SCIENCE CHINA
Mathematics

Special Issue on Pure Mathematics
• ARTICLES •
https://doi.org/10.1007/s11425-018-9493-1

Quantitative gradient estimates for harmonic maps into singular spaces

Dedicated to Professor Lo Yang on the Occasion of His 80th Birthday

Hui-Chun Zhang[1], Xiao Zhong[2] & Xi-Ping Zhu[1,*]

[1]*Department of Mathematics, Sun Yat-sen University, Guangzhou* 510275*, China;*
[2]*Department of Mathematics and Statistics, University of Helsinki, Helsinki FI*-00014*, Finland*

Email: zhanghc3@mail.sysu.edu.cn, xiao.x.zhong@helsinki.fi, stszxp@mail.sysu.edu.cn

Received November 26, 2018; accepted February 18, 2019

Abstract In this paper, we show the Yau's gradient estimate for harmonic maps into a metric space (X, d_X) with curvature bounded above by a constant κ ($\kappa \geqslant 0$) in the sense of Alexandrov. As a direct application, it gives some Liouville theorems for such harmonic maps. This extends the works of Cheng (1980) and Choi (1982) to harmonic maps into singular spaces.

Keywords harmonic maps, Bochner formula, $CAT(\kappa)$-spaces, Liouville theorem

MSC(2010) 58E20

Citation: Zhang H-C, Zhong X, Zhu X-P. Quantitative gradient estimates for harmonic maps into singular spaces. Sci China Math, 2019, 62, https://doi.org/10.1007/s11425-018-9493-1

1 Introduction

Let M and N be two smooth Riemannian manifolds. There is a natural concept of the energy functional for C^1-maps between M and N. The local minimizers (or more general critical points) of such an energy functional are called harmonic maps. Regularity of harmonic maps is an important topic in the field of geometric analysis. If $\dim M = 2$, the regularity of energy minimizing harmonic maps was established by Morrey [34]. If $\dim M \geqslant 3$, a beautiful regularity theory was established by Schoen and Uhlenbeck [36,37], and in a somewhat different context, by Giaquinta and Giusti [14, 15] (and by Hildebrandt et al. [19] when the image of the map is contained in a convex ball of N).

In 1975, Yau [43] established a seminal interior gradient estimate for harmonic functions on Riemanian manifolds with Ricci curvature bounded below. In 1980, Cheng [4] generalized the Yau's gradient estimate to harmonic maps.

Theorem 1.1 (See [4]). *Let M and N be complete Riemannian manifolds such that M has Ricci curvature $\mathrm{Ric}_M \geqslant -K$, $K \geqslant 0$, and that N is simply-connected and is having non-positive sectional curvature. Let $f : M \to N$ be a harmonic map. Assume that $f(B_a(x_0)) \subset B_b(y_0)$ for some $x_0 \in M$, $y_0 \in N$ and some $a, b > 0$. Then we have*

$$\sup_{B_{a/2}(x_0)} |\nabla f|^2 \leqslant C_n \cdot \frac{b^4}{a^4} \cdot \max\left\{\frac{Ka^4}{b^2}, \frac{a^2(1+Ka^2)}{b^2}, \frac{a^2}{b^2}\right\}, \tag{1.1}$$

where C_n is a constant depending only on $n = \dim(M)$.

* Corresponding author

In particular, when $K=0$, this implies a Liouville theorem: If f is bounded, then it is a constant map. Choi [5] further extended Cheng's work [4] as the following theorem.

Theorem 1.2 (See [5]). *Let M and N be complete Riemannian manifolds such that M has Ricci curvature $\mathrm{Ric}_M \geqslant -K$, $K \geqslant 0$, and that N has sectional curvature $\sec_N \leqslant \kappa$, $\kappa > 0$. Let $f: M \to N$ be a harmonic map. Assume that $f(M) \subset B_b(y_0)$ lies inside the cut locus of $y_0 \in N$ and some $b < \pi/(2\sqrt{\kappa})$. Then $|\nabla f|$ is bounded by a constant depending only on n, K, κ and b. If, furthermore, $K = 0$, then f is a constant map.*

It is well known from [19, 22] that the radius $b < \pi/(2\sqrt{\kappa})$ is sharp. Without the restriction that the image of u is contained in a ball with radius $\pi/(2\sqrt{\kappa})$, a harmonic map might not be even continuous.

1.1 Yau's gradient estimates for harmonic maps into metric spaces

The purpose of this paper is to extend the Yau's gradient estimate to harmonic maps into singular metric spaces.

In the seminal work of Gromov and Schoen [16], they initiated to study harmonic maps into singular spaces. A general theory of harmonic maps between singular spaces was developed by Korevaar and Schoen [28], Jost [23, 25] and Lin [30], independently.

If u is a map from a domain $\Omega \subset M$ of the Riemannian manifold to an arbitrarily metric space (X, d_X), which is unnecessary to be embedded into a Euclidean space, Korevaar and Schoen [28] introduced an intrinsic approach to generalize the concept of the energy of u. Given a map $u \in L^2(\Omega, X)$, for each $\epsilon > 0$, the *approximating energy* E^u_ϵ is defined as a functional on $C_0(\Omega)$:

$$E^u_\epsilon(\phi) := \int_\Omega \phi(x) e^u_\epsilon(x) dv_g(x),$$

where $\phi \in C_0(\Omega)$, the space of continuous functions compactly supported on Ω, and e^u_ϵ is *approximating energy density* defined by

$$e^u_\epsilon(x) := \frac{n(n+2)}{\omega_{n-1} \cdot \epsilon^n} \int_{B_\epsilon(x) \cap \Omega} \frac{d^2_X(u(x), u(y))}{\epsilon^2} dv_g(y),$$

where ω_{n-1} is the volume of $(n-1)$-sphere $\mathbb{S}^{n-1}$ with the standard metric. In [28], Korevaar and Schoen proved that

$$\lim_{\epsilon \to 0^+} E^u_\epsilon(\phi) = E^u(\phi)$$

for some positive functional $E^u(\phi)$ on $C_0(\Omega)$. The limit functional E^u is called the energy (functional) of u. By the Riesz representation theorem, the non-negative functional E^u is a Radon measure on Ω. Moreover, Korevaar and Schoen [28] proved that the measure is absolutely continuous respect to the Riemannian volume vol_g. Denote $e_u := \frac{dE^u}{d\mathrm{vol}_g}$, the energy density of u. For a smooth map f between two smooth Riemannian manifolds, we have $e_f = \mathrm{const} \cdot |\nabla f|^2$.

The (local) minimizing maps, in the sense of calculus of variations, of such an energy functional E^u are called *harmonic maps*.

If (X, d_X) is a locally compact Riemannian simplicial complex with (globally) non-positive curvature in the sense of Alexandrov, Gromov and Schoen [16] established the local Lipschitz regularity for harmonic maps from Ω to X. Korevaar and Schoen [28] extended to the case where X is a general $CAT(0)$-space, a metric space with non-positive curvature in the sense of Alexandrov. A further extension was given by Serbinowski [39]. Let us put these regularity results together as follows.

Theorem 1.3 (See [28, Theorem 2.4.6] and [39, Corollary 2.18]). *Let $\Omega \subset M$ be a bounded domain (with smooth boundary) of a Riemannian manifold (M, g) and let (X, d_X) be a $CAT(\kappa)$-space for some $\kappa \geqslant 0$. Suppose that $u: \Omega \to X$ is a harmonic map. Assume that the image of u is contained in a ball with radius $\rho < \pi/(2\sqrt{\kappa})$. Here and in the sequel, if $\kappa = 0$, we always understand $\pi/(2\sqrt{\kappa}) = +\infty$.*

Then u is locally Lipschitz continuous in Ω. Moreover, for any ball $B_R(o) \subset\subset \Omega$, it holds the following Bernstein-type gradient estimate:

$$\sup_{B_{R/2}(o)} e_u \leqslant C \fint_{B_R(o)} e_u dv_g, \tag{1.2}$$

where the constant C depends on $n = \dim(M)$, R, the injectivity radius of o, $\pi/(2\sqrt{\kappa}) - \rho$, and the C^1-norm of metric coefficients g on $B_R(o)$. Here and in the sequel, $\fint_E := \frac{1}{\mathrm{vol}_g(E)} \int_E$ denotes the average integral over the measurable set E.

In the last two decades, many regularity results have been obtained for (energy minimizing) harmonic maps into or between singular spaces (see, for example, [3, 9, 12, 21, 24, 25, 28, 42], and [1, 7, 8, 17, 30, 31, 45] and so on).

For the case when the domain Ω has non-negative sectional curvature and the target X is a $CAT(0)$-simplicial complex, Chen [3] showed that the constant C in (1.2) depends only on n. When the target X is a general $CAT(0)$-space, Jost [26] gave an approach to deduce an explicit bound of the constant in (1.2) in terms of the sectional curvature of M, n and R. Other quantitative Lipschitz estimates of u were also given in [7, 8].

In [26, Section 6, p. 167], Jost proposed an open problem, in the case when the target X is a $CAT(0)$-space, to ask if the $\sup_{B_{R/2}(o)} e_u$ can be dominated by a constant depending only on the lower bound for the Ricci curvature of M, the dimension of M, and the energy of u. Furthermore, a natural problem was arisen from the combination of the Jost's problem and the Cheng's work [4] to ask if a Yau-type interior gradient estimate holds for the harmonic map into a $CAT(0)$-space. The first result in this paper answers this problem affirmatively.

Theorem 1.4. *Let Ω be a bounded domain (with smooth boundary) of an n-dimensional Riemannian manifold (M, g) with $\mathrm{Ric}_M \geqslant -K$ for some $K \geqslant 0$, and let (X, d_X) be a $CAT(0)$-space. Suppose that $u : \Omega \to X$ is a harmonic map. Given any ball $B_R(x_0)$ with $B_{2R}(x_0) \subset\subset \Omega$, if $u(B_R(x_0)) \subset B_\rho(Q_0)$ for some $Q_0 \in X$ and some $\rho > 0$, then we have*

$$\sup_{B_{R/2}(x_0)} \mathrm{Lip}u \leqslant C_{n,\sqrt{K}R} \cdot \frac{\rho}{R},$$

where $\mathrm{Lip}u$ *is the pointwise Lipschitz constant given by*

$$\mathrm{Lip}u(x) := \limsup_{y \to x} \frac{d_X(u(x), u(y))}{d(x, y)},$$

and where $d(x, y)$ is the distance with respect to the Riemannian metric g on M, and $C_{n,\sqrt{K}R}$ is a constant depending only on n and $\sqrt{K}R$.

Remark 1.5. (1) It is clear from the definitions of e_u and $\mathrm{Lip}u$ that $e_u(x) \leqslant (n+2)\mathrm{Lip}^2 u(x)$ for almost all $x \in \Omega$.

(2) By the fact $\Delta d_X^2(u(x), u(x_0)) \geqslant 2e_u \geqslant 0$ (see [25] or Lemma 2.6), it is well known that the $\sup_{x \in B_{R/2}(x_0)} d_X^2(u(x), u(x_0))$ can be dominated by $C_{n,\sqrt{K}R} R^2 \cdot \fint_{B_R(x_0)} e_u dv_g$ (see, for example, (2.7)). So, by choosing $Q_0 = u(x_0)$, Theorem 1.4 implies that

$$\sup_{B_{R/2}(x_0)} \mathrm{Lip}u \leqslant C_{n,\sqrt{K}R} \left(\fint_{B_R(x_0)} e_u dv_g \right)^{1/2}. \tag{1.3}$$

It answers the Jost's problem (see [26, Section 6]) affirmatively. Recently, (1.3) was used by Sidler and Wenger [40] to find the harmonic quasi-isometric maps into Gromov hyperbolic $CAT(0)$-spaces.

As an immediate application of Theorem 1.4, by letting $R \to \infty$, we have the following Liouville theorem (see [41, Theorem 1.4] and [20, Theorem 1.2] for several other Liouville theorems).

Corollary 1.6. *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold with non-negative Ricci curvature, and let (X, d_X) be a $CAT(0)$-space. Let $u : M \to X$ be a harmonic map. If u*

satisfies sub-linear growth

$$\liminf_{R\to\infty}\frac{\sup_{y\in B_R(x_0)} d_X(u(y),Q_0)}{R}=0$$

for some $Q_0\in X$, then u must be a constant map.

For the case when the target space has curvature less than or equal to κ for some $\kappa>0$, we have the following gradient estimate.

Theorem 1.7. *Let Ω be a bounded domain (with smooth boundary) of an n-dimensional Riemannian manifold (M,g) with $\mathrm{Ric}_M\geqslant -K$ for some $K\geqslant 0$, and let (X,d_X) be a $CAT(\kappa)$-space, $\kappa>0$. Suppose that $u:\Omega\to X$ is a harmonic map with the image $u(\Omega)\subset B_\rho(Q_0)$ for some $Q_0\in X$ and $\rho<\pi/(2\sqrt{\kappa})$. Then we have*

$$\sup_{B_{R/2}(x_0)}\mathrm{Lip}u\leqslant\frac{C_{n,\sqrt{K}R,\pi/(2\sqrt{\kappa})-\rho}}{R},$$

where $C_{n,\sqrt{K}R,\pi/(2\sqrt{\kappa})-\rho}$ is a constant depending only on $n,\sqrt{K}R$ and $\pi/(2\sqrt{\kappa})-\rho$.

This implies the following Liouville theorem, by letting $R\to\infty$.

Corollary 1.8. *Let (M,g) be an n-dimensional complete non-compact Riemannian manifold with non-negative Ricci curvature, and let (X,d_X) be a $CAT(1)$-space. Let $u:M\to X$ be a harmonic map. If $u(M)\subset B_\rho(Q_0)$ for some $Q_0\in X$ and $\rho<\pi/2$, then u must be a constant map.*

Remark 1.9. If $u(M)\subset B_{\pi/2}(Q_0)$ and if $d_X^2(Q_0,u(x))\in L^1(M)$, then the same conclusion, u is a constant map, has been proved recently by Freidin and Zhang [11].

1.2 A sharp Bochner inequality for the harmonic maps into metric spaces

Cheng's argument in [4] is based on the classical Bochner formula of Eells and Sampson, i.e., for a smooth harmonic map u between two Riemannian manifolds M and N, it holds that

$$\begin{aligned}\frac{1}{2}|\nabla u|^2&=|d\nabla u|^2+\mathrm{Ric}_M(\nabla u,\nabla u)-\langle R^N(u_*e_\alpha,u_*e_\beta)u_*e_\alpha,u_*e_\beta\rangle\\&\geqslant|\nabla|\nabla u||^2-K|\nabla u|^2-\kappa|\nabla u|^4,\end{aligned}\tag{1.4}$$

where the Ricci curvature of M is bounded below by $-K$ and the sectional curvature of N is bounded above by κ. It is clear that the classical Bochner formula relies heavily on the smoothness of the target space X (requiring to have at least second order derivatives).

For harmonic maps into singular spaces, it is a basic problem to deduce a Bochner type formula. For the case when the domain Ω has non-negative sectional curvature and the target X is a non-positively curved simplicial complex, Chen [3] used the method in [16] to show that e_u is a sub-harmonic function on Ω. In [28], Korevaar and Schoen developed a finite difference technique to prove the following weak form of the Bochner type inequality: there exists a constant C, depending on the C^1-norm of g, such that

$$\int_\Omega e_u(\Delta\eta+C|\nabla\eta|+C\eta)dv_g\geqslant 0$$

for all $\eta\in C_0^\infty(\Omega)$. Korevaar-Schoen's method in [28] has been extended by Serbinowski [39] to the case when the target space is of $CAT(\kappa)$ for any $\kappa>0$. Mese [32] showed that $\Delta e_u\geqslant -2\kappa e_u^2$, in the sense of distributions, for a harmonic map from a flat domain to a $CAT(\kappa)$-space. Recently, Freidin [10] and Freidin and Zhang [11] improved the method in [16] to deduce the following Bochner type inequality for a harmonic map from a Riemannian manifold into a $CAT(\kappa)$-space:

$$\frac{1}{2}\Delta e_u\geqslant -Ke_u-\kappa e_u^2,\tag{1.5}$$

in the sense of distributions.

Recalling the arguments of Cheng [4] and Choi [5], the key intergradient is the positive term $|\nabla|\nabla u||^2$ on the right-hand side of (1.4). The Bochner inequality (1.5) is not enough to get Theorems 1.4 and 1.7. In this paper, we will establish a generalized Bochner inequality keeping such a positive term as follows.

Theorem 1.10. *Let Ω be a smooth domain of an n-dimensional Riemannian manifold (M,g) with $\mathrm{Ric}_M \geqslant -K$ for some $K \geqslant 0$, and let (X,d_X) be a $CAT(\kappa)$-space for some $\kappa \geqslant 0$. Suppose that the map $u:\Omega\to X$ is harmonic and that its image $u(\Omega)$ is contained in a ball $B_\rho \subset X$ with radius $\rho < \frac{\pi}{2\sqrt{\kappa}}$ if $\kappa > 0$.*

Then $\mathrm{Lip}u$ *is in* $W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ *and satisfies the following*:

$$\frac{1}{2}\Delta \mathrm{Lip}^2 u \geqslant |\nabla \mathrm{Lip}u|^2 - K\cdot \mathrm{Lip}^2 u - \kappa e_u \cdot \mathrm{Lip}^2 u \tag{1.6}$$

in the sense of distributions.

1.3 The outline of the proof of the Bochner inequality

In the following, we give an outline of the proof of Theorem 1.10. First, by the chain rule, one easily checks that (1.6) is equivalent to

$$\Delta \mathrm{Lip}u \geqslant -K\cdot \mathrm{Lip}u - \kappa e_u\cdot \mathrm{Lip}u \tag{1.7}$$

in the sense of distributions. We will first show that, for any $q\in(1,2]$,

$$\Delta(\mathrm{Lip}^q u/q) \geqslant -K\cdot \mathrm{Lip}^q u - \kappa e_u\cdot \mathrm{Lip}^q u \tag{1.7q}$$

in the sense of distributions, and then we check the limit as $q\to 1$ to get (1.7).

The proof of (1.7q) is inspired by the classical Hamilton-Jocabi flow. Recall that the classical Hamilton-Jacobi equation, given a function f:

$$\frac{\partial v(x,t)}{\partial t} = -|\nabla v(x,t)|^2$$

with $v(x,0)=f(x)$, has a solution (by the Hopf-Lax formula)

$$\mathscr{H}_t f(x) := \inf_{y\in B_R}\left\{\frac{d^2(x,y)}{2t} + f(y)\right\},\quad t>0.$$

The difference of "time t" to the Hamilton-Jacobi flow $\mathscr{H}_t f(x)$ at $t=0$ gives the gradient $-|\nabla f(x)|^2$, i.e., as $t\to 0^+$,

$$\frac{\mathscr{H}_t f(x) - f(x)}{t} \to -|\nabla f(x)|^2.$$

This suggests to study the Hamilton-Jacobi flow $\mathscr{H}_t f(x)$ for the gradient estimates of f.

In order to obtain (1.7q), we introduce a family of functions $(f_t)_{t>0}$ by: on a fixed ball $B_R := B_R(o)$ with $B_{2R}\subset\subset\Omega$, for any $q\in(1,2]$,

$$f_t(x) := \inf_{y\in B_{2R}}\left\{\frac{d^p(x,y)}{pt^{p-1}} - \phi(d_X(u(x),u(y)))\right\},\quad \forall\, x\in B_R,\quad \forall\, t>0, \tag{1.8}$$

where $p=q/(q-1)$ and $\phi:[0,1/10]\to\mathbb{R}$ is a suitable smooth convex function with $\phi(0)=0$ and $\phi'(0)=1$.

It is easy to check that, for any $x\in B_R$ and any sufficiently small t, the "inf" of (1.8) can be realized by some point $y_{t,x}\in B_{2R}$. The set of all such points is denoted by $S_t(x)$. Then we put

$$L_t(x) := \min_{y_{t,x}\in S_t(x)} d(x,y_{t,x}) \quad\text{and}\quad D_t(x) := \frac{L_t^p(x)}{pt^{p-1}} - f_t(x). \tag{1.9}$$

The proof of (1.7q) contains two parts. Without loss of generality, we may assume $\kappa=1$. Firstly, we shall show that, for any given $\varepsilon>0$, f_t satisfies an elliptic inequality

$$\Delta f_t(x) \leqslant \frac{K}{t^{p-1}}\cdot L_t^p(x) + (1+\varepsilon)\cdot e_u(x)D_t(x) \tag{1.10}$$

on B_R, for any sufficiently small $t>0$, in the sense of distributions. Secondly, we want to show that, for almost all $x\in B_R$,

$$\lim_{t\to 0}\frac{f_t}{t}=-\frac{1}{q}\mathrm{Lip}^q u,\quad \lim_{t\to 0^+}\frac{L_t}{t}=\mathrm{Lip}^{q/p}u,\quad \lim_{t\to 0^+}\frac{D_t}{t}=\mathrm{Lip}^q u. \tag{1.11}$$

The combination of (1.10) and (1.11) will yield the inequality (1.7q).

In order to prove (1.11), we recall a generalized Rademacher theorem in [27]. Let $f:\Omega\to X$ be a Lipschitz map. Kirchheim [27] proved for almost all $x\in\Omega$, that there exists a semi-norm, denoted by mdf_x and called *metric differential*, such that

$$d_X(f(\exp_x(t\xi)),f(x))-t\cdot\mathrm{mdf}_x(\xi)=o(t),$$

for all $\xi\in\mathbb{S}^{n-1}\subset T_xM$. By using this result, one can deduce a representative of point-wise Lipschitz constant of f: for almost all $x\in\Omega$,

$$\mathrm{Lip}f(x):=\max_{\xi\in\mathbb{S}^{n-1}}\mathrm{mdf}_x(\xi).$$

This suffices to show (1.11) (see Lemmas 2.10 and 4.4 for the details).

Now, we explain the proof of (1.10), which is inspired by the recent work [45] of the first and the third authors. For simplicity, we assume $\mathrm{Ric}_M\geqslant 0$. We need to show that

$$\Delta f_t(x)\leqslant(1+\varepsilon)e_u(x)D_t(x)+\theta$$

for sufficiently small $t>0$ and any $\theta>0$ in the sense of distributions. It is a local property. Then we need only to consider the case when R is small. We argue by contradictions. Suppose that it fails, by the maximum principle, we have that there exists a domain U and a positive number θ_0 such that f_t-v achieves a strict minimum in U, where v is the solution of the Dirichlet problem

$$\Delta v=(1+\varepsilon)e_u(x)D_t(x)+\theta_0\quad\text{in}\quad U,\quad v=f_t\quad\text{on}\quad\partial U.$$

From the construction of f_t, we know that the function

$$H(x,y):=\frac{d^p(x,y)}{pt^{p-1}}-F(x,y)-v(x)$$

has a minimum in $U\times B_R$, where $F(x,y):=\phi(d_X(u(x),u(y)))$. We denote one of these minimum points by $(\bar{x},\bar{y})$.

Let $T:T_{\bar{x}}M\to T_{\bar{y}}M$ be the parallel transportation from $\bar{x}$ to $\bar{y}$. We want to consider the asymptotic behavior of the average

$$\fint_{B_\epsilon(0)}H(\exp_{\bar{x}}(\eta),\exp_{\bar{y}}(T\eta))d\eta$$

as $\epsilon\to 0$. According to $\mathrm{Ric}_M\geqslant 0$, by integrating the second variation of arc-length over $B_\epsilon(0)$, we have that

$$\fint_{B_\epsilon(0)}(d^p(\exp_{\bar{x}}(\eta),\exp_{\bar{y}}(T\eta))-d^p(\bar{x},\bar{y}))d\eta\leqslant o(\epsilon^2). \tag{1.12}$$

Notice that $\Delta v=(1+\varepsilon)e_u(x)D_t(x)+\theta_0$ implies that v is smooth near $\bar{x}$, it follows that

$$-\fint_{B_\epsilon(0)}(v(\exp_{\bar{x}}(\eta))-v(\bar{x}))d\eta\leqslant-\frac{1}{2(n+2)}[(1+\varepsilon)e_u(\bar{x})D_t(\bar{x})+\theta_0]\cdot\epsilon^2+o(\epsilon^2). \tag{1.13}$$

Thus, we only need to show an asymptotic mean value inequality that

$$-\fint_{B_\epsilon(0)}(F(\exp_{\bar{x}}(\eta),\exp_{\bar{y}}(T\eta))-F(\bar{x},\bar{y}))d\eta\leqslant\frac{1+\varepsilon}{2(n+2)}e_u(\bar{x})D_t(\bar{x})\cdot\epsilon^2+o(\epsilon^2). \tag{1.14}$$

Indeed, once one has proved (1.14), the combination of (1.12)–(1.14) contradicts the fact that $H(x,y)$ has a minimum at $(\bar{x},\bar{y})$, and hence it follows (1.10).

In order to show (1.14), we need to choose a suitable function $\phi(s)$ in (1.8). In the simplest case that $\kappa=0$ and $p=q=2$, we can choose directly $\phi(s)=s$, as in [45].

In the case of $\kappa=1$ (and general $q\in(1,2]$), the definition of $CAT(1)$ suggests us to choose $\phi(s)=2\sin(s/2)$. However, this is not convex for small $s>0$. An exact relation in $CAT(1)$-spaces, Lemma 2.4, suggests us to perturb $2\sin(s/2)$ to

$$\phi(s)=2\sin(s/2)+4\sin^2(s/2).$$

Fortunately, this is convex for small $s>0$.

Given any $a,b\in\mathbb{R}$ with $a,b\geqslant 0$, and fixing any $q\in\Omega$, $Q\in X$, we define a function near q by

$$w_{a,b,Q,q}(x):=a\cdot d_X^2(u(x),u(q))+b\cdot\cos(d_X(u(x),Q)).$$

Since (X,d_X) has curvature $\leqslant 1$, by combining that e_ϵ^u converge to e_u as $\epsilon\to 0$ and the fact

$$\Delta\cos(d_X(u(x),Q))\leqslant-\cos(d_X(u(x),Q))\cdot e_u(x),$$

we will be able to deduce that, for almost all q, an asymptotic mean value inequality for $w_{a,b,Q,q}$ holds (for some subsequence $\epsilon_j\to 0$, see Lemma 3.3 for the explicit statements).

On the other hand, the assumption (X,d_X) having curvature less than or equal to 1 implies also that, for any q_1 and q_2, the function $w_{a_2,b,Q_m,q_1}+w_{a_1,b,Q_m,q_2}$ touches $-F(\cdot,\cdot)$ by above at (q_1,q_2) for some suitable constants $a_1,a_2,b\geqslant 0$, where Q_m is the mid-point of $u(q_1)$ and $u(q_2)$ (the details is given in Lemma 2.4). Therefore, we conclude that an asymptotic mean value inequality for $-F(\cdot,\cdot)$ at almost all (q_1,q_2) holds (see (4.12) and Lemma 3.3 for the explicit formulas). First, let us assume briefly that it holds the mentioned asymptotic mean value inequality for $-F(\cdot,\cdot)$ at $(\bar{x},\bar{y})$. Then we conclude (1.14) in this case. The primary issue is that there is no reason we can assume that it holds the asymptotic mean value inequality for $-F(\cdot,\cdot)$ at $(\bar{x},\bar{y})$. In this case, we will perturb the function $H(x,y)$ to $H_1(x,y):=H(x,y)+\gamma_\delta(x,y)$ by a smooth function $\gamma_\delta(x,y)$, which is arbitrarily small up to two order derivatives, such that the mentioned asymptotic mean value inequality for $-F(\cdot,\cdot)$ holds at one of minimum of $H_1(x,y)$. This argument of perturbation can be ensured by a generalized Jensen's lemma in the theory of viscosity solutions of second order partial differential equations.

2 Preliminaries

2.1 Energy and Sobolev spaces of maps into metric spaces

Let Ω be a bounded open domain of an n-dimensional smooth Riemannian manifold (M,g), and let (X,d_X) be a complete metric space. We will write

$$|xy|:=d(x,y),\quad\forall\,x,y\in M.$$

Several equivalent notions of the Sobolev space for maps into metric spaces have been introduced in [18, 25, 28, 29, 35]. Fix any $p\in[1,\infty)$. A Borel measurable map $u:\ \Omega\to X$ is said to be in the space $L^p(\Omega,X)$ if it has separable range and, for some (hence, for all) $P\in X$,

$$\int_\Omega d_X^p(u(x),P)dv_g(x)<\infty.$$

We equip with a distance in $L^p(\Omega,X)$ by

$$d_{L^p}^p(u,v):=\int_\Omega d_X^p(u(x),v(x))dv_g(x),\quad\forall\,u,v\in L^p(\Omega,X).$$

Denote by $C_0(\Omega)$ the set of continuous functions compactly supported on Ω. Given $p \in [1,\infty)$ and a map $u \in L^p(\Omega, X)$, for each $\epsilon > 0$, the *approximating energy* $E^u_{p,\epsilon}$ is defined as a functional on $C_0(\Omega)$:

$$E^u_{p,\epsilon}(\phi) := \int_\Omega \phi(x) e^u_{p,\epsilon}(x) dv_g(x), \quad \forall\, \phi \in C_0(\Omega),$$

where the *approximating energy density* is defined by

$$e^u_{p,\epsilon}(x) = e^u_{p,\epsilon,g}(x) := \frac{n+p}{c_{n,p} \cdot \epsilon^n} \int_{B_\epsilon(x)\cap\Omega} \frac{d^p_X(u(x), u(y))}{\epsilon^p} dv_g(y),$$

and the constant $c_{n,p} = \int_{\mathbb{S}^{n-1}} |x^1|^p \sigma(dx)$, and σ is the canonical Riemannian volume on $\mathbb{S}^{n-1}$. In particular, $c_{n,2} = \omega_{n-1}/n$, where ω_{n-1} is the volume of $(n-1)$-sphere $\mathbb{S}^{n-1}$ with standard metric. Next, a map $u \in L^p(\Omega, X)$ is said to be in $W^{1,p}(\Omega, X)$ if the energy satisfies $E^u_p < \infty$, where

$$E^u_p := \sup_{\phi \in C_0(\Omega), 0 \leqslant \phi \leqslant 1} \Big(\limsup_{\epsilon \to 0} E^u_{p,\epsilon}(\phi) \Big).$$

If $1 < p < \infty$ and $u \in W^{1,p}(\Omega, X)$, it was proved in [28] that, for each $\phi \in C_0(\Omega)$, the limit

$$E^u_p(\phi) := \lim_{\epsilon \to 0^+} E^u_{p,\epsilon}(\phi)$$

exists (called p-th energy functional of u), and that E^u_p is absolutely continuous with respect to the Riemannian volume vol_g. Denote the density by $e_{u,p} \in L^1_{\mathrm{loc}}(\Omega)$. Moreover, from [28, Lemma 1.4.2], there exists a constant $C > 0$, independent of ϵ such that

$$E^u_{p,\epsilon}(\phi) \leqslant E^u_p \Big(C\epsilon\phi + \max_{y \in B_\epsilon(x)} |\phi(y) - \phi(x)| \Big)$$

for any sufficiently small $\epsilon > 0$. Thus, by the Dunfold-Pettis theorem, it implies that

$$e^u_{p,\epsilon} \to e_{u,p} \quad \text{in} \quad L^1_{\mathrm{loc}}(\Omega), \quad \text{as} \quad \epsilon \to 0.$$

For the special case $p = 2$, we write $e_u := e_{u,2}$ and $E^u := E^u_2$ for any $u \in W^{1,2}(\Omega, X)$. We summarize some main properties of $W^{1,2}(\Omega, X)$, which can be found in [28, 29].

Proposition 2.1. *Let $u \in W^{1,2}(\Omega, X)$.*

(1) (Lower semi-continuity) *For any sequence $u_j \to u$ in $L^2(\Omega, X)$ as $j \to \infty$, we have*

$$E^u(\phi) \leqslant \liminf_{j\to\infty} E^{u_j}(\phi), \quad \forall\, 0 \leqslant \phi \in C_0(\Omega).$$

(2) (Equivalence for $X = \mathbb{R}$) *If $X = \mathbb{R}$, the above space $W^{1,2}(\Omega, \mathbb{R})$ is equivalent to the usual Sobolev space $W^{1,2}(\Omega)$.*

(3) (Weak Poincaré inequality, see, for example, [29, Theorem 4.2]) *For any ball $B_R(q)$ with $B_{6R}(q) \subset\subset \Omega$, there exists a constant $C_{n,K,R} > 0$ such that the following holds: for any $z \in B_R(q)$ and any $r \in (0, R/2)$, we have*

$$\int_{B_r(z)} \int_{B_r(z)} d^2_X(u(x), u(y)) dv_g(x) dv_g(y) \leqslant C_{n,K,R} \cdot r^{n+2} \cdot \int_{B_{6r}(z)} e_u(x) dv_g(x). \tag{2.1}$$

Remark 2.2. By a rescaling argument, one can easily improve the constant $C_{n,K,R}$ in (2.1) to a constant C_{n,KR^2} depending only on n and KR^2. Indeed, let us consider the rescaling the metric on M by $g_R := R^{-2}g$. Then we have $\mathrm{Ric}_{g_R} \geqslant -KR^2$ and $dv_{g_R} = R^{-n} dv_g$. By the definition of $e^u_{p,\epsilon,g}$, we get $e^u_{p,R^{-1}\epsilon,g_R} = R^p \cdot e^u_{p,\epsilon,g}$. Therefore, by the definition of $e_{u,p}$, the Poincaré constant in (2.1) is invariant with respect to the rescaling $g \mapsto g_R$.

2.2 $CAT(\kappa)$-spaces

Let us review firstly the concept of spaces with curvature bounded above (globally) in the sense of Alexandrov.

Definition 2.3 (See, for example, [2, 9])**.** A geodesic space (X, d_X) is called to be globally *curvature bounded above by* κ in the sense of Alexandrov, for some $\kappa \in \mathbb{R}$, denoted by $CAT(\kappa)$, if the following comparison property holds: Given any triangle $\triangle PQR \subset X$ such that

$$d_X(P,Q) + d_X(Q,R) + d_X(R,P) < 2\pi/\sqrt{\kappa}$$

if $\kappa > 0$ and point $S \in QR$ with

$$d_X(Q,S) = d_X(R,S) = \frac{1}{2} d_X(Q,R),$$

then there exists a comparison triangle $\triangle\bar{P}\bar{Q}\bar{R}$ in the simply connected 2-dimensional space form $\mathbb{S}^2_\kappa$ with standard metric with sectional curvature equal κ and the point $\bar{S} \in \bar{Q}\bar{R}$ with

$$d_{\mathbb{S}^2_\kappa}(\bar{Q},\bar{S}) = d_{\mathbb{S}^2_\kappa}(\bar{R},\bar{S}) = \frac{1}{2} d_{\mathbb{S}^2_\kappa}(\bar{Q},\bar{R})$$

such that

$$d_X(P,S) \leqslant d_{\mathbb{S}^2_\kappa}(\bar{P},\bar{S}).$$

It is obvious that (X, d_X) is a $CAT(\kappa)$-space if and only if the rescaled space $(X, \sqrt{\kappa} \cdot d_X)$ is a $CAT(1)$-space, for any $\kappa > 0$.

We need a lemma, which follows from [28, Corollary 2.1.3].

Lemma 2.4. *Let (X, d_X) be an $CAT(1)$ space. Take any ordered sequence $\{P,Q,R,S\} \subset X$, and let point Q_m be the mid-point of QR. we denote the distance $d_X(A,B)$ abbreviatedly by d_{AB}. Then, for any $0 \leqslant \alpha \leqslant 1$ and $\beta > 0$, we have*

$$\begin{aligned}
&\frac{1-\alpha}{2}\left(\left(2\sin\frac{d_{QR}}{2}\right)^2 - \left(2\sin\frac{d_{PS}}{2}\right)^2\right) + \alpha\left(2\sin\frac{d_{QR}}{2}\right)\left(2\sin\frac{d_{QR}}{2} - 2\sin\frac{d_{PS}}{2}\right) \\
&\quad \leqslant \left[1 - \frac{1-\alpha}{2}\left(1-\frac{1}{\beta}\right)\right]\left(2\sin\frac{d_{PQ}}{2}\right)^2 + 2\cos\frac{d_{QR}}{2}(\cos d_{PQ_m} - \cos d_{QQ_m}) \\
&\quad\quad + \left[1 - \frac{1-\alpha}{2}(1-\beta)\right]\left(2\sin\frac{d_{RS}}{2}\right)^2 + 2\cos\frac{d_{QR}}{2}(\cos d_{SQ_m} - \cos d_{RQ_m}).
\end{aligned} \tag{2.2}$$

Proof. Consider the embedding X into the cone $C(X)$ with the cone metric $|\cdot\cdot|_C$. Then $C(X)$ has non-positive curvature in the sense of Alexandrov. Denote

$$\bar{P} = (P,1), \quad \bar{Q} = (Q,1), \quad \bar{S} = (S,1), \quad \bar{R} = (R,1) \quad \text{and} \quad \bar{Q}_m = (Q_m,1).$$

It is clear that the midpoint of $\bar{Q}$ and $\bar{R}$ in $C(X)$ is

$$\bar{T} = \left(Q_m, \cos\frac{d_{QR}}{2}\right).$$

From the equation (2.1v) in [28, Corollary 2.1.3] (by taking $t = 1/2$ there), we get, for each $\alpha \in [0,1]$, that

$$\begin{aligned}
|\bar{T}\bar{P}|_C^2 + |\bar{T}\bar{S}|_C^2 &\leqslant |\bar{P}\bar{Q}|_C^2 + |\bar{R}\bar{S}|_C^2 + \frac{1}{2}(|\bar{S}\bar{P}|_C^2 - |\bar{Q}\bar{R}|_C^2) + \frac{1}{2}|\bar{Q}\bar{R}|_C^2 \\
&\quad - \frac{1}{2}(\alpha(|\bar{S}\bar{P}|_C - |\bar{Q}\bar{R}|_C)^2 + (1-\alpha)(|\bar{R}\bar{S}|_C - |\bar{P}\bar{Q}|_C)^2).
\end{aligned}$$

Notice that $|\bar{Q}\bar{R}|_C = 2|\bar{T}\bar{Q}|_C = 2|\bar{T}\bar{R}|_C$ and that

$$(|\bar{S}\bar{P}|_C^2 - |\bar{Q}\bar{R}|_C^2) - \alpha(|\bar{S}\bar{P}|_C - |\bar{Q}\bar{R}|_C)^2 = (1-\alpha)(|\bar{S}\bar{P}|_C^2 - |\bar{Q}\bar{R}|_C^2) + 2\alpha|\bar{Q}\bar{R}|_C(|\bar{S}\bar{P}|_C - |\bar{Q}\bar{R}|_C).$$

Therefore, we obtain

$$
\begin{aligned}
&\frac{1-\alpha}{2}(|\bar{Q}\bar{R}|_C^2-|\bar{S}\bar{P}|_C^2)+\alpha|\bar{Q}\bar{R}|_C(|\bar{Q}\bar{R}|_C-|\bar{S}\bar{P}|_C)\\
&\quad\leqslant |\bar{P}\bar{Q}|_C^2+|\bar{T}\bar{Q}|_C^2-|\bar{T}\bar{P}|_C^2+|\bar{S}\bar{R}|_C^2+|\bar{T}\bar{R}|_C^2-|\bar{T}\bar{S}|_C^2\\
&\qquad-\frac{1-\alpha}{2}(|\bar{R}\bar{S}|_C-|\bar{P}\bar{Q}|_C)^2\\
&\quad\leqslant |\bar{P}\bar{Q}|_C^2+|\bar{T}\bar{Q}|_C^2-|\bar{T}\bar{P}|_C^2+|\bar{S}\bar{R}|_C^2+|\bar{T}\bar{R}|_C^2-|\bar{T}\bar{S}|_C^2\\
&\qquad-\frac{1-\alpha}{2}\left(|\bar{R}\bar{S}|_C^2+|\bar{P}\bar{Q}|_C^2-\beta|\bar{R}\bar{S}|_C^2-\frac{1}{\beta}|\bar{P}\bar{Q}|_C^2\right)\\
&\quad=|\bar{P}\bar{Q}|_C^2\left(1-\frac{1-\alpha}{2}\left(1-\frac{1}{\beta}\right)\right)+|\bar{T}\bar{Q}|_C^2-|\bar{T}\bar{P}|_C^2\\
&\qquad+|\bar{S}\bar{R}|_C^2\left(1-\frac{1-\alpha}{2}(1-\beta)\right)+|\bar{T}\bar{R}|_C^2-|\bar{T}\bar{S}|_C^2
\end{aligned}
\tag{2.3}
$$

for any $\beta>0$, where we have used $2|\bar{R}\bar{S}|_C\cdot|\bar{P}\bar{Q}|_C\leqslant\beta|\bar{R}\bar{S}|_C^2+\frac{1}{\beta}|\bar{P}\bar{Q}|_C^2$. By recalling the definition of the cone metric $|\cdot\cdot|_C$, we have

$$
\begin{aligned}
&|\bar{Q}\bar{R}|_C=2\sin\frac{d_{QR}}{2},\quad |\bar{S}\bar{P}|_C=2\sin\frac{d_{SP}}{2},\\
&|\bar{P}\bar{Q}|_C=2\sin\frac{d_{PQ}}{2},\quad |\bar{R}\bar{S}|_C=2\sin\frac{d_{RS}}{2},\\
&|\bar{T}\bar{Q}|_C=|\bar{T}\bar{R}|_C=\frac{|\bar{Q}\bar{R}|_C}{2}=\sin\frac{d_{QR}}{2}
\end{aligned}
$$

and (by noticing that $|O\bar{T}|_C=\cos\frac{d_{QR}}{2}$)

$$
\begin{aligned}
&|\bar{T}\bar{P}|_C^2=1+\cos^2\frac{d_{QR}}{2}-2\cos\frac{d_{QR}}{2}\cos d_{PQ_m},\\
&|\bar{T}\bar{S}|_C^2=1+\cos^2\frac{d_{QR}}{2}-2\cos\frac{d_{QR}}{2}\cos d_{SQ_m}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|\bar{T}\bar{Q}|_C^2-|\bar{T}\bar{P}|_C^2&=\sin^2\frac{d_{QR}}{2}-1-\cos^2\frac{d_{QR}}{2}+2\cos\frac{d_{QR}}{2}\cos d_{PQ_m}\\
&=2\cos\frac{d_{QR}}{2}\left(\cos d_{PQ_m}-\cos\frac{d_{QR}}{2}\right)\\
&=2\cos\frac{d_{QR}}{2}(\cos d_{PQ_m}-\cos d_{QQ_m}),
\end{aligned}
$$

where we have used $d_{QQ_m}=\frac{d_{QR}}{2}$. Similarly, we have

$$
|\bar{T}\bar{R}|_C^2-|\bar{T}\bar{S}|_C^2=2\cos\frac{d_{QR}}{2}(\cos d_{SQ_m}-\cos d_{RQ_m}).
$$

Therefore, the combination of these and the estimate (2.3) implies the desired (2.2). The proof is completed. □

2.3 Harmonic maps

In the following, we always assume that Ω is a bounded domain in an n-dimensional smooth Riemannian manifold (M,g) with $\mathrm{Ric}_M\geqslant -K$ for some $K\geqslant 0$ and that (X,d_X) is a $CAT(\kappa)$ space for some $\kappa\geqslant 0$.

Given any $\phi\in W^{1,2}(\Omega,X)$, we set

$$
W^{1,2}_\phi(\Omega,X):=\{u\in W^{1,2}(\Omega,X): d_X(u(x),\phi(x))\in W^{1,2}_0(\Omega)\}.
$$

By using the variation method, it was proved in [25,30] that there exists a unique $u \in W^{1,2}_{\phi}(\Omega, X)$ which is a minimizer of energy E^u_2 in $W^{1,2}_{\phi}(\Omega, X)$, i.e., the energy $E^u := E^u_2 = E^u_2(\Omega)$ of u satisfies

$$E^u = \inf_w \{E^w : w \in W^{1,2}_{\phi}(\Omega, X)\}.$$

Such an energy minimizing map is called a *harmonic map*.

The basic existence and regularity were given by Korevaar and Schoen [28] for $\kappa \leqslant 0$ and by Serbinowski [39] for $\kappa > 0$. We state their regularity result in the case $\kappa > 0$ (see also [33, Theorem 2.3]).

Theorem 2.5 (See [28,39])**.** *Let u be a harmonic map from Ω to X. Assume that its image $u(\Omega)$ is contained in a ball $B_\rho \subset X$ with radius $\rho < \frac{\pi}{2\sqrt{\kappa}}$ if $\kappa > 0$. Then u is locally Lipschitz continuous in the interior of Ω.* (*Note that the local Lipschitz constant of u near a point $x \in \Omega$ depends on the C^1-norm of metric g near x.*)

We need also the following property.

Lemma 2.6 (See [39, Proposition 1.17] and [13, Lemma 2])**.** *Let $\kappa > 0$. Assume that its image $u(\Omega)$ is contained in a ball $B_\rho(P) \subset X$ with radius $\rho < \frac{\pi}{2\sqrt{\kappa}}$. Then the function $f_P(x) := \cos(\sqrt{\kappa} \cdot d_X(u(x), P))$ satisfies $f_P \in W^{1,2}(\Omega)$ and*

$$\Delta f_P \leqslant -\kappa \cdot f_P \cdot e_u \tag{2.4}$$

in the sense of distributions. If $\kappa = 0$, then for any $P \in X$ we have $\Delta d^2_X(P, u(x)) \geqslant 2e_u$ in the sense of distributions.

Recall that

$$\mathrm{Lip} u(x) = \limsup_{y \to x} \frac{d_X(u(x), u(y))}{|xy|} = \limsup_{r \to 0} \sup_{y \in B_x(r)} \frac{d_X(u(x), u(y))}{r}.$$

The above lemma implies the following point-wise estimates, which is a corollary of the mean value inequality for subharmonic functions.

Corollary 2.7. *Let u be a harmonic map from Ω to X. Assume that its image $u(\Omega)$ is contained in a ball $B_\rho \subset X$ with radius $\rho < \frac{\pi}{2\sqrt{\kappa}}$ if $\kappa > 0$. Then there exists a constant $C = C(n, \sqrt{K}R)$ depending only on n and $\sqrt{K}R$ such that: for any ball B_R with $B_{2R} \subset\subset \Omega$, we have*

$$\mathrm{Lip}^2 u(x) \leqslant C \cdot e_u(x), \quad \textit{for almost all} \quad x \in B_{R/6}.$$

Proof. For the case $\kappa = 0$, this is Theorem 5.5 in [45]. We need only to show the assertion for the case $\kappa > 0$. Without loss of the generality, we can assume $\kappa = 1$ in this case. The argument is similar to the proof of Theorem 5.5 in [45].

(i) Fix any z with $B_{2R}(z) \subset\subset \Omega$. From the continuity of u, there exists a small neighborhood O of z such that $\mathrm{diam} u(O) < \pi/2$ and $O \subset B_R(z)$, where $\mathrm{diam} u(O)$ is the diameter of $u(O)$.

By using Lemma 2.6 and the fact that $|\nabla d_X(u(x), P)| \leqslant e_u$ for any fixed $P \in X$, it is easy to check that $\Delta d_X(u(x), u(y_0)) \geqslant 0$ on O for any fixed $y_0 \in O$, in the sense of distributions. Let $\Delta^{(2)}$ be the Laplace-Beltrami operator on $M \times M$, the product manifold (with the product metric and the product measure). Consider the function $\rho_u := d_X(u(x), u(y))$ on $O \times O$. Hence, we obtain

$$\Delta^{(2)} \rho_u(x, y) \geqslant 0 \quad \text{on} \quad O \times O \tag{2.5}$$

in the sense of distributions (see the step (iii) in the proof of [45, Proposition 5.4] for the details).

From the mean value inequality for subharmonic functions on $O \times O$ (see [38, Theorem 6.2 of Chapter II]), we conclude that, for any ball $B_r((z_1, z_2))$ with $B_{2r}((z_1, z_2)) \subset\subset O \times O$,

$$\begin{aligned} \sup_{(x,y) \in B_r((z_1,z_2))} \rho^2_u(x, y) &\leqslant C_1 2^{C_2(1+\sqrt{K}r)} \cdot \fint_{B_{2r}((z_1,z_2))} \rho^2_u(x, y) dv_g(x) dv_g(y) \\ &\leqslant C_3(n, \sqrt{K}R) \cdot \fint_{B_{2r}((z_1,z_2))} \rho^2_u(x, y) dv_g(x) dv_g(y), \end{aligned} \tag{2.6}$$

where the constants C_1 and C_2 depend only on n, and $C_3(n,\sqrt{K}R)=C_1 2^{C_2(1+\sqrt{K}R)}$.

(ii) Since $B_{2r}((z,z))\subset B_{2r}(z)\times B_{2r}(z)$, the Poincaré inequality for $W^{1,2}(\Omega,X)$-maps (see [29], and also Proposition 2.1(3) and Remark 2.2) states that the right-hand side of (2.6) for $z_1=z_2=z$ can be dominated by $C_4(n,\sqrt{K}R)\cdot r^{n+2}\int_{B_{12r}(z)}e_u(x)dv_g(x)$. Therefore, we have

$$\begin{aligned}\sup_{y\in B_r(z)}\frac{\rho_u^2(z,y)}{r^2}&\leqslant\sup_{(x,y)\in B_r((z,z))}\frac{\rho_u^2(x,y)}{r^2}\\&\leqslant C_3C_4\cdot\frac{r^n\cdot\mathrm{vol}(B_z(12r))}{\mathrm{vol}(B_{2r}((z,z))\subset\Omega\times\Omega)}\fint_{B_{12r}(z)}e_u(x)dv_g(x)\\&\leqslant C_5(n,\sqrt{K}R)\cdot\fint_{B_{12r}(z)}e_u(x)dv_g(x),\end{aligned}\tag{2.7}$$

where we have used Bishop-Gromov inequality and $\mathrm{vol}(B_{2r}((z,z)))\geqslant\mathrm{vol}^2(B_r(z))$.

Notice that $\lim_{r\to0}\fint_{B_{12r}(z)}e_u(x)dv_g(x)=e_u(z)$ for almost all $z\in B_{R/6}$ and that

$$\mathrm{Lip}u(z)=\limsup_{r\to0}\sup_{y\in B_r(z)}\rho_u(y,z)/r.$$

By letting $r\to0$ in (2.7), it follows the desired estimate. □

2.4 Generalized Rademacher theorem for Lipschitz maps

Let Ω be a bounded domain of an n-dimensional Riemannian manifold (M,g). Recall that the classical Rademacher theorem states that any Lipschitz function $f:\Omega\to\mathbb{R}$ is differentiable at almost all $x\in\Omega$.

For our purpose, we have to consider the differentiability of maps into a metric space (X,d_X). Let us recall the notion of *metric differential* for maps from Ω into a metric space, which was introduced by Kirchheim [27].

Definition 2.8. We say that a map $f:\Omega\to X$ is *metrically differentiable* at x_0 if there exists a semi-norm $\|\cdot\|_{x_0}$ in $T_{x_0}M:=\mathbb{R}^n$ such that

$$d_X(f(\exp_{x_0}(t\xi)),f(x_0))-t\cdot\|\xi\|_{x_0}=o(t),$$

for all $\xi\in\mathbb{S}^{n-1}\subset T_{x_0}M$. This semi-norm will be called the *metric differential* and be denoted by mdf_{x_0}.

The following generalized Rademacher's theorem for maps was given in [27].

Theorem 2.9 (See [27])**.** *Any Lipschitz map $f:\Omega\to X$ is metrically differentiable at almost all $x\in\Omega$.*

If a Lipschitz continuous map $f:\Omega\to X$ is metrically differentiable at x, we put

$$G_f(x):=\max_{\xi\in\mathbb{S}^{n-1}}\mathrm{mdf}_x(\xi).\tag{2.8}$$

Lemma 2.10. *Let $f:\Omega\to X$ be a Lipschitz function. If f is metrically differentiable at x, then we have*

$$G_f(x)=\mathrm{Lip}f(x).\tag{2.9}$$

Proof. From the definition of $G_f(x)$, it is clear that $G_f(x)\leqslant\mathrm{Lip}f(x)$.

For the converse, we choose a sequence of points $\{y_j:=\exp_x(t_j\xi_j)\}_{j=1}^\infty\subset\Omega$ such that $\lim_{j\to\infty}t_j=0$, $|\xi_j|=1$, and

$$\mathrm{Lip}f(x)=\lim_{j\to\infty}\frac{d_X(f(y_j),f(x))}{t_j}.$$

Since f is metrically differentiable at x, we have

$$d_X(f(y_j),f(x))=\mathrm{mdf}_x(\xi_j)\cdot t_j+o(t_j),$$

From the definition of $G_f(x)$, we have

$$\mathrm{Lip}f(x)=\lim_{j\to\infty}\mathrm{mdf}_x(\xi_j)\leqslant G_f(x).$$

The proof is completed. □

3 An asymptotic mean value inequality

We will consider some asymptotic behaviors of harmonic maps from a domain of the smooth Riemannian manifold to a $CAT(1)$-space. Let us begin with the following mean value property, which is similar to Proposition 2.1 of Chapter I in [38].

Lemma 3.1. *Let (M,g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geqslant -K$ for some $K\in\mathbb{R}$. Suppose that f is a Lipschitz function on an open subset $\Omega\subset M$, $f\geqslant 0$, and $\Delta f\leqslant g\in L^1_{\mathrm{loc}}(\Omega)$ in the sense of distributions. Then for any $p\in\Omega$ and $R>0$ with $B_R(p)\subset\subset\Omega$,*

$$\frac{1}{A_K(R)}\int_{\partial B_R(p)} f\leqslant f(p)+\int_0^R\frac{\int_{B_r(p)}g(x)dv_g(x)}{A_K(r)}dr, \tag{3.1}$$

where $A_K(r)$ is the area of a geodesic sphere of radius r in the simply connected space form of constant curvature $-K/(n-1)$.

Proof. Since $\Delta f\leqslant g$, we have by the divergence theorem that

$$\begin{aligned}\int_{B_r(p)} g(x)dv_g(x)\geqslant\int_{B_r(p)}\Delta f dv_g&=\int_{\partial B_r(p)}\frac{\partial f}{\partial r}\\ &=\frac{\partial}{\partial r}\int_{\partial B_r(p)} f-\int_{\partial B_r(p)} Hf,\end{aligned} \tag{3.2}$$

where $0<r<R$, and H is the mean curvature of $\partial B_r(p)$ with resect to $\partial/\partial r$. The standard comparison theorem asserts that

$$H(x)\leqslant(n-1)\cot_K(r)=\frac{A_K'(r)}{A_K(r)},\quad \forall\, x\in\partial B_r(p).$$

Therefore, it follows from (3.2) and the assumption $f\geqslant 0$ that

$$\frac{\int_{B_r(p)}g(x)dv_g(x)}{A_K(r)}\geqslant\frac{\partial}{\partial r}\frac{\int_{\partial B_r(p)}f}{A_K(r)}.$$

Notice that $\lim_{r\to 0}\frac{\int_{\partial B_r(p)}f}{A_K(r)}=f(p)$. Integrating both sides of the above inequality with respect to r over $(0,R)$, we conclude that (3.1) holds. □

Now we consider the case that f needs not to be non-negative.

Corollary 3.2. *Let (M,g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}_M\geqslant -K$ for some $K\in\mathbb{R}$. Suppose that f is a Lipschitz function on an open subset $\Omega\subset M$, and $\Delta f\leqslant g\in L^1_{\mathrm{loc}}(\Omega)$ in the sense of distributions. Then for any $p\in\Omega$ and $R>0$ with $B_R(p)\subset\subset\Omega$,*

$$\begin{aligned}&\frac{1}{V_K(R)}\int_{B_R(p)}(f(x)-f(p))dv_g(x)\\ &\quad\leqslant R\cdot\mathrm{Lip}_{B_R(p)}f\cdot\left(1-\frac{\mathrm{vol}(B_R(p))}{V_K(R)}\right)\\ &\qquad+\frac{1}{V_K(R)}\int_0^R A_K(r)\int_0^r\frac{\int_{B_s(p)}g(x)dv_g(x)}{A_K(s)}dsdr,\end{aligned} \tag{3.3}$$

where $V_K(r)$ is the is the volume of a geodesic ball of radius r in the space form of constant curvature $-K/(n-1)$, and

$$\mathrm{Lip}_{B_R(p)}f:=\sup_{x,y\in B_R(p)}\frac{|f(x)-f(y)|}{|xy|}.$$

In particular, if p is a Lebesgue point of g, then the following asymptotic mean value inequality holds:

$$\frac{1}{V_K(R)}\int_{B_R(p)}(f(x)-f(p))dv_g(x)\leqslant\frac{g(p)}{2(n+2)}\cdot R^2+o(R^2)\quad\text{as}\quad R\to 0. \tag{3.4}$$

Proof. We consider the function $h(x) := f(x) - f(p)$. By applying Lemma 3.1 to the non-negative function

$$h_r(x) := h(x) - \inf_{y\in B_r(p)} h(y)$$

on $B_r(p)$, $0 < r < R$, we have

$$\frac{1}{A_K(r)}\int_{\partial B_r(p)} h_r \leqslant h_r(p) + G_K(r) = -\inf_{y\in B_r(p)} h(y) + G_K(r),$$

where

$$G_K(r) := \int_0^r \frac{\int_{B_s(p)} g(x)dv_g(x)}{A_K(s)}ds.$$

Denote $A(r) := \mathrm{vol}_{n-1}(\partial B_r(P)\subset M)$. We get

$$\int_{\partial B_r(p)} h \leqslant -\inf_{y\in B_r(p)} h(y)\cdot (A_K(r) - A(r)) + G_K(r)\cdot A_K(r).$$

Remark that $h(p) = 0$ and $\mathrm{Lip}_{B_r(p)} h \leqslant \mathrm{Lip}_{B_R(p)} h = \mathrm{Lip}_{B_R(p)} f$, so we have

$$-\inf_{y\in B_r(p)} h(y) \leqslant -\inf_{y\in B_R(p)} h(y) \leqslant R\cdot \mathrm{Lip}_{B_R(p)} f.$$

The above two inequalities imply

$$\int_{\partial B_r(p)} h \leqslant R\cdot \mathrm{Lip}_{B_R(p)} f\cdot (A_K(r) - A(r)) + G_K(r)\cdot A_K(r),$$

where we have used the Bishop inequality $A(r) \leqslant A_K(r)$ for all $r \in (0, R)$. Integrating both sides of the above inequality with respect to r over $(0, R)$, and then dividing by $V_K(R)$, we get (3.3).

Suppose that p is a Lebesgue point of g, i.e.,

$$\lim_{R\to 0} \fint_{B_R(p)} g(x)dv_g(x) = g(p). \tag{3.5}$$

Notice that

$$\frac{\mathrm{vol}(B_R(p))}{V_K(R)} = 1 + O(R^2) \quad \text{as} \quad R \to 0. \tag{3.6}$$

It follows that

$$R\cdot \mathrm{Lip}_{B_R(p)} f\cdot \left(1 - \frac{\mathrm{vol}(B_R(p))}{V_K(R)}\right) = O(R^3) \quad \text{as} \quad R\to 0$$

and that

$$\frac{1}{V_K(R)}\int_{B_R(p)} g(x)dv_g(x) = g(p) + o(1) \quad \text{as} \quad R \to 0.$$

Thus, by a direct calculation (noticing that $A_K(t) = \omega_{n-1}\cdot t^{n-1} + O(t^n)$ as $t \to 0$), we get

$$\begin{aligned}
&\frac{1}{V_K(R)}\int_0^R A_K(r)\int_0^r \frac{\int_{B_s(p)} g(x)dv_g(x)}{A_K(s)}dsdr\\
&= \frac{1}{V_K(R)}\int_0^R A_K(r)\int_0^r \frac{V_K(s)\cdot (g(p)+o(1))}{A_K(s)}dsdr\\
&= \frac{1}{V_K(R)}\int_0^R A_K(r)\int_0^r \frac{s}{n}(1+o(1))(g(p)+o(1))dsdr\\
&= \frac{g(p)}{2(n+2)}\cdot R^2 + o(R^2)
\end{aligned}$$

as $R \to 0$. The proof is finished. □

At last in this section, we want to use the above asymptotic mean value inequality to harmonic maps to a metric space with curvature bounded above.

Lemma 3.3. *Let Ω be a bounded domain in an n-dimensional smooth Riemannian manifold (M,g) and that (X,d_X) is a $CAT(1)$-space. Suppose that u is a harmonic map from Ω to X. Given any $a,b\in\mathbb{R}$ with $a,b\geqslant 0$, and any $q\in\Omega$, $Q\in X$, we put*

$$w_{a,b,Q,q}(x):=a\cdot d_X^2(u(x),u(q))+b\cdot\cos(d_X(u(x),Q)). \tag{3.7}$$

Then there exist a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ with $\epsilon_j\to 0$ as $j\to\infty$ and a subset $\mathscr{N}\subset\Omega$ with zero measure such that

$$\begin{aligned}&\int_{B_{\epsilon_j}(0)}[w_{a,b,P,x_0}(\exp_{x_0}(\eta))-w_{a,b,P,x_0}(x_0)]d\eta\\&\qquad\leqslant(2a-b\cdot\cos(d_X(u(x_0),P)))\frac{\omega_{n-1}}{2n(n+2)}\cdot e_u(x_0)\cdot\epsilon_j^{n+2}+o(\epsilon_j^{n+2})\end{aligned} \tag{3.8}$$

for any $x_0\in\Omega\backslash\mathscr{N}$ and for any $P\in X$ such that the image $u(\Omega)$ is contained in a ball $B_\rho(P)\subset X$ with radius $\rho<\frac{\pi}{2}$, and for every $a,b\geqslant 0$.

Proof. Recall that $e^u_{2,\epsilon}\to e_u$ in $L^1_{\rm loc}(\Omega)$ as $\epsilon\to 0$. Therefore, there exists a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ with $\epsilon_j\to 0$ as $j\to\infty$ such that

$$\lim_{j\to\infty}e^u_{2,\epsilon_j}(x_0)=e_u(x_0)\quad\text{for almost all}\quad x_0\in\Omega.$$

By the definition of the approximating energy density, it follows that

$$\int_{B_{\epsilon_j}(x_0)}d_X^2(u(x),u(x_0))dv_g(x)=\frac{\omega_{n-1}}{n(n+2)}\cdot e_u(x_0)\cdot\epsilon_j^{n+2}+o(\epsilon_j^{n+2})$$

for almost all points $x_0\in\Omega$. On the other hand, we have by Lemma 2.6 and Corollary 3.2 that

$$\begin{aligned}&\int_{B_{\epsilon_j}(x_0)}[\cos(d_X(u(x),P))-\cos(d_X(u(x_0),P))]dv_g(x)\\&\qquad\leqslant V_K(\epsilon_j)\cdot\left[-\frac{\cos(d_X(u(x_0),P))}{2(n+2)}\cdot e_u(x_0)\cdot\epsilon_j^2+o(\epsilon_j^2)\right]\\&\qquad=-\frac{\cos(d_X(u(x_0),P))\cdot\omega_{n-1}}{2n(n+2)}\cdot e_u(x_0)\cdot\epsilon_j^{n+2}+o(\epsilon_j^{n+2})\end{aligned}$$

for all Lebesgue points x_0 of e_u, and for all $P\in X$ such that the image $u(\Omega)$ is contained in a ball $B_\rho(P)\subset X$ with radius $\rho<\frac{\pi}{2}$. Here, we have used that $V_K(\epsilon_j)=\frac{\omega_{n-1}}{n}\cdot\epsilon_j^n+o(\epsilon_j^n)$. Thus, for almost all $x_0\in\Omega$, we have

$$\begin{aligned}&\int_{B_{\epsilon_j}(x_0)}[w_{a,b,P,x_0}(x)-w_{a,b,P,x_0}(x_0)]dv_g(x)\\&\qquad\leqslant(2a-b\cdot\cos(d_X(u(x_0),P)))\frac{\omega_{n-1}}{2n(n+2)}\cdot e_u(x_0)\cdot\epsilon_j^{n+2}+o(\epsilon_j^{n+2})\end{aligned} \tag{3.9}$$

and any $P\in X$, and for every $a,b\in\mathbb{R}$ with $b\geqslant 0$.

At last, we consider the exponential map $\exp_{x_0}:B_{\epsilon_j}(0)\subset\mathbb{R}^n\to B_{\epsilon_j}(x_0)$. It is well known

$$\frac{dv_g}{d\eta}=1+o(\epsilon_j).$$

Thus, for any general Lipschitz function h, we have, as $r \to 0$,

$$\begin{aligned}&\int_{B_r(0)} (h(\exp_{x_0}(\eta)) - h(x_0))d\eta \\ &\quad\leqslant \int_{B_r(x_0)} (h(x) - h(x_0))(1 + o(r))dv_g(x) \\ &\quad\leqslant \int_{B_r(x_0)} (h(x) - h(x_0))dv_g(x) + o(r) \cdot \int_{B_r(x_0)} |h(x) - h(x_0)|dv_g(x) \\ &\quad\leqslant \int_{B_r(x_0)} (h(x) - h(x_0))dv_g(x) + o(r) \cdot O(r) \cdot O(r^n).\end{aligned}$$

By using this to w_{a,b,P,x_0} and combining with (3.9), we obtain (3.8). □

4 The Bochner inequality for harmonic maps into $CAT(\kappa)$-spaces

Let Ω be a bounded domain in an n-dimensional smooth Riemannian (M, g) with $\mathrm{Ric}_M \geqslant -K$ for some $K \geqslant 0$, and let (X, d_X) be a complete $CAT(\kappa)$-space for some $\kappa > 0$.

In this section, we always assume that $u : \Omega \to X$ is a harmonic map with the image $\mathrm{Im}(u)$ containing in a ball $B_\rho(Q_0) \subset Y$ with $\rho < \pi/(2\sqrt{\kappa})$. From Theorem 2.5, we know that u is local Lipschitz continuous on Ω.

4.1 Auxiliary functions

In this subsection, we will introduce a family of auxiliary functions.

Fix $p \in (1, \infty)$ and a ball $B_R(o)$ such that $B_{2R}(o) \subset\subset \Omega$. Denote by $B_R := B_R(o)$ and by

$$\ell_0 := \mathrm{Lip}_{B_{2R}} u = \sup_{x,y \in B_{2R}, x \neq y} \frac{d_X(u(x), u(y))}{|xy|} < \infty.$$

We introduce a family of auxiliary functions $f_t(x)$ on B_R as follows: for any $t > 0$, we define

$$f_t(x) := \inf_{y \in B_{2R}} \left\{ \frac{|xy|^p}{pt^{p-1}} - F(x, y) \right\}, \quad x \in B_R, \tag{4.1}$$

where

$$F(x, y) := 2 \sin \frac{d_X(u(x), u(y))}{2} + 4 \sin^2 \frac{d_X(u(x), u(y))}{2}.$$

It is clear that $F(x, y) \leqslant 6$ and that (by taking $y = x$)

$$0 \geqslant f_t(x) \geqslant -6. \tag{4.2}$$

For any $0 < t < t_*(:= (R^p/6p)^{1/(p-1)})$, it is clear that the "inf" of (4.1) can be achieved, i.e., for any $x \in B_R$,

$$S_t(x) := \left\{ y \in B_{2R} \,\middle|\, f_t(x) = \frac{|xy|^p}{pt^{p-1}} - F(x, y) \right\} \neq \emptyset.$$

Since $F(x, \cdot)$ is continuous on B_{2R}, it follows that $S_t(x)$ is close. Fix any small $t \in (0, t_*)$. We define two functions on B_R,

$$L_t(x) := \min_{y \in S_t(x)} |xy| \quad \text{and} \quad D_t(x) := \frac{L_t^p(x)}{pt^{p-1}} - f_t(x). \tag{4.3}$$

We give some basic properties of these functions.

Lemma 4.1. *For any $t \in (0, t_*)$, we have the following properties*:

(1) *f_t is Lipschitz continuous on B_R*;

(2) *both L_t and D_t are lower semi-continuous on B_R*;

(3) *there exists a constant $C = C(p, \ell_0, \kappa) > 0$ such that, for any $t \in (0, t_*)$,*

$$L_t \leqslant Ct, \quad D_t \leqslant Ct \quad \text{and} \quad -f_t \leqslant Ct \quad \text{on} \quad B_R.$$

Proof. (1) Take any $x_1, x_2 \in B_R$. From the definition of f_t, we by choosing some $y_2 \in S_t(x_2)$ have that

$$f_t(x_1) - f_t(x_2) \leqslant \frac{|x_1y_2|^p - |x_2y_2|^p}{pt^{p-1}} - (F(x_1, y_2) - F(x_2, y_2)).$$

Noticing that both $|\cdot\cdot|^p$ and $F(\cdot,\cdot)$ are Lipschitz, we conclude that there exists some constant $C_t > 0$ such that $f_t(x_1) - f_t(x_2) \leqslant C_t|x_1x_2|$, i.e., f_t is Lipschitz continuous.

(2) From the definition of L_t, we know that L_t is lower semi-continuous. By using that f_t is continuous, we get that D_t is also lower semi-continuous.

(3) We take some $y_t \in S_t(x)$ such that $L_t(x) = |xy_t|$. We have

$$\begin{aligned} D_t(x) &= \frac{L_t^p(x)}{pt^{p-1}} - f_t(x) = F(x, y_t) \leqslant d_X(u(x), u(y_t)) + d_X^2(u(x), u(y_t)) \\ &\leqslant (1 + \pi/\sqrt{\kappa}) \cdot d_X(u(x), u(y_t)) \leqslant (1 + \pi/\sqrt{\kappa}) \cdot \ell_0 \cdot L_t(x) \\ &:= C_1(\ell_0, \kappa) L_t(x), \end{aligned} \tag{4.4}$$

where we have used

$$d_X(u(x), u(y_t)) \leqslant d_X(u(x), Q_0) + d_X(Q_0, u(y_t)) < \pi/\sqrt{\kappa}.$$

Noticing that $f_t \leqslant 0$, we get $D_t \geqslant \frac{L_t^p(x)}{pt^{p-1}}$. By combining with (4.4), we have

$$L_t \leqslant (p \cdot C_1)^{1/(p-1)} \cdot t := C_2(p, \ell_0, \kappa) \cdot t.$$

By using (4.4) again, we get $D_t \leqslant C_1C_2 \cdot t$. At last, $f_t = L_t^p/(pt^{p-1}) - D_t \geqslant -D_t \geqslant -C_1C_2 \cdot t$. The proof is finished. □

The main result in this subsection is the following elliptic inequality for f_t.

Proposition 4.2. *Assume that $\kappa = 1$, $p \in [2, \infty)$ and $\mathrm{diam}(u(B_{2R})) < \pi/2$, where $\mathrm{diam}(u(B_{2R}))$ is the diameter of the $u(B_{2R})$. Given any $\varepsilon > 0$, we have that, for any $t > 0$ sufficiently small, the inequality holds*

$$\Delta f_t(x) \leqslant \frac{K}{t^{p-1}} \cdot L_t^p(x) + (1 + \varepsilon) \cdot e_u(x) D_t(x) \quad on \quad B_R, \tag{4.5}$$

in the sense of distributions.

In order to prove this lemma, we need the following lemma.

Lemma 4.3 (Perturbation lemma). *Let $U \subset M$ be a convex domain of M and let $h \in W^{1,2}(U) \cap C(U)$ satisfy $\Delta h \leqslant \lambda$ on U for some constant $\lambda \in \mathbb{R}$. Assume that the point $\hat{x} \in U$ is one of minimum points of the function h on U. Assume a subset $A \subset U$ has full measure.*

Then for any $r, \delta > 0$ sufficiently small, (they are smaller than a constant δ_0 depending on the bounds of sectional curvature on U,) there exists a smooth function ϕ on a neighborhood of $\hat{x}$, $B_{r_0}(\hat{x})$, such that $h + \phi$ has a local minimum point in $B_r(\hat{x}) \cap A$ and that

$$|\phi| + |\nabla\phi| + |\mathrm{Hess}(\phi)| \leqslant \delta, \quad \forall\, x \in B_{r_0}(\hat{x}).$$

Proof. This comes from a slight extension of the classical Jensen's lemma [6, Lemma A.3]. We will give the details of the proof in Appendix A. □

Proof of Proposition 4.2. Denote $\rho_0 := \mathrm{diam}(u(B_{2R}))$. It suffices to prove the following claim.

Claim. There exists $\bar{t} = \bar{t}(p, \varepsilon, t_*, \pi/2 - \rho_0)$ such that for each $t \in (0, \bar{t})$, the function $f_t(\cdot)$ satisfies

$$\Delta f_t(x) \leqslant \frac{K}{t^{p-1}} \cdot L_t^p(x) + (1 + \varepsilon) e_u(x) D_t(x) + \theta \quad \text{on} \quad B_R$$

for any $\theta \in (0, 1)$, in the sense of distributions.

We shall prove this claim by a contradiction argument. Suppose that the claim fails for some sufficiently small $t \in (0, t_*)$ and some $\theta_0 \in (0,1)$. According to the maximum principle, there exists a domain $U \subset\subset B_R$ such that the function $f_t(\cdot) - v(\cdot)$ satisfies

$$\min_{x\in U}(f_t(x) - v(x)) < 0 = \min_{x\in \partial U}(f_t(x) - v(x)),$$

where v is the (unique) solution of the Dirichlet problem

$$\begin{cases} \Delta v = \dfrac{K}{t^{p-1}} \cdot L_t^p(x) + (1+\varepsilon)e_u(x)D_t(x) + \theta_0 & \text{in} \quad U, \\ v = f_t & \text{on} \quad \partial U. \end{cases}$$

This means that $f_t(\cdot) - v(\cdot)$ has a *strict minimum* in the interior of U.

Let us define a function $H(x,y)$ on $U \times B_{2R}$, by

$$H(x,y) := \frac{|xy|^p}{pt^{p-1}} - F(u(x), u(y)) - v(x).$$

Let $\bar{x} \in U$ be a minimum of $f_t(\cdot) - v$ on U, and let $\bar{y} := y_t(\bar{x}) \in S_t(\bar{x})$ such that $L_t(\bar{x}) = |\bar{x}\bar{y}|$. By the definition of f_t, we conclude that $H(x,y)$ has a minimum at $(\bar{x}, \bar{y})$.

Let $\mathscr{A} \subset U \times B_{2R}$ be the set of all points $(x^o, y^o) \in U \times B_{2R}$ satisfying the following two properties:

(1) $x^o \neq y^o$, and $x^o \notin \mathscr{N}$, $y^o \notin \mathscr{N}$, where the set $\mathscr{N}$ is given as in Lemma 3.3;

(2) the point x^o is a Lebesgue point of $\frac{K}{t^{p-1}} \cdot L_t^p(x) + (1+\varepsilon)e_u(x)D_t(x)$.

It is clear that $(U \times B_{2R})\backslash\mathscr{A}$ has zero measure.

Noting that the function $f(t) = 2\sin t + 4\sin^2 t$ satisfies $f'(t) > 0, f''(t) > 0$ for $t < \pi/6$ and (see the proof of Corollary 2.7)

$$\Delta^{(2)} d_X(u(x), u(y)) \geqslant 0,$$

we have $\Delta^{(2)}F(x,y) \geqslant 0$, where $\Delta^{(2)}$ is the Laplace-Beltrami operator on the product manifold $M \times M$. The Laplacian comparison theorem on the product space $M \times M$ implies that $\Delta^{(2)}(|xy|^2) \leqslant C_{n,K,\mathrm{diam}(U)}$ for some constant $C_{n,K,\mathrm{diam}(U)} > 0$. By using the assumption $p \geqslant 2$, we obtain $\Delta^{(2)}(|xy|^p) \leqslant C_{n,K,p,\mathrm{diam}(U)}$. Then, by Lemma 4.3, we conclude that, for any sufficiently small $\delta > 0$, there exists a smooth function $\gamma_\delta(x,y)$ such that $|\gamma_\delta| + |\nabla\gamma_\delta| + |\mathrm{Hess}\gamma_\delta| \leqslant \delta$ and that the function

$$H_1(x,y) = H(x,y) + \gamma_\delta(x,y) = \frac{|xy|^p}{pt^{p-1}} - F(u(x), u(y)) - v(x) + \gamma_\delta(x,y)$$

has a minimal point $(x^o, y^o) \in \mathscr{A}$ with

$$d^2_{X\times X}((\bar{x}, \bar{y}), (x^o, y^o)) = |\bar{x}x^o|^2 + |\bar{y}y^o|^2 < \delta^2. \tag{4.6}$$

Let $\sigma : [0, |x^o y^o|] \to M$ be a shortest geodesic with $\sigma(0) = x^o$ and $\sigma(|x^o y^o|) = y^o$ and let $T_{\sigma(t)} : T_{x^o}M \to T_{\sigma(t)}M$ be the parallel transport along $\sigma(t)$. Denote by $T := T_{y^o}$. We want to consider the asymptotic behavior of

$$\begin{aligned} I(\varepsilon_j) &:= \int_{B_{\varepsilon_j}(0)\subset T_{x^o}M=\mathbb{R}^n} [H_1(\exp_{x^o}(\eta), \exp_{y^o}(T\eta)) - H_1(x^o, y^o)]d\eta \\ &=: I_1(\varepsilon_j) - I_2(\varepsilon_j) - I_3(\varepsilon_j) + I_4(\varepsilon_j), \end{aligned} \tag{4.7}$$

where the sequence $\{\varepsilon_j\}$ is given in Lemma 3.3 and

$$\begin{aligned} I_1(\varepsilon_j) &:= \frac{1}{pt^{p-1}} \cdot \int_{B_{\varepsilon_j}(0)} (|\exp_{x^o}(\eta)\ \exp_{y^o}(T\eta)|^p - |x^o y^o|^p)d\eta, \\ I_2(\varepsilon_j) &:= \int_{B_{\varepsilon_j}(0)} (F(\exp_{x^o}(\eta), \exp_{y^o}(T\eta)) - F(x^o, y^o))d\eta, \\ I_3(\varepsilon_j) &:= \int_{B_{\varepsilon_j}(0)} (v(\exp_{x^o}(\eta)) - v(x^o))dH^n(\eta), \\ I_4(\varepsilon_j) &:= \int_{B_{\varepsilon_j}(0)} (\gamma_\delta(\exp_{x^o}(\eta), \exp_{y^o}(T\eta))) - \gamma_\delta(x^o, y^o)d\eta. \end{aligned}$$

The minimal property of point (x^o, y^o) implies that

$$I(\varepsilon_j) \geqslant 0. \tag{4.8}$$

We need to estimate I_1, I_2, I_3 and I_4.

(i) *The estimate of* I_1 *and* I_4.

Let T be the parallel transportation, the first and the second variation of arc-length implies that

$$|\exp_{x^o}(\eta)\ \exp_{y^o}(T\eta)| - |x^o y^o| \leqslant \frac{\epsilon^2}{2}\int_0^{|x^o y^o|} -R(\sigma'(t), T_{\gamma(t)}\eta, \sigma'(t), T_{\gamma(t)}\eta)dt + o(\epsilon^2)$$

for all $\eta \in B_\epsilon(0)$, where $R(\cdot,\cdot,\cdot,\cdot)$ is the Riemannian curvature tensor. By taking $\epsilon = \varepsilon_j$ and using the fact that $a \leqslant b + \delta$ implies $a^p \leqslant b^p + p\delta b^{p-1} + o(\delta)$ as $\delta \to 0$, and then integrating over $B_{\varepsilon_j(0)}$ we obtain

$$I_1(\varepsilon_j) \leqslant \frac{1}{pt^{p-1}} \cdot \frac{pK\cdot\omega_{n-1}}{2n(n+2)} \cdot |x^o y^o|^p \cdot \varepsilon_j^{n+2} + o(\varepsilon_j^{n+2}) \tag{4.9}$$

for any $j \in \mathbb{N}$, where we have used $\mathrm{Ric}(\sigma', \sigma') \geqslant -K$ and Fatou's lemma (since $|\exp_{x^o}(\eta)\exp_{y^o}(T\eta)| - |x^o y^o| \leqslant C\epsilon^2$ for some constant C depending on the sectional curvature on $B_{10|x^o y^o|}(x^o)$).

Since γ_δ is smooth and that $|\mathrm{Hess}\gamma_\delta| \leqslant \delta$, it is easy to check that

$$I_4(\varepsilon_j) \leqslant C(n)\cdot\delta\cdot\varepsilon_j^{n+2} + o(\varepsilon_j^{n+2}), \tag{4.10}$$

for any $j \in \mathbb{N}$, and for some constant $C(n) > 0$.

(ii) *The estimate of* I_2.

We put

$$P = u(\exp_{x^o}(\eta)), \quad Q = u(x^o), \quad W = u(y^o) \quad \text{and} \quad S = u(\exp_{y^o}(T\eta)),$$

and

$$l_0 := 2\sin\frac{d_{QW}}{2}, \quad l_1 := 2\cos\frac{d_{QW}}{2}, \quad \alpha = \frac{1}{1+2l_0} \in (0,1). \tag{4.11}$$

Denote the midpoint of Q and W by Q_m. Note that $\frac{1-\alpha}{2} = \alpha\cdot l_0$. Then, by Lemma 2.4 we have that for any $\beta > 0$,

$$\begin{aligned}
&\alpha l_0 \cdot (F(x^o, y^o) - F(\exp_{x^o}(\eta), \exp_{y^o}(T\eta)))\\
&= \alpha l_0 \left(\left(2\sin\frac{d_{QR}}{2}\right)^2 - \left(2\sin\frac{d_{PS}}{2}\right)^2 \right) + \alpha l_0 \left(2\sin\frac{d_{QR}}{2} - 2\sin\frac{d_{PS}}{2}\right)\\
&\leqslant \left[1 - \frac{1-\alpha}{2}\left(1-\frac{1}{\beta}\right)\right]\left(2\sin\frac{d_{PQ}}{2}\right)^2 + l_1\cdot(\cos d_{PQ_m} - \cos d_{QQ_m})\\
&\quad + \left[1 - \frac{1-\alpha}{2}(1-\beta)\right]\left(2\sin\frac{d_{RS}}{2}\right)^2 + l_1\cdot(\cos d_{SQ_m} - \cos d_{RQ_m})\\
&\leqslant [w_{a_1,b,Q_m,x^o}(\exp_{x^o}(\eta)) - w_{a_1,b,Q_m,x^o}(x^o)]\\
&\quad + [w_{a_2,b,Q_m,y^o}(\exp_{y^o}(T\eta)) - w_{a_2,b,Q_m,y^o}(y^o)],
\end{aligned} \tag{4.12}$$

where the function w_{a,b,Q_m,x^o} is given in Lemma 3.3 with the constants

$$a_1 := 1 - \frac{1-\alpha}{2}\left(1 - \frac{1}{\beta}\right), \quad b := l_1, \quad a_2 := 1 - \frac{1-\alpha}{2}(1-\beta), \tag{4.13}$$

and we have used $2\sin(t/2) \leqslant t$ for any $t \in (0,\pi)$. From $\rho_0 = \mathrm{diam}(u(B_{2R}))$ we have

$$u(B_{2R}) \subset \overline{B_{\rho_0}(u(x_0))} \cap \overline{B_{\rho_0}(u(y_0))}.$$

By the assumption $\rho_0 < \pi/2$ and that X is a $CAT(1)$-space, we obtain

$$u(B_{2R}) \subset \overline{B_{\rho_0}(Q_m)}.$$

By integrating over $B_{\varepsilon_j}(x^o)$, Lemma 3.3 implies that

$$\begin{aligned}\alpha l_0\cdot\int_{B_{\varepsilon_j}(x^o)}&(F(x^o,y^o)-F(\exp_{x^o}(\eta),\exp_{y^o}(T\eta)))d\eta\\ &\leqslant[2a_1-b\cdot\cos(d_X(u(x^o),Q_m))]\cdot\frac{\omega_{n-1}\cdot e_u(x^o)}{2n(n+2)}\cdot\epsilon_j^{n+2}\\ &\quad+[2a_2-b\cdot\cos(d_X(u(y^o),Q_m))]\cdot\frac{\omega_{n-1}\cdot e_u(y^o)}{2n(n+2)}\cdot\epsilon_j^{n+2}+o(\epsilon_j^{n+2}).\end{aligned}$$

Noticing that $\cos d_X(Q_m,u(x^o))=l_1/2$ and that $1-l_1^2/4=l_0^2/4$, we choose β such that

$$a_2=\frac{l_1^2}{4}\quad\left(\Leftrightarrow\beta=1-\frac{l_0(1+2l_0)}{4}\right),\tag{4.14}$$

where we have used $\alpha=1/(1+2l_0)$. Notice that $\beta>0$ provided $l_0\leqslant1$. Then we have

$$\begin{aligned}-I_2(\varepsilon_j)&=\int_{B_{\varepsilon_j}(0)}(F(x^o,y^o)-F(\exp_{x^o}(\eta),\exp_{y^o}(T\eta)))d\eta\\ &\leqslant\frac{a_1-l_1^2/4}{\alpha l_0}\cdot\frac{\omega_{n-1}e_u(x^o)}{n(n+2)}\cdot\epsilon_j^{n+2}++o(\varepsilon_j^{n+2}).\end{aligned}\tag{4.15}$$

From $1-\beta=\frac{l_0}{4\alpha}$ and $\frac{1}{\alpha}=1+2l_0$, we have, if $l_0\leqslant\varepsilon/6$, that

$$\begin{aligned}\frac{a_1-l_1^2/4}{\alpha l_0}&=\frac{1-\frac{1-\alpha}{2}(1-\frac{1}{\beta})-l_1^2/4}{\alpha l_0}\\ &=l_0(1+2l_0)\left(\frac{1}{4}+\frac{1}{4-l_0(1+2l_0)}\right)\\ &\leqslant\frac{l_0}{2}(1+\varepsilon).\end{aligned}$$

When both t and δ are small enough, the combination of (4.6) and Lemma 4.1(3) implies that $l_0\leqslant\varepsilon/6$. Therefore, we by (4.15) get that

$$-I_2(\varepsilon_j)\leqslant(1+\varepsilon)\frac{l_0}{2}\cdot\frac{\omega_{n-1}}{n(n+2)}e_u(x^o)\cdot\varepsilon_j^{n+2}+o(\varepsilon_j^{n+2}).$$

(iii) *The estimate of* I_3.

By Corollary 3.2 and the definition of v, we have

$$-I_3(\varepsilon_j)\leqslant\left(\frac{-K}{t^{p-1}}\cdot L_t^p(x^o)-(1+\varepsilon)e_u(x^o)D_t(x^o)-\theta_0\right)\cdot\frac{\omega_{n-1}}{2n(n+2)}\cdot\varepsilon_j^{n+2}+o(\varepsilon_j^{n+2}).$$

By combining these estimates for I_1,I_2,I_3 and I_4, we have

$$\begin{aligned}&\frac{K}{t^{p-1}}(|x^oy^o|^p-L_t^p(x^o))+(1+\varepsilon)e_u(x^o)\left(2\sin\frac{d_X(u(x^o),u(y^o))}{2}-D_t(x^o)\right)\\ &\quad-\theta_0+C(n)\frac{2n(n+2)}{\omega_{n-1}}\delta\geqslant0.\end{aligned}\tag{4.16}$$

Equation (4.6) implies that (x^o,y^o) converge to $(\bar{x},\bar{y})$ as $\delta\to0$. We by the lower semi-continuity of L_t and D_t have that

$$\liminf_{\delta\to0}L_t(x^o)\geqslant L_t(\bar{x})=\lim_{\delta\to0}|x^oy^o|$$

and

$$\liminf_{\delta\to0}D_t(x^o)\geqslant D_t(\bar{x})=F(\bar{x},\bar{y})=\lim_{\delta\to0}2\sin\frac{d_X(u(x^o),u(y^o))}{2}.$$

A contradiction appears in (4.16) when we take $\delta\to0$ (noticing that $K\geqslant0$). The proof is finished. □

4.2 The Bochner inequality

We will prove Theorem 1.10 in this subsection.

Let $p \in (1, \infty)$ and let f_t be the auxiliary functions defined as in (4.1) in the previous subsection, on a ball B_R with $B_{2R} \subset\subset \Omega$.

Lemma 4.4. (i) *Let $q \in (1, \infty)$ with $1/q + 1/p = 1$. For any $x \in B_R$, we have*

$$\liminf_{t\to 0} \frac{f_t(x)}{t} \geqslant -\frac{1}{q}\mathrm{Lip}^q u(x). \tag{4.17}$$

(ii) *If u is metrically differentiable at x, then we have*

$$\lim_{t\to 0^+} \frac{f_t(x)}{t} = -\frac{G_u^q(x)}{q} \tag{4.18}$$

and

$$\lim_{t\to 0^+} \frac{L_t(x)}{t} = G_u^{q/p}(x), \quad \lim_{t\to 0^+} \frac{D_t(x)}{t} = G_u^q(x). \tag{4.19}$$

Proof. (i) By the basic inequality $a^p/p - ab \geqslant -b^q/q$ for any $a, b \in \mathbb{R}$, we have

$$\frac{1}{p}\cdot\frac{|xy|^p}{t^p} - \frac{F(x,y)}{t} \geqslant -\frac{1}{q}\left(\frac{F(x,y)}{|xy|}\right)^q, \quad \forall\, x, y \in B_{2R}.$$

Taking $y_t \in S_t(x)$ with $|xy_t| = L_t(x)$, we obtain from the definition of f_t that

$$\begin{aligned}\liminf_{t\to 0} \frac{f_t}{t} &\geqslant -\frac{1}{q}\limsup_{y_t\to x}\left(\frac{F(x,y_t)}{|xy_t|}\right)^q\\ &= -\frac{1}{q}\limsup_{t\to 0}\left(\frac{D_t(x)}{L_t(x)}\right)^q\\ &\geqslant -\frac{1}{q}\mathrm{Lip}^q u(x),\end{aligned} \tag{4.20}$$

where we have used $\lim_{y\to x} F(x,y)/d_X(u(x),u(y)) = 1$. This proves (4.17).

(ii) Let u be metrically differentiable at x. Take a unit vector $\xi \in T_xM$ such that

$$G_u(x) = mdu_x(\xi).$$

For each small $t > 0$, we put $y_{t,x} := \exp_x(tG_u^{q/p}\cdot\xi)$. Then

$$|xy_{t,x}| = t\cdot G_u^{q/p}(x)$$

and

$$\begin{aligned}d_X(u(x), u(y_{t,x})) &= |xy_{t,x}|\cdot mdu_x(\xi) + o(|xy_{t,x}|)\\ &= t\cdot G_u^{q/p+1}(x) + o(t) = t\cdot G_u^q(x) + o(t),\end{aligned}$$

as $t \to 0$. Thus, by the definition of f_t, we obtain

$$\frac{f_t(x)}{t} \leqslant \frac{|xy_{t,x}|^p}{pt^p} - \frac{F(x,y_{t,x})}{t} = \frac{G_u^q(x)}{p} - G_u^q(x) + o(1) = -\frac{G_u^q(x)}{q} + o(1),$$

as $t \to 0$, i.e.,

$$\limsup_{t\to 0^+} \frac{f_t(x)}{t} \leqslant -\frac{G_u^q(x)}{q}. \tag{4.21}$$

Recall that $G_u(x) = \mathrm{Lip}u(x)$ by Lemma 2.10. The combination of (4.17) and (4.21) yields (4.18).

Combining (4.18) and (4.20) gives

$$\limsup_{t\to 0}\left(\frac{D_t(x)}{L_t(x)}\right)^q = \mathrm{Lip}^q u(x).$$

On the other hand, notice that

$$-\frac{f_t(x)}{t} = -\frac{L_t^p(x)}{pt^p} + \frac{D_t(x)}{L_t(x)} \cdot \frac{L_t(x)}{t} \leqslant \frac{1}{q}\left(\frac{D_t(x)}{L_t(x)}\right)^q.$$

Thus, we get

$$\liminf_{t\to 0} \frac{D_t(x)}{L_t(x)} \geqslant \mathrm{Lip}u(x).$$

Therefore, we obtain

$$\lim_{t\to 0} \frac{D_t(x)}{L_t(x)} = \mathrm{Lip}u(x).$$

By using $f_t/t = \frac{L_t^p}{pt^p} - \frac{D_t}{L_t} \cdot \frac{L_t}{t}$ again, it follows

$$\lim_{t\to 0} \frac{L_t(x)}{t} = \mathrm{Lip}^{\frac{1}{p-1}} u(x) = \mathrm{Lip}^{q/p} u(x),$$

and then

$$\lim_{t\to 0} \frac{D_t(x)}{t} = \mathrm{Lip}^{1+q/p} u(x) = \mathrm{Lip}^q u(x).$$

The proof is finished. □

Now we are in the place to prove Theorem 1.10.

Proof of Theorem 1.10. We have known that $\mathrm{Lip} \in L^\infty_{\mathrm{loc}}(\Omega)$ from Theorem 1.3. By a rescaling argument of the target space, we can assume that (X, d_X) is a $CAT(1)$-space. In this case, since $\mathrm{Lip}u = G_u$ for almost all $x \in \Omega$, we need to prove $G_u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and that

$$\Delta G_u \geqslant -KG_u - e_u \cdot G_u \tag{4.22}$$

on Ω, in the sense of distributions. It suffices to show that: for any $o \in \Omega$, there exists a neighborhood $B_R(o)$ with $B_{2R}(o) \subset \Omega$ such that $G_u \in W^{1,2}_{\mathrm{loc}}(B_R(o))$ and that (4.22) holds on $B_R(o)$, in the sense of distributions.

Since u is continuous (from Theorem 2.5), for any $o \in \Omega$, there exists a neighborhood $B_R(o)$ such that $\mathrm{Image}(u(B_{2R}(o))) \subset B_{\pi/4}(u(o))$. In particular, the triangle inequality yields $\mathrm{diam}u(B_{2R}(o)) < \pi/2$. Fix such a neighborhood $B_R = B_R(o)$.

Fix any $p \in [2, \infty)$. From Lemma 4.1(3), we get $\Delta f_t/t \leqslant C_{\ell_0}$ on $B_{3R/2}$ for any $t \in (0, t_*)$, where the constant C_{ℓ_0} is uniformly with respect to $t \in (0, t_*)$. By combining Lemma 4.4 and Proposition 4.2, we have, for any $\varepsilon > 0$ that $G_u^q/q \in W^{1,2}_{\mathrm{loc}}(B_{3R/2})$ and that

$$\Delta(G_u^q/q) \geqslant -KG_u^q - (1+\varepsilon) \cdot e_u \cdot G_u^q$$

on $B_{3R/2}$, in the sense of distributions, where $q = p/(p-1) \in (1, 2]$. From the arbitrariness of ε, we conclude that

$$\Delta(G_u^q/q) \geqslant -KG_u^q - e_u \cdot G_u^q \tag{4.23}$$

on $B_{3R/2}$, in the sense of distributions, where $q = p/(p-1) \in (1, 2]$.

In order to take the limit of (4.23) as $q \to 1$, we want to show that the energies $\|\nabla(G_u^q/q)\|_{L^2(B_R)}$ are bounded uniformly with respect to q. By the local Lipschitz continuity of u, there exists a constant $C_1 > 1$ such that $G_u, e_u \leqslant C_1$ on $B_{3R/2}$. Hence, we have

$$\Delta(G_u^q/q) \geqslant -K \cdot C_1^q - C_1^{q+1} \geqslant -K \cdot C_1^2 - C_1^3$$

on $B_{3R/2}$, in the sense of distributions, where we have used $K \geqslant 0$ and $q \in (1, 2]$. By applying the Caccioppoli inequality, we conclude that the energies $\|\nabla(G_u^q/q)\|_{L^2(B_R)}$ are bounded uniformly. Hence, there exists a sequence $\{q_j\}_{j\in\mathbb{N}}$ with $q_j \in (1, 2]$ and $q_j \searrow 1$ such that $\Delta(G_u^{q_j}/q_j) \rightharpoonup \Delta G_u$. Now, by letting $q_j \to 1$ in (4.23), we conclude that $G_u \in W^{1,2}(B_R)$ and (4.22). The proof is finished. □

5 Yau's gradient estimates

We will continue to assume that Ω is a smooth domain of an n-dimensional Riemannian manifold (M,g) with Ric $\geqslant -K$ for some $K \geqslant 0$, and that (X,d_X) is a $CAT(\kappa)$-space for some $\kappa \geqslant 0$. Let u be a harmonic map from Ω to X. Assume that its image $u(\Omega)\subset X$ is contained in a ball with radius $< \frac{\pi}{2\sqrt{\kappa}}$.

When the target space has non-positive curvature, we have the following a consequence of the Bochner inequality (see Theorem 1.10).

Lemma 5.1. *Let $\kappa = 0$. Suppose that $B_{2R}(x_0) \subset\subset \Omega$ and that $u(B_R(x_0)) \subset B_\rho(Q_0)$ for some $\rho > 0$ and $Q_0 \in X$. We put*

$$h = 2\rho^2 - d_X^2(Q_0, u(x)) \quad \textit{and} \quad F = \frac{\mathrm{Lip}u}{h}.$$

Then $F \in W^{1,2} \cap L^\infty(B_R(x_0))$ and

$$\Delta F + 2\langle \nabla F, \nabla \log h\rangle \geqslant C\rho^2 \cdot F^3 - K\cdot F \tag{5.1}$$

in the sense of distributions, where $C = C_{n,\sqrt{K}R}$.

Proof. From Theorem 1.10, we have $\mathrm{Lip}u \in W^{1,2}\cap L^\infty(B_R(x_0))$. Noticing that h is Lipschitz continuous and $h \geqslant \rho^2$, we obtain $F \in W^{1,2}\cap L^\infty(B_R(x_0))$.

By applying the chain rule to $\mathrm{Lip}u = hF$, we have

$$h\cdot \Delta F + 2\langle \nabla F, \nabla h\rangle + F\cdot \Delta h = \Delta \mathrm{Lip}u.$$

Multiplying both sides of this inequality by h^{-1} and substituting (1.7) and then $-\Delta h \geqslant 2e_u$ (see Lemma 2.6), we get

$$\Delta F + 2\langle \nabla F, \nabla \log h\rangle = -F\cdot\frac{\Delta h}{h} + \frac{\Delta \mathrm{Lip}u}{h} \geqslant F\cdot \frac{2e_u}{h} - KF \tag{5.2}$$

in the sense of distributions. As Corollary 2.7, we have

$$\begin{aligned}\Delta F + 2\langle \nabla F, \nabla \log h\rangle &\geqslant C_{n,\sqrt{K}R}\frac{F}{h}\cdot \mathrm{Lip}^2 u - KF = C_{n,\sqrt{K}R} hF^3 - KF\\ &\geqslant C_{n,\sqrt{K}R}\cdot \rho^2\cdot F^3 - KF,\end{aligned}$$

where we have used $h \geqslant \rho^2$ again. The proof is finished. □

Similarly, in the case of where the target is a $CAT(1)$-space, we have the following property.

Lemma 5.2. *Let $\kappa = 1$. Suppose that $B_{2R}(x_0) \subset\subset \Omega$ and that $u(B_R(x_0)) \subset B_\rho(Q_0)$ for some $\rho < \pi/2$ and $Q_0 \in X$. We put $\rho_0 = \frac{\rho+\pi/2}{2} = \rho + \frac{\pi/2-\rho}{2}$ and*

$$h_1 = \cos d_X(Q_0, u(x)) - \cos \rho_0 \quad \text{and} \quad F = \frac{\mathrm{Lip}u}{h_1}.$$

Then $F \in W^{1,2}\cap L^\infty(B_R(x_0))$ and

$$\Delta F + 2\langle \nabla F, \nabla \log h_1\rangle \geqslant C'\cdot F^3 - KF \tag{5.3}$$

in the sense of distributions, where $C' = C'(n, \sqrt{K}R, \pi/2-\rho)$ is a constant depending on $n, \sqrt{K}R$ and $\pi/2-\rho$.

Proof. It is easy to see that h_1 is Lipschitz continuous and

$$h_1 \geqslant \cos\rho - \cos\rho_0 =: C_1' > 0$$

for some positive number C_1' depends only on $\pi/2-\rho$. From Lemma 2.6, we have

$$-\Delta h_1 \geqslant \cos d_X^2(Q_0, u(x))\cdot e_u \geqslant \cos\rho\cdot e_u.$$

For the rest of the proof, by a similar argument to that in Lemma 5.1, we have

$$\Delta F + 2\langle \nabla F, \nabla \log h_1\rangle \geqslant C_{n,\sqrt{K}R}\cdot C_1'\cdot \cos\rho\cdot F^3 - KF \tag{5.4}$$

in the sense of distributions. □

In general, the function Lip$u$ is not smooth, even may not be continuous. It is difficult to employ the argument in [4] directly. So we will use the following the approximating version of the maximum principle.

Theorem 5.3 (See [44, Theorem 1.4]). *Let $f(x) \in W^{1,2}_{\rm loc} \cap L^\infty_{\rm loc}(\Omega)$ such that Δf is a signed Radon measure with $\Delta^{\rm sing} f \geqslant 0$, where $\Delta f = \Delta^{\rm ac} f \cdot {\rm vol}_g + \Delta^{\rm sing} f$ is the Radon-Nikodym decomposition with respect to ${\rm vol}_g$. Suppose that f achieves one of its strict maximum in Ω in the sense that: there exists a neighborhood $U \subset\subset \Omega$ such that*

$$\sup_U f > \sup_{\Omega\setminus U} f.$$

Then, given any $w \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, there exists a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset U$ such that they are the approximate continuity points of $\Delta^{\rm ac} f$ and $\langle \nabla f, \nabla w\rangle$, and that

$$f(x_j) \geqslant \sup_\Omega f - 1/j \quad \textit{and} \quad \Delta^{\rm ac} f(x_j) + \langle \nabla f, \nabla w\rangle (x_j) \leqslant 1/j.$$

Here and in the sequel, $\sup_U f$ *means* ess-$\sup_U f$.

Proof. It was proved in [44] in the setting of metric measure spaces with generalized Ricci curvature bounded from below. In particular, it holds for Riemannian manifolds with Ricci curvature bounded from below (with the Riemannian measure). □

The proofs of Theorems 1.4 and 1.7 are both based on the following lemma.

Lemma 5.4. *Let $B_R(x_0) \subset \Omega$ and let $F \in W^{1,2}_{\rm loc} \cap L^\infty_{\rm loc}(B_R(x_0))$ be a non-negative function. Assume that F satisfies*

$$\Delta F + \langle \nabla F, \nabla v\rangle \geqslant a_1 F^3 - a_2 F, \tag{5.5}$$

in the sense of distributions, where $v \in W^{1,2}_{\rm loc} \cap L^\infty_{\rm loc}(B_R(x_0))$ such that $|\nabla v| \leqslant a_3 F$, and the constants $a_1, a_3 > 0$ and $a_2 \geqslant 0$. Then there exists a constant $C_{n,\sqrt{K}R}$ such that

$$\sup_{B_{R/2}(x_0)} F^2 \leqslant \frac{2a_2}{a_1} + \frac{C_{n,\sqrt{K}R}}{R^2}\left(\frac{1}{a_1} + \frac{a_3^2}{a_1^2}\right).$$

Proof. Fix any a small number δ such that $0 < \delta < \frac{1}{2}\frac{\sup_{B_{R/2}} F}{\sup_{B_{3R/4}} F}$. Let us choose $\eta(x) = \eta(r(x))$ to be a function of the distance r to the fixed point x_0 with the following property that:

$$\delta \leqslant \eta \leqslant 1 \quad \text{on} \quad B_R, \quad \eta = 1 \quad \text{on} \quad B_{R/2}, \quad \eta = \delta \quad \text{on} \quad B_R\setminus B_{3R/4},$$

and

$$-\frac{C_1}{R} \leqslant \eta'(r) \leqslant 0 \quad \text{and} \quad |\eta''(r)| \leqslant \frac{C_1}{R^2}, \quad \forall\, r \in (0, 3R/4)$$

for some universal constant C_1 (which is independent of n, K and R). Then we have

$$|\nabla \eta| = |\eta'||\nabla r| \leqslant \frac{C_1}{R} \quad \text{on} \quad B_{3R/4}, \tag{5.6}$$

and, by the Laplacian comparison theorem, that

$$\begin{aligned}\Delta \eta = \eta'\Delta r + \eta''|\nabla r|^2 &\geqslant -\frac{C_1}{R}(\sqrt{(n-1)K}\coth(r\sqrt{K/(n-1)})) - \frac{C_1}{R^2}\\ &\geqslant -\frac{C_1}{R}\left(\sqrt{(n-1)K} + \frac{n-1}{R}\right) - \frac{C_1}{R^2} = -\frac{C_1\sqrt{(n-1)K}R + nC_1}{R^2} \geqslant -\frac{C_2}{R^2}\end{aligned} \tag{5.7}$$

on $B_{3R/4}$, in the sense of distributions, where we have used that

$$\coth(r\sqrt{K/(n-1)}) \leqslant \coth(R\sqrt{K/(n-1)}) \leqslant 1 + \frac{1}{R\sqrt{K/(n-1)}}.$$

Here and in the sequel of this proof, we denote $C_1, C_2, C_3, \dots$ the various constants which depend only on n and $\sqrt{K}R$.

Now we put $G = \eta F$. Then G is in $W^{1,2}(B_{3R/4}) \cap L^\infty(B_{3R/4})$ and G achieves one of its strict maximum in $B_{R/2}$ in the sense of Theorem 5.3, i.e.,

$$\begin{aligned}\Delta G + \langle \nabla G, \nabla v\rangle &= \Delta\eta \cdot F + 2\langle \nabla\eta, \nabla(G/\eta)\rangle + \eta\cdot\Delta F + \eta\langle\nabla F, \nabla v\rangle + F\langle\nabla\eta,\nabla v\rangle\\ &\geqslant \Delta\eta\cdot\frac{G}{\eta} + 2\langle\nabla\log\eta, \nabla G\rangle - 2\frac{|\nabla\eta|^2}{\eta}\cdot\frac{G}{\eta} + \eta(\Delta F + \langle\nabla F,\nabla v\rangle) + \frac{G}{\eta}\langle\nabla\eta,\nabla v\rangle\\ &\geqslant -\frac{G}{\eta}\frac{C_2}{R^2} + 2\langle\nabla\log\eta,\nabla G\rangle - \frac{G}{\eta^2}\frac{2C_1^2}{R^2} + \eta\cdot(a_1F^3 - a_2F) + \frac{G}{\eta}\langle\nabla\eta,\nabla v\rangle.\end{aligned}$$

By setting $w = v - 2\log\eta \in W^{1,2}(B_{3R/4}) \cap L^\infty(B_{3R/4})$ and using $|\nabla v| \leqslant a_3 F = a_3\frac{G}{\eta}$, we have

$$\begin{aligned}\Delta G + \langle\nabla G,\nabla w\rangle &\geqslant -\frac{C_2}{R^2}\frac{G}{\eta} - \frac{2C_1^2}{R^2}\frac{G}{\eta^2} + \eta(a_1(G/\eta)^3 - a_2(G/\eta)) - a_3\frac{G}{\eta}|\nabla\eta|\frac{G}{\eta}\\ &\geqslant -\frac{C_2}{R^2}\frac{G}{\eta} - \frac{2C_1^2}{R^2}\frac{G}{\eta^2} + a_1\frac{G^3}{\eta^2} - a_2G - a_3\frac{G^2}{\eta^2}\frac{C_1}{R}\\ &\geqslant -\frac{C_2}{R^2}\frac{G}{\eta}\frac{1}{\eta} - \frac{2C_1^2}{R^2}\frac{G}{\eta^2} + a_1\frac{G^3}{\eta^2} - a_2G\frac{1}{\eta^2} - a_3\frac{G^2}{\eta^2}\frac{C_1}{R}\\ &\geqslant \frac{G}{\eta^2}\bigg[-\frac{C_2}{R^2} - \frac{2C_1^2}{R^2} + a_1G^2 - a_2 - a_3\frac{C_1}{R}G\bigg],\end{aligned}$$

where we have used $G \geqslant 0$, $1/\eta \geqslant 1$ and $a_2 \geqslant 0$. Let $C_3 := C_2 + 2C_1^2$. Substituting

$$\frac{a_3C_1}{R}\cdot G \leqslant \frac{a_1}{2}G^2 + \frac{1}{2a_1}\bigg(\frac{a_3C_1}{R}\bigg)^2$$

into the above inequality, we obtain

$$\Delta G + \langle\nabla G,\nabla w\rangle \geqslant \frac{G}{\eta^2}\bigg[-\frac{C_3}{R^2} + \frac{a_1}{2}G^2 - a_2 - \frac{C_1^2a_3^2}{2a_1R^2}\bigg] \tag{5.8}$$

in the sense of distributions. According to Theorem 5.3, there exist a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$,

$$G_j := G(x_j) \geqslant \sup_{B_{3R/2}} G - 1/j$$

and that

$$\frac{G_j}{\eta^2(x_j)}\bigg[\frac{a_1}{2}G_j^2 - a_2 - \frac{C_3}{R^2} - \frac{C_1^2a_3^2}{2a_1R^2}\bigg] \leqslant \frac{1}{j}.$$

As $\eta \geqslant \delta > 0$, by letting $j \to \infty$, we have

$$\sup_{B_{3R/4}} G^2 = \lim_{j\to\infty} G_j^2 \leqslant \frac{2a_2}{a_1} + \frac{2C_3}{a_1R^2} + \frac{C_1^2a_3^2}{a_1^2R^2}.$$

This yields

$$\sup_{B_{R/2}} F^2 \leqslant \sup_{B_{3R/4}} G^2 \leqslant \frac{2a_2}{a_1} + \frac{2C_3}{a_1R^2} + \frac{C_1^2a_3^2}{a_1^2R^2} \leqslant \frac{2a_2}{a_1} + \frac{C_4}{R^2}\bigg(\frac{1}{a_1} + \frac{a_3^2}{a_1^2}\bigg),$$

where $C_4 := \max\{2C_3, C_1^2\}$. The proof is finished. □

Now we are in the place to show the main results.

Proof of Theorem 1.4. By applying Lemma 5.4 to (5.1) in Lemma 5.1 with $v = 2\log h$ and

$$a_1 = C_5\rho^2, \quad a_2 = K, \quad a_3 = 2\rho,$$

and noticing that

$$|\nabla v| = 2\frac{|\nabla h|}{h} = 2\frac{d_X(Q_0,u(x))\cdot|\nabla d_X(Q_0,u(x))|}{h} \leqslant 2\frac{\rho\cdot \mathrm{Lip}u}{h} = 2\rho F,$$

we conclude that, for some constants C_5, C_6 and C_7 depending only on n and $\sqrt{K}R$, it holds

$$\sup_{B_{R/2}(x_0)} \frac{\mathrm{Lip}^2 u}{(2\rho^2 - d_X^2(Q_0,u(x)))^2} \leqslant \frac{2K}{C_5\rho^2} + \frac{C_6}{R^2}\left(\frac{1}{C_5\rho^2} + \frac{4\rho^2}{C_5^2\rho^4}\right) \leqslant \frac{C_7}{\rho^2}\left(K + \frac{1}{R^2}\right). \tag{5.9}$$

This implies

$$\begin{aligned}\sup_{B_{R/2}(x_0)} \mathrm{Lip}^2 u &\leqslant \frac{C_7}{\rho^2}\left(K+\frac{1}{R^2}\right)\cdot \sup_{B_{R/2}(x_0)} (2\rho^2 - d_X^2(Q_0,u(x)))^2\\ &\leqslant C_7\frac{4\rho^2}{R^2}(KR^2+1) = C_8\cdot\frac{\rho^2}{R^2},\end{aligned}$$

for some constant C_8 depending only on n and $\sqrt{K}R$. The proof is finished. □

Proof of Theorem 1.7. By applying Lemma 5.4 to Lemma 5.2 with $v = 2\log h$ and noticing that

$$|\nabla v| = 2\frac{|\nabla h|}{h} = 2\frac{\sin d_X(Q_0,u(x))\cdot|\nabla d_X(Q_0,u(x))|}{h} \leqslant 2\frac{\mathrm{Lip}u}{h} = 2F,$$

and choosing

$$a_1 = C_1', \quad a_2 = K, \quad a_3 = 2,$$

here and in the sequel of this proof, we denote $C_1', C_2', C_3', \dots$ the various constants which depend only on n, $\sqrt{K}R$ and $\pi/2-\rho$, we conclude that

$$\sup_{B_{R/2}(x_0)} \frac{\mathrm{Lip}^2 u}{(\cos d_X(Q_0,u) - \cos\rho_0)^2} \leqslant \frac{2K}{C_1'} + \frac{C_2'}{R^2}\left(\frac{1}{C_1'} + \frac{4}{(C_1')^2}\right) \leqslant C_3'\left(K+\frac{1}{R^2}\right),$$

where $\rho_0 = \pi/4 + \rho/2$. By noticing that $\cos(d_X\circ u) - \cos\rho_0 \leqslant \cos\rho$, this implies

$$\sup_{B_{R/2}(x_0)} \mathrm{Lip}^2 u \leqslant C_3' \cos\rho\left(K+\frac{1}{R^2}\right) \leqslant \frac{C_4'}{R^2}.$$

The proof is finished. □

Acknowledgements The first and third authors were supported by National Natural Science Foundation of China (Grant No. 11521101). The first author was also supported by National Natural Science Foundation of China (Grant No. 11571374) and National Program for Support of Top-Notch Young Professionals. The second author was supported by the Academy of Finland. Part of the work was done when the first author visited the Department of Mathematics and Statistics, University of Jyväskylä for one month in 2016. The first author thanks the department for the hospitality.

References

1 Breiner C, Fraser A, Huang L-H, et al. Regularity of harmonic maps from polyhedra to $CAT(1)$ spaces. ArXiv:1610.07829, 2017

2 Burago D, Burago Y, Ivanov S. A Course in Metric Geometry. Graduate Studies in Mathematics, vol. 33. Providence: Amer Math Soc, 2001

3 Chen J. On energy minimizing mappings between and into singular spaces. Duke Math J, 1995, 79: 77–99

4 Cheng S Y. Liouville theorem for harmonic maps. Proc Sympos Pure Math, 1980, 36: 147–151

5 Choi H I. On the Liouville theorem for harmonic maps. Proc Amer Math Soc, 1982, 85: 91–94

6 Crandall M, Ishii H, Lions P-L. User's guide to viscosity sulutions of second order partial differential equations. Bull Amer Math Soc (NS), 1992, 27: 1–67

7 Daskalopoulos G, Mese C. Harmonic maps from a simplicial complex and geometric rigidity. J Differential Geom, 2008, 78: 269–293
8 Daskalopoulos G, Mese C. Harmonic maps between singular spaces I. Comm Anal Geom, 2010, 18: 257–337
9 Eells J, Fuglede B. Harmonic Maps between Riemannian Polyhedra. Cambridge Tracts in Mathematics, vol. 142. Cambridge: Cambridge University Press, 2001
10 Freidin B. A Bochner formula for harmonic maps into non-positively curved metric spaces. ArXiv:1605.08461v2, 2016
11 Freidin B, Zhang Y. A Liouville-type theorem and Bochner formula for harmonic maps into metric spaces. ArXiv:1805.04192v1, 2018
12 Fuglede B. Hölder continuity of harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature. Calc Var Partial Differential Equations, 2003, 16: 375–403
13 Fuglede B. Harmonic maps from Riemannian polyhedra to geodesic spaces with curvature bounded from above. Calc Var Partial Differential Equations, 2008, 31: 99–136
14 Giaquinta M, Giusti E. On the regularity of the minima of variational integrals. Acta Math, 1982, 148: 31–46
15 Giaquinta M, Giusti E. The singular set of the minima of certain quadratic functionals. Ann Sc Norm Super Pisa Cl Sci (5), 1984, 11: 45–55
16 Gromov M, Schoen R. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Publ Math Inst Hautes Études Sci, 1992, 76: 165–246
17 Guo C-Y. Harmonic mappings between singular metric spaces. ArXiv:1702.05086, 2017
18 Heinonen J, Koskela P, Shanmugalingam N, et al. Sobolev classes of Banach space-valued functions and quasiconformal mappings. J Anal Math, 2001, 85: 87–139
19 Hildebrandt S, Kaul H, Widman K-O. An existence theorem for harmonic mappings of Riemannian manifolds. Acta Math, 1977, 138: 1–16
20 Hua B, Liu S, Xia C. Liouville theorems for f-harmonic maps into Hadamard spaces. Pacific J Math, 2017, 290: 381–402
21 Ishizuka W, Wang C Y. Harmonic maps from manifolds of L^∞-Riemannian metrics. Calc Var Partial Differential Equations, 2008, 32: 287–405
22 Jäger W, Kaul H. Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems. J Reine Angew Math, 1983, 343: 146–161
23 Jost J. Equilibrium maps between metric spaces. Calc Var Partial Differential Equations, 1994, 2: 173–204
24 Jost J. Generalized harmonic maps between metric spaces. In: Geometric Analysis and the Calculus of Variations for Stefan Hildebrandt. Somerville: Int Press, 1996, 143–174
25 Jost J. Generalized Dirichlet forms and harmonic maps. Calc Var Partial Differential Equations, 1997, 5: 1–19
26 Jost J. Nonlinear Dirichlet forms. In: New Directions in Dirichlet Forms. Somerville: Int Press, 1998, 1–47
27 Kirchheim B. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. Proc Amer Math Soc, 1994, 121: 113–123
28 Korevaar N, Schoen R. Sobolev spaces and harmonic maps for metric space targets. Comm Anal Geom, 1993, 1: 561–659
29 Kuwae K, Shioya T.Sobolev and Dirichlet spaces over maps between metric spaces. J Reine Angew Math, 2003, 555: 39–75
30 Lin F H. Analysis on singular spaces. In: Collection of Papers on Geometry, Analysis and Mathematical Physics. River Edge: World Scientific, 1997, 114–126
31 Lin F H, Wang C Y. The Analysis of Harmonic Maps and Their Heat Flows. River Edge: World Scientific, 2008
32 Mese C. The curvature of minimal surfaces in singular spaces. Comm Anal Geom, 2001, 9: 3–34
33 Mese C. Harmonic maps into spaces with an upper curvature bound in the sense of Alexandrov. Math Z, 2002, 242: 633–661
34 Morrey C B. The problem of Plateau on an Riemannian manifold. Ann of Math (2), 1948, 49: 807–851
35 Ohta S-I. Cheeger type Sobolev spaces for metric space targets. Potential Anal, 2004, 20: 149–175
36 Schoen R, Uhlenbeck K. A regularity theory for harmonic maps. J Differential Geom, 1982, 17: 307–335
37 Schoen R, Uhlenbeck K. Regularity of minimizing harmonic maps into the sphere. Invent Math, 1984, 78: 89–100
38 Schoen R, Yau S T. Lectures on Differential Geometry. Boston: Int Press, 1994
39 Serbinowski T. Harmonic Maps into Metric Spaces with Curvature Bounded from above. PhD Thesis. Salt Lake City: The University of Utah, 1995
40 Sidler H, Wneger S. Harmonic quasi-isometric maps into Gromov hyperbolic $CAT(0)$-spaces. ArXiv:1804.06286v1, 2018
41 Sinaei Z. Riemannian polyhedra and Liouville-type theorems for harmonic maps. Anal Geom Metric Spaces, 2014, 2: 294–318
42 Sturm K-T. A semigroup approach to harmonic maps. Potential Anal, 2005, 23: 225–277
43 Yau S-T. Harmonic functions on complete Riemannian manifolds. Comm Pure Appl Math, 1975, 28: 201–228

44 Zhang H C, Zhu X P. Local Li-Yau's estimates on $RCD^*(K,N)$ metric measure spaces. Calc Var Partial Differential Equations, 2016, 55: 93

45 Zhang H C, Zhu X P. Lipschitz contunuity of harmonic maps between Alexandrov spaces. Invent Math, 2018, 211: 863–934

Appendix A A generalized Jensen's lemma and the proof of Lemma 4.3

We need a simple lemma for symmetric matrices as follows.

Lemma A.1. *Let $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ be two symmetric matrices. Assume that B is non-negative definite, and that $|a_{ij}-\delta_{ij}|\leqslant\frac{1}{2n^2}$ for any $1\leqslant i,j\leqslant n$, where $I=(\delta_{ij})_{n\times n}$ is the identity matrix. If* $\mathrm{trace}(AB)\leqslant C$ *for some $C>0$, then we have*

$$|\mathrm{det}B|\leqslant(2C)^n.$$

Proof. We put $\bar\mu=$ the maximum eigenvalue of B. Then by non-negative definiteness of B we have $\bar\mu\leqslant\|B\|:=(\sum_{i,j=1}^n b_{ij}^2)^{1/2}\leqslant\sqrt{n}\bar\mu$. Hence we have

$$\begin{aligned}\bar\mu\leqslant\mathrm{trace}(B)&=\mathrm{trace}((I-A)B)+\mathrm{trace}(AB)\\&\leqslant\|I-A\|\cdot\|B\|+\mathrm{trace}(AB)\\&\leqslant\left[n^2\cdot\left(\frac{1}{2n^2}\right)^2\right]^{1/2}\cdot\sqrt{n}\bar\mu+C=\frac{\bar\mu}{2\sqrt{n}}+C.\end{aligned}$$

This implies that $\bar\mu\leqslant 2C$. At last, by the assumption that B is non-negative definite, we have $0\leqslant\mathrm{det}B\leqslant\bar\mu^n\leqslant(2C)^n$. The proof is finished. □

The following lemma is a slight extension of Jensen's lemma (see, for example, [6, Lemma A.3]).

Lemma A.2. *Let $U\subset M$ be a convex domain of M and let $h\in W^{1,2}(U)\cap C(U)$ satisfy $\Delta h\leqslant\lambda$ on U for some constant $\lambda\in\mathbb{R}$. Assume that the point $\hat x\in U$ is a uniquely local minimum point of the function h on U. Let $\{y^j\}_{1\leqslant j\leqslant n}$ be a local geodesic coordinate system around $\hat x$. For any $p=(p^1,\dots,p^n)\in\mathbb{R}^n$, we set*

$$h_p(x):=h(x)+\sum_{i,j=1}^n p^jy^j(x).$$

Then for any $r,\delta>0$ sufficiently small, (namely, they are smaller than a constant δ_0 depending on the bounds of sectional curvature on U,) the set

$$K=\{x\in B_r(\hat x):\exists p\in B_\delta(0)\ \textit{for which}\ h_p\ \textit{has a local minimum at}\ x\}$$

has positive measure, where $B_\delta(0):=\{p\in\mathbb{R}^n:\|p\|<\delta,\ \forall\,1\leqslant j\leqslant n\}$.

Proof. Fixed any sufficiently small r, if δ is small enough, then for any $p\in B_\delta(0)$, there exists a local minimum of h_p lying in the interior of $B_r(\hat x)$, since h has the unique minimum at $\hat x$. We will split the proof into two steps, as in the argument of [6, Lemma A.3].

(i) We assume for the moment that h is of C^2 near $\hat x$. Let $(g_{ij})_{n\times n}$ be the local Riemannian metric around $\hat x$ with respect to the coordinate system $\{y^j\}_{1\leqslant j\leqslant n}$. There exists a number $r_0>0$, depending on the curvature on U, such that, for all $1\leqslant i,j,k\leqslant n$,

$$|g_{ij}(x)-\delta_{ij}|\leqslant\frac{1}{10n^2},\quad|\partial_k g_{ij}(x)|\leqslant 1,\quad\forall\,x\in B_{r_0}(\hat x).$$

Fixed any $r\in(0,r_0)$, it suffices to show that K has positive measure with respect to the Euclidean measure on $(B_r(\hat x),\delta_{ij})$.

Now we consider the elliptic operator

$$Lh:=\sum_{1\leqslant i,j\leqslant n}\partial_i(a_{ij}\partial_j h)\quad\text{with}\quad a_{ij}=g^{ij}\sqrt{\det(g_{ij})}.$$

It is easily seen that there exists a constant $C(n)$ such that for all $x \in B_r(\hat{x})$ and for all $1 \leqslant i, j, k \leqslant n$, it holds that

$$|a_{ij}(x) - \delta_{ij}| \leqslant \frac{1}{4n^2}, \quad |\partial_k a_{ij}| \leqslant C(n), \quad Lh \leqslant C(n) \cdot \lambda. \tag{A.1}$$

Since $Dh + p = Dh_p = 0$ holds for every minimum points of h_p, we have

$$Dh(K) \supseteq B_\delta(0).$$

Here, Dh (and the following D^2h) is the (2-order) differential of h with respect to the Euclidean metric on $B_r(\hat{x})$. Moreover, for every minimum points of h_p, we have that $D^2h_p = (\partial_i\partial_j h_p)$ is non-negative definite and that

$$\begin{aligned}\sum_{1\leqslant i,j\leqslant n} a_{ij}\partial_i\partial_j h_p &= \sum_{1\leqslant i,j\leqslant n} \partial_i(a_{ij}\partial_j h_p) \quad (\text{since } \partial_j h_p = 0)\\ &= Lh + L\bigg(\sum_{k=1}^n p_k \cdot x_k\bigg) = Lh + \sum_{1\leqslant i,j\leqslant n} p_j \cdot \partial_i a_{ij}\\ &\leqslant C(n)\cdot\lambda + C(n)n^2\delta \leqslant C_1(n,\lambda),\end{aligned}$$

for a constant $C_1 > 0$. By using Lemma A.1 for $B = D^2h_p$, we have $|\det D^2h| = |\det D^2h_p| \leqslant (2C_1)^n$ for all $x \in K$. Thus,

$$\mathcal{L}^n(B_\delta(0)) \leqslant \mathcal{L}^n(Dh(K)) \leqslant \int_K |\det D^2h| dx \leqslant \mathcal{L}^n(K)\cdot(2C_1)^n,$$

where $\mathcal{L}^n(K)$ is the Euclidean measure of K. This completes the proof for the smooth case.

(ii) In the general case, in which h need not be smooth, we will approximate it via heat flows. This is the reason that we have to assume that U is convex.

Let $\{P_t h\}_{t\geqslant 0}$ be the heat flow with the Neumann boundary value condition on U, with the initial data $P_0 h = h$. It is clear that $P_t h$ is smooth for any $t > 0$. By maximum principle, we have

$$\Delta P_t h = P_t \Delta h \leqslant \lambda, \quad \forall\, t > 0.$$

The corresponding set K_t obeys the above estimates in (i) for small $t > 0$. In particular, the measure of K_t is bounded from below by a constant $C(\delta, \lambda, n) > 0$ uniformly on $t > 0$.

At last, by using the convexity of the boundary of U and that the curvature of M is bounded on U, (in particular, the Ricci curvature on U is bounded from below,) the Li-Yau gradient estimates for solutions of the heat flow implies that $P_t h$ converges uniformly to h on $B_r(\hat{x}) \subset\subset U$. Notice that $K \supset \liminf_{t\to 0} K_t$. The result now follows. □

Now the perturbation Lemma 4.3 is a corollary as follows.

Corollary A.3. *Let $U \subset M$ be a convex domain of M and let*

$$h \in W^{1,2}(U) \cap C(U)$$

satisfy $\Delta h \leqslant \lambda$ on U for some constant $\lambda \in \mathbb{R}$. Assume that the point $\hat{x} \in U$ is one of minimum points of function h on U. Assume a subset $A \subset U$ has full measure. Then for any $r, \delta > 0$ sufficiently small, there exists a smooth function ϕ on a neighborhood of $\hat{x}$, $B_{r_0}(\hat{x})$, such that $h + \phi$ has a local minimum point in $B_r(\hat{x}) \cap A$ and that

$$|\phi| + |\nabla\phi| + |\mathrm{Hess}(\phi)| \leqslant \delta, \quad \forall\, x \in B_{r_0}(\hat{x}).$$

Proof. Fix any r, δ sufficiently small. We put

$$h_1 := h + \delta|\hat{x}x|^2/(10n).$$

Then h_1 has a unique minimum at $\hat{x}$. Since $\Delta h_1 \leqslant \Delta h + C(n,k_0)\cdot\delta$ by the Laplacian comparison on M, the above lemma implies that $h_1 + \sum_{j=1}^n p^j y^j$ has a local minimum at a point in $B_r(\hat{x}) \cap A$ and that $0 \leqslant p^j \leqslant \delta/2$, $1 \leqslant j \leqslant n$. Now, the function

$$\phi := \delta|\hat{x}x|^2/(10n) + \sum_{j=1}^n p^j y^j$$

is defined on a coordinate neighborhood $B_{r_0}(\hat{x})$. Notice that $|\mathrm{Hess} y^j| \leqslant C(k_0)$ for some constant $C(k_0)$ depending on k_0, a bound of $|\sec_M|$ on U. This implies that

$$|\phi| + |\nabla\phi| + |\mathrm{Hess}(\phi)| \leqslant \delta\cdot C(n,k_0), \quad \forall\, x \in B_{r_0}(\hat{x}).$$

The proof is finished. □