_2') \in E_{III}$ in a sufficiently small neighborhood of (α_1, α_2) such that $u(r; \alpha_1', \alpha_2')$ has a local maximum at $r = r' \in (r_1, r_2)$ and

$$u(r'; \alpha_1', \alpha_2') > u(r_1; \alpha_1', \alpha_2') > u(r_2; \alpha_1', \alpha_2'), \quad v(r_2; \alpha_1', \alpha_2') > 0.$$

Since $u(r; \alpha_1', \alpha_2') \to \infty$ as $r \to \infty$, there is $R \geq r_2$ such that

$$u(R; \alpha_1', \alpha_2') = \inf\{u(r; \alpha_1', \alpha_2') : r \in [r_1, \infty)\}.$$

Note that $u''(r'; \alpha_1', \alpha_2') \leq 0$ and $u''(R; \alpha_1', \alpha_2') \geq 0$. From (10), we deduce that $v(r'; \alpha_1', \alpha_2') < v(R; \alpha_1', \alpha_2')$. Since $v(r; \alpha_1', \alpha_2') \to -2a/\lambda(a + 1) < 0$ as $r \to \infty$, it is possible to pick $R' > r'$ such that

$$v(R'; \alpha_1', \alpha_2') = \sup\{v(r; \alpha_1', \alpha_2') : r \in [r', \infty)\}.$$

Hence $v''(R'; \alpha_1', \alpha_2') \leq 0$. On the other hand, since $u(R'; \alpha_1', \alpha_2') \geq u(R; \alpha_1', \alpha_2')$ and $v(R'; \alpha_1', \alpha_2') \geq v(R; \alpha_1', \alpha_2')$, it follows from (10) that

$$\Big(\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)}\Big)(R'; \alpha_1', \alpha_2') \geq \Big(\lambda v + \frac{2a(e^u - 1)}{(a+1)(e^u + a)}\Big)(R; \alpha_1', \alpha_2') = u''(R; \alpha_1', \alpha_2') \geq 0.$$

From $v(R';\alpha_1',\alpha_2') \geq v(r_2;\alpha_1',\alpha_2') > 0$ it implies that

$$v''(R';\alpha_1',\alpha_2') = \left(2q\left[\lambda v + \frac{2a(e^u-1)}{(a+1)(e^u+a)}\right] + 4q\frac{ae^u}{(e^u+a)^2}v\right)(R';\alpha_1',\alpha_2') > 0,$$

which leads to a contradiction. Therefore, E_{II} cannot contain any boundary point of E_{III}. □

Lemma 3.3. Let $L(\alpha_1) = \{(\alpha_1,\alpha_2) : \alpha_2 \in \mathbb{R}\}$ *and* $E = E_I \cup E_{II} \cup E_{III}$.

(a) *If* $(\alpha_1,\alpha_2) \in E_i$, *then* $L(\alpha_1) \cap E \subset E_i$ *for* $i = I, II, III$.
(b) *If* $(\alpha_1,\alpha_2), (\alpha_1,\alpha_2') \in E_i$, *then* $(\alpha_1, c) \in E_i$ *whenever* $c \in (\alpha_2,\alpha_2')$ *for* $i = I, II, III$.

Proof. It is an immediate consequence of the monotonicity property (33). □

Proof of Theorem 3.1. In view of Lemmas 3.1 and 3.3, it suffices to show that $L(\alpha_1)\cap E$ contains no interval for $i = I, II, III$. Since E_I contains exactly one point by Theorem 1.1, the assertion is immediate for $i = I$. Assume there are α_2, α_2' such that $(\alpha_1, c) \in E_{II}$ for all $c \in (\alpha_2,\alpha_2')$. In terms of Lemma 2.4.,

$$\frac{\partial\beta}{\partial\alpha_2} = -\lim_{r\to\infty} r\frac{\partial u}{\partial\alpha_2}(r) < 0 \quad \text{in the interval } (\alpha_2,\alpha_2'). \tag{34}$$

However, since $v > 0$, $0 < e^u < 0$ for type II solutions, this together with (18) and (33) indicates that $\beta(\alpha_1, s) < \beta(\alpha_1, t)$ whenever $s < t$, $s, t \in (\alpha_2,\alpha_2')$, which contradicts (34). Now assume that there are α_2, α_2' such that $(\alpha_1, c) \in E_{III}$ for all $c \in (\alpha_2,\alpha_2')$. Then, we have $rv'(r,\alpha_1,c) \to 0$ as $r \to \infty$, that is,

$$S(c) := \int_0^\infty 2q\left[\lambda v + \frac{2a(e^u-1)}{(a+1)(e^u+a)} + \frac{2ae^u}{(e^u+a)^2}v\right](r,\alpha_1,c)\, r\, dr = 0$$

for all $c \in (\alpha_2,\alpha_2')$. Thus, $dS/dc \equiv 0$ on (α_2,α_2'). On the other hand, by the definition of ϕ, ψ,

$$\frac{dS}{dc}(c') = \int_0^\infty \left\{4q\frac{ae^u}{(e^u+a)^2}\left[1 + \frac{a-e^u}{a+e^u}v\right]\phi + 2q\left[\lambda + \frac{2ae^u}{(e^u+a)^2}\right]\psi\right\}(r,\alpha_1,c')r\, dr > 0.$$

That is a contradiction. Here, we make use of the facts that $|(a-t)/(a+t)| < 1$ whenever $t \in \mathbb{R}$, and that $|v| \leq 1$ [via (17) with $\lambda = \kappa^2 q/2 \geq 2$]. The theorem is proved. □

For $\alpha_1 \in \mathbb{R}$, we let $\Gamma(\alpha_1)$ be such that $(\alpha_1, \Gamma(\alpha)) \in L(\alpha_1) \cap (E_I \cup E_{II} \cup E_{III})$. From Theorem 3.1, Γ is a function from $\mathbb{R}$ to $\mathbb{R}$. From (17), $-2a/\lambda(a+1) \leq \Gamma \leq 2/\lambda(a+1)$. Besides, we conclude the following property.

Lemma 3.4. The function Γ *is continuous. Moreover,* $\Gamma(\alpha) \to -2a/\lambda(a+1)$ *as* $\alpha \to \infty$ *and* $\Gamma(\alpha) \to 2/\lambda(a+1)$ *as* $\alpha \to -\infty$.

Proof. The continuity of Γ follows immediately from Lemma 3.1. Now we define auxiliary functions as follows:

$$\bar{u}(r) = \bar{u}(r,\alpha) = u(r;\alpha,\Gamma(\alpha)) - \alpha,$$
$$\bar{v}(r) = \bar{v}(r,\alpha) = v(r;\alpha,\Gamma(\alpha)) - \Gamma(\alpha).$$

Then $\bar{u}, \bar{v}$ satisfy the following equation:

$$\begin{cases} \bar{u}'' + r^{-1}\bar{u}' = 2q\left[\lambda(\bar{v}+\Gamma(\alpha)) + \dfrac{2a(e^\alpha e^{\bar{u}}-1)}{(a+1)(e^\alpha e^{\bar{u}}+a)}\right], \\ \bar{v}'' + r^{-1}\bar{v}' = 2q\left[\lambda(\bar{v}+\Gamma(\alpha)) + \dfrac{2a(e^\alpha e^{\bar{u}}-1)}{(a+1)(e^\alpha e^{\bar{u}}+a)}\right] + 4q\dfrac{ae^\alpha e^{\bar{u}}}{(e^\alpha e^{\bar{u}}+a)^2}(\bar{v}+\Gamma(\alpha)), \\ \bar{u}(r) = 2d\log r + o(1), \quad \text{as } r \to 0, \\ \bar{v}(0) = 0. \end{cases} \tag{35}$$

Assume $\Gamma(\alpha) \not\to -2a/\lambda(a+1)$ as $\alpha \to \infty$. We can obtain a number

$$-2a/\lambda(a+1) < c < 2/\lambda(a+1) \tag{36}$$

and a sequence $\alpha_j \to \infty$ such that $\Gamma(\alpha_j) \to c$ as $j \to \infty$. Let $u_j = \bar{u}(\cdot, \alpha_j)$, $v_j = \bar{v}(\cdot, \alpha_j)$. From (35), u_j, v_j approach $\hat{u}$, $\hat{v}$ (on any compact interval), which solves the equation

$$\begin{cases} \hat{u}'' + r^{-1}\hat{u}' = 2q\lambda(\hat{v} + C), \\ \hat{v}'' + r^{-1}\hat{v}' = 2q\lambda(\hat{v} + C), \\ \hat{u}(r) = 2d \log r + o(1), \quad \text{as } r \to 0, \\ \hat{v}(0) = 0, \end{cases}$$

where $C = c + (2a/\lambda(a+1)) > 0$. In terms of (17), v_j is uniformly bounded. However, $\hat{v}(r) \to \infty$ as $r \to \infty$. So there is a contradiction. Hence $\Gamma(\alpha) \to -2a/\lambda(a+1)$ as $\alpha \to \infty$. Similarly, if $\Gamma(\alpha) \not\to 2/\lambda(a+1)$ as $\alpha \to -\infty$, we pick c' [satisfying (36)] and $\alpha_j \to -\infty$ with $\Gamma(\alpha_j) \to c'$ as $j \to \infty$. Thus, $u_j, v_j \to \hat{u}, \hat{v}$ and

$$\begin{cases} \hat{u}'' + r^{-1}\hat{u}' = 2q\lambda(\hat{v} + C'), \\ \hat{v}'' + r^{-1}\hat{v}' = 2q\lambda(\hat{v} + C'), \\ \hat{u}(r) = 2d \log r + o(1), \quad \text{as } r \to 0, \\ \hat{v}(0) = 0, \end{cases}$$

where $C' = c - (2/\lambda(a+1)) < 0$. Note that $\hat{v}(r) \to -\infty$ as $r \to \infty$. This also contradicts the uniform boundedness of v_j. The lemma is concluded. □

Proof of Theorem 1.2. From Theorem 3.1, the curve $\mathcal{C} = (\alpha, \Gamma(\alpha))$, $\alpha \in \mathbb{R}$, compartmentalize the plane $\mathbb{R}^2$ into two 2-dimensional regions. The upper region is E_V and the lower region is E_{IV} according to the monotonicity property (33). $\mathcal{C}$ consists of the sets E_I, E_{II}, and E_{III}. Lemma 3.2 reveals that the boundary points of E_{II}, E_{III} can only belong to E_I. By the uniqueness of the topological solutions, there exists $\alpha^* \in \mathbb{R}$ such that $E_I = \{(\alpha^*, \Gamma(\alpha^*)\}$. So E_{II} and E_{III} are connected sets. Since $v(\cdot, \alpha_1, \alpha_2) > 0$ for $(\alpha_1\alpha_2) \in E_{II}$, it follows from Lemma 3.4 that

$$E_{II} = \{(\alpha, \Gamma(\alpha)) : \alpha < \alpha^*\}, \qquad E_{III} = \{(\alpha, \Gamma(\alpha)) : \alpha > \alpha^*\}.$$

With the substitution

$$s = \frac{a-1}{a+1}, \qquad \alpha_1 = c_1 + \log a, \qquad \alpha_2 = -\frac{2}{\kappa^2 q}c_2, \qquad \lambda = \frac{\kappa^2 q}{2},$$

we conclude Theorem 1.2. □

ACKNOWLEDGMENTS

The authors would like to express their gratitude to the anonymous referees for helpful suggestions and comments to improve the exposition of the article.

Work was partially supported by the Ministry of Science and Technology of Taiwan under Grant Nos. MOST 105-2115-M-008-012-MY3, MOST 103-2115-M-018-002-MY3, and MOST 104-2115-M-008-010-MY3.

[1] Chae, D. and Imanuvilov, O. Yu., "The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory," Commun. Math. Phys. **215**, 119–142 (2000).
[2] Chae, D. and Kim, N., "Topological multivortex solutions of the self-dual Maxwell-Chern-Simons-Higgs system," J. Differ. Equations **134**, 154–182 (1997).
[3] Chae, D. and Imanuvilov, O. Yu., "Non-topological multivortex solutions to the self-dual Maxwell-Chern-Simons-Higgs systems," J. Funct. Anal. **196**, 87–118 (2002).
[4] Chern, J.-L., Chen, Z.-Y., and Lin, C.-S., "Uniqueness of topological solutions and the structure of solutions for the Chern-Simons system with two Higgs particles," Commun. Math. Phys. **296**, 323–351 (2010).
[5] Chern, J.-L. and Yang, S.-G., "Evaluating solutions on an elliptic problem in a gravitational gauge field theory," J. Funct. Anal. **265**, 1240–1263 (2013).
[6] Chern, J.-L. and Yang, S.-G., "The non-topological fluxes of a two-particle system in the Chern-Simons theory," J. Differ. Equations **256**, 3417–3439 (2014).
[7] Choe, K. and Han, J., "Existence and properties of radial solutions in the self-dual Chern-Simons $O(3)$ sigma model," J. Math. Phys. **52**, 082301 (2011).
[8] Choe, K., Han, J., Lin, C.-S., and Lin, T.-C., "Uniqueness and solution structure of nonlinear equations arising from the Chern-Simons gauged $O(3)$ sigma models," J. Differ. Equations **255**, 2136–2166 (2013).

[9] Han, J. and Song, K., "Existence and asymptotics of topological solutions in the self-dual Maxwell-Chern-Simons $O(3)$ sigma model," J. Differ. Equations **250**, 204–222 (2011).
[10] Han, J., "Radial symmetry of topological one-vortex solutions in the Maxwell-Chern-Simons-Higgs model," Commun. Korean Math. Soc. **19**, 283–291 (2004).
[11] Han, J. and Nam, H.-S., "On the topological multivortex solutions of the self-dual Maxwell-Chern-Simons gauged $O(3)$ sigma model," Lett. Math. Phys. **73**, 17–31 (2005).
[12] Huang, H.-Y. and Lin, C.-S., "Classification of the entire radial self-dual solutions to non-Abelian Chern-Simons systems," J. Funct. Anal. **266**, 6796–6841 (2014).
[13] Kimm, K., Lee, K., and Lee, T., "Anyonic Bogomolnyi solitons in a gauged $O(3)$ sigma model," Phys. Rev. D **53**, 4436–4440 (1996).
[14] Nirenberg, L., *Topics in Nonlinear Analysis*, Courant Lecture Notes in Mathematics Vol. 6 (American Mathematical Society, 2001).
[15] Ricciardi, T. and Tarantello, G., "Vortices in the Maxwell-Chern-Simons theory," Commun. Pure Appl. Math. **53**, 811–851 (2000).