The Principle of the Fermionic Projector

Felix Finster

Abstract

The "principle of the fermionic projector" provides a new mathematical framework for the formulation of physical theories and is a promising approach for physics beyond the standard model. The book begins with a brief review of relativity, relativistic quantum mechanics and classical gauge theories, with the emphasis on the basic physical concepts and the mathematical foundations. The external field problem and Klein's paradox are discussed and then resolved by introducing the so-called fermionic projector, a global object in space-time which generalizes the notion of the Dirac sea. The mathematical core of the book is to give a precise definition of the fermionic projector and to employ methods of hyperbolic differential equations for its detailed analysis. The fermionic projector makes it possible to formulate a new type of variational principles in space-time. The mathematical tools for the analysis of the corresponding Euler-Lagrange equations are developed. A particular variational principle is proposed which gives rise to an effective interaction showing many similarities to the interactions of the standard model.

The main chapters of the book are easily accessible for beginning graduate students in mathematics or physics. Several appendices provide supplementary material which will be useful to the experienced researcher.

Contents

Preface to the Second Online Edition

In the almost twelve years since this book was completed, the fermionic projector approach evolved to what is known today as the theory of causal fermion systems. There has been progress in several directions: the mathematical setting was generalized, the mathematical methods were improved and enriched, and the physical applications have been concretized and worked out in more detail. The current status of the theory is presented in a coherent way in the recent monograph [**5**]. An untechnical physical introduction is given in [**9**].

Due to these developments, parts of the present book are superseded by the more recent research papers or the monograph [**5**]. However, other parts of this book have not been developed further and are still up to date. For some aspects not covered in [**5**], the present book is still the best reference. Furthermore, the present book is still of interest as being the first publication in which the causal action principle was presented. Indeed, comparing the presentation in the present book to the later developments should give the reader a deeper understanding of why certain constructions were modified and how they were improved. In order to facilitate such a study, we now outline the developments which led from the present book to the monograph [**5**]. In order not to change the original bibliography, a list of references to more recent research papers is given at the end of this preface, where numbers are used (whereas the original bibliography using letters is still at the end of the book). Similar as in the first online edition, I took the opportunity to correct a few typos. Also, I added a few footnotes beginning with "Online version:". Apart from these changes, the online version coincides precisely with the printed book in the AMS/IP series. In particular, all equation numbers are the same.

Maybe the most important change in the mathematical setup was the move from indefinite inner product spaces to Hilbert spaces, as we now explain in detail. Clearly, the starting point of all my considerations was Dirac theory. On Dirac wave functions in Minkowski space, one can introduce the two inner products

$$(\Psi|\Phi) = \int_{\mathbb{R}^3} (\overline{\Psi}\gamma^0\Phi)(t,\vec{x})\, d^3x \tag{1}$$

$$<\Psi|\Phi> = \int_{\mathbb{R}^4} \overline{\Psi(x)}\Phi(x)\, d^4x\,. \tag{2}$$

The first inner product (1) is positive definite and thus defines a scalar product. For solutions of the Dirac equation, it is time independent due to current conservation, making the solution space of the Dirac equation to a Hilbert space (more generally, the scalar product can be computed by integrating the normal component of the Dirac current over any Cauchy surface). The inner product (2), on the other hand, is indefinite. It is well-defined and covariant even on wave functions which do not satisfy the Dirac equation, giving rise to an indefinite inner product space (which can be given a Krein space structure). It should be pointed out that the time integral in (2) in

general diverges for solutions of the Dirac equation, a problem which I always considered to be more of technical than of fundamental nature (this technical problem can be resolved for example by working as in (2.2.26)–(2.2.29) with a δ-normalization in the mass parameter or by making use of the mass oscillation property as introduced in [**12**]).

The fermionic projector approach is based on the belief that on a microscopic scale (like the Planck scale), space-time should not be modeled by Minkowski space but should have a different, possibly discrete structure. Consequently, the Dirac equation in Minkowski space should not be considered as being fundamental, but it should be replaced by equations of different type. For such a more fundamental description, the scalar product (1) is problematic, because it is not clear how the analog of an integral over a hypersurface should be defined, and why this integral should be independent of the choice of the hypersurface. The indefinite inner product (2), however, can easily be generalized to for example a discrete space-time if one simply replaces the integral in (2) by a sum over all space-time points. Such considerations led me to regard the indefinite inner product (2) as being more fundamental than (1). This is the reason why throughout this book, we work preferably with indefinite inner product spaces. In particular, the structure of "discrete space-time" is introduced on an underlying indefinite inner product space (see §3.3).

My views changed gradually over the past few years. The first input which triggered this process was obtained when developing the existence theory for the causal action principle. While working on this problem in the simplest setting of a finite number of space-time points [**1**], it became clear that in order to ensure the existence of minimizers, one needs to assume that the image of the fermionic projector P is a *negative definite* subspace of the indefinite inner product space $(H, <.|.>)$. The fact that P has a definite image makes it possible to introduce a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ by setting $\langle .|. \rangle_{\mathcal{H}} = -<.|P\,.>$ and dividing out the null space. This construction, which was first given in [**7**, Section 1.2.2], gave an underlying Hilbert space structure. However, at this time, the connection of the corresponding scalar product to integrals over hypersurfaces as in (1) remained obscure.

From the mathematical point of view, having an underlying Hilbert space structure has the major benefit that functional analytic methods in Hilbert spaces become applicable. When thinking about how to apply these methods, it became clear that also measure-theoretic methods are useful. This led me to generalize the mathematical setting such as to allow for the description of not only discrete, but also continuous space-times. This setting was first introduced in [**3**] when working out the existence theory. This analysis also clarified which constraints one must impose in order to obtain a mathematically well-posed variational problem.

The constructions in [**3**] also inspired the notion of the *universal measure*, as we now outline. When working out the existence theory, it became clear that instead of using the kernel of the fermionic projector, the causal action principle can be formulated equivalently in terms of the local correlation operators $F(x)$ which relate the Hilbert space scalar product to the spin scalar product by

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = - \prec \psi(x) | \phi(x) \succ_x .$$

In this formulation, the only a-priori structure of space-time is that of being a measure space (M, μ). The local correlation operators give rise to a mapping

$$F \;:\; M \to \mathcal{F}\,, \quad x \mapsto F(x)\,,$$

where $\mathcal{F}$ is the subset of finite rank operators on $\mathcal{H}$ which are symmetric and (counting multiplicities) have at most $2N$ positive and at most $2N$ negative eigenvalues (where N denotes the number of sectors). Then, instead of working with the measure μ, the causal action can be expressed in terms of the push-forward measure $\rho = F_* \mu$, being a measure on $\mathcal{F}$ (defined by $\rho(\Omega) = \mu(F^{-1}(\Omega))$). As a consequence, it seems natural to leave out the measure space (M, μ) and to work instead directly with the measure ρ on $\mathcal{F}$, referred to as the universal measure. We remark that working with (M, μ) has the potential benefit that it is possible to prescribe properties of the measure ρ. In particular, if μ is a discrete measure, then so is ρ (for details see [**3**, Section 1.2]). However, the analysis of the causal action principle in [**13**] suggests that minimizing measures are always discrete, even if one varies over all regular Borel measures (which may have discrete and continuous components). With this in mind, it seems unnecessary to arrange the discreteness of the measure ρ by starting with a discrete measure space (M, μ). Then the measure space (M, μ) becomes obsolete. These considerations led me to the conviction that one should work with the universal measure ρ, which should be varied within the class of all regular Borel measures. Working with general regular Borel measures also has the advantage that it becomes possible to take convex combinations of universal measures, which seems essential for getting the connection to second-quantized bosonic fields (see the notions of decoherent replicas of space-time and of microscopic mixing of wave functions in [**4**] and [**15**]).

Combining all the above results led to the framework of *causal fermion systems*, where a physical system is described by a Hilbert space $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ and the universal measure ρ on $\mathcal{F}$. This framework was first introduced in [**7**]. Subsequently, the analytic, geometric and topological structures encoded in a causal fermion system were worked out systematically; for an overview see [**5**, Chapter 1].

From the conceptual point of view, the setting of causal fermion systems and the notion of the universal measure considerably changed both the role of the causal action principle and the concept of what space-time is. Namely, in the causal action principle in this book, one varies the fermionic projector P in a given discrete space-time. In the setting of causal fermion systems, however, one varies instead the universal measure ρ, being a measure on linear operators on an abstract Hilbert space. In the latter formulation, there is no space-time to begin with. On the contrary, space-time is introduced later as the support of the universal measure. In this way, the causal action principle evolved from a variational principle for wave functions in space-time to a variational principle for space-time itself as well as for all structures therein.

In order to complete the summary of the conceptual modifications, we remark that the connection between the scalar product $\langle .|. \rangle_{\mathcal{H}}$ and surface integrals as in (1), which had been obscure for quite a while, was finally clarified when working out Noether-like theorems for causal variational principles [**10**]. Namely, surface integrals now have a proper generalization to causal fermion systems in terms of so-called *surface layer integrals*. It was shown that the symmetry of the causal action under unitary transformations acting on $\mathcal{F}$ gives rise to conserved charges which can be expressed by surface layer integrals. For Dirac sea configurations, these conserved charges coincide with the surface integrals (1).

Another major development concerns the description of *neutrinos*. In order to explain how these developments came about, we first note that in this book, neutrinos are modelled as left-handed massless Dirac particles (see §5.1). This has the benefit that the neutrinos drop out of the closed chain due to chiral cancellations (see §5.3

and §5.4). When writing this book, I liked chiral cancellations, and I even regarded them as a possible explanation for the fact that neutrinos appear only with one chirality. As a side remark, I would like to mention that I was never concerned about experimental observations which indicate that neutrinos do have a rest mass, because I felt that these experiments are too indirect for making a clear case. Namely, measurements only tell us that there are fewer neutrinos on earth than expected from the number of neutrinos generated in fusion processes in the sun. The conventional explanation for this seeming disappearance of solar neutrinos is via neutrino oscillations, making it necessary to consider massive neutrinos. However, it always seemed to me that there could be other explanations for the lack of neutrinos on earth (for example, a modification of the weak interaction or other, yet unknown fundamental forces), in which case the neutrinos could well be massless.

My motivation for departing from massless neutrinos was not related to experimental evidence, but had to do with problems of mathematical consistency. Namely, I noticed that left-handed neutrinos do not give rise to stable minimizers of the causal action (see [**5**, Section 4.2]). This general result led me to incorporate right-handed neutrino components, and to explain the fact that only the left-handed component is observed by the postulate that the regularization breaks the chiral symmetry. This procedure cured the mathematical consistency problems and had the desired side effect that neutrinos could have a rest mass, in agreement with neutrino oscillations.

We now comment on other developments which are of more technical nature. These developments were mainly triggered by minor errors or shortcomings in the present book. First, Andreas Grotz noticed when working on his master thesis in 2007 that the normalization conditions for the fermionic projector as given in (2.6.11) and (2.6.12) are in general violated to higher order in perturbation theory. This error was corrected in [**6**] by a rescaling procedure. This construction showed that there are two different perturbation expansions: with and without rescaling. The deeper meaning of these two expansions became clearer later when working out different normalizations of the fermionic projector. This study was initiated by the quest for a non-perturbative construction of the fermionic projector, as was carried out in globally hyperbolic space-times in [**11, 12**]. It turned out that in space-times of finite lifetime, one cannot work with the δ-normalization in the mass parameter as used in (2.2.26)–(2.2.29) (the "mass normalization"). Instead, a proper normalization is obtained by using a scalar product $(.|.)$ which is represented similar to (1) by an integral over a spacelike hypersurface (the "spatial normalization"). As worked out in detail in [**14**] with a convenient contour integral method, the causal perturbation expansion without rescaling realizes the spatial normalization condition, whereas the rescaling procedure in [**6**] gives rise to the mass normalization. The constructions in curved space-time in [**11, 12**] as well as the general connection between the scalar product $(.|.)$ and the surface layer integrals in [**10**] showed that the physically correct and mathematically consistent normalization condition is the spatial normalization condition. With this in mind, the combinatorics of the causal perturbation expansion in this book is indeed correct, but the resulting fermionic projector does not satisfy the mass but the spatial normalization condition.

Clearly, the analysis of the continuum limit in Chapters 6–8 is superseded by the much more detailed analysis in [**5**, Chapters 3-5]. A major change concerns the treatment of the logarithmic singularities on the light cone, as we now point out. In the present book, some of the contributions involving logarithms are arranged to vanish by imposing that the regularization should satisfy the relation (6.2.9). I tried for quite

a while to construct an example of a regularization which realizes this relation, until I finally realized that there is no such regularization, for the following reason:

LEMMA I. There is no regularization which satisfies the condition (6.2.9).

Proof. The linear combination of monomials M in (6.2.6) involves a factor $T^{(1)}_{[2]}$, which has a logarithmic pole on the light cone (see (2.5.43), (2.5.42) and (2.5.41)). Restricting attention to the corresponding contribution $\sim \log|\vec{\xi}|$, we have

$$M \asymp -\frac{1}{16\pi^3}\, T^{(-1)}_{[0]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}}\, \log|\vec{\xi}|\,.$$

As a consequence,

$$\begin{aligned}(M-\overline{M})\,\overline{T^{(0)}_{[0]}}^{-1} &= -\frac{\log|\vec{\xi}|}{16\pi^3}\,\frac{\big|T^{(-1)}_{[0]}\big|^2}{\overline{T^{(0)}_{[0]}}}\big(\overline{T^{(0)}_{[0]}} - T^{(0)}_{[0]}\big) \\ &= -\frac{\log|\vec{\xi}|}{16\pi^3}\left|\frac{T^{(-1)}_{[0]}}{T^{(0)}_{[0]}}\right|^2\Big(\big|T^{(0)}_{[0]}\big|^2 - \big(T^{(0)}_{[0]}\big)^2\Big) = -\frac{\log|\vec{\xi}|}{8\pi^3}\left|\frac{T^{(-1)}_{[0]}}{T^{(0)}_{[0]}}\right|^2\Big(\operatorname{Im} T^{(0)}_{[0]}\Big)^2 \leq 0\,.\end{aligned}$$

Since this expression has a fixed sign, it vanishes in a weak evaluation on the light cone only if it vanishes identically to the required degree. According to (2.5.41), the function $\operatorname{Im} T^{(0)}_{[0]}$ is a regularization of the distribution $-i\pi\delta(\xi^2)\,\varepsilon(\xi^0)/(8\pi^3)$ on the scale ε. Hence on the light cone it is of the order ε^{-1}. This gives the claim. ∎

This no-go result led me to reconsider the whole procedure of the continuum limit. At the same time, I tried to avoid imposing relations between the regularization parameters, which I never felt comfortable with because I wanted the continuum limit to work for at least a generic class of regularizations. Resolving this important issue took me a lot of time and effort. My considerations eventually led to the method of compensating the logarithmic poles by a *microlocal chiral transformation*. These construction as well as many preliminary considerations are given in [**5**, Section 3.7].

Finally, I would like to make a few comments on each chapter of the book. Chapters 1–3 are still up to date, except for the generalizations and modifications mentioned above. Compared to the presentation in [**5**], I see the benefit that these chapters might be easier to read and might convey a more intuitive picture of the underlying physical ideas. Chapter 4 is still the best reference for the general derivation of the formalism of the continuum limit. In [**5**, Chapter 2] I merely explained the regularization effects in examples and gave an overview of the methods and results in Chapter 4, but without repeating the detailed constructions. Chapter 5 is still the only reference where the form of the causal action is motivated and derived step by step. Also, the notion of state stability is introduced in detail, thus providing the basis for the later analysis in [**2, 8**]. As already mentioned above, the analysis in Chapters 6–8 is outdated. I recommend the reader to study instead [**5**, Chapters 3–5]. The Appendices are still valuable. I added a few footnotes which point to later improvements and further developments.

Felix Finster, Regensburg, August 2016

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Preface to the First Online Edition

In the few years since the book appeared, I was frequently asked if the introductory chapters were also available online. Also, I heard complaints that the preprints on the arXiv on the "principle of the fermionic projector" were preliminary versions which were not quite compatible with the book and with subsequent papers. In order to improve the situation, I decided to replace my original preprints hep-th/0001048, hep-th/0202059 and hep-th/0210121 by the corresponding chapters of the book.

I took the opportunity to correct some typos. I also added a few footnotes beginning with "Online version:" which point out to later developments. Apart from these changes, the present online version coincides precisely with the book in the AMS/IP series. In particular, all equation numbers are the same.

Felix Finster, Regensburg, October 2009

Preface

The basic ideas behind the "principle of the fermionic projector" go back to the years 1990-91 when I was a physics student in Heidelberg. At that time, I was excited about relativity and quantum mechanics, in particular about classical Dirac theory, but I felt uncomfortable with quantum field theory. The dissatisfaction with second quantized fields, which was in part simply a beginner's natural skepticism towards an unfamiliar physical concept, was my motivation for thinking about possible alternatives. Today I clearly understand quantum field theory much better, and many of my early difficulties have disappeared. Nevertheless, some of my objections remain, and the idea of formulating physics in a unified way based on Dirac's original concept of a "sea of interacting particles" seems so natural and promising to me that I have pursued this idea ever since. It took quite long to get from the first ideas to a consistent theory, mainly because mathematical methods had to be developed in order to understand the "collective behavior" of the particles of the Dirac sea.

This book gives me the opportunity to present the main ideas and methods in a somewhat broader context, with the intention of making this area of mathematical physics accessible to both theoretical physicists and applied mathematicians. The emphasis of the main chapters is on the conceptual part, whereas the more technical aspects are worked out in the appendices.

I am grateful to Claus Gerhardt, Joel Smoller, Shing-Tung Yau and Eberhard Zeidler for their encouragement and support. I would like to thank Stefan Hoch, Niky Kamran, Johann Kronthaler, Wätzold Plaum and Joel Smoller for helpful comments, and Eva Rütz for the typesetting. Finally, I am grateful to the Max Planck Institute for Mathematics in the Sciences, Leipzig, and the Morningside Center, Beijing, for their hospitality while I was working on the manuscript.

Felix Finster, Regensburg, November 2004

CHAPTER 0

The Principle of the Fermionic Projector – A New Mathematical Model of Space-Time

The mathematical model of space-time has evolved in history. In Newtonian mechanics, space is described by a Euclidean vector space. In special relativity, space and time were combined to Minkowski space, a vector space endowed with a scalar product of signature $(+---)$. In general relativity, the vector space structure of space-time was given up on the large scale and was replaced by that of a Lorentzian manifold. The first hint that the notions of space and time should be modified also on the microscopic scale was obtained by Planck, who observed that the gravitational constant, Planck's constant and the speed of light give rise to a quantity of the dimension of length,

$$l_P \;=\; \sqrt{\frac{\hbar\,\kappa}{c^3}} \;\approx\; 1.6 \cdot 10^{-35}\,\text{m}\,,$$

and he conjectured that for distances as tiny as this so-called Planck length, the conventional laws of physics should no longer hold, and yet unknown physical effects might become significant. Later, this picture was confirmed by quantum field theory. Namely, due to the ultraviolet divergences, perturbative QFT is well-defined only after regularization, and the regularization is then removed using the renormalization procedure. While renormalization ensures that the observable quantities do not depend on the regularization, the theoretical justification for the renormalization program lies in the assumption that the continuum theory should be valid only down to some microscopic length scale, and it seems natural to associate this length scale to the Planck length.

Today most physicists agree that in order to make progress in fundamental physics, one should go beyond the continuum field theory and try to get a better understanding of the microscopic structure of space-time. However, giving up the usual space-time continuum leads to serious problems, and this is one reason why there is no consensus on what the correct mathematical model for "Planck scale physics" should be. Let us illustrate the difficulties by briefly discussing a few of the many approaches. The simplest and maybe most natural approach is to assume that on the Planck scale space-time is no longer a continuum but becomes in some way "discrete." This idea is for example used in lattice gauge theories, where space-time is modeled by a four-dimensional lattice (see Figure 0.1(a)). Using the specific structures of a lattice like the nearest-neighbor relation and the lattice spacing d, one can set up a quantum field theory which is ultraviolet finite [**Ro**]. Lattice gauge theories are very useful for numerical simulations [**K**]. However, they are not fully satisfying from a conceptual point of view because a space-time lattice is not consistent with the equivalence principle of general relativity. Namely, if one considers the lattice in the reference frame of an accelerated observer (denoted in in Figure 0.1(b) by (t', x')), the lattice points are no longer in a regular configuration. Thus the structure of a lattice is not invariant under

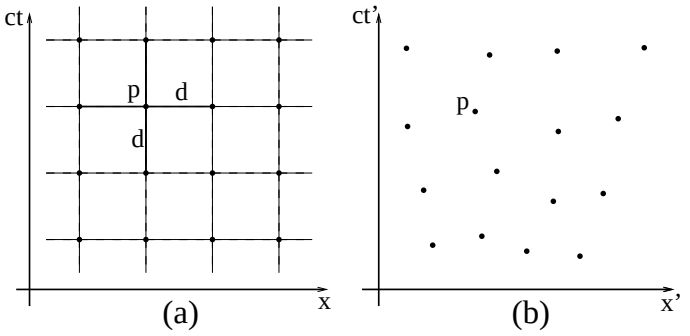

FIGURE 0.1. A lattice regularization for two different observers.

general coordinate transformations and hence is not compatible with the equivalence principle.

An alternative approach is to hold on to a space-time continuum, but to work with objects which are spread out in space-time, thereby avoiding the ultraviolet problems which occur for very small distances. The most prominent example in this direction is string theory, where physics is on a fundamental level described by so-called strings, which are extended in space-time and are therefore ultraviolet finite. The basic problem with such theories is that they are formulated using the structures of an underlying continuum space-time (like the vector space structure, the topology or even a metric), although all observable quantities (like the Lorentz metric, particles, fields, etc.) are to be derived from the non-localized objects, without referring to the underlying space-time continuum. Therefore, the structures of the underlying "continuum background" may seem artificial, and serious conceptual problems arise when these background structures are not compatible with basic physical principles (for example, a background vector space is not compatible with the equivalence principle). For short, one says that the theory is not background-free (for a more detailed discussion see [**Ba**] and the references therein).

Thus one difficulty in finding a promising model for Planck scale physics is that it should be background-free and should respect the basic physical principles (like the equivalence principle, the local gauge principle, etc.). There are approaches which actually meet these requirements. One is Connes' noncommutative geometry. As pointed out by Grothendieck, there is a one-to-one correspondence between the points of a manifold and the prime ideals of the (commutative) algebra of functions on this manifold. Thus the geometry of a manifold can be recovered from an underlying algebraic structure, and this makes it possible to extend the notions of space and time by considering more general, noncommutative algebras (see [**Co, CC**] for details and physical applications). The other approach is quantum gravity as pursued by Ashtekar and his school [**ARS, Th**]. The hope is that the ultraviolet divergences of QFT should disappear as soon as gravity, quantized in a non-perturbative way, is included.

Ultimately, a model for space-time on the Planck scale must be verified or falsified by physical experiments. Unfortunately, experiments on the Planck scale would require such enormously high energies that they are at present out of reach. Indirect experiments seem possible in principle [**ARA**] but have so far not been carried out. In my opinion, one should not hope for important new experimental input in the near future, but one should try to make due with the experimental data which is now available.

Indeed, this situation is not as hopeless as it might appear at first sight. Namely, in present physical models like the standard model, a lot of information from experiments is built in empirically, like the masses of the elementary particles, the gauge groups, the coupling constants, etc. Therefore, one can get a connection to experiments simply by trying to reproduce this well known empirical data. If successful, this procedure could give strong physical evidence for a model. For example, even if based on ad-hoc assumptions on the microscopic structure of space-time (which cannot be verified directly), a model would be very convincing and physically interesting if it gave the correct value for the fine structure constant and explained e.g. why the strong gauge group is $SU(3)$ or why neutrinos do not couple to the electromagnetic field. Thus the goal of a mathematical model for space-time on the Planck scale is to give a more fundamental explanation for the structures and empirical parameters in the standard model. To our opinion, only such concrete results can justify the model. Clearly, it is far from obvious how a promising model should look, or even in which mathematical framework it should be formulated. But at least for a mathematician, this makes the problem only more interesting, and it seems a challenging program to search for such models and to develop the mathematical methods needed for their analysis.

Our point of view that the mathematical model needs justification by known experimental data is not just a requirement which the model should fulfill at the very end, but it also gives a few hints on how one should proceed in order to find a promising model. First of all, one can expect concrete results only if one makes specific assumptions. Therefore, generalizing the notion of a Lorentzian manifold does not seem to be sufficient, but one should make a concrete ansatz for the microscopic structure of space-time (as it is done e.g. in string theory and lattice gauge theories). Furthermore, in order to make it easier to get a connection to well-established theories like classical field theory and quantum mechanics, it seems a good idea to take these theories as the starting point and to try to work as closely as possible with the objects used in these theories. Namely, if one drops important structures of the classical theories and/or introduces too many new structures ad hoc, it might become very difficult if not impossible to obtain a relation to observable data.

In our model of space-time we have tried to follow the above considerations. Our starting point is relativistic quantum mechanics and classical field theory. We assume that space-time is discrete on the Planck scale. But our notion of "discrete space-time" is much more general than a four-dimensional lattice; in particular, we do not assume any discrete symmetries in space-time, we keep the local gauge freedom, and we also extend the diffeomorphism invariance of a manifold in such a way that the equivalence principle is respected in discrete space-time. Furthermore, our model is background-free. In contrast to string theory, we do not introduce any new objects, but hold on to the structures already present in classical Dirac theory. We build in our physical ideas simply by prescribing which of these structures we consider as being fundamental, and then carry over these structures to discrete space-time. In the resulting mathematical framework, it is impossible to formulate the conventional physical equations, and thus we propose instead new equations of different type, referred to as the equations of discrete space-time. In a certain limiting case, the so-called continuum limit, we get a connection to the conventional formulation of physics in a space-time continuum. We point out that, in contrast to the Ashtekar program, we do not work with second quantized fields. But our concept is that the equations of discrete space-time should

also account for the physical effects of quantized fields if one goes beyond the continuum limit.

More specifically, we describe the physical system by the *fermionic projector* $P(x,y)$, which can be regarded as the projector on all occupied fermionic states of the system, including the states of the Dirac sea. After carrying over the fermionic projector to discrete space-time, we can set up variational principles like our "model variational principle"

$$\sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)] \;\rightarrow\; \min ,$$

where the "Lagrangian" $\mathcal{L}$ is given by

$$\mathcal{L}[A] \;=\; |A^2| - \mu |A|^2 ,$$

with μ a Lagrangian multiplier. Here $|A|$ is the so-called spectral weight defined as the sum of the absolute values of the eigenvalues of the matrix A (or, in case that A is not diagonalizable, of the zeros of its characteristic polynomial). We study the above variational principle for a fermionic projector which in the vacuum is the direct sum of seven identical massive sectors and one massless left-handed sector, each of which is composed of three Dirac seas. Analyzing the continuum limit for an interaction via general chiral and (pseudo)scalar potentials, we find that the sectors spontaneously form pairs, which are referred to as blocks. The resulting so-called effective interaction can be described by chiral potentials corresponding to the effective gauge group

$$SU(2) \times SU(3) \times U(1)^3 .$$

This model has striking similarity to the standard model if the block containing the left-handed sector is identified with the leptons and the three other blocks with the quarks. Namely, the effective gauge fields have the following properties.

- The $SU(3)$ corresponds to an unbroken gauge symmetry. The $SU(3)$ gauge fields couple to the quarks exactly as the strong gauge fields in the standard model.
- The $SU(2)$ potentials are left-handed and couple to the leptons and quarks exactly as the weak gauge potentials in the standard model. Similar to the CKM mixing in the standard model, the off-diagonal components of these potentials must involve a non-trivial mixing of the generations. The $SU(2)$ gauge symmetry is spontaneously broken.
- The $U(1)$ of electrodynamics can be identified with an Abelian subgroup of the effective gauge group.

The effective gauge group is larger than the gauge group of the standard model, but this is not inconsistent because a more detailed analysis of our variational principle should give further constraints for the Abelian gauge potentials. Moreover, there are the following differences to the standard model, which we derive mathematically without working out their physical implications.

- The $SU(2)$ gauge field tensor F must be simple in the sense that $F = \Lambda\, s$ for a real 2-form Λ and an $su(2)$-valued function s.
- In the lepton block, the off-diagonal $SU(2)$ gauge potentials are associated with a new type of potential, called nil potential, which couples to the right-handed component.

These results give a strong indication that the principle of the fermionic projector is of physical significance. Nevertheless, the goal of this book is not to work out our model

variational principle in all details. Our intention is to develop the general concepts and methods from the basics, making them easily accessible to the reader who might be interested in applying them to other equations of discrete space-time or to related problems.

These notes are organized as follows. In order to make the presentation self-contained, Chapter 1 gives a brief account of the mathematical and physical preliminaries. Chapter 2 introduces the fermionic projector in the continuum and provides the mathematical methods needed for its detailed analysis. In Chapter 3 we go beyond the continuum description and introduce our mathematical model for space-time on the Planck scale. In Chapter 4 we develop a mathematical formalism suitable for the analysis of the continuum limit. In Chapter 5 we present and discuss different equations of discrete space-time in the vacuum, and we choose the most promising equations as our "model equations". In the last Chapters 6-8 we analyze interacting systems in the continuum limit. The appendices contain additional material and will be referred to from the main chapters.

CHAPTER 1

Preliminaries

1.1. Relativity

In this section we briefly outline the mathematical framework of special and general relativity (for a more detailed introduction see [**Wo**] and [**Wa**]). We always work in *normal units* where $\hbar = c = 1$. In special relativity, space-time is described by Minkowski space $(M, \langle .,. \rangle)$, a real 4-dimensional vector space endowed with an inner product of signature $(+---)$. Thus, choosing a pseudo-orthonormal basis $(e_i)_{i=0,\ldots,3}$ and representing the vectors of M in this basis, $\xi = \sum_{i=0}^3 \xi^i e_i$, the inner product takes the form

$$\langle \xi, \eta \rangle \;=\; \sum_{j,k=0}^{3} g_{jk}\, \xi^j\, \eta^k \,, \tag{1.1.1}$$

where g_{ij}, the *Minkowski metric*, is the diagonal matrix $g = \text{diag}\,(1, -1, -1, -1)$. In what follows we usually omit the sums using Einstein's summation convention (i.e. we sum over all indices which appear twice, once as an upper and once as a lower index). Also, we sometimes abbreviate the Minkowski scalar product by writing $\xi\eta := \langle \xi, \eta \rangle$ and $\xi^2 := \langle \xi, \xi \rangle$. A pseudo-orthonormal basis $(e_i)_{i=0,\ldots,3}$ is in physics called a *reference frame*, because the corresponding coordinate system (x^i) of Minkowski space gives the time and space coordinates for an observer in a system of inertia. We also refer to $t := x^0$ as time and denote the spatial coordinates by $\vec{x} = (x^1, x^2, x^3)$.

The sign of the Minkowski metric encodes the causal structure of space-time. Namely, a vector $\xi \in M$ is said to be

$$\left.\begin{array}{cc} \textit{timelike} & \text{if } \langle \xi, \xi \rangle > 0 \\ \textit{spacelike} & \text{if } \langle \xi, \xi \rangle < 0 \\ \textit{null} & \text{if } \langle \xi, \xi \rangle = 0\,. \end{array}\right\}$$

Likewise, a vector is called *non-spacelike* if it is timelike or null. The null vectors form the double cone $L = \{\xi \in M \,|\, \langle \xi, \xi \rangle = 0\}$, referred to as the *light cone*. Physically, the light cone is formed of all rays through the origin of M which propagate with the speed of light. Similarly, the timelike vectors correspond to velocities slower than light speed; they form the *interior light cone* $I = \{\xi \in M \,|\, \langle \xi, \xi \rangle > 0\}$. Finally, we introduce the *closed light cone* $J = \{\xi \in M \,|\, \langle \xi, \xi \rangle \geq 0\}$. The space-time trajectory of a moving object describes a curve $q(\tau)$ in Minkowski space (with τ an arbitrary parameter). We say that the space-time curve q is timelike if the tangent vector to q is everywhere timelike. Spacelike, null, and non-spacelike curves are defined analogously. The usual statement of causality that no information can travel faster than with the speed of light can then be expressed as follows,

causality: information can be transmitted only along non-spacelike curves.

The set of all points which can be joined with a given space-time point x by a non-spacelike curve is precisely the closed light cone centered at x, denoted by $J_x := J - x$. It is the union of the two single cones

$$\begin{aligned} J_x^\vee &= \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \geq 0\} \\ J_x^\wedge &= \{y \in M \,|\, (y-x)^2 \geq 0,\ (y^0 - x^0) \leq 0\}\,, \end{aligned}$$

which have the interpretation as the points in the causal future and past of x, respectively. Thus we refer to $J_x^\vee$ and $J_x^\wedge$ as the closed *future* and *past light cones* centered at x, respectively. Similarly, we also introduce the sets $I_x^\vee$, $I_x^\wedge$ and $L_x^\vee$, $L_x^\wedge$.

A linear transformation of Minkowski space which leaves the form of the Minkowski metric (1.1.1) invariant is called a *Lorentz transformation*. The Lorentz transformations form a group, the *Lorentz group*. The Lorentz transformations which preserve both the time direction and the space orientation form a subgroup of the Lorentz group, the *orthochronous proper Lorentz group*.

The physical equations should be *Lorentz invariant*, meaning that they must be formulated in Minkowski space, independent of the reference frame. A convenient way of making Lorentz invariance manifest is to bring the equations in tensorial form (see [**L**] for a good introduction). Writing out the tensor indices, we get upper "contravariant" and lower "covariant" indices, which can be transformed into each other by contracting with the metric, e.g. $\xi_j = g_{jk}\xi^k$ and $\xi^k = g^{kl}\xi_l$ with $g^{kl} = (g_{kl})^{-1}$. In order to formulate electrodynamics in a manifestly Lorentz invariant form, one combines the electric potential and the vector potential to a 1-form $A = A_j dx^j$, the *electromagnetic potential*. The electric and magnetic fields are then components of the electromagnetic *field tensor* F defined by

$$F = dA \qquad \text{or, in components,} \qquad F_{jk} \;=\; \partial_j A_k - \partial_k A_j\,.$$

The Maxwell equations take the form

$$\partial_k F^{kl} \;=\; C\, j^l\,, \tag{1.1.2}$$

where j is the so-called 4-current and the constant $C = 4\pi e$ involves the electromagnetic coupling constant (we use the sign convention where $e > 0$). For an observer in a reference frame, the time and spatial components j^0 and $\vec{j}$ of the 4-current have the interpretation as the electric charge density and the electric current density, respectively. Since we shall always work in the 4-dimensional setting, it is unambiguous to refer to j simply as the *electromagnetic current*. Now consider a point particle of mass m and unit charge e in a given (=external) electromagnetic field. Since by causality its velocity is always smaller than light speed, its trajectory is a timelike curve $q(\tau)$. Thus we can parametrize it by the arc length, i.e.

$$\langle u, u\rangle \equiv 1 \qquad \text{with} \qquad u(\tau) := \frac{d}{d\tau} q(\tau)\,.$$

In a reference frame, the time and spatial components of the vector $m \cdot u(\tau)$ are the energy and momentum of the particle at the space-time point $q(\tau)$. We refer to mu as the *momentum* of the particle. The parameter τ has the interpretation as the *proper time* of an observer moving along q. The equation of motion is the tensor equation $m\frac{d}{d\tau}u^j = -eF^{jk}u_k$. Since we shall only consider particles of unit charge, it is convenient to remove the parameter e from the equation of motion. To this end, we rescale the electromagnetic potential according to $A \to e^{-1}A$. Then the equation of

motion simplifies to

$$m \frac{d}{d\tau} u^j \;=\; -F^{jk}\, u_k \,, \tag{1.1.3}$$

whereas the constant C in the Maxwell equations (1.1.2) becomes $C = 4\pi e^2$.

The starting point for general relativity is the observation that a physical process involving gravity can be understood in different ways. Consider for example an observer at rest on earth looking at a freely falling person (e.g. a diver who just jumped from a diving board). The observer at rest argues that the earth's gravitational force, which he can feel himself, also acts on the freely falling person and accelerates him. The person at free fall, on the other hand, does not feel gravity. He can take the point of view that he himself is at rest, whereas the earth is accelerated towards him. He then concludes that there are no gravitational fields, and that the observer on earth merely feels the force of inertia corresponding to his acceleration. Einstein postulated that these two points of view should be equivalent descriptions of the physical process. More generally, it depends on the observer whether one has a gravitational force or an inertial force. In other words,

equivalence principle: no physical experiment can distinguish between gravitational and inertial forces.

In mathematical language, observers correspond to coordinate systems, and so the equivalence principle states that the physical equations should be formulated in general (i.e. "curvilinear") coordinate systems, and should in all these coordinate systems have the same mathematical structure. This means that the physical equations should be invariant under diffeomorphisms, and thus space-time is to be modeled by a *Lorentzian manifold* (M, g).

A Lorentzian manifold is "locally Minkowski space" in the sense that at every space-time point $p \in M$, the corresponding *tangent space* T_pM is a vector space endowed with a scalar product $\langle .,. \rangle_p$ of signature $(+ - - -)$. Therefore, we can distinguish between spacelike, timelike and null tangent vectors. Defining a non-spacelike curve $q(\tau)$ by the condition that its tangent vector $u(\tau) \in T_{q(\tau)}M$ be everywhere non-spacelike, our above definition of light cones and the notion of causality immediately carry over to a Lorentzian manifold. In a coordinate chart, the scalar product $\langle .,. \rangle_p$ can be represented in the form (1.1.1) where g_{jk} is the so-called *metric tensor*. In contrast to Minkowski space, the metric tensor is not a constant matrix but depends on the space-time point, $g_{jk} = g_{jk}(p)$. Its ten components can be regarded as the relativistic analogue of Newton's gravitational potential. For every $p \in M$ there are coordinate systems in which the metric tensor coincides with the Minkowski metric up to second order,

$$g_{jk}(p) \;=\; \mathrm{diag}(1, -1, -1, -1) \,, \qquad \partial_j g_{kl}(p) \;=\; 0 \,. \tag{1.1.4}$$

Such *Gaussian normal coordinates* correspond to the reference frame of a "freely falling observer" who feels no gravitational forces. However, it is in general impossible to arrange that also $\partial_{jk} g_{lm}(p) = 0$. This means that by going into a suitable reference frame, the gravitational field can be transformed away locally (=in one point), but not globally. With this in mind, a reference frame corresponding to Gaussian normal coordinates is also called a *local inertial frame*.

The equation of motion (1.1.3) and the Maxwell equations (1.1.2) can easily be formulated on a Lorentzian manifold by the prescription that they should in a local

inertial frame have the same form as in Minkowski space; this is referred to as the *strong equivalence principle*. It amounts to replacing all partial derivatives by the corresponding *covariant derivatives* ∇ of the Levi-Civita connection; we write symbolically

$$\partial \;\longrightarrow\; \nabla\,. \tag{1.1.5}$$

We thus obtain the equations

$$m\,\nabla_\tau u^j \;=\; -F^{jk} u_k\,, \qquad\qquad \nabla_k F^{kl} \;=\; 4\pi e^2\, j^l \tag{1.1.6}$$

with $F_{jk} = (dA)_{jk} = \nabla_j A_k - \nabla_k A_j$.

The gravitational field is described via the curvature of space-time. More precisely, the Riemannian *curvature tensor* is defined by the relations

$$R^i_{jkl}\, u^l \;=\; \nabla_j \nabla_k u^i - \nabla_k \nabla_j u^i\,. \tag{1.1.7}$$

Contracting indices, one obtains the *Ricci tensor* $R_{jk} = R^i_{jik}$ and *scalar curvature* $R = R^j_j$. The relativistic generalization of Newton's gravitational law are the Einstein equations

$$R_{jk} - \frac{1}{2}\, R\, g_{jk} \;=\; 8\pi\kappa\, T_{jk}\,, \tag{1.1.8}$$

where κ is the gravitational constant. Here the *energy-momentum tensor* T_{jk} gives the distribution of matter and energy in space-time.

It is very convenient that the physical equations can all be derived from a variational principle. To this end, one considers the *action* (see e.g. [**LL**])

$$S \;=\; \int \Big(m\, g_{jk}\, u^j u^k + A_j u^j\Big)\, d\tau \;+\; \int_M \left(-\frac{1}{16\pi e^2}\, F_{jk} F^{jk} - \frac{1}{16\pi\kappa}\, R\right) d\mu\,, \tag{1.1.9}$$

where $u = c'(\tau)$ is the tangent vector of a timelike curve, and $d\mu := \sqrt{-\det g}\; d^4x$ is the integration measure on M. This action is not bounded below, but one can nevertheless look for stationary points and derive the corresponding Euler-Lagrange equations. Varying the space-time curve, the electromagnetic potential and the metric yield the equations of motion, the Maxwell equations and the Einstein equations, respectively.

1.2. Relativistic Quantum Mechanics

We now give an elementary introduction to relativistic quantum mechanics in Minkowski space (for more details see [**BD1, T**]). According to the Heisenberg Uncertainty Principle, the position and momentum of a quantum mechanical particle cannot be determined simultaneously, making it impossible to describe the particle by a trajectory in space-time. Instead, one uses a *wave function* $\Psi(t, \vec{x})$, whose absolute square has the interpretation as the probability density that the particle is at position $\vec{x}$. The simplest relativistic wave equation is the *Klein-Gordon equation*

$$(-\Box - m^2)\, \Psi \;=\; 0\,, \tag{1.2.1}$$

where $\Box \equiv \partial_j \partial^j$ is the wave operator. This equation describes a scalar particle (=particle without spin) of mass m. If the particle has electric charge, one needs to suitably insert the electromagnetic potential A into the Klein-Gordon equation. More precisely, one finds empirically that the equation

$$-\,(\partial_k - iA_k)(\partial^k - iA^k)\, \Psi \;=\; m^2 \Psi \tag{1.2.2}$$

describes a scalar particle of mass m and charge e in the presence of an electromagnetic field.

In order to describe a particle with spin, it was Dirac's idea to work with a first order differential operator whose square is the wave operator. One introduces the *Dirac matrices* γ^j as 4×4-matrices which satisfy the *anti-commutation relations*

$$2\, g^{jk}\, \mathbb{1} \;=\; \{\gamma^j,\, \gamma^k\} \;\equiv\; \gamma^j \gamma^k + \gamma^k \gamma^j \,. \tag{1.2.3}$$

Then the square of the operator $\gamma^j \partial_j$ is

$$(\gamma^j \partial_j)^2 \;=\; \gamma^j \gamma^k\, \partial_j \partial_k \;=\; \frac{1}{2}\, \{\gamma^j, \gamma^k\}\, \partial_{jk} \;=\; \Box \,. \tag{1.2.4}$$

For convenience, we shall always work in the Dirac representation

$$\gamma^0 \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \vec{\gamma} \;=\; \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{1.2.5}$$

where $\vec{\sigma}$ are the three Pauli matrices

$$\sigma^1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac equation in the vacuum reads

$$\left(i\gamma^k \frac{\partial}{\partial x^k} - m \right) \Psi(x) \;=\; 0 \,, \tag{1.2.6}$$

where $\Psi(x)$, the *Dirac spinor*, has four complex components. The leptons and quarks in the standard model are Dirac particles, and thus one can say that all matter is on the fundamental level described by the Dirac equation. Multiplying (1.2.6) by the operator $(i\gamma^j \partial_j + m)$ and using (1.2.4), one finds that each component of Ψ satisfies the Klein-Gordon equation (1.2.1). In the presence of an electromagnetic field, the Dirac equation must be modified to

$$i\gamma^k (\partial_k - iA_k) \Psi \;=\; m\Psi \,. \tag{1.2.7}$$

Multiplying by the operator $(i\gamma^j(\partial_j - iA_j) + m)$ and using the anti-commutation relations, we obtain the equation

$$\left[-(\partial_k - iA_k)(\partial^k - iA^k) + \frac{i}{2}\, F_{jk} \gamma^j \gamma^k - m^2 \right] \Psi \;=\; 0 \,.$$

This differs from the Klein-Gordon equation (1.2.2) by the extra term $\frac{i}{2} F_{jk} \gamma^j \gamma^k$, which describes the coupling of the spin to the electromagnetic field. We also denote the contraction with Dirac matrices by a slash, i.e. $\not{u} = \gamma^j u_j$ for u a vector of Minkowski space and $\not{\partial} = \gamma^j \partial_j$.

The wave functions at every space-time point are endowed with an indefinite scalar product of signature $(2,2)$, which we call *spin scalar product* and denote by

$$\prec \Psi \,|\, \Phi \succ (x) \;=\; \sum_{\alpha=1}^{4} s_\alpha\, \Psi^\alpha(x)^*\, \Phi^\alpha(x) \,, \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1 \,, \tag{1.2.8}$$

where Ψ^* is the complex conjugate wave function (this scalar product is often written as $\overline{\Psi}\Phi$ with the so-called adjoint spinor $\overline{\Psi} \equiv \Psi^* \gamma^0$). By the *adjoint* A^* of a matrix A we always mean the adjoint with respect to the spin scalar product as defined via the relations

$$\prec A^* \Psi \,|\, \Phi \succ \;=\; \prec \Psi \,|\, A\Phi \succ \qquad \text{for all } \Psi, \Phi.$$

In an obvious way, this definition of the adjoint gives rise to the notions "*selfadjoint*," "anti-selfadjoint" and "*unitary*." With these notions, the Dirac matrices are selfadjoint, meaning that

$$\prec \gamma^l \Psi \,|\, \Phi \succ \;=\; \prec \Psi \,|\, \gamma^l \Phi \succ \qquad \text{for all } \Psi, \Phi.$$

To every solution Ψ of the Dirac equation we can associate a time-like vector field J by

$$J^k \;=\; \prec \Psi \,|\, \gamma^k\, \Psi \succ\,, \tag{1.2.9}$$

which is called the *Dirac current*. The Dirac current is divergence-free,

$$\begin{aligned} \partial_k J^k &= \partial_k \prec \Psi \,|\, \gamma^k\, \Psi \succ \;=\; \prec \partial_k \Psi \,|\, \gamma^k\, \Psi \succ + \prec \Psi \,|\, \gamma^k \partial_k\, \Psi \succ \\ &= i\,(\prec i \partial\!\!\!/ \Psi \,|\, \Psi \succ - \prec \Psi \,|\, i \partial\!\!\!/ \Psi \succ) \\ &= i\,(\prec (i \partial\!\!\!/ + A\!\!\!/ - m) \Psi \,|\, \Psi \succ - \prec \Psi \,|\, (i \partial\!\!\!/ + A\!\!\!/ - m) \Psi \succ) \;=\; 0\,, \end{aligned}$$

this is referred to as *current conservation*.

So far Dirac spinors were introduced in a given reference frame. In order to verify that our definitions are coordinate independent, we consider two reference frames (x^j) and $(\tilde{x}^l)$ with the same orientation of time and space. Then the reference frames are related to each other by an orthochronous proper Lorentz transformation Λ, i.e. in components

$$\tilde{x}^l \;=\; \Lambda^l_j\, x^j\,, \qquad\qquad \Lambda^l_j\, \frac{\partial}{\partial \tilde{x}^l} \;=\; \frac{\partial}{\partial x^j}\,,$$

and Λ leaves the Minkowski metric invariant,

$$\Lambda^l_j\, \Lambda^m_k\, g_{lm} \;=\; g_{jk}\,. \tag{1.2.10}$$

Under this change of space-time coordinates, the Dirac operator $i\gamma^j(\partial_{x^j} - iA_j)$ transforms to

$$i\tilde{\gamma}^l \left(\frac{\partial}{\partial \tilde{x}^l} - i\tilde{A}_l \right) \qquad \text{with} \qquad \tilde{\gamma}^l \;=\; \Lambda^l_j \gamma^j\,. \tag{1.2.11}$$

This transformed Dirac operator does not coincide with the Dirac operator $i\gamma^l(\partial_{\tilde{x}^l} - i\tilde{A}_l)$ as defined in the reference frame $(\tilde{x}^l)$ because the Dirac matrices have a different form. However, the next lemma shows that the two Dirac operators do coincide up to a suitable unitary transformation of the spinors.

LEMMA 1.2.1. *For any orthochronous proper Lorentz transformation Λ there is a unitary matrix $U(\Lambda)$ such that*

$$U(\Lambda)\, \Lambda^l_j \gamma^j\, U(\Lambda)^{-1} \;=\; \gamma^l\,.$$

Proof. Since Λ is orthochronous and proper, we can write it in the form $\Lambda = \exp(\lambda)$, where λ is a suitable generator of a rotation and/or a Lorentz boost. Then $\Lambda(t) := \exp(t\lambda)$, $t \in \mathbb{R}$, is a family of Lorentz transformations, and differentiating (1.2.10) with respect to t at $t = 0$, we find that

$$\lambda^l_j\, g_{lk} \;=\; -g_{jm}\, \lambda^m_k\,.$$

Using this identity together with the fact that the Dirac matrices are selfadjoint, it is straightforward to verify that the matrix

$$u \;:=\; \frac{1}{4}\, \lambda^l_k\, \gamma_l \gamma^k$$

is anti-selfadjoint. As a consequence, the family of matrices

$$U(t) \;:=\; \exp\left(tu\right)$$

is unitary. We now consider for a fixed index l the family of matrices

$$A(t) \;:=\; U(t)\,\Lambda(t)^l_j \gamma^j\, U(t)^{-1}\,.$$

Clearly, $A(0) = \gamma^l$. Furthermore, differentiating with respect to t gives

$$\frac{d}{dt}\,A(t) \;=\; U\,\Lambda^l_j \left\{u\,\gamma^j - \gamma^j\,u + \lambda^j_k \gamma^k\right\} U^{-1}\,,$$

and a short calculation using the commutation relations

$$\left[\gamma_l \gamma_k, \gamma^j\right] \;=\; 2\left(\gamma_l\, g^{kj} - \delta^j_l\,\gamma^k\right)$$

shows that the curly brackets vanish. We conclude that $A(1) = A(0)$, proving the lemma. ∎

Applying this lemma to the Dirac operator in (1.2.11), one sees that the Dirac operator is invariant under the joint transformation of the space-time coordinates and the spinors

$$x^j \;\longrightarrow\; \Lambda^j_k x^k\,, \qquad \Psi \;\longrightarrow\; U(\Lambda)\,\Psi\,. \tag{1.2.12}$$

Moreover, since the matrix $U(\Lambda)$ is unitary, the representation of the spin scalar product (1.2.8) is valid in any reference frame. We conclude that our definition of spinors is indeed coordinate invariant.

Out of the Dirac matrices one can form the *pseudoscalar matrix* ρ by

$$\rho \;=\; \frac{i}{4!}\,\epsilon_{jklm}\gamma^j\gamma^k\gamma^l\gamma^m \tag{1.2.13}$$

(this matrix in the physics literature is usually denoted by γ^5). Here ϵ_{jklm} is the totally antisymmetric symbol (i.e. ϵ_{jklm} is equal to ± 1 if (j,k,l,m) is an even and odd permutation of $(0,1,2,3)$, respectively, and vanishes otherwise). A short calculation shows that the pseudoscalar matrix is anti-selfadjoint and $\rho^2 = \mathbb{1}$. As a consequence, the matrices

$$\chi_L \;=\; \frac{1}{2}\,(\mathbb{1} - \rho)\;, \qquad \chi_R \;=\; \frac{1}{2}\,(\mathbb{1} + \rho) \tag{1.2.14}$$

satisfy the relations

$$\chi^2_{L/R} = \chi_{L/R}\;, \qquad \rho\,\chi_L = -\chi_L\;, \qquad \rho\,\chi_R = \chi_R\;, \qquad \chi_L^* \;=\; \chi_R\;, \qquad \chi_L + \chi_R = \mathbb{1}\,.$$

They can be regarded as the spectral projectors of the matrix ρ and are called the *chiral projectors* . The projections $\chi_L\Psi$ and $\chi_R\Psi$ are referred to as the *left-* and *right-handed* components of the spinor. A matrix is said to be *even* and *odd* if it commutes and anti-commutes with ρ, respectively. It is straightforward to verify that the Dirac matrices are odd, and therefore

$$\gamma^j\,\chi_{L/R} \;=\; \chi_{R/L}\,\gamma^j\,.$$

Using this relation, one can rewrite the Dirac equation (1.2.7) as a system of equations for the left- and right-handed components of Ψ,

$$i\gamma^k(\partial_k - iA_k)\,\chi_L\Psi \;=\; m\,\chi_R\Psi\;, \qquad i\gamma^k(\partial_k - iA_k)\,\chi_R\Psi \;=\; m\,\chi_L\Psi\,.$$

If $m = 0$, these two equations decouple, and we get separate equations for the left- and right-handed components of Ψ. This observation is the starting point of the 2-component Weyl spinor formalism. We shall not use this formalism here, but will instead describe chiral massless particles (like neutrinos) by the left- or right-handed component of a Dirac spinor.

For the probabilistic interpretation of the Dirac wave function, we need to distinguish a direction of time and work in a particular reference frame. Then the zero component of the Dirac current $J^0(t, \vec{x})$ has for a given time t the interpretation as the *probability density* of the particle to be at position $\vec{x}$ (and is thus the relativistic analogue of the absolute square $|\Psi|^2$ of the Schrödinger or Pauli wave functions). Clearly, for this probabilistic interpretation the wave function must be properly normalized. More precisely, physical states must satisfy the *normalization condition*

$$\int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Psi \succ (t, \vec{x})\, d\vec{x} \;=\; 1\,. \tag{1.2.15}$$

The integral in (1.2.15) is also called the *probability integral*. Using Gauss' (divergence) theorem and the current conservation, one sees that the normalization integral is time independent,

$$\begin{aligned} &\int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Psi \succ (t_2, \vec{x})\, d\vec{x} - \int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Psi \succ (t_1, \vec{x})\, d\vec{x} \\ &= \int_{t_1}^{t_2} dt \int_{\mathbb{R}^3} d\vec{x}\; \partial_k \prec \Psi \,|\, \gamma^k\, \Psi \succ (t, \vec{x}) \;=\; 0\,, \end{aligned} \tag{1.2.16}$$

and thus it suffices to satisfy (1.2.15) for example at $t = 0$.

In a given reference frame, it is convenient to introduce a positive scalar product by polarizing the normalization integral,

$$(\Psi \,|\, \Phi) \;:=\; \int_{\mathbb{R}^3} \prec \Psi \,|\, \gamma^0\, \Phi \succ (t, \vec{x})\, d\vec{x}\,. \tag{1.2.17}$$

We denote the Hilbert space corresponding to this scalar product by $\mathcal{H} = L^2(\mathbb{R}^3)^4$. Multiplying the Dirac equation (1.2.7) by γ^0 and bringing the t-derivative on a separate side of the equation, we can write the Dirac equation as

$$i\partial_t \Psi \;=\; h\Psi \tag{1.2.18}$$

with a purely spatial operator h. Clearly, this equation is not manifestly covariant. In analogy to nonrelativistic quantum mechanics, it is referred to as the Dirac equation in *Hamiltonian form*, and h is the *Hamiltonian*. If Ψ and Φ are solutions of the Dirac equation, one sees similar to (1.2.16) that the scalar product (1.2.17) is independent of time. Hence

$$0 \;=\; \partial_t (\Psi \,|\, \Phi) \;=\; i\,((h\Psi \,|\, \Phi) - (\Psi \,|\, h\Phi))\;.$$

This shows that the Hamiltonian is a symmetric operator on $\mathcal{H}$.

We conclude this section by a brief discussion of the solutions of the free Dirac equation (=the Dirac equation without electromagnetic field) in the case $m \neq 0$. Taking the Fourier transform of the wave function,

$$\Psi(x) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{\Psi}(k)\, e^{-ikx}\;, \qquad \hat{\Psi}(k) \;=\; \int d^4x\; \Psi(x)\, e^{ikx}\;,$$

the Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$ reduces to the algebraic equation in momentum space

$$(k\!\!\!/ - m)\, \hat{\Psi}(k) \;=\; 0\,. \tag{1.2.19}$$

Multiplying by $k\!\!\!/ + m$ and using the identity $(k\!\!\!/ - m)(k\!\!\!/ + m) = k^2 - m^2$, one sees that if $k^2 \neq m^2$, the matrix $k\!\!\!/ - m$ is invertible and thus (1.2.19) has no solutions. If conversely $k^2 = m^2$, we have the relation $(k\!\!\!/ - m)^2 = -2m(k\!\!\!/ - m)$, showing that the matrix $k\!\!\!/ - m$ is diagonalizable and that its eigenvalues are either $-2m$ or zero. Taking the trace, $\mathrm{Tr}(k\!\!\!/ - m) = -4m$, it follows that the matrix $k\!\!\!/ - m$ has a two-dimensional kernel. A short calculation shows that the projector onto this kernel is given by

$$\Pi(k) \;:=\; \frac{k\!\!\!/ + m}{2m}\,. \tag{1.2.20}$$

We conclude that (1.2.19) has a solution only if k is on the *mass shell* $\{k \mid k^2 = m^2\}$. For each k on the mass shell, (1.2.19) has exactly two linearly independent solutions. In order to give these solutions more explicitly, we choose a reference frame $(t, \vec{x})$ and denote the corresponding momentum variables by $k = (\omega, \vec{p})$. The momenta on the mass shell are then given by

$$\omega \;=\; \omega(\vec{p}, \epsilon) := \epsilon \sqrt{|\vec{p}|^2 + m^2}$$

with parameters $\vec{p} \in \mathbb{R}^3$ and $\epsilon \in \{\pm 1\}$. The momenta with $\epsilon = 1$ and $\epsilon = -1$ are said to be on the *upper* and *lower mass shell*, respectively. For any given $(\vec{p}, \epsilon)$, we label the two linearly independent solutions of (1.2.19) by a parameter $s \in \{1, 2\}$ and denote them by $\chi_{\vec{p}s\epsilon}$. It is most convenient to choose them pseudo-orthonormal with respect to the spin scalar product,

$$\prec \chi_{\vec{p}s\epsilon} | \chi_{\vec{p}s'\epsilon} \succ \;=\; \epsilon\, \delta_{s,s'} \qquad \text{for all } \vec{p} \in \mathbb{R}^3, \epsilon \in \{\pm 1\} \text{ and } s, s' \in \{1, 2\}\,. \tag{1.2.21}$$

Here the factor ϵ reflects that the solution spaces on the upper and lower mass shell are positive and negative definite, respectively. Using a bra/ket notation in the spin scalar product, we get a simple representation of the projector (1.2.20),

$$\sum_{s=1,2} \epsilon\, |\chi_{\vec{p}s\epsilon}\succ \prec \chi_{\vec{p}s\epsilon}| \;=\; \Pi(\omega(\vec{p}, \epsilon), \vec{p})\,. \tag{1.2.22}$$

The spinors $\chi_{\vec{p}s\epsilon}$ form a complete set of solutions of (1.2.19). Taking their suitably normalized Fourier transform, we obtain the *plane wave solutions*

$$\Psi_{\vec{p}s\epsilon}(t, \vec{x}) \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{-i\omega(\vec{p},\epsilon)t + i\vec{p}\vec{x}}\, \chi_{\vec{p}s\epsilon}\,. \tag{1.2.23}$$

Each solution of the free Dirac equation is a superposition of plane wave solutions.

In the Hamiltonian framework (1.2.18), the plane wave solutions are eigenfunctions of the Hamiltonian with eigenvalue $\omega(\vec{p}, \epsilon)$,

$$h\, \Psi_{\vec{p}s\epsilon} \;=\; \omega(\vec{p}, \epsilon)\, \Psi_{\vec{p}s\epsilon}\,.$$

Since the eigenvalue of the Hamiltonian has the interpretation as the physical energy of the state, we conclude that the plane wave solutions on the upper and lower mass shell have positive and negative energy, respectively. Expressed more mathematically, the plane wave solutions correspond to points in the essential spectrum of the Hamiltonian, and thus

$$\sigma_{\text{ess}}(h) \;=\; \{\pm\sqrt{\vec{p}^2 + m^2},\, \vec{p} \in \mathbb{R}^3\} \;=\; (-\infty, -m] \cup [m, \infty)\,.$$

In particular, we conclude that the Dirac equation has solutions of negative energy and that the Hamiltonian is not bounded below. This was originally considered a serious problem of Dirac theory, mainly because a system with unbounded Hamiltonian has no stable ground state. Dirac resolved these problems by introducing the so-called Dirac sea [**D2**]. The concept of the Dirac sea plays a crucial role in the present work. At this point, we merely explain Dirac's idea in words (in the next section we shall explain how it is implemented mathematically in the framework of quantum field theory, and in Chapter §2 we will come back to it in greater detail). Thinking of many-particle quantum mechanics (and assuming that the particles do not interact with each other), the solutions of the Dirac equation can be regarded as one-particle states, which can be occupied by the particles of the system. According to the Pauli Exclusion Principle, each state may be occupied by at most one particle. If one assumes that no states are occupied in the vacuum, a system of n particles is unstable because the energy of the system can be made negative and arbitrarily small by occupying n negative-energy states. However, this problem disappears if one assumes that in the vacuum all states of negative energy are already occupied. Then the n additional particles must occupy states of positive energy, and the system becomes stable. This consideration led Dirac to no longer think of the vacuum as "empty space," but instead to conjecture that the vacuum should be formed of a "sea" of quantum mechanical particles of negative energy. Dirac's conception was that the effects of all the particles of the sea should compensate each other in such a way that the sea cannot be observed. Likewise, in this picture an interacting system of n particles corresponds to the Dirac sea and n additional particles of positive energy which all interact with each other. This intuitive concept of the Dirac sea as a "sea of interacting particles" was not only useful for resolving the problem of the negative-energy solutions, but furthermore led to the prediction of anti-particles and pair creation/annihilation. To this end, Dirac considered an interacting system which at initial time $t = 0$ is the vacuum. Then at a later time, one of the particles of the sea may no longer occupy a state of negative energy, but be instead in a positive-energy state. In this case, the system consists of one particle and one "hole" in the Dirac sea. Since the completely filled Dirac sea should be invisible, the hole appears as a virtual particle of energy and electric charge opposite to that of the unoccupied negative-energy state. Thus the virtual particle has positive energy, but its charge is opposite to that of an ordinary particle. This virtual particle is referred to as anti-particle. In the above process, particles and anti-particles are always generated in pairs, this explains the physical effect of pair creation. Conversely, a particle and a hole can recombine in a process called pair annihilation.

1.3. Fock Space Quantization of the Free Dirac Field

In this section we outline the canonical quantization of the free Dirac field (for details see [**BD2, IZ**]). For clarity, we first quantize without taking into account the Dirac sea and explain afterwards how the construction is to be modified in order to cure the problem of the negative-energy states. We begin with the one-particle Hilbert space $(\mathcal{H}, (.,.))$ in the Hamiltonian framework (1.2.18). Clearly, the plane-wave solutions (1.2.23) are not square integrable, but we can normalize them in the distributional sense. More precisely,

$$(\Psi_{\vec{p}s\epsilon} \,|\, \Psi_{\vec{p}'s'\epsilon'}) \;=\; \delta^3(\vec{p}-\vec{p'})\, \prec\chi_{\vec{p}s\epsilon} \,|\, \gamma^0 \chi_{\vec{p}s'\epsilon'}\succ . \tag{1.3.1}$$

In order to compute the inner product $\prec \chi_{\vec{p}s\epsilon} \,|\, \gamma^0 \chi_{\vec{p}s'\epsilon'} \succ$, we first plug in the spectral projectors (1.2.20), which for convenience we now denote by $\Pi_{\vec{p}\epsilon} := \Pi(\omega(\vec{p},\epsilon),\vec{p})$,

$$\prec \chi_{\vec{p}s\epsilon} \,|\, \gamma^0 \chi_{\vec{p}s'\epsilon'} \succ \;=\; \prec \Pi_{\vec{p}\epsilon} \chi_{\vec{p}s\epsilon} \,|\, \gamma^0\, \Pi_{\vec{p}\epsilon'} \chi_{\vec{p}s'\epsilon'} \succ \;=\; \prec \chi_{\vec{p}s\epsilon} \,|\, \Pi_{\vec{p}\epsilon}\, \gamma^0\, \Pi_{\vec{p}\epsilon'} \chi_{\vec{p}s'\epsilon'} \succ .$$

The matrix product $\Pi_{\vec{p}\epsilon} \gamma^0 \Pi_{\vec{p}\epsilon'}$ is computed in the cases $\epsilon = \epsilon'$ and $\epsilon \neq \epsilon'$ as follows,

$$\begin{aligned}
\Pi_{\vec{p}\epsilon}\, \gamma^0\, \Pi_{\vec{p}\,-\epsilon} &= \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m}\, \gamma^0\, \frac{-\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m} \\
&= \frac{(\not{k} + m)(-\not{k} + m)}{4m^2}\, \gamma^0 \;=\; 0 \\
\Pi_{\vec{p}\epsilon}\, \gamma^0\, \Pi_{\vec{p}\epsilon} &= \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m}\, \gamma^0\, \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{2m} \\
&= \frac{\omega\gamma^0 - \vec{p}\vec{\gamma} + m}{4m^2}\, 2\omega \;+ \frac{(\not{k} + m)(-\not{k} + m)}{4m^2}\, \gamma^0 \;=\; \frac{\omega}{m}\, \Pi_{\vec{p}\epsilon}\,,
\end{aligned}$$

where we set $\omega = |\omega(\vec{p},\epsilon)|$ and $k = (\omega, \vec{p})$. Hence the matrix products reduce to a multiple of the identity, and we can use the normalization (1.2.21) to obtain

$$(\Psi_{\vec{p}s\epsilon} \,|\, \Psi_{\vec{p}'s'\epsilon'}) \;=\; \frac{\omega(\vec{p})}{m}\, \delta^3(\vec{p} - \vec{p}')\, \delta_{\epsilon\epsilon'}\, \delta_{ss'}\,, \tag{1.3.2}$$

with $\omega(\vec{p}) := |\omega(\vec{p},\epsilon)| = \sqrt{|\vec{p}|^2 + m^2}$. Readers who dislike this δ-normalization can also state (1.3.2) by saying that, similar to a Fourier transformation, the mapping

$$L^2\Big(\mathbb{R}^3, \frac{d\vec{p}}{2\omega(\vec{p})}\Big)^4 \;\longrightarrow\; \mathcal{H} \;:\; f_{s\epsilon}(\vec{p}) \;\longmapsto\; \sqrt{2m} \sum_{s,\epsilon} \int_{\mathbb{R}^3} \frac{d\vec{p}}{2\omega(\vec{p})}\, f_{s\epsilon}(\vec{p})\, \Psi_{\vec{p}s\epsilon}(t,\vec{x})$$

is an isometry of Hilbert spaces. The factor $d\vec{p}/(2\omega(\vec{p}))$ appearing here can be interpreted as the Lorentz invariant measure on the mass shell (i.e. formally $d\vec{p}/(2\omega(\vec{p})) = \delta(k^2 - m^2)\, d^4k$), and we abbreviate it in what follows by $d\mu_{\vec{p}}$.

In many-particle quantum mechanics, the system where the n one-particle states $\Psi_{\vec{p}_1 s_1 \epsilon_1}, \ldots, \Psi_{\vec{p}_n s_n \epsilon_n}$ are occupied is described by the *Hartree-Fock state*

$$\Psi \;=\; \Psi_{\vec{p}_1 s_1 \epsilon_1} \,\wedge\, \cdots \,\wedge\, \Psi_{\vec{p}_n s_n \epsilon_n}\,. \tag{1.3.3}$$

Here the wedge product $\wedge$ is the anti-symmetrized tensor product. Due to the anti-symmetry, the wedge product vanishes if two of the one-particle wave functions $\Psi_{\vec{p}_i, s_i, \epsilon_i}$ coincide. This corresponds to the

Pauli Exclusion Principle: each quantum mechanical state can be occupied by at most one particle.

Particles which obey the Pauli Exclusion Principle are called *fermions* (whereas for bosons one uses instead of (1.3.3) the symmetric tensor product). Working with the n-particle state (1.3.3) also implies that the n particles are *indistinguishable* in the sense that if we exchange two particles, the wave function Ψ changes only by a physically irrelevant phase.

A general n-particle state corresponds to a linear combination of Hartree-Fock states and is thus a vector of the Hilbert space $\mathcal{F}^n = \wedge^n \mathcal{H}$. In quantum field theory, the number of particles is not fixed, and therefore the Dirac particles are described more generally by a vector of the *fermionic Fock space* $\mathcal{F} = \oplus_{n=0}^\infty \mathcal{F}^n$. Notice that the

scalar product on $\mathcal{F}$ is induced by that on $\mathcal{H}$; we denote it for clarity by $(.|.)_{\mathcal{F}}$. On the Fock space, we introduce the field operators $\hat{\Psi}^\dagger_{\vec{p}s\epsilon}$ by

$$\hat{\Psi}^\dagger_{\vec{p}s\epsilon} \;:\; \mathcal{F}^n \;\longrightarrow\; \mathcal{F}^{n+1} \;:\; \Psi \;\longmapsto\; \Psi_{\vec{p}s\epsilon} \wedge \Psi$$

and denote their adjoint with respect to the scalar product $(.|.)_{\mathcal{F}}$ by $\hat{\Psi}_{\vec{p}s\epsilon}$, $\hat{\Psi}_{\vec{p}s\epsilon} = (\hat{\Psi}^\dagger_{\vec{p}s\epsilon})^*$. The operators $\hat{\Psi}^\dagger_{\vec{p}s\epsilon}$ and $\hat{\Psi}_{\vec{p}s\epsilon}$ are referred to as the *creation* and *annihilation operators*, respectively. A straightforward calculation using our definitions and the normalization condition (1.3.2) yields that the field operators satisfy the *canonical anticommutation relations*

$$\begin{aligned} \{\hat{\Psi}_{\vec{p}s\epsilon}, \hat{\Psi}_{\vec{p}'s'\epsilon'}\} &= 0 = \{\hat{\Psi}^\dagger_{\vec{p}s\epsilon}, \hat{\Psi}^\dagger_{\vec{p}'s'\epsilon'}\} \\ \{\hat{\Psi}_{\vec{p}s\epsilon}, \hat{\Psi}^\dagger_{\vec{p}'s'\epsilon'}\} &= 2\omega(\vec{p})\, \delta^3(\vec{p}-\vec{p}')\, \delta_{\epsilon\epsilon'}\, \delta_{ss'}\;. \end{aligned} \tag{1.3.4}$$

The vacuum corresponding to these field operators, denoted by $|0>$, is a unit vector of $\mathcal{F}$ on which all annihilation operators vanish,

$$\hat{\Psi}_{\vec{p}s\epsilon}\,|0> \;=\; 0 \qquad \text{for all } \vec{p}, s, \epsilon. \tag{1.3.5}$$

The Hartree-Fock states can be obtained from it by applying the creation operators,

$$(1.3.3) \;=\; \hat{\Psi}^\dagger_{\vec{p}_1 s_1 \epsilon_1} \cdots \hat{\Psi}^\dagger_{\vec{p}_n s_n \epsilon_n}\,|0>\,,$$

and taking linear combinations, we can build up the whole Fock space from the vacuum. The energy of the Hartree-Fock state (1.3.3) is the sum of the energies $\omega(\vec{p},\epsilon)$ of all particles, and a short calculation shows that this coincides with the eigenvalue of the following operator on the Fock space,

$$H_0 \;=\; \sum_{\epsilon,s} \int_{\mathbb{R}^3} \omega(\vec{p},\epsilon)\; \hat{\Psi}^\dagger_{\vec{p}s\epsilon} \hat{\Psi}_{\vec{p}s\epsilon}\, d\mu_{\vec{p}}\,. \tag{1.3.6}$$

Thus H_0 is the Hamiltonian of the free many-particle theory.

Since the factor $\omega(\vec{p},\epsilon)$ in the integrand can be negative, the Hamiltonian (1.3.6) is not bounded from below. This is precisely the problem of the negative-energy solutions of the Dirac equation which we described at the end of the previous section. This problem disappears in quantum field theory as follows. According to the concept of the Dirac sea, all negative-energy states should be occupied in the vacuum. This is implemented here by redefining the vacuum; namely we replace (1.3.5) by the conditions

$$\hat{\Psi}_{\vec{p}s+}\,|0> \;=\; 0 \;=\; \hat{\Psi}^\dagger_{\vec{p}s-}\,|0> \qquad \text{for all } \vec{p}, s. \tag{1.3.7}$$

Since the anti-particles correspond to "holes" in the Dirac sea, we reinterpret the creation operators for the negative-energy states as annihilation operators and vice versa, i.e. we perform the formal replacements

$$\hat{\Psi}_{\vec{p}s-} \;\longleftrightarrow\; \hat{\Psi}^\dagger_{\vec{p}s-}\,. \tag{1.3.8}$$

This is convenient because after the reinterpretation, the new vacuum (1.3.7) again satisfies the usual conditions (1.3.5). The Hamiltonian (1.3.6) transforms under the replacements (1.3.8) into

$$H_0 \;=\; \sum_{\epsilon,s} \int_{\mathbb{R}^3} \omega(\vec{p})\; \hat{\Psi}^\dagger_{\vec{p}s\epsilon} \hat{\Psi}_{\vec{p}s\epsilon}\, d\mu_{\vec{p}} \;-\; \sum_{s} \int_{\mathbb{R}^3} \omega(\vec{p})\; \{\hat{\Psi}^\dagger_{\vec{p}s-}, \hat{\Psi}_{\vec{p}s-}\}\, d\mu_{\vec{p}}\,. \tag{1.3.9}$$

The first part of this Hamiltonian is positive. Using the anti-commutation relations (1.3.4), one sees that the second term in (1.3.9) is an infinite negative constant. Using the argument that adding a constant to the total energy of a system is nothing more than introducing a new convention for the zero point of energy measurements, one drops this second term and redefines the Hamiltonian by

$$H_0 \;=\; \sum_{\epsilon,s} \int_{\mathbb{R}^3} |\omega(\vec{p},s)|\; \hat{\Psi}^\dagger_{\vec{p}s\epsilon} \hat{\Psi}_{\vec{p}s\epsilon}\, d\mu_{\vec{p}}\,. \tag{1.3.10}$$

This Hamiltonian is positive and vanishes on the vacuum, giving rise to a satisfying physical theory. However, dropping the second summand in (1.3.9) was a problematic step in the construction. We postpone the discussion of this point to §2.2.

1.4. Classical Gauge Theories

We now briefly introduce the framework of local gauge theories (for a more detailed introduction see for example [**Ga**]). In order to avoid confusion between covariant derivatives ∇ and gauge-covariant derivatives D we restrict attention to Minkowski space. The generalization to curved space-time will be described in connection with the Dirac equation in §1.5. The starting point for gauge theories is the observation that changing the electromagnetic potential by the gradient of a real-valued function Λ,

$$A \;\longrightarrow\; A + \partial\Lambda\,, \tag{1.4.1}$$

leaves the field tensor unchanged,

$$F_{jk} \;\longrightarrow\; F_{jk} + \partial_j\partial_k\Lambda - \partial_k\partial_j\Lambda \;=\; F_{jk}\,.$$

The equations of classical electrodynamics (1.1.2, 1.1.3) do not involve the electromagnetic potential, only its field tensor. Therefore, these equations are obviously invariant under the transformation (1.4.1). In the quantum mechanical wave equations (1.2.2, 1.2.7) the electromagnetic potential does appear, but only in combination with a partial derivative in the operators $\partial_k - iA_k$. These operators transform under (1.4.1) as follows,

$$\partial_k - iA_k \;\longrightarrow\; \partial_k - iA_k - i\partial_k\Lambda \;=\; e^{i\Lambda}\,(\partial_k - iA_k)\,e^{-i\Lambda}\,.$$

Writing the transformation law with the multiplication operators $e^{\pm i\Lambda}$ reveals that the equations of quantum mechanics are invariant under (1.4.1) if at the same time the local phase of the wave functions is transformed according to

$$\Psi \;\longrightarrow\; e^{i\Lambda}\,\Psi\,. \tag{1.4.2}$$

Finally, these local phase transformations leave the Dirac current (1.2.9) unchanged. We conclude that classical field theory and relativistic quantum mechanics are invariant under the transformation (1.4.1, 1.4.2), which is referred to as a *local gauge transformation* of electrodynamics. The invariance of the physical equations under local gauge transformations can be interpreted as a physical symmetry, the *local gauge symmetry*.

Extending the above concept leads to the mathematical framework of gauge theories. We first note that the phase factor $e^{i\Lambda}$ in (1.4.2) can be interpreted as the operation of an element of the Lie group $U(1)$ on Ψ. Likewise, the factors $\partial_j\Lambda = ie^{i\Lambda}\partial_j e^{-i\Lambda}$ can be regarded as elements of the corresponding Lie algebra $u(1)$. Since in (1.4.1) this factor is added to the components A_j of the electromagnetic potential, it is natural to also consider the A_j as $u(1)$-valued functions. In generalization, we let the

gauge group $\mathcal{G}$ be an arbitrary Lie group in a given matrix representation on the wave functions (the wave functions may have more than four components; a typical example is $\mathcal{G} = U(p)$ and $\Psi(x) \in \mathbb{C}^4 \otimes \mathbb{C}^p$). The corresponding Lie algebra in its representation on the wave functions is denoted by $\mathfrak{g}$. We introduce the *gauge potentials* A_j as $\mathfrak{g}$-valued functions on M. For any smooth function $U : M \to \mathcal{G}$, the transformation of the wave functions

$$\Psi(x) \;\longrightarrow\; U(x)\, \Psi(x) \tag{1.4.3}$$

is referred to as a *local gauge transformation*. Clearly, partial derivatives of Ψ do not behave well under gauge transformations because we pick up derivatives of U. This problem disappears if instead of partial derivatives we consider *gauge-covariant derivatives*

$$D_j \;=\; \partial_j - iA_j\,, \tag{1.4.4}$$

provided that the gauge potentials transform according to

$$A_j \;\longrightarrow\; U A_j U^{-1} + iU\,(\partial_j U^{-1})\,. \tag{1.4.5}$$

Namely, a short calculation shows that the gauge-covariant derivative behaves under gauge transformations according to

$$D_j \;\longrightarrow\; U\, D_j\, U^{-1}\,, \tag{1.4.6}$$

and thus the gauge-covariant derivatives of Ψ obey the simple transformation rule

$$D_j \Psi \;\longrightarrow\; U\, D_j \Psi\,.$$

Next we need to introduce the gauge potentials into the physical equations and formulate the equations that describe the dynamics of the gauge fields. We just saw that in order to ensure gauge invariance, one should work with gauge-invariant derivatives instead of partial derivatives. The simplest method for making the physical theory gauge invariant is to replace all partial derivatives by the corresponding gauge-invariant derivatives,

$$\partial \;\longrightarrow\; D\,. \tag{1.4.7}$$

This ad-hoc method is in physics called the *minimal coupling* procedure. For the equations of quantum mechanics it can be motivated if one keeps in mind that with a local gauge transformation of the form $U(x) = \mathbb{1} - iA_j\,(x-p)^j + o(x-p)$ we can always arrange that $A(p) = 0$. In this gauge, the gauge-covariant derivatives coincide at p with the partial derivatives, and thus we can state minimal coupling as follows,

> Around each space-time point p there is a gauge such that the quantum mechanical equations coincide at p with the equations without gauge fields. (1.4.8)

In this formulation, minimal coupling can be understood similar to the strong equivalence principle; we only need to replace "coordinate system" by "gauge" and "gravitational field" by "gauge field." In the example of the free Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$, minimal coupling yields the equation

$$i\gamma^j(\partial_j - iA_j)\,\Psi \;=\; m\,\Psi\,,$$

which describes a behavior of a Dirac particle in the presence of the gauge field. This equation can also be derived by varying Ψ in the corresponding Dirac action

$$S_{\text{D}} \;=\; \int_M \prec\!\Psi \,|\, \big(i\gamma^j(\partial_j - iA_j) - m\big)\,\Psi\!\succ d\mu.$$

In order to get the equations for the gauge field, we construct out of the gauge-covariant derivative the *field tensor* by

$$F_{jk} \;=\; i\,[D_j, D_k] \;=\; \partial_j A_k - \partial_k A_j - i[A_j, A_k]\,.$$

Since its behavior under gauge transformation is simply

$$F_{jk} \;\longrightarrow\; U\,F_{jk}\,U^{-1}\,,$$

we can generalize the action of the electromagnetic field in (1.1.9) by the *Yang-Mills action*

$$S_{\text{YM}} \;=\; -\frac{1}{16\pi e^2}\int_M \text{Tr}(F_{jk}F^{jk})\,d\mu\,,$$

where "Tr" is a suitably normalized matrix trace. The total action is simply the sum of the Dirac and Yang-Mills actions,

$$S \;=\; S_{\text{D}} + S_{\text{YM}}\,.$$

Varying (Ψ, A) we obtain the coupled Dirac-Yang/Mills equations which describe the classical dynamics.

1.5. Dirac Spinors in Curved Space-Time

Dirac spinors are often formulated on a manifold using frame bundles, either an orthonormal frame [**B, Fr**] or a Newman-Penrose null frame [**PR, Ch**]. We here outline an equivalent formulation of spinors in curved space-time in the framework of a $U(2,2)$ gauge theory (for details see [**F2**]). We restrict attention to the Dirac operator in local coordinates; for global issues like topological obstructions for the existence of spin structures see e.g. [**LM**]. We let M be a 4-dimensional manifold (without Lorentz metric) and define the *spinor bundle* SM as a vector bundle over M with fibre $\mathbb{C}^4$. The fibres are endowed with a scalar product $\prec .|.\succ$ of signature $(2,2)$, which is again referred to as the *spin scalar product*. Sections in the spinor bundle are called *spinors* or wave functions. In local coordinates, a spinor is represented by a 4-component complex function on space-time, usually denoted by $\Psi(x)$. Choosing at every space-time point a pseudo-orthonormal basis $(e_\alpha)_{\alpha=1,\ldots,4}$ in the fibres,

$$\prec e_\alpha | e_\beta \succ \;=\; s_\alpha\,\delta_{\alpha\beta}\,, \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1 \tag{1.5.1}$$

and representing the spinors in this basis, $\Psi = \Psi^\alpha e_\alpha$, the spin scalar product takes again the form (1.2.8). Clearly, the basis (e_α) is not unique, but at every space-point can be transformed according to

$$e_\alpha \;\longrightarrow\; (U^{-1})^\beta_\alpha\, e_\beta\,,$$

where U is an isometry of the spin scalar product, $U \in U(2,2)$. Under this basis transformation the spinors behave as follows,

$$\Psi^\alpha(x) \;\longrightarrow\; U^\alpha_\beta(x)\,\Psi^\beta(x)\,. \tag{1.5.2}$$

Due to the analogy to (1.4.3) we interpret this transformation of the wave functions as a local gauge transformation with gauge group $\mathcal{G} = U(2,2)$. We refer to a choice of the spinor basis (e_α) as a *gauge*.

Our goal is to formulate classical Dirac theory in such a way that the above $U(2,2)$ gauge transformations correspond to a physical symmetry, the $U(2,2)$ *gauge symmetry*. To this end, we shall introduce the Dirac operator as the basic object on M, from which we will later deduce the Lorentz metric and the gauge potentials. We define a

differential operator $\mathcal{D}$ of first order on the wave functions by the requirement that in a chart and gauge it should be of the form

$$\mathcal{D} \;=\; iG^j(x)\frac{\partial}{\partial x^j} + B(x) \tag{1.5.3}$$

with suitable (4×4)-matrices G^j and B. This definition does not depend on coordinates and gauge, although the form of the matrices G^j and B clearly does. More precisely, under a change of coordinates $x^i \to \tilde{x}^i$ the operator (1.5.3) transforms into

$$i\left(G^k(\tilde{x})\,\frac{\partial \tilde{x}^j}{\partial x^k}\right)\frac{\partial}{\partial \tilde{x}^j} + B(\tilde{x})\,, \tag{1.5.4}$$

whereas a gauge transformation $\Psi \to U\Psi$ yields the operator

$$U\mathcal{D}U^{-1} \;=\; i\left(UG^jU^{-1}\right)\frac{\partial}{\partial x^j} + \left(UBU^{-1} + iUG^j(\partial_j U^{-1})\right). \tag{1.5.5}$$

We define the Dirac operator by the requirement that by choosing suitable coordinates and gauge, one can arrange that the matrices G^j in front of the partial derivatives "coincide locally" with the Dirac matrices of Minkowski space.

DEF. 1.5.1. *A differential operator $\mathcal{D}$ of first order is called* **Dirac operator** *if for every $p \in M$ there is a chart (x^i, U) around p and a gauge $(e_\alpha)_{\alpha=1,\ldots,4}$ such that $\mathcal{D}$ is of the form (1.5.3) with*

$$G^j(p) \;=\; \gamma^j\,, \tag{1.5.6}$$

where the γ^j are the Dirac matrices of Minkowski space in the Dirac representation (1.2.5).

It may seem unconventional that in this definition the zero order term B of the Dirac operator is not at all specified. Furthermore, our formulation as a gauge theory seems incomplete because we introduced local gauge transformations (1.5.2, 1.4.3), but not a corresponding gauge-covariant derivative (1.4.4). In order to clarify the situation, we shall now construct from the Dirac operator a gauge-covariant derivative D, also referred to as *spin derivative*. To this end, we must find matrices A_j which transform under local gauge transformations according to (1.4.5). This construction will also reveal the structure of the matrix B, and this will finally lead us to the definition of the so-called *physical Dirac operator*, which involves precisely the gravitational and electromagnetic fields.

In the chart and gauge where (1.5.6) holds, it is obvious from (1.2.3) that the anti-commutator of the matrices $G^j(p)$ gives the Minkowski metric. Using the transformation rules (1.5.4, 1.5.5), one sees that in a general coordinate system and gauge, their anti-commutator defines a Lorentz metric,

$$g^{jk}(x)\,\mathbb{1} \;=\; \frac{1}{2}\left\{G^j(x),\, G^k(x)\right\}. \tag{1.5.7}$$

In this way, the Dirac operator induces on the manifold a Lorentzian structure. We refer to the matrices G^j as the Dirac matrices in curved space-time. Since we can arrange that these matrices coincide locally with the Dirac matrices of Minkowski space, all manipulations of Dirac matrices can be performed at any given space-time point in an obvious way. In particular, the pseudoscalar matrix (1.2.13) now takes the more general form

$$\rho(x) \;=\; \frac{i}{4!}\,\varepsilon_{jklm}\,G^jG^kG^lG^m\,,$$

where the anti-symmetric tensor ε_{jklm} differs from the anti-symmetric symbol ϵ_{jklm} by the volume density, $\varepsilon_{jklm} = \sqrt{|\det g|}\,\epsilon_{jklm}$. The pseudoscalar matrix gives us again the notion of even and odd matrices and of chirality (1.2.14). Furthermore, we introduce the *bilinear matrices* σ^{jk} by

$$\sigma^{jk}(x) \;=\; \frac{i}{2}\,[G^j,\, G^k]\,.$$

As in Minkowski space, the matrices

$$G^j\,, \quad \rho G^j\,, \quad \mathbb{1}\,, \quad i\rho\,, \quad \sigma^{jk} \tag{1.5.8}$$

form a basis of the 16-dimensional (real) vector space of selfadjoint matrices (with respect to $\prec.|.\succ$). The matrices G^j and ρG^j are odd, whereas $\mathbb{1}$, $i\rho$ and σ^{jk} are even.

For the construction of the spin connection we must clearly consider derivatives. The Lorentzian metric (1.5.7) induces the Levi-Civita connection ∇, which defines the covariant derivative of tensor fields. Taking covariant derivatives of the Dirac matrices, $\nabla_k G^j = \partial_k G^j + \Gamma^j_{kl}\, G^l$, we obtain an expression which behaves under coordinate transformations like a tensor. However, it is not gauge covariant, because a gauge transformation (1.4.3) yields contributions involving first derivatives of U. More precisely, according to (1.5.5),

$$\begin{aligned} \nabla_k G^j \;\longrightarrow\; \nabla_k (U G^j U^{-1}) \;&=\; U(\nabla_k G^j) U^{-1} \,+\, (\partial_k U) G^j U^{-1} \,+\, U G^j (\partial_k U^{-1}) \\ &=\; U(\nabla_k G^j) U^{-1} \,-\, \big[U(\partial_k U^{-1}),\, U G^j U^{-1}\big]\,. \end{aligned} \tag{1.5.9}$$

We can use the second summand in (1.5.9) to partially fix the gauge.

LEMMA 1.5.2. *For every space-time point $p \in M$ there is a gauge such that*

$$\nabla_k G^j(p) \;=\; 0 \tag{1.5.10}$$

(for all indices j, k).

Proof. We start with an arbitrary gauge and construct the desired gauge with two subsequent gauge transformations:

(1) The matrix $\partial_j \rho$ is odd, because

$$0 \;=\; \partial_j \mathbb{1} \;=\; \partial_j(\rho\rho) \;=\; (\partial_j \rho)\rho + \rho(\partial_j \rho)\,.$$

As a consequence, the matrix $i\rho(\partial_j \rho)$ is selfadjoint. We can thus perform a gauge transformation U with $U(p) = \mathbb{1}$, $\partial_j U(p) = \frac{1}{2}\rho(\partial_j \rho)$. In the new gauge the matrix $\partial_j \rho(p)$ vanishes,

$$\partial_j \rho_{|p} \;\longrightarrow\; \partial_j (U \rho U^{-1})_{|p} \;=\; \partial_j \rho_{|p} + \frac{1}{2}\,[\rho(\partial_j \rho),\, \rho]_{|p} \;=\; \partial_j \rho_{|p} - \rho^2 (\partial_j \rho)_{|p} \;=\; 0\,.$$

Differentiating the relation $\{\rho, G^j\} = 0$, one sees that the matrix $\nabla_k G^j_{|p}$ is odd. We can thus represent it in the form

$$\nabla_k G^j_{|p} \;=\; \Lambda^j_{km}\, G^m_{|p} \,+\, \Theta^j_{km}\, \rho G^m \tag{1.5.11}$$

with suitable coefficients Λ^j_{km} and Θ^j_{km}.

This representation can be further simplified: According to Ricci's Lemma, $\nabla_n g^{jk} = 0$. Expressing the metric via the anti-commutation relations and differentiating through with the Leibniz rule, we obtain

$$\begin{aligned} 0 \;&=\; \{\nabla_n G^j,\, G^k\} \,+\, \{G^j,\, \nabla_n G^k\} \\ &=\; 2\Lambda^j_{nm}\, g^{mk} \,-\, \Theta^j_{nm}\, 2i\rho\sigma^{mk} \,+\, 2\Lambda^k_{nm}\, g^{mj} \,-\, \Theta^k_{nm}\, 2i\rho\sigma^{mj} \end{aligned} \tag{1.5.12}$$

and thus

$$\Lambda^j_{nm}\, g^{mk}_{|p} \;=\; -\Lambda^k_{nm}\, g^{mj}_{|p}\,. \tag{1.5.13}$$

In the case $j = k \neq m$, (1.5.12) yields that $\Theta^j_{nm} = 0$. For $j \neq k$, we obtain $\Theta^j_{nj}\,\sigma^{jk} + \Theta^k_{nk}\,\sigma^{kj} = 0$ and thus $\Theta^j_{nj} = \Theta^k_{nk}$ (j and k denote fixed indices, no summation is performed). We conclude that there are coefficients Θ_k with

$$\Theta^j_{km} \;=\; \Theta_k\,\delta^j_m\,. \tag{1.5.14}$$

(2) We perform a gauge transformation U with $U(p) = \mathbb{1}$ and

$$\partial_k U \;=\; -\frac{1}{2}\,\Theta_k\,\rho \;-\; \frac{i}{4}\,\Lambda^m_{kn}\, g^{nl}\,\sigma_{ml}\,.$$

Using the representation (1.5.11) together with (1.5.13, 1.5.14), the matrix $\nabla_k G^j$ transforms into

$$\begin{aligned}
\nabla_k G^j \;&\longrightarrow\; \nabla_k G^j \;+\; [\partial_k U,\, G^j] \\
&=\; \Lambda^j_{km}\, G^m \;+\; \Theta_k\,\rho G^j \;-\; \Theta_k\,\rho G^j \;-\; \frac{i}{4}\,\Lambda^m_{kn}\, g^{nl}\,[\sigma_{ml},\, G^j] \\
&=\; \Lambda^j_{km}\, G^m \;-\; \frac{i}{4}\,\Lambda^m_{kn}\, g^{nl}\, 2i\,(G_m\,\delta^j_l - G_l\,\delta^j_m) \\
&=\; \Lambda^j_{km}\, G^m \;+\; \frac{1}{2}\,\Lambda^m_{kn}\, g^{nj}\, G_m \;-\; \frac{1}{2}\,\Lambda^j_{km}\, G^m \;=\; 0\,. \qquad \blacksquare
\end{aligned}$$

We call a gauge satisfying condition (1.5.10) a *normal gauge* around p. In order to analyze the remaining gauge freedom, we let U be a transformation between two normal gauges. Then according to (1.5.9) and (1.5.10), the commutator $[U(\partial_k U^{-1}),\, UG^jU^{-1}]$ vanishes at p or, equivalently,

$$[i(\partial_k U^{-1})\,U,\, G^j]_{|p} \;=\; 0\,.$$

As is easily verified in the basis (1.5.8) using the commutation relations between the Dirac matrices, a matrix which commutes with all Dirac matrices is a multiple of the identity matrix. Moreover, the matrix $i(\partial_j U^{-1})\,U$ is selfadjoint because $(i(\partial_j U^{-1})U)^* = -iU^{-1}(\partial_j U) = -i\partial_j(U^{-1}U) + i(\partial_j U^{-1})U = i(\partial_j U^{-1})U$. We conclude that the matrix $i(\partial_j U^{-1})\,U$ is a real multiple of the identity matrix, and transforming it unitarily with U we see that it also coincides with the matrix $iU\,(\partial_j U^{-1})$. Under this strong constraint for the gauge transformation it is easy to find expressions with the required behavior (1.4.5) under gauge transformations. Namely, setting

$$a_j \;=\; \frac{1}{4}\,\mathrm{Re}\;\mathrm{Tr}(G_j\,B)\,\mathbb{1}\,, \tag{1.5.15}$$

where "Tr" denotes the trace of a 4×4-matrix, one sees from (1.5.5) that

$$a_j \;\longrightarrow\; a_j \;+\; \frac{1}{4}\,\mathrm{Re}\;\mathrm{Tr}\left(G_j G^k\, i(\partial_k U^{-1})\,U\right)\mathbb{1} \;=\; a_j + iU(\partial_j U^{-1})\,.$$

We can identify the a_j with the gauge potentials A_j and use (1.4.4) as the definition of the spin connection.

Def. 1.5.3. *The* **spin derivative** D *is defined by the condition that it behaves under gauge transformations (1.4.3) according to (1.4.6) and in normal gauges around p has the form*

$$D_j(p) \;=\; \frac{\partial}{\partial x^j} \;-\; ia_j \tag{1.5.16}$$

with the potentials a_j according to (1.5.15).

In general gauges, the spin derivative can be written as

$$D_j \;=\; \frac{\partial}{\partial x^j} \;-\; iE_j \;-\; ia_j \tag{1.5.17}$$

with additional matrices $E_j(x)$, which involve the Dirac matrices and their first derivatives. A short calculation shows that the trace of the matrix E_j does not change under gauge transformations, and since it vanishes in normal gauges, we conclude that the matrices E_j are trace-free. A straightforward calculation yields that they are explicitly given by

$$E_j \;=\; \frac{i}{2}\,\rho\,(\partial_j \rho) \;-\; \frac{i}{16}\,\mathrm{Tr}(G^m\,\nabla_j G^n)\,G_m G_n \;+\; \frac{i}{8}\,\mathrm{Tr}(\rho G_j\,\nabla_m G^m)\,\rho\,.$$

In the next two theorems we collect the basic properties of the spin connection.

THEOREM 1.5.4. *The spin derivative satisfies for all wave functions Ψ, Φ the equations*

$$[D_k, G^j] \;+\; \Gamma^j_{kl}\,G^l \;=\; 0 \tag{1.5.18}$$

$$\partial_j \prec\!\Psi \,|\, \Phi\!\succ \;=\; \prec\!D_j\Psi \,|\, \Phi\!\succ \;+\; \prec\!\Psi \,|\, D_j\Phi\!\succ\,. \tag{1.5.19}$$

Proof. The left side of (1.5.18) behaves under gauge transformations according to the adjoint representation $. \to U\,.\,U^{-1}$ of the gauge group. Thus it suffices to check (1.5.18) in a normal gauge, where

$$[D_k, G^j] + \Gamma^j_{kl}\,G^l \;=\; \nabla_k G^j \;-\; \frac{i}{4}\,\mathrm{Re}\;\mathrm{Tr}(G_j B)\,[\mathbb{1}, G^j] \;=\; 0\,.$$

Since both sides of (1.5.19) are gauge invariant, it again suffices to consider a normal gauge. The statement is then an immediate consequence of the Leibniz rule for partial derivatives and the fact that the spin derivative differs from the partial derivative by an imaginary multiple of the identity matrix (1.5.16). ∎

The identity (1.5.18) means that the coordinate and gauge invariant derivative of the Dirac matrices vanishes. The relation (1.5.19) shows that the spin connection is compatible with the spin scalar product.

We define *torsion* $\mathcal{T}$ and *curvature* $\mathcal{R}$ of the spin connection as the following 2-forms,

$$\mathcal{T}_{jk} \;=\; \frac{i}{2}\,([D_j, G_k] - [D_k, G_j])\;, \qquad\qquad \mathcal{R}_{jk} \;=\; \frac{i}{2}\,[D_j, D_k]\,.$$

THEOREM 1.5.5. *The spin connection is torsion-free. Curvature has the form*

$$\mathcal{R}_{jk} \;=\; \frac{1}{8}\,R_{mnjk}\,\sigma^{mn} \;+\; \frac{1}{2}\,(\partial_j a_k - \partial_k a_j) \tag{1.5.20}$$

where R_{mnjk} is the the Riemannian curvature tensor and the a_j are given by (1.5.15).

Proof. The identity (1.5.18) yields that

$$[D_j,\, G_k] \;=\; [D_j,\, g_{kl}\,G^l] \;=\; (\partial_j g_{kl})\,G^l \;-\; g_{kl}\,\Gamma^l_{jm}\,G^m \;=\; \Gamma^m_{jk}\,G_m$$

and thus, using that the Levi-Civita connection is torsion-free,

$$\mathcal{T}_{jk} \;=\; \frac{i}{2}\,(\Gamma^m_{jk} - \Gamma^m_{kj})\,G_m \;=\; 0\,.$$

Again using (1.5.18), we can rewrite the covariant derivative as a spin derivative,

$$G_l \,\nabla_k u^l \;=\; [D_k,\, G_l u^l]\,.$$

Iterating this relation, we can express the Riemann tensor (1.1.7) by

$$\begin{aligned} G_i\, R^i_{jkl}\, u^l &= [D_j,\, [D_k,\, G_l u^l]] - [D_k,\, [D_j,\, G_l u^l]] \\ &= [[D_j,\, D_k],\, G_l u^l] \;=\; -2i\,[\mathcal{R}_{jk},\, G_l u^l]\,. \end{aligned}$$

This equation determines curvature up to a multiple of the identity matrix,

$$\mathcal{R}_{jk}(x) \;=\; \frac{1}{8}\, R_{mnjk}\, \sigma^{mn} \;+\; \lambda_{jk} \mathbb{1}\,.$$

Thus it remains to compute the trace of curvature,

$$\frac{1}{4}\,\mathrm{Tr}(\mathcal{R}_{jk})\,\mathbb{1} \;=\; \frac{1}{8}\,\mathrm{Tr}(\partial_j A_k - \partial_k A_j)\,\mathbb{1} \;=\; \frac{1}{2}\,(\partial_j a_k - \partial_k a_j)\,,$$

where we used (1.5.17) and the fact that the matrices E_j are trace-free. ∎

We come to the physical interpretation of the above construction. According to Lemma 1.5.2 we can choose a gauge around p such that the covariant derivatives of the Dirac matrices vanish at p. Moreover, choosing normal coordinates and making a global (=constant) gauge transformation, we can arrange that $G(p) = \gamma^j$ and $\partial_j g_{kl}(p) = 0$. Then the covariant derivatives at p reduce to partial derivatives, and we conclude that

$$G^j(p) \;=\; \gamma^j\,, \qquad \partial_k G^j(p) \;=\; 0\,. \tag{1.5.21}$$

These equations have a large similarity with the conditions for normal coordinates (1.1.4), only the role of the metric is now played by the Dirac matrices. Indeed, differentiating (1.5.7) one sees that (1.5.21) implies (1.1.4). Therefore, (1.5.21) is a stronger condition which not only gives a constraint for the coordinates, but also for the gauge. We call a coordinate system and gauge where (1.5.21) is satisfied a *normal reference frame* around p.

In a normal reference frame, the Dirac matrices, and via (1.5.7) also the metric, are the same as in Minkowski space up to the order $o(x-p)$. According to the strong equivalence principle, the Dirac equation at p should coincide with that in Minkowski space. Now we use minimal coupling in the formulation (1.4.8) to conclude that there should be a normal gauge such that all gauge potentials vanish at p, and thus the Dirac operator at p should coincide with the free Dirac operator $i\partial\!\!\!/$. This physical argument allows us to specify the zero order term in (1.5.3).

DEF. 1.5.6. *A Dirac operator $\mathcal{D}$ is called* **physical Dirac operator** *if for any $p \in M$ there is a normal reference frame around p such that $B(p) = 0$.*

Equivalently, the physical Dirac operator could be defined as a differential operator of first order (1.5.3) with the additional structure that for any $p \in M$ there is a coordinate chart and gauge such that the following three conditions are satisfied,

$$G^j(p) = \gamma^j\,, \qquad \partial_k G^j(p) = 0\,, \qquad B(p) = 0\,.$$

This alternative definition has the disadvantage that it is a-priori not clear whether the second condition $\partial_k G^j(p) = 0$ can be satisfied for a general metric. This is the reason why we preferred to begin with only the first condition (Def. 1.5.1), then showed that

the second condition can be arranged by choosing suitable coordinates and gauge, and satisfied the third condition at the end (Def. 1.5.6).

In general coordinates and gauge, the physical Dirac operator can be written as

$$\mathcal{D} \;=\; iG^j D_j \;=\; iG^j \left(\partial_j - iE_j - ia_j\right),$$

where D is the spin connection of Def. 1.5.3. The matrices E_j take into account the gravitational field and are called *spin coefficients*, whereas the a_j can be identified with the *electromagnetic potential* (compare (1.2.7)). We point out that the gravitational field cannot be introduced into the Dirac equation by the simple replacement rule (1.4.7) because gravity has an effect on both the Dirac matrices and the spin coefficients. But factorizing the gauge group as $U(2,2) = U(1) \times SU(2,2)$, the $SU(2,2)$-gauge transformations are linked to the gravitational field because they influence G^j and E_j, whereas the $U(1)$ can be identified with the gauge group of electrodynamics. In this sense, we obtain a unified description of electrodynamics and general relativity as a $U(2,2)$ gauge theory. The Dirac equation

$$(\mathcal{D} - m)\, \Psi \;=\; 0$$

describes a Dirac particle in the gravitational and electromagnetic field. According to Theorem 1.5.5, the curvature of the spin connection involves both the Riemann tensor and the electromagnetic field tensor. We can write down the classical action in terms of these tensor fields, and variation yields the classical Einstein-Dirac-Maxwell equations.

For the probabilistic interpretation of the Dirac equation in curved space-time, we choose a space-like hypersurface $\mathcal{H}$ (corresponding to "space" for some observer) and consider in generalization of (1.2.17) on solutions of the Dirac equation the scalar product

$$(\Psi \,|\, \Phi) \;=\; \int_{\mathcal{H}} \prec \Psi \,|\, G^j \nu_j \, \Phi \succ d\mu_{\mathcal{H}}\,, \tag{1.5.22}$$

where ν is the future-directed normal on $\mathcal{H}$ and $d\mu_{\mathcal{H}}$ is the invariant measure on the Riemannian manifold $\mathcal{H}$. Then $(\Psi|\Psi)$ is the *normalization integral*, which we again normalize to one. Its integrand has the interpretation as the *probability density*. In analogy to (1.2.9) the *Dirac current* is introduced by $J^k = \prec \Psi \,|\, G^k \Psi \succ$. Using Theorem 1.5.4 one sees similar as in Minkowski space that the Dirac current is divergence-free, $\nabla_k J^k = 0$. From Gauss' theorem one obtains that the scalar product (1.5.22) does not depend on the choice of the hypersurface $\mathcal{H}$.

We finally remark that using Theorem 1.5.4 together with Gauss' theorem, one easily verifies that the physical Dirac operator is Hermitian with respect to the inner product

$$<\Psi \,|\, \Phi> \;:=\; \int_M \prec \Psi \,|\, \Phi \succ d\mu\,, \tag{1.5.23}$$

in which the wave functions (which need not satisfy the Dirac equation but must have a suitable decay at infinity) are integrated over the whole space-time. This inner product is not positive, but it will nevertheless play an important conceptual role in the next chapters.

CHAPTER 2

The Fermionic Projector in the Continuum

In the previous chapter we introduced the concept of the Dirac sea in order to give the negative-energy solutions of the free Dirac equation a physical meaning as anti-particle states (see §1.2, §1.3). Now we shall extend this concept to the case with interaction. We will see that the Dirac sea can still be introduced as a universal object in space-time, described mathematically by the so-called *fermionic projector* (§2.3). We develop the mathematical methods for an explicit analysis of the fermionic projector in position space (§2.5) and finally consider the normalization of the fermionic states (§2.6).

2.1. The External Field Problem

We begin with the simplest interaction: a classical external field in Minkowski space. In this situation the Dirac wave function is a solution of the Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\Psi \;=\; 0\,, \tag{2.1.1}$$

where the operator $\mathcal{B}$ is composed of the external potentials (as an example one may choose $\mathcal{B} = A\!\!\!/$ with A the electromagnetic potential (1.2.7)). In order to have current conservation (i.e. the identity $\partial_k J^k = 0$ with J according to (1.2.9)), we always assume that $\mathcal{B}$ is Hermitian (with respect to the spin scalar product). If $\mathcal{B}$ is *static* (=time independent), we can separate the time dependence of the wave function with a plane wave ansatz,

$$\Psi(t, \vec{x}) \;=\; e^{-i\omega t}\, \psi(\vec{x})\,. \tag{2.1.2}$$

The separation constant ω has the interpretation as the energy of the solution. Thus the energy is a conserved quantity, and its sign distinguishes between solutions of positive and negative energy. In more mathematical terms, for a static potential the Hamiltonian in (1.2.18) is time independent, and the sign of the spectrum of h gives a splitting of the solution space of the Dirac equation into the subspaces of positive and negative energy, respectively. As a consequence, our previous construction of the Dirac sea can be adapted: When building up the Fock space from the one-particle states, we cure the problem of the negative-energy solutions similar to (1.3.7–1.3.9) by redefining the vacuum and by formally exchanging the creation and annihilation operators corresponding to the negative-energy solutions.

The situation becomes much more difficult when $\mathcal{B}$ is *time-dependent*. In this case, the separation ansatz (2.1.2) no longer works. The energy is not conserved, and it is even possible that a solution which has positive energy at initial time will have negative energy at a later time. Expressed more mathematically, the Hamiltonian in (1.2.18) now depends explicitly on time, and therefore the sign of the spectrum of h no longer gives a canonical splitting of the solution space of the Dirac equation (this splitting would also depend on time). As a consequence, it is not clear which solutions have the interpretation as "negative-energy solutions" and thus correspond to anti-particle

states (1.3.8). For this reason, it is no longer obvious how to quantize the Dirac field in a canonical way. This difficulty is usually referred to as the *external field problem*. It becomes most evident in the setting of Klein's paradox, where one considers a step potential whose amplitude is larger than the mass gap (see [**BD1, T**]). However, we point out that the external field problem appears already for arbitrarily weak external fields, simply because the time dependence of $\mathcal{B}$ leads to a complicated mixing of the solutions of positive and negative energy. We could speak of a "solution of negative energy" only if it were a superposition of states which *all* had negative energy, and there seems no reason why such solutions should exist.

It is instructive to discuss the external field problem in the setting of pertubation theory. Consider a first order perturbation of the plane-wave solution $\Psi_{\vec{p}s\epsilon}$,

$$\Psi \;=\; \Psi_{\vec{p}s\epsilon} + \Delta\Psi + \mathcal{O}(\mathcal{B}^2)\,. \tag{2.1.3}$$

Substituting this ansatz into the Dirac equation (2.1.1), we obtain to first order in $\mathcal{B}$ the inhomogeneous Dirac equation

$$(i\partial\!\!\!/ - m)\,\Delta\Psi \;=\; -\mathcal{B}\,\Psi_{\vec{p}s\epsilon}\,. \tag{2.1.4}$$

If $s_m(x,y)$ is a *Green's function* of the free Dirac equation, characterized by the distributional equation

$$(i\partial\!\!\!/_x - m)\,s_m(x,y) \;=\; \delta^4(x-y)\,, \tag{2.1.5}$$

we can construct a solution of (2.1.4) by

$$\Delta\Psi \;=\; -\int d^4y\; s(x,y)\,\mathcal{B}(y)\,\Psi_{\vec{p}s\epsilon}(y) \tag{2.1.6}$$

(in order not to distract from the main ideas, we here calculate on a formal level; the analytic justification will be given at the end of §2.2). If the Green's function were unique, (2.1.3, 2.1.6) would give a unique procedure for perturbing the negative-energy solutions of the vacuum, making it possible to extend the notion of "negative-energy state" to the interacting theory (at least in first order perturbation theory).

The problem is that the Green's function is *not unique*, as we now briefly recall (for details see [**BD1**]). Taking the Fourier transform of (2.1.5),

$$s_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\; s_m(k)\, e^{-ik(x-y)}\,, \tag{2.1.7}$$

we obtain the algebraic equation

$$(k\!\!\!/ - m)\,s_m(k) \;=\; \mathbb{1}\,.$$

Since the matrix $k\!\!\!/ - m$ is singular on the mass shell (see the argument after (1.2.19)), this equation can be solved for $s_m(k)$ only after using a $\pm i\varepsilon$-regularization on the mass shell. The most popular choices are the *advanced* and the *retarded* Green's functions defined by

$$s_m^\vee(k) \;=\; \lim_{\varepsilon \searrow 0} \frac{k\!\!\!/ + m}{k^2 - m^2 - i\varepsilon k^0} \qquad \text{and} \qquad s_m^\wedge(k) \;=\; \lim_{\varepsilon \searrow 0} \frac{k\!\!\!/ + m}{k^2 - m^2 + i\varepsilon k^0}\,, \tag{2.1.8}$$

respectively (with the limit $\varepsilon \searrow 0$ taken in the distributional sense). Computing their Fourier transform (2.1.7) with residues, one sees that they are *causal* in the sense that their supports lie in the upper and lower light cone, respectively,

$$\text{supp}\, s_m^\vee(x,.) \subset J_x^\vee\,, \qquad \text{supp}\, s_m^\wedge(x,.) \subset J_x^\wedge\,. \tag{2.1.9}$$

Another commmon choice is the *Feynman propagator*

$$s_m^F(k) \;:=\; \lim_{\varepsilon \searrow 0} \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon}\,. \tag{2.1.10}$$

Taking the Fourier transform (2.1.7) with residues, one finds

$$s_m^F(x,y) \;=\; \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} (\not{k} + m)\, e^{-ik(x-y)}\Big|_{k=(\epsilon(t)\,\omega(\vec{p}),\,\vec{p})}\, d\mu_{\vec{p}} \tag{2.1.11}$$

with $t \equiv (y-x)^0$ and $\omega(\vec{p})$, $d\mu_{\vec{p}}$ as introduced after (1.3.2). Here ϵ denotes the step function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise. Thus for positive t we get the integral over the upper mass shell, whereas for negative t we integrate over the lower mass shell. As a consequence, the Feynman propagator is not causal, but it is instead characterized by the *frequency conditions* that it is for positive and negative time t composed only of positive and negative frequencies, respectively. More systematically, the defining equation for the Green's function (2.1.5) determines s_m only up to a solution of the homogeneous Dirac equation. Thus we can write a general Green's function s_m in the form

$$s_m(x,y) \;=\; s_m^\vee(x,y) \;+\; a(x,y)\,, \tag{2.1.12}$$

where $a(x,y)$ is a linear combination of plane-wave solutions in the variable x, i.e.

$$a(x,y) \;=\; \sum_{s,\epsilon} \int_{\mathbb{R}^3} \Psi_{\vec{p}s\epsilon}(x)\, c_{\vec{p}s\epsilon}(y)\, d\mu_{\vec{p}}$$

with a suitable complex-valued function $c_{\vec{p}s\epsilon}(y)$.

Due to the non-uniqueness of the Green's function (2.1.12), the relations (2.1.3, 2.1.6) do *not* give a unique procedure for perturbing the plane-wave solutions $\Psi_{\vec{k}s\epsilon}$. In particular, it is not clear how to extend the notion of "negative energy state" to the interacting theory. This corresponds precisely to the external field problem. We conclude that in a perturbative approach, the external field problem becomes manifest in the non-uniqueness of the perturbation expansion.

Feynman [**Fe**] gave the frequency conditions in the Feynman propagator the physical interpretation as "positive-energy states moving to the future" and "negative-energy states moving to the past". Identifying "negative-energy states moving to the past" with "antiparticle states" he concluded that the Feynman propagator is distinguished from all other Green's functions in that it takes into account the particle/antiparticle interpretation of the Dirac equation in the physically correct way. Feynman proposed to perform the perturbation expansion exclusively with the Feynman propagator, thereby making the perturbation expansion unique. The flaw is that the frequency conditions are not invariant under general coordinate and gauge transformations (simply because such transformations "mix" positive and negative frequencies), and therefore Feynman's method is not compatible with the equivalence principle and the local gauge principle. This is not a problem for most calculations in physics, but it is not satisfying conceptually. Another approach is to work with the so-called *Hadamard states* [**H, Wa2**]. The disadvantage of this approach is that the states of the quantum field no longer have a particle interpretation. In other words, the notions of "particle" and "anti-particle" depend on the local observer, and therefore also the notion of the Dirac sea loses its universal meaning. We proceed in §2.2 by showing that the Dirac sea can indeed be introduced as a global object of space-time, even in the presence of a general interaction.

We finally remark that the above arguments apply in the same way for second quantized fields: we only need to replace $\mathcal{B}$ by an operator on a suitable bosonic Fock space. Also, our assumption that $\mathcal{B}$ is an *external* field is merely a technical simplification (more precisely, we disregard the dynamical equations for the bosonic fields, thereby also avoiding the divergences of QFT and the renormalization procedure), but it is not essential for our arguments. Namely, in a time-dependent interacting system, the Dirac wave functions satisfy (2.1.1), where $\mathcal{B}(t, \vec{x})$ is determined by the dynamical equations of the whole system. Solving these equations we can (at least in principle) compute $\mathcal{B}$, and applying our above arguments with this $\mathcal{B}$ as an external field, we conclude that the notion of "negative-energy state" ceases to exist. In what follows we will for simplicity again consider an external field, but we shall come back to coupled systems in §2.4.

2.2. The Causal Perturbation Expansion

We saw in the previous section that the external field problem for the Dirac equation (2.1.1) is equivalent to the non-uniqueness of the perturbation expansion for the individual states of the Dirac sea. We shall now solve this problem by considering the collection of all states of the Dirac sea. This will reveal an underlying causal structure, which will enable us to make the perturbation expansion unique[1]. We closely follow the constructions given in [**F4**].

In the vacuum, out of all plane-wave solutions of negative energy we form the object

$$P^{\text{sea}}(x,y) \;:=\; -\frac{m}{\pi} \sum_s \int_{\mathbb{R}^3} |\Psi_{\vec{p}s-} \succ\!\prec \Psi_{\vec{p}s-}|\; d\mu_{\vec{p}}\,. \tag{2.2.1}$$

Using the explicit form of the plane-wave solutions (1.2.23, 1.2.22, 1.2.20), we obtain the covariant formula

$$P^{\text{sea}}(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,, \tag{2.2.2}$$

which also shows that $P^{\text{sea}}(x,y)$ is a well-defined distribution. In order to get a connection to causality, we decompose $P^{\text{sea}}(x,y)$ in the form

$$P^{\text{sea}}(x,y) \;=\; \frac{1}{2}\,(p_m(x,y) \;-\; k_m(x,y))\,, \tag{2.2.3}$$

where p_m and k_m are the distributions

$$p_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, e^{-ik(x-y)} \tag{2.2.4}$$

$$k_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \epsilon(k^0)\, e^{-ik(x-y)}\,. \tag{2.2.5}$$

[1] We remark for clarity that our "causal perturbation expansion" does not seem to be related to Scharf's "causal approach" to QED [**S**]. Scharf uses causality to avoid the ultraviolet divergences of perturbative QED, whereas in our setting of an external field all Feynman diagrams are finite anyway. On the other hand, Scharf is interested only in the scattering states, whereas our goal is to describe the dynamics also for intermediate times.

In order to relate the distribution k_m to the advanced and retarded Green's functions, we substitute the distributional equation

$$\lim_{\varepsilon \searrow 0} \left(\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) = 2\pi i\, \delta(x)$$

into the formula for k_m in momentum space,

$$\begin{aligned} k_m(p) &= (\not{p} + m)\, \delta(p^2 - m^2)\, \epsilon(p^0) \\ &= \frac{1}{2\pi i}\, (\not{p} + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{p^2 - m^2 - i\varepsilon} - \frac{1}{p^2 - m^2 + i\varepsilon} \right] \epsilon(p^0) \\ &= \frac{1}{2\pi i}\, (\not{p} + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{p^2 - m^2 - i\varepsilon p^0} - \frac{1}{p^2 - m^2 + i\varepsilon p^0} \right]. \end{aligned} \tag{2.2.6}$$

Using (2.1.8) we get the simple formula

$$k_m = \frac{1}{2\pi i} \left(s_m^\vee - s_m^\wedge \right). \tag{2.2.7}$$

The support property (2.1.9) yields that k_m is causal in the sense that

$$\operatorname{supp} k_m(x, .) \subset J_x .$$

The distribution p_m is *not* causal, but it can be deduced from k_m as follows. For a diagonalizable matrix A with real eigenvalues we can uniquely define its *absolute value* $|A|$ as the diagonalizable matrix with non-negative eigenvalues and $|A|^2 = A^2$. The matrix $\not{k} + m$ in (2.2.4, 2.2.5) is diagonalizable with non-negative eigenvalues and thus

$$\left| \epsilon(k^0)\, (\not{k} + m) \right| = \not{k} + m . \tag{2.2.8}$$

Before applying this relation to (2.2.4, 2.2.5), it is useful to consider the above distributions P^{sea}, $p_m, \ldots$ as integral kernels of corresponding operators on the wave functions in space-time, for example

$$(P^{\text{sea}}\, \Psi)(x) := \int P^{\text{sea}}(x, y)\, \Psi(y)\, d^4y . \tag{2.2.9}$$

Then the operators k_m and p_m are diagonal in momentum space, and so (2.2.8) gives rise to the formal identity

$$p_m = |k_m| , \tag{2.2.10}$$

where $|\,.\,|$ now is the absolute value of an operator on wave functions in Minkowski space. With (2.2.3) and (2.2.7, 2.2.10) we have related the fermionic projector to the causal Green's functions in a way which can be generalized to the interacting theory, as we shall now make precise.

We begin with the perturbation expansion for the causal Green's functions. The retarded Green's function in the presence of the external field $\mathcal{B}$, denoted by $\tilde{s}_m^\wedge$, is characterized by the conditions

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{s}_m^\wedge(x, y) = \delta^4(x - y) , \qquad \operatorname{supp} \tilde{s}_m^\wedge(x, .) \subset J_x^\wedge . \tag{2.2.11}$$

The existence and uniqueness of the advanced Green's functions follows from the general theory of linear hyperbolic PDEs [**J, Ta**]. In short, for the existence proof one considers the solution of the Cauchy problem

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi = f \in C_0^\infty((t_0, \infty) \times \mathbb{R}^3)^4 , \qquad \Psi(t_0, \vec{x}) = 0 ;$$

by linearity it can be expressed as an integral over the inhomogeneity,

$$\Psi(x) \;=\; \int_{\mathbb{R}^4} \tilde{s}^\wedge_m(x,y)\, f(y)\, d^4y\,.$$

To prove uniqueness, one considers the difference of two retarded Green's functions $\tilde{s}^\wedge_{m,1}$ and $\tilde{s}^\wedge_{m,2}$,

$$\Psi(x) \;=\; \tilde{s}^\wedge_{m,1}(x,y) - \tilde{s}^\wedge_{m,2}(x,y)\,.$$

Then $\Psi(x)$ is for fixed y a solution of the homogeneous Dirac equation which vanishes identically on the half space $x^0 < y^0$. The uniqueness of the solution of the Cauchy problem yields that $\Psi \equiv 0$.

Expanding (2.2.11) in powers of $\mathcal{B}$, one obtains the perturbation series $\tilde{s}^\wedge_m = \sum_{n=0}^\infty s^\wedge_{(n)}$, where $s^\wedge_{(0)} = s^\wedge_m$ is the advanced Green's function of the vacuum, and the other summands are determined by the conditions that they are causal, $\text{supp}\, s^\wedge_{(n)}(x,.) \in J^\wedge_x$, and satisfy the inductive relations

$$(i\partial\!\!\!/ - m)\, s^\wedge_{(n)} \;=\; -\mathcal{B}\, s^\wedge_{(n-1)} \qquad\qquad (n \geq 1).$$

Here we again used the operator notation (2.2.9) and considered $\mathcal{B}$ as a multiplication operator. The operator product

$$(-s^\wedge_m\, \mathcal{B}\, s^\wedge_m)(x,y) \;=\; -\int d^4z\; s^\wedge_m(x,z)\, \mathcal{B}(z)\, s^\wedge_m(z,y) \tag{2.2.12}$$

is causal in the sense that $\mathcal{B}(z)$ enters only for $z \in L^\wedge_x \cap L^\vee_y$ (the analytic justification of this and all other operator products in this section will be given in Lemma 2.2.2 below). In particular, the support of (2.2.12) is again in the past light cone. Furthermore, it satisfies the relation

$$(i\partial\!\!\!/ - m)\, (-s^\wedge_m\, \mathcal{B}\, s^\wedge_m) \;=\; -\mathcal{B}\, s^\wedge_m\,,$$

and can thus be identified with the operator $s^\wedge_{(1)}$. By iteration, we obtain for the other terms of the perturbation series the explicit formulas

$$s^\wedge_{(n)} \;=\; (-s^\wedge_m\, \mathcal{B})^n\, s^\wedge_m\,.$$

We conclude that the retarded Green's function can be represented as

$$\tilde{s}^\wedge_m \;=\; \sum_{k=0}^\infty \big(-s^\wedge_m\, \mathcal{B}\big)^k\, s^\wedge_m\,. \tag{2.2.13}$$

Similarly, we introduce the advanced Green's function $\tilde{s}^\vee_m$ by the conditions

$$(i\partial\!\!\!/ - m + \mathcal{B})\, \tilde{s}^\vee_m(x,y) \;=\; \delta^4(x-y)\,, \qquad\qquad \text{supp}\, \tilde{s}^\vee_m(x,.) \;\subset\; J^\vee_x\,. \tag{2.2.14}$$

It has the perturbation expansion

$$\tilde{s}^\vee_m \;=\; \sum_{k=0}^\infty \big(-s^\vee_m\, \mathcal{B}\big)^k\, s^\vee_m\,. \tag{2.2.15}$$

Having uniquely introduced the causal Green's functions, we can now extend (2.2.7) to the case with interaction. Namely, we define the operator $\tilde{k}_m$ by

$$\tilde{k}_m \;=\; \frac{1}{2\pi i}\, \big(\tilde{s}^\vee_m - \tilde{s}^\wedge_m\big) \tag{2.2.16}$$

with the causal Green's functions as given by (2.2.13, 2.2.15). Finally, we also extend (2.2.10) to the case with interaction by setting

$$\tilde{p}_m \overset{\text{formally}}{:=} \sqrt{\tilde{k}_m^2}\,. \tag{2.2.17}$$

In the next theorem we will give this last relation a precise mathematical meaning and show that it gives rise to a unique perturbation expansion. It is most convenient to work with the Green's function

$$s_m := \frac{1}{2}(s_m^\vee + s_m^\wedge)\,. \tag{2.2.18}$$

Furthermore, we introduce the series of operator products

$$b_m^< = \sum_{k=0}^\infty (-s_m\,\mathcal{B})^k \quad , \qquad b_m = \sum_{k=0}^\infty (-\mathcal{B}\,s_m)^k\,\mathcal{B} \quad , \qquad b_m^> = \sum_{k=0}^\infty (-\mathcal{B}\,s_m)^k$$

and set for $Q \subset \mathbb{N}$

$$F_m(Q,n) = \begin{cases} p_m & \text{if } n \in Q \\ k_m & \text{if } n \notin Q \end{cases}\,.$$

THEOREM 2.2.1. *The relations (2.2.16, 2.2.17) uniquely determine the perturbation expansions for k_m and p_m. We have the explicit formulas*

$$\tilde{k}_m = \sum_{\beta=0}^\infty (-i\pi)^{2\beta}\, b_m^<\, k_m\, (b_m\, k_m)^{2\beta}\, b_m^> \tag{2.2.19}$$

$$\tilde{p}_m = \sum_{\beta=0}^\infty \sum_{\alpha=0}^{\left[\frac{\beta}{2}\right]} c(\alpha,\beta)\, G_m(\alpha,\beta) \tag{2.2.20}$$

with the coefficients

$$c(0,0) = 1 \tag{2.2.21}$$

$$c(\alpha,\beta) = \sum_{n=\alpha+1}^{\beta} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\, 2^n} \binom{\beta-\alpha-1}{n-\alpha-1} \quad \text{for } \beta \geq 1 \tag{2.2.22}$$

and the operator products

$$G_m(\alpha,\beta) = \sum_{Q \in \mathcal{P}(\beta+1),\ \#Q = 2\alpha+1} (-i\pi)^{2\beta}$$
$$\times\, b_m^<\, F_m(Q,1)\, b_m k_m b_m\, F_m(Q,2)\, b_m k_m b_m \,\cdots\, b_m k_m b_m\, F_m(Q,\beta+1)\, b_m^>\,, \tag{2.2.23}$$

where $\mathcal{P}(n)$ denotes the set of subsets of $\{1,\ldots,n\}$ (we use the convention $l!! = 1$ for $l \leq 0$).

Proof. An explicit calculation using (2.1.5) shows that $(i\partial\!\!\!/ + \mathcal{B} - m)\, b_m^< = 0$. Since all operator products in (2.2.19) and (2.2.23) have a factor $b_m^<$ at the left, the operators $\tilde{p}_m$, $\tilde{k}_m$ are solutions of the Dirac equation,

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{p}_m = 0 = (i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{k}_m\,.$$

It remains to verify that the conditions (2.2.16, 2.2.17) are satisfied and to show uniqueness.

According to (2.2.7, 2.2.18), the advanced and retarded Green's function can be written as

$$s_m^\vee \;=\; s_m + i\pi\, k_m \quad , \qquad s_m^\wedge \;=\; s_m - i\pi\, k_m \,. \tag{2.2.24}$$

We substitute the series (2.2.13, 2.2.15) into (2.2.16),

$$\tilde{k}_m \;=\; \frac{1}{2\pi i} \sum_{k=0}^\infty \Big((-s_m^\vee\, \mathcal{B})^k\, s_m^\vee \;-\; (-s_m^\wedge\, \mathcal{B})^k\, s_m^\wedge \Big) \,, \tag{2.2.25}$$

insert (2.2.24) and expand. This gives a sum of operator products of the form

$$C_1\, \mathcal{B}\, C_2\, \mathcal{B} \cdots \mathcal{B}\, C_{l+1} \qquad \text{with} \qquad C_j = k_m \text{ or } C_j = s_m \,.$$

The contributions with an even number of factors k_m have the same sign for the advanced and retarded Green's functions and cancel in (2.2.25). The contributions with an odd number of k_m's occur in each Green's function exactly once and have the opposite sign. Using the notation

$$C_m(Q,n) \;=\; \begin{cases} k_m & \text{if } n \in Q \\ s_m & \text{if } n \notin Q \end{cases} \quad , \qquad Q \subset \mathbb{N} \,,$$

we can thus rewrite (2.2.25) in the form

$$\begin{aligned} \tilde{k}_m \;=\; \sum_{l=0}^\infty (-1)^l \sum_{Q \in \mathcal{P}(l+1),\; \#Q \text{ odd}} & (i\pi)^{\#Q-1} \\ & \times C_m(Q,1)\, \mathcal{B}\, C_m(Q,2)\, \mathcal{B} \cdots \mathcal{B}\, C_m(Q,l)\, \mathcal{B}\, C_m(Q,l+1) \,. \end{aligned}$$

After reordering the sums, this coincides with (2.2.19).

Next we want to give the relation (2.2.10) a mathematical meaning. To this end, we consider $m \geq 0$ as a variable mass parameter. Then we can form products of the operators p_m, k_m by manipulating the arguments of the distributions in momentum space. For example, using (2.2.4) we obtain

$$\begin{aligned} p_m(k)\, p_{m'}(k) \;&=\; (\not{k} + m)\, \delta(k^2 - m^2)\, (\not{k} + m')\, \delta(k^2 - (m')^2) \\ &=\; (k^2 + (m+m')\not{k} + mm')\, \delta(m^2 - (m')^2)\, \delta(k^2 - m^2) \\ &=\; (k^2 + (m+m')\not{k} + mm')\, \frac{1}{2m}\, \delta(m - m')\, \delta(k^2 - m^2) \\ &=\; \delta(m-m')\, p_m(k) \,, \end{aligned} \tag{2.2.26}$$

and similarly from (2.2.5),

$$p_m\, k_{m'} \;=\; k_{m'}\, p_m \;=\; \delta(m-m')\, k_m \tag{2.2.27}$$

$$\tag{2.2.28}$$

$$k_m\, k_{m'} \;=\; \delta(m-m')\, p_m \,. \tag{2.2.29}$$

We remark that this formalism has some similarity with the bra/ket notation in quantum mechanics, if the position variable $\vec{x}$ is replaced by the mass parameter m. Equation (2.2.26) can be interpreted that the p_m are the spectral projectors of the free Dirac operator; the relations (2.2.28, 2.2.29) reflect the relative minus sign in k_m for the states on the upper and lower mass shell. In particular, one sees that $k_m\, k_{m'} = p_m\, p_{m'}$. This relation can be extended to the case with interaction,

$$\tilde{p}_m\, \tilde{p}_{m'} \;=\; \tilde{k}_m\, \tilde{k}_{m'} \,, \tag{2.2.30}$$

and gives a meaningful square of (2.2.10) (we will see in a moment that $\tilde{k}_m \tilde{k}_{m'}$ vanishes for $m \neq m'$). If our construction ensures that $\tilde{p}_m$ is a positive operator, (2.2.30) is even equivalent to (2.2.10).

Let us compute the product $\tilde{k}_m \, \tilde{k}_{m'}$ explicitly. The definitions (2.2.4, 2.2.5) and (2.2.18, 2.1.8) yield in analogy to (2.2.26) the formulas[2]

$$p_m \, s_{m'} \;=\; s_{m'} \, p_m \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) p_m \tag{2.2.31}$$

$$k_m \, s_{m'} \;=\; s_{m'} \, k_m \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) k_m \tag{2.2.32}$$

$$s_m \, s_{m'} \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) (s_m - s_{m'})\,, \tag{2.2.33}$$

where $\mathrm{PP}(x^{-1}) = \frac{1}{2} \lim_{\varepsilon \searrow 0} [(x+i\varepsilon)^{-1} + (x-i\varepsilon)^{-1}]$ denotes the principal value. As a consequence, the operator products involving the factor $s_m \cdot s_{m'}$ are telescopic,

$$\sum_{p=0}^{n} k_m \, (\mathcal{B}\, s_m)^p \, (s_{m'}\, \mathcal{B})^{n-p} \, k_{m'} \;=\; 0 \qquad \text{for } n \geq 1. \tag{2.2.34}$$

This allows us to evaluate the following product,

$$k_m \, b_m^> \, b_{m'}^< \, k_{m'} \;=\; \delta(m-m')\, p_m\,. \tag{2.2.35}$$

With this formula, we can compute the square of (2.2.19),

$$\tilde{k}_m \, \tilde{k}_{m'} \;=\; \delta(m-m') \sum_{\beta_1, \beta_2 = 0}^{\infty} (-i\pi)^{2\beta_1 + 2\beta_2} \, b_m^< \, (k_m \, b_m)^{2\beta_1} \, p_m \, (b_m \, k_m)^{2\beta_2} \, b_m^>\,. \tag{2.2.36}$$

We could continue the proof by verifying explicitly that the product $\tilde{p}_m \tilde{p}_{m'}$ with $\tilde{p}_m$ according to (2.2.20) coincides with (2.2.36). This is a straightforward computation, but it is rather lengthy and not very instructive. We prefer to describe how the operator products (2.2.23) and the coefficients (2.2.22) can be derived. In order to make the proof more readable, we make the following simplifications. Since the factors $b_m^<$, $b_m^>$ cancel similar to (2.2.35) in telescopic sums, we can omit them in all formulas without changing the multiplication rules for the operator products. Then all operator products have k_m or p_m as their first and last factor, and we can multiply them with the rules (2.2.26–2.2.29). Since all these rules give a factor $\delta(m-m')$, we will in any case get the prefactor $\delta(m-m')$ as in (2.2.36). Therefore, we can just forget about all factors $\delta(m-m')$ and consider all expressions at the same value of m. Furthermore, we will omit the subscript '$_m$' and write the intermediate factors b as a dot '.'. After these simplifications, we end up with formal products of the form

$$F_1 \,.\, F_2 \,.\, F_3 \,.\, \cdots \,.\, F_n \qquad \text{with} \qquad F_j = k \text{ or } F_j = p \tag{2.2.37}$$

and have the multiplication rules

$$p^2 \;=\; k^2 \;=\; 1 \quad , \qquad p\,k \;=\; k\,p \;=\; k\,. \tag{2.2.38}$$

[2] Online version: As noticed by A. Grotz, in (2.2.33) the summand $\pi^2 \delta(m-m')$ is missing. This error is corrected in the paper [**6**] (listed in the references in the preface to the second online edition).

We must find a positive operator $\tilde{p}$ being a formal sum of operator products (2.2.37) such that

$$\tilde{p}^2 \;=\; \sum_{\beta_1,\beta_2=0}^{\infty} (-i\pi)^{2\beta_1+2\beta_2}\, (k\,.)^{2\beta_1}\, p\,(.\,k)^{2\beta_2}\,. \tag{2.2.39}$$

In this way, we have reduced our problem to the combinatorics of the operator products. As soon as we have found a solution $\tilde{p}$ of (2.2.39), the expression for $\tilde{p}_m$ is obtained by adding the subscripts '$_m$' and by inserting the factors $b_m^<$, b_m, $b_m^>$. Relation (2.2.30) follows as an immediate consequence of (2.2.39).

The basic step for the calculation of $\tilde{p}$ is to rewrite (2.2.39) in the form

$$\tilde{p}^2 \;=\; p + A \qquad \text{with} \qquad A \;=\; \sum_{(\beta_1,\beta_2)\neq(0,0)} (-i\pi)^{2\beta_1+2\beta_2}\, (k\,.)^{2\beta_1}\, p\,(.\,k)^{2\beta_2}\,. \tag{2.2.40}$$

The operator p is idempotent and acts as the identity on A, $Ap = pA = A$. Therefore, we can take the square root of $p + A$ with a formal Taylor expansion,

$$\tilde{p} \;=\; \sqrt{p+A} \;=\; p \;+\; \sum_{n=1}^{\infty} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\, 2^n}\, A^n\,, \tag{2.2.41}$$

which uniquely defines $\tilde{p}$ as a positive operator.

It remains to calculate A^n. If we take the nth power of the sum in (2.2.40) and expand, we end up with one sum over more complicated operator products. We first consider how these operator products look like: The operator products in (2.2.40) all contain an even number of factors k and exactly one factor p. The factor p can be the 1st, 3rd,... factor of the product. Each combination of this type occurs in A exactly once. If we multiply n such terms, the resulting operator product consists of a total odd number of factors p, k. It may contain several factors p, which all occur at odd positions in the product. Furthermore, the total number of factors p is odd, as one sees inductively. We conclude that A^n consists of a sum of operator products of the form

$$(k\,.\,k\,.)^{q_1}\, p\,.\,k\,.\,(k\,.\,k\,.)^{q_2}\, p\,.\,k\,.\,(k\,.\,k\,.)^{q_3}\, \cdots\, (k\,.\,k\,.)^{q_{2\alpha+1}}\, p\,(.\,k\,.\,k)^{q_{2\alpha+2}} \tag{2.2.42}$$

with $\alpha, q_j \geq 0$. We set $\beta = 2\alpha + \sum_j q_j$. Notice that the number of factors p in (2.2.42) is $2\alpha+1$; the total number of factors p, k is $2\beta+1$. The form of the operator product gives the only restriction $0 \leq 2\alpha \leq \beta$ for the choice of the parameters α, β.

Next we count how often each operator product (2.2.42) occurs in the sum: The easiest way to realize (2.2.42) is to form the product of the $\alpha+1$ factors

$$\begin{aligned}&\big[(k.k.)^{q_1}\, p\,(.k.k)^{q_2+1}\big]\;\big[(k.k.)^{q_3+1}\, p\,(.k.k)^{q_4+1}\big]\\ &\qquad\qquad \cdots\;\big[(k.k.)^{q_{2\alpha+1}+1}\, p\,(.k.k)^{q_{2\alpha+2}}\big]\,.\end{aligned} \tag{2.2.43}$$

However, this is not the only way to factor (2.2.42). More precisely, to each factor in (2.2.43) we can apply the identities

$$\begin{aligned}(k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^r &= [(k\,.\,k\,.)^q\, p]\;[p\,(.\,k\,.\,k)^r]\\ (k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^r &= [(k\,.\,k\,.)^s\, p]\;\big[(k\,.\,k\,.)^{q-s}\, p\,(.\,k\,.\,k)^r\big]\\ (k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^r &= \big[(k\,.\,k\,.)^q\, p\,(.\,k\,.\,k)^{r-s}\big]\;[p\,(.\,k\,.\,k)^s]\,.\end{aligned}$$

By iteratively substituting these identities into (2.2.43), we can realize every factorization of (2.2.42). Each substitution step increases the number of factors by one. The steps are independent in the sense that we can fix at the beginning at which positions

in (2.2.43) the product shall be split up, and can then apply the steps in arbitrary order. There are $(\alpha+1)+(q_1-1)+\sum_{j=2}^{2\alpha+1} q_j + (q_{2\alpha+2}-1) = \beta-(\alpha+1)$ positions in (2.2.43) where we could split up the product (in the case $q_1=0$ or $q_{2\alpha+2}=0$, the counting of the positions is slightly different, but yields the same result). Since we want to have n factors at the end, we must choose $n-(\alpha+1)$ of these positions, which is only possible for $\alpha+1 \leq n \leq \beta$ and then gives $(\beta-\alpha-1)!/((n-\alpha-1)!\,(\beta-n)!)$ possibilities.

Combining these combinatorial factors with the constraints $0 \leq 2\alpha \leq \beta$ and $\alpha+1 \leq n \leq \beta$, we obtain for $n \geq 1$ the identity

$$\begin{aligned} A^n &= \sum_{\beta=n}^{\infty} \sum_{\alpha=0}^{\min\left(n-1,\left[\frac{\beta}{2}\right]\right)} \binom{\beta-\alpha-1}{n-\alpha-1} \sum_{Q \in \mathcal{P}(\beta+1),\ \#Q=2\alpha+1} \\ &\quad \times (-i\pi)^{2\beta}\, F(Q,1)\,.\,k\,.\,F(Q,2)\,.\,k\,.\,\cdots\,.\,k\,.\,F(Q,\beta+1) \end{aligned} \tag{2.2.44}$$

with $F(Q,n)=p$ for $n \in Q$ and $F(Q,n)=k$ otherwise. Notice that the last sum in (2.2.44) runs over all possible configurations of the factors p,k in the operator product (2.2.42) for fixed α,β. We finally substitute this formula into (2.2.41) and pull the sums over α,β outside. This gives the desired formula for $\tilde{p}$. ∎

We call the perturbation expansion of the above theorem the *causal perturbation expansion*. It allows us to define the Dirac sea in the presence of an external field canonically by

$$P^{\text{sea}}(x,y) \;=\; \frac{1}{2}\,(\tilde{p}_m - \tilde{k}_m)(x,y)\,.$$

In the next section the causal perturbation expansion will be extended to systems of Dirac seas, and in §2.4 we will discuss it in detail.

We conclude this section by showing that, under suitable regularity and decay assumptions on the external potentials, all operator products which appeared in this section are well-defined and finite.

LEMMA 2.2.2. *Let (C_j), $0 \leq j \leq n$, be a choice of operators $C_j \in \{k_m, p_m, s_m\}$. If the external potential $\mathcal{B}$ is smooth and decays so fast at infinity that the functions $\mathcal{B}(x)$, $x^i\mathcal{B}(x)$, and $x^ix^j\mathcal{B}(x)$ are integrable, then the operator product*

$$(C_n\,\mathcal{B}\,C_{n-1}\,\mathcal{B}\cdots\mathcal{B}\,C_0)(x,y) \tag{2.2.45}$$

is a well-defined tempered distribution on $\mathbb{R}^4 \times \mathbb{R}^4$.

Proof. Calculating the Fourier transform of (2.2.45) gives the formal expression

$$\begin{aligned} M(q_2,q_1) \;&:=\; \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_{n-1}}{(2\pi)^4} C_n(q_2)\,\hat{\mathcal{B}}(q_2-p_{n-1}) \\ &\quad \times C_{n-1}(p_{n-1})\,\hat{\mathcal{B}}(p_{n-1}-p_{n-2})\;\cdots\;C_1(p_1)\,\hat{\mathcal{B}}(p_1-q_1)\,C_0(q_1)\,, \end{aligned} \tag{2.2.46}$$

where we consider the C_j as multiplication operators in momentum space and where $\hat{\mathcal{B}}$ denotes the Fourier transform of the function $\mathcal{B}$ (it is more convenient to work in momentum space because the operators C_j are then diagonal). We will show that $M(q_2,q_1)$ is a well-defined tempered distribution; the Lemma then immediately follows by transforming back to position space.

The assumptions on $\mathcal{B}$ yield that $\hat{\mathcal{B}}$ is C^2 and has rapid decay at infinity, i.e.

$$\sup_{q \in \mathbb{R}^4,\, |\kappa| \leq 2} |q^{i_1} \cdots q^{i_n}\, \partial_\kappa \hat{\mathcal{B}}(q)| \;<\; \infty$$

for all n, all tensor indices $i_1, \ldots, i_n$ and multi-indices κ (with $\kappa = (\kappa^1, \ldots, \kappa^q)$, $|\kappa| := q$). As is verified explicitly in momentum space, the distributions k_m, p_m or s_m are bounded in the Schwartz norms of the test functions involving derivatives of only first order, more precisely

$$|C(f)| \;\leq\; \text{const}\, \|f\|_{4,1} \qquad \text{with } C = k_m,\, p_m \text{ or } s_m \text{ and } f \in \mathcal{S},$$

where the Schwartz norms are as usual defined by

$$\|f\|_{p,q} \;=\; \max_{|I| \leq p,\, |J| \leq q} \sup_{x \in \mathbb{R}^4} |x^I\, \partial_J f(x)|\,.$$

As a consequence, we can apply the corresponding operators even to functions with rapid decay which are only C^1. Furthermore, we can form the convolution of such functions with C; this gives continuous functions (which will no longer have rapid decay, however). Since C involves first derivatives, a convolution decreases the order of differentiability of the function by one.

We consider the combination of multiplication and convolution

$$F(p_2) \;:=\; \int \frac{d^4 p_1}{(2\pi)^4}\, f(p_2 - p_1)\, C(p_1)\, g(p_1)\,, \tag{2.2.47}$$

where we assume that $f \in C^2$ has rapid decay and $g \in C^1$ is bounded together with its first derivatives, $\|g\|_{0,1} < \infty$. For any fixed p_2, the integral in (2.2.47) is well-defined and finite because $f(p_2 - .)\, g(.)$ is C^1 and has rapid decay. The resulting function F is C^1 and bounded together with its first derivatives, more precisely

$$\|F\|_{0,1} \quad\leq\quad \text{const}\, \|f\|_{4,2}\, \|g\|_{0,1}\,. \tag{2.2.48}$$

After these preparations, we can estimate the integrals in (2.2.46) from the right to the left: We choose two test functions $f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$ and introduce the functions

$$F_1(p_1) \;=\; \int \frac{d^4 q_2}{(2\pi)^4}\, \hat{\mathcal{B}}(p_1 - q_1)\, C_0(q_1)\, g(q_1) \tag{2.2.49}$$

$$F_j(p_j) \;=\; \int \frac{d^4 p_{j-1}}{(2\pi)^4}\, \hat{\mathcal{B}}(p_j - p_{j-1})\, C_{j-1}(p_{j-1})\, F_{j-1}(p_{j-1}) \;\;,\;\; 1 < j \leq n\,. \tag{2.2.50}$$

The integral (2.2.49) is of the form (2.2.47) and satisfies the above considered assumptions on the integrand. Using the bound (2.2.48), we can proceed inductively in (2.2.50). Finally, we perform the q_2-integration,

$$M(f, g) \;=\; \int \frac{d^4 q_2}{(2\pi)^4}\, f(q_2)\, C_n(q_2)\, F_n(q_2)\,.$$

We conclude that M is a linear functional on $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^4) \times \mathcal{S}(\mathbb{R}^4, \mathbb{C}^4)$, which is bounded in the Schwartz norm $\|.\|_{4,1}$ of the test functions. ∎

Using the language of quantum field theory, we also refer to the summands of the perturbation expansions as *Feynman diagrams*. Then the result of the last lemma can be understood from the fact that in an external field one only encounters tree diagrams, which are all finite. Clearly, the existence of the perturbation expansion

to every order does not imply the convergence of the perturbation series, and we will come back to this problem in §2.5.

2.3. Definition of the Fermionic Projector

In this section we introduce the mathematical framework for describing a many-fermion system in the presence of an external field. To this end, we first extend the construction of §2.2 to a system of Dirac seas of in general different masses, which may involve chiral massless Dirac seas. Then we introduce particles and anti-particles by occupying additional states and creating "holes" in the Dirac seas, respectively. Our construction is intended to be so general that it allows us to model the fermion configuration of the standard model (see §5.1). For clarity, we postpone the question of how the fermionic states are to be normalized to §2.6.

First, we need to introduce a distribution $P^{\text{sea}}(x,y)$ which describes the system in the vacuum. The most general ansatz is to take a direct sum of sums of Dirac seas,

$$P^{\text{sea}} \;=\; \bigoplus_{a=1}^{N} \sum_{\alpha=1}^{g(a)} P^{\text{sea}}_{a\alpha}\,, \tag{2.3.1}$$

where $g(a)$ are positive integers and the summands $P^{\text{sea}}_{a\alpha}$ are Dirac seas of a form similar to (2.2.2). The direct sum increases the total number of components of the wave functions, the so-called *spin dimension*, to $4N$. The direct summands are called sectors, and we refer to the indices a and α as the *sector* and *generation index*, respectively. For each Dirac sea we introduce a mass parameter $m_{n\alpha} \geq 0$. In order to allow for chiral massless Dirac seas, we introduce (4×4)-matrices $X_{a\alpha}$ with

$$X_{a\alpha} \;=\; \begin{cases} \mathbb{1} & \text{if } m_{a\alpha} > 0 \\ \mathbb{1}, \chi_L \text{ or } \chi_R & \text{if } m_{a\alpha} = 0 \end{cases}$$

and set

$$P^{\text{sea}}_{a\alpha} \;=\; \frac{1}{2}\, X_{a\alpha}\, (p_{m_{a\alpha}} - k_{m_{a\alpha}})\,. \tag{2.3.2}$$

We refer to P^{sea} as defined by (2.3.1, 2.3.2) as the *fermionic projector of the vacuum*. It is sometimes useful to consider P^{sea} as a matrix in the sectors indices,

$$(P^{\text{sea}})^a_b \;=\; \delta^a_b\, \frac{1}{2} \sum_{\alpha=1}^{g(a)} X_{a\alpha}\, (p_{m_{a\alpha}} - k_{m_{a\alpha}})$$

with $a, b = 1, \ldots, N$.

Since each sector may involve several Dirac seas of different masses, it seems impossible to write the fermionic projector of the vacuum as a solution of a suitable Dirac equation, and thus we have no starting point for a perturbation expansion. In order to bypass this problem, we replace the sum in (2.3.1) by a direct sum and introduce the so-called *auxiliary fermionic projector* by

$$P^{\text{sea}} \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} P^{\text{sea}}_{a\alpha}\,. \tag{2.3.3}$$

Using the same notation as for the fermionic projector is usually no problem because it will be clear from the context whether the fermionic projector or the auxiliary

fermionic projector is meant. In case of potential confusion we write the auxiliary fermionic projector as a matrix in the sector and generation indices,

$$(P^{\text{sea}})^{(a\alpha)}_{(b\beta)} \;=\; \delta^a_b\, \delta^\alpha_\beta \frac{1}{2}\, X_{a\alpha}\, (p_{m_{a\alpha}} - k_{m_{a\alpha}})$$

with $a, b = 1, \ldots, N$, $\alpha = 1, \ldots, g(a)$, $\beta = 1, \ldots, g(b)$). In this notation, one also sees that the fermionic projector can be obtained from the auxiliary fermionic projector by taking the so-called *partial trace*[3] over the generations,

$$(P^{\text{sea}})^a_b \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P^{\text{sea}})^{(a\alpha)}_{(b\beta)}\,. \tag{2.3.4}$$

We introduce the operators

$$p \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} p_{m_{a\alpha}}\,, \qquad\qquad k \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} k_{m_{a\alpha}}$$

and define the matrices

$$X \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} X_{a\alpha}\,, \qquad\qquad Y \;=\; \frac{1}{m} \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} m_{a\alpha}\,,$$

which are called *chiral asymmetry matrix* and *mass matrix*, respectively. Here m is an arbitrary mass parameter; a convenient choice is $m = \max_{a,\alpha} m_{a\alpha}$. These operators act on direct sums of Dirac wave functions, i.e. on functions of the form $\Psi = (\Psi^{(a\alpha)}(x))$ with $\Psi^{(a\alpha)}$ a 4-component Dirac spinor. On these wave funtions, we introduce the *spin scalar product* by

$$\prec\!\Psi \,|\, \Phi\!\succ(x) \;=\; \sum_{a=1}^{n} \sum_{\alpha=1}^{g(a)} \prec\!\Psi^{(a\alpha)} \,|\, \Phi^{(a\alpha)}\!\succ_{\text{Dirac}}\,, \tag{2.3.5}$$

where $\prec\!.|.\!\succ_{\text{Dirac}}$ is the usual spin scalar product on Dirac spinors (1.2.8). In generalization of (1.5.23) we also introduce the indefinite inner product

$$<\!\Psi \,|\, \Phi\!> \;=\; \int_M \prec\!\Psi \,|\, \Phi\!\succ\, d\mu\,. \tag{2.3.6}$$

Then the operators p and k are Hermitian with respect to $<\!.|.\!>$, and the mass matrix Y is Hermitian with respect to the spin scalar product. Using the above notation, we can write the auxiliary fermionic projector as

$$P^{\text{sea}}(x,y) \;=\; X\, \frac{1}{2}\, (p(x,y) - k(x,y))\,. \tag{2.3.7}$$

Since $m_l = 0$ for $X_l \neq \mathbb{1}$ and since the operators $p_{m=0}$, $k_{m=0}$ are odd, we have alternatively

$$P^{\text{sea}}(x,y) \;=\; \frac{1}{2}\, (p(x,y) - k(x,y))\, X^*\,, \tag{2.3.8}$$

where X^* is the adjoint with respect to the spin scalar product. The auxiliary fermionic projector is a solution of the free Dirac equation

$$(i\partial\!\!\!/_x - mY)\, P^{\text{sea}}(x,y) \;=\; 0\,. \tag{2.3.9}$$

[3] Online version: In more recent works on the fermionic projector, the partial trace is referred to as the *sectorial projection*.

Our strategy is to extend the definition of the auxiliary fermionic projector to the interacting system and then to get back to the fermionic projector by taking the partial trace (2.3.4).

In order to describe the system of Dirac seas in the presence of an external field, we insert a differential operator $\mathcal{B} = (\mathcal{B}^{(a\alpha)}_{(b\beta)})$ into the Dirac equation (2.3.9),

$$(i\partial\!\!\!/_x + \mathcal{B} - mY)\, P^{\text{sea}}(x,y) \;=\; 0\,. \tag{2.3.10}$$

We always assume that $\mathcal{B}$ is Hermitian with respect to the inner product $<.|.>$. The causal perturbation expansion for the operators k and p can be carried out exactly as in §2.2: We define the advanced and retarded Green's functions by

$$s^\vee \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} s^\vee_{m_{a\alpha}}\,, \qquad\qquad s^\wedge \;=\; \bigoplus_{a=1}^{N} \bigoplus_{\alpha=1}^{g(a)} s^\wedge_{m_{a\alpha}}\,.$$

Their perturbation expansion is, in analogy to (2.2.13, 2.2.15), uniquely given by

$$\tilde{s}^\vee \;=\; \sum_{k=0}^{\infty} (-s^\vee\, \mathcal{B})^k\, s^\vee\,, \qquad\qquad \tilde{s}^\wedge \;=\; \sum_{k=0}^{\infty} (-s^\wedge\, \mathcal{B})^k\, s^\wedge\,. \tag{2.3.11}$$

The method of Theorem 2.2.1 now yields the following result.

THEOREM 2.3.1. *The perturbation expansion for p and k is uniquely determined by the conditions*

$$\tilde{k} \;=\; \frac{1}{2\pi i}\,(\tilde{s}^\vee - \tilde{s}^\wedge)\,, \qquad\qquad \tilde{p} \overset{\text{formally}}{=} \sqrt{\tilde{k}^2}\,. \tag{2.3.12}$$

We have the explicit formulas

$$\tilde{k} \;=\; \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\; b^<\, k\, (b\, k)^{2\beta}\, b^>\,, \qquad\qquad \tilde{p} \;=\; \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\left[\frac{\beta}{2}\right]} c(\alpha,\beta)\, G(\alpha,\beta)$$

with

$$\begin{aligned} c(0,0) \;&=\; 1\,, \\ c(\alpha,\beta) \;&=\; \sum_{n=\alpha+1}^{\beta} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\, 2^n} \binom{\beta-\alpha-1}{n-\alpha-1} \qquad \text{for } \beta \geq 1 \end{aligned}$$

and

$$\begin{aligned} G(f,g) \;=\; & \sum_{Q \in \mathcal{P}(\beta+1),\; \#Q = 2\alpha+1} (-i\pi)^{2\beta} \\ & \times b^<\, F(Q,1)\, bkb\, F(Q,2)\, bkb \,\cdots\, bkb\, F(Q,\beta+1)\, b^>\,, \end{aligned}$$

where $\mathcal{P}(n)$ is the set of subsets of $\{1,\ldots,n\}$ and where we use the notation

$$\begin{aligned} s \;&=\; \frac{1}{2}\,(s^\vee + s^\wedge)\,, \qquad F(Q,n) \;=\; \begin{cases} p & \text{if } n \in Q \\ k & \text{if } n \notin Q \end{cases} \\ b^< \;&=\; \sum_{k=0}^{\infty} (-s\,\mathcal{B})^k\,, \qquad b \;=\; \sum_{k=0}^{\infty} (-\mathcal{B}\, s)^k\, \mathcal{B}\,, \qquad b^> \;=\; \sum_{k=0}^{\infty} (-\mathcal{B}\, s)^k\,. \end{aligned}$$

The contributions to this perturbation expansion are all well-defined according to Lemma 2.2.2.

After this straightforward generalization, we come to the more subtle question of how to define P^{sea} when a chiral asymmetry is present. The obvious idea is to set in generalization of (2.3.7)

$$P^{\text{sea}}(x,y) \;=\; X\,\frac{1}{2}\,(\tilde{p}-\tilde{k})(x,y)\,. \tag{2.3.13}$$

This is not convincing, however, because we could just as well have defined $P^{\text{sea}}(x,y)$ in analogy to (2.3.8) by $P^{\text{sea}} = \frac{1}{2}(\tilde{p}-\tilde{k})\,X^*$, which does not coincide with (2.3.13) as soon as X, X^* do not commute with $\mathcal{B}$. Actually, this arbitrariness in defining the Dirac sea reflects a basic problem of the causal perturbation expansion for systems with chiral asymmetry. In order to describe the problem in more detail, we consider the perturbation calculation for k to first order. According to (2.3.11, 2.3.12),

$$\begin{aligned} \tilde{k} &= k \;-\; \frac{1}{2\pi i}\,(s^\vee\,\mathcal{B}\,s^\vee \;-\; s^\wedge\,\mathcal{B}\,s^\wedge) \;+\; \mathcal{O}(\mathcal{B}^2) \\ &= k \;-\; s\,\mathcal{B}\,k \;-\; k\,\mathcal{B}\,s \;+\; \mathcal{O}(\mathcal{B}^2)\,. \end{aligned} \tag{2.3.14}$$

This expansion is causal in the sense that $\tilde{k}(x,y)$ only depends on $\mathcal{B}$ in the "diamond" $(L_x^\vee \cap L_y^\wedge) \cup (L_y^\vee \cap L_x^\wedge)$, as is obvious from (2.3.14). It is not clear, however, how to insert the chiral asymmetry matrix into this formula. It seems most natural to replace all factors k by Xk,

$$(\tilde{Xk}) \;=\; Xk \;-\; s\,\mathcal{B}\,Xk \;-\; Xk\,\mathcal{B}\,s \;+\; \mathcal{O}(\mathcal{B}^2)\,. \tag{2.3.15}$$

Unfortunately, this expression cannot be written similar to (2.3.14) with the advanced and retarded Green's functions, which means that the causality of the expansion is in general lost. In order to avoid this problem, one might want to insert X at every factor s, k,

$$\begin{aligned} (\tilde{Xk}) &= Xk \;-\; Xs\,\mathcal{B}\,Xk \;-\; Xk\,\mathcal{B}\,Xs \;+\; \mathcal{O}(\mathcal{B}^2) \\ &= Xk \;-\; \frac{1}{2\pi i}\,(Xs^\vee\,\mathcal{B}\,Xs^\vee \;-\; Xs^\wedge\,\mathcal{B}\,Xs^\wedge) \;+\; \mathcal{O}(\mathcal{B}^2)\,. \end{aligned} \tag{2.3.16}$$

This expansion is causal similar to (2.3.14). In general, however, it does not satisfy the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\,(\tilde{Xk}) = 0$, which does not seem to be what we want.

The only way to resolve this problem is to impose that the perturbation expansions (2.3.15) and (2.3.16) should coincide. This yields a condition for the operator $\mathcal{B}$, which can be characterized as follows. We demand that

$$Xs^\vee\,\mathcal{B}\,Xs^\vee \;=\; s^\vee\,\mathcal{B}\,Xs^\vee \;=\; Xs^\vee\,\mathcal{B}\,s^\vee\,. \tag{2.3.17}$$

Since the operator $s^\vee_{m=0}$ is odd, we have $Xs^\vee = s^\vee X^*$. Substituting into the second equation of (2.3.17) yields the condition $X^*\,\mathcal{B} \;=\; \mathcal{B}\,X$. Since X is idempotent, this condition automatically implies the first equation of (2.3.17). We formulate the derived condition for the whole Dirac operator $i\partial\!\!\!/ + \mathcal{B} - mY$ and thus combine it with the fact that chiral fermions are massless (i.e. $X^*Y = YX = Y$) and that X is composed of chiral projectors (which implies that $X^*\partial\!\!\!/ = \partial\!\!\!/ X$).

DEF. 2.3.2. *The Dirac operator is called* **causality compatible** *with X if*

$$X^*\,(i\partial\!\!\!/ + \mathcal{B} - mY) \;=\; (i\partial\!\!\!/ + \mathcal{B} - mY)\,X\,. \tag{2.3.18}$$

In the perturbation expansion to higher order, the condition (2.3.18) allows us to commute X through all operator products. Using idempotence $X^2 = X$, we can moreover add factors X to the product; in particular,

$$X\, C_1 \mathcal{B} C_1 \mathcal{B} \cdots \mathcal{B} C_n \;=\; X C_1 \mathcal{B} X C_1 \mathcal{B} \cdots \mathcal{B} X C_n \qquad \text{with} \qquad C_j = p,\, C_j = k \text{ or } C_j = s\,.$$

This ensures that the perturbation expansion is also well-defined to higher order. For a Dirac operator which is causality compatible with X, the *auxiliary fermionic projector* is defined canonically by (2.3.13).

So far the auxiliary fermionic projector describes a system of Dirac seas in the presence of an external field. In order to insert particles and anti-particles into the system, we add the projectors on particle states and substract the projectors on anti-particle states,

$$\begin{aligned} P(x,y) \;&=\; P^{\text{sea}}(x,y) \\ &\quad + c_{\text{norm}} \sum_{k=1}^{n_{\text{p}}} |\Psi_k(x) \succ\!\prec \Psi_k(y)| \;-\; c_{\text{norm}} \sum_{l=1}^{n_{\text{a}}} |\Phi_l(x) \succ\!\prec \Phi_l(y)|\,, \end{aligned} \tag{2.3.19}$$

where Ψ_k and Φ_l are an orthogonal set of solutions of the Dirac equation, and the Φ_l must lie in the image of P^{sea} (for the normalization constant c_{norm} see §2.6). The parameters n_{p} and n_{a} denote the total number of particles and anti-particles, respectively. We usually avoid the issue of convergence of the sums in (2.3.19) by assuming that $n_{\text{p}}, n_{\text{a}} < \infty$, but one could clearly also consider an infinite number of particles and/or anti-particles. Finally, the *fermionic projector* is obtained from this expression by taking similar to (2.3.4) the partial trace,

$$(P)_b^a \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P)_{(b\beta)}^{(a\alpha)}\,. \tag{2.3.20}$$

Theorem 2.3.1 together with (2.3.13, 2.3.19, 2.3.20) yields a mathematical framework for describing a general many-fermion system in the presence of an external field. Our construction makes Dirac's concept of a "sea of interacting particles" mathematically precise. Apart from the causality compatibility condition (2.3.18) and the regularity conditions in Lemma 2.2.2, the operator $\mathcal{B}$ is completely arbitrary. We point out that we do not use the fermionic Fock space formalism of canonical quantum field theory; the connection to this formalism will be explained in §3.2 and Appendix A.

2.4. Interpretation and Consequences

With the definition of the fermionic projector we radically departed from the usual concept of the Dirac sea as "all negative-energy solutions" of the Dirac equation. Instead, we used causality in a particular way. In order to clarify the connection between our definition and the usual concept of the Dirac sea, we now describe how the above constructions simplify in the special situation that $\mathcal{B}$ is *static*. If considered as multiplication operators, static potentials map functions of positive and negative frequency into functions of positive and negative frequency, respectively. Since the operators p, k and s are diagonal in momentum space, they clearly preserve the sign of the frequency too. Thus

$$[\Pi^{\pm}, p] \;=\; [\Pi^{\pm}, k] \;=\; [\Pi^{\pm}, s] \;=\; [\Pi^{\pm}, \mathcal{B}] \;=\; 0\,, \tag{2.4.1}$$

where the operators $\Pi^\pm$ are the projectors onto the states of positive and negative frequency, respectively (i.e. in momentum space, $\Pi^\pm$ are the operators of multiplication by the functions $\Theta(\pm k^0)$). The operators p and k differ only by a relative minus sign for the states of positive and negative frequency,

$$\Pi^\pm p \;=\; \pm\, \Pi^\pm k\,.$$

Using this relation together with (2.4.1), we can replace pairs of factors p by pairs of factors k. For example,

$$\begin{aligned} \cdots p\,\mathcal{B} \cdots p\,\mathcal{B} \cdots &= \cdots p\,\mathcal{B} \cdots p\,\mathcal{B} \cdots\, (\Pi^+ + \Pi^-) \\ &= \Pi^+ (\cdots k\,\mathcal{B} \cdots k\,\mathcal{B} \cdots) \;+\; \Pi^- (\cdots (-k)\,\mathcal{B} \cdots (-k)\,\mathcal{B} \cdots) \\ &= \cdots k\,\mathcal{B} \cdots k\,\mathcal{B} \cdots\,, \end{aligned} \tag{2.4.2}$$

where the dots '$\cdots$' denote any combination of the operators s, k, p and $\mathcal{B}$. This allows us to simplify the formula for $\tilde{p}$ by using only one factor p in every operator product. After going through the details of the combinatorics, one obtains the formula

$$\tilde{p} \;=\; \sum_{b=0}^{\infty} (-i\pi)^{2b}\, b^< \,p\, (b\,k)^{2b}\, b^>\,.$$

Thus the fermionic projector (2.3.13) can be written as

$$P^{\text{sea}}(x,y) \;=\; \sum_{b=0}^{\infty} (-i\pi)^{2b}\, b^< \left[\frac{1}{2}\, X\,(p-k)\right] (b\,k)^{2b}\, b^>\,.$$

This equation shows that P^{sea} is composed of all negative-frequency eigenstates of the Dirac operator (notice that the expression in the brackets $[\cdots]$ is the fermionic projector of the vacuum and that all other factors preserve the sign of the frequency). We conclude that for static potentials our definition reduces to the usual concept of the Dirac sea as "all negative-energy states."

In order to get a better understanding of the time-dependent situation, we next consider a *scattering process*. For simplicity, we consider a system of one Dirac sea and assume that the scattering takes place in finite time $t_0 < t < t_1$. This means that the wave functions Ψ satisfy the Dirac equation (2.1.1) with $\mathcal{B}$ supported in a finite time interval,

$$\mathcal{B}(t,\vec{x}) \;=\; 0 \qquad \text{if } t \notin [t_0, t_1]\,. \tag{2.4.3}$$

As a consequence, $\Psi(t,\vec{x})$ is for $t < t_0$ a solution of the free Dirac equation. We uniquely extend this free solution to the whole Minkowski space and denote it by Ψ_{in},

$$(i\partial\!\!\!/ - m)\,\Psi_{\text{in}} \;=\; 0 \qquad \text{and} \qquad \Psi_{\text{in}}(t,\vec{x}) \;=\; \Psi(t,\vec{x}) \quad \text{for } t < t_0.$$

Similarly, $\Psi(t,\vec{x})$ is also for $t > t_1$ a solution of the free Dirac equation; we denote its extension by Ψ_{out},

$$(i\partial\!\!\!/ - m)\,\Psi_{\text{out}} \;=\; 0 \qquad \text{and} \qquad \Psi_{\text{out}}(t,\vec{x}) \;=\; \Psi(t,\vec{x}) \quad \text{for } t > t_1.$$

The wave functions Ψ_{in} and Ψ_{out} are called the incoming and outgoing *scattering states*, respectively. Recall that the dynamics of the wave functions is described infinitesimally by the Dirac equation in the Hamiltonian form (1.2.18), where h is a symmetric operator on the Hilbert space $(\mathcal{H}, (.,.))$ with scalar product (1.2.17). Integrating this equation from t_0 to t_1, we obtain a unitary operator S which maps the incoming scattering states to the corresponding outgoing states,

$$\Psi_{\text{out}} \;=\; S\,\Psi_{\text{in}}\,. \tag{2.4.4}$$

The operator S is called scattering operator or *S-matrix*.

Using the scattering states, we can introduce fermionic projectors which describe the vacua in the asymptotic past and future: For an observer in the past $t < t_0$, the external potential is zero. Thus it is natural for him to describe the vacuum with the free Dirac sea (2.2.1). If this Dirac sea is extended to the whole Minkowski space with external potential, one gets the object

$$P^\wedge(x,y) \;=\; -\frac{m}{\pi} \sum_s \int_{\mathbb{R}^3} |\Psi^\wedge_{\vec{p}s-} \succ\!\prec \Psi^\wedge_{\vec{p}s-}|\; d\mu_{\vec{p}}\,, \tag{2.4.5}$$

where the wave functions $\Psi^\wedge_{\vec{p}s\epsilon}$ are the solutions of the Dirac equation (2.1.1) whose incoming scattering states are the plane wave solutions $\Psi_{\vec{p}s\epsilon}$,

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi^\wedge_{\vec{p}s\epsilon} \;=\; 0 \qquad \text{and} \qquad (\Psi^\wedge_{\vec{p}s\epsilon})_{\text{in}} \;=\; \Psi_{\vec{p}s\epsilon}\,.$$

Using the support conditions (2.4.3, 2.1.9), we can express the state $\Psi^\wedge_{\vec{p}s\epsilon}$ in a perturbation series,

$$\Psi^\wedge_{\vec{p}s\epsilon} \;=\; \sum_{n=0}^\infty (-s^\wedge\, \mathcal{B})^n\, \Psi_{\vec{p}s\epsilon}\,.$$

Substituting this formula into (2.4.5) we obtain for $P^\wedge$ a perturbation expansion which involves only the retarded Green's functions,

$$P^\wedge \;=\; \sum_{n_1,n_2=0}^\infty (-s^\wedge\, \mathcal{B})^{n_1}\, P^{\text{vac}}\, (-\mathcal{B}\, s^\wedge)^{n_2}\,, \tag{2.4.6}$$

where P^{vac} stands for the free Dirac sea (2.2.1). Accordingly, an observer in the future $t > t_0$ describes the vacuum by the fermionic projector

$$P^\vee(x,y) \;=\; -\frac{m}{\pi} \sum_s \int_{\mathbb{R}^3} |\Psi^\vee_{\vec{p}s-} \succ\!\prec \Psi^\vee_{\vec{p}s-}|\; d\mu_{\vec{p}}\,, \tag{2.4.7}$$

where

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \Psi^\vee_{\vec{p}s\epsilon} \;=\; 0 \qquad \text{and} \qquad (\Psi^\vee_{\vec{p}s\epsilon})_{\text{out}} \;=\; \Psi_{\vec{p}s\epsilon}\,.$$

Its perturbation expansion involves only the advanced Green's function,

$$P^\vee \;=\; \sum_{n_1,n_2=0}^\infty (-s^\vee\, \mathcal{B})^{n_1}\, P^{\text{vac}}\, (-\mathcal{B}\, s^\vee)^{n_2}\,. \tag{2.4.8}$$

Using (2.4.4) in (2.4.5, 2.4.7), we can describe the fermionic projectors in the asymptotic past and future with the S-matrix by

$$P^\wedge_{\text{in}} \;=\; P^{\text{vac}} \;=\; P^\vee_{\text{out}}\,, \qquad P^\wedge_{\text{out}} \;=\; S P^{\text{vac}} S^{-1}\,, \qquad P^\vee_{\text{in}} \;=\; S^{-1} P^{\text{vac}} S\,. \tag{2.4.9}$$

What makes the scattering process interesting is the fact that the vacua in the asymptotic past and future in general do not coincide. Consider for example the physical system described by the fermionic projector $P := P^\wedge$. For the observer in the past, the system is in the vacuum. However, if $P^\wedge \neq P^\vee$, the system will *not* be in the vacuum for the observer in the future. This means that for him, positive frequency states are occupied and negative frequency states are unoccupied and thus the system

contains particles and anti-particles. More precisely, if we write the fermionic projector in analogy to (2.3.19) as

$$\begin{aligned} P(x,y) \;&=\; P^\vee(x,y) \\ &\quad + c_{\text{norm}} \sum_{k=1}^{n_{\text{p}}} |\Psi_k(x) \succ\!\prec \Psi_k(y)| \;-\; c_{\text{norm}} \sum_{l=1}^{n_{\text{a}}} |\Phi_l(x) \succ\!\prec \Phi_l(y)| \,, \end{aligned} \tag{2.4.10}$$

then the Ψ_k and Φ_l are the wave functions of the particles and anti-particles, respectively. These particles and anti-particles are physical reality; the observer in the future can detect them by making suitable experiments. This is the physical effect of *pair creation*. Using (2.4.9) one can express the pair creation completely in terms of the S-matrix. Other scattering processes are described similarly.

We point out that describing the scattering process with the two observers in the past and future is merely a matter of convenience. The physical process can be described equivalently (although maybe less conveniently) in the reference frame of any other observer. To give a concrete example, we consider an observer in the future who is in a reference frame moving with constant acceleration. This leads to the so-called *Unruh effect* , which we now briefly outline (for details see e.g. [**Wa2**]). For the accelerated observer, space-time is stationary (i.e. his time direction is a Killing field, but it is not a unit normal to the hypersurfaces $t = \text{const}$), and this allows him to use the separation ansatz (2.1.2) with t his proper time. The sign of ω gives him a splitting of the solution space into solutions of positive and negative energy. Using Dirac's hole interpretation corresponding to this splitting, he finds for the many-fermion system described by P an infinite number of particles and anti-particles in a thermal equilibrium. This bizarre effect shows that the interpretation of the physical system in terms of particles and anti-particles does depend on the observer. Nevertheless, the Unruh effect does not contradict the pair creation experiments made by the future observer at rest. Namely, if the accelerated observer wants to explain the experiments by the future observer at rest, he must take into account that he himself is feeling a gravitational field, and that for him the experimental apparatus used by the observer at rest is in accelerated motion. It turns out that these additional effects just compensate the Unruh effect, so that the predictions by the accelerated observer are in complete agreement with the observations of the particles and anti-particles in (2.4.10) by the future observer at rest. More generally, all quantities which can be measured in experiments can be expressed in terms of the S-matrix. Since the S-matrix does not depend on the particle/anti-particle interpretation, it is clear that all experiments can be explained equivalently in any reference frame.

We just saw that the particle/anti-particle interpretation of a fermionic system may depend on the observer. Actually, the situation is even worse for an observer in the time period $t_0 < t < t_1$ when the interaction takes place. For him, the system is neither static nor stationary. Therefore, he has no notion of "negative-energy state", and thus for him the particle/anti-particle interpretation completely breaks down. Taking into account that a scattering process is an idealized process and that in a real physical situation there will be no region of space-time where no interaction takes place, we come to the disillusioning conclusion that for a local observer under generic conditions, a many-fermion system has no interpretation in terms of particles and anti-particles.

The causal perturbation expansion yields a canonical object P^{sea} which describes the Dirac sea in the scattering process, even in the region with interaction $t_0 < t < t_1$.

Its construction is explicitly covariant and independent of a local observer. Decomposing the fermionic projector in the form (2.3.19), we obtain a canonical interpretation of the many-fermion system in terms of particles and anti-particles. One should keep in mind that P^{sea} does not correspond to the vacuum of any local observer, but is a global object of space-time. As a consequence, also the particle/anti-particle interpretation in (2.3.19) can be associated only to an abstract "global observer" in space-time. More specifically, comparing Theorem 2.3.1 and (2.3.13) with (2.4.6, 2.4.8), one sees that P^{sea} coincides neither with $P^\wedge$ nor with $P^\vee$. Since its perturbation expansion involves both retarded and advanced Green's functions, it can be considered as being some kind of "interpolation" between $P^\wedge$ and $P^\vee$.

Let us now discuss our assumption on the potential $\mathcal{B}$ as being *external*. As explained at the end of §2.1, this is no restriction in principle because one can first solve the physical equations of the coupled system and then can define P^{sea} for the external potential $\mathcal{B}$ as given by the solution of the coupled system. Clearly, this procedure cannot be carried out in practice, but this is of no relevance for the theoretical considerations here. The important point is that P^{sea} is not defined locally; for its definition we need to know $\mathcal{B}$ in the whole space-time. This is puzzling because the conventional physical equations are local and causal, and this is the first time that an object appears which is defined in a non-local way. One might conclude that P^{sea} is an object which is not compatible with causality and should therefore have no physical significance. Our concept is the opposite: We regard the appearance of a non-local object as a first hint that locality and causality should be given up in the strict sense. In order to formulate physical equations which could replace the conventional local and causal equations, we shall consider the fermionic projector as the fundamental object in space-time.

Before we can make these ideas precise in Chapter 3, we need to analyze the fermionic projector in the continuum in more detail. One task is to understand what "causality" of the causal perturbation expansion means precisely. At the moment, we know that causality was used for the definition of P^{sea}, but that nevertheless the fermionic projector is a nonlocal and non-causal object. We need to find out how these seemingly contradicting facts fit together. Also, we must understand better how P^{sea} depends on $\mathcal{B}$. More specifically, we need to analyze what information on the external potentials is encoded in P^{sea}, and how this information can be extracted. Finally, we must specify how the fermionic states in (2.3.19) are to be normalized. The next sections provide the mathematical tools for answering these questions.

2.5. The Light-Cone Expansion

The light-cone expansion is a very useful technique for analyzing the fermionic projector near the light cone. In order to give a brief but self-contained introduction, we will explain the methods and results of [**F6**] leaving out many proofs and technical details. Our setting is that of §2.3 with several sectors and generations (2.3.1). It suffices to consider the *auxiliary fermionic projector* because the fermionic projector is obtained from it simply by taking the partial trace (2.3.20). We again assume that the Dirac operator in (2.3.10) is causality compatible (2.3.18) and that the operator $\mathcal{B}$ is Hermitian with respect to the inner product (2.3.6). Furthermore, we assume as in [**F6**] that $\mathcal{B}$ is a multiplication operator composed of *chiral* and *scalar/pseudoscalar potentials*,

$$\mathcal{B}(x) \;=\; \chi_L \,\A_R(x) \;+\; \chi_R \,\A_L(x) \;+\; \Phi(x) \;+\; i\rho\,\Xi(x)\,. \tag{2.5.1}$$

We note for clarity that these potentials may act non-trivially on the sectors and generations (e.g. writing the right-handed potential as a matrix, $A_R = (A_R)^{(a\alpha)}_{(b\beta)}$, the matrix elements are independent vector fields with the only constraint that the matrix must be Hermitian). In particular, the potentials in (2.5.1) do in general not commute, and thus we must in products be careful with the order of multiplication. For an operator $\mathcal{B}$ which involves bilinear and gravitational potentials see [**F5**].

DEF. 2.5.1. *A distribution $A(x,y)$ on $\mathbb{R}^4 \times \mathbb{R}^4$ is of the order $\mathcal{O}((y-x)^{2p})$, $p \in \mathbb{Z}$, if the product*

$$(y-x)^{-2p}\, A(x,y)$$

is a regular distribution (=a locally integrable function). It has the **light-cone expansion**

$$A(x,y) \;=\; \sum_{j=g}^{\infty} A^{[j]}(x,y) \tag{2.5.2}$$

with $g \in \mathbb{Z}$ if the distributions $A^{[j]}(x,y)$ are of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the sense that for all $p \geq g$,

$$A(x,y) \;-\; \sum_{j=g}^{p} A^{[j]}(x,y) \qquad \text{is of the order } \mathcal{O}((y-x)^{2p+2})\,. \tag{2.5.3}$$

The light-cone expansion describes the behavior of a distribution near the light cone. More precisely, the expansion parameter $(y-x)^2$ vanishes if y lies on the light cone centered at x, and thus the distributions $A^{[j]}(x,y)$ approximate $A(x,y)$ for y in a neighborhood of L_x. The first summand $A^{[g]}(x,y)$ gives the leading order of $A(x,y)$ on the light cone, $A^{[g+1]}$ gives the next order on the light cone, etc. If the distribution A is singular on the light cone, the parameter g will be negative. Note that the distributions $A^{[j]}$ are determined only up to contributions of higher order $\mathcal{O}((y-x)^{2j+2})$, but this ambiguity will not lead to any problems in what follows. We point out that we do not demand that the infinite series in (2.5.2) converges. This series is defined only via the approximation by the partial sums (2.5.3). Despite this formal character of the series, the light-cone expansion completely describes the behavior of $A(x,y)$ near the light cone. This situation can be seen in analogy to the Taylor expansion of a smooth, nonanalytic function. Although the Taylor series does in general not converge, the Taylor polynomials give local approximations of the function. An important difference to a Taylor expansion is that the $A^{[j]}(x,y)$ approximate $A(x,y)$ even for points x and y which are far apart. We only need that y is close to the light cone L_x, which is an unbounded hypersurface in $\mathbb{R}^4$. In this sense, the light-cone expansion is a *non-local expansion*.

For clarity, we begin with the light-cone expansion for the causal Green's functions, and we will later extend the results to the fermionic projector. In order to get a first idea of how the light-cone expansion can be carried out, we consider the free advanced Green's function $s^\vee_m$ as defined by (2.1.7, 2.1.8). We can pull the Dirac matrices out of the Fourier integral by setting

$$s^\vee_m(x,y) \;=\; (i\partial\!\!\!/_x + m)\, S^\vee_{m^2}(x,y)\,, \tag{2.5.4}$$

where $S^\vee_{m^2}$ is the advanced Green's function of the Klein-Gordon operator,

$$S^\vee_{m^2}(x,y) \;=\; \lim_{\varepsilon \searrow 0} \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2 - m^2 - i\varepsilon k^0}\, e^{-ik(x-y)}\,. \tag{2.5.5}$$

This Fourier integral can be computed explicitly; we expand the resulting Bessel function in a power series,

$$\begin{aligned} S^\vee_{m^2}(x,y) &= -\frac{1}{2\pi}\,\delta(\xi^2)\,\Theta(\xi^0) + \frac{m^2}{4\pi}\,\frac{J_1(\sqrt{m^2\xi^2})}{\sqrt{m^2\xi^2}}\,\Theta(\xi^2)\,\Theta(\xi^0) \\ &= -\frac{1}{2\pi}\,\delta(\xi^2)\,\Theta(\xi^0) + \frac{m^2}{8\pi}\sum_{j=0}^\infty \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j}\,\Theta(\xi^2)\,\Theta(\xi^0)\,, \end{aligned} \tag{2.5.6}$$

where we used the abbreviation $\xi \equiv y-x$. This calculation shows that $S^\vee_{m^2}(x,y)$ has a $\delta((y-x)^2)$-like singularity on the light cone. Furthermore, one sees that $S^\vee_{m^2}$ is a power series in m^2. The important point for us is that the contributions of higher order in m^2 contain more factors $(y-x)^2$ and are thus of higher order on the light cone. More precisely,

$$\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|m^2=0}(x,y) \qquad \text{is of the order} \quad \mathcal{O}((y-x)^{2n-2})\,. \tag{2.5.7}$$

According to (2.5.4), the Dirac Green's function is obtained by taking the first partial derivatives of (2.5.6). Thus $s^\vee_m(x,y)$ has a singularity on the light cone which is even $\sim \delta'((y-x)^2)$. The higher order contributions in m are again of increasing order on the light cone. This means that we can view the Taylor expansion of (2.5.4) in m,

$$s^\vee_m(x,y) = \sum_{n=0}^\infty \frac{m^{2n}}{n!}\,(i\partial\!\!\!/ + m)\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|\,m^2=0}(x,y)\,, \tag{2.5.8}$$

as a light-cone expansion of the Green's function.

Writing the light-cone expansion of $s^\vee_m$ in the form (2.5.8) clearly is more convenient than working with the explicit formula (2.5.6). This is our motivation for using an expansion with respect to the mass parameter also in the presence of the external field. Expanding the perturbation expansion (2.3.11) in m gives a double series in powers of m and $\mathcal{B}$. In order to combine these two expansions in a single perturbation series, we write the mass matrix and the scalar/pseudoscalar potentials together by setting

$$Y_L(x) = Y - \frac{1}{m}\,(\Phi(x) + i\Xi(x))\,, \qquad Y_R(x) = Y - \frac{1}{m}\,(\Phi(x) - i\Xi(x))\,. \tag{2.5.9}$$

The matrices $Y_{L/R}(x)$ are called *dynamical mass matrices*; notice that $Y_L^* = Y_R$. With this notation, we can rewrite the Dirac operator as

$$i\partial\!\!\!/ + \mathcal{B} - mY = i\partial\!\!\!/ + B \qquad \text{with} \tag{2.5.10}$$

$$B = \chi_L\,(A\!\!\!/_R - m\,Y_R) + \chi_R\,(A\!\!\!/_L - m\,Y_L)\,. \tag{2.5.11}$$

For the light-cone expansion of the Green's functions, we shall always consider B as the perturbation of the Dirac operator. This has the advantage that the free theory consists of zero-mass fermions, and thus the Green's functions of the free Dirac operator have the simple form

$$s^\vee(x,y) = i\partial\!\!\!/_x\, S^\vee_{m^2=0}(x,y)\,, \qquad s^\wedge(x,y) = i\partial\!\!\!/_x\, S^\wedge_{m^2=0}(x,y)\,. \tag{2.5.12}$$

The Green's functions with interaction are given in analogy to (2.3.11) by the perturbation series

$$\tilde{s}^\vee = \sum_{k=0}^\infty (-s^\vee\,B)^k s^\vee\,, \qquad \tilde{s}^\wedge = \sum_{k=0}^\infty (-s^\wedge\,B)^k s^\wedge\,. \tag{2.5.13}$$

We remark that this perturbation expansion around zero mass is most convenient, but not essential for the light-cone expansion; see [**F5**] for a light cone expansion of a massive Dirac sea.

Our first goal is to perform the light-cone expansion of each Feynman diagram in (2.5.13). Using an inductive construction based on Lemma 2.5.2 below, this will give us the result of Theorem 2.5.3. Since the construction is exactly the same for the advanced and retarded Green's functions, we will omit all superscripts '$\vee$' and '$\wedge$'. The formulas for the advanced and retarded Green's functions are obtained by adding either superscripts '$\vee$' or '$\wedge$' to all operators s and S. For the mass expansion of the operator S_{m^2}, we set $a = m^2$ and introduce the notation

$$S^{(l)} \;=\; \left(\frac{d}{da}\right)^l S_{a|a=0} \qquad\qquad (l \geq 0). \tag{2.5.14}$$

Let us derive some computation rules for the $S^{(l)}$. S_a satisfies the defining equation of a Klein-Gordon Green's function

$$(-\Box_x - a)\, S_a(x,y) \;=\; \delta^4(x-y)\,.$$

Differentiating with respect to a yields

$$-\Box_x S^{(l)}(x,y) \;=\; \delta_{l,0}\, \delta^4(x-y) \;+\; l\, S^{(l-1)}(x,y) \qquad\qquad (l \geq 0). \tag{2.5.15}$$

For $l = 0$, this formula does not seem to make sense because $S^{(-1)}$ is undefined. However, the expression is meaningful if one keeps in mind that in this case the factor l is zero, and thus the whole second summand vanishes. We will also use this convention in the following calculations. Next, we differentiate the formula for S_a in momentum space,

$$S_a^\vee(p) \;=\; \frac{1}{p^2 - a - i\varepsilon p^0}\,, \qquad\qquad S_a^\wedge(p) \;=\; \frac{1}{p^2 - a + i\varepsilon p^0}\,, \tag{2.5.16}$$

with respect to both p and a. Comparing the results gives the relation

$$\frac{\partial}{\partial p^k} S_a(p) \;=\; -2p_k\, \frac{d}{da} S_a(p)\,,$$

or, after expanding in the parameter a,

$$\frac{\partial}{\partial p^k} S^{(l)}(p) \;=\; -2p_k\, S^{(l+1)}(p) \qquad\qquad (l \geq 0). \tag{2.5.17}$$

This formula also determines the derivatives of $S^{(l)}$ in position space. Namely,

$$\begin{aligned}
\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;&=\; \int \frac{d^4p}{(2\pi)^4}\, S^{(l)}(p)\, (-ip_k)\, e^{-ip(x-y)} \\
&\overset{(2.5.17)}{=}\; \frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, \frac{\partial}{\partial p^k} S^{(l-1)}(p)\, e^{-ip(x-y)} \\
&=\; -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, S^{(l-1)}(p)\, \frac{\partial}{\partial p^k} e^{-ip(x-y)} \\
&=\; \frac{1}{2}\, (y-x)_k\, S^{(l-1)}(x,y) \qquad\qquad (l \geq 1).
\end{aligned} \tag{2.5.18}$$

We iterate this relation to compute the Laplacian,

$$\begin{aligned} -\Box_x S^{(l)}(x,y) &= -\frac{1}{2}\,\frac{\partial}{\partial x^k}\Big((y-x)^k\, S^{(l-1)}(x,y)\Big) \\ &= 2\, S^{(l-1)}(x,y) \;-\; \frac{1}{4}\,(y-x)^2\, S^{(l-2)}(x,y) \qquad (l \geq 2). \end{aligned}$$

After comparing with (2.5.15), we conclude that

$$(y-x)^2\, S^{(l)}(x,y) \;=\; -4l\, S^{(l+1)}(x,y) \qquad (l \geq 0). \tag{2.5.19}$$

Furthermore, $S^{(l)}(x,y)$ is only a function of $(y-x)$, and thus

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;=\; -\frac{\partial}{\partial y^k} S^{(l)}(x,y) \qquad (l \geq 0). \tag{2.5.20}$$

Finally, it is convenient to use the identity (2.5.18) also in the case $l = 0$ and to use it as the definition of S^{-1},

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;=\; \frac{1}{2}\,(y-x)_k\, S^{(l-1)}(x,y) \qquad (l \geq 0). \tag{2.5.21}$$

Notice that $S^{(-1)}$ itself remains undefined, only the combination $(y-x)_k S^{(-1)}$ is given a mathematical meaning as the partial derivative of the distribution $2S^{(0)}$.

The next lemma gives the light-cone expansion of an operator product where a potential V is sandwiched between two mass-derivatives of the Green's function. This expansion is the key for the subsequent iterative light-cone expansion of all Feynman diagrams. We always assume without saying that the potentials satisfy the regularity conditions of Lemma 2.2.2.

LEMMA 2.5.2. *The operator product $S^{(l)}\, V\, S^{(r)}$ with $l, r \geq 0$ has the light-cone expansion*

$$\begin{aligned} (S^{(l)}\, V\, S^{(r)})(x,y) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n \\ &\qquad \times (\Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y)\,. \end{aligned} \tag{2.5.22}$$

The fact that line integrals appear in this lemma can be understood in analogy to the method of *integration along characteristics* (see e.g. [**Ta, Fl**]) for a solution of an inhomogeneous wave equation (for a more detailed discussion of this point see [**F5**]). The advantage of the above lemma is that it gives a whole series of line integrals. A further difference is that the left side is an operator product, making it unnecessary to specify initial or boundary values.

Proof of Lemma 2.5.2. Our method is to first compute the Laplacian of both sides of (2.5.22). Comparing the structure of the resulting formulas, it will be possible to proceed by induction in l.

On the left side of (2.5.22), we calculate the Laplacian with the help of (2.5.15) to

$$-\Box_x (S^{(l)}\, V\, S^{(r)})(x,y) \;=\; \delta_{l,0}\, V(x)\, S^{(r)}(x,y) \;+\; l\,(S^{(l-1)}\, V\, S^{(r)})(x,y)\,. \tag{2.5.23}$$

The Laplacian of the integral on the right side of (2.5.22) can be computed with (2.5.21) and (2.5.15),

$$\begin{aligned}
&-\Box_x \int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r+1)}(x,y) \qquad (2.5.24)\\
&= -\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^n\,(\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r+1)}(x,y)\\
&\quad -\int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\\
&\qquad\qquad \times (\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)^k\; S^{(n+l+r)}(x,y)\\
&\quad +(n+l+r+1)\\
&\qquad \times \int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n (\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r)}(x,y)\,.
\end{aligned}$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α and integrate by parts,

$$\begin{aligned}
&\int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,(\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)^k\\
&= \int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,\frac{d}{d\alpha}(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&= -\delta_{n,0}\,\delta_{l,0}\,V(x)\\
&\quad -(n+l)\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&\quad +(n+r+1)\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&= -\delta_{n,0}\,\delta_{l,0}\,V(x)\\
&\quad -n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&\quad +(n+l+r+1)\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\\
&\quad -l\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,.
\end{aligned}$$

We substitute back into the original equation to obtain

$$\begin{aligned}
(2.5.24) \;&=\; \delta_{n,0}\,\delta_{l,0}\,V(x)\,S^{(r)}(x,y)\\
&\quad +l\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r)}(x,y)\\
&\quad -\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^n\,(\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r+1)}(x,y)\\
&\quad +n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\; S^{(n+l+r)}(x,y)\,.
\end{aligned}$$

After dividing by $n!$ and summing over n, the last two summands are telescopic and cancel each other. Thus we get

$$\begin{aligned} -\Box \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y) \\ = \;\; \delta_{l,0}\, V(x)\, S^{(r)}(x,y) \;+\; l \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^{l-1}\, (1-\alpha)^r\, (\alpha-\alpha^2)^n \\ \times\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, d\alpha\; S^{(n+l+r)}(x,y)\,. \end{aligned} \tag{2.5.25}$$

We now compare the formulas (2.5.23) and (2.5.25) for the Laplacian of both sides of (2.5.22). In the special case $l = 0$, these formulas coincide, and we can use a uniqueness argument for the solutions of the wave equation to prove (2.5.22): We assume that we consider the advanced Green's function (for the retarded Green's function, the argument is analogous). For given y, we denote the difference of both sides of (2.5.22) by $F(x)$. Since the support of $F(x)$ is in the past light cone $x \in L_y^\wedge$, F vanishes in a neighborhood of the hypersurface $\mathcal{H} = \{z \in \mathbb{R}^4 \,|\, z^0 = y^0 + 1\}$. Moreover, the Laplacian of F is identically equal to zero according to (2.5.23) and (2.5.25). We conclude that

$$\Box F \;=\; 0 \qquad \text{and} \qquad F_{|\mathcal{H}} \;=\; \partial_k F_{|\mathcal{H}} \;=\; 0\,.$$

Since the wave equation has a unique solution for given initial data on the Cauchy surface $\mathcal{H}$, F vanishes identically.

The general case follows by induction in l: Suppose that (2.5.22) holds for given $\hat{l}$ (and arbitrary r). Then, according to (2.5.23, 2.5.25) and the induction hypothesis, the Laplacian of both sides of (2.5.22) coincides for $l = \hat{l} + 1$. The above uniqueness argument for the solutions of the wave equation again gives (2.5.22). ∎

The above lemma can be used iteratively for the light-cone expansion of more complicated operator products. To explain the method, we look at the example of three factors $S^{(0)}$ and two potentials V, W,

$$(S^{(0)}\, V\, S^{(0)}\, W\, S^{(0)})(x,y) \;=\; \int d^4z\, S^{(0)}(x,z)\, V(z)\, (S^{(0)}\, W\, S^{(0)})(z,y)\,. \tag{2.5.26}$$

Having split up the operator product in this form, we can apply Lemma 2.5.2 to the factor $S^{(0)} W S^{(0)}$,

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4z\, S^{(0)}(x,z) \left\{ V(z) \int_0^1 (\alpha-\alpha^2)^n\, (\Box^n W)_{|\alpha y+(1-\alpha)z}\, d\alpha \right\} S^{(n+1)}(z,y)\,.$$

Now we rewrite the z-integral as the operator product $(S^{(0)} g_y S^{(0)})(x,y)$, where $g_y(z)$ is the function in the curly brackets. The y-dependence of g_y causes no problems because we can view y as a fixed parameter throughout. Thus we can simply apply Lemma 2.5.2 once again to obtain

$$\begin{aligned} = \;\; \sum_{m,n=0}^{\infty} \frac{1}{m!\, n!} \int_0^1 d\beta\, (1-\beta)^{n+1}\, (\beta-\beta^2)^m \int_0^1 d\alpha\, (\alpha-\alpha^2)^n \\ \times\, \Box_z^m \left(V(z)\, (\Box^n W)_{|\alpha y+(1-\alpha)z} \right)_{|z=\beta y+(1-\beta)x}\; S^{(m+n+2)}(x,y)\,. \end{aligned}$$

Now the Laplacian $\Box_z^m$ can be carried out with the Leibniz rule. Notice that the manipulations of the infinite series are not problematic because the number of terms is finite to every order on the light cone.

The Feynman diagrams in (2.5.13) can be expanded with this iterative method, but they are a bit more difficult to handle. One complication is that pulling the Dirac matrices out of the Green's functions according to (2.5.12) gives one additional partial derivative per Green's function. However, this causes no major problems, because these partial derivatives can be carried out after each induction step by differentiating through the light-cone expansion of Lemma 2.5.2. Another issue is to keep track of the *chirality* of the potentials. To this end, one must use that the zero mass Green's function s and the factors $\not{A}_{L/R}$ are odd, whereas the dynamical mass matrices are even. Thus the chirality of the potentials changes each time a dynamical mass matrix appears, as in the example of the operator product

$$\chi_L \, s \, \not{A}_L \, s \cdots s \, \not{A}_L \, s \, Y_L \, s \, \not{A}_R \, s \cdots s \, \not{A}_R \, s \, Y_R \, s \, \not{A}_L \, s \cdots . \tag{2.5.27}$$

The last difficulty is that partial derivatives inside the line integrals may be contracted with a factor $(y-x)$, like for example in the expression

$$\int_0^1 (y-x)^j \, \partial_j V_{|\alpha y + (1-\alpha) x} \, d\alpha \,.$$

Such derivatives act in direction of the line integral and are thus called *tangential*. Writing them as derivatives with respect to the integration variable, we can integrate by parts, e.g.

$$\int_0^1 (y-x)^j \, \partial_j V_{|\alpha y + (1-\alpha) x} \, d\alpha \;=\; \int_0^1 \frac{d}{d\alpha} V(\alpha y + (1-\alpha) x) \, d\alpha \;=\; V(y) - V(x) \,.$$

Going through the calculations and the combinatorics in detail, one finds that with such integrations by parts we can indeed get rid of all tangential derivatives, and one ends up with terms of th following structure (for the proof see [**F6**]).

THEOREM 2.5.3. **(light-cone expansion of the k^{th} order Feynman diagram)** *Using a multi-index notation and the abbreviation*

$$\int_x^y [l, r \,|\, n] \, f(z) \, dz \;:=\; \int_0^1 d\alpha \; \alpha^l \, (1-\alpha)^r \, (\alpha - \alpha^2)^n \; f(\alpha y + (1-\alpha) x) \,, \tag{2.5.28}$$

the light-cone expansion of the k^{th} order contribution to the perturbation series (2.5.13) can be written as an infinite sum of expressions of the form

$$\begin{aligned} \chi_c \, C \, (y-x)^K \, W^{(0)}(x) \int_x^y [l_1, r_1 \,|\, n_1] \, dz_1 \, W^{(1)}(z_1) \int_{z_1}^y [l_2, r_2 \,|\, n_2] \, dz_2 \, W^{(2)}(z_2) \\ \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \,|\, n_\alpha] \, dz_\alpha \, W^{(\alpha)}(z_\alpha) \, \gamma^J \, S^{(h)}(x,y) \quad , \; \alpha \leq k \,. \end{aligned} \tag{2.5.29}$$

Here the factors $W^{(\beta)}$ are composed of the potentials and their partial derivatives,

$$W^{(\beta)} \;=\; (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} V^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) \cdots (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} V^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) \tag{2.5.30}$$

with $a_1 = 1$, $a_{\beta+1} = b_\beta + 1$, $b_\beta \geq a_\beta - 1$ (in the case $b_\beta = a_\beta - 1$, $W^{(\beta)}$ is identically equal to one) and $b_\alpha = k$. Furthermore, $c, c_a \in \{L, R\}$ are chiral indices, C is a complex number, and the parameters l_a, r_a, n_a, and p_a are non-negative integers. The

functions $V^{(a)}_{J_a,c_a}$ coincide with any of the individual potentials in (2.5.11) with chirality c_a, i.e.

$$\begin{aligned} V^{(a)}_{c_a} &= A_{c_a} \qquad \textit{(in which case } |J_a|=1\textit{)} \quad \textit{or} \\ V^{(a)}_{c_a} &= m Y_{c_a} \qquad \textit{(in which case } |J_a|=0\textit{)}\,. \end{aligned} \tag{2.5.31}$$

The chirality c_a of the potentials is determined by the following rules:

(i) The chirality c_1 of the first potential coincides with the chirality of the factor χ_c.

(ii) The chirality of the potentials is reversed at every mass matrix, i.e.

$$c_a \text{ and } c_{a+1} \quad \begin{cases} \textit{coincide} & \textit{if } V^{(a)}_{c_a} = A_{c_a} \\ \textit{are opposite} & \textit{if } V^{(a)}_{c_a} = m Y_{c_a}\,. \end{cases}$$

The tensor indices of the multi-indices are all contracted with each other, according to the following rules:

(a) No two tensor indices of the same multi-index are contracted with each other.

(b) The tensor indices of the factor γ^J are all contracted with different multi-indices.

(c) The tensor indices of the factor $(y-x)^K$ are all contracted with the tensor indices of the factors $V^{(a)}_{J_a}$ or γ^J, but not with the partial derivatives ∂^{K_a}.

To every order h on the light cone, the number of terms of the form (2.5.29) is finite. Furthermore,

$$2h \;=\; k - 1 - |K| + \sum_{a=1}^{k} (|K_a| + 2p_a)\,. \tag{2.5.32}$$

The rules *(i)* and *(ii)* correspond precisely to our observation that the chirality changes at each dynamical mass matrix (2.5.27). The restrictions *(a)* and *(b)* on the possible contractions of tensor indices prevent an abuse of our multi-index notation. More precisely, *(a)* avoids factors $(y-x)^2$ in $(y-x)^I$, an unnecessary large number of γ-matrices in γ^J and "hidden" Laplacians inside the partial derivatives $\partial^{I_a}_{z_a}$. The rule *(b)*, on the other hand, prevents factors $(y-x)^2$ and hidden Laplacians in combinations of the form $(y-x)_i\,(y-x)_j\,\gamma^i\,\gamma^j$ and $\partial_{ij} V^{(a)}_{J_a}\,\gamma^i\,\gamma^j$, respectively. The rule *(c)* means that no tangential derivatives appear. The rules *(a)–(c)* imply that the tensor indices of the multi-index K are all contracted with the chiral potentials, except for one index which may be contracted with the factor γ^J. Since at most k chiral potentials appear, we get the inequality $|K| \leq k+1$. Using this inequality in (2.5.32) we get the bound

$$h \;\geq\; -1 + \frac{1}{2} \sum_{a=1}^{k} (|K_a| + 2p_a)\,. \tag{2.5.33}$$

This shows that h never becomes smaller than -1 and that derivatives of the potentials increase the order on the light cone. In the case $h = -1$, it follows from (2.5.32) that $|K| \geq 1$, meaning that at least one factor $(y-x)$ appears in (2.5.29). We conclude that the factor $S^{(h)}$ in (2.5.29) is always well-defined by either (2.5.14) or (2.5.21).

So far the Green's function was described perturbatively by a sum of Feynman diagrams (2.5.13). In order to get from this perturbative description to *non-perturbative* formulas of the light-cone expansion, we shall now carry out the sum over all Feynman diagrams to any fixed order on the light cone. This procedure is called *resummation* of

the light-cone expansion. In order to give a first idea of how the resummation works, we consider the leading singularity on the light cone by neglecting all terms of the order $\mathcal{O}((y-x)^{-2})$. According to (2.5.7), we need to take into account only the contributions (2.5.29) with $h = -1$. The inequality (2.5.33) implies that no derivatives of the potentials appear. Moreover, we obtain from (2.5.32) that $|K| = k+1$. Again using the contraction rules *(a)–(c)*, we conclude that one tensor index of the multi-index K is contracted with a Dirac matrix, whereas the remaining k indices of K are all contracted with chiral potentials. Therefore, all k potentials are chiral, and no dynamical mass matrices appear. A detailed calculation yields for the k^{th} order Feynman diagram a term of precisely this structure,

$$\begin{aligned}\chi_c\,((-s\,B)^k s)(x,y) \;&=\; \chi_c\,(-i)^k \int_x^y dz_1\;(y-x)_{j_1}\,A_c^{j_1}(z_1) \\ &\times \int_{z_1}^y dz_2\;(y-z_1)_{j_2}\,A_c^{j_2}(z_2) \cdots \int_{z_{k-1}}^y dz_k\;(y-z_k)_{j_2}\,A_c^{j_k}(z_k)\;s(x,y) \\ &+ \mathcal{O}((y-x)^{-2})\,,\end{aligned}$$

where we used for the line integrals the short notation

$$\int_x^y f(z)\,dz \;:=\; \int_x^y [0,0\,|\,0]\,f(z)\,dz \;=\; \int_0^1 f(\alpha y + (1-\alpha)x)\,d\alpha\,. \tag{2.5.34}$$

The obtained nested line integrals can be identified with the summands of the familiar Dyson series. This allows us to carry out the sum over all Feynman diagrams,

$$\chi_c\,\tilde{s}(x,y) \;=\; \chi_c\,\text{Pexp}\left(-i\int_x^y (y-x)_j\,A_c^j(z)\,dz\right) s(x,y) \;+\; \mathcal{O}((y-x)^{-2})\,, \tag{2.5.35}$$

where we again used the notation (2.5.34) and the following definition of Pexp.

DEF. 2.5.4. *For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the* **ordered exponential** $\text{Pexp}(\int F(\alpha)\,d\alpha)$ *is given by the Dyson series*

$$\begin{aligned}\text{Pexp}\left(\int_a^b F(\alpha)\,d\alpha\right) \;&=\; \mathbb{1} \;+\; \int_a^b dt_0\,F(t_0)\,dt_0 \;+\; \int_a^b dt_0\,F(t_0)\int_{t_0}^b dt_1\,F(t_1) \\ &+ \int_a^b dt_0\,F(t_0)\int_{t_0}^b dt_1\,F(t_1)\int_{t_1}^b dt_2\,F(t_2) \;+\; \cdots\,.\end{aligned} \tag{2.5.36}$$

The appearance of the ordered exponential in (2.5.35) can be understood from the local gauge invariance. We explain this relation for simplicity in the example of dynamical mass matrices and chiral potentials of the form

$$Y_L \;=\; Y_R \;=\; 0\,, \qquad A_L(x) \;=\; A_R(x) \;=\; iU(x)\,(\partial U^{-1}(x))\,,$$

where $U = U^{(a\alpha)}_{(b\beta)}$ is a unitary matrix on the sectors and generations. In this case, the Dirac operator is related to the free Dirac operator simply by a local unitary transformation,

$$i\partial\!\!\!/ + B \;=\; U\,i\partial\!\!\!/\,U^{-1}\,.$$

Interpreting this local transformation as in §1.4 as a gauge transformation (1.4.3), we can say that the external potential can be transformed away globally by choosing a suitable gauge. Using the well-known behavior of the Green's function under gauge transformations, we obtain the simple formula

$$\tilde{s}(x,y) \;=\; U(x)\,s(x,y)\,U^{-1}(y)\,. \tag{2.5.37}$$

Let us verify that this is consistent with (2.5.35). Setting $V(\alpha) = U(\alpha y + (1-\alpha)x)$ and using the relation $V(V^{-1})' = (VV^{-1})' - V'V^{-1} = -V'V^{-1}$, we can write the integrand of the ordered exponential as

$$-i(y-x)_j\, A^j_c(z) \;=\; -V'(\alpha)\, V^{-1}(\alpha)\,. \tag{2.5.38}$$

Differentiating (2.5.36) with respect to a as well as evaluating it for $a = b$, one sees that the ordered exponential can be characterized as the solution of the initial value problem

$$\frac{d}{da}\text{Pexp}\left(\int_a^b F\right) \;=\; -F(a)\,\text{Pexp}\left(\int_a^b F\right), \qquad \text{Pexp}\left(\int_b^b F\right) \;=\; \mathbb{1}\,.$$

In the case $F = -V'V^{-1}$, it is easily verified that the solution to this initial value problem is

$$\text{Pexp}\left(-\int_a^b V'(\alpha)\, V^{-1}(\alpha)\, d\alpha\right) \;=\; V(a)\, V^{-1}(b)\,.$$

Using (2.5.38), we conclude that

$$\text{Pexp}\left(-i\int_x^y (y-x)_j\, A^j_c(z)\, dz\right) \;=\; U(x)\, U^{-1}(y)\,.$$

Thus the ordered exponential in (2.5.35) gives precisely the factor $U(x)\, U(y)^{-1}$ needed for the correct behavior under gauge transformations (2.5.37).

To higher order on the light cone, the situation clearly is more complicated. Nevertheless, it is very helpful to imagine that after rearranging the summands of the light-cone expansion in a suitable way, certain subseries can be summed up explicitly giving rise to ordered exponentials of the chiral potentials. As in (2.5.27), the chirality of the potentials should change each time a dynamical mass matrix appears. This conception is made precise by the following definition and theorem, proving that the light-cone expansion of the Green's function can be obtained to any given order on the light cone by taking a finite number of terms of the form (2.5.28) and inserting suitable ordered exponentials.

DEF. 2.5.5. *A contribution (2.5.28) to the light-cone expansion of Theorem 2.5.3 is called* **phase-free** *if all the tangential potentials $V^{(a)}_{J_a}$ are differentiated, i.e.*

$$|K_a| + 2p_a > 0 \qquad \textit{whenever} \qquad J_a \textit{ is contracted with } (y-x)^K.$$

From every phase-free contribution the corresponding **phase-inserted** *contribution is obtained as follows: We insert ordered exponentials according to the replacement rule*

$$W^{(\beta)}(z_\beta) \;\longrightarrow\; W^{(\beta)}(z_\beta)\,\text{Pexp}\left(-i\int_{z_\beta}^{z_{\beta+1}} A^{j_\beta}_{c_\beta}\,(z_{\beta+1}-z_\beta)\right), \quad \beta = 0,\ldots,\alpha\,,$$

where we set $z_0 = x$ and $z_{\alpha+1} = y$. The chiralities c_β are determined by the relations $c_0 = c$ and

$$c_{\beta-1} \textit{ and } c_\beta \left\{\begin{matrix} \textit{coincide} \\ \textit{are opposite}\end{matrix}\right\} \textit{ if } W^{(\beta-1)} \textit{ contains an} \left\{\begin{matrix}\textit{even}\\ \textit{odd}\end{matrix}\right\} \textit{number of factors } Y_\cdot.$$

THEOREM 2.5.6. *To every order on the light cone, the number of phase-free contributions is finite. The light-cone expansion of the Green's function $\tilde{s}(x,y)$ is given by the sum of the corresponding phase-inserted contributions.*

For the proof we refer to [**F6**]. In short, the first statement follows directly from the contraction rules *(a)–(c)* and (2.5.32), whereas for the second part one uses for fixed x and y the behavior of the Green's function under non-unitary local transformations of the spinors.

Our constructions have led to a convenient procedure for performing the light-cone expansion of the Green's function. One only needs to compute to any order on the light cone the finite number of phase-free contributions. Then one inserts ordered exponentials according to Def. 2.5.5. For the computation of the phase-free contributions we use a computer program. Appendix B lists those phase-free contributions which will be of relevance in Chapters 6–8.

In the remainder of this section we describe how the above methods can be adapted to the *fermionic projector*. We begin for simplicity with the fermionic projector corresponding to one Dirac sea in the vacuum (2.2.1). Similar to (2.5.4) we pull out the Dirac matrices,

$$P^{\text{sea}}(x,y) \;=\; (i\partial\!\!\!/_x + m)\, T_{m^2}(x,y)\,, \tag{2.5.39}$$

where T_{m^2} is the Fourier transform of the lower mass shell,

$$T_{m^2}(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{2.5.40}$$

Computing this Fourier integral and expanding the resulting Bessel functions gives

$$\begin{aligned} T_{m^2}(x,y) \;=\; & -\frac{1}{8\pi^3}\left(\frac{\text{PP}}{\xi^2} \;+\; i\pi\delta(\xi^2)\,\varepsilon(\xi^0)\right) \\ & + \frac{m^2}{32\pi^3}\left(\log|m^2\xi^2| + C_E + i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0)\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j} \\ & - \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j}\,(\Phi(j+1)+\Phi(j))\,. \end{aligned} \tag{2.5.41}$$

Here $\xi \equiv y-x$, $C_E = 2C - 2\log 2$ with Euler's constant C, and Φ is the function

$$\Phi(0) = 0\,, \qquad \Phi(n) = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \geq 1\,.$$

Similar to (2.5.6), this expansion involves distributions which are singular on the light cone. But in addition to singularities $\sim \delta((y-x)^2$ and $\sim \Theta((y-x)^2$, now also singularities of the form form $\text{PP}/(y-x)^2$ and $\log|(y-x)^2|$ appear. In particular, T_{m^2} is *not causal* in the sense that $\operatorname{supp} T_{m^2}(x,.) \not\subset L_x$. Another similarity to (2.5.6) is that power series in $m^2(y-x)^2$ appear. This suggests that in analogy to (2.5.7), the higher orders in m^2 should be of higher order on the light cone. However, due to the term $\log|m^2\xi^2|$, the distribution T_{m^2} is *not* a power series in m^2. This means that the higher mass derivatives of T_{m^2} do not exist, and the analog of (2.5.7) breaks down. This so-called *logarithmic mass problem* reflects a basic infrared problem in the light-cone expansion of the fermionic projector. In the vacuum, it can be resolved with the following simple construction. We subtract the problematic $\log|m^2|$-terms by setting

$$T^{\text{reg}}_{m^2}(x,y) \;=\; T_{m^2}(x,y) \;-\; \frac{m^2}{32\pi^3}\,\log|m^2| \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2\xi^2)^j}{4^j}\,. \tag{2.5.42}$$

This distribution is a power series in m^2, and in analogy to (2.5.14) we can set

$$T^{(l)} \;=\; \left(\frac{d}{da}\right)^l T_a^{\text{reg}}|_{a=0} \qquad\qquad (l \geq 0). \tag{2.5.43}$$

Furthermore, we introduce $T^{(-1)}$ similar to (2.5.21) as the distributional derivative of $T^{(0)}$. Similar to (2.5.7),

$$T^{(n)}(x,y) \qquad \text{is of the order} \quad \mathcal{O}((y-x)^{2n-2})\,,$$

and thus the mass expansion of $T^{\text{reg}}_{m^2}$ gives us its light-cone expansion. The point is that the difference of T_{m^2} and $T^{\text{reg}}_{m^2}$,

$$T_{m^2} - T^{\text{reg}}_{m^2} \;=\; \frac{m^2}{32\pi^3}\, \log|m^2| \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\, \frac{(m^2\xi^2)^j}{4^j}\,,$$

is an absolutely convergent power series in ξ^2 and is thus a *smooth* function in position space. This smooth contribution is of no relevance as long as the singularities on the light cone are concerned. This leads us to write the fermionic projector in the form

$$P^{\text{sea}}(x,y) \;=\; \sum_{n=0}^{\infty} \frac{m^{2n}}{n!} (i\partial\!\!\!/_x + m)\, T^{(n)}(x,y) \;+\; P^{\text{lc}}(x,y)\,, \tag{2.5.44}$$

with

$$P^{\text{lc}} \;:=\; (i\partial\!\!\!/_x + m)\left(T_{m^2} - T^{\text{reg}}_{m^2}\right).$$

The series in (2.5.44) is a light-cone expansion which completely describes the singular behavior of the fermionic projector on the light cone. The so-called *low-energy contribution* P^{lc}, on the other hand, is a smooth function.

This method of performing a light-cone expansion modulo smooth functions on the light cone also works for the general fermionic projector with interaction. But the situation is more complicated and at the same time more interesting, in particular because the space-time dependence of the involved external potentials reveals the causal structure of the fermionic projector. The first construction step is to use the identity on the left side of (2.3.12) to carry over the light-cone expansion from the Green's function to the distribution $\tilde{k}$. Next, comparing (2.2.6) with the formula

$$\begin{aligned} p_m(k) &= (k\!\!\!/ + m)\, \delta(k^2 - m^2) \\ &= \frac{1}{2\pi i}\, (k\!\!\!/ + m)\, \lim_{\varepsilon \searrow 0} \left[\frac{1}{k^2 - m^2 - i\varepsilon} \;-\; \frac{1}{k^2 - m^2 + i\varepsilon} \right], \end{aligned}$$

one sees that the distributions p and k differ from each only by the $i\varepsilon$-regularization in momentum space. The key step of the construction is the so-called *residual argument*, which relates the light-cone expansion of $\tilde{k}$ to an expansion in momentum space. Using that the latter expansion remains unchanged if the poles of the distributions are suitably shifted in momentum space, one obtains the light-cone expansion for an operator $\tilde{p}^{\text{res}}$, which can be regarded as a perturbation of p, but with a different combinatorics than in the expansion of Theorem 2.3.1. More precisely, $\tilde{p}$ can be obtained from $\tilde{p}^{\text{res}}$ by replacing pairs of factors k in the perturbation expansion by corresponding factors p. The argument (2.4.2) shows that the difference $\tilde{p} - \tilde{p}^{\text{res}}$ vanishes in the static case, and more generally one can say that $\tilde{p} - \tilde{p}^{\text{res}}$ will be of

significance only if the frequency of the external potentials is of the order of the mass gap. Therefore, the operator

$$P^{\text{he}} := \frac{1}{2} X (\tilde{p} - \tilde{p}^{\text{res}})$$

is called the *high-energy contribution* to the fermionic projector. Moreover, resolving the logarithmic mass problem we obtain again a *low-energy contribution* P^{le}. We thus obain a representation of the fermionic projector of the form

$$\begin{aligned} P^{\text{sea}}(x,y) &= \sum_{n=-1}^{\infty} (\text{phase-inserted line integrals}) \times T^{(n)}(x,y) \\ &\quad + P^{\text{le}}(x,y) + P^{\text{he}}(x,y) \,. \end{aligned} \tag{2.5.45}$$

Here the series is a light-cone expansion which describes the singular behavior of the fermionic projector on the light cone non-perturbatively. It is obtained from the light-cone expansion of the Green's functions by the simple replacement rule

$$S^{(n)} \longrightarrow T^{(n)} \,.$$

In particular, the phase-inserted line integrals are exactly the same as those for the Green's functions (see Def. 2.5.5). The contributions P^{le} and P^{he}, on the other hand, are given perturbatively by a series of terms which are all smooth on the light cone (we expect that the perturbation series for P^{le} and P^{le} should converge, but this has not yet been proven). The "causality" of the causal perturbation expansion can be understood from the fact that the phase-inserted line integrals in (2.5.45) are all bounded integrals along the line segment joining the points x and y (whereas the light-cone expansion of general operator products involves unbounded line integrals [**F1**]). In particular, when y lies in the causal future or past of x, the light-cone expansion in (2.5.45) depends on the external potential only inside the "diamond" $(J_x^\vee \cap J_y^\wedge) \cup (J_x^\wedge \cap J_y^\vee)$. Nevertheless, the light-cone expansion is not causal in the strict sense because there are contributions for $y \notin J_x$. Furthermore, the low- and high-energy contributions cannot be described with line integrals and violate locality as well as causality. This non-locality can be understood from the fact that the fermionic projector is a global object in space-time (see the discussion in §2.4). Mathematically, it is a consequence of the non-local operation of taking the absolute value of an operator (2.2.17) in the definition of the fermionic projector. We conclude that the singular behavior of the fermionic projector on the light-cone can be described explicitly by causal line integrals, whereas the smooth contributions to the fermionic projector are governed by non-local effects.

Inspecting the explicit formulas of Appendix B, one sees immediately that from the line integrals of the light-cone expansion one can reconstruct the chiral and scalar/pseudoscalar potentials. In this sense, P^{sea} encodes all information on the external potential. Furthermore, the fermionic projector gives via its representation (2.3.19) all information on the fermions and anti-fermions of the system. We thus come to the important conclusion that the fermionic projector describes the physical system completely.

2.6. Normalization of the Fermionic States

In §2.2 we disregarded that the fermionic states are in general not normalizable in infinite volume. We avoided this problem using a δ-normalization in the mass

parameter (see e.g. (2.2.26)). In this section we will analyze the normalization in detail by considering the system in finite volume and taking the infinite volume limit. Apart from justifying the formalism introduced in §2.2, this will clarify in which sense the fermionic projector is idempotent (Theorem 2.6.1). Furthermore, we will see that the probability integral has a well-defined infinite volume limit (Theorem 2.6.2), and this will also determine the normalization constant c_{norm} in (2.3.19) (see (2.6.24)). We postpone the complications related to the chiral fermions to Appendix C and thus assume here that the chiral asymmetry matrix $X = \mathbb{1}$. We work again in the setting of §2.3 and assume that the external potential $\mathcal{B}$ satisfies the regularity assumptions of Lemma 2.2.2. Furthermore, we make the physically reasonable assumption that the *masses* are *non-degenerate in the generations*, meaning that

$$m_{a\alpha} \neq m_{a\beta} \qquad \text{for all } a \text{ and } \alpha \neq \beta\,. \tag{2.6.1}$$

In order to ensure that all normalization integrals are finite, we need to introduce an *infrared regularization*. For clarity, we explain the construction for a single Dirac sea of mass m in the vacuum. First, we replace space by the three-dimensional box

$$T^3 \;=\; [-l_1, l_1] \times [-l_2, l_2] \times [-l_3, l_3] \qquad \text{with } 0 < l_i < \infty \tag{2.6.2}$$

and set $V = |T^3| = 8\, l_1 l_2 l_3$. We impose periodic boundary conditions; this means that we restrict the momenta $\vec{k}$ to the lattice L^3 given by

$$L^3 \;=\; \frac{\pi\,\mathbb{Z}}{l_1} \times \frac{\pi\,\mathbb{Z}}{l_2} \times \frac{\pi\,\mathbb{Z}}{l_3} \;\subset\; \mathbb{R}^3\,.$$

In order to carry over the operators p_m, k_m and s_m (see §2.2) to finite volume, we leave the distributions in momentum space unchanged. In the transformation to position space, we replace the Fourier integral over 3-momentum by a Fourier series according to

$$\int \frac{d\vec{k}}{(2\pi)^3} \;\longrightarrow\; \frac{1}{V} \sum_{\vec{k} \in L^3}\,. \tag{2.6.3}$$

When taking products of the resulting operators, we must take into account that the spatial integral is now finite. For example, we obtain that

$$\begin{aligned}
(p_m\, p_{m'})(x,y) &= \int_{\mathbb{R}\times T^3} p_m(x,z)\, p_{m'}(z,y)\, d^4z \\
&= \int_{\mathbb{R}\times T^3} d^4z \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, e^{-ik(x-z)} \\
&\qquad \times \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{l}\in L^3} p_{m'}(l)\, e^{-il(z-y)} \\
&= \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, e^{-ikx} \\
&\qquad \times \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{l}\in L^3} p_{m'}(l)\, e^{ily}\, 2\pi\, \delta(k^0 - l^0)\, V\, \delta_{\vec{k},\vec{l}} \\
&= \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k}\in L^3} p_m(k)\, p_{m'}(k)\, e^{-ik(x-y)}
\end{aligned}$$

$$= \quad \delta(m-m')\, p_m(x,y)\;, \tag{2.6.4}$$

where $p_m(k) = (\not{k}+m)\,\delta(k^2-m^2)$. More generally, the calculation rules (2.2.26–2.2.33) for products of the operators k_m, p_m and s_m remain valid in finite 3-volume.

In (2.6.4) we are still using a δ-normalization in the mass parameter. In order to go beyond this formalism and to get into the position where we can multiply operators whose mass parameters coincide, we "average" the mass over a small interval $[m, m+\delta]$. More precisely, we set

$$\bar{p}_m \;=\; \frac{1}{\delta} \int_m^{m+\delta} p_\mu \, d\mu \qquad \text{and} \qquad \bar{k}_m \;=\; \frac{1}{\delta} \int_m^{m+\delta} k_\mu \, d\mu\;. \tag{2.6.5}$$

Then

$$\begin{aligned} \bar{p}_m\, \bar{p}_m \;&=\; \frac{1}{\delta^2} \int_m^{m+\delta} d\mu \int_m^{m+\delta} d\mu'\; p_\mu\, p_{\mu'} \\ &=\; \frac{1}{\delta^2} \int_m^{m+\delta} d\mu \int_m^{m+\delta} d\mu'\; \delta(\mu-\mu')\, p_\mu \;=\; \frac{1}{\delta^2} \int_m^{m+\delta} p_\mu \, d\mu \;=\; \frac{1}{\delta}\, \bar{p}_m\;, \end{aligned}$$

and thus, apart from the additional factor δ^{-1}, $\bar{p}_m$ is idempotent. Similarly, we have the relations

$$\bar{k}_m\, \bar{k}_m \;=\; \frac{1}{\delta}\, \bar{p}_m \qquad \text{and} \qquad \bar{k}_m\, \bar{p}_m \;=\; \bar{p}_m\, \bar{k}_m \;=\; \frac{1}{\delta}\, \bar{k}_m\;.$$

Thus, introducing the *infrared regularized fermionic projector* corresponding to a Dirac sea of mass m by

$$P^{\text{sea}} \;=\; \frac{\delta}{2}\, (\bar{p}_m - \bar{k}_m)\;, \tag{2.6.6}$$

this operator is indeed a projector,

$$(P^{\text{sea}})^* \;=\; P^{\text{sea}} \qquad \text{and} \qquad (P^{\text{sea}})^2 \;=\; P^{\text{sea}}\;. \tag{2.6.7}$$

The *infinite volume limit* corresponds to taking the limits $l_1, l_2, l_3 \to \infty$ and $\delta \searrow 0$.

Let us discuss the above construction. Clearly, our regularization method relies on special assumptions (3-dimensional box with periodic boundary conditions, averaging of the mass parameter). This is partly a matter of convenience, but partly also a necessity, because much more general regularizations would lead to unsurmountable technical difficulties. Generally speaking, infrared regularizations change the system only on the macroscopic scale near spatial infinity and possibly for large times. Due to the decay assumptions on the external potentials in Lemma 2.2.2, in this region the system is only weakly interacting. This should make infrared regularizations insensitive to the details of the regularization procedure, and it is reasonable to expect (although it is certainly not proven) that if the infinite volume limit exists, it should be independent of which regularization method is used. Here we simply take this assumption for granted and thus restrict attention to a special regularization scheme. But at least we will see that the infinite volume limit is independent of how the limits $l_i \to \infty$ and $\delta \searrow 0$ are taken.

It is worth noting that not *every* infrared regularization has a well-defined infinite volume limit. To see this, we consider the example of a regularization in a 4-dimensional box. Restricting the time integral to the finite interval $t \in [-T, T]$, we

obtain

$$\begin{aligned}
&(p_m\, p_m)(x,y) \\
&\quad = \int_{-T}^{T} dt \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\, \vec{k}=\vec{l}} e^{-i(k^0-l^0)t}\, p_m(k)\, p_m(l)\, e^{ikx-ily} \\
&\quad = \int_{-T}^{T} dt \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\, \vec{k}=\vec{l}} e^{-i(k^0-l^0)t} \\
&\qquad\qquad \times (\not{k} + m)\, (\not{l} + m)\, \delta((k^0)^2 - (l^0)^2)\, \delta(k^2 - m^2)\, e^{ikx-ily} \\
&\quad = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{V} \sum_{\vec{k} \in L^3} \frac{mT}{|k^0|}\, p_m(k)\, e^{-ik(x-y)} \;+\; \mathcal{O}(T^0)\,,
\end{aligned}$$

and due to the factor $|k^0|^{-1}$ in the last line, this is not a multiple of $p_m(x,y)$. This problematic factor $|k^0|^{-1}$ also appears under more general circumstances (e.g. when we introduce boundary conditions at $t = \pm T$ and/or take averages of the mass parameter), and thus it seems impossible to arrange that the fermionic projector is idempotent. We conclude that a 4-dimensional box does not give a suitable regularization scheme.

The mass averaging in (2.6.5) leads to the bizarre effect that for fixed $\vec{k}$, a whole continuum of states of the fermionic projector, namely all states with

$$k^0 \in \left[-\sqrt{|\vec{k}|^2 + (m+\delta)^2},\; -\sqrt{|\vec{k}|^2 + m^2} \right], \tag{2.6.8}$$

are occupied. If one prefers to occupy only a finite number of states for every $\vec{k}$, one can achieve this by taking the mass averages for the bra- and ket-states separately. For example, instead of (2.6.6) we could define the fermionic projector by

$$P(x,y) \;=\; \delta \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int_{-\infty}^{\infty} \frac{dl^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k},\vec{l} \in L^3,\, \vec{k}=\vec{l}} \bar{t}_m(k)\, \bar{t}_m(l)\, e^{-ikx+ily} \tag{2.6.9}$$

with $\bar{t}_m = \frac{1}{2}(\bar{p}_m - \bar{k}_m)$. This fermionic projector is for every $\vec{k}$ composed of a finite number of states. Furthermore, it is a projector in the sense of (2.6.7). In contrast to (2.6.6), (2.6.9) is not homogeneous in time, but decays on the scale $t \sim \delta^{-1}$. However, if we restrict attention to a fixed region of space-time for which $t \ll \delta^{-1}$, then (2.6.6) and (2.6.9) differ only by terms of higher order in δ, and therefore we can expect that (2.6.5) and (2.6.7) should have the same infinite volume limit. The definition (2.6.6) has the advantage that it will be easier to introduce the interaction.

After these preparations, we come to the general construction of the fermionic projector in the three-dimensional box T^3. Since we want to "smear out" the mass similar to (2.6.5), the mass parameter needs to be variable. To this end, we introduce a parameter $\mu > 0$ which shifts all masses by the same amount. Thus we describe the system in the vacuum by the Dirac operator

$$i\partial\!\!\!/ - mY - \mu \mathbb{1}\,. \tag{2.6.10}$$

The external field is described by an operator $\mathcal{B}$ in the space-time $\mathbb{R} \times T^3$, which we again insert into the Dirac operator,

$$i\partial\!\!\!/ + \mathcal{B} - mY - \mu \mathbb{1}\,.$$

Now we can introduce the operators p, k and their perturbation expansions exactly as in §2.3. For clarity, we denote the dependence on the parameter μ by an additional index $+\mu$. In particular, we denote the operator products in Theorem 2.3.1 by $\tilde{p}_{+\mu}$ and $\tilde{k}_{+\mu}$. Since the multiplication rules (2.2.26–2.2.33) also hold in finite 3-volume, all the computations of §2.3 are still true. In particular, the operators $\tilde{p}_{+\mu}$ and $\tilde{k}_{+\mu}$ satisfy the δ-normalization conditions[4]

$$\tilde{p}_{+\mu}\, \tilde{p}_{+\mu'} \;=\; \tilde{k}_{+\mu}\, \tilde{k}_{+\mu'} \;=\; \delta(\mu-\mu')\, \tilde{p}_{+\mu} \tag{2.6.11}$$

$$\tilde{p}_{+\mu}\, \tilde{k}_{+\mu'} \;=\; \tilde{k}_{+\mu}\, \tilde{p}_{+\mu'} \;=\; \delta(\mu-\mu')\, \tilde{k}_{+\mu}\,. \tag{2.6.12}$$

In analogy to (2.6.5) and (2.6.6) we define the auxiliary fermionic projector by

$$P^{\text{sea}} \;=\; \frac{1}{2} \int_0^\delta (\tilde{p}_{+\mu} - \tilde{k}_{+\mu})\, d\mu\,, \tag{2.6.13}$$

and the fermionic projector is again obtained by taking the partial trace (2.3.20),

$$(P^{\text{sea}})_b^a \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P^{\text{sea}})_{(b\beta)}^{(a\alpha)}\,. \tag{2.6.14}$$

The next theorem shows that the fermionic projector is idempotent in the infinite volume limit, independent of how the limits $l_i \to \infty$ and $\delta \searrow 0$ are taken.

THEOREM 2.6.1. **(idempotence of the fermionic projector)** *Consider a system composed of massive fermions with non-degenerate masses (2.6.1). Then the fermionic projector defined by (2.6.13) and (2.6.14) satisfies the relations*

$$\int_{\mathbb{R}\times T^3} d^4z \sum_{b=1}^{N} (P^{sea})_b^a(x,z)\, (P^{sea})_c^b(z,y) \;=\; (P^{sea})_c^a(x,y) \;+\; \delta^2\, Q_c^a(x,y)\,,$$

where Q has an expansion as a sum of operators which all have a well-defined infinite volume limit.

Proof. For simplicity we omit the superscript 'sea'. It follows immediately from (2.6.11–2.6.13) that the auxiliary fermionic projector is idempotent,

$$\sum_{b,\beta} P_{(b\beta)}^{(a\alpha)}\; P_{(c\gamma)}^{(b\beta)} \;=\; P_{(c\gamma)}^{(a\alpha)}\,. \tag{2.6.15}$$

Thus it remains to show that

$$\sum_b \sum_{\alpha,\gamma} \sum_{\beta,\beta' \text{ with } \beta\neq\beta'} P_{(b\beta)}^{(a\alpha)}\; P_{(c\gamma)}^{(b\beta)} \;=\; \delta^2\, Q_c^a(x,y)\,. \tag{2.6.16}$$

According to the non-degeneracy assumption (2.6.1), there are constants $c, \delta > 0$ such that for all sufficiently small δ,

$$|(m_{b\beta}+\mu) - (m_{b\beta'}+\mu')| \;\geq\; c \qquad \text{for all } b,\ \beta\neq\beta',\ \text{and } 0<\mu,\mu'<\delta\,. \tag{2.6.17}$$

[4]Online version: As noticed by A. Grotz, these relations are in general violated to higher order in perturbation theory. In order to cure the problem, one needs to rescale the states of the fermionic projector, as is worked out in the paper [**6**] (listed in the references in the preface to the second online edition).

On the left side of (2.6.16) we substitute (2.6.13) and the operator product expansions of Theorem 2.3.1. Using (2.6.17), the resulting operator products are all finite and can be estimated using the relations

$$\int_0^\delta d\mu \int_0^\delta d\mu' \,(\cdots A_{+\mu})^{(a\alpha)}_{(b\beta)} \,\big(A_{+\mu'} \cdots\big)^{(b\beta')}_{(c\gamma)} \;=\; c^{-1}\, \mathcal{O}(\delta^2)\,, \tag{2.6.18}$$

where each factor A stands for p, k or s. This gives (2.6.16). ∎

At this point we can make a remark on the name "partial trace." The notion of a trace suggests that two matrix indices should be set equal and summed over; thus one may want to define the fermionic projector instead of (2.3.20) by[5]

$$P^a_b(x,y) \;=\; \sum_{\alpha=1}^{3} P^{(a\alpha)}_{(b\alpha)}(x,y)\,. \tag{2.6.19}$$

This alternative definition suffers from the following problem. The off-diagonal elements of $P^{(a\alpha)}_{(b\beta)}$, $\alpha \neq \beta$, are important to make the auxiliary fermionic projector idempotent, because

$$P^{(a\alpha)}_{(c\gamma)} \overset{(2.6.15)}{=} \sum_{b=1}^{8} \sum_{\beta=1}^{3} P^{(a\alpha)}_{(b\beta)}\, P^{(b\beta)}_{(c\gamma)} \overset{\text{in general}}{\neq} \sum_{b=1}^{8} P^{(a\alpha)}_{(b\alpha)}\, P^{(b\gamma)}_{(c\gamma)}\,.$$

But these off-diagonal elements do not enter the definition (2.6.19), and this makes it difficult to arrange that P^a_b is idempotent. In more technical terms, defining the fermionic projector by (2.6.19) would in the proof of the above theorem lead instead of (2.6.16) to the conditions

$$\sum_{b=1}^{8} \sum_{\alpha,\beta=1}^{3} \Big(P^{(a\alpha)}_{(b\alpha)}\, P^{(b\beta)}_{(c\beta)} \;-\; P^{(a\alpha)}_{(b\beta)}\, P^{(b\beta)}_{(c\alpha)} \Big) \;=\; \delta^2\, Q^a_c(x,y)\,.$$

As a consequence, we would in (2.6.18) get contributions with $\beta = \beta'$, which are singular. The only way to avoid these singular contributions would be to consider perturbations which are diagonal on the generations. But in this special case, also the auxiliary fermionic projector is diagonal on the generations, and so the definitions (2.6.19) and (2.3.20) coincide. We conclude that (2.3.20) is the more general and thus preferable definition of the partial trace.

We next consider the normalization of the individual states of the fermionic projector. In finite 3-volume in the vacuum, a Dirac sea of mass m is composed of a discrete number of fermionic states. More precisely,

$$\begin{aligned} P^{\text{sea}}(x,y) &= \int \frac{dk^0}{2\pi}\, \frac{1}{V} \sum_{\vec{k} \in L^3} (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)} \\ &= \frac{1}{2\pi V} \sum_{\vec{k} \in L^3} \frac{1}{2\,|k^0|}\, (\not{k} + m)\, e^{-ik(x-y)}\Big|_{k^0 = -\sqrt{|\vec{k}|^2 + m^2}}\,. \end{aligned} \tag{2.6.20}$$

[5] Online version: This potential source of confusion is the reason why in more recent works on the fermionic projector, the partial trace is referred to as the *sectorial projection*.

Here the image of $(\not{k} + m)$ is two-dimensional; it is spanned by the two plane-wave solutions of the Dirac equation of momentum k with spin up and down, respectively. Thus we can write the fermionic projector in analogy to (2.2.1) as

$$P^{\text{sea}}(x,y) \;=\; \sum_{\vec{k} \in L^3} \sum_{s=\pm 1} -|\Psi_{\vec{k}s}(x)\succ\!\prec \Psi_{\vec{k}s}(y)|\;, \tag{2.6.21}$$

where $\Psi_{\vec{k}s}$ are the suitably normalized negative-energy plane-wave solutions of the Dirac equation, and s denotes the two spin orientations. If an external field is present, it is still possible to decompose the fermionic projector similar to (2.6.21) into individual states. But clearly, each of these states is perturbed by $\mathcal{B}$; we denote these perturbed states by a tilde. The next theorem shows that the probability integral for these states is independent of the interaction and of the size of T^3.

THEOREM 2.6.2. **(probability integral)** *Under the assumptions of Theorem 2.6.1, every state $\tilde{\Psi}$ of the fermionic projector is normalized according to*

$$\int_{T^3} \prec\!\tilde{\Psi}|\gamma^0\, \tilde{\Psi}\succ\!(t,\vec{x})\, d\vec{x} \;=\; \frac{1}{2\pi}\;. \tag{2.6.22}$$

Proof. Since $\tilde{\Psi}$ is a solution of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - mY - \mu\mathbb{1})\tilde{\Psi} = 0$, it follows from current conservation (see 1.2.16) that the probability integral (2.6.22) is time independent. Thus it suffices to compute it in the limits $t \to \pm\infty$, in which according to our decay assumptions on $\mathcal{B}$ the system is non-interacting. Since in the vacuum, the fermionic projector splits into a direct sum of Dirac seas, we may restrict attention to a single Dirac sea (2.6.21). Using that the probability integral is the same for both spin orientations,

$$\int_{T^3} \prec\!\Psi_{\vec{k}s}|\gamma^0\, \Psi_{\vec{k}s}\succ\!(t,\vec{x})\, d\vec{x} \;=\; \int_{T^3} \frac{1}{2} \sum_{s=\pm} \text{Tr}\left(\gamma^0\, |\Psi_{\vec{k}s}\succ\!\prec \Psi_{\vec{k}s}|\right) d\vec{x}\;,$$

and comparing with (2.6.20) gives

$$\begin{aligned}\int_{T^3} \prec\!\Psi_{\vec{k}s}|\gamma^0 \Psi_{\vec{k}s}\succ\!(t,\vec{x})\, d\vec{x} \;&=\; \frac{1}{4\pi\, V} \int_{T^3} \frac{1}{2k^0}\, \text{Tr}(\gamma^0\, (\not{k} + m))\big|_{k^0 = -\sqrt{|\vec{k}|^2 + m^2}}\, d\vec{x} \\ &=\; \frac{1}{4\pi\, V} \int_{T^3} \frac{4k^0}{2k^0}\, d\vec{x} \;=\; \frac{1}{2\pi}\;.\end{aligned}$$

∎

Let us discuss what this result means for the states of the fermionic projector (2.6.13, 2.6.14). As pointed out in the paragraph of (2.6.8), the fermionic projector of the vacuum is composed for each $\vec{k} \in L^3$ of a continuum of states (2.6.8). However, if we choose the space-time points in the fixed time interval $-T < t < T$ and let $\delta \searrow 0$, we need not distinguish between the frequencies in (2.6.8) and obtain that only the discrete states with $\vec{k} \in L^3$, $k^0 = -\sqrt{|\vec{k}|^2 + m^2}$ are occupied (see the discussion after (2.6.8)). In the causal perturbation expansion, each of these states is perturbed, and thus also the interacting fermionic projector for small δ can be regarded as being composed of discrete states. We write in analogy to (2.6.21),

$$P(x,y) \;=\; \sum_a -|\tilde{\Psi}_a\succ\!\prec \tilde{\Psi}_a|\;,$$

where a runs over all the quantum number of the fermions. According to (2.6.13) and Theorem 1.5.5, the probability integral is

$$\int_{T^3} \prec \tilde{\Psi}_a \,|\, \gamma^0\, \tilde{\Psi}_a \succ (t, \vec{x})\, d\vec{x} \;=\; \frac{\delta}{2\pi}\,. \tag{2.6.23}$$

By substituting the formulas of the light-cone expansion of §2.5 into (2.6.13), one sees that the contributions of the light-cone expansion to the fermionic projector all involve at least one factor δ. Thus after rescaling P by a factor δ^{-1}, the probability integral (2.6.20) as well as the formulas of the light-cone expansion have a well-defined and non-trivial continuum limit. In particular, using that the particle and anti-particle states are to be normalized in accordance with the states of the sea, we can specify the normalization constant c_{norm} in (2.3.19). If the wave functions Ψ_k and Φ_l in (2.3.19) are normalized according to (1.2.15), we must choose

$$c_{\text{norm}} \;=\; -\frac{1}{2\pi}\,. \tag{2.6.24}$$

We finally remark that Theorem 1.5.5 can be generalized in a straightforward way to include a gravitational field, if (2.6.22) is replaced by (1.5.22), with $\mathcal{H}$ a space-like hypersurface with future-directed normal ν. However, we need to assume that the gravitational field decays at infinity in the sense that space-time is asymptotically flat and for $t \to \pm\infty$ goes over asymptotically to Minkowski space. In particular, realistic cosmological models like the Friedman-Robertson-Walker space-times are excluded. We do not expect that the large-scale structure of space-time should have an influence on the normalization[6], but this is an open problem which remains to be investigated.

[6]Online version: This picture has been confirmed by the paper arXiv:0901.0602 [math-ph].

CHAPTER 3

The Principle of the Fermionic Projector

In this chapter we introduce a new mathematical framework intended for the formulation of physical theories. We first generalize the notions of relativistic quantum mechanics and classical field theory in several construction steps. This will be done in a very intuitive way. The aim is to work out the essence of the underlying physical principles by dropping all additional and less important structures. This will lead us to a quite abstract mathematical framework. The "principle of the fermionic projector" states that the fundamental physical equations should be formulated within this framework. We conclude this chapter with a brief physical overview and discussion.

3.1. Connection between Local Gauge Freedom and the Measurability of Position and Time

In this section we give a possible explanation as to why local gauge freedom occurs in nature. This physical consideration will provide a formalism which will be the starting point for the constructions leading to the principle of the fermionic projector. We begin for simplicity with the example of the $U(1)$ gauge transformations of the magnetic field for a Schrödinger wave function Ψ in nonrelativistic quantum mechanics. Since it will be sufficient to consider the situation for fixed time, we only write out the spatial dependence of the wave function, $\Psi = \Psi(\vec{x})$ with $\vec{x} \in \mathbb{R}^3$. In the nonrelativistic and static limit, the gauge freedom of electrodynamics (1.4.1, 1.4.2) reduces to the transformations

$$\vec{A}(\vec{x}) \;\longrightarrow\; \vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x}) \tag{3.1.1}$$

$$\Psi(\vec{x}) \;\longrightarrow\; e^{i\Lambda(\vec{x})}\, \Psi(\vec{x})\,, \tag{3.1.2}$$

where the so-called vector potential $\vec{A}$ consists of the three spatial components of the electromagnetic potential A. Similar to (1.4.4), we introduce the gauge-covariant derivative by

$$\vec{D} \;=\; \vec{\nabla} - i\vec{A}\,. \tag{3.1.3}$$

With the transformation (3.1.2) we can arbitrarily change the phase of the wave function Ψ at any point $\vec{x}$. This is consistent with the quantum mechanical interpretation of the wave function, according to which the phase of a wave function is not an observable quantity, only its absolute square $|\Psi(\vec{x})|^2$ has a physical meaning as the probability density of the particle. One can even go one step further and take the point of view that the inability to determine the local phase of a quantum mechanical wave function is the physical reason for the local gauge freedom (3.1.1, 3.1.2). Then the $U(1)$ gauge transformations of the magnetic field become a consequence of the principles of quantum mechanics. This argument becomes clearer when stated in more mathematical

terms: We consider on the Schrödinger wave functions the usual scalar product

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^3} \overline{\Psi(\vec{x})}\, \Phi(\vec{x})\, d\vec{x}$$

and denote the corresponding Hilbert space by H. On H, the position operators $\vec{X}$ are given as the multiplication operators with the coordinate functions,

$$\vec{X}\, \Psi(\vec{x}) \;=\; \vec{x}\, \Psi(\vec{x})\,.$$

As it is common in quantum mechanics, we consider H as an abstract Hilbert space (i.e. we forget about the fact that H was introduced as a space of functions). Then the wave function $\Psi(\vec{x})$ corresponding to a vector $\Psi \in H$ is obtained by constructing a *position representation* of the Hilbert space. In bra/ket notation, this is done by choosing an "eigenvector basis" $|\vec{x}>$ of the position operators,

$$\vec{X}\, |\vec{x}> \;=\; \vec{x}\, |\vec{x}>\,, \qquad\qquad <\vec{x} \,|\, \vec{y}> \;=\; \delta^3(\vec{x}-\vec{y})\,, \tag{3.1.4}$$

and the wave function is then introduced by

$$\Psi(\vec{x}) \;=\; <\vec{x} \,|\, \Psi> \tag{3.1.5}$$

(we remark that the formal bra/ket notation can be made mathematically precise using spectral measures [**F3**]). The important point for us is that the "eigenvectors" $|\vec{x}>$ of the position operators are determined only up to a phase. Namely, the transformation

$$|\vec{x}> \;\longrightarrow\; \exp\left(-i\Lambda(\vec{x})\right) |\vec{x}> \tag{3.1.6}$$

leaves invariant the conditions (3.1.4) for the "eigenvector basis." If we substitute (3.1.6) into (3.1.5), we obtain precisely the transformation (3.1.2) of the wave function. The transformation properties of the gauge-covariant derivative (3.1.3) and of the gauge potentials in (3.1.1) follow from (3.1.2) if one assumes that the gauge-covariant derivatives $\vec{D}$ are operators on H (and thus do not depend on the representation of H as functions in position space). In physics, the operators $\vec{\pi} = -i\vec{D}$ are called the "canonical momentum operators."

The relation just described between the position representation of quantum mechanical states and the $U(1)$ gauge transformations of the magnetic field was noticed long ago. However, the idea of explaining local gauge freedom from quantum mechanical principles was not regarded as being of general significance. In particular, it was never extended to the relativistic setting or to more general gauge groups. The probable reason for this is that these generalizations are not quite straightforward; they make it necessary to formulate relativistic quantum mechanics in a particular way as follows. We again consider on the four-component Dirac spinors $(\Psi^\alpha(x))_{\alpha=1,\ldots,4}$ in Minkowski space the spin scalar product (1.2.8) and denote the vector space of all Dirac wave functions by H. Integrating the spin scalar product over space-time, we obtain an indefinite scalar product on H,

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^4} \prec\!\Psi \,|\, \Phi\!\succ(x)\, d^4x\,. \tag{3.1.7}$$

Furthermore, we introduce on H time/position operators $(X^i)_{i=0,\ldots,3}$ by multiplication with the coordinate functions,

$$X^i\, \Psi(x) \;=\; x^i\, \Psi(x)\,.$$

We now consider $(H,\, <.|.>)$ as an abstract scalar product space. In order to construct a *time/position representation* of H, we must choose an "eigenvector basis"

of the time/position operators. Since the wave functions have four components, an "eigenvector basis" has in bra/ket notation the form $|x\alpha>$, $x \in \mathbb{R}^4$, $\alpha = 1, \ldots, 4$; it is characterized by the conditions

$$X^i\,|x\alpha> \;=\; x^i\,|x\alpha>\,, \qquad\qquad <x\alpha\,|\,y\beta> \;=\; s_\alpha\,\delta_{\alpha\beta}\,\delta^4(x-y) \tag{3.1.8}$$

with s_α as in (1.2.8). The wave function corresponding to a vector $\Psi \in H$ is defined by

$$\Psi^\alpha(x) \;=\; <x\alpha\,|\,\Psi>\,. \tag{3.1.9}$$

The conditions (3.1.8) determine the basis $|x\alpha>$ only up to local isometries of a scalar product of signature $(2,2)$, i.e. up to transformations of the form

$$|x\alpha> \;\longrightarrow\; \sum_{\beta=1}^{4} (U(x)^{-1})^\alpha_\beta\,|x\beta> \qquad \text{with} \qquad U(x) \in U(2,2)\,. \tag{3.1.10}$$

If we identify these transformations with gauge transformations and substitute into (3.1.9), we obtain local gauge freedom of the form

$$\Psi(x) \;\longrightarrow\; U(x)\,\Psi(x)\,. \tag{3.1.11}$$

Since gauge transformations correspond to changes of the "eigenvector basis" $|x\alpha>$, we also refer to $|x\alpha>$ as a *gauge*.

From the mathematical point of view, (3.1.8–3.1.10) is a straightforward generalization of (3.1.4–3.1.6) to the four-dimensional setting and four-component wave functions, taking into account that the spin scalar product has signature $(2,2)$. However, our construction departs from the usual description of physics, because the time operator X^0 is not commonly used in relativistic quantum mechanics and because the scalar product (3.1.7) is unconventional. In particular, one might object that the scalar product (3.1.7) may be infinite for physical states, because the time integral diverges. However, this is not a serious problem, which could be removed for example by considering the system in finite 4-volume and taking a suitable infinite volume limit. Furthermore, one should keep in mind that the scalar product (3.1.7) gives us the spin scalar product, and using the spin scalar product one can introduce the usual positive scalar product $(.|.)$ by integrating over a space-like hypersurface (see (1.2.17) or more generally (1.5.22)). Therefore, it causes no principal problems to consider $<.|.>$ instead of $(.|.)$ as the fundamental scalar product. We conclude that (3.1.7–3.1.10) is certainly an unconventional point of view, but it is nevertheless consistent and indeed mathematically equivalent to the usual description of relativistic quantum mechanics as outlined in §1.2.

The above explanation of local gauge freedom fits together nicely with our description of Dirac spinors in the gravitational field in §1.5: We let $(H, <.|.>)$ be the vector space of wave functions on a manifold M, endowed with the indefinite scalar product (1.5.23). For every coordinate system x^i we introduce the corresponding multiplication operators X^i. Considering H as an abstract vector space, the arbitrariness of the time/position representation of H again yields the local $U(2,2)$ gauge freedom (3.1.11). We thus obtain precisely the gauge transformations (1.5.2). In this way, (3.1.8–3.1.10) is not only consistent with all the constructions in §1.5, but it also gives a simple explanation for the gauge group $U(2,2)$.

The $U(2,2)$ gauge symmetry describes gravitation and electrodynamics, but it does not include the weak and strong interactions. In order to obtain additional gauge freedom, we must extend the gauge group. Since our gauge group is the isometry group

of the spin scalar product, this can be accomplished only by increasing the number of components of the wave functions. In general, one can take wave functions with $p+q$ components and a spin scalar product of signature (p,q),

$$\prec \Psi \,|\, \Phi \succ (x) \;=\; \sum_{\alpha=1}^{p+q} s_\alpha\, \Psi^\alpha(x)^*\, \Phi^\alpha(x) \qquad \text{with}$$
$$s_1 = \cdots = s_p = 1\,, \qquad s_{p+1} = \cdots = s_{p+q} = -1\,. \tag{3.1.12}$$

We call (p,q) the *spin dimension*. Repeating the above construction (3.1.7–3.1.9) for this spin scalar product yields local gauge freedom with gauge group $U(p,q)$. Unfortunately, it is not possible to introduce the Dirac operator in this generality. Therefore, we will always assume that the spin dimension is $(2N,2N)$ with $N \geq 1$. In this case, one can regard the $4N$ component wave functions as the direct sum of N Dirac spinors, exactly as we did in the general definition of the fermionic projector (2.3.1). Then our above argument yields the gauge group $U(2N,2N)$. The interaction can be described for example as in §2.3 by inserting a multiplication operator $\mathcal{B}$ into the Dirac operator (2.3.10) and taking the partial trace (2.3.20). More generally, one can modify the first order terms of the Dirac operator in analogy to Def. 1.5.1. Our concept is that the $U(2n,2n)$ gauge symmetry should be related to corresponding gauge potentials in the Dirac operator, and that this should, in the correct model, give rise to the gravitational, strong and electroweak gauge fields.

For clarity, we finally point out the differences of our approach to standard gauge theories. Usually, the gauge groups (e.g. the $SU(2)_{\mathrm{W}}$ or $SU(3)_{\mathrm{S}}$ in the standard model) act on separate indices of the wave functions (called the isospin and color indices, respectively). In contrast to this, our $U(2,2)$ gauge transformations simply act on the spinor index. In our generalization to higher spin dimension (3.1.12), we make no distinction between the spinor index and the index of the gauge fields; they are both combined in one index $\alpha = 1, \ldots, 4N$. In our setting, the gauge group and the coupling of the gauge fields to the Dirac particles are completely determined by the spin dimension. Compared to standard gauge theories, where the gauge groups and their couplings can be chosen arbitrarily, this is a strong restriction for the formulation of physical models.

3.2. Projection on Fermionic States

The fermionic projector was introduced in Chapter 2 in order to resolve the external field problem, and we used it to describe a general many-fermion system (2.3.19). We now discuss the concept of working with a "projector" in a more general context. A single Dirac particle is clearly described by its wave function $\Psi^\alpha(x) = <x\alpha|\Psi>$, or, in a gauge-independent way, by a vector $\Psi \in H$. Since the phase and normalization of Ψ have no physical significance, we prefer to describe the Dirac particle by the one-dimensional subspace $<\Psi> \equiv \{\lambda\Psi,\ \lambda \in \mathbb{C}\} \subset H$. Now consider the system of n Dirac particles, which occupy the one-particle states $\Psi_1, \ldots, \Psi_n \in H$. Generalizing the subspace $<\Psi>$ of the one-particle system, it seems natural to describe the many-particle system by the subspace $<\Psi_1, \ldots, \Psi_n> \subset H$ spanned by $\Psi_1, \ldots, \Psi_n$. We consider for simplicity only the generic case that this subspace is non-degenerate (i.e. there should be no vectors $0 \neq \Psi \in Y$ with $<\Psi|\Phi> = 0$ for all $\Phi \in Y$). Just as in positive definite scalar product spaces, every non-degenerate subspace $Y \subset H$ uniquely determines a projector P_Y on this subspace, characterized by the conditions

$P_Y^* = P_Y = P_Y^2$ and $\mathrm{Im}(P_Y) = Y$, where the star denotes the adjoint with respect to the scalar product $<.|.>$. Instead of working with the subspace $<\Psi_1, \ldots, \Psi_n> \subset H$, it is more convenient for us to consider the corresponding projector P,

$$P \;=\; P_{<\Psi_1,\ldots,\Psi_n>} \,.$$

We call P the *fermionic projector*. In this work we will always describe the Dirac particles of the system by a fermionic projector.

The concept of the fermionic projector departs from the usual description of a many-particle state by a vector of the fermionic Fock space (as introduced in §1.3). Let us discuss this difference in detail. In many-particle quantum mechanics, the system of Dirac particles $\Psi_1, \ldots, \Psi_n$ is described by the anti-symmetric product wave function

$$\Psi \;=\; \Psi_1 \wedge \cdots \wedge \Psi_n \,. \tag{3.2.1}$$

The wave functions of the form (3.2.1) are called n-particle Hartree-Fock states. They span the n-particle Fock space $F^n = \bigwedge^n H$. In the fermionic Fock space formalism, a quantum state is a linear combination of Hartree-Fock states, i.e. a vector of the Fock space $F = \bigoplus_{n=0}^\infty F^n$ (see §1.3 for details). In order to connect the fermionic projector with the Fock space formalism, we can associate to a projector P_Y on a subspace $Y = <\Psi_1, \ldots, \Psi_n> \subset H$ the wave function (3.2.1). This mapping clearly depends on the choice of the basis of Y. More precisely, choosing another basis $\Phi_i = \sum_{j=1}^n \kappa_i^j \Psi_j$, we have

$$\Phi_1 \wedge \cdots \wedge \Phi_n \;=\; \det(\kappa)\; \Psi_1 \wedge \cdots \wedge \Psi_n \,.$$

This shows that, due to the anti-symmetrization, the mapping is unique up to a complex factor. Therefore, with the mapping

$$P_{<\Psi_1,\ldots,\Psi_n>} \;\to\; <\Psi_1 \wedge \cdots \wedge \Psi_n> \subset F^n$$

we can associate to every projector a unique one-dimensional subspace of the Fock space. Since the image of this mapping is always a Hartree-Fock state, we obtain a one-to-one correspondence between the projectors P_Y on n-dimensional subspaces $Y \subset H$ and n-particle Hartree-Fock states. In this way, one sees that the description of a many-particle state with the fermionic projector is equivalent to using a Hartree-Fock state. With this correspondence, the formalism of the fermionic projector becomes a special case of the Fock space formalism, obtained by restricting to Hartree-Fock states. In particular, we conclude that the physical concepts behind fermionic Fock spaces, namely the Pauli Exclusion Principle and the fact that quantum mechanical particles are indistinguishable (see page 17), are also respected by the fermionic projector. However, we point out that the the fermionic projector is not mathematically equivalent to a state of the Fock space, because a vector of the Fock space can in general be represented only by a linear combination of Hartree-Fock states.

Let us analyze what this mathematical difference means physically. If nature is described by a fermionic projector, the joint wave function of all fermions of the Universe must be a Hartree-Fock state. However, this condition cannot be immediately verified in experiments, because measurements can never take into account all existing fermions. In all realistic situations, one must restrict the observations to a small subsystem of the Universe. As is worked out in Appendix A, the effective wave function of a subsystem need *not* be a Hartree-Fock state; it corresponds to an arbitrary vector of the Fock space of the subsystem, assuming that the number of particles of the whole system is sufficiently large. From this we conclude that the description of

the many-particle system with the fermionic projector is indeed physically equivalent to the Fock space formalism. For theoretical considerations, it must be taken into account that the fermionic projector merely corresponds to a Hartree-Fock state; for all practical purposes, however, one can just as well work with the whole Fock space.

We saw after (3.2.1) that the description of a many-particle state with the fermionic projector implies the Pauli Exclusion Principle. This can also be understood directly in a non-technical way as follows. For a given state $\Psi \in H$, we can form the projector $P_{<\Psi>}$ describing the one-particle state, but there is no projector which would correspond to a two-particle state (notice that the naive generalization $2P_{<\Psi>}$ is not a projector). More generally, every vector $\Psi \in H$ either lies in the image of P, $\Psi \in P(H)$, or it does not. Via these two conditions, the fermionic projector encodes for every state $\Psi \in H$ the occupation numbers 1 and 0, respectively, but it is impossible to describe higher occupation numbers. In this way, the fermionic projector naturally incorporates the Pauli Exclusion Principle in its formulation that each quantum mechanical state may be occupied by at most one fermion.

As explained at the end of §2.5, the fermionic projector contains all the information about the physical system in the sense that from a given fermionic projector one can uniquely reconstruct the fermionic states as well as the Dirac operator with interaction. Therefore, it is consistent to consider the fermionic projector as the basic object in space-time and to regard the Dirac operator merely as an auxiliary object which is useful in describing the interaction of the fermions via classical fields.

3.3. Discretization of Space-Time

The ultraviolet divergences of perturbative QFT indicate that the current description of physics should break down at very small distances. It is generally believed that the length scale where yet unknown physical effects should become important is given by the Planck length. Here we will assume that space-time consists on the Planck scale of discrete space-time points. The simplest way to discretize space-time would be to replace the space-time continuum by a four-dimensional lattice (as it is e.g. done in lattice gauge theories). In the following construction, we go much further and discretize space-time in a way where notions like "lattice spacing" and "neighboring lattice points" are given up. On the other hand, we will retain the principles of general relativity and our local gauge freedom.

We first consider the situation in a given coordinate system x^i in space-time[1]. For the discretization we replace the time/position operators X^i by mutually commuting operators with a *purely discrete spectrum*. We take the joint spectrum of these operators, i.e. the set

$$M \;=\; \{x \in \mathbb{R}^4 \,|\, \text{there is } u \in H \text{ with } X^i u = x^i u \text{ for all } i = 0, \ldots, 3\}\,,$$

as our discrete space-time points. We assume that the joint eigenspaces e_x of the X^i,

$$e_x \;=\; \{u \,|\, X^i u = x^i u \text{ for all } i = 0, \ldots, 3\}\,, \qquad\qquad x \in M\,,$$

[1]We assume for simplicity that the chart x^i describes all space-time. The generalization to a non-trivial space-time topology is done in a straightforward way by gluing together different charts; for details see [**F3**].

are $4N$-dimensional subspaces of H, on which the scalar product $<.|.>$ has the signature $(2N, 2N)$. Then we can choose a basis $|x\alpha>$, $x \in M$, $\alpha = 1, \ldots, 4N$ satisfying

$$\begin{aligned} X^i\,|x\alpha> &= x^i\,|x\alpha>\,, \qquad <x\alpha\,|\,y\beta> = s_\alpha\,\delta_{\alpha\beta}\,\delta_{xy} \qquad \text{with} \\ s_1 = \cdots = s_{2N} = 1\,, &\quad s_{2N+1} = \cdots = s_{4N} = -1\,. \end{aligned} \tag{3.3.1}$$

These relations differ from (3.1.8) only by the replacement $\delta^4(x-y) \to \delta_{xy}$. It is useful to introduce the projectors E_x on the eigenspaces e_x by

$$E_x \;=\; \sum_{\alpha=1}^{p+q} s_\alpha\,|x\alpha><x\alpha|\,; \tag{3.3.2}$$

they satisfy the relations

$$X^i\,E_x \;=\; x^i\,E_x \qquad \text{and} \tag{3.3.3}$$

$$E_x^* \;=\; E_x\,, \qquad E_x\,E_y \;=\; \delta_{xy}\,E_x\,, \qquad \sum_{x\in M} E_x \;=\; \mathbb{1}\,, \tag{3.3.4}$$

where the star denotes the adjoint with respect to the scalar product $<.|.>$ (these relations immediately follow from (3.3.1) and the fact that $|x\alpha>$ is a basis). The operators E_x are independent of the choice of the basis $|x\alpha>$, they are uniquely characterized by (3.3.3, 3.3.4) as the spectral projectors of the operators X^i.

If we change the coordinate system to $\tilde{x}^i = \tilde{x}^i(x)$, the discrete space-time points $M \subset \mathbb{R}^4$ are mapped to different points in $\mathbb{R}^4$, more precisely

$$\tilde{M} = \tilde{x}(M)\,, \qquad \tilde{E}_{\tilde{x}(x)} = E_x\,. \tag{3.3.5}$$

With such coordinate transformations, the relative position of the discrete space-time points in $\mathbb{R}^4$ can be arbitrarily changed. Taking general coordinate invariance seriously on the Planck scale, this is consistent only if we forget about the fact that M and $\tilde{M}$ are subsets of $\mathbb{R}^4$ and consider them merely as index sets for the spectral projectors. In other words, we give up the ordering of the discrete space-time points, which is inherited from the ambient vector space $\mathbb{R}^4$, and consider M and $\tilde{M}$ only as point sets. After this generalization, we can identify M with $\tilde{M}$ (via the equivalence relation $\tilde{x}(x) \simeq x$). According to (3.3.5), the spectral projectors $(E_p)_{p\in M}$ are then independent of the choice of coordinates.

We regard the projectors $(E_p)_{p\in M}$ as the basic objects describing space-time. The time/position operators can be deduced from them. Namely, every coordinate system yields an injection of the discrete space-time points

$$x \;:\; M \;\hookrightarrow\; \mathbb{R}^4\,, \tag{3.3.6}$$

and the corresponding time/position operators X^i can be written as

$$X^i \;=\; \sum_{p\in M} x^i(p)\,E_p\,. \tag{3.3.7}$$

Since every injection of the discrete space-time points into $\mathbb{R}^4$ can be realized by a suitable choice of coordinates (i.e. for every injection $\iota : M \hookrightarrow \mathbb{R}^4$ there is a chart x^i such that $x(M) = \iota(M)$), we can drop the condition that x is induced by a coordinate system. We can thus take for x in (3.3.6, 3.3.7) any embedding of M into $\mathbb{R}^4$.

Let us summarize the result of our construction. We shall describe space-time by an indefinite scalar product space $(H, <.|.>)$ and projectors $(E_p)_{p\in M}$ on H, where M is a (finite or countable) index set. The projectors E_p are characterized by the conditions

(3.3.4). Furthermore, we assume that the *spin dimension* is $(2N, 2N)$, i.e. $E_p(H) \subset H$ is for all $p \in M$ a subspace of signature $(2N, 2N)$. We call $(H, <.|.>, (E_p)_{p \in M})$ *discrete space-time*. The equivalence principle is taken into account via the freedom in choosing the embeddings (3.3.6, 3.3.7) of the discrete space-time points. Moreover, one can choose a basis $|p\alpha>$, $p \in M$, $\alpha = 1, \ldots, 4N$, of H satisfying the conditions

$$E_p\, |q\alpha> \;=\; \delta_{pq}\, |p\alpha>\,, \qquad\qquad <p\alpha \,|\, q\beta> \;=\; s_\alpha\, \delta_{\alpha\beta}\, \delta_{pq}$$

with s_α as in (3.3.1); such a basis is called a *gauge*. It is determined only up to transformations of the form

$$|p\alpha> \;\to\; \sum_{\beta=1}^{2N} (U(p)^{-1})^\alpha_\beta\, |p\beta> \quad \text{with} \quad U(p) \in U(2N, 2N)\,. \tag{3.3.8}$$

These are the local gauge transformations of discrete space-time.

3.4. The Principle of the Fermionic Projector

For the complete description of a physical system we must introduce additional objects in discrete space-time $(H, <.|.>, (E_p)_{p \in M})$. As mentioned at the end of §3.2, one can in the space-time continuum regard the fermionic projector as the basic physical object. Therefore, it seems promising to carry over the fermionic projector to discrete space-time. We introduce the *fermionic projector of discrete space-time* P as a projector acting on the vector space H of discrete space-time.

In analogy to the situation for the continuum, we expect that a physical system can be completely characterized by a fermionic projector in discrete space-time. At this stage, however, it is not at all clear whether this description makes any physical sense. In particular, it seems problematic that neither the Dirac equation nor the classical field equations can be formulated in or extended to discrete space-time; thus it becomes necessary to replace them by equations of completely different type. We take it as an ad-hoc postulate that this can actually be done; namely we assert

The Principle of the Fermionic Projector:

> A physical system is completely described by the fermionic projector in discrete space-time. The physical equations should be formulated exclusively with the fermionic projector in discrete space-time, i.e. they must be stated in terms of the operators P and $(E_p)_{p \in M}$ on H.

Clearly, the validity and consequences of this postulate still need to be investigated; this is precisely the aim of the present work. The physical equations formulated with P and $(E_p)_{p \in M}$ are called the *equations of discrete space-time*.

3.5. A Variational Principle

Before coming to the general discussion of the principle of the fermionic projector, we want to give an example of a variational principle in discrete space-time. This is done to give the reader an idea of how one can formulate equations in discrete space-time. This example will serve as our model variational principle, and we will often come back to it. A more detailed motivation of our Lagrangian is given in Chapter 5.

Let us first discuss the general mathematical form of possible equations in discrete space-time. The operators P and $(E_p)_{p \in M}$ all have a very simple structure in that they are projectors acting on H. Therefore, it is not worth studying these operators separately; for physically promising equations, we must combine the projectors P

and $(E_p)_{p \in M}$ in a mathematically interesting way. Composite expressions in these operators can be manipulated using the idempotence of P and the relations (3.3.4) between the projectors $(E_p)_{p \in M}$: First of all, the identities $\sum_{p \in M} E_p = \mathbb{1}$ and $E_p^2 = E_p$ allow us to insert factors E_p into the formulas; e.g.

$$E_x\, P\, \Psi \;=\; E_x\, P \left(\sum_{y \in M} E_y \right) \Psi \;=\; \sum_{y \in M} (E_x\, P\, E_y)\, E_y\, \Psi\,.$$

Writing

$$P(x,y) \;\equiv\; E_x\, P\, E_y\,,$$

we obtain the identity

$$E_x\, (P\, \Psi) \;=\; \sum_{y \in M} P(x,y)\, E_y\, \Psi\,.$$

This representation of P by a sum over the discrete space-time points resembles the integral representation of an operator in the continuum with an integral kernel. Therefore, we call $P(x,y)$ the *discrete kernel* of the fermionic projector. The discrete kernel can be regarded as a canonical representation of the fermionic projector of discrete space-time, induced by the projectors $(E_p)_{p \in M}$. Now consider a general product of the operators P and $(E_p)_{p \in M}$. Using the relations $P^2 = P$ and $E_x\, E_y = \delta_{xy}\, E_x$, every operator product can be simplified to one with alternating factors P and E_p, i.e. to an operator product of the form

$$E_{x_1}\, P\, E_{x_2}\, P\, E_{x_3} \cdots E_{x_{n-1}}\, P\, E_{x_n} \qquad \text{with } x_j \in M. \tag{3.5.1}$$

Again using that $E_p^2 = E_p$, we can rewrite this product with the discrete kernel as

$$P(x_1, x_2)\, P(x_2, x_3)\; \cdots\; P(x_{n-1}, x_n)\,. \tag{3.5.2}$$

We conclude that the equations of discrete space-time should be formed of products of the discrete kernel, where the second argument of each factor must coincide with the first argument of the following factor. We refer to (3.5.2) as a *chain*.

In analogy to the Lagrangian formulation of classical field theory, we want to set up a variational principle. Our "action" should be a scalar functional depending on the operators P and E_p. Most scalar functionals on operators (like the trace or the determinant) can be applied only to endomorphisms (i.e. to operators which map a vector space into itself). The chain (3.5.2) is a mapping from the subspace $E_{x_n}(H) \subset H$ to $E_{x_1}(H)$. This makes it difficult to form a scalar, unless $x_1 = x_n$. Therefore, we will only consider *closed chains*

$$P(x, y_1)\, P(y_1, y_2) \cdots P(y_k, x) \;:\; E_x(H) \to E_x(H)\,.$$

In the simplest case $k = 0$, the closed chain degenerates to a single factor $P(x,x)$. This turns out to be too simple for the formulation of a physically interesting action, mainly because the light-cone structure of the fermionic projector (see §2.5) would then not enter the variational principle. Thus we are led to considering closed chains of two factors, i.e. to the operator product $P(x,y)P(y,x)$. Suppose that we are given a real-valued functional $\mathcal{L}$ on the endomorphisms of $E_x(H) \subset H$ (this will be discussed and specified below). Then $\mathcal{L}[P(x,y)\; P(y,x)]$ is a real function depending on two space-time arguments, and we get a scalar by summing over x and y. Therefore, we

take for our action S the ansatz

$$S \;=\; \sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)]\,. \tag{3.5.3}$$

This ansatz is called a *two-point action*, and in analogy to classical field theory we call $\mathcal{L}$ the corresponding *Lagrangian*.

We shall now introduce a particular Lagrangian $\mathcal{L}$. The requirement which will lead us quite naturally to this Lagrangian is that $\mathcal{L}$ should be *positive*. Positivity of the action is desirable because it is a more convincing concept to look for a local minimum of the action than merely for a critical point of an action which is unbounded below.

Let us first consider how one can form a positive functional on $P(x,y)\,P(y,x)$. The closed chain $P(x,y)\,P(y,x)$ is an endomorphism of $E_x(H)$; we abbreviate it in what follows by A. In a given gauge, A is represented by a $4N \times 4N$ matrix. Under gauge transformations (3.3.8), this matrix transforms according to the adjoint representation,

$$A \;\to\; U(x)\, A\, U(x)^{-1}\,.$$

Furthermore, A is Hermitian on $E_x(H)$, i.e.

$$<A\,\Psi \,|\, \Phi> \;=\; <\Psi \,|\, A\,\Phi> \qquad\qquad \text{for } \Psi, \Phi \in E_x(H)\,, \tag{3.5.4}$$

or simply $A^* = A$. In positive definite scalar product spaces, the natural positive functional on operators is an operator norm, e.g. the Hilbert-Schmidt norm $\|B\|_2 = \mathrm{tr}(B^* B)^{\frac{1}{2}}$. In our setting, the situation is more difficult because our scalar product $<.|.>$ is indefinite on $E_x(H)$ (of signature $(2N, 2N)$). As a consequence, Hermitian matrices do not have the same nice properties as in positive definite scalar product spaces; in particular, the matrix A might have complex eigenvalues, and it is in general not even diagonalizable. Also, the operator product $A^* A$ need not be positive, so that we cannot introduce a Hilbert-Schmidt norm. In order to analyze the situation more systematically, we decompose the characteristic polynomial of A into linear factors

$$\det(\lambda - A) \;=\; \prod_{k=1}^{K} (\lambda - \lambda_k)^{n_k}\,. \tag{3.5.5}$$

This decomposition is useful because every functional on A can be expressed in terms of the roots and multiplicities of the characteristic polynomial; thus it is sufficient to consider the λ_k's and n_k's in what follows. Each root λ_k corresponds to an n_k-dimensional A-invariant subspace of $E_x(H)$, as one sees immediately from a Jordan representation of A. The roots λ_k may be complex. But since A is Hermitian (3.5.4), we know at least that the characteristic polynomial of A is real,

$$\overline{\det(\lambda - A)} \;=\; \det(\lambda - A) \qquad\qquad \text{for } \lambda \in \mathbb{R}.$$

This means that the complex conjugate of every root is again a root with the same multiplicity (i.e. for every λ_k there is a λ_l with $\overline{\lambda_k} = \lambda_l$ and $n_k = n_l$). The reality of the characteristic polynomial is verified in detail as follows. In a given gauge, we can form the transposed, complex conjugated matrix of A, denoted by $A^\dagger$. For clarity, we point out that $A^\dagger$ is *not* an endomorphism of $E_x(H)$, because it has the wrong behavior under gauge transformations (in particular, the trace $\mathrm{tr}(A^\dagger A)$ depends on the gauge and is thus ill-defined). Nevertheless, the matrix $A^\dagger$ is useful because we can write the adjoint of A in the form $A^* = S A^\dagger S$, where S is the spin signature matrix,

$S = \mathrm{diag}((s_\alpha)_{\alpha=1,\ldots,4N})$. Since $S^2 = \mathbb{1}$, and since the determinant is multiplicative, we conclude that for any real λ,

$$\begin{aligned}\overline{\det(\lambda - A)} &= \det(\lambda - A^\dagger) = \det(\lambda - S^2\, A^\dagger) \\ &= \det(\lambda - S A^\dagger S) = \det(\lambda - A^*) = \det(\lambda - A)\,.\end{aligned}$$

It is worth noting that every Lagrangian is symmetric in the two arguments x and y, as the following consideration shows. For any two quadratic matrices B and C, we choose ε not in the spectrum of C and set $C^\varepsilon = C - \varepsilon\mathbb{1}$. Taking the determinant of the relation $C^\varepsilon(BC^\varepsilon - \lambda) = (C^\varepsilon B - \lambda)C^\varepsilon$, we can use that the determinant is multiplicative and that $\det C^\varepsilon \neq 0$ to obtain the equation $\det(BC^\varepsilon - \lambda) = \det(C^\varepsilon B - \lambda)$. Since both determinants are continuous in ε, this equation holds even for all $\varepsilon \in \mathbb{R}$, proving the elementary identity

$$\det(BC - \lambda\mathbb{1}) = \det(CB - \lambda\mathbb{1})\,. \tag{3.5.6}$$

Applying this identity to the closed chain,

$$\det(P(x,y)\,P(y,x) - \lambda\mathbb{1}) = \det(P(y,x)\,P(x,y) - \lambda\mathbb{1})\,,$$

we find that the characteristic polynomial of the matrix A remains the same if the two arguments x and y are interchanged. Hence

$$\mathcal{L}[P(x,y)\,P(y,x)] = \mathcal{L}[P(y,x)\,P(x,y)]\,. \tag{3.5.7}$$

An obvious way to form a positive functional is to add up the absolute values of the roots, taking into account their multiplicities. We thus define the *spectral weight* $|A|$ of A by

$$|A| = \sum_{k=1}^K n_k\,|\lambda_k|\,. \tag{3.5.8}$$

This functional depends continuously on the λ_k, and also it behaves continuously when the roots of the characteristic polynomial degenerate and the multiplicities n_k change. Thus the spectral weight $|\,.\,|$ is a continuous functional. Furthermore, the spectral weight is zero if and only if the characteristic polynomial is trivial, $\det(\lambda - A) = \lambda^{4N}$. This is equivalent to A being nilpotent (i.e. $A^k = 0$ for some k). Thus, in contrast to an operator norm, the vanishing of the spectral weight does not imply that the operator is zero. On the other hand, it does not seem possible to define an operator norm in indefinite scalar product spaces, and therefore we must work instead with the spectral weight.

Using the spectral weight, one can write down many positive Lagrangians. The simplest choice would be $\mathcal{L}[A] = |A|$. Minimizing the corresponding action (3.5.3) yields a variational principle which attempts to make the absolute values of the roots $|\lambda_k|$ as small as possible. This turns out to be a too strong minimizing principle. It is preferable to formulate a variational principle which aspires to equalize the absolute values of all roots. This can be accomplished by combining the expressions $|A^2|$ and $|A|^2$. Namely, using that the sum of the multiplicities equals the dimension of the vector space, $\sum_{k=1}^K n_k = 4N$, the Schwarz inequality yields that

$$|A^2| = \sum_{k=1}^K n_k\,|\lambda_k|^2 \geq \frac{1}{4N}\left(\sum_{k=1}^K n_k\,|\lambda_k|\right)^2 = \frac{1}{4N}\,|A|^2\,,$$

and equality holds only if the absolute values of all roots are equal. Thus it is reasonable to minimize $|A^2|$, keeping $|A|^2$ fixed. This is our motivation for considering the two-point action:

$$\text{minimize} \quad S \;=\; \sum_{x,y \in M} \big|(P(x,y)\, P(y,x))^2\big| \tag{3.5.9}$$

under the constraint

$$T \;:=\; \sum_{x,y \in M} |P(x,y)\, P(y,x)|^2 \;=\; \text{const}\,. \tag{3.5.10}$$

This is our model variational principle.

We next consider a stationary point of the action and derive the corresponding "Euler-Lagrange equations." For simplicity, we only consider the case that the matrix $P(x,y)\, P(y,x)$ can be diagonalized. This is the generic situation; the case of a non-diagonalizable matrix can be obtained from it by an approximation procedure. Having this in mind, we may assume that the endomorphism $A = P(x,y)P(y,x)$ has a spectral decomposition of the form

$$A \;=\; \sum_{k=1}^{K} \lambda_k\, F_k\,, \tag{3.5.11}$$

where λ_k are the roots in (3.5.5), and the F_k are operators mapping onto the corresponding eigenspaces (A, K, the λ_k, and the F_k clearly depend on x and y, but we will, for ease in notation, usually not write out this dependence). Since the underlying scalar product space is indefinite, the spectral decomposition (3.5.11) requires a brief explanation. Suppose that we choose a basis where A is diagonal. In this basis, the operators F_k are simply the diagonal matrices with diagonal entries 1 if the corresponding diagonal elements of A are λ_k, and 0 otherwise. Clearly, these operators map onto the eigenspaces and are orthonormal and complete, i.e.

$$A\, F_k \;=\; \lambda_k\, F_k\,, \quad F_k\, F_l \;=\; \delta_{kl}\, F_k \quad \text{and} \quad \sum_{k=1}^{K} F_k \;=\; \mathbb{1}_{E_x(H)}\,.$$

However, the F_k are in general *not* Hermitian (with respect to the spin scalar product). More precisely, taking the adjoint swaps the operators corresponding to complex conjugated eigenvalues,

$$F_k^* \;=\; F_l \qquad \text{when} \qquad \overline{\lambda_k} = \lambda_l\,. \tag{3.5.12}$$

These relations can be understood immediately because they ensure that the spectral decomposition (3.5.11) is Hermitian,

$$\left(\sum_{k=1}^{K} \lambda_k\, F_k\right)^* \;=\; \sum_{k=1}^{K} \overline{\lambda_k}\, F_k^* \overset{(3.5.12)}{=} \sum_{k=1}^{K} \lambda_k\, F_k\,.$$

Since the eigenvalues are in general complex, we can introduce a new matrix by taking the complex conjugate of the eigenvalues but leaving the spectral projectors unchanged,

$$\overline{A} \;=\; \sum_{k=1}^{K} \overline{\lambda_k}\, F_k \tag{3.5.13}$$

We refer to $\overline{A}$ as the *spectral adjoint* of A.

We now consider continuous variations $P(\tau)$ and $(E_p(\tau))_{p \in M}$, $-\varepsilon < \tau < \varepsilon$, of our operators. The structure of the operators must be respected by the variations; this means that $P(\tau)$ should be a projector and that the relations (3.3.4) between the operators $(E_p)_{p \in M}$ should hold for all τ. Continuity of the variation implies that the rank of P and the signature of its image do not change. This implies that the variation of P can be realized by a unitary transformation

$$P(\tau) \;=\; U(\tau)\, P\, U(\tau)^{-1}\,, \tag{3.5.14}$$

where $U(\tau)$ is a unitary operator on H with $U(0) = \mathbb{1}$. Similarly, the variations of the projectors $(E_p)_{p \in M}$ are also unitary. From (3.3.4) we can conclude the stronger statement that the variations of all operators $(E_p)_{p \in M}$ can be realized by one unitary transformation, i.e.

$$E_p(\tau) \;=\; V(\tau)\, E_p\, V(\tau)^{-1}$$

with a unitary operator $V(\tau)$ and $V(0) = \mathbb{1}$. Since our action is invariant under unitary transformations of the vector space H, we can, instead of unitarily transforming both P and $(E_p)_{p \in M}$, just as well keep the $(E_p)_{p \in M}$ fixed and *vary only the fermionic projector* by (3.5.14). To first order in τ, this variation becomes

$$\delta P \;\equiv\; \frac{d}{d\tau} P(\tau)_{|\tau=0} \;=\; i\,[B,\, P]\,, \tag{3.5.15}$$

where $B = -iU'(0)$ is a Hermitian operator on H. We will only consider variations where B has *finite support* , i.e. where the kernel $B(x,y) \equiv E_x\, B\, E_y$ of B satisfies the condition

$$B(x,y) \;=\; 0 \qquad \text{except for } x, y \in N \subset M \text{ with } \#N \text{ finite.}$$

This condition can be regarded as the analogue of the assumption in the classical calculus of variations that the variation should have compact support.

Let us compute the variation of the action (3.5.9) (the constraint (3.5.10) will be considered afterwards). Writing out the action with the eigenvalues λ_k and multiplicities n_k, we obtain

$$S \;=\; \sum_{x,y \in M} \sum_{k=1}^{K} n_k\, |\lambda_k|^2\,.$$

The variation can be computed in perturbation theory to first order,

$$\begin{aligned} \delta S \;&=\; 2\,\mathrm{Re} \sum_{x,y \in M} \sum_{k=1}^{K} \overline{\lambda_k}\, \mathrm{tr}(F_k\, \delta A) \\ &=\; 2\,\mathrm{Re} \sum_{x,y \in M} \sum_{k=1}^{K} \overline{\lambda_k}\, \mathrm{tr}\left(F_k\, (\delta P(x,y)\, P(y,x) \;+\; P(x,y)\, \delta P(y,x))\right), \end{aligned}$$

where "tr" denotes the trace in the vector space H. Exchanging the names of x and y in the first summand in the trace and using cyclicity of the trace, this expression can be written as an operator product,

$$\delta S \;=\; 2\,\mathrm{Re}\; \mathrm{tr}(Q_1\, \delta P)\,, \tag{3.5.16}$$

where the kernel $Q_1(x,y) \equiv E_x\, Q_1\, E_y$ of Q_1 has the form

$$Q_1(x,y) \;=\; \left[\sum_{k=1}^{K} \overline{\lambda_k}\, F_k\right]_{xy} P(x,y) \;+\; P(x,y) \left[\sum_{k=1}^{K} \overline{\lambda_k}\, F_k\right]_{yx}, \tag{3.5.17}$$

and the subscripts "$_{xy}$" and "$_{yx}$" indicate that the corresponding brackets contain the spectral decomposition of the operators $P(x,y)\,P(y,x)$ and $P(y,x)\,P(x,y)$, respectively. Note that the trace in (3.5.16) is well-defined because the trace is actually taken only over a finite-dimensional subspace of H. At this point the following lemma is useful.

LEMMA 3.5.1. *Let B and C be two matrices and assume that their products $A := BC$ and $\tilde{A} := CB$ are both diagonalizable. Then they have the same eigenvalues $\lambda_1, \ldots, \lambda_K$ with the same multiplicities $n_1, \ldots, n_K$. The corresponding spectral projectors F_k and $\tilde{F}_k$ satisfy the relations*

$$F_k\, B \;=\; B\, \tilde{F}_k\,. \tag{3.5.18}$$

Proof. It immediately follows from (3.5.6) that the matrices A and $\tilde{A}$ have the same eigenvalues with the same multiplicities. For any λ not in the spectrum of A, we multiply the identity $B(CB-\lambda) = (BC-\lambda)B$ from the left and right by $(A-\lambda)^{-1}$ and $(\tilde{A}-\lambda)^{-1}$, respectively. This gives

$$(A-\lambda)^{-1}\, B \;=\; B\, (\tilde{A}-\lambda)^{-1}\,.$$

Integrating λ over a contour around any of the eigenvalues λ_k and using the Cauchy integral formulas

$$F_k \;=\; -\frac{1}{2\pi i} \oint_{\partial B_\epsilon(\lambda_k)} (A-\lambda)^{-1}\, d\lambda\,, \qquad \tilde{F}_k \;=\; -\frac{1}{2\pi i} \oint_{\partial B_\epsilon(\lambda_k)} (\tilde{A}-\lambda)^{-1}\, d\lambda\,,$$

we obtain (3.5.18). ∎

This lemma allows us to simplify (3.5.17) to

$$Q_1(x,y) \;=\; 2 \left[\sum_{k=1}^K \overline{\lambda_k}\, F_k \right]_{xy} P(x,y)\,. \tag{3.5.19}$$

A short straightforward computation using (3.5.12) and Lemma 3.5.18 shows that the operator Q_1 is Hermitian. Thus the trace in (3.5.16) is real, and we conclude that

$$\delta S \;=\; 2\,\mathrm{tr}(Q_1\, \delta P) \quad .$$

The variation of our constraint (3.5.10) can be computed similarly, and one gets

$$\begin{aligned} \delta T \;&=\; 2\,\mathrm{tr}(Q_2\, \delta P) \qquad \text{with} \\ Q_2(x,y) \;&=\; 2 \left[\Big(\sum_{l=1}^K n_l\, |\lambda_l| \Big) \sum_{k=1}^K \frac{\overline{\lambda_k}}{|\lambda_k|}\, F_k \right]_{xy} P(x,y)\,. \end{aligned}$$

Now consider a local minimum of the action. Handling the constraint with a Lagrange multiplier μ, we obtain the condition

$$0 \;=\; \delta S - \mu\, \delta T \;=\; 2\,\mathrm{tr}\left((Q_1 - \mu Q_2)\, \delta P\right) \overset{(3.5.15)}{=} 2i\,\mathrm{tr}\left((Q_1 - \mu Q_2)\, [B, P]\right).$$

Assume that the products $(Q_1 - \mu Q_2)\, P$ and $P\, (Q_1 - \mu Q_2)$ are well-defined operators. Since B has finite support, we can then cyclically commute the operators in the trace and obtain

$$0 \;=\; 2i\,\mathrm{tr}\left(B\, [P,\, Q_1 - \mu Q_2]\right).$$

Since B is arbitrary, we conclude that $[P,\, Q_1 - \mu Q_2] = 0$, where our notation with the commutator implicitly contains the condition that the involved operator products must be well-defined. Thus our *Euler-Lagrange equations* are the commutator equations

$$[P,\, Q] \;=\; 0 \qquad \text{with} \qquad Q(x,y) \;=\; 2\, C_{xy}\, P(x,y)\,, \tag{3.5.20}$$

$$C_{xy} \;=\; \sum_{k=1}^{K} \left[\overline{\lambda_k} \;-\; \mu\, \frac{\overline{\lambda_k}}{|\lambda_k|} \sum_{l=0}^{K} n_l\, |\lambda_l| \right]_{xy} F_k\,. \tag{3.5.21}$$

In the formula (3.5.21) for C_{xy}, we consider the spectral decomposition (3.5.5, 3.5.11) of the closed chain $P(x,y)\, P(y,x)$. The equations (3.5.20, 3.5.21) are the equations of discrete space-time corresponding to the variational principle (3.5.9, 3.5.10).

3.6. Discussion

In the previous sections the principle of the fermionic projector was introduced in a rather abstract mathematical way. Our constructions departed radically from the conventional formulation of physics, so much so that the precise relation between the principle of the fermionic projector and the notions of classical and quantum physics is not obvious. In order to clarify the situation, we now describe the general physical concept behind the principle of the fermionic projector and explain in words the connection to classical field theory, relativistic quantum mechanics and quantum field theory. Since we must anticipate results which will be worked out later, the description in this section is clearly not rigorous and is intended only to give a brief qualitative overview.

The constructions in §3.1 and §3.2 are merely a reformulation of classical field theory and relativistic quantum mechanics. Although they are an important preparation for the following construction steps, they do not by themselves have new physical implications. Therefore, we need not consider them here and begin by discussing the concept of discrete space-time of §3.3. With our definition of discrete space-time, the usual space-time continuum is given up and resolved into discrete space-time points. A-priori, the discrete space-time points are merely a point set without any relations (like for example the nearest-neighbor relation on a lattice) between them. Thus one may think of discrete space-time as a "disordered accumulation of isolated points." There exists no time parameter, nor does it make sense to speak of the "spatial distance" between the space-time points. Clearly, this concept of a pure point set is too general for a reasonable description of space-time. Namely, we introduced discrete space-time with the intention of discretizing the space-time continuum on the Planck scale. Thus, for systems which are large compared to the Planck length, the discrete nature of space-time should not be apparent. This means that discrete space-time should, in a certain *continuum limit*, go over to a Lorentzian manifold. However, since M is merely a point set, discrete space-time $(H,\, <.|.>,\, (E_p)_{p \in M})$ is symmetric under permutations of the space-time points. Taking a naive continuum limit would imply that the points of space-time could be arbitrarily exchanged, in clear contradiction to the topological and causal structure of a Lorentzian manifold.

In order to avoid this seeming inconsistency, one must keep in mind that we introduced an additional object space-time: the fermionic projector P. Via its discrete kernel $P(x,y)$, the fermionic projector yields relations between the discrete space-time points. Our idea is that the discrete kernel should provide all structures needed for a

reasonable continuum limit. In more detail, our concept is as follows. In the space-time continuum (see Chapter 2), the fermionic projector is built up of all quantum mechanical states of the fermionic particles of the system. Closely following Dirac's original concept, we describe the vacuum by the "sea" of all negative-energy states; systems with particles and anti-particles are obtained by occupying positive-energy states and removing states from the Dirac sea, respectively. The fermionic projector of the continuum completely characterizes the physical system. In particular, its integral kernel $P(x, y)$ is singular if and only if y lies on the light cone centered at x. In this way, the fermionic projector of the continuum encodes the causal, and thus also topological, structure of the underlying space-time. We have in mind that the fermionic projector of discrete space-time should, similar to a regularization on the Planck scale, approximate the fermionic projector of the continuum. This means that on a macroscopic scale (i.e. for systems comprising a very large number of space-time points), the fermionic projector of discrete space-time can, to good approximation, be identified with a fermionic projector of the continuum. Using the just-mentioned properties of the continuum kernel, we conclude that the discrete kernel induces on discrete space-time a structure which is well-approximated by a Lorentzian manifold. However, on the Planck scale (i.e. for systems involving only few space-time points), the discrete nature of space-time becomes manifest, and the notions of space, time and causality cease to exit.

The critical step for making this concept precise is the formulation of the physical equations intrinsically in discrete space-time. Let us describe in principle how this is supposed to work. In the continuum description of Chapter 2, the fermionic projector satisfies the Dirac equation (2.3.10); furthermore the potentials entering the Dirac equation obey classical field equations. As a consequence of these equations, the fermionic projector of the continuum is an object with very specific properties. Our idea is that, using the special form of the fermionic projector, it should be possible to restate the Dirac equation and classical field equations directly in terms of the fermionic projector. Thus we wish to formulate equations into which the fermionic projector enters as the basic object, and which are equivalent to, or a generalization of, both the Dirac equation and the classical field equations. It turns out that it is impossible to state equations of this type in the space-time continuum, because composite expressions in the fermionic projector are mathematically ill-defined. But one can formulate mathematically meaningful equations in discrete space-time, removing at the same time the ultraviolet problems of the continuum theory. The variational principle (3.5.9, 3.5.10) leading to the Euler-Lagrange equations (3.5.20, 3.5.21) is an example for such equations. Note that this variational principle and the corresponding Euler-Lagrange equations in discrete space-time are clearly not causal, but, for consistency with relativistic quantum mechanics and classical field theory, we demand that they should, in the continuum limit, reduce to local and causal equations (namely, to the Dirac and classical field equations). Since the fermionic projector is not an object which is commonly considered in physics, it is difficult to give an immediate physical interpretation for the equations of discrete space-time; only a detailed mathematical analysis can provide an understanding of the variational principle. If one wishes, one can regard the equations of discrete space-time as describing a direct particle-particle interaction between all the states of the fermionic projector. The collective interaction of the fermions of the Dirac sea with the additional particles and holes should, in the continuum limit, give rise to an effective interaction of fermions and anti-fermions via

classical fields. Ultimately, the collective particle-particle interaction should even give a microscopic justification for the appearance of a continuous space-time structure.

Let us now describe the relation to quantum field theory. Since the coupled Dirac and classical field equations, combined with the pair creation/annihilation of Dirac's hole theory, yield precisely the Feynman diagrams of QFT (see e.g. [**BD1**]), it is clear that all results of perturbative quantum field theory, in particular the high precision tests of QFT, are also respected by our ansatz (provided that the equations of discrete space-time have the correct continuum limit). Thus the only question is if the particular effects of quantized fields, like the Planck radiation and the photo electric effect, can be explained in our framework. The basic physical assumption behind Planck's radiation law is that the energy levels of an electromagnetic radiation mode do not take continuous values, but are quantized in steps of $E = \hbar\omega$. While the quantitative value $\hbar\omega$ of the energy steps can be understood via the quantum mechanical identification of energy and frequency (which is already used in classical Dirac theory), the crucial point of Planck's assumption lies in the occurrence of discrete energy levels. The photo electric effect, on the other hand, can be explained by a "discreteness" of the electromagnetic interaction: the electromagnetic wave tends not to transmit its energy continuously, but prefers to excite few atoms of the photographic material. We have the conception (which will, however, not be worked out in this book) that these different manifestations of "discreteness" should follow from the equations of discrete space-time if one goes beyond the approximation of an interaction via classical fields.

If this concept of explaining the effects of quantized fields from the equations of discrete space-time were correct, it would even have consequences for the interpretation of quantum mechanics. Namely, according to the statistical interpretation, quantum mechanical particles are point-like; the absolute value $|\Psi(\vec{x})|^2$ of the wave function gives the probability density for the particle to be at the position $\vec{x}$. Here, we could regard the wave function itself as the physical object; the particle character would come about merely as a consequence of the "discreteness" of the interaction of the wave function with e.g. the atoms of a photographic material. The loss of determinism could be explained naturally by the non-causality of the equations of discrete space-time.

We conclude that the principle of the fermionic projector raises quite fundamental questions on the structure of space-time, the nature of field quantization and the interpretation of quantum mechanics. Before entering the study of these general questions, however, it is most essential to establish a quantitative connection between the equations of discrete space-time and the Dirac and classical field equations. Namely, the principle of the fermionic projector can make physical sense only if it is consistent with classical field theory and relativistic quantum mechanics; thus it is of importance to first check this consistency. Even this comparatively simple limiting case is of highest physical interest. Indeed, the principle of the fermionic projector provides a very restrictive framework for the formulation of physical models; for example there is no freedom in choosing the gauge groups, the coupling of the gauge fields to the fermions, or the masses of the gauge bosons. This means that, if a connection could be established to relativistic quantum mechanics and classical field theory, the principle of the fermionic projector would give an explanation for the interactions observed in nature and would yield theoretical predictions for particle masses and coupling constants. We begin with this study in the next chapters.

CHAPTER 4

The Continuum Limit

According to the principle of the fermionic projector, we want to formulate physics with the fermionic projector P in discrete space-time $(H, <.|.>, (E_p)_{p \in M})$. In this chapter we will establish a mathematically sound connection between this description and the usual formulation of physics in a space-time continuum. More precisely, we will develop a general technique with which equations in discrete space-time, like for example the Euler-Lagrange equations (3.5.20, 3.5.21), can be analyzed within the framework of relativistic quantum mechanics and classical field theory. Our approach is based on the assumption that the fermionic projector of discrete space-time can be obtained from the fermionic projector of the continuum by a suitable regularization process on the Planck scale. The basic difficulty is that composite expressions in the fermionic projector (like in (3.5.20)) depend essentially on how the regularization is carried out; our task is to analyze this dependence in detail. We will show that, if we study the behavior close to the light cone, the dependence on the regularization simplifies considerably and can be described by a finite number of parameters. Taking these parameters as free parameters, we will end up with a well-defined effective continuum theory.

We point out that, since we deduce the fermionic projector of discrete space-time from the fermionic projector of the continuum, the causal and topological structure of the space-time continuum, as well as the Dirac equation and Dirac's hole theory, will enter our construction from the very beginning. Thus the continuum limit cannot give a justification or even derivation of these structures from the equations of discrete space-time (for such a justification one must go beyond the continuum limit; see §5.6 for a first attempt in this direction). The reason why it is nevertheless interesting to analyze the continuum limit is that we do not need to specify the classical potentials which enter the Dirac equation; in particular, we do not assume that they satisfy the classical field equations. Thus we can hope that an analysis of the equations of discrete space-time should give constraints for the classical potentials; this means physically that the equations of discrete space-time should in the continuum limit yield a quantitative description of the interaction of the Dirac particles via classical fields. This quantitative analysis of the continuum limit of interacting systems will be explained in Chapters 6–8.

For clarity we will mainly restrict attention to a fermionic projector consisting of one Dirac sea of mass m. The generalizations to systems of fermions with different masses and to chiral fermions (as introduced in §2.3) are given in §4.5. Having gauge fields in mind, which are in quantum field theory described by bosons, we often refer to the external potentials contained in the operator $\mathcal{B}$ in the Dirac equation (2.3.10) as *bosonic potentials* and the corresponding fields as *bosonic fields*.

4.1. The Method of Variable Regularization

Let us consider how one can get a relation between the continuum fermionic projector and the description of physics in discrete space-time. As discussed in §3.6, discrete space-time should for macroscopic systems go over to the usual space-time continuum. For consistency with relativistic quantum mechanics, the fermionic projector of discrete space-time should in this limit coincide with the continuum fermionic projector. Using furthermore that the discretization length should be of the order of the Planck length, we conclude that the fermionic projector of discrete space-time should correspond to a certain "regularization" of the continuum fermionic projector on the Planck scale. Thus it seems a physically reasonable method to construct the fermionic projector of discrete space-time from the fermionic projector of the continuum by a suitable regularization process on the Planck scale.

Regularizations of the continuum theory are also used in perturbative QFT in order to make the divergent Feynman diagrams finite. However, there is the following major difference between the regularizations used in QFT and our regularization of the fermionic projector. In contrast to QFT, where the regularization is merely a mathematical technique within the renormalization procedure, we here consider the regularized fermionic projector as the object describing the physical reality. The regularized fermionic projector should be a model for the fermionic projector of discrete space-time, which we consider as the basic physical object. As an important consequence, it is not inconsistent for us if the effective continuum theory depends on how the regularization is carried out. In this case, we must regularize in such a way that the regularized fermionic projector is a good microscopic approximation to the "physical" fermionic projector of discrete space-time; only such a regularization can yield the correct effective continuum theory. This concept of giving the regularization a physical significance clearly suffers from the shortcoming that we have no detailed information about the microscopic structure of the fermionic projector in discrete space-time, and thus we do not know how the correct regularization should look like. In order to deal with this problem, we shall consider a general class of regularizations. We will analyze in detail how the effective continuum theory depends on the regularization. Many quantities will depend sensitively on the regularization, so much so that they are undetermined and thus ill-defined in the continuum limit. However, certain quantities will be independent of the regularization and have a simple correspondence in the continuum theory; we call these quantities *macroscopic*. We will try to express the effective continuum theory purely in terms of macroscopic quantities. We cannot expect that the effective continuum theory will be completely independent of the regularization. But for a meaningful continuum limit, it must be possible to describe the dependence on the regularization by a small number of parameters, which we consider as empiric parameters modelling the unknown microscopic structure of discrete space-time. We refer to this general procedure for constructing the effective continuum theory as the *method of variable regularization*.

In order to illustrate the method of variable regularization, we mention an analogy to solid state physics. On the microscopic scale, a solid is composed of atoms, which interact with each other quantum mechanically. On the macroscopic scale, however, a solid can be regarded as a continuous material, described by macroscopic quantities like the density, the pressure, the conductivity, etc. The macroscopic quantities satisfy macroscopic physical equations like the equations of continuum mechanics, Ohm's law,

etc. Both the macroscopic characteristics of the solid and the macroscopic physical laws can, at least in principle, be derived microscopically from many-particle quantum mechanics. However, since the details of the microscopic system (e.g. the precise form of the electron wave functions) are usually not known, this derivation often does not completely determine the macroscopic physical equations. For example, it may happen that a macroscopic equation can be derived only up to a proportionality factor, which depends on unknown microscopic properties of the solid and is thus treated in the macroscopic theory as an empirical parameter. The physical picture behind the method of variable regularization is very similar to the physics of a solid, if one considers on the microscopic scale our description of physics in discrete space-time and takes as the macroscopic theory both relativistic quantum mechanics and classical field theory. Clearly, the concept of discrete space-time is more hypothetical than atomic physics because it cannot at the moment be verified directly in experiments. But we can nevertheless get indirect physical evidence for the principle of the fermionic projector by studying whether or not the method of variable regularization leads to interesting results for the continuum theory.

In the remainder of this section we will specify for which class of regularizations we shall apply the method of variable regularization. Our choice of the regularization scheme is an attempt to combine two different requirements. On one hand, we must ensure that the class of regularizations is large enough to clarify the dependence of the effective continuum theory on the regularization in sufficient detail; on the other hand, we must keep the technical effort on a reasonable level. Consider the integral kernel of the continuum fermionic projector (2.3.19, 2.5.45). Under the reasonable assumption that the fermionic wave functions Ψ_k and Φ_l are smooth, the projectors on the particle/anti-particle states in (2.3.19) are smooth in x and y. The non-causal low- and high-energy contributions P^{le} and P^{he} as well as the phase-inserted line integrals in (2.5.45) also depend smoothly on x and y. The factors $T^{(n)}$, however, have singularities and poles on the light cone (see (2.5.42) and (2.5.43)). Let us consider what would happen if we tried to formulate a variational principle similar to that in §3.5 with the continuum kernel (instead of the discrete kernel). The just-mentioned smooth terms in the kernel would not lead to any difficulties; we could just multiply them with each other when forming the closed chain $P(x,y)\,P(y,x)$, and the resulting smooth functions would influence the eigenvalues $\lambda_k(x,y)$ in (3.5.5) in a continuous way. However, the singularities of $T^{(n)}$ would cause severe mathematical problems because the multiplication of $T^{(n)}(x,y)$ with $T^{(n)}(y,x)$ leads to singularities which are ill-defined even in the distributional sense. For example, the naive product $P(x,y)\,P(y,x)$ would involve singularities of the form $\sim \delta'((y-x)^2)\,\delta((y-x)^2)$ and $\sim \delta((y-x)^2)^2$. This simple consideration shows why composite expressions in the fermionic projector make mathematical sense only after regularization. Furthermore, one sees that the regularization is merely needed to remove the singularities of $T^{(n)}$. Hence, it seems reasonable to regularize only the factors $T^{(n)}$ in (2.5.45), but to leave the fermionic wave functions Ψ_a, Φ_a as well as the bosonic potentials unchanged. This regularization method implies that the fermionic wave functions and the bosonic potentials are well-defined also for the regularized fermionic projector; using the notation of page 90, they are macroscopic quantities. Therefore, we call our method of only regularizing $T^{(n)}$ the *assumption of macroscopic potentials and wave functions.*

The assumption of macroscopic potentials and wave functions means physically that energy and momentum of all bosonic fields and of each particle/anti-particle of

the physical system should be small compared to the Planck energy. In other words, we exclude the case that the physical potentials and wave functions have oscillations or fluctuations on the Planck scale. Namely, such microscopic inhomogeneities could not be described by smooth functions in the continuum limit and are thus not taken into account by our regularization method. If, conversely, the potentials and wave functions are nearly constant on the Planck scale, the unregularized and the (no matter by which method) regularized quantities almost coincide, and it is thus a good approximation to work in the regularized fermionic projector with the unregularized potentials and wave functions.

According to the assumption of macroscopic potentials and wave functions, it remains to regularize the factors $T^{(n)}$ in (2.5.45). Recall that we constructed the distributions $T^{(n)}$ from the continuum kernel of the fermionic projector of the vacuum (2.2.1) via (2.5.39) and the expansion in the mass parameter (2.5.43). An essential step for getting a meaningful regularization scheme is to extend this construction to the case with regularization. Namely, this extension makes it sufficient to specify the regularization of the fermionic projector of the vacuum; we can then deduce the regularized $T^{(n)}$ and obtain, by substitution into (2.5.45), the regularized fermionic projector with interaction (if it were, on the contrary, impossible to derive the regularized $T^{(n)}$ from the regularized fermionic projector of the vacuum, the independent regularizations of all functions $T^{(n)}$, $n = -1, 0, 1, \ldots$, would involve so many free parameters that the effective continuum theory would be under-determined). Having in mind the extension of (2.5.39) and (2.5.43) to the case with regularization (which will be carried out in §4.5 and Appendix D), we now proceed to describe our regularization method for the fermionic projector of the vacuum. In the vacuum and for one Dirac sea, the kernel of the continuum fermionic projector $P(x, y)$ is given by the Fourier integral (2.2.2),

$$P(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\, e^{-ik(x-y)}\,. \tag{4.1.1}$$

This distribution is invariant under translations in space-time, i.e. it depends only on the difference vector $y - x$. It seems natural and is most convenient to preserve the translation symmetry in the regularization. We thus assume that the kernel of the regularized fermionic projector of the vacuum, which we denote for simplicity again by $P(x, y)$, is translation invariant,

$$P(x,y) \;=\; P(y-x) \quad \text{for} \quad x, y \in M \subset \mathbb{R}^4\,. \tag{4.1.2}$$

We refer to (4.1.2) as a *homogeneous regularization of the vacuum*. Notice that the assumption (4.1.2) allows for both discrete and continuum regularizations. In the first case, the set M is taken to be a discrete subset of $\mathbb{R}^4$ (e.g. a lattice), whereas in the latter case, $M = \mathbb{R}^4$. According to our concept of discrete space-time, it seems preferable to work with discrete regularizations. But since continuous regularizations give the same results and are a bit easier to handle, it is worth considering them too. The assumption of a homogeneous regularization of the vacuum means physically that the inhomogeneities of the fermionic projector on the Planck scale should be irrelevant for the effective continuum theory. Since such microscopic inhomogeneities can, at least in special cases, be described by microscopic gravitational or gauge fields, this assumption is closely related to the assumption of macroscopic potentials and wave functions discussed above.

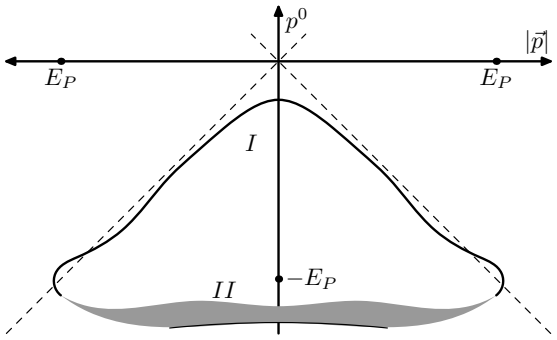

FIGURE 4.1. Example for $\hat{P}$, the regularized fermionic projector of the vacuum in momentum space.

Taking the Fourier transform in the variable $y-x$, we write (4.1.2) as the Fourier integral

$$P(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{P}(p)\, e^{-ip(x-y)} \tag{4.1.3}$$

with a distribution $\hat{P}$. If one considers a discrete regularization, $\hat{P}$ may be defined only in a bounded region of $\mathbb{R}^4$ (for a lattice regularization with lattice spacing d, for example, one can restrict the momenta to the "first Brillouin zone" $p \in (-\frac{\pi}{d}, \frac{\pi}{d})^4$). In this case, we extend $\hat{P}$ to all $\mathbb{R}^4$ by setting it to zero outside this bounded region. Although it will be of no relevance for what follows, one should keep in mind that for a discrete regularization, x and y take values only in the discrete set M. Let us briefly discuss the properties of the distribution $\hat{P}$. First of all, $P(x,y)$ should be the kernel of a Hermitian operator; this implies that $P(x,y)^* = P(y,x)$ and thus

$$\hat{P}(p)^* \;=\; \hat{P}(p) \qquad \text{for all } p \tag{4.1.4}$$

(where the star again denotes the adjoint with respect to the spin scalar product). For consistency with the continuum theory, the regularized kernel (4.1.3) should, for macroscopic systems, go over to the continuum kernel (4.1.1). Thus we know that $\hat{P}(p)$ should, for small energy-momentum p (i.e. when both the energy p^0 and the momentum $|\vec{p}|$ are small compared to the Planck energy), coincide with the distribution $(\not{p} + m)\, \delta(p^2 - m^2)\, \Theta(-p^0)$. This is illustrated in the example of Figure 4.1. In the region I close to the origin, $\hat{P}$ looks similar to a hyperbola on the lower mass shell. Furthermore, we know that $\hat{P}$ is a regularization on the Planck scale. This means that, in contrast to the integrand in (4.1.1), $\hat{P}$ should decay at infinity, at least so rapidly that the integral (4.1.3) is finite for all x and y. The length scale for this decay in momentum space should be of the order of the Planck energy $E_P = l_P^{-1}$. However, the precise form of $\hat{P}$ for large energy or momentum is completely arbitrary, as is indicated in Figure 4.1 by the "high energy region" II. This arbitrariness reflects our freedom in choosing the regularization.

We finally make an ansatz for $\hat{P}$ which seems general enough to include all relevant regularization effects and which will considerably simplify our analysis. According to (4.1.4), $\hat{P}(p)$ is a Hermitian 4×4 matrix and can thus be written as a real linear combination of the basis of the Dirac algebra $\mathbb{1}$, $i\rho$, $\rho\gamma^j$ and σ^{jk} (with the pseudoscalar matrix $\rho = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the bilinear covariants $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$). The integrand of the

continuum kernel (4.1.1) contains only vector and scalar components. It is reasonable to assume that the regularized kernel also contains no pseudoscalar and pseudovector components, because the regularization would otherwise break the symmetry under parity transformations. The inclusion of a bilinear component in $\hat{P}$, on the other hand, would cause technical complications but does not seem to give anything essentially new. Thus we make an ansatz where $\hat{P}$ is composed only of a vector and a scalar component, more precisely

$$\hat{P}(p) \;=\; (v_j(p)\,\gamma^j \;+\; \phi(p)\,\mathbb{1})\,f(p) \tag{4.1.5}$$

with a vector field v and a scalar field ϕ; f is a distribution. We also need to assume that $\hat{P}$ is reasonably regular and well-behaved; this will be specified in the following sections. We refer to the ansatz (4.1.5) as the assumption of a *vector-scalar structure* for the fermionic projector of the vacuum.

4.2. The Regularized Product $P(x,y)\,P(y,x)$ in the Vacuum

According to the method of variable regularization, we must analyze how the effective continuum theory depends on the choice of the regularization. We shall now consider this problem for the simplest composite expression in the fermionic projector, the closed chain $P(x,y)\,P(y,x)$ in the vacuum. The discussion of this example will explain why we need to analyze the fermionic projector on the light cone. Working out this concept mathematically will eventually lead us to the general formalism described in §4.5.

Using the Fourier representation (4.1.3), we can calculate the closed chain to be

$$\begin{aligned} P(x,y)\,P(y,x) &= \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4}\, \hat{P}(k_1)\,\hat{P}(k_2)\, e^{-i(k_1-k_2)(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\int \frac{d^4q}{(2\pi)^4}\, \hat{P}(p+q)\,\hat{P}(q) \right] e^{-ip(x-y)}\,, \end{aligned} \tag{4.2.1}$$

where we introduced new integration variables $p = k_1 - k_2$ and $q = k_2$. Thus the Fourier transform of the closed chain is given by the convolution in the square brackets. This reveals the following basic problem. The convolution in the square bracket involves $\hat{P}$ for small and for large energy-momentum. Even when p is small, a large q leads to a contribution where both factors $\hat{P}(p+q)$ and $\hat{P}(q)$ are evaluated for large energy-momenta. If we look at the example of Figure 4.1, this means that (4.2.1) depends essentially on the behavior of $\hat{P}$ in the high-energy region II and can thus have an arbitrary value. More generally, we conclude that, since the form of $\hat{P}$ for large energy or momentum is unknown, the value of (4.2.1) is undetermined.

At first sight, it might seem confusing that the pointwise product $P(x,y)\,P(y,x)$ of the regularized fermionic projector should be undetermined, although the unregularized kernel (4.1.1) is, for $y-x$ away from the light cone, a smooth function, and so pointwise multiplication causes no difficulties. In order to explain the situation in a simple example, we briefly discuss the fermionic projector $\tilde{P}$ obtained by adding to P a plane wave,

$$\tilde{P}(x,y) \;=\; P(x,y) \;+\; e^{-ik(x-y)}\,\mathbb{1}\,.$$

If the energy or the momentum of the plane wave is of the order of the Planck energy, the plane wave is highly oscillatory in space-time. Such an oscillatory term is irrelevant on the macroscopic scale. Namely, if $\tilde{P}$ acts on a macroscopic function η, the oscillatory term is evaluated in the weak sense, and the resulting integral $\int \exp(iky)\,\eta(y)\,d^4y$ gives

almost zero because the contributions with opposite signs compensate each other. This "oscillation argument" can be made mathematically precise using integration by parts, e.g. in the case of high energy $k^0 \sim E_P$,

$$\int e^{iky}\, f(y)\, d^4y \;=\; -\frac{1}{ik^0} \int e^{iky}\, (\partial_t f)\, d^4y \;\sim\; \frac{1}{E_P}\,.$$

In the corresponding closed chain

$$\tilde{P}(x,y)\, \tilde{P}(y,x) \;=\; P(x,y)\, P(y,x) \;+\; P(x,y)\, e^{-ik(y-x)} \;+\; e^{-ik(x-y)}\, P(y,x) \;+\; \mathbb{1}\,,$$

the second and third summands are also oscillatory. In the last summand, however, the oscillations have dropped out, so that this term affects the macroscopic behavior of the closed chain. This elementary consideration illustrates why the unknown high-energy contribution to the fermionic projector makes it impossible to determine the closed chain pointwise. We remark that for very special regularizations, for example the regularization by convolution with a smooth "mollifier" function having compact support, the pointwise product makes sense away from the light cone and coincides approximately with the product of the unregularized kernels. But such regularizations seem too restrictive. We want to allow for the possibility that the fermionic projector describes non-trivial (yet unknown) high-energy effects. Therefore, the high-energy behavior of the fermionic projector should not be constrained by a too simple regularization method.

The fact that the product $P(x,y)\,P(y,x)$ is undetermined for fixed x and y does not imply that a pointwise analysis of the closed chain is mathematically or physically meaningless. But it means that a pointwise analysis would essentially involve the unknown high-energy behavior of $\hat{P}$; at present this is a problem out of reach. Therefore, our strategy is to find a method for evaluating the closed chain in a way where the high-energy behavior of $\hat{P}$ becomes so unimportant that the dependence on the regularization can be described in a simple way. We hope that this method will lead us to a certain limiting case in which the equations of discrete space-time become manageable.

The simplest method to avoid the pointwise analysis is to evaluate the closed chain in the weak sense. The Fourier representation (4.2.1) yields that

$$\int P(x,y)\, P(y,x)\, \eta(x)\, d^4x \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{\eta}(p) \left[\int \frac{d^4q}{(2\pi)^4}\, \hat{P}(p+q)\, \hat{P}(q) \right], \qquad (4.2.2)$$

where $\hat{\eta}$ is the Fourier transform of a smooth function η. For macroscopic η (i.e. a function which is nearly constant on the Planck scale), the function $\hat{\eta}(p)$ is localized in a small neighborhood of $p = 0$ and has rapid decay. Thus, exactly as (4.2.1), the integral (4.2.2) depends on the form of $\hat{P}$ for large energy-momentum. Hence this type of weak analysis is not helpful. In order to find a better method, we consider again the Fourier integral (4.1.3) in the example of Figure 4.1. We want to find a regime for $y - x$ where the "low energy region" I plays an important role, whereas the region II is irrelevant. This can be accomplished only by exploiting the special form of $\hat{P}$ in the low-energy region as follows. The hyperbola of the lower mass shell in region I comes asymptotically close to the cone $C = \{p^2 = 0\}$. If we choose a vector $(y-x) \neq 0$ on the light cone $L = \{(y-x)^2 = 0\}$, then the hypersurface $\mathcal{H} = \{p \,|\, p(y-x) = 0\}$ is null and thus tangential to the cone C. This means that for all states on the hyperbola which are close to the straight line $C \cap \mathcal{H}$, the exponential in (4.1.3) is approximately one. Hence all these states are "in phase" and thus yield a large contribution to the Fourier

integral (4.1.3). The states in the high-energy region II, however, are not in phase; they will give only a small contribution to (4.1.3), at least when the vector $(y-x) \in L$ is large, so that the exponential in (4.1.3) is highly oscillatory on the scale $p \sim E_P$. This qualitative argument shows that by considering the fermionic projector on the light cone, one can filter out information on the behavior of $\hat{P}$ in the neighborhood of a straight line along the cone C. This should enable us to analyze the states on the lower mass shell without being affected too much by the unknown high-energy behavior of $\hat{P}$. We point out that if $P(x,y)$ depends only on the behavior of $\hat{P}$ close to the cone C, then the same is immediately true for composite expressions like the product $P(x,y)\, P(y,x)$. Thus restricting our analysis to the light cone should simplify the dependence on the regularization considerably, also for composite expressions like the closed chain. Our program for the remainder of this chapter is to make this qualitative argument mathematically precise and to quantify it in increasing generality.

4.3. The Regularized Vacuum on the Light Cone, Scalar Component

For simplicity we begin the analysis on the light cone for the scalar component of (4.1.5), i.e. we consider the case

$$\hat{P}(p) \;=\; \phi(p)\, f(p) \tag{4.3.1}$$

(the vector component will be treated in the next section). We can assume that the spatial component of the vector $y-x$ in (4.1.3) points in the direction of the x-axis of our Cartesian coordinate system, i.e. $y-x = (t, r, 0, 0)$ with $r > 0$. Choosing cylindrical coordinates ω, k, ρ and φ in momentum space, defined by $p = (\omega, \vec{p})$ and $\vec{p} = (k,\ \rho\cos\varphi,\ \rho\sin\varphi)$, the Fourier integral (4.1.3) takes the form

$$P(x,y) \;=\; \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; \hat{P}(\omega,k,\rho,\varphi)\; e^{i\omega t - ikr}\,. \tag{4.3.2}$$

Since the exponential factor in this formula is independent of ρ and φ, we can write the fermionic projector as the two-dimensional Fourier transform

$$P(x,y) \;=\; 2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk\; h(\omega,k)\; e^{i\omega t - ikr} \tag{4.3.3}$$

of a function h defined by

$$h(\omega,k) \;=\; \frac{1}{2\,(2\pi)^4} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; (\phi\, f)(\omega,k,\rho,\varphi)\,. \tag{4.3.4}$$

We want to analyze $P(x,y)$ close to the light cone $(y-x)^2 = 0$ away from the origin $y = x$. Without loss of generality, we can restrict attention to the upper light cone $t = r$. Thus we are interested in the region $t \approx r > 0$. The "light-cone coordinates"

$$s \;=\; \frac{1}{2}\,(t-r)\,, \qquad\qquad l \;=\; \frac{1}{2}\,(t+r) \tag{4.3.5}$$

are well-suited to this region, because the "small" variable s vanishes for $t = r$, whereas the "large" variable l is positive and non-zero. Introducing also the associated momenta

$$u \;=\; -k-\omega\,, \qquad\qquad v \;=\; k-\omega\,, \tag{4.3.6}$$

we can write the fermionic projector as

$$P(s,l) \;=\; \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h(u,v)\, e^{-i(us+vl)}\,. \tag{4.3.7}$$

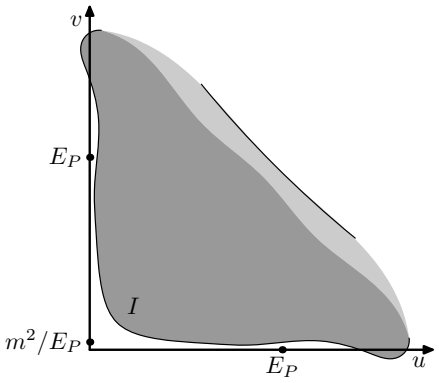

FIGURE 4.2. Example for $h(u,v)$, the reduced two-dimensional distribution.

Let us briefly discuss the qualitative form of the function h, (4.3.4). According to the continuum kernel (4.1.1), the scalar component (4.3.1) should, for energy and momentum small compared to the Planck energy E_P, go over to the δ-distribution on the lower mass shell $\hat{P} = m\,\delta(p^2 - m^2)\,\Theta(-p^0)$. In this limit, the integral (4.3.4) can be evaluated to be

$$\begin{aligned} h &= \frac{m}{2\,(2\pi)^4} \int_0^\infty \rho\,d\rho \int_0^{2\pi} d\varphi\; \delta(\omega^2 - k^2 - \rho^2 - m^2)\,\Theta(-\omega) \\ &= \frac{m}{4\,(2\pi)^3}\,\Theta(\omega^2 - k^2 - m^2)\,\Theta(-\omega) \;=\; \frac{m}{32\pi^3}\,\Theta(uv - m^2)\,\Theta(u)\,; \end{aligned} \tag{4.3.8}$$

thus integrating out ρ and φ yields a constant function in the interior of the two-dimensional "lower mass shell" $\omega^2 - k^2 = m^2$, $\omega < 0$. From this we conclude that for $u, v \ll E_P$, $h(u,v)$ should have a discontinuity along the hyperbola $\{uv = m^2,\, u > 0\}$, be zero below (i.e. for $uv < m^2$) and be nearly constant above. Furthermore, we know that h decays at infinity on the scale of the Planck energy. Similar to our discussion of $\hat{P}$ after (4.1.4), the precise form of h for large energy or momentum is completely arbitrary. The function $h(u,v)$ corresponding to the example of Figure 4.1 is shown in Figure 4.2. The two branches of the hyperbola asymptotic to the u and v axes are labeled by "A" and "B," respectively.

It is instructive to consider the energy scales of our system. The scale for high energies is clearly given by the Planck energy E_P. The relevant low-energy scale, on the other hand, is m^2/E_P (it is zero for massless fermions). This is because the hyperbola $uv = m^2$ comes as close to the v-axis as as $v \sim m^2/E_P$ before leaving the low-energy region. These two energy scales are also marked in Figure 4.2. Since we want to analyze the situation close to the light cone, we choose the "small" light-cone parameter s on the Planck scale, i.e.

$$s \sim E_P^{-1} \qquad \text{or} \qquad s < E_P^{-1}\,. \tag{4.3.9}$$

The "large" light-cone parameter l, on the other hand, is non-zero and thus yields a third energy scale. We shall always choose this scale between the two extremal energy scales, more precisely

$$\frac{1}{E_P} \;\ll\; l \;<\; l_{\max} \;\ll\; \frac{E_P}{m^2}\,. \tag{4.3.10}$$

The parameter $l_{\max}$ was introduced here in order to avoid l being chosen too large. Namely, we will always regard l as being small compared to the length scales of macroscopic physics (a reasonable value for $l_{\max}$ would e.g. be the Fermi length). One should keep in mind that the quotient of the two fundamental energy scales is in all physical situations extremely large; namely $E_P^2/m^2 \gg 10^{35}$. Thus the constraints (4.3.10) can be easily satisfied and still leave us the freedom to vary l on many orders of magnitude.

In the remainder of this section we shall evaluate the Fourier integral (4.3.7) using the scales (4.3.9) and (4.3.10). In preparation, we discuss and specify the function $h(u,v)$ for fixed u, also denoted by $h_u(v)$. As one sees in Figure 4.2, h_u will in general not be continuous. More precisely, in the example of Figure 4.2, h_u has a discontinuous "jump" from zero to a finite value on the hyperbola (and its extension to the high-energy region) and maybe has a second jump to zero for large v (e.g. on line "a"). For simplicity, we assume that h_u is always of this general form, i.e.

$$h_u(v) \;=\; \begin{cases} 0 & \text{for } v < \alpha_u \text{ or } v > \beta_u \\ \text{smooth} & \text{for } \alpha_u \leq v \leq \beta_u \end{cases} \tag{4.3.11}$$

with parameters $\alpha_u < \beta_u$. The case of less than two discontinuities can be obtained from (4.3.11) by setting $h_u(\alpha_u)$ or $h_u(\beta_u)$ equal to zero, or alternatively by moving the position of the discontinuities α_u or β_u to infinity. We remark that the discontinuity at $v = \beta_u$ will become irrelevant later; it is here included only to illustrate why the behavior of the fermionic projector on the light cone is independent of many regularization details. Without regularization, $h_u(v)$ is for $v \geq \alpha_u$ a constant function, (4.3.8). Thus the v-dependence of $h_u(v)$ for $\alpha_u \leq v \leq \beta_u$ is merely a consequence of the regularization, and it is therefore reasonable to assume that the v-derivatives of $h_u(v)$ scale in inverse powers of the regularization length E_P. More precisely, we demand that there is a constant $c_1 \ll l E_P$ with

$$|h_u^{(n)}(v)| \;\leq\; \left(\frac{c_1}{E_P}\right)^n \max|h_u| \qquad \text{for } \alpha_u \leq v \leq \beta_u\,, \tag{4.3.12}$$

where the derivatives at $v = \alpha_u$ and β_u are understood as the right- and left-sided limits, respectively. This regularity condition is typically satisfied for polynomial, exponential and trigonometric functions, but it excludes small-scale fluctuations of h_u. Clearly, we could also consider a more general ansatz for h_u with more than two discontinuities or weaker regularity assumptions. But this does not seem to be the point because all interesting effects, namely the influence of discontinuities for small and large v, as well as of smooth regions, can already be studied in the setting (4.3.11, 4.3.12).

Let us analyze the v-integral of the Fourier transform (4.3.7),

$$P_u(l) \;:=\; \int_{-\infty}^{\infty} h_u(v)\, e^{-ivl}\, dv\,. \tag{4.3.13}$$

According to the first inequality in (4.3.10), the exponential factor in (4.3.13) is highly oscillatory on the scale $v \sim E_P$. Thus we can expect that the smooth component of h_u gives only a small contribution to the integral (4.3.13), so that the discontinuities at α_u and β_u play the dominant role. In order to make this picture mathematically

precise, we iteratively integrate in (4.3.13) K times by parts,

$$\begin{aligned} P_u(l) &= \int_{\alpha_u}^{\beta_u} h_u(v)\, e^{-ivl}\, dv \;=\; -\frac{1}{il} \int_{\alpha_u}^{\beta_u} dv\; h_u(v)\, \frac{d}{dv} e^{-ivl} \\ &= -\frac{1}{il}\, h_u(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} + \frac{1}{il} \int_{\alpha_u}^{\beta_u} h_u'(v)\, e^{-ivl}\, dl \;=\; \cdots \;= \\ &= -\frac{1}{il} \sum_{n=0}^{K-1} \left(\frac{1}{il}\right)^n h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} + \left(\frac{1}{il}\right)^K \int_{\alpha_u}^{\beta_u} h_u^{(K)}(v)\, e^{-ivl}\, dl\,. \end{aligned} \tag{4.3.14}$$

If we bound all summands in (4.3.14) using the first inequality in (4.3.10) and the regularity condition (4.3.12), each v-derivative appears in combination with a power of l^{-1}, and this gives a factor $c_1/(lE_P) \ll 1$. Thus we can in the limit $K \to \infty$ drop the integral in (4.3.14) and obtain

$$P_u(l) \;=\; -\frac{1}{il} \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^n h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u}\,. \tag{4.3.15}$$

This expansion converges, and its summands decay like $(c_1/(lE_P))^n$.

Using (4.3.13), we can write the Fourier transform (4.3.7) as

$$P(s,l) \;=\; \int_{-\infty}^{\infty} P_u(l)\, e^{-ius}\, du\,. \tag{4.3.16}$$

Notice that, apart from the constraints (4.3.10), the "large" variable l can be freely chosen. We want to study the functional dependence of (4.3.16) on the parameter l. In preparation, we consider an integral of the general form

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du\,, \tag{4.3.17}$$

where we assume that $(u, \gamma(u))$ is a curve in the high-energy region, more precisely $\gamma \sim E_P$. Assume furthermore that γ is monotone with $|\gamma'| \sim 1$ and that $(b-a) \sim E_P$. By transforming the integration variable, we can then write (4.3.17) as the Fourier integral

$$\int_{\gamma(a)}^{\gamma(b)} f\, |\gamma'|^{-1}\, e^{-i\gamma l}\, d\gamma\,. \tag{4.3.18}$$

If the function $f\,|\gamma'|^{-1}$ is smooth, its Fourier transform (4.3.18) has rapid decay in the variable l. Under the stronger assumption that $f\,|\gamma'|^{-1}$ varies on the scale E_P, we conclude that the length scale for this rapid decay is of the order $l \sim E_P^{-1}$. As a consequence, the rapid decay can be detected even under the constraint $l < l_{\max}$ imposed by (4.3.10), and we say that (4.3.18) has *rapid decay in* l. The reader who feels uncomfortable with this informal definition can immediately make this notion mathematically precise by an integration by parts argument similar to (4.3.14) imposing for $f\,|\gamma'|^{-1}$ a condition of type (4.3.12). The precise mathematical meaning of rapid decay in l for the integral (4.3.17) is that for every integer k there should be constants $c \sim 1$ and $l_{\min} \ll l_{\max}$ such that for all $l \in (l_{\min}, l_{\max})$,

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du \;\leq\; c\, (lE_P)^{-k} \int_a^b |f(u)|\, du\,.$$

We return to the analysis of the integral (4.3.16). The boundary terms of (4.3.15) at β_u yield contributions to $P(s,l)$ of the form

$$-\left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\beta_u)\, e^{-i\beta_u l - ius}\, du\,. \tag{4.3.19}$$

Recall that the points (u, β_u) are in the high-energy region (in the example of Figure 4.2, these points lie on curve "a"). According to (4.3.9), the length scale for the oscillations of the factor $\exp(-ius)$ is $u \sim E_P$. Under the reasonable assumption that β_u is monotone and that the functions $|\beta'(u)|^{-1}$ and $h_u^{(n)}(\beta_u)$ vary on the scale E_P, the integral (4.3.19) is of the form (4.3.18), and the above consideration yields that (4.3.19) has rapid decay in l. We remark that this argument could be extended to the case where β_u has extremal points (basically because the extrema give contributions only for isolated momenta u and thus can be shown to be negligible), but we will not go into this here. Having established rapid decay in l for (4.3.19), it remains to consider the boundary terms in (4.3.19) at α_u, more precisely

$$\begin{aligned} P(s,l) &= \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-i\alpha_u l - ius}\, du \\ &\quad + (\text{rapid decay in } l)\,. \end{aligned} \tag{4.3.20}$$

We cannot again apply our "oscillation argument" after (4.3.17), because α_u tends asymptotically to zero on branch "A" of the hyperbola (see Figure 4.2), so that the factor $\exp(-i\alpha_u l)$ is non-oscillating in this region. We expand this factor in a Taylor series,

$$P(s,l) = \sum_{n,k=0}^{\infty} \frac{1}{k!}\, (il)^{k-n-1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, (-\alpha_u)^k\, e^{-ius}\, du\,. \tag{4.3.21}$$

In the region where $l\alpha_u \ll 1$, this expansion might seem problematic and requires a brief explanation. First of all, α_u becomes large near $u = 0$ (on branch "B" of the hyperbola in Figure 4.2). In the case without regularization, the power expansion of the factor $\exp(-i\alpha_u\, l)$ corresponds to an expansion in the mass parameter (recall that in this case, $\alpha_u = m^2/u$ according to (4.3.8)), and in (4.3.21) it would lead to a singularity of the integrand at the origin. Indeed, this difficulty is a special case of the logarithmic mass problem which was mentioned in §2.5 and was resolved by working with the "regularized" distribution T_a^{reg} (2.5.42). Using these results, the behavior of the unregularized $P(s,l)$ for small momenta $u \ll E_P$ is well understood. Our oscillation argument after (4.3.17) yields that the regularization for $u \ll E_P$ (i.e. the form of the extension of branch "B" of the hyperbola to the high-energy region) affects $P(s,l)$ merely by rapidly decaying terms. Thus it is sufficient to consider here the integrand in (4.3.21) away from the origin $u = 0$. When combined with the results in §2.5, our analysis will immediately yield a complete description of the regularized fermionic projector near the light cone. Furthermore, the function α_u might become large for $u \sim E_P$, and this is a more subtle point. One way of justifying (4.3.21) would be to simply assume that $l_{\max}\alpha_u \ll 1$ along the whole extension of branch "A" to the high-energy region. A more general method would be to split up the curve (u, α_u) in the high-energy region $u \sim E_P$ into one branch where the expansion (4.3.21) is justified and another branch where our oscillation argument after (4.3.17) applies. The intermediate region $l\alpha_u \sim 1$, where none of the two methods can be

used, is generically so small that it can be neglected. In order to keep our analysis reasonably simple, we here assume that α_u is sufficiently small away from the origin, more precisely

$$\alpha_u \;<\; \alpha_{\max} \;\ll\; l_{\max}^{-1} \qquad \text{for } u \sim E_P. \tag{4.3.22}$$

For a fixed value of $k-n$, all summands in (4.3.21) have the same l-dependence. Let us compare the relative size of these terms. According to our regularity assumption (4.3.12), the derivatives of h scale like $h_u^{(n)} \sim E_P^{-n}$. Using the bound (4.3.22), we conclude that, for a fixed power of l, the summands in (4.3.21) decrease like $(\alpha_{\max}/E_P)^n$. Thus it is a very good approximation to drop the summands for large n. At first sight, it might seem admissible to take into account only the first summand $n=0$. But the situation is not quite so simple. For example, it may happen that, when restricted to the curve (u, α_u), the function $h(u,v)$ is so small that the summands for $n=0$ in (4.3.21) are indeed not dominant. More generally, we need to know that for some $n_0 \geq 0$, the function $h_u^{(n_0)}(\alpha_u)$ is really of the order given in (4.3.12), i.e.

$$|h_u^{(n_0)}(\alpha_u)| \;\geq\; c \left(\frac{c_1}{E_P}\right)^{n_0} \max |h_u| \qquad \text{for } u \sim E_P \tag{4.3.23}$$

with a positive constant c which is of the order one. If this condition is satisfied, we may neglect all summands for $n > n_0$, and collecting the terms in powers of l, we conclude that

$$\begin{aligned} &P(s,l) \\ &\quad= \frac{1}{(il)^{n_0+1}} \sum_{k=0}^{\infty} (-il)^k \sum_{n=\max(n_0-k,0)}^{n_0} \frac{(-1)^{n_0-n}}{(k-n_0+n)!} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, \alpha_u^{k-n_0+n}\, e^{-ius}\, du \\ &\qquad + \sum_{n=n_0+1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-ius}\, du \;+\; (\text{rapid decay in } l) \\ &\qquad + (\text{higher orders in } (\alpha_{\max}/E_P)\,. \end{aligned} \tag{4.3.24}$$

We point out that, according to (4.3.22),

$$\alpha_{\max}/E_P \ll (l_{\max} E_P)^{-1}\,,$$

and this explains why we disregard the higher orders in $\alpha_{\max}/E_P$. In our case, the function h_u has in the low-energy region according to (4.3.8) the form $h_u(\alpha_u) = m/(32\pi^3)\,\Theta(u)$. Hence it is natural to assume that (4.3.23) is satisfied for $n_0 = 0$. Introducing the shorter notation

$$h(u) \;:=\; h_u(\alpha(u))\,, \qquad h^{[n]}(u) \;:=\; h_u^{(n)}(\alpha_u)\,, \qquad \alpha(u) \;:=\; \alpha_u\,, \tag{4.3.25}$$

we have thus derived the following result.

Expansion of the scalar component: *Close to the light cone (4.3.9, 4.3.10), the*

scalar component (4.3.1) of the fermionic projector of the vacuum has the expansion

$$P(s,l) \;=\; \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} h\, \alpha^k\, e^{-ius}\, du \tag{4.3.26}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h^{[n]}\, e^{-ius}\, du \tag{4.3.27}$$

$$+ \,(\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \tag{4.3.28}$$

with suitable regularization functions h, $h^{[n]}$ and α. In the low-energy region $u \ll E_P$, the regularization functions are

$$h(u) \;=\; \frac{m}{32\pi^3}\, \Theta(u)\,, \qquad h^{[n]}(u) \;=\; 0\,, \qquad \alpha(u) \;=\; \alpha_u \;=\; \frac{m^2}{u}\,. \tag{4.3.29}$$

In this expansion, the l-dependence is written out similar to a Laurent expansion. The main simplification compared to our earlier Fourier representation (4.1.3) is that the dependence on the regularization is now described by functions of only one variable, denoted by h, $h^{[n]}$ and α. In composite expressions in $P(s,l)$, we will typically get convolutions of these functions; such one-dimensional convolutions are convenient and can be easily analyzed. The simplification to one-dimensional regularization functions became possible because many details of the regularization affect only the contribution with rapid decay in l, which we do not consider here. Notice that the summands in (4.3.26) and (4.3.27) decay like $(l\,\alpha_{\max})^k/k! \ll (l/l_{\max})^k/k!$ and $(lE_P)^{-n}$, respectively. In the low-energy limit (4.3.29), the expansion (4.3.26) goes over to a power series in m^2, and we thus refer to (4.3.26) as the *mass expansion*. In the mass expansion, the regularization is described by only two functions h and α. The series (4.3.27), on the other hand, is a pure regularization effect and is thus called the *regularization expansion*. It involves an infinite number of regularization functions $h^{[n]}$. Accordingly, we will use the notions of mass and regularization expansions also for other expansions of type (4.3.24).

In the expansion (4.3.24), the fermionic projector is described exclusively in terms of the function $h(u,v)$ in a neighborhood of the discontinuity along the curve (u, α_u). Let us go back to the definition of h, (4.3.4), and consider what this result means for the regularized fermionic projector in momentum space (4.3.1). In the case without regularization (4.3.8), we saw that integrating out the cylindrical coordinates ρ and φ yields a discontinuity of h whenever the 2-plane $(\omega, k) = \text{const}$ meets and is tangential to the hyperboloid $\omega^2 - k^2 - \rho^2 = m^2$. This picture is true in the general case in the sense that the discontinuity of h can be associated to a contribution to $\hat{P}$ which describes a hypersurface in four-dimensional momentum space. The simplest way to recover the discontinuity of h when integrating out the cylindrical coordinates would be to choose $\hat{P}$ of the form (4.3.1) with a function ϕ and the spherically symmetric distribution $f = \delta(|\vec{p}| - \omega - \alpha(-|\vec{p}| - \omega))$. Since spherically symmetric regularizations seem too restrictive, it is preferable to describe the discontinuity of h more generally by a contribution to $\hat{P}$ of the form

$$\phi(\vec{p})\; \delta(\omega - \Omega(\vec{p}))\,, \tag{4.3.30}$$

which is singular on the hypersurface $\omega = \Omega(\vec{p})$. For small momentum $|\vec{p}| \ll E_P$, the surface should clearly go over to the mass shell given by $\Omega = -\sqrt{|\vec{p}|^2 + m^2}$ and $\phi = m/|2\Omega|$; also, it is reasonable to assume that ϕ and Ω are smooth and sufficiently

regular. This consideration shows that for the behavior of the fermionic projector on the light cone (4.3.24), the essential role is played by states lying on a hypersurface. We refer to these one-particle states as the *surface states* of the fermionic projector of the vacuum. This result seems physically convincing because the surface states naturally generalize the states on the lower mass shell known from relativistic quantum mechanics. By integrating out the cylindrical coordinates for the ansatz (4.3.30), one can express the regularization functions $h_u^{(n)}$ in (4.3.24) in terms of ϕ and the geometry of the hypersurface. But we point out that, in contrast to the just discussed discontinuity of h, the partial derivatives of h depend also on states other than surface states. For example, a contribution to $\hat{P}$ of the form $b(\omega, \vec{p})\, \Theta(\omega - \Omega(\vec{p}))$ with Ω as in (4.3.30) and a smooth function b has a discontinuity on the surface Ω and affects all the regularization functions $h_u^{(n)}$ for $n \geq 1$ (as one verifies by a short computation). Thinking of the decomposition of the fermionic projector into the one-particle states, such non-surface contributions would consist of a large number of states and would thus make it necessary to introduce many additional fermions into our system. It does not seem quite reasonable or appropriate to considerably increase the number of particles of the system with the only purpose of having more freedom for the derivative terms of h in (4.3.24). It seems easiest and physically most convincing to assume that all the regularization functions in (4.3.24) come about as a consequence of surface states. We refer to this assumption as the *restriction to surface states*. It is of no relevance for the scalar component (4.3.26, 4.3.28), but it will yield an important relation between the regularization functions for the vector component in the next section. To avoid confusion, we point out that the restriction to surface states clearly does not imply that $\hat{P}$ is of the form (4.3.30). It imposes a condition only on the behavior of $\hat{P}$ in a neighborhood of our hypersurface; namely that the only distributional or non-regular contribution to $\hat{P}$ in this neighborhood should be the hypersurface itself.

For clarity, we finally review our assumptions on the regularization. Our first assumption was that the function $h(u, v)$ has, for every fixed u, at most two discontinuities at $\alpha(u)$ and $\beta(u)$, and is sufficiently regular otherwise (4.3.12). Furthermore, the function $\beta(u)$ had to be monotone and again sufficiently regular. For the function $\alpha(u)$, we assumed that (4.3.22) holds. Since h is obtained from $\hat{P}$, (4.3.1), by integrating out the cylindrical coordinates (4.3.4), these assumptions implicitly pose conditions on the fermionic projector of the vacuum. Although they could clearly be weakened with more mathematical effort, these conditions seem sufficiently general for the moment. In order to understand this better, one should realize that integrating out the cylindrical coordinates does generically (i.e. unless there are singularities parallel to the plane $(\omega, k) = \text{const}$) improve the regularity. The restriction to the generic case is in most situations justified by the fact that the direction $y - x$ and the coordinate system in (4.3.2) can be freely chosen. Using the above assumptions on $h(u, v)$, we showed that the dominant contribution to the fermionic projector on the light cone comes from states on a hypersurface in four-dimensional momentum space. With the "restriction to surface states" we assumed finally that the behavior on the light cone (4.3.24) is completely characterized by these states.

4.4. The Regularized Vacuum on the Light Cone, Vector Component

We shall now extend the previous analysis to the vector component in (4.1.5). More precisely, we will analyze the Fourier integral (4.1.3) for

$$\hat{P}(p) \;=\; v_j(p)\, \gamma^j\, f(p) \tag{4.4.1}$$

close to the light cone. We again choose light-cone coordinates (s, l, x_2, x_3) with $y-x = (s, l, 0, 0)$ (s and l are given by (4.3.5), while x_2 and x_3 are Cartesian coordinates in the orthogonal complement of the sl-plane). The associated momenta are denoted by $p = (u, v, p_2, p_3)$ with u and v according to (4.3.6). As in (4.3.3), we integrate out the coordinates perpendicular to u and v,

$$h_j(u,v) \;:=\; \frac{1}{2\,(2\pi)^4} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3\; (v_j\, f)(u, v, p_2, p_3)\,, \tag{4.4.2}$$

and obtain a representation of the fermionic projector involving two-dimensional Fourier integrals

$$P(s,l) \;=\; \gamma^j\, P_j(s,l)$$

with

$$P_j(s,l) \;:=\; \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h_j(u,v)\, e^{-i(us+vl)}\,. \tag{4.4.3}$$

The tensor indices in (4.4.2) and (4.4.3) refer to the coordinate system (s, l, x_2, x_3). For clarity, we denote the range of the indices by $j = s, l, 2, 3$; thus

$$\gamma^s \;=\; \frac{1}{2}\,(\gamma^0 - \gamma^1)\,, \qquad \gamma^l \;=\; \frac{1}{2}\,(\gamma^0 + \gamma^1)\,, \tag{4.4.4}$$

where $\gamma^0, \ldots, \gamma^3$ are the usual Dirac matrices of Minkowski space. According to the continuum kernel (4.1.1), $\hat{P}$ has in the case without regularization the form $\hat{P} = \not{p}\, \delta(p^2 - m^2)\, \Theta(-p^0)$ and h_j can be computed similar to (4.3.8) to be

$$\gamma^j\, h_j(u,v) \;=\; \frac{1}{32\pi^3}(-u\gamma^s - v\gamma^l)\, \Theta(uv - m^2)\, \Theta(u)\,. \tag{4.4.5}$$

This limiting case specifies the regularized $h_j(u,v)$ for small energy-momentum $u, v \ll E_P$. In order to keep the form of the functions h_j in the high-energy region sufficiently general, we merely assume in what follows that the h_j satisfy all the conditions we considered for the function h in the previous section (see the summary in the last paragraph of §4.3). Our main result is the following.

Expansion of the vector component: *Close to the light cone (4.3.9, 4.3.10), the vector component (4.4.1) of the fermionic projector of the vacuum has the expansion* $P = \gamma^j P_j$ *with*

$$\begin{aligned} P_s(s,l) &= \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} -u\, g_s\ \alpha^k\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} -u\, g_s^{[n]}\, e^{-ius}\, du \\ &\quad + (\text{rapid decay in } l) + (\text{higher orders in } (\alpha_{\max}/E_P) \qquad (4.4.6) \\ P_l(s,l) &= \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[(k-1)\, \alpha^k \;+\; k\, \frac{b}{u}\, \alpha^{k-1} \right] g_l\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} -(n+1)\, g_l^{[n]}\, e^{-ius}\, du \\ &\quad + (\text{rapid decay in } l) + (\text{higher orders in } (\alpha_{\max}/E_P) \qquad (4.4.7) \\ P_{2\!/\!3}(s,l) &= \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[\alpha^k \;+\; k\, \frac{b_{2\!/\!3}}{u}\, \alpha^{k-1} \right] g_{2\!/\!3}\, e^{-ius}\, du \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} g_{2\!/\!3}^{[n]}\, e^{-ius}\, du \\ &\quad + (\text{rapid decay in } l) + (\text{higher orders in } (\alpha_{\max}/E_P) \qquad (4.4.8) \end{aligned}$$

and suitable regularization functions g_j, $g_j^{[n]}$, b, $b_{2\!/\!3}$ and the mass regularization function α as in (4.3.26, 4.3.29). In the low energy region $u \ll E_P$, the regularization functions have the form

$$g_s(u) = \frac{1}{32\pi^3}\, \Theta(u)\,, \qquad g_s^{[n]}(u) = 0 \qquad (4.4.9)$$

$$g_l(u) = \frac{1}{32\pi^3}\, \Theta(u)\,, \qquad g_l^{[n]}(u) = b(u) = 0 \qquad (4.4.10)$$

$$g_{2\!/\!3}(u) = g_{2\!/\!3}(u) = b_{2\!/\!3}(u) = 0\,. \qquad (4.4.11)$$

Before entering the derivation, we briefly discuss these formulas. To this end, we consider the situation where, like in the case without regularization, the vector $v(p)$ in (4.4.1) points into the direction p. In this case we can write the vector component as

$$\hat{P}(p) = p_j \gamma^j\, (\phi f)(p)\,, \qquad (4.4.12)$$

where (ϕf) has the form of the scalar component considered in §4.3. Since multiplication in momentum space corresponds to differentiation in position space, we obtain for (4.4.3)

$$P(s,l) = -i \left(\gamma^s \frac{\partial}{\partial s} + \gamma^l \frac{\partial}{\partial l} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} \right) P_{\text{scalar}}(s,l)\,,$$

where P_{scalar} is the scalar component (4.3.7) with h as in (4.3.4). We now substitute for P_{scalar} the expansion on the light cone (4.3.26–4.3.28) and carry out the partial derivatives. For the s- and l-components, this gives exactly the expansions (4.4.6, 4.4.7) with

$$g_s = g_l = h\,, \qquad g_s^{[n]} = g_l^{[n]} = h^{[n]}\,, \qquad b = 0\,. \qquad (4.4.13)$$

For the components $j = 2, 3$, the calculation of the partial derivatives is not quite so straightforward because the expansion of the scalar component (4.3.26–4.3.28) was carried out for fixed x_2 and x_3. Nevertheless, one can deduce also the expansion (4.4.8) from (4.3.26–4.3.28) if one considers x_2 and x_3 as parameters of the regularization functions h, $h^{[n]}$ and α, and differentiates through, keeping in mind that differentiation yields a factor $1/l$ (to get the scaling dimensions right). In this way, the simple example (4.4.12) explains the general structure of the expansions (4.4.6–4.4.8). We point out that the regularization function b vanishes identically in (4.4.13). This means that b is non-zero only when the direction of the vector field v is modified by the regularization. Thinking in terms of the decomposition into the one-particle states, we refer to this regularization effect as the *shear of the surface states*.

We shall now derive the expansions (4.4.6–4.4.8). Since the Fourier integrals in (4.4.3) are of the form (4.3.7), they have the expansion (4.3.24), valid close to the light cone (4.3.9, 4.3.10). It remains to determine the parameter n_0 in (4.3.24). We consider the components $j = s, l, 2$ and 3 separately. According to (4.4.5), the function h_s in the low-energy region looks similar to the hyperbola depicted in Figure 4.2. The main difference to the low-energy behavior of the scalar component (4.3.8) is the additional factor u in h_s which grows linearly along branch "A" of the hyperbola. Thus in the low-energy region away from the origin,

$$(h_s)_u(\alpha_u) \;\sim\; E_P \qquad \text{and} \qquad \max_{v \in (0, E_P)} |(h_s)_u(v)| \;\sim\; E_P \,. \tag{4.4.14}$$

Hence it is natural to assume that h_s satisfies the bound (4.3.23) with $n_0 = 0$. Because of the linearly growing factor u in the low-energy region, it is convenient to write the regularization functions in the form

$$(h_s)_u(\alpha_u) \;=:\; -u\, g_s(u) \,, \qquad (h_s)_u^{(n)}(\alpha_u) \;=:\; -u\, g_s^{[n]}(u) \tag{4.4.15}$$

with suitable functions g_s and $g_s^{[n]}$ (this can be done because, as explained after (4.3.21), close to the origin $u = 0$, we can work with the unregularized fermionic projector). This yields the expansion (4.4.6). According to (4.4.5) and (4.4.15), the regularization functions have the low-energy limit (4.4.9). For the l-component, the situation is much different. According to (4.4.5), the function h_l in the low-energy limit has the form

$$h_l(u, v) \;=\; -\frac{1}{32\pi^3}\, v\, \Theta(uv - m^2) \,. \tag{4.4.16}$$

The factor v decreases like m^2/u along branch "A" of the hyperbola. Thus in the low-energy region away from the origin,

$$(h_l)_u(\alpha_u) \;\sim\; m^2/E_P \qquad \text{whereas} \qquad \max_{v \in (0, E_P)} |(h_l)_u(v)| \;\sim\; E_P \,. \tag{4.4.17}$$

Therefore, we cannot assume that h_l satisfies the bound (4.3.23) with $n_0 = 0$. But $(h_l)_u^{(1)}(\alpha_u) \sim 1$ in the low-energy region, and thus we may choose $n_0 = 1$. We conclude that it is necessary to take into account two inner summands in (4.3.24), more precisely

$$\begin{aligned} P_l(s,l) \;&=\; \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \Big[(h_l)'_u(\alpha_u)\, \alpha_u^k \;-\; k\, (h_l)_u(\alpha_u)\, \alpha_u^{k-1} \Big]\, e^{-ius}\, du \\ &\quad + \cdots , \end{aligned} \tag{4.4.18}$$

where "$\cdots$" stands for the regularization expansion and all terms neglected in (4.3.24). In the low-energy region, we have according to (4.4.16, 4.3.29),

$$(h_l)_u(\alpha_u) \;=\; -\frac{1}{32\pi^3}\,\frac{m^2}{u} \;=\; (h_l)'_u(\alpha_u)\,\alpha_u\,.$$

Thus in this region, the two summands in the square brackets of (4.4.18) are of the same order of magnitude, and none of them can be neglected. In view of the low-energy limit, we introduce the regularization functions as

$$\begin{aligned} (h_l)'_u(\alpha_u) &=: -g_l(u) \\ (h_l)^{[1+n]}_u(\alpha_u) &=: -(n+1)\,g_l^{[n]}(u) \\ (h_l)'_u(\alpha_u)\,\alpha_u - (h_l)_u(\alpha_u) &=: \frac{b(u)}{u}\,g_l(u)\,; \end{aligned} \tag{4.4.19}$$

this yields the expansion (4.4.7). According to (4.4.5), the regularization functions have the low-energy limit (4.4.10). We finally consider the components $j = 2$ and 3. According to (4.4.5), these components are identically equal to zero in the low-energy limit. But for $u \sim E_P$, they might behave similar to P_s or P_l. To be on the safe side, we choose $n_0 = 1$. Denoting the regularization functions by

$$\begin{aligned} (h_{2\!/\!3})'_u(\alpha_u) &=: g_{2\!/\!3}(u) \\ (h_{2\!/\!3})^{[1+n]}_u(\alpha_u) &=: g^{[n]}_{2\!/\!3}(u) \\ -(h_{2\!/\!3})_u(\alpha_u) &=: \frac{b_{2\!/\!3}(u)}{u}\,g_{2\!/\!3}(u)\,, \end{aligned} \tag{4.4.20}$$

we obtain the expansion (4.4.8). According to (4.4.5), the regularization functions $g_{2\!/\!3}$, $g^{[h]}_{2\!/\!3}$ and $b_{2\!/\!3}$ vanish in the low-energy region, (4.4.11).

For clarity, we point out that choosing $n_0 = 1$ (as in (4.4.7, 4.4.8)) is a generalization of setting $n_0 = 0$ (as in (4.4.6)), obtained by taking into account more summands of the expansion (4.3.21). Nevertheless, the different behavior in the low-energy region (4.4.14, 4.4.17) suggests that (4.4.7) and (4.4.8) should not be merely more general formulas than (4.4.6), but that the behavior of $P_j(s,l)$, $j = l, 2, 3$, should be really different from that of $P_s(s,l)$. We shall now make this difference precise. Comparing (4.4.14) and (4.4.17) (and using that $h_{2\!/\!3}$ vanishes in the low-energy region), it is reasonable to impose that there should be a constant $\varepsilon_{\text{shear}} > 0$ with

$$|(h_j)_u(\alpha_u)| \;<\; \varepsilon_{\text{shear}}\,|(h_s)_u(\alpha_u)| \qquad \text{for } u \sim E_P \text{ and } j = l, 2, \text{ or } 3. \tag{4.4.21}$$

In view of (4.4.14) and (4.4.17), $\varepsilon_{\text{shear}}$ should be as small as

$$\varepsilon_{\text{shear}} \;\sim\; \frac{m^2}{E_P^2}\,. \tag{4.4.22}$$

However, if the surface states have shear (as defined earlier in this section), the constant $\varepsilon_{\text{shear}}$ must in general be chosen larger. In order to keep our analysis as general as possible, we will not specify here how $\varepsilon_{\text{shear}}$ scales in the Planck energy, but merely assume that $m^2/E_P^2 < \varepsilon_{\text{shear}} \ll 1$. Using (4.4.15), (4.4.19) and (4.4.20), the condition (4.4.21) can be expressed in terms of the regularization functions g_j and b_j as

$$\left(\frac{b}{u} + \alpha_u\right) g_l,\; \frac{b_{2\!/\!3}}{u}\,g_{2\!/\!3} \;<\; \varepsilon_{\text{shear}}\,u\,g_s \qquad \text{for } u \sim E_P. \tag{4.4.23}$$

It is interesting to discuss what the condition (4.4.21) means for the functions P_j. We begin with the case without regularization. In this case, the vector component of $P(x,y)$ points into the direction $y-x$, more precisely $P(x,y) = i(y-x)_j \gamma^j S(x,y)$ with a scalar distribution S. In a composite expression like the closed chain $P(x,y)\,P(y,x)$, one can contract the tensor indices and obtains in a formal calculation $P(x,y)P(y,x) = (y-x)^2\, S(x,y)\, S(y,x)$ with a scalar factor $(y-x)^2$ which vanishes on the light cone. Let us consider this contraction in our light-cone coordinates. Before the contraction, each factor $(y-x)_j \gamma^j = 2l\,\gamma^s + 2s\,\gamma^l \approx 2l\gamma^s$ is, if we take only the leading contribution on the light cone (i.e. the lowest order in s/l), proportional to l. After the contraction, however, the product $(y-x)^2 = 4ls$ is proportional to both l and s. Thus the contraction yields, to leading order on the light cone, a dimensionless factor s/l. While the factor l^{-1} changes the scaling behavior in the "large" variable, the factor s tends to make the composite expression "small" near the light cone. The analysis of the scaling behavior in l can immediately be extended to the case with regularization by looking at the expansions (4.4.6) and (4.4.7). Let us consider as an example the leading term of the mass expansion. For the expansion (4.4.6), this is the summand $k = 0$, and it scales like $P_s(s,l) \sim 1/l$. If we assume that (4.4.21) holds with $\varepsilon_{\text{shear}}$ according to (4.4.22), then (4.4.23) shows that $b(u) \sim 1$, and the summands in the square bracket in (4.4.7) are of comparable size. Hence the leading term of the expansion (4.4.7) is also the summand $k = 0$, and it scales in l like $P_l(s,l) \sim 1/l^2$. Hence the leading term of the sum $\gamma^l P_l + \gamma^s P_s$ behaves like $P \sim 1/l + \mathcal{O}(1/l^2)$. Since s and l are null directions, a contraction of the tensor indices in the closed chain leads only to mixed products of the form $P_s\, P_l$, and this mixed product scales in l like $P_s\, P_l \sim 1/l^3$. Thus, exactly as in the case without regularization, the contraction of the tensor indices yields an additional factor l^{-1}. If on the other hand, the condition (4.4.21) were violated, the regularization function b could be chosen arbitrarily large. But if b becomes large enough, the cominant contribution to (4.4.7) is the summand $k = 1$ (notice that b does not appear in the summand $k = 0$), and hence $P_l(s,l) \sim 1/l$. This implies that $P_s\, P_l \sim 1/l^2$, and the contraction does no longer yield an additional factor l^{-1}. This consideration is immediately extended to the components $P_{2/3}$ by considering the l-dependence of the summands in (4.4.8). We conclude that the condition (4.4.21) with $\varepsilon_{\text{shear}} \ll 1$ means that the contraction of the tensor indices yields a scalar factor which is small on the light cone. We refer to this condition by saying that the *vector component is null on the light cone*. If one wishes, one can simply take this condition as an additional assumption on the fermionic projector of the vacuum. However, the property of the vector component being null on the light cone also arises in the study of composite expressions in the fermionic projector as a compatibility condition and can thus be derived from the equations of discrete space-time (see Remark 6.2.4)).

The next question is if our regularization functions α, g_j, $g_j^{[n]}$ and b, which appear in our expansions (4.4.6–4.4.8), are all independent of each other, or whether there are some relations between them. Recall that the regularization functions are given in terms of the boundary values of the functions $\partial_v^n h_j(u,v)$, $n \geq 0$, on the curve (u, α_u) (see (4.4.15, 4.4.19, 4.4.20)). Since the $(h_j)_{j=s,l,2,3}$ were treated in our two-dimensional Fourier analysis as four independent and (apart from our regularity assumptions) arbitrary functions, we can certainly not get relations between the regularization functions by looking at the situation in the uv-plane. But we can hope that when we consider the surface states in four-dimensional momentum space (as introduced in §4.3), the geometry of the hypersurface defined by these states might yield useful restrictions for

the regularization functions. First of all, we mention that our discussion of surface states of the previous section applies without changes also to the vector component; we will in what follows make use of the *restriction to surface states*. Since in the low-energy region the regularization is irrelevant and the results of §2.5 apply, we can furthermore restrict attention to large energy and momentum $\omega, |\vec{k}| \sim E_P$. We choose polar coordinates $(\omega, k = |\vec{k}|, \vartheta, \varphi)$ in momentum space and introduce the "mass shell coordinates"

$$U \;=\; -|\vec{k}| - \omega\,, \qquad\qquad V \;=\; |\vec{k}| - \omega\,. \tag{4.4.24}$$

Notice that, in contrast to the coordinates u and v, (4.3.6), the variables (4.4.24) are the spherically symmetric part of a four-dimensional coordinate system $(U, V, \vartheta, \varphi)$. Extending also the notation (4.4.4) in a spherically symmetric way, we introduce the Dirac matrix

$$\gamma^S \;=\; \frac{1}{2}\left(\gamma^0 \;-\; \frac{\vec{\gamma}\vec{k}}{k}\right).$$

Let us consider what the expansions (4.4.6–4.4.8) tell us about the surface states. Similar as explained before (4.3.30), the discontinuities of h_j come about in (4.4.2) when the plane $(u, v) = \text{const}$ meets and is tangential to the hypersurface of the surface states. We denote the tangential intersection point of the surface $(u, v) = \text{const}$ with the hypersurface by $Q = (U, V, \vartheta, \varphi)$. In the high-energy region under consideration, the variable U is of the order E_P. The variable V, on the other hand, will be of order $\alpha(U) < \alpha_{\max}$. Thus our hypersurface is close to the mass cone in the sense that $V/U \sim \alpha_{\max}/E_P \ll 1$. As a consequence, the coordinate ϑ of the intersection point Q must be small (more precisely, $\vartheta \leq \sqrt{\alpha_{\max}/E_P}$), and we conclude that, to leading order in $\alpha_{\max}/E_P$, $V = \alpha(U)$. Hence we can write the hypersurface as a graph $V = A(U, \vartheta, \varphi)$ with a function A satisfying the condition

$$A(U, \vartheta = 0) \;=\; \alpha(U) \;+\; (\text{higher orders in } \alpha_{\max}/E_P)\,.$$

One can think of the function $A(u, \vartheta, \varphi)$ as the extension of α to the four-dimensional setting. In order to determine the structure of the Dirac matrices, we first recall that the assumption that the vector component is null on the light cone implied in our consideration after (4.4.6) that the parameter n_0 corresponding to P_l, P_2 and P_3 was equal to one. This means that to leading order in $\alpha_{\max}/E_P$, only the function $h_s(u, v)$ is discontinuous on the curve (u, α_u), and we conclude that the distribution $\hat{P}$ is on the hypersurface at the point Q a scalar multiple of γ^s; we use the short notation $\hat{P}(Q) \sim \gamma^s$. Using again that ϑ is small, we obtain that to leading order in $\alpha_{\max}/E_P$, $\hat{P}(U, A(U, \vartheta = 0), \vartheta = 0) \sim \gamma^s$. Since the spatial direction of the vector $y - x$ in (4.1.3) can be chosen arbitrarily, we can by rotating our coordinate system immediately extend this result to general ϑ and φ, and obtain that $\hat{P}(U, \alpha(U, \vartheta, \varphi), \vartheta, \varphi) \sim \gamma^S$. Hence the surface states are described by a contribution to $\hat{P}$ of the form

$$-\,32\pi^3\, g(U, \vartheta, \varphi)\, \gamma^S\, \delta(V - A(U, \vartheta, \varphi)) \;+\; (\text{higher orders in } \alpha_{\max}/E_P) \tag{4.4.25}$$

with some function g. It is reasonable to assume that the functions in (4.4.25) are sufficiently regular. Similar to our regularity condition (4.3.12) for h, we here assume that the derivatives of A and g_S have the natural scaling behavior in E_P. More precisely, for all $n_1, n_2, n_3 \geq 0$ there should exist a constant $c \ll l E_P$ with

$$|\partial_U^{n_1} \partial_\vartheta^{n_2} \partial_\varphi^{n_3} A(U, \vartheta, \varphi)| \;+\; |\partial_U^{n_1} \partial_\vartheta^{n_2} \partial_\varphi^{n_3} g(U, \vartheta, \varphi)| \;\leq\; c\, E_P^{-n_1}\, \max(|A| + |g|) \tag{4.4.26}$$

for all $U \sim E_P$.

The form of the surface states (4.4.25) allows us to calculate the regularization functions g_j, $g_j^{[n]}$ and b_j. For this, we first represent the matrix γ^S in (4.4.25) in the Dirac basis $(\gamma^j)_{j=s,l,2,3}$; this yields the contributions of the surface states to the distributions $(v_j\, f)$. By substituting into (4.4.2) and carrying out the integrals over p_2 and p_3, one obtains the functions h_j. Finally, the regularization functions can be computed via (4.4.15, 4.4.19, 4.4.20). This whole calculation is quite straightforward, and we only state the main results. To leading order in v/u, we can take A and g as constant functions, and thus the calculation of $\gamma^s h_s + \gamma^l h_l$ reduces to the integral

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \; \left(\gamma^s + \frac{v}{u}\, \gamma^l\right) g(u, \vartheta = 0)\, \delta\left(v - \alpha_u - \frac{p_2^2 + p_3^2}{u}\right)$$
$$+(\text{higher orders in } v/u,\, \alpha_{\max}/E_P)\,.$$

An evaluation in cylindrical coordinates yields that both $g_s(u)$ and $g_l(u)$ are equal to $g(u, \vartheta = 0)$, and we thus have the important relation

$$g_s(u) \;=\; g_l(u) \;=:\; g(u)\,. \tag{4.4.27}$$

In the case without shear of the surface states, this relation was already found in (4.4.13); we now see that it holds in a much more general setting. The calculation of the angular components $j = 2, 3$ gives for $g_{2/3}$ contributions proportional to $u\,\partial_{2/3} A$ and $u\,\partial_{2/3} g$. Unfortunately, this is not very helpful because we have no information on the derivatives of A and g. The computation of the regularization functions $g_j^{[n]}$ involves higher derivatives of the functions in (4.4.25) and becomes quite complicated. We remark that the above analysis of the surface states can be carried out similarly for the scalar component of the previous section and gives relations between the regularization functions h and $h^{[n]}$, (4.3.25), but these relations all depend on unknown details of the geometry of the hypersurface. We thus conclude that (4.4.27) is the only relation between the regularization functions which can be derived with our present knowledge on the surface states.

We finally mention two assumptions on the regularization which, although we will not use them in the present work, might be worth considering later. The first assumption is related to the fact that P should as a projector be idempotent, $P^2 = P$. A formal calculation using (4.1.3) and (4.1.5) yields that

$$(P^2)(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{P}(p)^2\, e^{-ip(x-y)} \qquad \text{with} \tag{4.4.28}$$

$$\hat{P}(p)^2 \;=\; \left(2\phi(p)\, v_j(p)\, \gamma^j \;+\; (v_j(p)\, v^j(p) + \phi(p)^2)\right) f(p)^2\,. \tag{4.4.29}$$

In order to make sense out of (4.4.29), one must regularize in momentum space, e.g. by considering the system in finite 3-volume and take a suitable limit. Since the results of this analysis depend sensitively on how the regularization in momentum space is carried out, (4.4.29) cannot give any detailed information on the functions ϕ, v, or f. The only simple conclusion independent of the regularization is that the scalars multiplying the factors $v_j \gamma^j$ in (4.1.5) and (4.4.29) should have the same sign, and thus $\phi(p)\, f(p)$ should be positive. According to (4.3.4), this implies that the regularization function h be positive,

$$h(u) \;\geq\; 0 \qquad \text{for all } u.$$

This assumption is called the *positivity of the scalar component*. The second assumption is obtained by considering the rank of $\hat{P}(p)$. The 4×4 matrix $(\not{p} + m)$ in the

integrand of the unregularized fermionic projector (4.1.1) has the special property of being singular of rank two. This means that the fermionic projector is composed of only two occupied fermionic states, for every momentum p on the mass shell. The natural extension of this property to the case with regularization is that for every p on the hypersurface defined by the surface states, the matrix $\hat{P}(p)$ corresponding to the vector-scalar structure (4.1.5) should be of rank two. We refer to this property as the assumption of *half occupied surface states*. In terms of the functions $h(u,v)$ and $h_j(u,v)$, it means that $h_s(u,\alpha(u))\, h_l(u,\alpha_u) = h(u,\alpha_u)^2$. Using (4.3.25, 4.4.15, 4.4.19, 4.4.27), the assumption of half occupied surface states yields the following relation between the regularization functions of the scalar and vector components,

$$(\alpha(u)\, u + b(u))\, g(u)^2 \;=\; h(u)^2\,. \tag{4.4.30}$$

4.5. The General Formalism

In this section we shall extend our previous analysis on the light cone in three ways: to the case with interaction, to systems of Dirac seas as introduced in §2.3 and to composite expressions in the fermionic projector. Our first step is to develop a method which allows us to introduce a regularization into the formulas of the light-cone expansion (2.5.45). We here only motivate and describe this method, the rigorous justification is given in Appendix D. Since the formulas of the light-cone expansion involve the factors $T^{(n)}$, (2.5.43, 2.5.42, 2.5.40), we begin by bringing these distributions into a form similar to our expansion of the regularized scalar component (4.3.26). By partly carrying out the Fourier integral (2.5.40) in the light-cone coordinates introduced in §4.3 (see (4.3.5, 4.3.6)), we can write the distribution T_a as

$$T_a(s,l) \;=\; \frac{1}{32\pi^3}\,\frac{1}{il}\int_0^\infty e^{-\frac{ial}{u} - ius}\,du\,. \tag{4.5.1}$$

This formula can be regarded as a special case of the expansion (4.3.20) (notice that the function $h(u,v)$ corresponding to T_a is computed similar to (4.3.8)), but (4.5.1) holds also away from the light cone. The distribution T_a is not differentiable in a at $a=0$, as one sees either directly in position space (2.5.41) or equivalently in (4.5.1), where formal differentiation leads to a singularity of the integrand at $u=0$. We bypassed this problem by working instead of T_a with the distribution T_a^{reg} (2.5.42). Let us briefly consider what this "regularization" means in the integral representation (4.5.1). The formal a-derivative of (4.5.1),

$$\frac{d}{da}T_a(s,l) \;=\; -\frac{1}{32\pi^3}\int_0^\infty \frac{1}{u}\, e^{-\frac{ial}{u} - ius}\,du\,,$$

is well-defined and finite for $a \neq 0$ because of the oscillatory factor $\exp(-ial/u)$. However, the limit $a \to 0$ leads to a logarithmic divergence. Thus one must subtract a logarithmic counter term before taking the limit; more precisely,

$$T^{(1)}(s,l) \;=\; -\frac{1}{32\pi^3}\,\lim_{a\to 0}\int_{-\infty}^\infty \left[\frac{1}{u}\, e^{-\frac{ial}{u}}\,\Theta(u) \;-\; (1+\log a)\,\delta(u)\right] e^{-ius}\,du\,.$$

The higher a-derivatives $T^{(n)}$, $n>1$, are defined similarly using suitable counterterms which are localized at $u=0$. Since we do not need the details in what follows, we simply write

$$T^{(n)}(s,l) \;=\; -\frac{1}{32\pi^3}\,(-il)^{n-1}\int_0^\infty \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\,du\,. \tag{4.5.2}$$

Consider a summand of the light-cone expansion (2.5.45),

$$\text{(phase-inserted line integrals) } T^{(n)}(s,l)\,. \tag{4.5.3}$$

According to our assumption of macroscopic potentials and wave functions described in §4.1, we shall regularize only the distribution $T^{(n)}$, keeping the iterated line integral unchanged. Let us briefly analyze what this assumption means quantitatively. Not regularizing the iterated line integral in (4.5.3), denoted in what follows by F, will be a good approximation if and only if F is nearly constant on the Planck scale. In other words, not regularizing F is admissible if we keep in mind that this method can describe the regularized fermionic projector only modulo contributions of the order $\partial_j F/E_P$. In the case that this last derivative acts on the bosonic potentials and fields contained in F, we obtain the limitation already mentioned in §4.1 that energy and momentum of the bosonic fields should be small compared to the Planck energy. More precisely, we can describe the fermionic projector only to leading order in $(l_{\text{macro}} E_P)^{-1}$, where l_{macro} is a typical length scale of macroscopic physics. A point we did not pay attention to earlier is that the iterated line integrals also involve factors $(y-x)$ which are contracted with the bosonic potentials and fields. Thus in light-cone coordinates, F will in general contain factors of l. If the derivative in $\partial_j F$ acts on a factor l, this factor is annihilated. Hence keeping the iterated line integrals in (4.5.3) unchanged can describe only the leading order in $(lE_P)^{-1}$ of the fermionic projector. We conclude that the assumption of macroscopic potentials and wave functions is justified if and only if we restrict our analysis to the *leading order in* $(lE_P)^{-1}$ *and* $(l_{macro} E_P)^{-1}$. We remark that going beyond the leading order in $(lE_P)^{-1}$ or $(l_{\text{macro}} E_P)^{-1}$ would make it impossible to describe the interaction by classical fields and is thus at present out of reach.

The restriction to the leading order in $(lE_P)^{-1}$ is a considerable simplification. First of all, we can neglect all regularization expansions (which are just expansions in powers of $(lE_P)^{-1}$; see e.g. (4.3.27) and the discussion thereafter), and thus we do not need to consider the regularization functions $h^{[n]}$ and $g_j^{[n]}$. Next we compare for given k the summands in (4.4.6–4.4.8) (the analysis for fixed k is justified assuming that the vector component is null on the light cone; see (4.4.21) and the discussion thereafter). One sees that the tensor index $j = s$ gives the leading contribution in $(lE_P)^{-1}$ to the vector component. This is a great simplification when tensor indices are contracted in composite expressions. Namely, when the vector component is contracted with the bosonic potentials or fields, it suffices to consider the contribution P_s, (4.4.6). If vector components are contracted with each other, the products of type $P_{2/3} P_{2/3}$ are according to (4.4.6–4.4.8) of higher order in $(lE_P)^{-1}$ or $\varepsilon_{\text{shear}}$ than corresponding products of type $P_s\, P_l$. Hence in such contractions, we must take into account both P_s and P_l, but we can again neglect the components P_2 and P_3. We conclude that the only regularization functions which should be of relevance here are those appearing in (4.3.26) and in the mass expansions of (4.4.6) and (4.4.7), i.e. the four functions

$$\alpha(u)\,, \qquad g(u)\,, \qquad h(u) \qquad \text{and} \qquad b(u) \tag{4.5.4}$$

with g given by (4.4.27).

Under the assumption of macroscopic potentials and wave functions, it suffices to regularize the factor $T^{(n)}$ in (4.5.3). Our method for regularizing $T^{(n)}$ is to go over to the integral representation (4.5.2) and to insert the regularization functions (4.5.4)

into the integrand. The procedure depends on whether the contribution to the light-cone expansion is of even or odd order in the mass parameter m. Furthermore, we must treat the factors $(y-x)_j \gamma^j$ in the light-cone expansion separately. The precise regularization method is the following.

Regularization of the light-cone expansion: *A summand of the light-cone expansion (2.5.45) which is proportional to m^p,*

$$m^p \text{ (phase-inserted line integrals) } T^{(n)}(s,l)\,, \tag{4.5.5}$$

has the regularization

$$\begin{aligned} &(-1) \text{ (phase-inserted line integrals)} \qquad (4.5.6)\\ &\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \\ &+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (lE_P)^{-1}, (l_{\text{macro}} E_P)^{-1}, \varepsilon_{\text{shear}})\,. \qquad (4.5.7) \end{aligned}$$

A contribution to the light-cone expansion (2.5.45) which is proportional to m^p and contains a factor $(y-x)_j \gamma^j$,

$$m^p \text{ (phase-inserted line integrals) } (y-x)_j \gamma^j\, T^{(n)}(s,l)\,, \tag{4.5.8}$$

is properly regularized according to

$$\begin{aligned} &(-1) \text{ (phase-inserted line integrals)} \\ &\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left[2l\, \gamma^s \left(\frac{1}{u^n}\right)^{\text{reg}} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} + 2l\, b(u)\, \gamma^l \left(\frac{1}{u^{n+2}}\right)^{\text{reg}} \right] \\ &\times e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} + (\text{contributions} \sim \gamma^2, \gamma^3) \\ &+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (lE_P)^{-1}, (l_{\text{macro}} E_P)^{-1}, \varepsilon_{\text{shear}})\,. \qquad (4.5.9) \end{aligned}$$

In these formulas, the regularization function a is given by

$$a(u) \;=\; u\, \alpha(u)\,, \tag{4.5.10}$$

ε_{shear} is defined via (4.4.21), and l_{macro} is a macroscopic length scale.

Let us briefly explain and motivate this regularization method (see Appendix D for the derivation). First of all, we note that, after writing the factor $(y-x)_j \gamma^j$ together with the iterated line integrals, the expression (4.5.8) is of the form (4.5.5), and the regularization rule (4.5.7) applies. Thus (4.5.9) is an extension of (4.5.7) giving additional information on the l-component of the factor $(y-x)_j \gamma^j$. As we shall see later, this information is essential when the factor $(y-x)_j$ in (4.5.8) is to be contracted with another factor $(y-x)^j$ in a composite expression. To explain the formula (4.5.7), we first point out that the expansions of the scalar and vector components (4.3.26–4.3.28, 4.4.6, 4.4.7) do not involve the mass parameter m. The reason is that m was absorbed into the regularization functions g, h and α, as one sees by considering the low-energy limit; see (4.3.29, 4.4.9, 4.4.10). Furthermore, we note that each contribution to the mass expansions of the scalar or vector components contains either a factor h or g (see (4.3.26, 4.4.6, 4.4.7)), and it is therefore reasonable that we should also use exactly one of these factors here. As a consequence, the power m^p in (4.5.5) uniquely determines how many factors of each regularization function we should take. Namely for even p, we must take one factor g and $p/2$ factors α, whereas

the case of odd p gives rise to one factor h and $(p-1)/2$ factors α. On the other hand, we know that the insertion of the regularization functions into (4.5.2) should modify the behavior of the integrand only for large $u \sim E_P$; in particular, we should for small u have a behavior $\sim u^{-n}$. In order to comply with all these conditions, one must insert the regularization functions precisely as in (4.5.7). In order to motivate (4.5.9), we consider the expansion of the vector component (4.4.6, 4.4.7). Recall that the regularization function b vanishes in the low-energy region (4.4.10) and describes the shear of the surface states (as explained after (4.4.13)). Since this effect is not related to the mass of the Dirac particle, it is plausible that we should not associate to b a power of m. For the mass expansion of the vector component, we should thus collect all terms to a given power of α. The contribution $\sim \alpha^k$ to $\gamma^s P_s + \gamma^l P_l$ takes according to (4.4.6, 4.4.7) the form

$$\frac{1}{il}\,\frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left(-u\,\gamma^s \;+\; \frac{k-1}{il}\,\gamma^l \;-\; \frac{b}{u}\,\gamma^l \right) g\,\alpha^k\, e^{-ius}\, du\,.$$

In order to obtain the correct behavior in the low-energy region, we must multiply this formula by $-2l$ and choose $k = n+1$. This explains the form of the square bracket in (4.5.9). The combination of the regularization functions g, h and a in (4.5.9) can be understood exactly as in (4.5.7) using power counting in m.

Our constructions so far were carried out for the case $N = 1$ of one Dirac sea. We will now generalize our regularization method to systems of Dirac seas as introduced in (2.3) and will also introduce a compact notation for the regularization. Exactly as in §2.5 we only consider the *auxiliary fermionic projector*, because the fermionic projector is then obtained simply by taking the partial trace (2.3.20). We first outline how chiral particles (e.g. neutrinos) can be described. Without regularization, a chiral Dirac sea is obtained by multiplying the Dirac sea of massless particles with the chiral projectors $\chi_{L/R} = \frac{1}{2}(\mathbb{1} \mp \rho)$; for example in the vacuum and left/right handed particles,

$$\hat{P}(p) \;=\; \chi_{L/R}\, \not{p}\, \delta(p^2)\, \Theta(-p^0)\,. \tag{4.5.11}$$

The most obvious regularization method is to deduce the regularized chiral Dirac sea from a Dirac sea regularized with our above methods again by multiplying from the left with a chiral projector. This simple method indeed works, under the following assumptions. First, we must ensure that the regularized fermionic projector of the vacuum is a Hermitian operator. To this end, we must assume that the scalar component ϕ in (4.1.5) be identically equal to zero (this generalizes the requirement of massless particles needed in the case without regularization). Hence we regularize (4.5.11) by setting

$$\hat{P}(p) \;=\; \chi_{L/R}\, v_j(p) \gamma^j\, f(p)\,.$$

The expansions near the light cone are then obtained from (4.3.27, 4.3.28) and (4.4.6–4.4.8) by setting the scalar regularization functions h and $h^{[n]}$ to zero and by multiplying with $\chi_{L/R}$. Assuming furthermore that the bosonic potentials are causality compatible (see Def. 2.3.2), the formulas of the light-cone expansion are regularized likewise by taking the regularizations (4.5.7, 4.5.9) with h set identically equal to zero, and by multiplying from the left by a chiral projector $\chi_{L/R}$.

We next consider the generalization to systems of Dirac seas. In the vacuum, we can describe a system of Dirac seas by taking a direct sum of regularized Dirac seas and by using instead of the chiral projectors $\chi_{L/R}$ the chiral asymmetry matrix X (see (2.3.7)). Since we may choose the regularization functions for each Dirac

sea independently, this procedure clearly increases the total number of regularization functions. However, it is natural to impose that the regularization should respect all symmetries among the Dirac seas. More precisely, if the fermionic projector of the vacuum contains identical Dirac seas (e.g. corresponding to an underlying color $SU(3)$ symmetry), then we will always use the same regularization functions for all of these Dirac seas. Once the regularization has been specified for the vacuum, we can again apply the rules (4.5.5–4.5.9) to regularize the light-cone expansion. In the special case that the bosonic potentials are diagonal in the Dirac sea index, we can simply take the direct sum of the contributions (4.5.7, 4.5.9), using in each summand the regularization functions of the corresponding vacuum Dirac sea. In the general case of a non-diagonal bosonic field, the regularization functions can be inserted uniquely if one uses that, according to the assumption of macroscopic potentials and wave functions of §4.1, the fermionic projector is modified by the bosonic fields only on the macroscopic scale, so that its microscopic structure is the same as in the vacuum. For example, one can in the case of a gravitational and Yang-Mills field make the bosonic potential locally to zero by transforming to a suitable coordinate sytem and gauge, can in this system insert the regularization functions as in the vacuum and can finally transform back to the original system. We conclude that the generalization of our regularization method to systems of Dirac seas is quite straightforward and canonical. Therefore we can introduce a short notation for the regularizations of the factors $T^{(n)}$ in the light-cone expansion by simply adding a label for the order in the mass parameter. More precisely, we introduce in the case $N = 1$ of one Dirac sea the following abbreviations for the Fourier integrals in (4.5.7) and (4.5.9),

$$\begin{aligned} T^{(n)}_{[p]} &\equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius} \\ &\qquad \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \qquad (4.5.12) \\ (\not\xi\, T^{(n)}_{[p]}) &\equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du\, e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \\ &\quad \times \left[2l\, \gamma^s \left(\frac{1}{u^n}\right)^{\text{reg}} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} + 2l\, b(u)\, \gamma^l \left(\frac{1}{u^{n+2}}\right)^{\text{reg}} \right] \qquad (4.5.13) \\ T^{(n)}_{\{p\}} &\equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\, b(u) \\ &\qquad \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even}\,. \end{cases} \qquad (4.5.14) \end{aligned}$$

In the case of a system of Dirac seas (2.3.3), we use the same notation for the corresponding direct sum. With this notation, the regularization of the light-cone expansion is carried out (modulo all the contributions neglected in (4.5.7) and (4.5.9)) merely by the replacement $m^p\, T^{(n)}(x,y) \to T^{(n)}_{[p]}$ and by marking with brackets that the factors $(y-x)_j \gamma^j$ and $T^{(n)}_{[p]}$ belong together (where we use the abbreviation $\xi \equiv y - x$) . We call a factor $\not\xi$ inside the brackets $(\not\xi T^{(n)}_{[p]})$ an *inner factor* $\not\xi$. Notice that the functions

$T^{(n)}_{\{p\}}$ in (4.5.14) involve the regularization function b; they will be needed below to handle contractions between the inner factors.

We finally come to the analysis of composite expressions in the fermionic projector. In §4.2 we already discussed the simplest composite expression, the closed chain $P(x,y)\,P(y,x)$ in the vacuum. In order to analyze the closed chain near the light cone, we substitute for $P(x,y)$ and $P(y,x)$ the regularized formulas of the light-cone expansion and multiply out. It is convenient to use that the fermionic projector is Hermitian and thus $P(y,x) = P(x,y)^*$; hence the light-cone expansion of $P(y,x)$ is obtained from that for $P(x,y)$ by taking the adjoint (with respect to the spin scalar product). The iterated line integrals can be multiplied with each other giving smooth functions; also we can simplify the resulting product of Dirac matrices using their anti-commutation relations. Denoting the adjoints of (4.5.12) and (4.5.13) by $\overline{T^{(n)}_{[p]}}$ and $(\overline{\not{\xi} T^{(n)}_{[p]}})$, respectively, we thus obtain for the closed chain a sum of terms of the following forms,

$$\begin{aligned} &F\, T^{(n_1)}_{[r_1]}\, \overline{T^{(n_2)}_{[r_2]}} \quad &,& \qquad F\, (\xi_{j_1} T^{(n_1)}_{[r_1]})\, \overline{T^{(n_2)}_{[r_2]}} \\ &F\, T^{(n_1)}_{[r_1]}\, \overline{(\xi_{j_2} T^{(n_2)}_{[r_2]})} \quad &,& \qquad F\, (\xi_{j_1} T^{(n_1)}_{[r_1]})\, \overline{(\xi_{j_2} T^{(n_2)}_{[r_2]})}\,, \end{aligned} \tag{4.5.15}$$

where F is a smooth function in x and y and n_j, r_j are integer parameters. Here the tensor indices of the inner factors ξ are contracted either with each other or with tensor indices in the smooth prefactor F. In order to analyze Euler-Lagrange equations like for example (3.5.20, 3.5.21), we need to consider more general expressions. More precisely, all Euler-Lagrange equations in this book can be written in terms of expressions being a product of a smooth function with a quotient of two monomials in $T^{(n)}_{[r]}$ and $\overline{T^{(n)}_{[r]}}$, possibly with inner factors ξ in the numerator. Thus our key problem is to mathematically handle expressions of the form

$$\begin{aligned} &(\text{smooth function}) \times \left[T^{(l_1)}_{[s_1]} \cdots T^{(l_f)}_{[s_f]}\, \overline{T^{(l_{f+1})}_{[s_{f+1}]} \cdots T^{(l_g)}_{[s_g]}} \right]^{-1} \\ &\quad \times (\xi_{j_1} T^{(n_1)}_{[r_1]}) \cdots (\xi_{j_a} T^{(n_a)}_{[r_a]})\, T^{(n_{a+1})}_{[r_{a+1}]} \cdots T^{(n_b)}_{[r_b]} \\ &\qquad \times \overline{(\xi_{j_{b+1}} T^{(n_{b+1})}_{[r_{b+1}]}) \cdots (\xi_{j_c} T^{(n_c)}_{[r_c]})\, T^{(n_{c+1})}_{[r_{c+1}]} \cdots T^{(n_d)}_{[r_d]}} \end{aligned} \tag{4.5.16}$$

with $0 \leq f \leq g$, $0 \leq a \leq b \leq c \leq d$, parameters l_j, s_i, n_i, p_i and tensor indices j_i. Here the tensor indices of the inner factors ξ_i are again contracted either with other inner factors or with tensor indices in the smooth prefactor. We mention for clarity that, since the factors in (4.5.16) are complex functions or, in the case $N > 1$ of systems of Dirac seas, direct sums of complex functions, the product (4.5.16) clearly is commutative.

The inner factors in (4.5.16) can be simplified using the particular form (4.5.12, 4.5.13) of $T^{(n)}_{[r]}$ and $(\xi_j T^{(n)}_{[r]})$. We begin with the case of an inner factor which is contracted with a tensor index in the smooth prefactor, i.e. with products of the form

$$\cdots F^j\, (\xi_j T^{(n)}_{[r]}) \cdots \qquad \text{or} \qquad \cdots F^j\, \overline{(\xi_j T^{(n)}_{[r]})} \cdots$$

and a smooth vector field F, where "$\cdots$" stands for any other factors of the form as in (4.5.16). According to (4.5.13), to leading order in $(lE_P)^{-1}$ it suffices to take into

account the s-component, and thus (4.5.12) yields that $(\not{\xi} T^{(n)}_{[r]}) \approx 2l\,\gamma^s\, T^{(n)}_{[r]}$. Since $2l\,\gamma^s$ coincides on the light cone with $\not{\xi}$, we conclude that, to leading order in $(lE_P)^{-1}$,

$$F^j\,(\xi_j T^{(n)}_{[r]}) \;=\; F^j \xi_j\, T^{(n)}_{[r]} \qquad \text{and} \qquad F^j\,(\overline{\xi_j\, T^{(n)}_{[r]}}) \;=\; F^j \xi_j \overline{T^{(n)}_{[r]}}\,. \tag{4.5.17}$$

These relations coincide with what one would have expected naively. We next consider the case of two inner factors which are contracted with each other, i.e. products of the following form,

$$\begin{aligned} &\cdots (\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \cdots \quad , \qquad \cdots (\xi_j T^{(n_1)}_{[r_1]})(\overline{\xi^j T^{(n_2)}_{[r_2]}}) \cdots \\ &\qquad\qquad \text{or} \qquad\qquad\qquad\quad \cdots (\overline{\xi_j T^{(n_1)}_{[r_1]}})(\xi^j T^{(n_2)}_{[r_2]}) \cdots\,. \end{aligned} \tag{4.5.18}$$

In this case, the product cannot be calculated naively because the factor $\xi_j \xi^j = \xi^2$ vanishes on the light cone. But we can still compute the product using the Fourier representation (4.5.13). Since the s- and l-directions are null, only the mixed products of the s- and l-components in (4.5.13) contribute, and we obtain

$$\begin{aligned} &(\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \\ &\quad= \quad (-il)^{n_1-1}\, l \int_{-\infty}^{\infty} du_1\, \frac{1}{u_1^{n_1}}\, e^{-iu_1 s} \times \begin{cases} h(u_1)\, a(u_1)^{\frac{r_1-1}{2}} & \text{for } r_1 \text{ odd} \\ g(u_1)\, a(u_1)^{\frac{r_1}{2}} & \text{for } r_1 \text{ even} \end{cases} \\ &\qquad \times (-il)^{n_2-1} \int_{-\infty}^{\infty} du_2 \left[\frac{2in_2}{u_2^{n_2+1}} + \frac{2l\, b(u_2)}{u_2^{n_2+2}} \right] e^{-iu_2 s} \\ &\qquad\qquad \times \begin{cases} h(u_2)\, a(u_2)^{\frac{r_2-1}{2}} & \text{for } r_2 \text{ odd} \\ g(u_2)\, a(u_2)^{\frac{r_2}{2}} & \text{for } r_2 \text{ even} \end{cases} \\ &\qquad + (-il)^{n_1-1} \int_{-\infty}^{\infty} du_1 \left[\frac{2in_1}{u_1^{n_1+1}} + \frac{2l\, b(u_1)}{u_1^{n_1+2}} \right] e^{-iu_1 s} \\ &\qquad\qquad \times \begin{cases} h(u_1)\, a(u_1)^{\frac{r_1-1}{2}} & \text{for } r_1 \text{ odd} \\ g(u_1)\, a(u_1)^{\frac{r_1}{2}} & \text{for } r_1 \text{ even} \end{cases} \\ &\qquad \times (-il)^{n_2-1}\, l \int_{-\infty}^{\infty} du_2\, \frac{1}{u_2^{n_2}}\, e^{-iu_2 s} \times \begin{cases} h(u_2)\, a(u_2)^{\frac{r_2-1}{2}} & \text{for } r_2 \text{ odd} \\ g(u_2)\, a(u_2)^{\frac{r_2}{2}} & \text{for } r_2 \text{ even}\,, \end{cases} \end{aligned}$$

and similarly for the two other products in (4.5.18). In the case of systems of Dirac seas, this calculation can be done for each summand of the direct sum separately. Rewriting the Fourier integrals using the notation (4.5.12) and (4.5.14), we get the following result.

Contraction rules: *To leading order in* $(lE_P)^{-1}$,

$$\begin{aligned} &(\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \\ &\quad= \quad -2\, T^{(n_1)}_{[r_1]}\, (n_2\, T^{(n_2+1)}_{[r_2]} + T^{(n_2+2)}_{\{r_2\}}) \;-\; 2\, (n_1\, T^{(n_1+1)}_{[r_1]} + T^{(n_1+2)}_{\{r_1\}})\, T^{(n_2)}_{[r_2]} \end{aligned} \tag{4.5.19}$$

$$\begin{aligned}
&(\xi_j T^{(n_1)}_{[r_1]})(\overline{\xi^j T^{(n_2)}_{[r_2]}}) \\
&\quad = \; -2\, T^{(n_1)}_{[r_1]}\, (n_2\, \overline{T^{(n_2+1)}_{[r_2]}} + \overline{T^{(n_2+2)}_{\{r_2\}}}) \;-\; 2\, (n_1\, T^{(n_1+1)}_{[r_1]} + T^{(n_1+2)}_{\{r_1\}})\, \overline{T^{(n_2)}_{[r_2]}} \qquad (4.5.20) \\
&(\overline{\xi_j T^{(n_1)}_{[r_1]}})(\overline{\xi^j T^{(n_2)}_{[r_2]}}) \\
&\quad = \; -2\, \overline{T^{(n_1)}_{[r_1]}}\, (n_2\, \overline{T^{(n_2+1)}_{[r_2]}} + \overline{T^{(n_2+2)}_{\{r_2\}}}) \;-\; 2\, (n_1\, \overline{T^{(n_1+1)}_{[r_1]}} + \overline{T^{(n_1+2)}_{\{r_1\}}})\, \overline{T^{(n_2)}_{[r_2]}}\,. \qquad (4.5.21)
\end{aligned}$$

By iteratively applying (4.5.17) and the contraction rules (4.5.19–4.5.21), we can in (4.5.16) eliminate all inner factors ξ to end up with products of the form

$$(\text{smooth function})\, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \qquad (4.5.22)$$

with parameters $\alpha, \beta \geq 1$, $\gamma, \delta \geq 0$ and a_i, b_i, c_i, d_i (if $\gamma = 0 = \delta$ the denominator clearly is equal to one). Here each subscript "$\circ$" stands for an index $[r]$ or $\{r\}$. The quotient of the two monomials in (4.5.22) is called a *simple fraction*.

We point out that the above transformation rules for the inner factors (4.5.17) and (4.5.19–4.5.21) are identities valid pointwise (i.e. for fixed x and y) close to the light cone. We anticipate that Euler-Lagrange equations like (3.5.20, 3.5.21) do not lead us to evaluate the products of the form (4.5.16) pointwise, but merely in the weak sense. Therefore, we now go over to a weak analysis of the simple fraction. In the case of a continuous regularization, we thus consider the integral

$$\int d^4x\, \eta(x)\, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \qquad (4.5.23)$$

with a test function η. Before coming to the derivation of calculation rules for the integrand in (4.5.23), we must think about how the test function η is to be chosen. As explained in §4.2 in the example of the closed chain (4.2.2), a weak integral in general depends essentially on the unknown high-energy behavior of the fermionic projector and is therefore undetermined. To avoid this problem, we must evaluate (4.5.23) in such a way that our expansions near the light cone become applicable. To this end, we assume that η has its *support near the light cone*, meaning that in light-cone coordinates (s, l, x_2, x_3), the "large" variable l satisfies on the support of η the conditions (4.3.10). For clarity, we remark that this definition does not state that the support of η should be in a small neighborhood of the light cone, but merely in a strip away from the origin. This is sufficient because we shall extract information on the behavior near the light cone by considering the singularities of the integral for $E_P \to \infty$ (see (4.5.29) below). Furthermore, we assume that η is *macroscopic* in the sense that its partial derivatives scale in powers of l^{-1} or l^{-1}_{macro}. Under these assumptions, the integrand in (4.5.23) is macroscopic in l, and carrying out the s- and l-integrals in (4.5.23) gives a function which is macroscopic in the "transversal" variables x_2 and x_3. Therefore, in the three variables (l, x_2, x_3), a weak analysis is equivalent to a pointwise analysis, and thus it suffices to consider the s-integral in (4.5.23), i.e. the expression

$$\int_{-\infty}^{\infty} ds\, \eta\, \frac{T^{(a_1)}_\circ \cdots T^{(a_\alpha)}_\circ\, \overline{T^{(b_1)}_\circ \cdots T^{(b_\beta)}_\circ}}{T^{(c_1)}_\circ \cdots T^{(c_\gamma)}_\circ\, \overline{T^{(d_1)}_\circ \cdots T^{(d_\delta)}_\circ}} \qquad (4.5.24)$$

for fixed l, x_2 and x_3. In the case of a discrete regularization, the integral in (4.5.23) must be replaced by a sum over all space-time points, i.e. we must consider instead of (4.5.23) the weak sum

$$\sum_{x \in M} \eta(x)\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}}\,, \tag{4.5.25}$$

where $M \subset \mathbb{R}^4$ are the discrete space-time points and η is a macroscopic function in $\mathbb{R}^4$ with support near the light cone. Up to a normalization factor, (4.5.25) can be regarded as a Riemann sum which approximates the integral (4.5.23). Assuming that the space-time points have a generic position in $\mathbb{R}^4$ and keeping in mind that the function inside the sum (4.5.25) is macroscopic in the variables l, x_2, and x_3, the Riemann sum and the integral indeed coincide to leading order in $(lE_P)^{-1}$ and $(l_{\text{macro}} E_P)^{-1}$. Hence it is admissible to work also in the discrete case with the one-dimensional integral (4.5.24).

Let us analyze the integral (4.5.24) in more detail. We first consider how (4.5.24) scales in the Planck energy. In the limit $E_P \to \infty$, the factors $T_\circ^{(n)}$ go over to distributions which are in general singular on the light cone. Hence their product in (4.5.24) becomes ill-defined for $E_P \to \infty$ even in the distributional sense, and thus we expect that the integral (4.5.24) should diverge for $E_P \to \infty$. The order of this divergence can be determined using the following power counting argument. Keeping in mind that the regularization functions decay on the Planck scale $u \sim E_P$, the Fourier integrals (4.5.12) and (4.5.14) behave on the light cone (i.e. for $s = 0$) like

$$T_\circ^{(n)} \;\sim\; \log^g(E_P)\, E_P^{-n+1}$$

with $g = 1$ in the case $n = 1$ and $g = 0$ otherwise. Hence the product in the integrand of (4.5.24) scales on the light cone as

$$\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \;\sim\; \log^g(E_P)\, E_P^L \tag{4.5.26}$$

with $g \in \mathbb{Z}$ and

$$L \;=\; \alpha + \beta - \gamma - \delta - \sum_{j=1}^{\alpha} a_j - \sum_{j=1}^{\beta} b_j + \sum_{j=1}^{\gamma} c_j + \sum_{j=1}^{\delta} d_j\,. \tag{4.5.27}$$

We call L the *degree* of the simple fraction. We will here restrict attention to the case $L > 1$. In this case, the simple fraction (4.5.26) diverges in the limit $E_P \to \infty$ at least quadratically. If s is not zero, the oscillations of the factor $\exp(-ius)$ in (4.5.12, 4.5.14) lead to a decay of $T_\circ^{(n)}$ on the scale $s \sim E_P^{-1}$. This consideration shows that the dominant contribution to the integral (4.5.24) when $E_P \to \infty$ is obtained by evaluating η on the light cone, and the scaling behavior of this contribution is computed by multiplying (4.5.26) with a factor E_P^{-1}. We conclude that (4.5.24) diverges in the limit $E_P \to \infty$, and its leading divergence scales in E_p like

$$\eta(s=0)\, \log^g(E_P)\, E_P^{L-1}\,. \tag{4.5.28}$$

Collecting the logarithmic terms in the light-cone expansion, one can easily compute the parameter g. We remark that due to possible zeros in the denominator, the integral (4.5.24) might diverge even for finite E_P. In this case we can still use (4.5.28) if we set the proportionality factor equal to plus or minus infinity. We also note that, by

substituting the Fourier representations (4.5.12, 4.5.14) into (4.5.24), one can rewrite the products in (4.5.24) in terms of the regularization functions (this is explained in detail in Appendix E for a particular choice of regularization functions). Collecting the factors of l in (4.5.12) and (4.5.14), we end up with the following result.

Weak evaluation on the light cone: *Consider the integral (4.5.24) for a simple fraction of degree $L > 1$. Then there is an integer $g \geq 0$ and a real coefficient c_{reg} independent of s and l such that for every macroscopic test function η,*

$$\int_{-\infty}^{\infty} ds\, \eta\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \;=\; \frac{c_{\text{reg}}}{(il)^L}\, \eta(s=0)\, \log^g(E_P)\, E_P^{L-1} + (\text{higher orders in } (lE_P)^{-1} \text{ and } (l_{\text{macro}} E_P)^{-1})\,. \tag{4.5.29}$$

The coefficient c_{reg} clearly depends on the indices of the simple fraction and on the details of the regularization. We call c_{reg} a *regularization parameter*.

Integrals of type (4.5.24) can be transformed using integration by parts. For clarity we begin with the special case of a monomial,

$$\int_{-\infty}^{\infty} ds\, \left(\frac{d}{ds}\eta\right) T_\circ^{(a_1)} \cdots \overline{T_\circ^{(b_q)}} \;=\; -\int_{-\infty}^{\infty} ds\, \eta\, \frac{d}{ds}\left(T_\circ^{(a_1)} \cdots \overline{T_\circ^{(b_q)}}\right) \tag{4.5.30}$$

$$= \;-\int_{-\infty}^{\infty} ds\, \eta \left[\left(\frac{d}{ds} T_\circ^{(a_1)}\right) T_\circ^{(a_2)} \cdots \overline{T_\circ^{(b_q)}} + \cdots + T_\circ^{(a_1)} \cdots \overline{T_\circ^{(b_{q-1})}} \left(\frac{d}{ds} \overline{T_\circ^{(b_q)}}\right)\right], \tag{4.5.31}$$

where in the last step we applied the Leibniz rule. Differentiating (4.5.12) and (4.5.14) with respect to s yields that

$$\frac{d}{ds} T_\circ^{(n)} \;=\; -l\, T_\circ^{(n-1)} \qquad \text{and} \qquad \frac{d}{ds} \overline{T_\circ^{(n)}} \;=\; -l\, \overline{T_\circ^{(n-1)}}\,. \tag{4.5.32}$$

With these relations, we can carry out the derivatives in (4.5.31). Notice that the differentiation rules (4.5.32) decrease the index n by one. According to (4.5.27) and (4.5.29), decrementing the upper index of a factor $T_\circ^{(a_j)}$ or $\overline{T_\circ^{(b_k)}}$ increments the degree of the monomial and yields in the weak integral a factor of the order E_P/l. Using furthermore that η is macroscopic (as defined after (4.5.23)), we conclude that each summand in (4.5.31) dominates the left side of (4.5.30) by one order in lE_P or $l_{\text{macro}} E_P$. We have thus derived the following result.

Integration-by-parts rule for monomials: *Consider a monomial of degree $L > 1$. In a weak analysis near the light cone, we have to leading order in $(lE_P)^{-1}$ and $(l_{macro} E_P)^{-1}$,*

$$0 \;=\; T_\circ^{(a_1-1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} + \cdots + T_\circ^{(a_1)} \cdots T_\circ^{(a_p-1)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} + T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1-1)}} \cdots \overline{T_\circ^{(b_q)}} + \cdots + T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q-1)}}\,. \tag{4.5.33}$$

The integration-by-parts method works similarly for simple fractions. For ease in notation we state it more symbolically.

Integration-by-parts rule for simple fractions: *Consider a simple fraction of*

degree $L > 1$. In a weak analysis near the light cone and to leading order in $(lE_P)^{-1}$ and $(l_{macro}E_P)^{-1}$,

$$\nabla \left(\frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \right) \;=\; 0\,. \tag{4.5.34}$$

Here ∇ acts on all factors like a derivation (i.e. it is linear and satisfies the Leibniz rule), commutes with complex conjugations and

$$\nabla T_\circ^{(n)} \;=\; T_\circ^{(n-1)}\,, \qquad \nabla \frac{1}{T_\circ^{(n)}} \;=\; -\frac{T_\circ^{(n-1)}}{(T_\circ^{(n)})^2}\,.$$

The integration-by-parts rule gives us relations between simple fractions. We say that simple fractions are *independent* if the integration-by-parts rules gives no relations between them. More systematically, we consider the vector space of linear combinations of simple fractions. We say that two such linear combinations are *equivalent* if they can be transformed into each other with the integration-by-parts rules. We refer to the equivalence classes as the *basic fractions*. Taking the linear combination of the corresponding regularization parameters c_{reg}, we can associate to every basic fraction a so-called *basic regularization parameter*. In Appendix E it is shown for all simple fractions which will appear in this book that the corresponding basic fractions are linearly independent in the sense that there are no further identities between them. Therefore it seems a reasonable method to take the basic regularization parameters as empirical parameters modeling the unknown microscopic structure of space-time.

We remark that the notion of the basic fraction can be made more concrete by choosing from each equivalence class one representative. Then one can identify every basic fraction with the distinguished simple fraction in its equivalence class. For simplicity we give this construction in the special case that the simple fractions are monomials of the form

$$T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}}$$

(the construction can immediately be extended to simple fractions, but it becomes a bit complicated and we do not need it here). If only one factor $T_\circ^{(a)}$ appears ($p = 1$), one can by applying the integration-by-parts rule iteratively increment the parameter a_1; this clearly decreases the other parameters $b_1, \ldots, b_q$. In order to avoid that any of the parameters $b_1, \ldots, b_q$ becomes smaller than -1, we stop the integration-by-parts procedure as soon as one of the b_j equals -1. In this way, we can express every monomial as a unique linear combination of monomials of the form

$$T_\circ^{(a_1)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} \qquad \text{with} \qquad -1 = b_1 \leq \cdots \leq b_q. \tag{4.5.35}$$

Similarly for $p > 1$, the integration-by-parts rule allows us to increment the smallest of the parameters a_j, unless either one of the parameters b_j equals -1 or there are two factors $T^{(a_j)}$ with $a_j = \min(a_1, \ldots, a_p)$. By iteration, we can thus transform any monomial into a linear combination of monomials of the following type,

$$\begin{aligned} T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)}} \cdots \overline{T_\circ^{(b_q)}} \quad & \text{with} \quad a_1 \leq \cdots \leq a_p,\;\; b_1 \leq \cdots \leq b_q \\ & \text{and} \quad a_1 = a_2 \text{ or } b_1 = -1\,. \end{aligned} \tag{4.5.36}$$

We can now consider (4.5.35) and (4.5.36) as the basic monomials.

With the above constructions we have developed the mathematical framework for analyzing composite expressions in the fermionic projector in the continuum. Our procedure is outlined as follows. We first substitute for the fermionic projector the regularized formulas of the light-cone expansion; this yields sums of products of the form (4.5.16), where the smooth prefactor involves the bosonic potentials and fields as well as the wave functions of the Dirac particles and anti-particles of the system. Applying our contraction rules, we then eliminate all inner factors and obtain terms of the form (4.5.22). When evaluated in the weak sense (4.5.29), the l-dependence determines the degree L of the simple fraction, and the dependence on the regularization is described for each simple fraction by the corresponding regularization parameters c_{reg}. Using our integration-by-parts rule, we can furthermore restrict attention to the basic fractions and the corresponding basic regularization parameters. Taking the basic regularization parameters as free empirical parameters, the composite expressions in the fermionic projector reduce to expressions in the bosonic fields and fermionic wave functions, involving a small number of free parameters. This procedure for analyzing composite expressions in the fermionic projector is called the *continuum limit*.

CHAPTER 5

The Euler-Lagrange Equations in the Vacuum

In this chapter we discuss a general class of equations of discrete space-time in the vacuum, for a fermionic projector which is modeled according to the configuration of the fermions in the standard model. Our goal is to motivate and explain the model variational principle introduced in §3.5 in more detail and to give an overview of other actions which might be of physical interest. The basic structure of the action will be obtained by considering the continuum limit (§5.3–§5.5), whereas the detailed form of our model variational principle will be motivated by a consideration which also uses the behavior of the fermionic projector away from the light cone (§5.6).

5.1. The Fermion Configuration of the Standard Model

Guided by the configuration of the leptons and quarks in the standard model, we want to introduce a continuum fermionic projector which seems appropriate for the formulation of a realistic physical model. We proceed in several steps and begin for simplicity with the first generation of elementary particles, i.e. with the quarks d, u and the leptons e, ν_e. The simplest way to incorporate these particles into the fermionic projector as defined in §2.3 is to take the direct sum of the corresponding Dirac seas,

$$P^{\text{sea}} = \bigoplus_{a=1}^{4} X_a \,\frac{1}{2}\,(p_{m_a} - k_{m_a}) \tag{5.1.1}$$

with $m_1 = m_d$, $m_2 = m_u$, $m_3 = m_e$, $m_4 = 0$ and $X_1 = X_2 = X_3 = \mathbb{1}$, $X_4 = \chi_L$. The spin dimension in (5.1.1) is $(8, 8)$. Interpreting isometries of the spin scalar product as local gauge transformations (see §3.1), the gauge group is $U(8, 8)$. Clearly, the ordering of the Dirac seas in the direct sum in (5.1.1) is a pure convention. Nevertheless, our choice entails no loss of generality because any other ordering can be obtained from (5.1.1) by a suitable global gauge transformation.

In the standard model, the quarks come in three "colors," with an underlying $SU(3)$ symmetry. We can build in this symmetry here by taking three identical copies of each quark Dirac sea. This leads us to consider instead of (5.1.1) the fermionic projector

$$P^{\text{sea}} = \bigoplus_{a=1}^{N} X_a \,\frac{1}{2}\,(p_{m_a} - k_{m_a}) \tag{5.1.2}$$

with $N = 8$ and $m_1 = m_2 = m_3 = m_d$, $m_4 = m_5 = m_6 = m_u$, $m_7 = m_e$, $m_8 = 0$, and $X_1 = \cdots = X_7 = \mathbb{1}$, $X_8 = \chi_L$. Now the spin dimension is $(16, 16)$, and the gauge group is $U(16, 16)$.

Let us now consider the realistic situation of three generations. Grouping the elementary particles according to their lepton number and isospin, we get the four families (d, s, b), (u, c, t), $(\nu_e, \nu_\mu, \nu_\tau)$ and (e, μ, τ). In the standard model, the particles within each family couple to the gauge fields in the same way. In order to also arrange

this here, we take the (ordinary) sum of these Dirac seas. Thus we define the *fermionic projector of the vacuum* by

$$P(x,y) \;=\; \bigoplus_{a=1}^{N} \sum_{\alpha=1}^{3} X_a \,\frac{1}{2}\,(p_{m_{a\alpha}} - k_{m_{a\alpha}}) \tag{5.1.3}$$

with $N = 8$, $X_1 = \cdots = X_7 = \mathbb{1}$ and $X_8 = \chi_L$; furthermore $m_{11} = m_{21} = m_{31} = m_d$, $m_{12} = m_{22} = m_{32} = m_s$, $m_{13} = m_{23} = m_{33} = m_b$, $m_{41} = m_{51} = m_{61} = m_u$, …, $m_{71} = m_e$, $m_{72} = m_\mu$, $m_{73} = m_\tau$, and $m_{81} = m_{82} = m_{83} = 0$. We refer to the direct summands in (5.1.3) as *sectors*. The spin dimension in (5.1.3) is again $(16, 16)$.

The fermionic projector of the vacuum (5.1.3) fits into the general framework of §2.3 (it is a special case of (2.3.1) obtained by setting $g(a) = 3$). Thus the interaction can be introduced exactly as in §2.3 by defining the auxiliary fermionic projector (2.3.3), inserting bosonic potentials $\mathcal{B}$ into the auxiliary Dirac equation (2.3.10) and finally taking the partial trace (2.3.20). We point out that our only free parameters are the nine masses of the elementary leptons and quarks. The operator $\mathcal{B}$ which describes the interaction must satisfy the causality compatibility condition (2.3.18). But apart from this mathematical consistency condition, the operator $\mathcal{B}$ is arbitrary. Thus in contrast to the standard model, we do not put in the structure of the fundamental interactions here, i.e. we do not specify the gauge groups, the coupling of the gauge fields to the fermions, the coupling constants, the CKM matrix, the Higgs mechanism, the masses of the W- and Z-bosons, etc. The reason is that in our description, the physical interaction is to be determined and described by our variational principle in discrete space-time.

5.2. The General Two-Point Action

The model variational principle in §3.5 was formulated via a two-point action (3.5.3, 3.5.9, 3.5.10). In the remainder of this chapter, we shall consider the general two-point action

$$S \;=\; \sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)] \tag{5.2.1}$$

more systematically and study for which Lagrangians $\mathcal{L}$ the corresponding Euler-Lagrange (EL) equations are satisfied in the vacuum (a problem arising for actions other than two-point actions is discussed in Remark 6.2.5). Let us derive the EL equations corresponding to (5.2.1). We set

$$A_{xy} \;=\; P(x,y)\, P(y,x) \tag{5.2.2}$$

and for simplicity often omit the subscript "$_{xy}$" in what follows. In a gauge, A is represented by a $4N \times 4N$ matrix, with $N = 8$ for the fermion configuration of the standard model (5.1.3). We write the matrix components with Greek indices, $A = (A^\alpha_\beta)_{\alpha,\beta=1,\ldots,4N}$. The Lagrangian in (5.2.1) is a functional on $4N \times 4N$ matrices. Denoting its gradient by $\mathcal{M}$,

$$\mathcal{M}[A] \;=\; (\mathcal{M}[A]^\alpha_\beta)_{\alpha,\beta=1,\ldots,4N} \qquad \text{with} \qquad \mathcal{M}[A]^\alpha_\beta \;=\; \frac{\partial \mathcal{L}[A]}{\partial A^\beta_\alpha}\,,$$

the variation of $\mathcal{L}$ is given by

$$\delta \mathcal{L}[A] \;=\; \sum_{\alpha,\beta=1}^{4N} \mathcal{M}[A]^\alpha_\beta\, \delta A^\beta_\alpha \;=\; \mathrm{Tr}\,(\mathcal{M}[A]\, \delta A)\;, \tag{5.2.3}$$

where "Tr" denotes the trace of $4N \times 4N$ matrices. Summing over x and y yields the variation of the action,

$$\delta S \;=\; \sum_{x,y \in M} \delta \mathcal{L}[A_{xy}] \;=\; \sum_{x,y \in M} \mathrm{Tr}\left(\mathcal{M}[A_{xy}]\, \delta A_{xy}\right) .$$

We substitute the identity

$$\delta A_{xy} \;=\; \delta P(x,y)\, P(y,x) \;+\; P(x,y)\, \delta P(y,x) \tag{5.2.4}$$

and use the symmetry $x \leftrightarrow y$ as well as the fact that the trace is cyclic to obtain

$$\delta S \;=\; 4 \sum_{x,y \in M} \mathrm{Tr}\left(Q(x,y)\, \delta P(y,x)\right) \tag{5.2.5}$$

with

$$Q(x,y) \;=\; \frac{1}{4}\left(\mathcal{M}[A_{xy}]\, P(x,y) \;+\; P(x,y)\, \mathcal{M}[A_{yx}]\right) . \tag{5.2.6}$$

This equation can be simplified, in the same spirit as the transformation from (3.5.17) to (3.5.19) for our model variational principle.

LEMMA 5.2.1. *In the above setting,*

$$\mathcal{M}[A_{xy}]\, P(x,y) \;=\; P(x,y)\, \mathcal{M}[A_{yx}] .$$

Proof. We consider for fixed $x, y \in M$ variations of the general form

$$\delta P(x,y) \;=\; C , \qquad \delta P(y,x) \;=\; C^*$$

with C any $4N \times 4N$-matrix. Then, using (5.2.3, 5.2.4) and cyclically commuting the factors inside the trace, we obtain

$$\delta \mathcal{L}[A_{xy}] \;=\; \mathrm{Tr}\Big(\mathcal{M}[A_{xy}]\, P(x,y)\, C^* \;+\; P(y,x)\, \mathcal{M}[A_{xy}]\, C\Big).$$

Since the Lagrangian is symmetric (3.5.7), this is equal to

$$\delta \mathcal{L}[A_{yx}] \;=\; \mathrm{Tr}\Big(\mathcal{M}[A_{yx}]\, P(y,x)\, C \;+\; P(x,y)\, \mathcal{M}[A_{yx}]\, C^*\Big).$$

Subtracting these two equations, we get

$$\begin{aligned} &\mathrm{Tr}\Big(\left(\mathcal{M}[A_{xy}]\, P(x,y) - P(x,y)\, \mathcal{M}[A_{yx}]\right) C^*\Big) \\ &\quad= \;\mathrm{Tr}\Big(\left(\mathcal{M}[A_{yx}]\, P(y,x) - P(y,x)\, \mathcal{M}[A_{xy}]\right) C\Big) . \end{aligned}$$

Changing the phase of C according to $C \to e^{i\varphi} C$, $\varphi \in [0, 2\pi)$, one sees that both sides of the equation vanish separately, and thus

$$\mathrm{Tr}\Big(\left(\mathcal{M}[A_{xy}]\, P(x,y) - P(x,y)\, \mathcal{M}[A_{yx}]\right) C^*\Big) \;=\; 0 .$$

Since C is arbitrary, the claim follows. ∎

Using this lemma, we can simplify (5.2.6) to

$$Q(x,y) \;=\; \frac{1}{2}\, \mathcal{M}[A_{xy}]\, P(x,y) . \tag{5.2.7}$$

We can also write (5.2.5) in the compact form

$$\delta S \;=\; 4\, \mathrm{tr}(Q\, \delta P) , \tag{5.2.8}$$

where Q is the operator on H with kernel (5.2.7). Exactly as in §3.5 we consider unitary variations of P (3.5.14) with finite support, i.e.

$$\delta P \;=\; i\,[B, P]\,,$$

where B is a Hermitian operator of finite rank. Substituting into (5.2.8) and cyclically commuting the operators in the trace yields that

$$\delta S \;=\; 4i\,\mathrm{tr}(Q\,[B,P]) \;=\; 4i\,\mathrm{tr}([P,Q]\,B)\,.$$

Since B is arbitrary, we conclude that

$$[P, Q] \;=\; 0 \tag{5.2.9}$$

with Q according to (5.2.7). These are the EL equations.

5.3. The Spectral Decomposition of $P(x,y)\,P(y,x)$

As outlined in §3.5 in a model example, the EL equations (5.2.9) can be analyzed using the spectral decomposition of the matrix A, (3.5.11). On the other hand, we saw in Chapter 4 that A should be looked at in an expansion about the light cone. We shall now combine these methods and compute the eigenvalues and spectral projectors of A using the general formalism of the continuum limit introduced in §4.5.

Since the fermionic projector of the vacuum (5.1.3) is a direct sum, we can study the eight sectors separately. We first consider the *neutrino sector* $n = 8$, i.e.

$$P(x,y) \;=\; \sum_{\alpha=1}^{3} \chi_L\, \frac{1}{2}\,(p_{m_\alpha} - k_{m_\alpha}) \qquad \text{with} \qquad m_\alpha = 0.$$

Assuming that the regularized Dirac seas have a vector-scalar structure (4.1.5) and regularizing as explained after (4.5.11), the regularized fermionic projector, which with a slight abuse of notation we denote again by $P(x,y)$, is of the form

$$P(x,y) \;=\; \chi_L\, g_j(x,y)\, \gamma^j \tag{5.3.1}$$

with suitable functions g_j. Since P is Hermitian, $P(y,x)$ is given by

$$P(y,x) \;=\; P(x,y)^* \;=\; \chi_L\, \overline{g_j(x,y)}\, \gamma^j\,.$$

Omitting the arguments (x,y) of the functions g_j, we obtain for the 4×4 matrix A

$$A \;=\; \chi_L \,\not{g}\, \chi_L\, \overline{\not{g}} \;=\; \chi_L\, \chi_R\, \not{g}\, \overline{\not{g}} \;=\; 0\,. \tag{5.3.2}$$

Hence in the neutrino sector, A_{xy} is identically equal to zero. We refer to cancellations like in (5.3.2), which come about because the neutrino sector contains only left-handed particles, as *chiral cancellations*.

Next we consider the *massive sectors* $n = 1, \ldots, 7$ in (5.1.3), i.e.

$$P(x,y) \;=\; \sum_{\alpha=1}^{3} \frac{1}{2}\,(p_{m_\alpha} - k_{m_\alpha})\;. \tag{5.3.3}$$

Again assuming that the regularized Dirac seas have a vector-scalar structure, the regularized fermionic projector is

$$P(x,y) \;=\; g_j(x,y)\, \gamma^j + h(x,y) \tag{5.3.4}$$

with suitable functions g_j and h. Using that P is Hermitian, we obtain

$$P(y,x) \;=\; \overline{g_j(x,y)}\, \gamma^j + \overline{h(x,y)}\,. \tag{5.3.5}$$

Again omitting the arguments (x,y), we obtain for the 4×4 matrix A_{xy}

$$A \;=\; \not{g}\,\overline{\not{g}} \;+\; h\,\overline{\not{g}} \;+\; \not{g}\,\overline{h} \;+\; h\overline{h}\,. \tag{5.3.6}$$

It is useful to decompose A in the form

$$A \;=\; A_1 \;+\; A_2 \;+\; \mu$$

with

$$A_1 \;=\; \frac{1}{2}\,[\not{g}, \overline{\not{g}}]\,, \qquad A_2 \;=\; h\,\overline{\not{g}} \;+\; \not{g}\,\overline{h}\,, \qquad \mu \;=\; g\overline{g} \;+\; h\overline{h}$$

and $g\overline{g} \equiv g_j\,\overline{g^j}$. Then the matrices A_1 and A_2 anti-commute, and thus

$$(A-\mu)^2 \;=\; A_1^2 + A_2^2 \;=\; (g\overline{g})^2 \;-\; g^2\,\overline{g}^2 \;+\; (g\overline{h} + h\overline{g})^2\,. \tag{5.3.7}$$

The right side of (5.3.7) is a multiple of the identity matrix, and so (5.3.7) is a quadratic equation for A. The roots $\lambda_\pm$ of this equation,

$$\lambda_\pm \;=\; g\overline{g} \;+\; h\overline{h} \;\pm\; \sqrt{(g\overline{g})^2 \;-\; g^2\,\overline{g}^2 \;+\; (g\overline{h} + h\overline{g})^2}\,, \tag{5.3.8}$$

are the zeros of the characteristic polynomial of A. However, we must be careful about associating eigenspaces to $\lambda_\pm$ because A need not be diagonalizable. Let us first consider the case that the two eigenvalues in (5.3.8) are distinct. If we assume that A is diagonalizable, then $\lambda_\pm$ are the two eigenvalues of A, and the corresponding spectral projectors $F_\pm$ are computed to be

$$F_\pm \;=\; \frac{\mathbb{1}}{2} \;\pm\; \frac{1}{\lambda_+ - \lambda_-}\left(A \;-\; \frac{1}{2}\,(\lambda_+ + \lambda_-)\,\mathbb{1}\right) \tag{5.3.9}$$

$$\;=\; \frac{\mathbb{1}}{2} \;\pm\; \frac{\frac{1}{2}\,[\not{g}, \overline{\not{g}}] \;+\; h\overline{\not{g}} \;+\; \not{g}\overline{h}}{2\,\sqrt{(g\overline{g})^2 \;-\; g^2\,\overline{g}^2 \;+\; (g\overline{h} + h\overline{g})^2}}\,. \tag{5.3.10}$$

The explicit formula (5.3.10) even implies that A is diagonalizable. Namely, a short calculation yields that

$$A\,F_\pm \;=\; \lambda_\pm\,F_\pm \qquad\quad \text{and} \qquad\quad F_+ + F_- \;=\; \mathbb{1}\,,$$

proving that the images of F_+ and F_- are indeed eigenspaces of A which span $\mathbb{C}^4$. Moreover, a short computation using (5.3.4) and (5.3.10) yields that

$$F_\pm\,P(x,y) \;=\; \frac{\not{g} + h}{2} \;\pm\; \frac{\not{g}\,(g\overline{g} + h\overline{h}) \;-\; \overline{\not{g}}\,(g^2 - h^2) \;+\; (g\overline{g})\,h \;+\; g^2\,\overline{h}}{2\,\sqrt{(g\overline{g})^2 \;-\; g^2\,\overline{g}^2 \;+\; (g\overline{h} + h\overline{g})^2}}\,. \tag{5.3.11}$$

Writing out for clarity the dependence on x and y, the spectral decomposition of A is

$$A_{xy} \;=\; \sum_{s=\pm} \lambda_s^{xy}\,F_s^{xy}\,. \tag{5.3.12}$$

The following lemma relates the spectral representation of A_{xy} to that of A_{yx}; it can be regarded as a special case of Lemma 3.5.1.

LEMMA 5.3.1.

$$\lambda_\pm^{xy} \;=\; \lambda_\mp^{yx} \tag{5.3.13}$$

$$F_\pm^{xy}\,P(x,y) \;=\; P(x,y)\,F_\mp^{yx}\,, \tag{5.3.14}$$

Proof. According to (5.3.4, 5.3.5), A_{yx} is obtained from A_{xy} by the transformations $g_j \leftrightarrow \overline{g_j}$, $h \leftrightarrow \overline{h}$. The eigenvalues (5.3.8) are invariant under these transformations. Our convention for labeling the eigenvalues is (5.3.13). Using this convention, we obtain from (5.3.9) that

$$\begin{aligned} F_\pm^{xy}\, P(x,y) &= \frac{1}{2}\, P(x,y) \pm \frac{1}{\lambda_+^{xy} - \lambda_-^{xy}} \left(A_{xy}\, P(x,y) \;-\; \frac{1}{2}\, (\lambda_+^{xy} + \lambda_-^{xy})\, P(x,y) \right) \\ P(x,y)\, F_\mp^{yx} &= \frac{1}{2}\, P(x,y) \pm \frac{1}{\lambda_+^{xy} - \lambda_-^{xy}} \left(P(x,y)\, A_{yx} \;-\; \frac{1}{2}\, (\lambda_+^{xy} + \lambda_-^{xy})\, P(x,y) \right). \end{aligned}$$

The identity $P(x,y)\, A_{yx} = P(x,y)\, P(y,x)\, P(x,y) = A_{xy}\, P(x,y)$ yields (5.3.14). ∎

If the eigenvalues in (5.3.8) are equal, the matrix A need not be diagonalizable (namely, the right side of (5.3.7) may be zero without (5.3.6) being a multiple of the identity matrix). Since such degenerate cases can be treated by taking the limits $\lambda_+ - \lambda_- \to 0$ in the spectral representation (5.3.12), we do not worry about them here.

Before going on, we point out that according to (5.3.8), the 4×4 matrix A has at most two distinct eigenvalues. In order to understand better how this degeneracy comes about, it is useful to consider the space V of real vectors which are orthogonal to g_j and $\overline{g_j}$ (with respect to the Minkowski metric),

$$V \;=\; \{v \,|\, v_j\, g^j = 0 = v_j\, \overline{g^j}\}\,.$$

Since we must satisfy two conditions in four dimensions, $\dim V \geq 2$. Furthermore, a short calculation using (5.3.6) shows that for every $v \in V$,

$$[A,\, v_j \gamma^j \gamma^5] \;=\; 0\,. \tag{5.3.15}$$

Thus the eigenspaces of A are invariant subspaces of the operators $v_j \gamma^j \gamma^5$. In the case when the two eigenvalues (5.3.8) are distinct, the family of operators $(v_j \gamma^j \gamma^5)_{v \in V}$ acts transitively on the two-dimensional eigenspaces of A. Notice that the operators $v_j \gamma^j \gamma^5$ map left-handed spinors into right-handed spinors and vice versa. Thus one may regard (5.3.15) as describing a symmetry between the left- and right-handed component of A. We refer to the fact that A has at most two distinct eigenvalues as the *chiral degeneracy* of the massive sectors in the vacuum.

Our next step is to rewrite the spectral representation using the formalism of §4.5. Expanding in powers of m and regularizing gives for a Dirac sea the series

$$\sum_{n=0}^\infty \frac{1}{n!} \left(\frac{i \not{g}}{2}\, T_{[2n]}^{(n-1)}(x,y) \;+\; T_{[2n+1]}^{(n)}(x,y) \right). \tag{5.3.16}$$

In composite expressions, one must carefully keep track that every factor ξ is associated to a corresponding factor $T_{[p]}^{(n)}$. In §4.5 this was accomplished by putting a bracket around both factors. In order to have a more flexible notation, we here allow the two factors to be written separately, but in this case the pairing is made manifest by adding an index ${}^{(n)}$, and if necessary also a subscript ${}_{[r]}$, to the factors ξ. With this notation, the contraction rules (4.5.19–4.5.21) can be written as

$$(\xi_{[r_1]}^{(n_1)})_j\, (\xi_{[r_2]}^{(n_2)})^j \;=\; \frac{1}{2}\, (z_{[r_1]}^{(n_1)} + z_{[r_2]}^{(n_2)})\,, \quad (\xi_{[r_1]}^{(n_1)})_j\, (\overline{\xi_{[r_2]}^{(n_2)}})^j \;=\; \frac{1}{2}\, (z_{[r_1]}^{(n_1)} + \overline{z_{[r_2]}^{(n_2)}}) \tag{5.3.17}$$

(and similar for the complex conjugates), where we introduced factors $z_{[r]}^{(n)}$ which by definition combine with the corresponding factor $T_{[r]}^{(n)}$ according to

$$z_{[r]}^{(n)}\, T_{[r]}^{(n)} \;=\; -4\,(n\, T_{[r]}^{(n+1)} + T_{\{r\}}^{(n+2)}) \;+\; (\text{smooth functions})\,. \tag{5.3.18}$$

In our calculations, most separate factors ξ and z will be associated to $T_{[0]}^{(-1)}$. Therefore, we shall in this case often omit the indices, i.e.

$$\xi \;\equiv\; \xi_{[0]}^{(-1)}\,, \qquad\qquad z \;\equiv\; z_{[0]}^{(-1)}\,.$$

We point out that the calculation rule (5.3.18) is valid only modulo smooth functions. This is because in Chapter 4 we analyzed the effects of the ultraviolet regularization, but disregarded the "regularization" for small momenta related to the logarithmic mass problem. However, this is not a problem because the smooth contribution in (5.3.18) is easily determined from the behavior away from the light cone, where the factors $T_\circ^{(n)}$ are known smooth functions and $z_{[r]}^{(n)} = \xi^2$.

We remark that one must be a little bit careful when regularizing the sum in (5.3.3) because the regularization functions will in general be different for each Dirac sea. This problem is handled most conveniently by introducing new "effective" regularization functions for the sums of the Dirac seas. More precisely, in the integrands (4.5.12–4.5.14) we make the following replacements,

$$\left.\begin{aligned} h\, a^{\frac{p-1}{2}} \;&\to\; \sum_{\alpha=1}^3 h_\alpha\, a_\alpha^{\frac{p-1}{2}} \quad, \qquad h\, a^{\frac{p-1}{2}}\, b \;\to\; \sum_{\alpha=1}^3 h_\alpha\, a_\alpha^{\frac{p-1}{2}}\, b_\alpha \\ g\, a^{\frac{p}{2}} \;&\to\; \sum_{\alpha=1}^3 g_\alpha\, a_\alpha^{\frac{p}{2}} \quad, \qquad g\, a^{\frac{p}{2}}\, b \;\to\; \sum_{\alpha=1}^3 g_\alpha\, a_\alpha^{\frac{p}{2}}\, b_\alpha\,. \end{aligned}\right\} \tag{5.3.19}$$

As is easily verified, all calculation rules for simple fractions remain valid also when different regularization functions are involved. This implies that the contraction rules (5.3.17, 5.3.18) are valid for the sums of Dirac seas as well.

Using (5.3.16) and the contraction rules, we can expand the spectral decomposition around the singularities on the light cone. Our expansion parameter is the *degree on the light cone*, also denoted by "deg". It is defined in accordance with (4.5.27) by

$$\deg(T_\circ^{(n)}) \;=\; 1-n\,, \qquad\qquad \deg(z^{(n)}) = -1\,,$$

and the degree of a function which is smooth and non-zero on the light cone is set to zero. The degree of a product is obtained by adding the degrees of all factors, and of a quotient by taking the difference of the degrees of the numerator and denominator. The leading contribution to the eigenvalues is computed as follows,

$$\begin{aligned} g\overline{g} + h\overline{h} \;&=\; (\deg < 3) \\ &\quad + \frac{1}{4}\left((\xi_j\, T_{[0]}^{(-1)})(\overline{\xi^j\, T_{[0]}^{(-1)}}) \;+\; (\xi_j\, T_{[0]}^{(-1)})(\overline{\xi^j\, T_{[2]}^{(0)}}) \;+\; (\xi_j\, T_{[2]}^{(0)})(\overline{\xi^j\, T_{[0]}^{(-1)}}) \right) \\ &=\; \frac{1}{8}\,(z+\overline{z})\, T_{[0]}^{(-1)}\, \overline{T_{[0]}^{(-1)}} \;+\; (\deg < 3) \\ &=\; \frac{1}{2}\left(T_{[0]}^{(0)}\, \overline{T_{[0]}^{(-1)}} + T_{[0]}^{(-1)}\, \overline{T_{[0]}^{(0)}} \right) \;+\; (\deg < 3) \end{aligned}$$

$$
\begin{aligned}
(g\overline{g})^2 - g^2\,\overline{g}^2 &= \frac{1}{16}\left((\xi_j\, T^{(-1)}_{[0]})(\overline{\xi^j\, T^{(-1)}_{[0]}})\right)^2 \\
&\quad -\frac{1}{16}\,(\xi\, T^{(-1)}_{[0]})^2\,(\overline{\xi\, T^{(-1)}_{[0]}})^2 \;+\; (\deg < 6) \\
&= \frac{1}{4}\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right)^2 \\
&\quad -T^{(-1)}_{[0]}\,T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}\,T^{(0)}_{[0]}} \;+\; (\deg < 6) \\
&= \frac{1}{4}\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right)^2 \;+\; (\deg < 6) \\
(g\overline{h} + h\overline{g})^2 &= \frac{1}{4}\left((i\xi\, T^{(-1)}_{[0]})\,\overline{T^{(0)}_{[1]}} + T^{(0)}_{[1]}\,(\overline{i\xi\, T^{(-1)}_{[0]}})\right)^2 \;+\; (\deg < 6) \\
&= (\deg < 6)\,,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\lambda_\pm &= \frac{1}{2}\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right) \;+\; (\deg < 3) \\
&\quad \pm\frac{1}{2}\sqrt{\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right)^2 \;+\; (\deg < 6)} \\
&= \frac{1}{2}\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right) \;+\; (\deg < 3) \\
&\quad \pm\frac{1}{2}\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right) \pm \frac{(\deg < 6)}{4\left(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}\right)} \;+\; \cdots \\
&= \begin{cases} T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} & \text{for ``+''} \\ T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}} & \text{for ``$-$''} \end{cases} \;+\; (\deg < 3). \qquad (5.3.20)
\end{aligned}
$$

The spectral projectors are calculated similarly,

$$
\begin{aligned}
F_\pm &= \frac{\mathbb{1}}{2} \pm \frac{\frac{1}{8}\,[\not{g}\, T^{(-1)}_{[0]},\, \overline{\not{g}\, T^{(-1)}_{[0]}}] \;-\; \frac{i}{2}\left(T^{(0)}_{[1]}\,(\overline{\not{g} T^{(-1)}_{[0]}}) \;-\; (\not{g} T^{(-1)}_{[0]})\,\overline{T^{(0)}_{[1]}}\right)}{T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}} \\
&\quad +(\deg < 0) \qquad (5.3.21) \\
F_\pm\, P(x,y) &= \frac{i}{4}\,(\not{g}\, T^{(-1)}_{[0]}) \;+\; (\deg < 2) \\
&\quad \pm\frac{i}{4}\,\frac{(\not{g}\, T^{(-1)}_{[0]})(T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} + T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}) \;-\; 2\,(\overline{\not{g}\, T^{(-1)}_{[0]}})\,T^{(-1)}_{[0]}\,T^{(0)}_{[0]}}{T^{(0)}_{[0]}\,\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\,\overline{T^{(0)}_{[0]}}}. \qquad (5.3.22)
\end{aligned}
$$

The last expression contains inner factors ξ. In situations when these factors are not contracted to other inner factors in a composite expression, we can treat them as outer

factors. Then (5.3.22) simplifies to

$$F_\pm\, P(x,y) \;=\; \begin{cases} 0 & \text{for "+"} \\ \frac{i}{2}\, \not{\xi}\, T^{(-1)}_{[0]} & \text{for "$-$"} \end{cases} \;+\; (\deg < 2)\,. \tag{5.3.23}$$

By expanding, one can compute the eigenvalues and spectral projectors also to lower degree on the light cone. We do not want to enter the details of this calculation here because in this chapter we only need that the lower degrees involve the masses of the Dirac seas. This is illustrated by the following expansion of the eigenvalues,

$$\begin{aligned} \lambda_\pm \;=\; \frac{1}{4} \times \begin{cases} (z\, T^{(-1)}_{[0]})\, \overline{T^{(-1)}_{[0]}} \;+\; (z\, T^{(0)}_{[2]})\, \overline{T^{(-1)}_{[0]}} \;+\; (z\, T^{(-1)}_{[0]})\, \overline{T^{(0)}_{[2]}} & \text{for "+"} \\ T^{(-1)}_{[0]}\, (\overline{z\, T^{(-1)}_{[0]}}) \;+\; T^{(-1)}_{[0]}\, (\overline{z\, T^{(0)}_{[2]}}) \;+\; T^{(0)}_{[2]}\, (\overline{z\, T^{(-1)}_{[0]}}) & \text{for "$-$"} \end{cases} \\ + T^{(0)}_{[1]}\, \overline{T^{(0)}_{[1]}} \mp \frac{T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}}}{T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[0]}}}\, (T^{(0)}_{[1]}\, \overline{T^{(0)}_{[0]}} - T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}}) \\ + (\deg < 2)\,. \end{aligned} \tag{5.3.24}$$

Similarly, the contributions to degree < 2 involve even higher powers of the masses.

Let us specify how we can give the above spectral decomposition of A_{xy} a mathematical meaning. A priori, our formulas for $\lambda_\pm$ and $F_\pm$ are only formal expressions because the formalism of the continuum limit applies to simple fractions, but dividing by $(T^{(0)}_{[0]}\overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\overline{T^{(0)}_{[0]}})$ is not a well-defined operation. In order to make mathematical sense of the spectral decomposition and in order to ensure at the same time that the EL equations have a well-defined continuum limit, we shall only consider Lagrangians for which all expressions obtained by substituting the spectral representation of A into the EL equations are linear combinations of simple fractions. Under this assumption, working with the spectral representation of A_{xy} can be regarded merely as a convenient formalism for handling the EL equations, the latter being well-defined according to §4.5. Having Lagrangians of this type in mind, we can treat A in the massive sectors as a diagonalizable matrix with two distinct eigenvalues $\lambda_\pm$ and corresponding spectral projectors $F_\pm$.

The explicit formulas (5.3.20, 5.3.21) show that the eigenvalues of A are to leading degree not real, but appear in *complex conjugate pairs*, i.e.

$$\overline{\lambda_+} \;=\; \lambda_- \qquad \text{and} \qquad F_+^* \;=\; F_-\,. \tag{5.3.25}$$

If one considers perturbations of these eigenvalues by taking into account the contributions of lower degree, λ_+ and λ_- will clearly still be complex. This implies that the relations (5.3.25) remain valid (see the argument after (3.5.5) and the example (5.3.24)). We conclude that in our expansion about the singularities on the light cone, the eigenvalues appear to every degree in complex conjugate pairs (5.3.25).

We finally summarize the results obtained in the neutrino and massive sectors and introduce a convenient notation for the eigenvalues and spectral projectors of A_{xy}. We found that in the continuum limit, A can be treated as a diagonalizable matrix. We denote the distinct eigenvalues of A by $(\lambda_k)_{k=1,\ldots,K}$ and the corresponding spectral projectors by F_k. Since A vanishes in the neutrino sector, zero is an eigenvalue of A of multiplicity four; we choose the numbering such that $\lambda_1 = 0$. Due to the chiral degeneracy, all eigenspaces are at least two-dimensional. Furthermore, all non-zero eigenvalues of A are complex and appear in complex conjugate pairs (5.3.25). It is

useful to also consider the eigenvalues counting their multiplicities. We denote them by $(\lambda_\alpha)_{\alpha=1,\ldots,4N}$ or also by $(\lambda_{ncs})_{n=1,\ldots,8,\, c=L/R,\, s=\pm}$, where n refers to the sectors and c, s count the eigenvalues within each sector. More precisely,

$$\lambda_{8a} \;=\; 0 \qquad \text{and} \qquad \lambda_{nc\pm} = \lambda_\pm^{(n)}\,, \quad n = 1, \ldots, 7 \tag{5.3.26}$$

with $\lambda_\pm^{(n)}$ as given by (5.3.8) or (5.3.24), where the index "$^{(n)}$" emphasizes that the eigenvalues $\lambda_\pm$ depend on the regularization functions in the corresponding sector.

5.4. Strong Spectral Analysis of the Euler-Lagrange Equations

In this section we shall derive conditions which ensure that the EL equations (5.2.9, 5.2.7) are satisfied in the vacuum and argue why we want to choose our Lagrangian in such a way that these sufficient conditions are fulfilled. Since the Lagrangian $\mathcal{L}[A]$ must be independent of the matrix representation of A, it depends only on the eigenvalues λ_α,

$$\mathcal{L}[A_{xy}] \;=\; \mathcal{L}(\lambda_1^{xy}, \ldots, \lambda_{4N}^{xy})\,,$$

and furthermore $\mathcal{L}(\lambda_1^{xy}, \ldots, \lambda_{4N}^{xy})$ is symmetric in its arguments. In preparation, we consider the case when the eigenvalues of A are non-degenerate. Then the variation of the eigenvalues is given in first order perturbation theory by $\delta\lambda_\alpha = \mathrm{Tr}(F_\alpha\, \delta A)$. Let us assume that $\mathcal{L}$ depends smoothly on the λ_α, but is not necessarily holomorphic (in particular, $\mathcal{L}$ is allowed to be a polynomial in $|\lambda_\alpha|$). Then

$$\delta\mathcal{L} \;=\; \sum_{\alpha=1}^{4N} \left(\frac{\partial \mathcal{L}(\lambda)}{\partial\, \mathrm{Re}\lambda_\alpha}\, \mathrm{Re}\,\mathrm{Tr}(F_\alpha\, \delta A) \;+\; \frac{\partial \mathcal{L}(\lambda)}{\partial\, \mathrm{Im}\lambda_\alpha}\, \mathrm{Im}\,\mathrm{Tr}(F_\alpha\, \delta A) \right) \tag{5.4.1}$$

$$\;=\; \mathrm{Re} \sum_{\alpha=1}^{4N} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_\alpha}\, \mathrm{Tr}(F_\alpha\, \delta A)\,, \tag{5.4.2}$$

where we set

$$\frac{\partial \mathcal{L}(\lambda)}{\partial\, \mathrm{Re}\lambda_\alpha} \;=\; \lim_{\mathbb{R} \ni \varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathcal{L}(\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda_\alpha + \varepsilon, \lambda_{\alpha+1}, \ldots, \lambda_{4N}) - \mathcal{L}(\lambda_1, \ldots, \lambda_{4N}) \right)$$

$$\frac{\partial \mathcal{L}(\lambda)}{\partial\, \mathrm{Im}\lambda_\alpha} \;=\; \lim_{\mathbb{R} \ni \varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathcal{L}(\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda_\alpha + i\varepsilon, \lambda_{\alpha+1}, \ldots, \lambda_{4N}) - \mathcal{L}(\lambda_1, \ldots, \lambda_{4N}) \right)$$

and

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_\alpha} \;\equiv\; \frac{\partial \mathcal{L}(\lambda)}{\partial\, \mathrm{Re}\lambda_\alpha} \;-\; i\, \frac{\partial \mathcal{L}(\lambda)}{\partial\, \mathrm{Im}\lambda_\alpha}\,. \tag{5.4.3}$$

If some of the λ_αs coincide, we must apply perturbation theory with degeneracies. One obtains in generalization of (5.4.2) that

$$\delta\mathcal{L} \;=\; \mathrm{Re} \sum_{k=1}^{K} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_\alpha}\, \mathrm{Tr}(F_k\, \delta A) \Big|_{\lambda_\alpha = \lambda_k}\,. \tag{5.4.4}$$

Here our notation means that we choose the index α such that $\lambda_\alpha = \lambda_k$. Clearly, α is not unique if λ_k is a degenerate eigenvalue; in this case α can be chosen arbitrarily due to the symmetry of $\mathcal{L}$. We also write (5.4.4) in the shorter form

$$\delta\mathcal{L} \;=\; \mathrm{Re} \sum_{k=1}^{K} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_k}\, \mathrm{Tr}(F_k\, \delta A)\,. \tag{5.4.5}$$

In this formula it is not necessary to take the real part. Namely, as the Lagrangian is real and symmetric in its arguments, we know that $\overline{\mathcal{L}(\lambda_1, \ldots, \lambda_{4N})} = \mathcal{L}(\overline{\lambda_1}, \ldots, \overline{\lambda_{4N}})$, and thus, according to (5.4.3),

$$\overline{\frac{\partial \mathcal{L}(\lambda_1, \ldots, \lambda_{4N})}{\partial \lambda_k}} \;=\; \frac{\partial \mathcal{L}(\overline{\lambda_1}, \ldots, \overline{\lambda_{4N}})}{\partial \overline{\lambda_k}} \,.$$

Using furthermore that the eigenvalues of A appear in complex conjugate pairs, (5.3.25), one sees that the operator

$$\sum_{k=1}^{K} \frac{\partial \mathcal{L}}{\partial \lambda_k}\, F_k$$

is Hermitian. Hence we can write the sum in (5.4.5) as the trace of products of two Hermitian operators on H, being automatically real. We conclude that

$$\delta \mathcal{L} \;=\; \sum_{k=1}^{K} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_k}\, \mathrm{Tr}(F_k\, \delta A)\,. \tag{5.4.6}$$

Comparing (5.4.6) with (5.2.3) gives

$$\mathcal{M}[A_{xy}] \;=\; \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_k^{xy}}\, F_k^{xy}\,. \tag{5.4.7}$$

We substitute this identity into (5.2.7) and apply Lemma 5.3.1 in each sector to obtain

$$Q(x,y) \;=\; \frac{1}{2} \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}}\, F_k^{xy}\, P(x,y) \;=\; \frac{1}{2} \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}}\, P(x,y)\, F_k^{yx}\,. \tag{5.4.8}$$

Using these relations, we can write the EL equations (5.2.9) as

$$\int d^4z \left(P(x,z)\, P(z,y) \sum_{k=1}^{K_{zy}} \frac{\partial \mathcal{L}(\lambda^{zy})}{\partial \lambda_k^{zy}}\, F_k^{yz} \right. \tag{5.4.9}$$

$$\left. - \sum_{k=1}^{K_{xz}} \frac{\partial \mathcal{L}(\lambda^{xz})}{\partial \lambda_k^{xz}}\, F_k^{xz}\, P(x,z)\, P(z,y) \right) \;=\; 0\,. \tag{5.4.10}$$

This equation splits into separate equations on the eight sectors. In the neutrino sector, we have according to (5.3.1),

$$P(x,z)\, P(z,y) \;=\; \chi_L \,\not{g}(x,z)\, \chi_L \,\not{g}(z,y) \;=\; \chi_L\, \chi_R \,\not{g}(x,z)\, \not{g}(z,y) \;=\; 0\,. \tag{5.4.11}$$

Since both summands in (5.4.10) contain a factor $P(x,z)\, P(z,y)$, the EL equations are trivially satisfied in the neutrino sector due to chiral cancellations. In the massive sectors, there are no chiral cancellations. As shown in Appendix F, there are no further cancellations in the commutator if generic perturbations of the physical system are taken into account. This means that (5.4.10) will be satisfied if and only if Q vanishes in the massive sectors, i.e.

$$\sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}}\, F_k^{xy}\, P(x,y)\, XX^* \;=\; 0\,. \tag{5.4.12}$$

As explained on page 131, we shall only consider Lagrangians for which (5.4.12) is a linear combination of simple fractions. Thus we can evaluate (5.4.12) weakly on

the light cone and apply the integration-by-parts rules. We will now explain why it is reasonable to demand that (5.4.12) should be valid even in the *strong* sense, i.e. without weak evaluation and the integration-by-parts rule. First, one should keep in mind that the integration-by-parts rules are valid only to leading order in $(lE_P)^{-1}$. As is worked out in Appendix D, the restriction to the leading order in $(lE_P)^{-1}$ is crucial when perturbations by the bosonic fields are considered (basically because the microscopic form of the bosonic potentials is unknown). But in the vacuum, one can consider the higher orders in $(lE_P)^{-1}$ as well (see the so-called regularization expansion in §4.3 and §4.4). Therefore, it is natural to impose that in the vacuum the EL equations should be satisfied to all orders in $(lE_P)^{-1}$. Then the integration-by-parts rules do not apply, and weak evaluation becomes equivalent to strong evaluation. A second argument in favor of a strong analysis is that even if we restricted attention to the leading order in $(lE_P)^{-1}$ and allowed for integrating by parts, this would hardly simplify the equations (5.4.12), because the integration-by-parts rules depend on the indices n of the involved factors $T_\circ^{(n)}$ and $\overline{T_\circ^{(n)}}$. But the relations between the simple fractions given by the integration-by-parts rules are different to every degree, and thus the conditions (5.4.12) could be satisfied only by imposing to every degree conditions on the regularization parameters. It seems difficult to satisfy all these extra conditions with our small number of regularization functions. Clearly, this last argument does not rule out the possibility that there might be a Lagrangian together with a special regularization such that (5.4.12) is satisfied to leading order in $(lE_P)^{-1}$ only after applying the integration-by-parts rules. But such Lagrangians are certainly difficult to handle, and we shall not consider them here.

For these reasons, we here restrict attention to Lagrangians for which (5.4.12) is satisfied strongly. Then (5.4.12) simplifies to the conditions

$$\frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda^{xy}_{ncs}} \;=\; 0 \qquad \text{for } n = 1, \ldots, 7. \tag{5.4.13}$$

5.5. Motivation of the Lagrangian, the Mass Degeneracy Assumption

Let us discuss the conditions (5.4.13) in concrete examples. We begin with the class of Lagrangians which are *polynomial* in the eigenvalues λ_α of A. Since different powers of λ_α have a different degree on the light cone, there cannot be cancellations between them. Thus it suffices to consider polynomials which are *homogeneous of degree* h, $h \geq 1$. Furthermore, as the Lagrangian should be independent of the matrix representation of A, it can be expressed in terms of traces of powers of A. Thus we consider Lagrangians of the form

$$\mathcal{L}[A] \;=\; \mathcal{P}_h[A]\,, \tag{5.5.1}$$

where $\mathcal{P}_l$ denotes a polynomial in $\mathrm{Tr}(A^p)$ homogeneous in A of degree l, i.e.

$$\mathcal{P}_l \;=\; \sum_n c_n\, R_{p_1} \cdots R_{p_{\max(n)}} \qquad \text{with} \qquad \sum_{j=1}^{\max(n)} p_j \;=\; l, \tag{5.5.2}$$

$$R_p \;=\; \mathrm{Tr}(A^p) \;=\; \sum_{\alpha=1}^{4N} \lambda_\alpha^p \tag{5.5.3}$$

and coefficients c_n, which for simplicity we assume to be rational. For example, the general Lagrangian of degree $h = 3$ reads

$$\mathcal{L}[A] \;=\; c_1\, R_3 \;+\; c_2\, R_1\, R_2 \;+\; c_3\, R_1\, R_1\, R_1 \tag{5.5.4}$$

with three coefficients $c_n \in \mathbb{Q}$ (only two of which are of relevance because the normalization of $\mathcal{L}$ clearly has no effect on the EL equations). We assume that $\mathcal{L}$ is *non-trivial* in the sense that at least one of the coefficients c_n in the definition (5.5.1, 5.5.2) of $\mathcal{L}$ should be non-zero. The EL equations corresponding to (5.5.4) can easily be computed using that $\delta R_p = p\, \mathrm{Tr}(A^{p-1}\, \delta A)$ together with (5.2.4) and the fact that the trace is cyclic. The resulting operator Q in (5.2.9) is of the form

$$Q(x,y) \;=\; \Big[\mathcal{P}_0\, A^{h-1} \;+\; \mathcal{P}_1\, A^{h-2} \;+\; \cdots \;+\; \mathcal{P}_{h-1}\Big]\, P(x,y) \tag{5.5.5}$$

where $\mathcal{P}_l$ are homogeneous polynomials of the form (5.5.2) ($\mathcal{P}_0$ is a rational number). In the example (5.5.4),

$$Q(x,y) \;=\; \left[\frac{3}{2}\, c_1\, A^2 \;+\; c_2\, R_1\, A \;+\; \frac{1}{2}\,(c_2\, R_2 + 3 c_3\, R_1^2)\right] P(x,y)\,.$$

By substituting the regularized formulas of the light-cone expansion into (5.5.5), one sees that $Q(x,y)$ is to every degree on the light cone a polynomial in $T_\circ^{(n)}$ and $\overline{T_\circ^{(n)}}$, well-defined according to §4.5. Thus for polynomial actions, our spectral decomposition is not needed. But it is nevertheless a convenient method for handling the otherwise rather complicated combinatorics of the Dirac matrices.

For the polynomial Lagrangian (5.5.1), the conditions (5.4.13) become

$$\mathcal{P}_0\, \lambda^{h-1} \;+\; \mathcal{P}_1\, \lambda^{h-2} \;+\; \cdots \;+\; \mathcal{P}_{h-1} \;=\; 0 \qquad \text{for } \lambda = \lambda_{ncs},\; n = 1, \ldots, 7 \tag{5.5.6}$$

and the $\mathcal{P}_l$ as in (5.5.2). It is useful to analyze these conditions algebraically as polynomial equations with rational coefficients for the eigenvalues of A. To this end, we need to introduce an abstract mathematical notion which makes precise that, according to (5.3.26), the eigenvalues λ_{ncs} have certain degeneracies, but that there are no further relations between them. We say that the matrix A has n *independent eigenvalues* if A has n distinct eigenvalues, one of them being zero and the others being algebraically independent. The following lemma shows that the conditions (5.5.6) can be fulfilled only if the degree of the Lagrangian is sufficiently large.

Lemma 5.5.1. *For a non-trivial Lagrangian of the form (5.5.1) which satisfies the conditions (5.5.6),*

$$h \;\geq\; n\,, \tag{5.5.7}$$

where n denotes the number of independent eigenvalues of A.

Proof. First suppose that the $\mathcal{P}_l$ in (5.5.6) are not all zero. Then we can regard the left side of (5.5.6) as a polynomial in λ of degree at most $h-1$. According to (5.3.26), the eigenvalues λ_{8a} in the lepton sector all vanish. Thus the polynomial in (5.5.6) has at least $n-1$ distinct zeros, and thus its degree must be at least $n-1$. This proves (5.5.7).

It remains to consider the case when the coefficients $\mathcal{P}_l$ in (5.5.6) all vanish. Since the Lagrangian is non-trivial, at least one of the $\mathcal{P}_l$ is non-trivial, we write $\mathcal{P}_l \not\equiv 0$. On the other hand, $\mathcal{P}_l[A] = 0$ and furthermore $l \leq h-1$ from (5.5.6). Hence to conclude the proof it suffices to show that

$$\mathcal{P}_l \not\equiv 0 \;\text{ and }\; \mathcal{P}_l[A] = 0 \quad\Longrightarrow\quad l \geq n-1. \tag{5.5.8}$$

To prove (5.5.8), we proceed inductively in n. For $n = 1$ there is nothing to show. Assume that (5.5.8) holds for given n and any matrix A with n independent eigenvalues. Consider a matrix A with $n+1$ independent eigenvalues. A given non-trivial polynomial $\mathcal{P}_{l+1}$ can be uniquely decomposed in the form

$$\mathcal{P}_{l+1} \;=\; R_1\, \mathcal{P}_l \;+\; R_2\, \mathcal{P}_{l-1} \;+\; \cdots \;+\; R_{l+1}\, \mathcal{P}_0\,, \tag{5.5.9}$$

where the polynomials $\mathcal{P}_{l-k}$ contain only factors R_j with $j > k$. Since $\mathcal{P}_{l+1} \not\equiv 0$, at least one of the factors $\mathcal{P}_{l-k}$ is non-trivial. Let $k \geq 0$ be the smallest natural number such that $\mathcal{P}_{l-k} \not\equiv 0$. The functional $\mathcal{P}_{l+1}[A]$ is a homogeneous polynomial of degree $l+1$ in the eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$ of A. We pick those contributions to this polynomial which are homogeneous in $0 \neq \lambda_{n+1}$ of degree $n+1$. These contributions all come from the summand $R_{k+1}\mathcal{P}_{l-k}$ in (5.5.9) because the summands to its left are trivial and the summands to its right are composed only of factors R_l with $l > k+1$. Hence, apart from the prefactor λ_{n+1}^{k+1} and up to irrelevant combinatorial factors for each of the monomials, these contributions coincide with the polynomial $\mathcal{P}_{l-k}(\lambda_1, \ldots, \lambda_n)$ evaluated for a matrix with n independent eigenvalues. We conclude that if $\mathcal{P}_{l+1}(\lambda_1, \ldots, \lambda_{n+1})$ vanishes, then $\mathcal{P}_{l-k}(\lambda_1, \ldots, \lambda_n)$ must also be zero. The induction hypothesis yields that $l - k \geq n - 1$ and thus $l + 1 \geq n$. ∎

We seek a Lagrangian which is as simple as possible. One strategy is to consider the general polynomial Lagrangian (5.5.1) and to choose the degree h as small as possible. According to Lemma 5.5.1, the degree cannot be smaller than the number of independent eigenvalues of A. Thus if we treat the eigenvalues as being algebraically independent in the sectors containing the Dirac seas (d, s, b), (u, c, t), and (e, μ, τ), then h is bounded from below by $h \geq 3 \times 2 + 1 = 7$. Unfortunately, polynomials of degree ≥ 7 involve many coefficients c_n and are complicated. Therefore, it is desirable to reduce the number of independent eigenvalues. Since the eigenvalues depend on the masses and regularization functions of the particles involved (see (5.3.24)), we can reduce the number of distinct eigenvalues only by assuming that the massive sectors are identical. The best we can do is to assume that

$$m_u = m_d = m_e\,, \qquad m_c = m_s = m_\mu\,, \qquad m_t = m_b = m_\tau\,, \tag{5.5.10}$$

and that the regularization functions in the massive sectors coincide. Then the additional degeneracies in the massive sectors reduce the number of distinct eigenvalues to three. If (5.5.10) holds, the bound of Lemma 5.5.1 is even optimal. Namely, a simple calculation shows that the polynomial Lagrangian of degree three (5.5.4) with

$$c_1 \;=\; 14\,, \qquad c_2 \;=\; -\frac{3}{2}\,, \qquad c_3 \;=\; \frac{1}{28} \tag{5.5.11}$$

satisfies the conditions (5.4.13). According to Lemma 5.5.1, a degree $h < 2$ would make it necessary to impose relations between λ_+ and λ_-, which is impossible in our formalism (5.3.20). We conclude that (5.5.4, 5.5.11) is the polynomial Lagrangian of minimal degree which satisfies the conditions (5.4.13). We refer to (5.5.10) as the *mass degeneracy assumption*. In order to understand what this condition means physically, one should keep in mind that (5.5.10) gives conditions for the bare masses, which due to the self-interaction are different from the effective masses (this is a bit similar to the situation in the grand unified theories, where simple algebraic relations between the bare quark and lepton masses are used with some success [**Ro**]).

Another strategy for finding promising Lagrangians is to consider homogeneous polynomials of higher degree, but which are of a particularly simple form. A good example for such a Lagrangian is the determinant,

$$\mathcal{L}[A] \;=\; \det A\,. \tag{5.5.12}$$

By writing $\det A = \det(\mathbb{1} - (\mathbb{1} - A))$, expanding in powers of $(\mathbb{1} - A)$ and multiplying out, this Lagrangian can be brought into the form (5.5.1, 5.5.2) with $h = 4N$. The Lagrangian (5.5.12) is appealing because of its simple form. Furthermore, it has the nice property that whenever the eigenvalues of A appear in complex conjugate pairs (5.3.25), the product of these eigenvalues is positive, and thus $\mathcal{L} \geq 0$. Unfortunately, this Lagrangian has the following drawback. The matrix A vanishes identically in the neutrino sector (5.3.2), and so A has a zero eigenvalue of multiplicity four. As a consequence, $\mathcal{L}$ and its variations vanish until perturbations of at least fourth order are taken into account, making the analysis rather complicated. For this reason, (5.5.12) does not seem the best Lagrangian for developing our methods, and we shall not consider it here.

In the polynomial Lagrangian (5.5.4, 5.5.11) we did not use that the eigenvalues of A appear in complex conjugate pairs. This fact can be exploited to construct an even simpler Lagrangian. Assume again that the masses are degenerate (5.5.10). Then the absolute values $|\lambda_\alpha|$ of the eigenvalues of A take only the two values 0 and $|\lambda_+| = |\lambda_-|$, with multiplicities 4 and 28, respectively. Thus if we consider homogeneous polynomials in $|\lambda_\alpha|$, there is already a Lagrangian of degree two which satisfies the conditions (5.4.13), namely

$$\mathcal{L} \;=\; \sum_{\alpha=1}^{4N} |\lambda_\alpha|^2 \;-\; \frac{1}{28} \left(\sum_{\alpha=1}^{4N} |\lambda_\alpha| \right)^2 . \tag{5.5.13}$$

Using the notion of the spectral weight (3.5.8), this Lagrangian can be written as

$$\mathcal{L}[A] \;=\; |A^2| \;-\; \frac{1}{28}\, |A|^2\,. \tag{5.5.14}$$

The factor $1/28$ may be replaced by a Lagrange multiplier μ,

$$\mathcal{L}[A] \;=\; |A^2| \;-\; \mu\, |A|^2\,, \tag{5.5.15}$$

because the value of $\mu = 1/28$ is uniquely determined from the condition that the EL equations should be satisfied in the vacuum. The functional (5.5.15) can be regarded as the effective Lagrangian corresponding to the variational principle with constraint (3.5.9, 3.5.10). We conclude that (5.5.14) is precisely our model variational principle introduced in §3.5.

The above considerations give a motivation for our model Lagrangian (5.5.14) as well as for the mass degeneracy assumption (5.5.10). Also, it is nice that many special properties of the fermionic projector of the vacuum were used. Namely, the EL equations corresponding to (5.5.14) are fulfilled only because of chiral cancellations in the neutrino sector and the fact that the eigenvalues of A appear in complex conjugate pairs. But unfortunately, our arguments so far do not determine the action uniquely. In particular, variational principles formulated with the spectral weight of higher powers of A, like for example

$$\mathcal{L}[A] \;=\; |A^4| \;-\; \frac{1}{28}\, |A^2|^2$$

or, more generally,

$$\mathcal{L}[A] \;=\; |A^{2n}| \;-\; \frac{1}{28}\,|A^n|^2 \qquad (n \geq 1), \tag{5.5.16}$$

are all satisfied in the vacuum for exactly the same reasons as (5.5.14). The next section gives an argument which distinguishes (5.5.14) from the other Lagrangians.

5.6. Stability of the Vacuum

In §4.2 we argued that the pointwise product $P(x,y)\,P(y,x)$ depends essentially on the unknown high-energy behavior of the fermionic projector and is therefore undetermined. This argument, which led us to a weak analysis near the light cone and was the starting point for the formalism of the continuum limit in §4.5, must clearly be taken seriously if one wants to get information for general regularizations. On the other hand, the equations of discrete space-time (if analyzed without taking the continuum limit) should yield constraints for the regularization, and one might expect that for the special regularizations which satisfy these constraints, one can make statements on the fermionic projector even pointwise. In particular, it is tempting to conjecture that away from the light cone, where the fermionic projector of the continuum is smooth (see §2.5), the fermionic projector of discrete space-time should be well-approximated pointwise by the fermionic projector of the continuum. Since going rigorously beyond the continuum limit is difficult and requires considerable work, we cannot prove this conjecture here. But we can take it as an ad-hoc assumption that away from the light cone (i.e. for $|s|, |l| \gg E_P^{-1}$), the physical fermionic projector should coincide to leading orders in sE_P^{-1} and lE_P^{-1} with the continuum fermionic projector. We refer to this assumption that the fermionic projector is *macroscopic away from the light cone*.

In this section we will analyze the EL equations in the vacuum under the assumption that the fermionic projector is macroscopic away from the light cone. This will give us some insight into how causality enters the EL equations. More importantly, a stability analysis of the vacuum will make it possible to uniquely fix our variational principle. Our analysis here can be considered as being complementary to the continuum limit: whereas in the continuum limit we restrict attention to the singularities on the light cone, we here consider only the behavior away from the light cone, where the fermionic projector is smooth. This gives us smooth functions defined for $y \in M \setminus L_x$, which however have poles when y approaches the light cone around x. In this way, we will again encounter singularities on the light cone, but of different nature than those considered in the continuum limit. Unfortunately, our method gives no information on the behavior of these singularities, and therefore we must treat them with an ad-hoc "regularity assumption" (see Def. 5.6.3). For this reason, the arguments given here should be considered only as a first step towards an analysis beyond the continuum limit. The methods and results of this section will not be needed later in this book.

We begin by deriving the spectral decomposition of the closed chain away from the light cone. Due to the direct sum structure of the fermionic projector, we can again consider the sectors separately. In the neutrino sector, the product $P(x,y)\;P(y,x)$ vanishes identically due to chiral cancellations (5.3.2). Again assuming that the masses are degenerate (see §5.5), it remains to consider one massive sector (5.3.3) with three mass parameters m_α, $\alpha = 1,2,3$. Since we assume that the fermionic projector is macroscopic away from the light cone, we do not need a regularization.

LEMMA 5.6.1. *In a massive sector, the closed chain $A = P(x,y)\,P(y,x)$ has away from the light cone the following spectral decompositions. If $y-x$ is spacelike,*

$$A \;=\; \lambda\, \mathbb{1} \qquad\qquad \text{with } \lambda \in \mathbb{R}\,. \tag{5.6.1}$$

If on the other hand $y-x$ is timelike, A has either the spectral decomposition (5.6.1) with $\lambda \geq 0$ or

$$A \;=\; \sum_{s=\pm} \lambda_s\, F_s \tag{5.6.2}$$

with spectral projectors $F_\pm$ on two-dimensional eigenspaces and corresponding eigenvalues $\lambda_\pm$ which are real and positive,

$$\lambda_\pm \;\geq\; 0\,. \tag{5.6.3}$$

Proof. We write the fermionic projector (5.3.3) in the form (5.3.4) with

$$\begin{aligned} g_j(x,y) &= i\partial_j \,\frac{1}{2} \sum_{\alpha=1}^{3} (P_{m_\alpha^2} - K_{m_\alpha^2})(x,y) \\ h(x,y) &= \frac{1}{2} \sum_{\alpha=1}^{3} m_\alpha (P_{m_\alpha^2} - K_{m_\alpha^2})(x,y)\,, \end{aligned}$$

where P_a and K_a denote the fundamental solutions of the Klein-Gordon equation,

$$P_a(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2-a)\, e^{-ip(x-y)} \tag{5.6.4}$$

$$K_a(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \epsilon(p^0)\, \delta(p^2-a)\, e^{-ip(x-y)}\,. \tag{5.6.5}$$

Due to Lorentz symmetry, the vector field g is parallel to $y-x$. Thus we can write it as

$$g_j(x,y) \;=\; i(y-x)_j\, f(x,y) \tag{5.6.6}$$

with a complex scalar distribution f. Furthermore, the distribution K_a is causal in the sense that $\text{supp}\, K_a(x,.) \subset J_x$. This can be seen by decomposing it similar to (2.2.7) as

$$K_a \;=\; \frac{1}{2\pi i} \left(S_a^\vee - S_a^\wedge\right),$$

where $S^\vee$ and $S^\wedge$ are the causal Green's functions of the Klein-Gordon equation (see (2.5.5)). Alternatively, one can for any spacelike vector $y-x$ choose a reference frame where $y-x$ points in spatial direction, $y-x = (0,\vec{v})$. Then the Fourier integral (5.6.5) can be written as

$$K_a \;=\; \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega\, \epsilon(\omega) \int_{\mathbb{R}^3} d\vec{p}\, \delta(\omega^2 - |\vec{p}|^2 - a)\, e^{-i\vec{p}\vec{v}}\,,$$

and a symmetry argument shows that the integrals over the upper and lower mass shells cancel each other.

If $y-x$ is spacelike, K_a drops out of our formulas due to causality,

$$g_j(x,y) \;=\; i\partial_j\, \frac{1}{2} \sum_{\alpha=1}^{3} P_{m_\alpha^2}(x,y)\,, \qquad\qquad h(x,y) \;=\; \frac{1}{2} \sum_{\alpha=1}^{3} m_\alpha P_{m_\alpha^2}(x,y)\,.$$

Performing in (5.6.4) the transformation $p \to -p$, we find that P_a has even parity, $P_a(y,x) = P_a(x,y)$. As a consequence,

$$\overline{g(x,y)} \;=\; g(y,x) \;=\; -g(x,y)\,, \qquad \overline{h(x,y)} \;=\; h(y,x) \;=\; h(x,y)\,.$$

Plugging these relations into (5.3.6), we obtain

$$A \;=\; \not{g}\,\overline{\not{g}} + h\,\overline{h} \;=\; \big(\langle g, \overline{g}\rangle + |h|^2\big)\,\mathbb{1}\,,$$

where in the last step we used (5.6.6). This proves (5.6.1).

If $y-x$ is timelike, the situation is more complicated because both P_a and K_a contribute. Using the representation (5.3.4, 5.6.6), we obtain

$$A \;=\; i\not{\xi}\,\big(\overline{f}h + f\overline{h}\big) + \xi^2\,|f|^2\,\mathbb{1} + |h|^2\,\mathbb{1}\,.$$

Away from the light cone, the matrix $\not{\xi}$ has the two real eigenvalues $\pm\sqrt{\xi^2}$, each with multiplicity two. Hence the eigenvalues of A are given by

$$\lambda_\pm \;=\; \pm\sqrt{\xi^2}\,\big(\overline{f}h + f\overline{h}\big) + \xi^2\,|f|^2 + |h|^2 \;=\; \Big|\sqrt{\xi^2}f \pm h\Big|^2 \;\geq\; 0\,.$$

∎

This lemma has important consequences. We first recall that the conditions (5.4.13) obtained from a strong analysis near the light cone led us to only consider Lagrangians for which $\mathcal{M}$ vanishes if A has in the massive sector two eigenvalues appearing as complex conjugate pairs. According to (5.6.1), this last condition is satisfied if $y-x$ is space-like. This means that for all variational principles discussed in §5.5, the matrix $\mathcal{M}_{xy}$ vanishes for space-like $y-x$. According to (5.2.7), the same is true for $Q(x,y)$. In other words, Q is supported inside the (closed) light cone. This remarkable property can be understood as a manifestation of some kind of "causality" in the EL equations. However, we point out that this property is no longer true in an interacting system (see Chapter 6).

Using that Q is supported inside the light cone, we can in what follows restrict attention to *timelike* $y-x$. For convenience, we will also use (5.6.2) in the case that A is a multiple of the identity matrix (5.6.1); we then simply set $\lambda_+ = \lambda_- = \lambda$. The main statement of Lemma 5.6.1 for timelike $y-x$ is that, according to (5.6.3), A is a *positive* matrix. This means that the concept of the spectral weight, which was important in §5.5 (and which will also be crucial in Chapter 6), becomes trivial in the vacuum. The spectral weight (3.5.8) reduces to the ordinary trace, and so our Lagrangians (5.5.16) simplify away from the light cone to polynomial Lagrangians.

Suppose that P is a *stable minimum* of the action, in the sense that it is impossible to decrease the action by a variation of P,

$$S[P+\delta P] \;\geq\; S[P] \quad \text{for all variations } \delta P. \tag{5.6.7}$$

In order to derive necessary conditions for stability, we shall consider this inequality for special variations. Our idea is to vary P by changing the momentum and spin orientation of individual fermionic states. In order to have discrete states, we consider the system as in §2.6 in finite 3-volume. According to the replacement rule (2.6.3), the

fermionic projector of the vacuum then becomes

$$P(x,y) \;=\; \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{1}{V} \sum_{k \in L^3} \hat{P}(k) e^{-ik(x-y)} \tag{5.6.8}$$

$$\hat{P}(k) \;=\; \sum_{\alpha=1}^{3} (\not{k} + m_\alpha)\, \delta(k^2 - m_\alpha^2)\, \theta(-k^0)\,. \tag{5.6.9}$$

Since we want to preserve the vector-scalar structure, we take both fermionic states for any k on one of the mass shells and bring them into states of momentum q which are not on the mass shells,

$$\delta P \;=\; -c\,(\not{k} + m)\, e^{-ik(x-y)} + \tilde{c}\,(\not{l} + \tilde{m})\, e^{-iq(x-y)} \tag{5.6.10}$$

with $m \in \{m_1, m_2, m_3\} \not\ni \sqrt{q^2}$ and $k^2 = m^2$, $k^0 < 0$, $l^2 = \tilde{m}^2$. Here c and $\tilde{c}$ are normalization constants; carrying out the k^0-integral in (5.6.8) one sees that

$$c \;=\; \frac{1}{4\pi k^0} \frac{1}{V}\,. \tag{5.6.11}$$

We must make sure that we do not change the normalization of the fermionic states. First of all, this means that we must preserve the sign of the inner product of the states, and this implies that l must be on the lower mass shell, $l^0 < 0$. Furthermore, the phase factors $e^{-ik(x-y)}$ and $e^{-iq(x-y)}$ in (5.6.10) drop out of normalization integrals and thus have no influence on the normalization, but the normalization constants must be kept fixed. This means that $cm = \tilde{c}\tilde{m}$, and by rescaling $\tilde{c}$ and $\tilde{m}$ we can arrange that $\tilde{c} = c$ and $\tilde{m} = m$. We conclude that the variation of P must be of the form

$$\delta P = -c\,(\not{k} + m)\, e^{-ik(x-y)} + c\,(\not{l} + m)\, e^{-iq(x-y)} \tag{5.6.12}$$

with

$$m \in \{m_1, \ldots, m_g\} \not\ni \sqrt{q^2}\,, \quad k^2 = l^2 = m^2\,, \quad k^0, l^0 < 0\,. \tag{5.6.13}$$

This variation preserves the normalization of the fermionic states, as one sees easily from the fact that it can be realized by a unitary transformation U (3.5.14) (U picks the states of momentum k, multiplies them by the phase factor $e^{-i(q-k)x}$ and finally "Lorentz boosts" the spinors with a constant unitary transformation as considered in Lemma 1.2.1). We point out that the variation (5.6.12, 5.6.13) is discrete (and not a continuous family of variations $\delta P(\tau)$). But due to the factor V^{-1} in (5.6.11), we can make δP arbitrarily small by making the 3-volume sufficiently large. Therefore, it is admissible to treat δP perturbatively. The variation of the action (5.2.8) is (up to an irrelevant normalization constant) computed to be

$$\delta S \;=\; -\mathrm{Tr}(\hat{Q}(k)(\not{k} + m)) + \mathrm{Tr}(\hat{Q}(q)(\not{l} + m))\,, \tag{5.6.14}$$

where $\hat{Q}$ is the Fourier transform of Q,

$$\hat{Q}(p) \;=\; \int Q(\xi)\, e^{-ip\xi}\, d^4\xi\,.$$

Evaluating the stability inequality (5.6.7) gives rise to the notion of state stability. We first give the definition and explain it afterwards. We denote the *mass cone* by

$$\mathcal{C} \;=\; \{p \,|\, p^2 > 0\}$$

and label the *upper* and *lower mass cone* by superscripts $\vee$ and $\wedge$, respectively,

$$\mathcal{C}^\vee \;=\; \{p \,|\, p^2 > 0 \text{ and } p^0 > 0\}\,, \qquad \mathcal{C}^\wedge \;=\; \{p \,|\, p^2 > 0 \text{ and } p^0 < 0\}\,. \tag{5.6.15}$$

DEF. 5.6.2. *The fermionic projector of the vacuum is called* **state stable** *if the corresponding operator $\hat{Q}(p)$ is well-defined in the lower mass cone $\mathcal{C}^\wedge$ and can be written as*

$$\hat{Q}(p) \;=\; a\,\frac{\not{p}}{|p|} + b \tag{5.6.16}$$

with continuous real functions a and b on $\mathcal{C}^\wedge$ having the following properties:

(i) *a and b are Lorentz invariant,*

$$a \;=\; a(p^2)\,, \qquad b \;=\; b(p^2)\,.$$

(ii) *a is non-negative.*

(iii) *The function $a+b$ is minimal on the mass shells,*

$$(a+b)(m_\alpha^2) \;=\; \inf_{q \in \mathcal{C}^\wedge} (a+b)(q^2) \quad \text{for } \alpha = 1,2,3.$$

It is natural to assume that Q is Lorentz invariant because P has this property too. Thus the main point of (5.6.16) is that $\hat{Q}$ is finite and continuous inside the lower mass cone. This is needed because otherwise the term $\mathrm{Tr}(\hat{Q}(k)(\not{k}+m))$ in (5.6.14) would be ill-defined (strictly speaking, we only need that $\hat{Q}$ is finite on the lower mass shells, but this seems impossible to arrange without $\hat{Q}$ being well-defined and continuous inside the whole lower mass cone). Using (5.6.16), we obtain for any $q \in \mathcal{C}^\wedge$,

$$\frac{1}{4}\,\mathrm{Tr}(\hat{Q}(q)(\not{l}+m)) \;=\; a\,\frac{\langle q, l\rangle}{|q|} + b\,m\,. \tag{5.6.17}$$

Since the vectors q and l are both in the lower mass cone, their inner product $\langle q, l\rangle$ is positive, and it can be made arbitrarily large by choosing $|l^0| \gg 1$. Hence if a were negative, (5.6.17) would not be bounded from below, leading to an instability. This explains (ii). If a is non-negative,

$$\frac{1}{4}\,\inf_l\,\mathrm{Tr}(\hat{Q}(q)(\not{l}+m)) \;=\; a(q)\,m + b(q)\,m\,.$$

Comparing this with the first term in (5.6.14) gives (iii).

The question arises whether the conditions of Def. 5.6.2, which are clearly necessary for stability, are also sufficient. The fact that we vary pairs of fermions keeping the vector-scalar structure is indeed no restriction because if one varies one fermionic state, the additional pseudoscalar, axial and bilinear contributions (see (C.1.9) for details) drop out when taking similar to (5.6.17) the trace with $\hat{Q}$. Also, at least if a is strictly positive and if the function $a+b$ has no minima away from the mass shells, we find that the variation (5.6.14) really increases the action, and we obtain stability. Hence the main restriction of state stability is that we consider stability only within the class of homogenous fermionic projectors. The second restriction is that Def. 5.6.2 involves no condition for $\hat{Q}(p)$ if p is outside the lower mass cone. This is because we want to allow for the possibility that $\hat{Q}(q)$ is infinite for $q \notin \mathcal{C}^\wedge$, in such a way that the expression (5.6.17) equals $+\infty$. Treating such infinities would make it necessary to introduce an ultraviolet regularization and to analyze the divergences as the regularization is removed. Since this would go far beyond the treatment in this section, we here simply disregard those q for which ultraviolet divergences might appear.

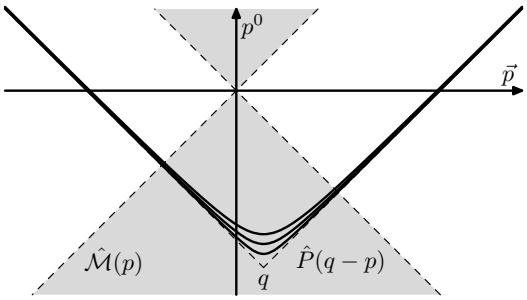

FIGURE 5.1. The convolution $\hat{\mathcal{M}} * \hat{P}$.

We come to the analysis of state stability. First, we rewrite Q using the spectral representation of A. Our starting point is (5.5.4),

$$Q(x,y) \;=\; \frac{1}{2}\,\mathcal{M}(x,y) P(x,y)\,. \tag{5.6.18}$$

Away from the light cone, we can apply Lemma 5.6.1 and write $\mathcal{M}(x,y)$ as

$$\mathcal{M}(x,y) \;=\; \begin{cases} \displaystyle\sum_{s=\pm} \frac{\partial \mathcal{L}}{\partial \lambda_s} F_s & \text{if } y-x \text{ is timelike} \\ 0 & \text{if } y-x \text{ is spacelike}\,. \end{cases} \tag{5.6.19}$$

The important point for us is that the integral kernel of Q has a product structure (5.6.18). This product in position space can also be expressed as a convolution in momentum space,

$$\hat{Q}(q) \;=\; \frac{1}{2} \int \frac{d^4p}{(2\pi)^4}\, \hat{\mathcal{M}}(p)\, \hat{P}(q-p)\,. \tag{5.6.20}$$

As is illustrated in Figure 5.1, the integration range is unbounded. Therefore, the existence of the integral as well as its value might depend sensitively on an ultraviolet regularization. Since in this section we work with the unregularized fermionic projector, we simply impose that (5.6.20) should hold even after the ultraviolet regularization has been removed.

DEF. 5.6.3. *The fermionic projector satisfies the assumption of a* **distributional $\mathcal{M}P$-product** *if for every q for which $\hat{Q}(q)$ exists, the convolution integral (5.6.20) is well-defined without regularization and coincides with $\hat{Q}(q)$.*

We point out that the above assumption is not merely a technical simplification, but it makes a highly non-trivial statement on the high-energy behavior of the physical fermionic projector. Ultimately, it needs to be justified by an analysis of the variational principle with ultraviolet regularization. At this stage, it can at least be understood from the fact that the assumption of a distributional $\mathcal{M}P$-product makes our stability analysis robust to regularization details. This means physically that the system should also be stable under "microscopic" perturbations of the fermionic projector on the Planck scale.

In what follows we will assume that the conditions in Def. 5.6.2 and Def. 5.6.3 are satisfied. Then the convolution integral (5.6.20) is well-defined for all $q \in \mathcal{C}^\wedge$. This implies that $\hat{\mathcal{M}}(p)$ must be a well-defined distribution on the set $\overline{\mathcal{C}^\wedge} \cup \{p \mid p^2 < 0\}$. Furthermore, since both $\hat{P}$ and $\hat{Q}$ are Lorentz invariant, we may assume that $\hat{\mathcal{M}}$ is

also Lorentz invariant (otherwise we could replace $\hat{\mathcal{M}}$ by its Lorentz invariant part, and (5.6.20) would remain true). But if $\mathcal{M}(x,y)$ is Lorentz invariant, it involves no bilinear contribution, and its vector component is a multiple of $y-x$. According to (5.6.19), the matrix $\mathcal{M}(x,y)$ commutes with A. This implies that, for all x, y for which $\mathcal{M}(x,y) \neq 0$, the matrix A involves no bilinear contribution. Since the bilinear part of A is given by the commutator of $P(x,y)$ and $P(y,x)$, we find that $P(x,y)$ and $P(y,x)$ commute. It follows that $A_{xy} = A_{yx}$ for all x, y for which $\mathcal{M}(x,y) \neq 0$. As a consequence, $\mathcal{M}(x,y) = \mathcal{M}(y,x)$, and thus

$$\hat{\mathcal{M}}(-p) \;=\; \hat{\mathcal{M}}(p)\,. \tag{5.6.21}$$

This identity yields that $\hat{\mathcal{M}}$ is a well-defined distribution even on $\mathcal{C}^\vee$. We thus come to the following conclusion.

PROPOSITION 5.6.4. *If the fermionic projector is state stable and the $\mathcal{M}P$-product is distributional, then $\hat{\mathcal{M}}(p)$ is a Lorentz invariant distribution of even parity (5.6.21).*

This proposition poses a strong constraint for the ultraviolet regularization of the fermionic projector. It is a difficult question whether there are regularizations which satisfy this constraint. But at least we can say that the result of Proposition 5.6.4 does not immediately lead to inconsistencies, as the following lemma shows.

LEMMA 5.6.5. *For all actions considered in §5.5, there is a distribution $\tilde{\mathcal{M}}$ on M which coincides with $\mathcal{M}$ away from the light cone,*

$$\tilde{\mathcal{M}}(y-x) \;=\; \mathcal{M}(x,y) \qquad \text{for all } x, y \in M \text{ with } (y-x)^2 \neq 0\,.$$

Proof. An explicit calculation using (5.6.19) and the representation of $P(x,y)$ with Bessel functions shows that for all actions considered in §5.5, $\mathcal{M}(x,y)$ is for $\xi \equiv y-x$ inside the upper light cone a smooth function with the following properties. It can be written as

$$\mathcal{M}(\xi) \;=\; i\not{\partial}_\xi f(\xi^2) + g(\xi^2) \qquad (\xi \in I^\wedge)$$

with complex-valued functions $f, g \in C^\infty(\mathbb{R}^+)$, which have at most polynomial growth as $\xi^2 \to \infty$ and at most a polynomial singularity on the light cone, i.e.

$$|f(\xi^2)| + |g(\xi^2| \;\leq\; c\left(\xi^{2n} + \frac{1}{\xi^{2n}}\right)$$

for suitable constants $c > 0$, $n \in \mathbb{N}$.

Setting $z = \xi^2$, the Laplacian of a Lorentz invariant function $h(\xi^2)$ is computed to be

$$\Box h(z) \;=\; 4zh''(z) + 8h'(z) = \frac{4}{z}(z^2 h'(z))'\,. \tag{5.6.22}$$

This allows us to invert the Laplacian explicitly,

$$\Box^{-1} h(z) \;=\; \frac{1}{4}\int_{z_0}^{z} \frac{d\tau}{\tau^2} \int_{\tau_0}^{\tau} \sigma\, h(\sigma)\, d\sigma \tag{5.6.23}$$

with two free constants τ_0 and z_0. We choose $z_0 = 1$ and set $\tau_0 = 0$ if $\int_0^1 \sigma |h(\sigma)| d\sigma < \infty$ and $\tau_0 = 1$ otherwise. Then applying $\Box^{-1}$ decreases the order of the pole on the light cone by one. Hence the functions

$$F^\vee(\xi) \;=\; \Box^{-n-1} f(\xi^2)\,, \qquad G^\vee(\xi) \;=\; \Box^{-n-1} g(\xi^2)$$

are smooth functions on $I^\vee$ which are locally bounded near the light cone and have at most polynomial growth as $\xi^2 \to \infty$. Repeating the above construction inside the

lower light cone gives rise to functions $F^\wedge$ and $G^\wedge$ on $I^\wedge$. Extending these functions by zero outside their respective domains of definition, the functions

$$F \;=\; F^\vee + F^\wedge\,, \qquad\qquad G \;=\; G^\vee + G^\wedge$$

are regular distributions on M. By construction, the distribution

$$\tilde{\mathcal{M}} \;=\; i\partial\!\!\!/ \Box^{n+1} F + \Box^{n+1} G$$

coincides inside the light cone with $\mathcal{M}$ and vanishes outside the light cone. ∎

The next lemma states a few important properties of the distribution $\hat{\mathcal{M}}$.

LEMMA 5.6.6. *Under the assumptions of Proposition 5.6.4, the vector component of $\hat{\mathcal{M}}$ is supported inside the mass cone $\{p \mid p^2 \geq 0\}$. If the scalar component of $\mathcal{M}$ has a nonvanishing contribution with the following asymptotics as $I \ni \xi \to 0$,*

$$\mathcal{M}(\xi) \;\sim\; \xi^{2r} \log^s \xi^2 \quad \text{with} \quad r, s \in \mathbb{Z},\; s \geq 1\,, \tag{5.6.24}$$

then its Fourier transform satisfies outside the mass cone for some $c > 0$ the bound

$$|\hat{\mathcal{M}}(p)| \;\geq\; \frac{1}{c}(-p^2)^{-2-r} \quad \text{if } p^2 < -c\,. \tag{5.6.25}$$

Proof. The statement for the vector component immediately follows from a symmetry argument: Due to Lorentz invariance, the vector component of $\hat{\mathcal{M}}(p)$ can be written as $p\!\!\!/ f(p)$ with a scalar distribution f. Since $\hat{\mathcal{M}}$ has even parity (5.6.21), it follows that $f(-p) = -f(p)$. Again using Lorentz invariance, we conclude that $f(p)$ vanishes if p is outside the mass cone.

For the scalar component of $\hat{\mathcal{M}}$, the above symmetry argument does not apply, and (as can easily be verified by an explicit calculation) there is indeed a contribution outside the mass cone. More specifically, in the case $r \geq 0$ we can rescale the Fourier transform by $\xi \to \tau^{-1}\xi$,

$$\int \xi^{2r} \log^s(\xi^2)\, e^{i\tau p\xi}\, d^4\xi \;=\; \tau^{-4-2r} \int \xi^{2r} (\log \xi^2 - \log \tau^2)^s\, e^{ip\xi}\, d^4\xi\,,$$

and obtain that (5.6.24) gives rise to a contribution to $\hat{\mathcal{M}}$ with the following asymptotics as $p^2 \to -\infty$,

$$\hat{\mathcal{M}}(p) \;\sim\; (-p^2)^{-2-r}\, (\log^s(-p^2) + \text{l.o.t.}) \qquad (r \geq 0)\,, \tag{5.6.26}$$

where 'l.o.t.' denotes lower order in $\log(-p^2)$. If $r < 0$, the singularity on the light cone cannot be treated with this scaling argument. But we can nevertheless compute the Fourier transform by iteratively applying the operator $\Box_p^{-1}$,

$$(-\Box_p)^{2r} \int \log^s(\xi^2)\, e^{ip\xi} d^4\xi \;=\; \int \xi^{2r} \log^s(\xi^2)\, e^{ip\xi} d^4\xi\,.$$

Using (5.6.23) and (5.6.26), we obtain a contribution to $\hat{\mathcal{M}}(p)$ with the following asymptotics as $p^2 \to -\infty$,

$$\hat{\mathcal{M}}(p) \;\sim\; (-p^2)^{-2-r}\, (\log^{s+1}(\xi^2) + \text{l.o.t.}) \qquad (r < 0) \tag{5.6.27}$$

(here we do not need to worry about the integration constants in (5.6.23) because these correspond to contributions localized on the light cone, which will be considered separately below, see (5.6.28)). The asymptotic formulas (5.6.26, 5.6.27) explain the estimate (5.6.25). However, the proof is not yet finished because the singular part

of $\mathcal{M}$ which is localized on light cone might give rise to a contribution to $\hat{\mathcal{M}}$ which cancels (5.6.26) or (5.6.27). Thus it remains to show that all contributions to $\mathcal{M}$ which are localized on the light cone have an asymptotics different from (5.6.26, 5.6.27) and thus cannot compensate these contributions.

A scalar Lorentz invariant distribution h which is localized on the light cone satisfies for some $n > 0$ the relation $\xi^{2n} h(\xi) = 0$. Hence its Fourier transform is a distributional solution of the equation

$$\Box^n \hat{h}(p) \;=\; 0\,.$$

In the case $n = 1$, we see from (5.6.22) that away from the mass cone, $\hat{h}(p)$ must be a linear combination of the functions 1 and p^{-2}. By iteratively applying (5.6.23) one finds that for general n, $\hat{h}$ is of the form

$$\frac{1}{p^2}\,(\text{polynomial in } p^2) \;+ \log(-p^2)\,(\text{polynomial in } p^2) \tag{5.6.28}$$

The asymptotics of these terms as $p^2 \to -\infty$ is clearly different from that in (5.6.26, 5.6.27). ∎

We are now ready to prove the main result of this section.

THEOREM 5.6.7. *Consider the action corresponding to a Lagrangian of type (5.5.16). Assume that the fermionic projector of the vacuum is state stable and that the $\mathcal{M}P$-product is distributional (see Def. 5.6.2 and Def. 5.6.3). Then $n = 1$.*

Proof. Since here we consider only one sector, we need to change the normalization of the spectral weight in (5.5.16),

$$\mathcal{L}[A] = |A^{2n}| - \frac{1}{4}|A^n|^2\,.$$

According to Lemma 5.6.1, $\mathcal{L}$ vanishes identically for spacelike $y - x$, whereas for timelike $y - x$ we may replace the spectral weight by an ordinary trace. Hence away from the light cone and up to an irrelevant constant factor $2n$,

$$\mathcal{M} \;=\; \left[A^n - \frac{1}{4}\mathrm{Tr}(A^n)\right] A^{n-1}\,. \tag{5.6.29}$$

According to Proposition 5.6.4, $\hat{\mathcal{M}}$ is a Lorentz invariant distribution of even parity. Following Def. 5.6.2 and Def. 5.6.3, the convolution integral (5.6.20) should be well-defined for any q inside the lower mass cone. According to Lemma 5.6.6, the vector component of $\hat{\mathcal{M}}$ is supported inside the closed mass cone. Thus for the corresponding contribution to (5.6.20), the integration range is indeed compact (see Figure 5.1), and so (5.6.20) is well-defined in the distributional sense.

It remains to consider the scalar component of $\hat{\mathcal{M}}$. This requires a more detailed analysis. We again write the fermionic projector in the form (5.3.4, 5.6.6),

$$P(x,y) \;=\; i\not{\xi} f + h\,.$$

Using the representation (2.5.41) of the distribution $T_{m^2} = \frac{1}{2}(P_{m^2} - K_{m^2})$ in position space, we can write f and g in the upper light cone as

$$\begin{aligned} f &= \frac{c_1}{\xi^4} + \frac{c_2}{\xi^2} + \alpha\,(\log \xi^2 + i\pi) + \beta \\ g &= \frac{c_3}{\xi^2} + \gamma\,(\log \xi^2 + i\pi) + \delta \end{aligned}$$

with constants $c_j \neq 0$ and smooth real function $\alpha, \beta, \gamma, \delta$ with $\gamma_0 := \gamma(0) \neq 0$. A short calculation yields

$$\begin{aligned} A &= \xi^2 |f|^2 + |h|^2 - \not{g}\,\mathrm{Im}(f \overline{h}) \\ &= \gamma_0^2 \log^2 \xi^2 + (\text{lower orders in } \log \xi^2) \;+\; \not{g} \left(\frac{1}{\xi^4} + (\text{higher orders in } \xi^2) \right) . \end{aligned}$$

The important point is that the vector component of A involves no logarithms, whereas the scalar component involves terms $\sim \log^2 \xi^2$. Substituting this expansion of A into (5.6.29), we obtain for the scalar component of $\mathcal{M}$ sums of expressions of the general form $\xi^{2r} \log^s \xi^2$. We select those terms for which s is maximal, and out of these terms for which r is minimal. We get a contribution only when both the square bracket and the factor A^{n-1} in (5.6.29) contain an odd number of Dirac matrices, and a short computation yields the asymptotics

$$\mathcal{M} \;\sim\; (n-1)\, \xi^{-6} \log^{4n-6} \xi^2 . \tag{5.6.30}$$

If $n = 1$, the scalar component of $\hat{\mathcal{M}}$ vanishes, and we get no further conditions. However if $n > 1$, Lemma 5.6.6 yields that $\hat{\mathcal{M}}(p)$ is outside the mass cone for large p bounded away from zero by

$$|\hat{\mathcal{M}}(p)| \;\geq\; -\frac{p^2}{c} .$$

As a consequence, the convolution integral (5.6.20) diverges. ∎

We close this section with a few remarks. First, we can now discuss the stability of the vacuum for the polynomial actions (5.5.1). The strong analysis on the light cone in §5.5 forced us to only consider polynomial Lagrangians which vanish identically if A has two independent eigenvalues. According to Lemma 5.6.1, this implies that Q vanishes identically away from the light cone, and so the above stability analysis becomes trivial. Indeed, the following general consideration shows that for polynomial actions, the vacuum is *not stable* in the strict sense: Under the only assumption that A has vector-scalar structure, the matrix A has at most two independent eigenvalues (see (5.3.12) and Lemma 5.3.1), and so the Lagrangian vanishes identically. Thus the variational principle is trivial; it gives no conditions on the structure of the fermionic projector of the vacuum.

It might seem confusing that in the above analysis, the operator $Q(\xi)$ had poles on the light cone (see (5.6.30, 5.6.18)), although it vanishes identically in the formalism of the continuum limit. This can be understood from the fact that on the light cone, the matrix A has a pair of *complex* conjugated eigenvalues (see (5.3.25)), whereas away from the light cone its eigenvalues are in general distinct *real* eigenvalues (see Lemma 5.6.1). If a matrix has non-real eigenvalues, this property remains valid if the matrix is slightly perturbed. Therefore, treating the eigenvalues of A perturbatively, it is impossible to get from the region on the light cone to the region away from the

light cone. This is the reason why the result of Lemma 5.6.1 cannot be obtained in the formalism of the continuum limit. The transition between the asymptotic regions near and away from the light cone can be studied only by analyzing the EL equations with regularization in detail, and this goes beyond the scope of this book. However, we point out that the order of the pole in (5.6.30) is lower than the order of all singularities which we will study in the continuum limit. This justifies that we can neglect (5.6.30) in what follows; taking into account (5.6.30) would have no effect on any of the results in this book.

The above argument shows that from all Lagrangians mentioned in §5.5, only for (5.5.14) the vacuum can be stable (in the sense made precise in Theorem 5.6.7). The important question arises whether the above necessary conditions are also sufficient, i.e. if for the Lagrangian (5.5.14) the vacuum is indeed state stable. This question cannot be answered here; in any case a more detailed analysis would yield additional constraints for the mass parameters m_α and the regularization. But at least we can say that for the Lagrangian (5.5.14), the vacuum has very nice properties which point towards stability: First of all, Q vanishes in the continuum limit. But it does not vanish identically away from the light cone, and thus the EL equations are non-trivial in the vacuum. Furthermore, $Q(\xi)$ is *supported in the light cone*, giving an interesting (although not yet fully understood) link to causality. The crucial point for Theorem 5.6.7 is that the *scalar component* of $\mathcal{M}$ *vanishes* (as is obvious from (5.6.29)), and therefore $\hat{\mathcal{M}}$ is *supported inside the mass cone* (see Lemma 5.6.6). The last property implies via a simple support argument that, for q inside the lower mass cone, the convolution integral (5.6.20) is finite (see Figure 5.1). This means that the stability conditions (ii) and (iii) of Def. 5.6.2 (which we did not consider here) could be analyzed without any assumption on the regularization. However, for q outside the lower mass cone, the convolution integral (5.6.20) will in general diverge (see again Figure 5.1), indicating that the vacuum could be stable even under perturbations where fermionic states with momenta outside the lower mass cone are occupied.

CHAPTER 6

The Dynamical Gauge Group

We now begin the analysis of the continuum limit of the EL equations with interaction. In order to work in a concrete example, we shall analyze our model variational principle of §3.5. But the methods as well as many of the results carry over to other variational principles, as will be discussed in the Remarks at the end of Chapter 6 and at the end of Chapter 7. We again consider the fermionic projector of the standard model §5.1. For the bosonic potentials in the corresponding auxiliary Dirac equation (2.3.10) we make the ansatz

$$\mathcal{B} \;=\; \not{C} + \gamma^5 \not{E} + \Phi + i\gamma^5\, \Xi \tag{6.0.1}$$

with a vector potential C, an axial potential E and scalar/pseudoscalar potentials ϕ and Ξ, which again in component notation $\mathcal{B} = \mathcal{B}^{(a\alpha)}_{(b\beta)}$ we assume to be of the form

$$C \;=\; C^a_b\, \delta^\alpha_\beta\,, \qquad E \;=\; E^a_b\, \delta^\alpha_\beta\,, \qquad \phi \;=\; \phi^{(a\alpha)}_{(b\beta)}\,, \qquad \Xi \;=\; \Xi^{(a\alpha)}_{(b\beta)}\,. \tag{6.0.2}$$

Exactly as in §2.5 it is convenient to introduce the chiral potentials

$$A_{L/R} \;=\; C \pm E\,, \tag{6.0.3}$$

and to define the *dynamical mass matrices* by

$$mY_{L/R} \;=\; mY - \phi \mp i\Xi\,. \tag{6.0.4}$$

Then the auxiliary Dirac equation takes the form

$$(i\partial\!\!\!/ + \chi_L(\not{A}_R - mY_R) + \chi_R(\not{A}_L - mY_L))\, P(x,y) \;=\; 0\,. \tag{6.0.5}$$

Clearly, the potentials in (6.0.1) must be causality compatible (2.3.18). We assume in what follows that this condition is satisfied, and we will specify what it means in the course of our analysis.

Let us briefly discuss the ansatz (6.0.1). The vector and axial potentials in (6.0.1) have a similar form as the gauge potentials in the standard model. Indeed, when combined with the chiral potentials (6.0.3), they can be regarded as the gauge potentials corresponding to the gauge group $U(8)_L \times U(8)_R$. This so-called *chiral gauge group* includes the gauge group of the standard model. At every space-time point, it has a natural representation as a pair of 8×8 matrices acting on the sectors; we will work in this representation throughout. Compared to the most general ansatz for the chiral potentials, the only restriction in (6.0.3, 6.0.2) is that the chiral potentials are the same for the three generations. This can be justified from the behavior of the fermionic projector under generalized gauge transformations, as will be explained in Remark 6.2.3 below. The scalar potentials in (6.0.1) do not appear in the standard model, but as we shall see, they will play an important role in our description of the interaction (here and in what follows, we omit the word "pseudo" and by a "scalar potential" mean a scalar or a pseudoscalar potential). We point out that we do not consider a gravitational field. The reason is that here we want to restrict attention

to the interactions of the standard model. But since the principle of the fermionic projector respects the equivalence principle, one could clearly include a gravitational field; we plan to do so in the future. Compared to a general multiplication operator, (6.0.1) does not contain bilinear potentials (i.e. potentials of the form $H_{jk}\sigma^{jk}$ with $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$). Clearly, bilinear potentials do not appear in the standard model, but it is not obvious why they should be irrelevant in our description. Nevertheless, we omit bilinear potentials in order to keep the analysis as simple as possible. To summarize, (6.0.1) is certainly not the most general ansatz which is worth studying. But since the potentials in (6.0.1) are considerably more general than the gauge potentials in the standard model, it seems reasonable to take (6.0.1) as the starting point for our analysis.

6.1. The Euler-Lagrange Equations to Highest Degree on the Light Cone

We come to the detailed calculations. We again work with the spectral decomposition of A_{xy} and proceed degree by degree on the light cone. In this section we consider the highest degree. Then the fermionic projector is influenced only by the chiral potentials (and not by the scalar potentials or the particle states), and the chiral potentials merely describe local phase transformations of the fermionic projector. More precisely, truncating all contributions of degree < 2 and denoting this "truncated fermionic projector" by $P_0(x,y)$, we have (see §2.5 and Appendix B)[1]

$$P_0(x,y) \;=\; \left(\chi_L\, X_L \int_x^y{}^{\mathrm{L}} + \chi_R\, X_R \int_x^y{}^{\mathrm{R}}\right) \frac{i}{2}\, \mathcal{T}_{[0]}^{(-1)}(x,y)\,, \tag{6.1.1}$$

where we used for the ordered exponentials the short notation

$$\int_x^y{}^{\mathrm{c}} \;=\; \mathrm{Pexp}\left(-i \int_0^1 A_c^j(\tau y + (1-\tau)x)\, (y-x)_j\, d\tau\right) \tag{6.1.2}$$

with $c = L$ or R. We also truncate the matrix A_{xy} by setting

$$A_0(x,y) \;=\; P_0(x,y)\, P_0(y,x)\,.$$

It follows from (6.1.1) that

$$A_0 \;=\; \left\{\chi_L\, X_L \int_x^y{}^{\mathrm{L}} \int_y^x{}^{\mathrm{R}}\, X_R \;+\; \chi_R\, X_R \int_x^y{}^{\mathrm{R}} \int_y^x{}^{\mathrm{L}}\, X_L\right\} \frac{1}{4}\, (\mathcal{T}_{[0]}^{(-1)})(\overline{\mathcal{T}_{[0]}^{(-1)}})\,. \tag{6.1.3}$$

We can assume that the matrix inside the curly brackets is diagonalizable; indeed, this is the generic situation, and the general case immediately follows from it by approximation. The matrix A_0 is invariant on the left- and right-handed spinors. If considered on one of these invariant subspaces, the curly brackets depend only on the sector indices $a, b = 1, \ldots, 8$, whereas the factors to their right involve only Dirac matrices. This allows us to factor the spectral decomposition of A_0 as follows. We first diagonalize the phase factor, i.e.

$$W_c \;\equiv\; X_c \int_x^y{}^{\mathrm{c}} \int_y^x{}^{\bar{c}}\, X_{\bar{c}} \;=\; \sum_{n=1}^{8} \nu_{nc}\, I_{nc} \tag{6.1.4}$$

[1] Online version: Here the factor g^2 obtained by summing over the generations when forming the partial trace (see (2.3.4)) is absorbed into the definition of the factors $T_{[0]}^{(-1)}$. This differs from the convention used in the book [**5**] (listed in the references in the preface to the second online edition), where for clarity the factors g which count the number of generations are always written out.

with eigenvalues ν_{nc} (counting multiplicities) and corresponding spectral projectors I_{nc}, where $\overline{c}$ is defined by $\overline{L} = R$ and $\overline{R} = L$. The matrices W_L and W_R are obtained from each other by taking their adjoint. Thus we can arrange that the same holds for their spectral decompositions,

$$\overline{\nu_{nc}} \;=\; \nu_{n\overline{c}}\,, \qquad\qquad I_{nc}^* \;=\; I_{n\overline{c}}\,. \tag{6.1.5}$$

The spectral representation of the second term in (6.1.3) is computed exactly as described in §5.3. More precisely, it is obtained from (5.3.8, 5.3.10) by setting h to zero, i.e. similar to (5.3.20, 5.3.21),

$$\frac{1}{4}\,(\not{\xi}\, T^{(-1)}_{[0]})\overline{(\not{\xi}\, T^{(-1)}_{[0]})} \;=\; \sum_{s=\pm} \lambda_s\, F_s \tag{6.1.6}$$

with

$$\lambda_s \;=\; \frac{1}{4}\, T^{(-1)}_{[0]}\, \overline{T^{(-1)}_{[0]}} \times \begin{cases} z & \text{if } s=+ \\ \overline{z} & \text{if } s=- \end{cases} \tag{6.1.7}$$

$$F_s \;=\; \frac{1}{z-\overline{z}} \times \begin{cases} \not{\xi}\not{\overline{\xi}} - \overline{z} & \text{if } s=+ \\ -\not{\xi}\not{\overline{\xi}} + z & \text{if } s=-\,. \end{cases} \tag{6.1.8}$$

Combining (6.1.4) and (6.1.6) gives

$$A_0 \;=\; \sum_{n=1}^{8} \sum_{c=L,R} \sum_{s=\pm} \lambda_{ncs}\, F_{ncs} \tag{6.1.9}$$

with

$$\lambda_{ncs} \;=\; \nu_{nc}\, \lambda_s\,, \qquad\qquad F_{ncs} \;=\; \chi_c\, I_{nc}\, F_s\,. \tag{6.1.10}$$

It might be surprising at first sight that, although A_0 clearly is a gauge-invariant expression, the phase shifts described by the ordered exponentials in (6.1.1) do not drop out in (6.1.3). Let us explain in detail how this comes about. We first recall that under gauge transformations, the truncated fermionic projector transforms like

$$P_0(x,y) \;\longrightarrow\; U(x)\, P_0(x,y)\, U(y)^{-1}\,, \tag{6.1.11}$$

where U is unitary with respect to the spin scalar product, $U(x)^* = U(x)^{-1}$. These $U(2N, 2N)$ gauge transformations correspond to a local symmetry of the system, which is related to the freedom in choosing a local basis for the spinors (see §3.1). When forming the closed chain, the gauge transformations at y drop out,

$$P_0(x,y)\, P_0(y,x) \;\longrightarrow\; U(x)\, P_0(x,y)\, P_0(y,x)\, U(x)^{-1}\,.$$

In order to see the relation between the phase transformations in (6.1.1) and the above gauge transformations, it is useful to consider the situation when the chiral potentials have the form of pure gauge potentials, i.e.

$$A_c^j \;=\; iV_c\, (\partial^j V_c^{-1}) \tag{6.1.12}$$

with unitary operators $V_L, V_R \in U(8)$. Then the ordered exponential (6.1.2) reduces to a product of unitary transformations at the two end points,

$$\int_x^y \!\!\mathrm{c} \;=\; V_c(x)\, V_c(y)^{-1}\,.$$

Using that the potentials are causality compatible, (6.1.1) becomes

$$P_0(x,y) \;=\; \sum_{c=L,R} \chi_c\, V_c(x)\, X_c \left(\frac{i}{2} \,\not{g}\, T^{(-1)}_{[0]}(x,y) \right) V_c(y)^{-1} \,. \tag{6.1.13}$$

Hence the left- and right-handed components of P_0 are transformed independently by V_L and V_R, respectively. In order to write these transformations in a form similar to (6.1.11), we combine V_L and V_R into one operator V,

$$V \;=\; \chi_L\, V_L + \chi_R\, V_R \,.$$

The effect of the chiral potentials in (6.1.13) is then described by the transformation

$$P_0(x,y) \;\longrightarrow\; V(x)\, P_0(x,y)\, V(y)^* \,,$$

and thus the closed chain transforms according to

$$P_0(x,y)\, P_0(y,x) \;\longrightarrow\; V(x)\, P_0(x,y)\, V(y)^*\, V(y)\, P_0(y,x)\, V(x)^* \,. \tag{6.1.14}$$

The point is that the transformation V is in general *not* unitary, because

$$V^* \;=\; \chi_R\, V_L^{-1} + \chi_L V_R^{-1} \quad \overset{\text{in general}}{\neq} \quad \chi_L\, V_L^{-1} + \chi_R V_R^{-1} \;=\; V^{-1} \,.$$

More precisely, V is unitary if and only if $V_L = V_R$ at every space-time point. According to (6.1.12), this implies the condition $A_L \equiv A_R$. From (6.0.3) we conclude that V is unitary if and only if the axial potentials E in (6.0.1) are identically equal to zero. This means that only the subgroup $U(8) \subset U(8)_L \times U(R)_R$ of the chiral gauge group, which gives rise to the vector potential C in (6.0.1), describes local unitary transformations of the fermionic projector and thus corresponds to a local gauge symmetry in the sense of §3.1. We refer to this subgroup of the chiral gauge group as the *free gauge group* $\mathcal{F}$; it can be identified with a subgroup of the gauge group, $\mathcal{F} \subset U(2N, 2N)$ (we remark for clarity that the other degrees of freedom of the gauge group $U(2N, 2N)$ are related to the gravitational field §1.5 and are thus not considered here). The axial potentials, however, describe local transformations which are not unitary and thus cannot be identified with gauge transformations in the sense of §3.1. These non-unitary transformations do not correspond to an underlying local symmetry of the system. The interpretation of these results is that the *chiral gauge group is spontaneously broken*, and only its subgroup $\mathcal{F}$ corresponds to an unbroken local symmetry of the system.

A simple way to understand why the chiral gauge group is spontaneously broken is that axial potentials describe relative phase shifts between the left- and right-handed components of the fermionic projector. Such relative phases do not drop out when we form composite expressions, as one sees in (6.1.14) or, more generally, in (6.1.3). By imposing that the relative phases be zero in all composite expressions, we can distinguish those systems in which the axial potentials vanish identically. In this way, one can fix the gauge up to global chiral gauge transformations (i.e. transformations of the form (6.1.13) with constant matrices V_c) and up to local free gauge transformations. Since this gauge fixing argument makes use of the phases which appear in $P_0(x,y)$, one may regard the chiral gauge symmetry as being spontaneously broken by the fermionic projector.

The spontaneous breaking of the chiral gauge symmetry by the fermionic projector has, at least on the qualitative level, some similarity to the Higgs mechanism in the standard model. We recall that in the Higgs mechanism one arranges by a suitable quartic potential in the classical Lagrangian that the Higgs field Φ has a non-trivial ground state, i.e. $\Phi \neq 0$ in the vacuum. The Higgs field is acted upon by a local gauge

group. Since $\Phi \neq 0$, one can, by prescribing the phase of Φ, fix the gauge globally. This shows that the local gauge symmetry is spontaneously broken by the Higgs field, a fact which can then be used to give the gauge bosons mass. In our setting, we also have in the vacuum a non-trivial object, namely the fermionic projector, which is composed of the Dirac seas corresponding to the leptons and quarks. Thus our situation is indeed quite similar to the Higgs mechanism, if one only keeps in mind that the role of the Higgs field in our description is played by the fermionic projector of the vacuum. Clearly, this analogy does not carry over to the mathematical details. But also in our description, the spontaneous symmetry breaking makes it possible that undifferentiated gauge potentials enter the EL-equations, giving the hope that the corresponding gauge bosons might be massive.

Since the chiral gauge symmetry is spontaneously broken, we cannot expect that the EL equations admit chiral potentials corresponding to the whole group $U(8)_L \times U(8)_R$. In order to quantify which restrictions for the chiral potentials we get, we must work in a more general setting and introduce a suitable mathematical notation. Contributions to the fermionic projector which involve the phases of the chiral potentials, but not the gauge fields, currents, or scalar potentials, are called *gauge terms*. Likewise, we refer to the contributions of the gauge terms to a composite expression in the fermionic projector as the gauge terms in the respective expression. The simplest examples for gauge terms are (6.1.1) or (6.1.3), but we shall encounter gauge terms to lower degree on the light cone as well.

DEF. 6.1.1. *A subgroup $\mathcal{G}$ of the chiral gauge group is called a* **dynamical gauge group** *if the gauge terms of the potentials corresponding to $\mathcal{G}$ vanish in the EL equations. The subgroup $\mathcal{G} \cap \mathcal{F}$ is the* **free dynamical gauge group**.

Clearly, this definition does not give a unique dynamical gauge group. In particular, every subgroup of a dynamical gauge group is again a dynamical gauge group. Since we want to choose the dynamical gauge group as large as possible, we will always restrict attention to dynamical gauge groups which are *maximal* in the sense that they are not contained in a larger dynamical gauge group.

We first analyze the gauge term (6.1.1) in the EL equations corresponding to our variational principle (5.5.14). We only consider the highest degree on the light cone, which is degree 5. This gives the following result.

THEOREM 6.1.2. *The eigenvalues ν_{nc} of W_c must satisfy the conditions*

$$\nu_{8c} = 0 \quad \text{and} \quad |\nu_{nc}| = |\nu_{n'c'}| \text{ for } n, n' = 1, \ldots, 7 \text{ and } c, c' = L, R. \tag{6.1.15}$$

The dynamical gauge group $\mathcal{G}$ is restricted by

$$\mathcal{G} \subset (U(7) \times U(1))_L \times (U(7) \times U(1))_R \,, \tag{6.1.16}$$

where the $U(7)$ are unitary matrices acting on the seven massive sectors, and the $U(1)$ act on the neutrino sector.

If conversely the conditions (6.1.15) or (6.1.16) are satisfied, then the EL equations are satisfied to degree 5 on the light cone.

It is easy to see that the conditions in the above theorem are sufficient for the EL equations to be satisfied, and this consideration also gives an idea of how these conditions come about. Namely, suppose that (6.1.16) holds. Then the dynamical gauge potentials are invariant on the massive sectors as well as on the neutrino sector. Using a block matrix notation where the first component refers to the massive sectors

and the second component to the neutrino sector, we see from (6.1.4) that the matrices W_L and W_R have the form

$$W_L \;=\; \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}, \qquad W_R \;=\; \begin{pmatrix} U^* & 0 \\ 0 & 0 \end{pmatrix} \tag{6.1.17}$$

with U a unitary $7{\times}7$ matrix. Hence their eigenvalues ν_{nc} satisfy the conditions (6.1.15). Using (5.4.7), the gradient of the Lagrangian is computed to be

$$\mathcal{M}[A] \;=\; 2\sum_{k=1}^{K} \left\{ \left(|\lambda_k| - \frac{1}{28}\,|A| \right) \frac{\overline{\lambda_k}}{|\lambda_k|} \right\} F_k\,. \tag{6.1.18}$$

We saw in §5.5 that the curly brackets vanish in the vacuum. If (6.1.15) is satisfied, the gauge terms change the eigenvalues λ_{ncs} only by a phase (6.1.10). Since these phases drop out when absolute values are taken, the curly brackets in (6.1.18) are zero even with interaction (to the highest degree on the light cone). According to (5.2.7), this implies that Q vanishes, and so the EL equations are satisfied.

It is more difficult to show that the conditions (6.1.15) and (6.1.16) are also necessary. We give the proof in detail.

Proof of Theorem 6.1.2. Using the argument given after the statement of the theorem, it remains to show that the conditions (6.1.15) and (6.1.16) are necessary. Substituting the spectral decomposition of A_0 into (5.5.4), we obtain, in analogy to (5.4.8), the following representation for Q,

$$Q(x,y) \;=\; \frac{1}{2} \sum_{n,c,s} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda^{xy}_{ncs}}\, F^{xy}_{ncs}\, P_0(x,y) \;+\; (\deg < 5).$$

Computing the Euler-Lagrange equations similar to (5.4.10) and keeping track of the chiral cancellations, we obtain in analogy to (5.4.12) the equation

$$\sum_{n,c,s} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda^{xy}_{ncs}}\, F^{xy}_{ncs}\, P_0(x,y) \Xint-_y^z \bar{c}\; X_{\bar{c}} \;+\; (\deg < 5) \;=\; 0\,,$$

and by multiplying from the right by the macroscopic unitary matrix $\Xint-_z^y \bar{c}$, we can arrange that $z = y$. We substitute (6.1.1) as well as the right of (6.1.10) and apply (5.3.23) to obtain

$$\sum_{nc} \frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{nc-}}\, \chi_c\, I_{nc} \left\{ X_c \Xint-_x^y c \Xint-_y^x \bar{c}\; X_{\bar{c}} \right\} \left(\frac{i}{2}\, \not{\xi}\, T^{(-1)}_{[0]} \right) \;+\; (\deg < 5) \;=\; 0\,.$$

The curly brackets coincide with the matrix W_c, and since I_{nc} is a spectral projector of this matrix, we simply get a scalar factor ν_{nc}. Furthermore, we use the particular form of our Lagrangian (5.5.14) as well as the first equation in (6.1.10). This gives

$$i \sum_{n,c} \left(|\nu_{nc}| - \frac{1}{14} \sum_{n',c'} |\nu_{n'c'}| \right) \chi_c\, \not{\xi}\, I_{nc}\, \overline{\lambda_-}\, T^{(-1)}_{[0]} \;=\; 0\,. \tag{6.1.19}$$

Using (5.3.20), the non-smooth factors are a monomial of degree five,

$$\overline{\lambda_-}\, T^{(-1)}_{[0]} \;=\; T^{(0)}_{[0]}\, T^{(-1)}_{[0]}\, \overline{T^{(-1)}_{[0]}}\,. \tag{6.1.20}$$

We cannot assume that this monomial is equal to zero. Namely, the fermionic projector differs to highest degree on the light cone from the fermionic projector of the vacuum

only by macroscopic phase factors (this is guaranteed by the gauge invariance of the regularized causal perturbation expansion, see Appendix D). Therefore, exactly as explained for the vacuum in §5.4, we can evaluate (6.1.19) even strongly. In particular, we can consider the regularization expansion of (6.1.19) (see §4.3, §4.4). This means that, in order to set the monomial (6.1.20) in (6.1.19) equal to zero, we would have to impose an infinite number of regularization conditions. We conclude that in order to satisfy (6.1.19), we must assume that the macroscopic prefactor vanishes. Using that the spectral projectors I_{nc} are linearly independent, we get the conditions

$$\left(|\nu_{nc}| - \frac{1}{14} \sum_{n',c'} |\nu_{n'c'}| \right) |\nu_{nc}|^2 \;=\; 0 \qquad \text{for all } n, c.$$

This implies that the absolute values of 14 of the eigenvalues ν_{nc} must coincide, and that the remaining two eigenvalues must be zero. The matrix W_L contains a factor X_R and is thus singular of rank one. Choosing the numbering such that $\nu_{L8} = 0$, it follows from (6.1.5) that also $\nu_{R8} = 0$. Hence the two zero eigenvalues are those for $n = 8$. This shows that (6.1.15) is a necessary condition.

Next we will show that (6.1.15) implies the constraint for the dynamical gauge group (6.1.16). We introduce for fixed x and y the abbreviations

$$U \;= \int_x^y \int_y^x \qquad \text{and} \qquad T \;=\; X_R \,. \tag{6.1.21}$$

We consider $U = (U^a_b)$ and $T = (T^a_b)$ as matrices on $\mathbb{C}^8$ endowed with the standard Euclidean scalar product $\langle .,. \rangle$. Then U is unitary, and T is a projector of rank 7. According to (6.1.5), the conditions (6.1.15) tell us that the matrix UT must have 7 eigenvalues on the unit circle and one zero eigenvalue. For a vector u in the kernel of UT,

$$0 \;=\; \langle UTu,\, UTu \rangle \;=\; \langle Tu,\, Tu \rangle \,,$$

where we used in the last step that U is unitary. Thus u is also in the kernel of T. On the other hand, if u is an eigenvector of UT corresponding to an eigenvalue on the unit circle,

$$|u|^2 \;=\; \langle UTu,\, UTu \rangle \;=\; \langle Tu,\, Tu \rangle \,.$$

Since for a projector, $|Tu| < |u|$ unless u is in the image of T, it follows that u is also an eigenvector of T, of eigenvalue one. We conclude that every eigenvalue of UT is also an eigenvalue of T, or equivalently that

$$[UT,\, T] \;=\; 0 \,. \tag{6.1.22}$$

Let us analyze what this commutator condition means for the chiral potentials. We already know from the causality compatibility condition that

$$[A_R, T] \;=\; 0 \,. \tag{6.1.23}$$

Hence substituting the definition of U, (6.1.21), into (6.1.22) and using that the resulting condition must hold for all x and y, we obtain that $[A_L T, T] = 0$. Subtracting the adjoint and using that T is idempotent gives the stronger statement

$$[A_L, T] \;=\; 0 \,. \tag{6.1.24}$$

From (6.1.23) and (6.1.24) we conclude that the chiral potentials must be block diagonal in the sense that $(A_c)^a_b = 0$ if $a = 8$ and $b \neq 8$ or vice versa. Such chiral potentials correspond precisely to the gauge group in (6.1.16). ∎

REMARK 6.1.3. We point out that the subgroup $U(1)_L \times U(1)_R$ of the gauge group in (6.1.16), which acts on the neutrino sector, is not uniquely determined and could be replaced by any other subgroup which contains $U(1)_L$. This can immediately be understood from the fact that the neutrino sector contains only left-handed particles, and thus the form of the potential A_R, which acts on the right-handed component, has no significance. To make this argument rigorous, we consider the Dirac equation (6.0.5). Since $P = XP$, we may replace the chiral potentials A_c in (6.0.5) by $A_c X_c$, and this indeed makes the component of A_R in the neutrino sector equal to zero, showing that this component is of no relevance. We conclude that we may arbitrarily change the subgroup $U(1)_R$ in (6.1.16); e.g. we could replace (6.1.16) by

$$(U(7) \times U(1))_L \times U(7)_R \qquad \text{or} \qquad (U(7)_L \times U(7)_R) \times U(1)\,. \tag{6.1.25}$$

We do not write out this obvious arbitrariness in what follows; instead we will simply give the gauge groups in the most convenient form.

6.2. The Gauge Terms in the Euler-Lagrange Equations

We come to the analysis of the EL equations to the next lower degree 4 on the light cone. According to the formulas of the light-cone expansion in Appendix B, the structure of the fermionic projector to the next lower degree is considerably more complicated than (6.1.1), because in addition to gauge terms, there are also contributions involving the chiral fields and currents as well as the scalar potentials. Fortunately, the following general argument allows us to distinguish these different types of contributions in the EL equations. In the formulas for $P(x,y)$, the gauge terms always involve ordered exponentials of the chiral potentials, integrated along the line segment $\overline{xy} = \{\alpha x + (1-\alpha)y,\, 0 \leq \alpha \leq 1\}$. We refer to such contributions as *line contributions*. The fields, currents and scalar potentials, however, are in the light-cone expansion evaluated at individual points, namely either at the end points x, y or at an intermediate point $z \in \overline{xy}$; we call the corresponding contributions to the light-cone expansion *point contributions*. In the case of an evaluation at an intermediate point z, the point contribution clearly involves an integral over z along $\overline{xy}$. But in contrast to the line contribution, where the chiral potentials at different points enter the ordered exponential in a nonlinear way, the line integrals in a point contribution simply takes averages of the potentials, fields, or currents along the line segment. For example by expanding the ordered exponential in a Dyson series (see Def. 2.5.4) and considering the higher order terms, one sees immediately that the line and point contributions are independent in the EL equations in the sense that the EL equations must be satisfied separately by the line and point contributions. Moreover, we can distinguish point contributions in the EL equations, provided that their configuration of the tensor indices is different. Therefore, the point contributions involving the scalar potentials, the chiral fields and the currents are independent in the EL equations as well.

Using the above arguments, we can study the gauge terms and the contributions involving the scalar potentials, the gauge fields and the currents separately. In the remainder of this section, we consider only the gauge terms. Thus we restrict attention to chiral potentials, i.e. instead of (6.0.5) we consider the Dirac equation

$$(i\partial\!\!\!/ - mY + \chi_L A\!\!\!/_R + \chi_R A\!\!\!/_L)\, P(x,y) \;=\; 0$$

with Y a fixed matrix. We will return to the general Dirac equation (6.0.5) in Chapter 7.

DEF. 6.2.1. *We introduce for $p \in \{4,\ldots,7\}$ the groups $B_p, F_p \subset U(8)$ by*

$$B_p \;=\; \{\underbrace{g \oplus \cdots \oplus g}_{p \text{ summands}} \oplus \underbrace{g^{-1} \oplus \cdots \oplus g^{-1}}_{7-p \text{ summands}} \text{ with } g \in U(1)\} \tag{6.2.1}$$

$$F_p \;=\; U(p) \times U(7-p) \times U(1) \tag{6.2.2}$$

and define corresponding subgroups $\mathcal{B}_p$ and $\mathcal{F}_p$ of the dynamical gauge group by

$$\mathcal{B}_p \;=\; B_p \times \mathbb{1} \;\subset\; U(8)_L \times U(8)_R\,, \qquad \mathcal{F}_p \;=\; \{(g,g) \text{ with } g \in F_p\} \;\subset\; \mathcal{F}\,. \tag{6.2.3}$$

Their product

$$\mathcal{G}_p \;=\; \mathcal{B}_p \cdot \mathcal{F}_p \;\equiv\; \{bf \text{ with } b \in \mathcal{B}_p,\, f \in \mathcal{F}_p\} \;\subset\; U(8)_L \times U(8)_R \tag{6.2.4}$$

is called the $\boldsymbol{p^{\text{th}}}$ **dynamical gauge group**.

In block matrix notation, the elements of B_p and F_p can be written as

$$\begin{pmatrix} z\,\mathbb{1}_p & 0 & 0 \\ 0 & z^{-1}\,\mathbb{1}_q & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} g & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & l \end{pmatrix}, \tag{6.2.5}$$

respectively, where the first component refers to the first p sectors, the second component to the next $q \equiv 7-p$ sectors and the last component to the neutrino sector. Here $z, l \in U(1)$, $g \in U(p)$ and $h \in U(q)$. Clearly, B_p and $\mathcal{B}_p$ are group isomorphic to $U(1)$. Notice that $\mathcal{B}_p$ acts only the left-handed component. The group $\mathcal{F}_p$ transforms the left- and right-handed components in the same way, and so its corresponding gauge potentials are vector potentials. The groups $\mathcal{B}_p$ and $\mathcal{F}_p$ commute, and this ensures that their product (6.2.4) is again a group. It is easy to verify that $\mathcal{F}_p$ is indeed the largest subgroup of $\mathcal{F}$ which commutes with $\mathcal{B}_p$,

$$\mathcal{F}_p \;=\; \left\{f \in \mathcal{F} \mid bfb^{-1} = f \text{ for all } b \in \mathcal{B}_p\right\}\,.$$

We introduce an abbreviation for the linear combination of monomials,

$$M \;\equiv\; T^{(0)}_{[1]}\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}} \;-\; T^{(-1)}_{[0]}\, T^{(1)}_{[2]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}}\,. \tag{6.2.6}$$

THEOREM 6.2.2. *There are precisely the following possibilities for the choice of the dynamical gauge group.*

(1) *Without assuming any relations between the basic fractions, the dynamical gauge group must be contained in the free gauge group,*

$$\mathcal{G} \;\subset\; \mathcal{F}_0 \;=\; U(7) \times U(1)\,. \tag{6.2.7}$$

(2) *If we allow for one relation between the basic fractions, the dynamical gauge group is (possibly after a global gauge transformation) restricted by*

$$\mathcal{G} \;\subset\; \mathcal{G}_p \qquad \text{for some } p \in \{4,\ldots,7\}. \tag{6.2.8}$$

In this case, the relation between the basic fractions is[2]

$$(M - \overline{M})\, \overline{T^{(0)}_{[0]}}^{-1} \;=\; 0 \tag{6.2.9}$$

with M according to (6.2.6).

[2] Online version: As shown in Lemma I in the preface to the second online edition, there is no regularization which realizes this relation.

(3) *If we allow for two relations between the basic fractions, we get no constraints for the dynamical gauge group besides those of Theorem 6.1.2. The two relations between the basic fractions are (6.2.9) and*

$$M\, \overline{T^{(0)}_{[0]}}^{-1} \;=\; 0\,. \tag{6.2.10}$$

In each of these cases, the gauge terms vanish in the EL equations to degree 4 on the light cone.

Proof. We first bring the EL equations to degree 4 into a more explicit form. Theorem 6.1.2 implies that the variation of our Lagrangian vanishes to highest degree on the light cone,

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_{ncs}} \;+\; (\deg < 3) \;=\; 0\,. \tag{6.2.11}$$

According to (5.4.7, 5.2.7), all contributions to the EL equations vanish for which the variation of the Lagrangian is considered to highest degree (even if the spectral projectors or the factors $P(x,y)$ are expanded to lower degree). This means that we only need to compute the Lagrangian to the next lower degree, whereas it suffices to take into account both the spectral projectors and the factors $P(x,y)$ to highest degree.

Since the Lagrangian is a function of the eigenvalues only, our task is to calculate the contribution to the eigenvalues to the next lower degree two, denote by $\Delta\lambda_{ncs}$. This calculation is carried out in a more general context in Appendix G (see Theorems G.4.1 and G.5.1). Specializing the obtained results gives

$$\Delta\lambda_{8cs} \;=\; 0\,,$$

whereas for $n = 1, \ldots, 7$,

$$\Delta\lambda^{xy}_{nc-} \;=\; T^{(0)}_{[1]}\, \overline{T^{(0)}_{[1]}} - T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \;+\; \nu_{nc} \left(T^{(0)}_{[2]}\, \overline{T^{(0)}_{[0]}} + T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \right) \tag{6.2.12}$$

$$+ \frac{T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}}}{\overline{\lambda^{xy}_{nc-}} - \lambda^{xy}_{nc-}} \left(\nu_{nc}\, T^{(0)}_{[1]}\, \overline{T^{(0)}_{[0]}} \;-\; \overline{\nu_{nc}}\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}} \right) \tag{6.2.13}$$

$$\Delta\lambda^{yx}_{nc+} \;=\; \overline{\Delta\lambda^{xy}_{n\bar{c}-}} \tag{6.2.14}$$

(here λ^{xy}_{nc-} denotes the eigenvalues of A_0, (6.1.10)). The EL equations take again the form (5.4.12). Substituting the asymptotic formula to highest degree (5.3.23) and expanding our Lagrangian (5.5.14) shows that the EL equations become to degree 4,

$$\sum_{nc} \Delta \left(|\lambda_{nc-}| - \frac{1}{28} \sum_{n',c',s'} |\lambda_{n'c's'}| \right) \frac{\overline{\lambda_-}}{|\lambda_-|}\, |\nu_{nc}|^2 \chi_c\, I_{nc}\, (i\not{g}\, T^{(-1)}_{[0]}) \;=\; 0\,.$$

Since the eigenvalues appear in complex conjugate pairs, we may replace the sum over s' by a factor two and set $s' = -$. Also, the non-vanishing macroscopic factor $\not{g}$ can be omitted. Furthermore, we use that the spectral projectors I_{nc} are macroscopic and linearly independent, and that $|\nu_{nc}|^2$ vanishes for $n = 8$ and is equal to one otherwise, (6.1.15). We thus obtain that the EL equations to degree 4 are equivalent to the

conditions that for all $n = 1, \ldots, 7$,

$$\left[\mathrm{Re}\left(\overline{\lambda_{nc-}}\, \Delta\lambda_{nc-}\right) - \frac{1}{14} \sum_{n',c'} \mathrm{Re}\left(\overline{\lambda_{n'c'-}}\, \Delta\lambda_{n'c'-}\right) \right] \frac{1}{\overline{T^{(0)}_{[0]}}} \;=\; 0\,.$$

It is more convenient to write this condition in the form that the expression

$$H_{nc} \;\equiv\; 2\,\mathrm{Re}\left(\overline{\lambda_{nc-}}\, \Delta\lambda_{nc-}\right) \overline{T^{(0)}_{[0]}}^{-1}$$

should be independent of n and c,

$$H_{nc} \;=\; H_{n'c'} \qquad \text{for all } n, n' = 1, \ldots, 7 \text{ and } c, c' = L, R. \tag{6.2.15}$$

Next we compute H_{nc} by substituting the formulas for λ_{nc-} and $\Delta\lambda_{nc-}$, (5.3.20) and (6.2.12, 6.2.13). Since the last summand of $\Delta\lambda_{nc-}$, (6.2.13) is imaginary, we can use that for $\alpha \in i\mathbb{R}$,

$$2\,\mathrm{Re}\left(\overline{\lambda_{nc-}}\, \alpha\right) \;=\; \alpha\, (\overline{\lambda_{nc-}} - \lambda_{nc-})\,,$$

and the denominator in (6.2.13) drops out. We thus obtain

$$H_{nc} \;=\; \left(\nu_{nc}\, M + \overline{\nu_{nc}}\, \overline{M} + L + \overline{L}\right) \overline{T^{(0)}_{[0]}}^{-1}\,, \tag{6.2.16}$$

where M is the linear combination of monomials (6.2.6), and L is given by

$$L \;\equiv\; T^{(0)}_{[0]}\, \overline{T^{(-1)}_{[0]}} \left(T^{(0)}_{[2]}\, \overline{T^{(0)}_{[0]}} + T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \right). \tag{6.2.17}$$

We conclude that the EL equations to degree 4 are equivalent to the conditions (6.2.15) with H_{nc} given by (6.2.16, 6.2.6, 6.2.17).

Let us analyze what the conditions (6.2.15) mean. First of all, the contributions to (6.2.16) which involve L or $\overline{L}$ are clearly independent of n, c and thus drop out in (6.2.15). In the case $n' = n$ and $c' = \overline{c}$, we can in (6.2.15) apply the first part of (6.1.5) to obtain the necessary conditions

$$(\nu_{nc} - \overline{\nu_{nc}})\, (M - \overline{M})\, \overline{T^{(0)}_{[0]}}^{-1} \;=\; 0\,. \tag{6.2.18}$$

If we assume no relations between the basic fractions, this implies that $\nu_{nc} = \overline{\nu_{nc}}$, and thus

$$\nu_{nc} \;=\; \pm 1 \qquad \text{for all } n = 1, \ldots, 7 \text{ and } c = L, R. \tag{6.2.19}$$

For $x = y$, the matrix W_c becomes $W_c = X_c X_{\overline{c}}$, and thus the eigenvalues in (6.2.19) are equal to one. Since these eigenvalues depend smoothly on x and y, we conclude that $\nu_{nc} = 1$ for all x and y. This means in the block matrix representation for W_c, (6.1.17), that the unitary matrix U is equal to the identity. Thus, according to (6.1.4),

$$X_c \int_x^y \!\!\mathrm{c} \int_y^x \overline{c}\; X_{\overline{c}} \;=\; W_c \;=\; X_c X_{\overline{c}}\,. \tag{6.2.20}$$

Differentiating with respect to y and setting $y = x$ gives

$$X_c\, (A_c - A_{\overline{c}})\, X_{\overline{c}} \;=\; 0\,.$$

Hence the left- and right-handed potentials must coincide on the massive sectors. Using the argument in Remark 6.1.3, we can arrange the same in the neutrino sector. This gives the dynamical gauge group in (6.2.7). Conversely, if (6.2.7) is satisfied, then the matrices W_c are of the form (6.2.20). It follows that $\nu_{nc} = 1$ for all $n = 1, \ldots, 7$ and $c = L, R$, and thus (6.2.15) holds.

We next consider the case when we allow for relations between the basic fractions. The only way to avoid the conditions (6.2.19) (which lead to the dynamical gauge group (6.2.7)) is to assume that the factor $M - \overline{M}$ in (6.2.18) vanishes. This gives precisely the relation (6.2.9). If (6.2.9) holds, H_{nc} simplifies to

$$H_{nc} \;=\; 2\,\mathrm{Re}\,(\nu_{nc})\; M \;+\; L + \overline{L}\,. \tag{6.2.21}$$

If we assume no further relations between the basic fractions, the conditions (6.2.15) are equivalent to

$$\mathrm{Re}\,(\nu_{nc}) \;=\; \mathrm{Re}\,(\nu_{n'c'}) \qquad\quad \text{for all } n = 1, \ldots, 7 \text{ and } c = L, R. \tag{6.2.22}$$

The only way to avoid these conditions is to impose in addition that (6.2.10) holds. If this is done, all terms involving ν_{nc} or $\overline{\nu_{nc}}$ drop out in (6.2.16), and (6.2.15) is satisfied.

It remains to show that the conditions (6.2.22) are equivalent to (6.2.8). Again using the argument in Remark 6.1.3, it is obvious that the dynamical gauge group has the required form on the neutrino sector, and thus we can in what follows restrict attention to the seven massive sectors. Then $\mathcal{G}$ is a subgroup of $U(7)_L \times U(7)_R$, and the matrices W_c are unitary and have the spectral representation

$$W_c(x,y) \;= \int_x^y \mathrm{c} \int_y^x \overline{\mathrm{c}} \;\;=\; \sum_{n=1}^{7} \nu_{nc}\, I_{nc}\,. \tag{6.2.23}$$

Suppose that (6.2.8) is satisfied. Then

$$\mathcal{G} \;\ni\; \left(\int_x^y \mathrm{L}\,, \int_y^x \mathrm{R} \right) \;=\; (bf, f)$$

with $b \in \mathcal{B}_p$ and $f \in \mathcal{F}_p$, and thus

$$W_c \;=\; bf\, f^{-1} \;=\; b\,.$$

As one sees immediately from (6.2.5), the eigenvalues ν_{nc} of b are equal to z and $\overline{z}$ with $z \in U(1) \subset \mathbb{C}$. Thus the conditions (6.2.22) are satisfied.

Suppose conversely that the conditions (6.2.22) hold. We denote the Lie algebra of the dynamical gauge group by $\mathfrak{g}$; it is a subalgebra of $su(7) \oplus su(7)$. Let π be the projection onto the axial part,

$$\pi \;:\; g \to su(7) \;:\; (A_L, A_R) \mapsto A_L - A_R\,. \tag{6.2.24}$$

Its image $\pi(\mathfrak{g})$ is a subspace of $su(7)$ (but it is in general no subalgebra). For the first part of our argument, we consider the situation "locally" for x near y. Expanding the ordered exponentials in (6.2.23) in a power series around x yields according to (6.1.2) that

$$W_{L/R}(x + \varepsilon u, x) \;=\; \mathbb{1} \;\pm\; i\varepsilon (A_j^L(x) - A_j^R(x))\, u^j \;+\; O(\varepsilon^2)\,. \tag{6.2.25}$$

Since the gauge potentials at x can be chosen freely with values in the dynamical gauge algebra, the term $A \equiv (A_j^L(x) - A_j^R(x))\, u^j$ can take any value in $\pi(\mathfrak{g})$. The eigenvalues of (6.1.4) have the expansion $\nu_{nc} = 1 \pm i\varepsilon\lambda_n + o(\varepsilon)$, where λ_n are the eigenvalues of A. We conclude from (6.2.22) that

$$\sigma(A) \;=\; \{\pm\lambda \text{ with } \lambda = \lambda(A) \in \mathbb{R}\} \qquad\quad \text{for all } A \in \pi(\mathfrak{g}). \tag{6.2.26}$$

We can assume in what follows that $\pi(\mathfrak{g})$ is non-trivial, $\pi(\mathfrak{g}) \neq 0$, because otherwise according to (6.2.24) the dynamical gauge potentials are pure vector potentials, giving rise to (6.2.7).

Next we consider the eigenvalues of W_c^{xy} "globally" for y far from x. Expanding the ordered exponentials in (6.2.23) along the line $x + \varepsilon\xi$, $\xi \equiv y - x$ gives

$$\begin{aligned} W_L(x+\varepsilon\xi, y) \;&=\; W_L(x,y) \\ &\quad + i\varepsilon \left(A_j^L(x)\, W_L(x,y) - W_L(x,y)\, A_j^R(y) \right) \xi^j \;+\; O(\varepsilon^2)\,. \end{aligned} \tag{6.2.27}$$

It would be nicer to have the potentials A^L and A^R on the same side of the factor W_L. Therefore, we perform a unitary transformation with $U_\varepsilon = \mathbb{1} - i\varepsilon A_j^L + O(\varepsilon^2)$ to obtain

$$U_\varepsilon\, W_L(x+\varepsilon\xi, y)\, U_\varepsilon^{-1} \;=\; W_L(x,y) \;+\; i\varepsilon\, W_L(x,y)\, A \;+\; O(\varepsilon^2)\,, \tag{6.2.28}$$

where we set $A \equiv (A_j^L - A_j^R)\, \xi^j$. Let us analyze what (6.2.22) and our information on A, (6.2.26), tell us about the form of W_L; for simplicity, we work rather elementary with matrices. As explained before (6.2.26), we are free to choose $A \in \pi(\mathfrak{g})$; we fix any $A \neq 0$. We diagonalize the matrix W_L for given x and y. This gives according to (6.2.22),

$$W_L(x,y) \;=\; \begin{pmatrix} z\,\mathbb{1}_p & 0 \\ 0 & \overline{z}\,\mathbb{1}_q \end{pmatrix} \tag{6.2.29}$$

with $z \in U(1)$, where we used a block matrix notation similar to that in (6.2.5) and again set $q = 7 - p$. We can without loss of generality assume that $p \in \{4, 5, 6\}$. We fist consider the case $z \neq \overline{z}$. Computing the eigenvalues of (6.2.28) in first order perturbation theory, the conditions (6.2.22) yield that A must be of the form

$$A \;=\; \begin{pmatrix} \lambda\,\mathbb{1}_p & C^* \\ C & -\lambda\,\mathbb{1}_q \end{pmatrix}$$

with a $q \times p$ matrix C and $\lambda \in \mathbb{R}$. By changing the basis on the eigenspaces of $W_L(x,y)$, we can even arrange that

$$A \;=\; \begin{pmatrix} \lambda\,\mathbb{1}_p & C^* & 0 \\ C & -\lambda\,\mathbb{1}_p & 0 \\ 0 & 0 & -\lambda\,\mathbb{1}_{7-2p} \end{pmatrix} \tag{6.2.30}$$

with a $p \times p$ matrix C. Thus $-\lambda$ is an eigenvalue of A. According to (6.2.26), the eigenvalues of A are precisely $\pm\lambda$. Since $A \neq 0$, we know too that $\lambda \neq 0$. It is a general result on self-adjoint matrices that if the expectation value of a unit vector coincides with the largest eigenvalue of the matrix, then this vector must be an eigenvector. Applied to (6.2.30), this result shows that the submatrix C is zero. Thus

$$A \;=\; \begin{pmatrix} \lambda\,\mathbb{1}_p & 0 \\ 0 & -\lambda\,\mathbb{1}_q \end{pmatrix} \qquad \text{with } p \in \{4,5,6\},\; \lambda \neq 0.$$

This means that W_L and A have the same eigenspaces. Repeating the above construction for general x and y while keeping A fixed, one sees that the matrices $W_L(x,y)$ all have the same eigenspaces as A (and this is trivially true even when W_L degenerates to a multiple of the identity matrix). This shows that in our basis, (6.2.29) holds even for all x and y. In the case $z = \overline{z}$ for our original matrix W_L (chosen before (6.2.29)), W_L is a multiple of the identity matrix. If this is true for all x and y, then (6.2.29) holds for $p = 0$. Otherwise, we choose x and y such that $W_L(x,y)$ is not a multiple of the identity matrix and repeat the above argument. We conclude that for some $p \in \{4, \ldots, 7\}$ and possibly after a global gauge transformation, the matrix W_L has the form (6.2.29) for all x and y.

Let us show that the representation (6.2.29) is surjective in the sense that for every $z \in U(1)$ we can choose the dynamical gauge potentials on the line segment $\overline{xy}$ such

that W_L is of the form (6.2.29) for this given z. To this end, we take the determinant of (6.2.29, 6.2.23),

$$\overline{z}^{7-2p} \;=\; \det W_L \;=\; \det\left(\int_x^y \mathrm{L} \int_y^x \mathrm{R}\right) .$$

Using that the determinant is multiplicative, we obtain from (6.1.2) that

$$\overline{z}^{7-p} \;=\; \exp\left(-i \int_0^1 \mathrm{Tr}\left(A(\tau y + (1-\tau)x)\, d\tau\right)\right) ,$$

where we again set $A = (A_j^L - A_j^R)\, \xi^j$. This shows that the phase of z is simply additive along the line segment $\overline{xy}$. It follows immediately that this phase can take arbitrary values, provided that there is an $A \in \pi(\mathfrak{g})$ with non-zero trace. Indeed, it follows from (6.2.26) and the fact that A is an odd-dimensional matrix that $\mathrm{Tr}(A) \neq 0$ for all $A \neq 0$.

We finally return to the expansion (6.2.27). Writing the chiral potentials as block matrices,

$$A_j^c\, \xi^j \;=\; \begin{pmatrix} a_{11}^c & a_{12}^c \\ a_{21}^c & a_{22}^c, \end{pmatrix} ,$$

and using that both $W_L(x + \varepsilon\xi, y)$ and $W_L(x, y)$ are of the form (6.2.29) with phases denoted by $z = z_\varepsilon$ and $z = z_0$, respectively, we obtain

$$\begin{aligned} \begin{pmatrix} z_\varepsilon\, \mathbb{1}_p & 0 \\ 0 & \overline{z_\varepsilon}\, \mathbb{1}_q \end{pmatrix} \;&=\; \begin{pmatrix} z_0\, \mathbb{1}_p & 0 \\ 0 & \overline{z_0}\, \mathbb{1}_q \end{pmatrix} \\ &\quad + i\varepsilon \begin{pmatrix} z_0\, (a_{11}^L - a_{11}^R) & \overline{z_0}\, a_{12}^L - z_0\, a_{12}^R \\ z_0\, a_{21}^L - \overline{z_0}\, a_{21}^R & \overline{z_0}\, (a_{22}^L - a_{22}^R) \end{pmatrix} \;+\; O(\varepsilon^2) . \end{aligned}$$

Since $z_0 \in U(1)$ can take arbitrary values, it follows that

$$a_{11}^L - a_{11}^R \;=\; \lambda\, \mathbb{1}_p , \qquad a_{22}^L - a_{22}^R \;=\; -\lambda\, \mathbb{1}_q , \qquad a_{12}^c \;=\; 0 \;=\; a_{21}^c .$$

Chiral potentials of this form correspond precisely to the dynamical gauge group $\mathcal{G}_p$ in (6.1.13). ∎

We finally make three remarks which give a better justification of the ansatz for the vector/axial potentials in (6.0.1), of the formalism used and of the variational principles to which this formalism applies.

Remark 6.2.3. *(The chiral potentials on the generations)* Compared to the most general ansatz for the vector and axial potentials,

$$C \;=\; C_{(b\beta)}^{(a\alpha)} , \qquad E \;=\; E_{(b\beta)}^{(a\alpha)} , \tag{6.2.31}$$

the potentials in (6.0.2) are restricted in that they must be the same for the three generations. We shall now justify the ansatz in (6.0.2) from the form of the gauge terms.

Recall that in §5.3 we combined the regularization functions of the three generations to new "effective" regularization functions in each sector (5.3.19). Here we write this procedure symbolically as

$$T_\circ^{(n)} \;=\; \sum_{\alpha=1}^{3} T_{\alpha\,\circ}^{(n)} ,$$

where $T^{(n)}_{\alpha\,\circ}$ involves the regularization functions for a single Dirac sea in generation α. Let us consider how the analog of the gauge term (6.1.1) looks like. In the case when the potentials are diagonal on the generations, i.e.

$$C \;=\; (C^\alpha)^a_b\, \delta^\alpha_\beta\,, \qquad\quad E \;=\; (E^\alpha)^a_b\, \delta^\alpha_\beta\,, \tag{6.2.32}$$

the generalization of (6.1.1) is straightforward, namely

$$P_0(x,y) \;=\; \sum_{\alpha=1}^3 \left(\chi_L\, X_L \fint_x^y{}_{\text{L}} + \chi_R\, X_R \fint_x^y{}_{\text{R}} \right)_\alpha \frac{i}{2}\, \not{\xi}\, T^{(-1)}_{\alpha\,[0]}(x,y)\,, \tag{6.2.33}$$

where the index α of the brackets means that we take the ordered exponentials of the chiral potentials in the corresponding generation. This gauge term involves relative phase shifts of the individual Dirac seas. If we substitute it into the EL equations, we get many contributions involving these relative phases, and if we want these contributions to drop out, we must introduce additional regularization conditions for certain polynomials in $T^{(n)}_{\alpha\,\circ}$, $\alpha = 1,2,3$. Thus unless we impose very strong additional conditions on the regularization, the only way to fulfill the EL equations is to set all the relative phases to zero. This gives precisely our ansatz (6.0.2).

If the potentials C and E are not diagonal on the generations, the form of the gauge terms is not obvious because there is no longer a canonical way to put in the factors $T^{(-1)}_{\alpha\,\circ}$. This point could be clarified by generalizing the regularized causal perturbation expansion of Appendix D to the case of systems of Dirac seas involving different regularizations, but we do not want to get into these technical details here. Qualitatively speaking, it is clear that if already the potentials (6.2.32) lead to strong additional conditions in the EL equations, this will be even more the case for the general ansatz (6.2.31).

REMARK 6.2.4. *(The vector component is null on the light cone)* In §4.4 we introduced the regularization condition that the vector component should be null on the light cone. We remarked that this condition need not be imposed ad hoc, but that it actually follows from the equations of discrete space-time. We are now in a position to justify this regularization condition from the EL equations.

In our formula for the perturbation of the eigenvalues (6.2.12, 6.2.13) we omitted all contributions involving factors $T^{(n)}_{\{.\}}$, assuming that they are of lower degree on the light cone. This is the only point where we used that the vector component is null on the light cone. Without imposing these regularization conditions, we get for $\Delta\lambda^{xy}_{nc-}$ the additional contributions

$$\begin{aligned}
&-\nu_{nc} \left(T^{(0)}_{[2]}\, \overline{T^{(1)}_{\{0\}}} + T^{(-1)}_{[0]}\, \overline{T^{(2)}_{\{2\}}} \right) \\
&+ \frac{T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} - T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}}}{\overline{\lambda^{xy}_{nc-}} - \lambda^{xy}_{nc-}} \left(\nu_{nc}\, T^{(0)}_{[1]}\, \overline{T^{(1)}_{\{0\}}} \;-\; \overline{\nu_{nc}}\, T^{(1)}_{\{0\}}\, \overline{T^{(0)}_{[1]}} \right).
\end{aligned}$$

This leads to an additional contribution to H_{nc}, (6.2.16), of the form

$$\left(\nu_{nc}\, K + \overline{\nu_{nc}}\, \overline{K} + L + \overline{L}\right)\, \overline{T^{(0)}_{[0]}}^{-1}$$

with polynomials K and L, where K is given explicitly by

$$K \;=\; T^{(0)}_{[1]}\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}\, T^{(1)}_{\{0\}}} - T^{(-1)}_{[0]}\, T^{(1)}_{\{0\}}\, \overline{T^{(0)}_{[0]}\, T^{(0)}_{[1]}}\,.$$

The monomials appearing here have a different homogeneity in the "large" light-cone coordinate l than those in (6.2.6) (more precisely, they involve an additional factor of l; note that the scaling in l is given by the upper index of $T_\circ^{(n)}$, see (4.5.12–4.5.14)). Using the different scaling behavior in l, we can distinguish between the contributions involving K and L in the EL equations in the sense that both of these contributions must vanish separately. But this means that we can just as well omit K in the EL equations, exactly as it was done in (6.2.12, 6.2.13) under the assumption that the vector component is null on the light cone. This argument applies similarly to other contributions to the EL equations, to every degree on the light cone.

REMARK 6.2.5. *(n-point actions)* We now discuss some difficulties which arise in the study of actions other than two-point actions. These difficulties are the reason why we do not consider such actions here. Let S be a general n-point action

$$S \;=\; \sum_{x_1,\ldots,x_n \in M} \mathcal{L}[P(x_1,x_2) \cdots P(x_{n-1},x_n)\, P(x_n,x_1)]] \,, \qquad\qquad n \geq 1.$$

If $n = 1$, the corresponding EL equations are of the form (5.2.9) with

$$Q(x,y) \;=\; \delta_{xy}\, f[P(x,x)] \,, \tag{6.2.34}$$

where f is a functional depending only on $P(x,x)$. Expressions like (6.2.34) do not have a well-defined continuum limit because the methods of Chapter 4 apply to composite expressions only away from the origin (i.e. for $x \neq y$). Even if one succeeded in giving (6.2.34) a mathematical meaning, this expression is local and does not involve any ordered exponentials of the chiral potentials. As a consequence, we would have no gauge terms, and the only constraint for the chiral potentials would be the causality compatibility condition. The resulting dynamical gauge group $\mathcal{G} = U(8)_L \times U(7)_R$ would be too large for physical applications. For these reasons, one-point actions do not seem worth considering.

If on the other hand $n > 2$, the operator Q in the EL equations takes the form

$$\begin{aligned} Q(x_1,x_2) \;=\; \sum_{x_2,\ldots,x_{n-1} \in M} & f[P(x_1,x_2) \cdots P(x_{n-1},x_n)\, P(x_n,x_1)] \\ & \times P(x_2,x_2) \cdots P(x_{n-1},x_n) \,. \end{aligned}$$

where f is a functional on the closed chain. Again, it is not clear how to make mathematical sense of this expression in the continuum limit, but in contrast to (6.2.34) it now seems possible in principle to adapt the methods of Chapter 4. We disregard these technical difficulties here and merely discuss the form of the gauge terms in the simplest example of a single Dirac sea and a $U(1)$ vector potential A. It might be that the only relevant contributions to the EL equations comes about when the points $x_1, \ldots, x_n$ all lie on a straight line. Generally speaking, the situation in this case would be quite similar to that for a two-point action, and does not seem to give anything essentially new (although the quantitative details would clearly be different). In particular, the gauge terms of type (6.1.1) drop out in the closed chain, in agreement with the fact that the $U(1)$ corresponds to an unbroken local gauge symmetry. However, the situation is much different if we assume that the points $x_1, \ldots, x_n$ do not necessarily lie on a straight line. Namely, in this case the phase shifts in the closed chain add up to an integral along the polygon C with vertices $x_1, \ldots, x_n$,

$$e^{-i \int_{x_1}^{x_2} A_j\, (x_2-x_1)^j} \cdots e^{-i \int_{x_n}^{x_1} A_k\, (x_1-x_n)^k} \;=\; \exp\left(-i \oint_C A_i\, ds^i \right) .$$

Stokes' theorem allows us to write this line integral as a surface integral. More precisely, choosing a two-dimensional surface S with $\partial S = C$,

$$\exp\left(-i \oint_C A_i \, ds^i\right) \;=\; \exp\left(-i \int_S F_{ij} \, d\sigma^{ij}\right) \,,$$

where $F = dA$ is the field tensor and $d\sigma$ is the area form on S. This simple consideration shows that the phase shift in the closed chain is in general not zero; indeed, it is zero for any position of the points x_k if and only if the field tensor vanishes identically. Thus in the EL equations we now expect additional contributions which involve surface integrals of the gauge field tensor; we refer to such contributions as *surface terms*. The appearance of surface terms seems a problem because they give constraints even for those gauge potentials which correspond to a local symmetry of the system.

CHAPTER 7

Spontaneous Block Formation

The dynamical gauge group introduced in the previous chapter cannot be identified with the physical gauge group, because the results of Theorem 6.2.2 are not compatible with the gauge groups in the standard model. Namely, if we allow for two relations between the basic fractions (case (3)), the resulting dynamical gauge group $(U(7) \times U(1))_L \times (U(7) \times U(1))_R$ is too large. The cases (1) and (2), on the other hand, seem too restrictive because either no chiral gauge fields are allowed (6.2.7), or else the chiral gauge fields must be Abelian and are diagonal on the sectors (6.2.8), in contrast to the weak $SU(2)$ gauge fields in the standard model. Fortunately, these seeming inconsistencies disappear when scalar potentials are taken into account, and it is indeed possible in case (2) to model an interaction involving non-Abelian chiral gauge fields. The point is that if scalar potentials are included, the dynamical mass matrices $Y_{L/R}$, (6.0.4), are in general not diagonal on the sectors. Using a local transformation of the fermionic projector, one can reformulate the interaction such that the dynamical mass matrices become diagonal, but then the resulting chiral fields have off-diagonal contributions and can be identified with so-called "effective" non-Abelian gauge fields. In this chapter we study the EL equations for an interaction involving both chiral and scalar potentials. After the preparations of §7.1, we show in §7.2 that the EL equations imply that the fermionic projector splits globally into four so-called blocks, which interact with each other only via free gauge fields. We can distinguish between three quark blocks and one lepton block; these will be analyzed in more detail in §7.3. In Chapter 8 we finally give the transformation to the corresponding effective interaction.

Since including the scalar potentials may give further constraints for the dynamical gauge potentials, we cannot expect that the dynamical gauge potentials of the previous chapter will all be admissible here. Therefore, we merely assume that the dynamical gauge potentials present in the system correspond to a subgroup of the dynamical gauge group of Theorem 6.2.2. In order not to get lost in analytical details which are of no physical relevance, we make the following additional assumptions.

(I) The system should contain chiral dynamical gauge fields.
(II) The chiral Dirac particles should enter the EL equations.

From the physical point of view, the last assumption is trivial because otherwise the chiral Dirac particles (=neutrinos) would be unobservable. Furthermore, we need to assume that our system involves several gauge fields which are sufficiently "independent" from each other. This assumption could be made precise in many different ways; our formulation seems particularly convenient.

(III) The free gauge fields should distinguish the chiral and massive Dirac particles in the sense that for every pair of a chiral and a massive Dirac particle there is a free dynamical gauge field which couples to the two particles differently.

For the interactions in the standard model, the last assumption is clearly satisfied because the electromagnetic field couples to all massive Dirac particles, but not to the neutrinos. Assumption (III) could be weakened, but this would make it necessary to rule out a number of exceptional cases in the analysis, and we do not want to consider this here. We finally give our guideline for dealing with the regularization.

(IV) Impose as few relations between the basic fractions as possible such that (I)–(III) can be fulfilled.

This method will uniquely determine all relations between the basic fractions.

7.1. The Partial Trace and the Dynamical Mass Matrices

We want to analyze the EL equations in the presence of chiral and scalar potentials (6.0.5) to the degree 4 on the light cone. Thus the only difference to the setting of Theorem 6.2.2 is that, instead of a constant matrix Y, we now allow more generally for dynamical mass matrices $Y_L(x)$ and $Y_R(x)$. One difficulty is that the scalar potentials may depend in a complicated way on the generation index (in contrast to the chiral potentials which we assumed to be constant on the generations; see (6.0.2)). In particular, the partial trace (2.3.20) becomes a non-trivial operation when dynamical mass matrices are present. In this section, we give a few general considerations on the partial trace of the dynamical mass matrices.

We first introduce a convenient notation. In our calculations so far, we omitted the mass matrix Y in all contributions to the fermionic projector. Now that we are working with the dynamical mass matrices $Y_{L/R}$, these clearly have to be written out. In composite expressions, we need to make clear how the partial traces are to be taken. To this end, we denote the sums over the upper and lower generation index by the tildes $\acute{}$ and $\grave{}$, respectively. Thus we introduce the matrices

$$\begin{aligned} \acute{Y}_{L/R} \;:\; \mathbb{C}^{8\times 3} \to \mathbb{C}^8\,, \qquad & (\acute{Y}_{L/R})^a_{(b\beta)} \;=\; \sum_{\alpha=1}^{3} (Y_{L/R})^{(a\alpha)}_{(b\beta)} \\ \grave{Y}_{L/R} \;:\; \mathbb{C}^{8} \to \mathbb{C}^{8\times 3}\,, \qquad & (\grave{Y}_{L/R})^{(a\alpha)}_b \;=\; \sum_{\beta=1}^{3} (Y_{L/R})^{(a\alpha)}_{(b\beta)}\,. \end{aligned}$$

Similarly, we denote the sum over both generation indices by the accent $\hat{}$,

$$\hat{Y}_{L/R} \;:\; \mathbb{C}^8 \to \mathbb{C}^8\,, \qquad (\hat{Y}_{L/R})^a_b \;=\; \sum_{\alpha,\beta=1}^{3} (Y_{L/R})^{(a\alpha)}_{(b\beta)}\,.$$

Clearly, $(\acute{Y}_{L/R})^* = \grave{Y}_{R/L}$ and $(\hat{Y}_{L/R})^* = \hat{Y}_{R/L}$. In a contribution to the fermionic projector which involves a product of dynamical mass matrices, the partial trace leads us to label the first and last factor $Y_{L/R}$ by $\acute{}$ and $\grave{}$, respectively. For example, in the presence of a homogeneous scalar potential, we write the light-cone expansion of the left-handed component of the fermionic projector in analogy to (5.3.16) as

$$\chi_L\, P(x,y) \;=\; \chi_L \left(X\, \frac{i\not{\xi}}{2}\, T^{(-1)}_{[0]} \;+\; \hat{Y}_L\, T^{(0)}_{[1]} \;+\; \frac{i\not{\xi}}{2}\, \acute{Y}_L\, \grave{Y}_R\, T^{(0)}_{[2]} \;+\; \cdots \right). \tag{7.1.1}$$

Furthermore, we denote the contraction in the sector index by Tr_S,

$$\mathrm{Tr}_S B \;\equiv\; \sum_{n=1}^{8} B^n_n\,.$$

One should keep in mind that the partial trace is *not* cyclic, because we sum over the upper and lower index independently. For example,

$$\mathrm{Tr}_S \acute{Y}_L \grave{Y}_R \overset{\text{in general}}{\neq} \mathrm{Tr}_S \acute{Y}_R \grave{Y}_L \,. \tag{7.1.2}$$

But both terms are clearly real and non-negative.

The EL equations are formulated in terms of the fermionic projector, which is obtained from the auxiliary fermionic projector by taking the partial trace (2.3.20). Therefore we regard the fermionic projector as a physical object only after the partial trace has been taken. Thus it is a reasonable point of view that we do not need to worry about noncausal line integrals in the light-cone expansion as long as the corresponding contributions to the auxiliary fermionic projector drop out when the partial trace is taken. This leads us to weaken the causality compatibility condition (2.3.18) by imposing a condition only on the partial trace of the spectral projectors.

DEF. 7.1.1. *The Dirac operator is* **weakly causality compatible** *if*

$$\sum_{\alpha,\beta=1}^{3} (X\,(\tilde{p}-\tilde{k}))^{(a\alpha)}_{(b\beta)} \;=\; \sum_{\alpha,\beta=1}^{3} ((\tilde{p}-\tilde{k})\,X^*)^{(a\alpha)}_{(b\beta)} \,.$$

Under this assumption, the fermionic projector is defined canonically by

$$P^a_b(x,y) \;=\; \sum_{\alpha,\beta=1}^{3} X\,\frac{1}{2}\,(\tilde{p}-\tilde{k}))^{(a\alpha)}_{(b\beta)}(x,y)\,. \tag{7.1.3}$$

In what follows, we shall assume that the weak causality compatibility condition is satisfied for all contributions to the fermionic projector which are of relevance to the degree on the light cone under consideration.

Our point of view that the fermionic projector has a physical meaning only after taking the partial trace also implies that we should consider different choices of dynamical mass matrices as being equivalent if taking the partial trace (2.3.20) gives the same fermionic projector. Furthermore, for this equivalence it is not necessary that the fermionic projectors be identical, but it suffices that all contributions to the fermionic projector which enter the EL equations are the same. More specifically, to the degree 4 on the light cone the EL equations will involve at most quadratic terms in m, and so every factor $Y_{L/R}$ carries an accent. Thus two dynamical mass matrices can be considered as being equivalent if their partial traces coincide. In other words, the dynamical mass matrices are determined only modulo the equivalence relation

$$B_1 \simeq B_2 \qquad \text{if} \qquad \acute{B}_1 = \acute{B}_2 \text{ and } \grave{B}_1 = \grave{B}_2.$$

This arbitrariness in choosing the dynamical mass matrices can be used to simplify these matrices. For example, we will set the matrix entries to zero whenever possible by applying for every $a, b \in \{1,\ldots,8\}$ and $c \in \{L,R\}$ the rule

$$(\acute{Y}_c)^a_{(b.)} = 0 = (\grave{Y}_c)^{(a.)}_b \quad \Longrightarrow \quad (Y_c)^{(a.)}_{(b.)} = 0\,.$$

Here the dot means that we are using a matrix notation in the generation index, i.e. $(\acute{Y}_c)^a_{(b.)}$ is (for fixed a, b) a 3-vector and $(Y_c)^{(a.)}_{(b.)}$ a 3×3 matrix. We refer to this method of simplifying the dynamical mass matrices that we *choose a convenient representation* of Y_c.

In order to rule out pathological cases, we need to impose a condition on the dynamical mass matrices. Note that in the vacuum the mass matrices are block diagonal in the sense that $(Y_{L/R})^{(a.)}_{(b.)} = \delta^a_b Y^a_{L/R}$ for suitable 3×3 matrices $Y^a_{L/R}$. Thus the off-diagonal elements $(Y_{L/R})^{(a.)}_{(b.)}$, $a \neq b$, contain scalar potentials. It would be too restrictive to assume that there are no cancellations when the partial trace is taken; i.e. we do want to allow for the possibility that $(\acute{Y}_c)^a_{(b.)} = 0$ or $(\grave{Y}_c)^{(a.)}_b = 0$ although $(Y_c)^{(a.)}_{(b.)} \neq 0$ (for some $a \neq b$). But such cancellations should occur only with a special purpose, for example in order to ensure that the Dirac operator be weakly causality compatible or in order to arrange that certain terms drop out of the EL equations. For such a purpose, it is not sufficient that one off-diagonal element of $\acute{Y}_c$ vanishes, but all the off-diagonal elements in the same row should be zero. This is the motivation for the following definition.

DEF. 7.1.2. *The dynamical mass matrices are* **non-degenerate** *if for all $a, b \in \{1, \ldots, 8\}$, $a \neq b$ and $c \in \{L, R\}$,*

$$(\grave{Y}_c)^{(a.)}_b \neq 0 \text{ and } (\acute{Y}_c)^a_{(b.)} = 0 \quad \Longrightarrow \quad (\acute{Y}_c)^a_{(d.)} = 0 \text{ for all } d \neq a.$$

The freedom to choose a convenient representation of the dynamical mass matrices reduces our problem to revealing the structure of the matrices $\acute{Y}_c$ and $\grave{Y}_c$. One difficulty is that the EL equations involve these matrices only in products of the form $\acute{Y}_{L/R}(y)\grave{Y}_{L/R}(x)$. The following elementary lemma will allow us to use information on the matrix product to derive properties of the individual factors.

LEMMA 7.1.3. **(Uniform Splitting Lemma)** *Let $\mathcal{B} \subset \text{Mat}(\mathbb{C}^{p_1}, \mathbb{C}^{p_2})$ be a set of $(p_2 \times p_1)$ matrices with the property that for all $B_1, B_2 \in \mathcal{B}$ there is $\lambda \in \mathbb{C}$ such that*

$$B_1^* B_2 \;=\; \lambda \, \mathbb{1}_{\mathbb{C}^{p_1}} \,. \tag{7.1.4}$$

Then there is a unitary $(p_2 \times p_2)$ matrix U and an integer $r \geq 0$ with $rp_1 \leq p_2$ such that every $B \in \mathcal{B}$ can be written in the form

$$B \;=\; U \begin{pmatrix} \overbrace{b \oplus \cdots \oplus b}^{p_1 \text{ summands}} \\ 0 \end{pmatrix} \begin{matrix} \}rp_1 \text{ rows} \\ \}p_2 - rp_1 \text{ rows} \end{matrix} \tag{7.1.5}$$

for a suitable $(r \times 1)$ matrix b.

We mention for clarity that $b \oplus \cdots \oplus b$ is a $(rp_1 \times p_1)$ matrix; it could also be written as a block matrix with diagonal entries b. The word "uniform" in the name of the lemma refers to the fact that the unitary transformation U is independent of $B \in \mathcal{B}$. In our applications, this will mean that U is constant in space-time. Such constant unitary transformations are irrelevant (e.g. they could be absorbed into a more general definition of the partial trace), and we can often simply ignore them.

Proof of Lemma 7.1.3. Let $(e_1, \ldots, e_{p_1})$ be an orthonormal basis of $\mathbb{C}^{p_1}$. We introduce the subspaces

$$E_i \;=\; <\{Be_i \text{ with } B \in \mathcal{B}\}> \;\subset\; \mathbb{C}^{p_2}$$

and the mappings

$$\pi_i \;:\; \mathcal{B} \to E_i \;:\; B \mapsto Be_i \,.$$

The property (7.1.4) implies that for all $B_1, B_2 \in \mathcal{B}$,

$$\langle B_2\, e_i,\, B_1\, e_j \rangle \;=\; \langle B_1^*\, B_2\, e_i,\, e_j \rangle \;=\; \lambda(B_1, B_2)\, \delta_{ij} \,. \tag{7.1.6}$$

If $i \neq j$, this relation shows that the subspaces $(E_i)_{i=1,\ldots,p_1}$ are orthogonal. In the case $i = j$, (7.1.6) yields that the inner product $\langle \pi_i(B_1),\, \pi_i(B_2) \rangle$ is independent of i. Thus the mappings π_i are unitarily equivalent, and so we can arrange by a unitary transformation that the π_i all have the same matrix representation $\pi(B) = b$. ∎

7.2. Analysis of Degeneracies

The operator Q corresponding to the Dirac operator (6.0.5) is again given by (5.2.7, 5.4.7),

$$Q(x,y) \;=\; \frac{1}{2} \sum_{k=1}^{K_{xy}} \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda_k^{xy}}\, F_k^{xy}\; P(x,y)\,. \tag{7.2.1}$$

Following (I) and (IV), we can restrict attention to case (2) of Theorem 6.2.2. In this case, the eigenvalues of A are highly degenerate. We must take into account that these degeneracies will in general be removed by the scalar perturbation. This subtle problem is treated in a more general context in Appendix G. We now specialize the obtained results using a notation which is adapted to the dynamical gauge group $\mathcal{G}_p$, (6.2.8). We let $\uparrow$ and $\downarrow$ be the sets

$$\uparrow \;=\; \{1,\ldots,p\}\,, \qquad\qquad \downarrow \;=\; \{p+1,\ldots,7\}$$

and introduce the corresponding projectors $I_{\uparrow/\downarrow}$ by

$$I_\uparrow \;=\; \sum_{n\in\uparrow} I_n\,, \qquad\qquad I_\downarrow \;=\; \sum_{n\in\downarrow} I_n\,,$$

where $(I_n)^a_b = \delta^a_b\, \delta^a_n$ are the projectors on the sectors (in the case $p = 7$, we set $\downarrow = \emptyset$ and $P_\downarrow = 0$). To the highest degree on the light cone (i.e. if the eigenvalues are treated as in Theorem 6.1.2), the chiral gauge fields corresponding to $\mathcal{G}_p$ lead to five distinct eigenvalues of A_{xy}, one of which is zero. The spectral projectors corresponding to the kernel and the non-zero eigenvalues are given by I_8 and

$$(\chi_c\, I_\uparrow + \chi_{\bar c}\, I_\downarrow)\, F_s \qquad\qquad \text{with } c = L/R,\; s = \pm, \tag{7.2.2}$$

respectively. To the next lower degree on the light cone, we need to take into account the perturbation of A by gauge terms and the scalar potentials. Theorem G.5.1 shows that the dimension of the kernel of A is not affected by the perturbation, and thus it suffices to consider the non-zero eigenvalues. According to Theorem G.4.1, the degeneracy of the non-zero eigenvalues is in general removed. In order to describe the splitting of the eigenvalues in the massive sectors, we first associate to each spectral projector (7.2.2) a projector on an invariant subspace of A (which is no longer necessarily an eigenspace), and the perturbed eigenvalues are then obtained by diagonalizing A on these invariant subspaces (see §G.1 and §G.4 for details). It is the main result of Theorem G.4.1 that the perturbation is block diagonal on the left- and right-handed components of the invariant subspaces. This means more precisely that the left- and right-handed components of $(F_k)_{k=2,\ldots,K}$, i.e. the image of the eight projectors

$$\chi_c\, I_\uparrow\, F_s \text{ and } \chi_c\, I_\downarrow\, F_s \qquad\qquad \text{with } c = L/R,\; s = \pm, \tag{7.2.3}$$

can be perturbed to obtain invariant subspaces of A, and thus it suffices to analyze A on these smaller subspaces. Since each of these subspaces carries fixed indices (c, s), a basis on each subspace may be labeled by the sector index n. We choose a

basis such that A is diagonal on the invariant spaces. We denote the corresponding eigenvalues (counting multiplicities) by $(\lambda_{ncs} + \Delta\lambda_{ncs})$ and the spectral projectors by $(F_{ncs} + \Delta F_{ncs})$. For clarity, we point out that the unperturbed spectral projectors F_{ncs} appearing here may differ from those in (6.1.10) in that we are using a different basis on the sectors, which need not be orthogonal and may depend on c, s and x, y. This slight abuse of notation cannot lead to confusion because in (6.1.10) we are free to choose any basis on the degenerate subspaces.

For our choice of the Lagrangian (5.5.13) and the dynamical gauge group according to (6.2.8), the factors $\partial\mathcal{L}/\partial\lambda_k$ in (7.2.1) vanish identically to the highest degree, see (6.2.11). Thus it suffices to take into account the perturbation of these factors. Using the above notation, we obtain

$$Q(x,y) \;=\; \frac{1}{2} \sum_{n,c,s} \left(\Delta \frac{\partial \mathcal{L}(\lambda^{xy})}{\partial \lambda^{xy}_{ncs}} \right) F^{xy}_{ncs}\, P(x,y) \;+\; (\deg < 4)\,. \tag{7.2.4}$$

Note that the perturbation of the spectral projectors ΔF_{ncs} does not appear here; this is a major simplification. Computing the perturbation of the Lagrangian, one sees that our task is to compute terms of the form

$$\sum_{n,c,s} \mathcal{P}(\lambda^{xy}_{ncs}, \overline{\lambda^{xy}_{ncs}})\; \Delta\lambda^{xy}_{ncs}\, F^{xy}_{ncs}\, P(x,y) \tag{7.2.5}$$

$$\sum_{n,c,s} \mathcal{P}(\lambda^{xy}_{ncs}, \overline{\lambda^{xy}_{ncs}})\; \overline{\Delta\lambda^{xy}_{ncs}}\, F^{xy}_{ncs}\, P(x,y)\,, \tag{7.2.6}$$

where $\mathcal{P}$ stands for a function in both arguments. The subtle point in computing expressions of the form (7.2.5, 7.2.6) is to carry out the sums over $n \in \uparrow$ and $n \in \downarrow$ (for fixed c, s), because the corresponding indices $\{(ncs), n \in \uparrow / \downarrow\}$ label our basis vectors on the invariant subspaces associated to the projectors $\chi_c I_{\uparrow/\downarrow} F_s$. We shall now give a procedure for explicitly computing these sums. First of all, it is helpful that the unperturbed eigenvalues do not depend on $n \in \uparrow$ or $\downarrow$. Thus the polynomials $\mathcal{P}$ in (7.2.5, 7.2.6) may be taken out of the sums. It is a complication that $\Delta\lambda_{ncs}$ and $P(x,y)$ involve the gauge potentials corresponding to the free gauge group $\mathcal{F}_p$. To bypass this difficulty, we choose x and y on a fixed null line $\mathcal{L}$ in Minkowski space,

$$x, y \in \mathcal{L} = u + \mathbb{R}\, v \qquad \text{with } v^2 = 0 \tag{7.2.7}$$

and arrange by a gauge transformation that the free gauge potentials vanish identically on $\mathcal{L}$ (this is possible because free gauge transformations are local unitary transformations; see page 151). After this transformation, the chiral potentials are Abelian on $\overline{xy}$ and diagonal in the sector index.

We first state the formulas for the perturbation of the eigenvalues in full generality; we shall discuss and analyze these formulas afterwards beginning with simple special cases. In order to keep the notation as simple as possible, we restrict attention to the case $c = L$ and $n \in \uparrow$, and we shall give symbolic replacement rules with which the analogous formulas are obtained in all other cases.

DEF. 7.2.1. *Let ν, μ and ν_8, μ_8 be the phase factors*

$$\nu \;=\; \mathrm{Tr}_S \left(I_7 \int_x^y{}_R \int_y^x{}_L \right), \qquad \mu \;=\; \mathrm{Tr}_S \left(I_1 \int_x^y{}_L \right) \mathrm{Tr}_S \left(I_7 \int_y^x{}_R \right) \tag{7.2.8}$$

$$\nu_8 \;=\; \mathrm{Tr}_S \left(I_8 \int_x^y{}_R \int_y^x{}_L \right), \qquad \mu_8 \;=\; \mathrm{Tr}_S \left(I_1 \int_x^y{}_L \right) \mathrm{Tr}_S \left(I_8 \int_y^x{}_R \right). \tag{7.2.9}$$

We introduce the $p \times p$ matrix Λ by

$$\begin{aligned}
\Lambda &= \nu \int_x^y dz\; I_\uparrow \acute{Y}_L \grave{Y}_R\, I_\uparrow T^{(0)}_{[2]}\, \overline{T^{(0)}_{[0]}} && (7.2.10)\\
&\quad + \nu \int_y^x dz\; I_\uparrow \acute{Y}_R \grave{Y}_L\, I_\uparrow T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} && (7.2.11)\\
&\quad \left.\begin{aligned}
&+ I_\uparrow \hat{Y}_L(y)\, I_\uparrow \hat{Y}_L(x)\, I_\uparrow T^{(0)}_{[1]}\, \overline{T^{(0)}_{[1]}} \\
&- I_\uparrow \acute{Y}_R(y)\, I_\uparrow \grave{Y}_L(x)\, I_\uparrow T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \\
&- \frac{1}{\nu\lambda_- - \overline{\nu}\lambda_+}\, I_\uparrow \Big(\hat{Y}_L(y)\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} \;-\; \hat{Y}_R(y)\, T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}} \Big) \\
&\qquad \times I_\uparrow \Big(\nu\, \hat{Y}_R(x)\, T^{(0)}_{[1]}\, \overline{T^{(0)}_{[0]}} \;-\; \overline{\nu}\, \hat{Y}_L(x)\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}} \Big)\, I_\uparrow
\end{aligned} \right\} && (7.2.12)\\
&\quad \left.\begin{aligned}
&+ \mu\nu\, I_\uparrow \hat{Y}_L(y)\, I_\downarrow \hat{Y}_L(x)\, I_\uparrow T^{(0)}_{[1]}\, \overline{T^{(0)}_{[1]}} \\
&- \mu\nu\, I_\uparrow \acute{Y}_R(y)\, I_\downarrow \grave{Y}_L(x)\, I_\uparrow T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \\
&- \frac{\mu\nu}{\lambda_- - \lambda_+}\, I_\uparrow \Big(\hat{Y}_L(y)\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} \;-\; \hat{Y}_R(y)\, T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}} \Big) \\
&\qquad \times I_\downarrow \Big(\hat{Y}_R(x)\, T^{(0)}_{[1]}\, \overline{T^{(0)}_{[0]}} \;-\; \hat{Y}_L(x)\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}} \Big)\, I_\uparrow
\end{aligned} \right\} && (7.2.13)\\
&\quad - \mu_8 \nu_8\, I_\uparrow \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}}\,. && (7.2.14)
\end{aligned}$$

We denote the spectral adjoint of Λ (as defined in (3.5.13)) by $\overline{\Lambda}$.

LEMMA 7.2.2. *Up to contributions of degree < 4,*

$$\sum_{n \in \uparrow} \Delta\lambda^{xy}_{nL+}\, F^{xy}_{nL+}\, P(x,y) \;=\; 0 \;=\; \sum_{n \in \uparrow} \overline{\Delta\lambda^{xy}_{nL+}}\, F^{xy}_{nL+}\, P(x,y) \tag{7.2.15}$$

$$\sum_{n \in \uparrow} \Delta\lambda^{xy}_{nL-}\, F^{xy}_{nL-}\, P(x,y) \;=\; \Lambda\, P(x,y) \tag{7.2.16}$$

$$\sum_{n \in \uparrow} \overline{\Delta\lambda^{xy}_{nL-}}\, F^{xy}_{nL-}\, P(x,y) \;=\; \overline{\Lambda}\, P(x,y)\,. \tag{7.2.17}$$

The corresponding formulas for the opposite chirality are obtained by the symbolic replacements

$$L \longleftrightarrow R\,, \qquad \nu \longleftrightarrow \overline{\nu}\,, \qquad \nu_8 \longleftrightarrow \overline{\nu_8}\,, \qquad \mu_8 \longleftrightarrow \overline{\nu}\nu_8\, \mu_8\,. \tag{7.2.18}$$

In the case $p < 7$, we may furthermore perform the replacements

$$\uparrow \longleftrightarrow \downarrow\,, \qquad \nu \longleftrightarrow \overline{\nu}\,, \qquad \mu \longleftrightarrow \overline{\mu} \qquad \text{and} \qquad \mu_8 \longleftrightarrow \overline{\mu\nu}\, \mu_8\,. \tag{7.2.19}$$

Proof. According to (5.3.23), to the highest degree on the light cone we have the identity $F_{nc+}P(x,y) = 0$. This gives (7.2.15), and (7.2.16) follows directly from Theorem G.4.1 and the results of §G.3. In a basis where Λ is diagonal, (7.2.17) is an immediate consequence of (7.2.16).

To derive the replacement rule (7.2.18), we first note that in the case $p = 7$, the projector $I_\downarrow$ vanishes, and thus all contributions to Λ involving μ are equal to zero. In

the case $p < 7$,

$$\mu \longrightarrow \mathrm{Tr}_S \left(I_1 \int_x^y \right)_{\mathrm{R}} \mathrm{Tr}_S \left(I_7 \int_y^x \right)_{\mathrm{L}} = \overline{\nu}\, \mathrm{Tr}_S \left(I_1 \int_x^y \right)_{\mathrm{L}} \nu\, \mathrm{Tr}_S \left(I_7 \int_y^x \right)_{\mathrm{R}} = \mu$$

$$\mu_8 \longrightarrow \mathrm{Tr}_S \left(I_1 \int_x^y \right)_{\mathrm{R}} \mathrm{Tr}_S \left(I_8 \int_y^x \right)_{\mathrm{L}} = \overline{\nu}\, \mathrm{Tr}_S \left(I_1 \int_x^y \right)_{\mathrm{L}} \nu_8\, \mathrm{Tr}_S \left(I_8 \int_y^x \right)_{\mathrm{R}} = \overline{\nu} \nu_8\, \mu\,.$$

Using the relations

$$\mathrm{Tr}_S \left(I_7 \int_x^y \right)_{\mathrm{L}} \mathrm{Tr}_S \left(I_1 \int_y^x \right)_{\mathrm{R}} = \overline{\mu} \quad \text{and} \quad \mathrm{Tr}_S \left(I_7 \int_x^y \right)_{\mathrm{L}} \mathrm{Tr}_S \left(I_8 \int_y^x \right)_{\mathrm{R}} = \overline{\mu\nu}\, \mu_8\,,$$

the replacement rule (7.2.19) is straightforward. ∎

A straightforward calculation using (7.2.4) and Lemma 7.2.2 shows that for our Lagrangian (5.5.13), the EL equations yield the conditions

$$\left[\overline{\lambda^{xy}_{\uparrow L-}}\, \Lambda + \lambda^{xy}_{\uparrow L-}\, \overline{\Lambda} \right] \frac{P(x,y)}{\lambda^{xy}_{\uparrow L-}} = f(x,y)\, I_\uparrow\, P(x,y) + (\deg < 4)\,, \tag{7.2.20}$$

where we set $\lambda_{\uparrow cs} = \lambda_{ncs}$, $n \in \uparrow$. Here $f(x,y)$ can be any scalar function; it takes into account that the average of all eigenvalues drops out in the EL equations when we take the difference of the contributions resulting from the two terms in (5.5.13). Similar conditions for the opposite chirality and for $\uparrow$ replaced by $\downarrow$ are obtained from (7.2.20) by applying the rules (7.2.19, 7.2.18). We point out that the resulting four equations must clearly be satisfied for the same function $f(x,y)$. These four equations together are even equivalent to the EL equations to degree 4.

The remaining problem is to analyze the obtained equations of types (7.2.20). At first sight, this seems a difficult problem because the matrix Λ has a complicated explicit form (see Def. 7.2.1) and because taking the spectral adjoints makes it necessary to diagonalize these matrices. Fortunately, the requirement that the EL equations be mathematically consistent will give us strong restrictions on the form of Λ, and this will indeed make it possible to reveal a relatively simple global structure of the admissible interactions. In order to explain how the mathematical consistency conditions come about, we first recall that for polynomial Lagrangians (5.5.1) we saw after (5.5.4) that the resulting operator Q is a polynomial in the fermionic projector and is thus well-defined within the formalism of the continuum limit. However, the situation is different for our Lagrangian (5.5.14) because the spectral weight is an operation which does not necessarily make sense in the continuum limit. More specifically, the mathematical problem in (7.2.20) is to make sense of the spectral adjoint. For clarity, we explain the difficulty and our basic argument in the simple example

$$\overline{B_1\, M_1 + B_2\, M_2}\,, \tag{7.2.21}$$

where B_1 and B_2 are matrices depending on the macroscopic potentials, and $M_{1/2}$ are two monomials. The monomials can be considered as scalar functions which are highly singular on the light cone, and which we can control in the continuum limit only in the weak sense. To form the spectral adjoint in (7.2.21), we need to know the eigenvalues and spectral projectors of the matrix $B_1 M_1 + B_2 M_2$. In general, the spectral decomposition of this matrix will depend nonlinearly on M_1 and M_2, because the zeros of the characteristic polynomials involve roots of the monomials. In this generic situation, the spectral adjoint is ill-defined in the formalism of the continuum

limit. The only case in which the eigenvalues are linear in M_1 and M_2 is when the eigenvectors can be chosen independent of the monomials. This is possible iff the matrices B_1 and B_2 have a common eigenvector basis, or equivalently, if they commute,

$$[B_1, B_2] \;=\; 0\,.$$

This simple argument shows that the requirement that the spectral adjoint be well-defined leads to commutator relations for the macroscopic potentials. In the next lemma we apply this argument to the matrix Λ. By the *contributions to* Λ we mean the individual summands obtained by multiplying out all the terms in (7.2.10–7.2.14).

LEMMA 7.2.3. *For any $x, y \in \mathcal{L}$ there is a basis on the sectors such that the contributions to Λ are all diagonal matrices.*

Proof. Clearly, our argument after (7.2.21) applies in the same way to the spectral adjoint of a finite sum. Thus in order to make mathematical sense of the spectral adjoint $\overline{\Lambda}$, we need to assume that the contributions to Λ all commute with each other. Hence we can choose a basis such that these contributions are all diagonal. In particular, one sees that in this basis the matrix products $\hat{Y}_{c_1}(x)$ and $\hat{Y}_{c_2}(y)$ are diagonal for all $c_1, c_2 \in \{L, R\}$. ∎

We proceed by analyzing the EL equations (7.2.20) for special choices of x and y, for which the matrix Λ becomes particularly simple. We begin with the situation where we choose x such that the scalar potentials vanish at x, i.e.

$$Y_L(x) \;=\; Y \;=\; Y_R(x) \tag{7.2.22}$$

with Y the mass matrix of the vacuum (for example, we can choose x close to infinity). Then the matrices $Y_{L/R}(x)$ are diagonal in the sector index and on the massive sectors are a multiple of the identity. Thus the "off-diagonal" contributions (7.2.13, 7.2.14) to Λ vanish. In the "diagonal" contributions (7.2.10–7.2.12), on the other hand, we can simplify our notation by omitting the factors $\hat{Y}_{L/R}(x)$. Then the matrix Λ takes the form

$$\begin{aligned}
\Lambda \;=\;& \nu \int_x^y dz\; I_\uparrow \acute{Y}_L \grave{Y}_R\, I_\uparrow\, T^{(0)}_{[2]}\, \overline{T^{(0)}_{[0]}} \\
&+\nu \int_y^x dz\; I_\uparrow \acute{Y}_R \grave{Y}_L\, I_\uparrow\, T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \\
&+I_\uparrow \hat{Y}_L(y)\, I_\uparrow\, T^{(0)}_{[1]}\, \overline{T^{(0)}_{[1]}} \\
&-I_\uparrow \acute{Y}_R(y)\, \grave{Y}\, I_\uparrow\, T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}} \\
&-\frac{\nu\, T^{(0)}_{[1]}\, \overline{T^{(0)}_{[0]}} \;-\; \overline{\nu}\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}}}{\nu\lambda_- - \overline{\nu}\lambda_+}\; I_\uparrow \left(\hat{Y}_L(y)\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}} \;-\; \hat{Y}_R(y)\, T^{(-1)}_{[0]}\, \overline{T^{(0)}_{[1]}} \right) I_\uparrow\,.
\end{aligned}$$

Evaluating the EL equations (7.2.20) for this choice of Λ yields the following result.

LEMMA 7.2.4. *Suppose that (I) holds. Without introducing any relations between the basic fractions (besides those of Theorem 6.2.2), we can choose for any $y \in \mathcal{L}$*

suitable parameters $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$ such that at y,

$$I_\uparrow \acute{Y}_L \grave{Y}_R \, I_\uparrow \;=\; a \, I_\uparrow \,, \qquad I_\uparrow \acute{Y}_R \grave{Y}_L \, I_\uparrow \;=\; b \, I_\uparrow \tag{7.2.23}$$

$$I_\uparrow \hat{Y}_L(y) \, I_\uparrow \;=\; c \, I_\uparrow \,, \qquad I_\uparrow \hat{Y}_R(y) \, I_\uparrow \;=\; \overline{c} \, I_\uparrow \,. \tag{7.2.24}$$

The analogous formulas for $I_\uparrow$ interchanged by $I_\downarrow$ are obtained by the replacements

$$\uparrow \longleftrightarrow \downarrow \qquad \text{and} \qquad L \longleftrightarrow R \tag{7.2.25}$$

with the parameters a, b and c unchanged.

Proof. The above Λ contains contributions which are scalar multiples of the matrices $I_\uparrow \hat{Y}_L(y) I_\uparrow$ and $I_\uparrow \hat{Y}_R(y) I_\uparrow$. Thus in the basis of Lemma 7.2.3, these matrices are both diagonal. Since one is the adjoint of the other, we conclude that these matrices are normal, and thus their spectral adjoints coincide with the usual adjoints,

$$\overline{I_\uparrow \hat{Y}_L \, I_\uparrow} \;=\; I_\uparrow \hat{Y}_R \, I_\uparrow \,, \qquad \overline{I_\uparrow \hat{Y}_R \, I_\uparrow} \;=\; I_\uparrow \hat{Y}_L \, I_\uparrow \,. \tag{7.2.26}$$

The matrices $\acute{Y}_L \grave{Y}_R$ and $\acute{Y}_R \grave{Y}_L$, on the other hand, are Hermitian and thus spectrally selfadjoint,

$$\overline{\acute{Y}_L \, \grave{Y}_R} \;=\; \acute{Y}_L \, \grave{Y}_R \,, \qquad \overline{\acute{Y}_R \, \grave{Y}_L} \;=\; \acute{Y}_R \, \grave{Y}_L \,. \tag{7.2.27}$$

Applying the relations (7.2.26) and (7.2.27), a straightforward calculation gives

$$\begin{aligned}
&\overline{\lambda_{\uparrow L-}} \, \Lambda + \lambda_{\uparrow L-} \, \overline{\Lambda} \\
&= \int_x^y dz \; I_\uparrow \acute{Y}_L \grave{Y}_R \, I_\uparrow \left(T^{(0)}_{[0]} \, T^{(0)}_{[2]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \;+\; T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(0)}_{[0]} \, T^{(0)}_{[2]}} \right) \\
&\quad + \int_y^x dz \; I_\uparrow \acute{Y}_R \grave{Y}_L \, I_\uparrow \left(T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(-1)}_{[0]} \, T^{(1)}_{[2]}} \;+\; T^{(-1)}_{[0]} \, T^{(1)}_{[2]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \right) \\
&\quad + \overline{\nu} \; I_\uparrow \hat{Y}_L \, I_\uparrow \, T^{(0)}_{[0]} \, T^{(0)}_{[1]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[1]}} \;+\; \nu \; I_\uparrow \hat{Y}_R \, I_\uparrow \, T^{(-1)}_{[0]} \, T^{(0)}_{[1]} \, \overline{T^{(0)}_{[0]} \, T^{(0)}_{[1]}} \\
&\quad - \overline{\nu} \; I_\uparrow \acute{Y}_R \grave{Y} \, I_\uparrow \, T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(-1)}_{[0]} \, T^{(1)}_{[2]}} \;-\; \nu \, \overline{I_\uparrow \acute{Y}_R \grave{Y} \, I_\uparrow} \, T^{(-1)}_{[0]} \, T^{(1)}_{[2]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \\
&\quad + \left(\nu \, T^{(0)}_{[1]} \, \overline{T^{(0)}_{[0]}} \;-\; \overline{\nu} \, T^{(0)}_{[0]} \, \overline{T^{(0)}_{[1]}} \right) \; I_\uparrow \left(\hat{Y}_L \, T^{(0)}_{[1]} \, \overline{T^{(-1)}_{[0]}} \;-\; \hat{Y}_R \, T^{(-1)}_{[0]} \, \overline{T^{(0)}_{[1]}} \right) I_\uparrow \\
&= \int_x^y dz \; I_\uparrow \acute{Y}_L \grave{Y}_R \, I_\uparrow \left(T^{(0)}_{[0]} \, T^{(0)}_{[2]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \;+\; T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(0)}_{[0]} \, T^{(0)}_{[2]}} \right) \\
&\quad + \int_y^x dz \; I_\uparrow \acute{Y}_R \grave{Y}_L \, I_\uparrow \left(T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(-1)}_{[0]} \, T^{(1)}_{[2]}} \;+\; T^{(-1)}_{[0]} \, T^{(1)}_{[2]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \right) \\
&\quad + \nu \left(I_\uparrow \hat{Y}_L \, I_\uparrow \, T^{(0)}_{[1]} \, T^{(0)}_{[1]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \;-\; \overline{I_\uparrow \acute{Y}_R \grave{Y} \, I_\uparrow} \, T^{(-1)}_{[0]} \, T^{(1)}_{[2]} \, \overline{T^{(-1)}_{[0]} \, T^{(0)}_{[0]}} \right) \\
&\quad + \overline{\nu} \left(I_\uparrow \hat{Y}_R \, I_\uparrow \, T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(0)}_{[1]} \, T^{(0)}_{[1]}} \;-\; I_\uparrow \acute{Y}_R \grave{Y} \, I_\uparrow \, T^{(-1)}_{[0]} \, T^{(0)}_{[0]} \, \overline{T^{(-1)}_{[0]} \, T^{(1)}_{[2]}} \right) ,
\end{aligned}$$

where for simplicity the arguments y were omitted. We substitute this formula into (7.2.20). Since we do not allow for additional relations between the basic fractions, we can simplify the resulting simple fractions only by applying (6.2.9). This implies that (7.2.20) is satisfied for suitable $f(x,y)$ if and only if the following five matrices

are multiples of $I_\uparrow$,

$$\int_x^y I_\uparrow \acute{Y}_L \grave{Y}_R I_\uparrow , \qquad \int_y^x I_\uparrow \acute{Y}_R \grave{Y}_L I_\uparrow \tag{7.2.28}$$

$$\left.\begin{aligned} &\overline{\nu}\left(I_\uparrow \hat{Y}_R I_\uparrow - I_\uparrow \acute{Y}_R \grave{Y} I_\uparrow\right), \qquad \nu\, I_\uparrow \hat{Y}_L I_\uparrow - \overline{\nu}\, I_\uparrow \acute{Y}_R \grave{Y} I_\uparrow , \\ &\qquad\qquad \nu\, \overline{I_\uparrow \acute{Y}_L I_\uparrow} - \overline{\nu}\, I_\uparrow \acute{Y}_R I_\uparrow . \end{aligned}\right\} \tag{7.2.29}$$

We can assume that $y \neq x$, because otherwise (7.2.23) and (7.2.24) follow immediately from (7.2.22). Differentiating (7.2.28) with respect to y along the line $\mathcal{L}$ gives (7.2.23) (a and b are real because the matrices on the left of (7.2.23) are Hermitian). According to (I), the phase factor ν can take any value on the unit circle. Thus in (7.2.29) the contributions involving ν and $\overline{\nu}$ must separately be multiples of $I_\uparrow$. This gives the left relation in (7.2.24), and the relation on the right is obtained by taking the adjoint.

The analogous relations for $I_\uparrow$ replaced by $I_\downarrow$ are derived in the same way. The replacements (7.2.25) leave the phase factor ν unchanged (see (7.2.8) and (7.2.9)). Thus the EL equation (7.2.20) remains valid under (7.2.25) for the same function f only if the parameters a, b, and c are unchanged. ∎

Next we consider the the degeneracies in the limit $y \to x$. In this case, the formulas of Definition 7.2.1 simplify in that all phase factors drop out. We obtain the following result.

LEMMA 7.2.5. *Without introducing any relations between the basic fractions (besides those of Theorem 6.2.2), the dynamical mass matrices must satisfy the relations*

$$I_\uparrow \hat{Y}_L I_\downarrow \;=\; 0 \;=\; I_\uparrow \hat{Y}_R I_\downarrow . \tag{7.2.30}$$

Proof. According to the replacement rule (7.2.18), it suffices to derive the second part of (7.2.30). We compute the matrix Λ modulo scalar multiples of $I_\uparrow$. Using (7.2.23) and (5.3.20), we obtain

$$\begin{aligned} \Lambda \;&=\; I_\uparrow \hat{Y}_L I_\downarrow \hat{Y}_L I_\uparrow T^{(0)}_{[1]} \overline{T^{(0)}_{[1]}} \\ &\quad -\frac{1}{\lambda_- - \lambda_+}\, I_\uparrow \left(\hat{Y}_L T^{(0)}_{[1]} \overline{T^{(-1)}_{[0]}} - \hat{Y}_R T^{(-1)}_{[0]} \overline{T^{(0)}_{[1]}} \right) \\ &\qquad \times I_\downarrow \left(\hat{Y}_R T^{(0)}_{[1]} \overline{T^{(0)}_{[0]}} - \hat{Y}_L T^{(0)}_{[0]} \overline{T^{(0)}_{[1]}} \right) I_\uparrow \\ &=\; \left(I_\uparrow \hat{Y}_L I_\downarrow \hat{Y}_L I_\uparrow + I_\uparrow \hat{Y}_R I_\downarrow \hat{Y}_R I_\uparrow \right) \frac{\lambda_-}{\lambda_- - \lambda_+}\, T^{(0)}_{[1]} \overline{T^{(0)}_{[1]}} \\ &\quad -I_\uparrow \hat{Y}_L I_\downarrow \hat{Y}_R I_\uparrow \;\frac{1}{\lambda_- - \lambda_+}\, T^{(0)}_{[1]} T^{(0)}_{[1]} \overline{T^{(-1)}_{[0]} T^{(0)}_{[0]}} \\ &\quad -I_\uparrow \hat{Y}_R I_\downarrow \hat{Y}_L I_\uparrow \frac{1}{\lambda_- - \lambda_+}\, T^{(-1)}_{[0]} T^{(0)}_{[0]} \overline{T^{(0)}_{[1]} T^{(0)}_{[1]}} . \end{aligned}$$

Next we compute the square bracket in (7.2.20),

$$\begin{aligned}
&\overline{\lambda_{\uparrow L-}}\,\Lambda + \lambda_{\uparrow L-}\,\overline{\Lambda} \\
&\quad = \; -\frac{I_\uparrow \hat{Y}_L\, I_\downarrow \hat{Y}_R\, I_\uparrow}{\lambda_- - \lambda_+} \left(\lambda_+\, T^{(0)}_{[1]}\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}} \;-\; \lambda_-\, T^{(-1)}_{[0]}\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}\, T^{(0)}_{[1]}} \right) \\
&\qquad + \frac{I_\uparrow \hat{Y}_R\, I_\downarrow \hat{Y}_L\, I_\uparrow}{\lambda_- - \lambda_+} \left(\lambda_-\, T^{(0)}_{[1]}\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}} \;-\; \lambda_+\, T^{(-1)}_{[0]}\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}\, T^{(0)}_{[1]}} \right).
\end{aligned}$$

Since $I_\downarrow$ projects onto a subspace of dimension $7-p<p$, the rank of the matrices $I_\uparrow \hat{Y}_R I_\downarrow \hat{Y}_L I_\uparrow$ and $I_\uparrow \hat{Y}_L I_\downarrow \hat{Y}_R I_\uparrow$ is smaller than p, and therefore these matrices cannot be scalar multiples of $I_\uparrow$. Thus the EL equations have a well-defined continuum limit only if the factors $(\lambda_- - \lambda_+)^{-1}$ in the above expression drop out. This is the case only if

$$I_\uparrow \hat{Y}_R\, I_\downarrow \hat{Y}_L\, I_\uparrow \;=\; I_\uparrow \hat{Y}_L\, I_\downarrow \hat{Y}_R\, I_\uparrow \,.$$

If these necessary conditions are satisfied, the above formula simplifies to

$$\overline{\lambda_{\uparrow L-}}\,\Lambda + \lambda_{\uparrow L-}\,\overline{\Lambda} \;=\; I_\uparrow \hat{Y}_L\, I_\downarrow \hat{Y}_R\, I_\uparrow \left(T^{(0)}_{[1]}\, T^{(0)}_{[1]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}} \;+\; T^{(-1)}_{[0]}\, T^{(0)}_{[0]}\, \overline{T^{(0)}_{[1]}\, T^{(0)}_{[1]}} \right) .$$

Now the EL equations have a well-defined continuum limit, and assuming for the regularization parameters only the relation (6.2.9), we conclude that

$$I_\uparrow \hat{Y}_R\, I_\downarrow \hat{Y}_L\, I_\uparrow \;=\; 0\,. \tag{7.2.31}$$

The matrix product in this equation can be written in the form BB^* with $B \equiv I_\uparrow \hat{Y}_R I_\downarrow$. Hence (7.2.31) implies that $B=0$. ∎

The previous two lemmas simplify considerably the structure of the perturbation on the degenerate subspaces. Namely, we can write Λ in the form

$$\Lambda \;=\; \rho(\nu,\overline{\nu})\, I_\uparrow \;-\; I_\uparrow \acute{Y}_R(y)\, (I_\uparrow + \mu\nu\, I_\downarrow + \mu_8\nu_8\, I_8)\; \grave{Y}_L(x)\, I_\uparrow\, T^{(-1)}_{[0]}\, \overline{T^{(1)}_{[2]}}\,,$$

where ρ is a complex function which is invariant under the replacements (7.2.25). A short calculation yields

$$\begin{aligned}
&\overline{\lambda_{\uparrow L-}}\,\Lambda + \lambda_{\uparrow L-}\,\overline{\Lambda} \;=\; (a + \nu\, b + \overline{\nu}\,\overline{b})\, I_\uparrow && (7.2.32)\\
&\quad -\overline{\nu}\, I_\uparrow \acute{Y}_R(y)\, I_\uparrow \grave{Y}_L(x)\, I_\uparrow\, \overline{N} \;-\; \nu\, \overline{I_\uparrow \acute{Y}_R(y)\, I_\uparrow \grave{Y}_L(x)\, I_\uparrow}\, N && (7.2.33)\\
&\quad -\mu\, I_\uparrow \acute{Y}_R(y)\, I_\downarrow \grave{Y}_L(x)\, I_\uparrow\, \overline{N} \;-\; \overline{\mu}\, \overline{I_\uparrow \acute{Y}_R(y)\, I_\downarrow \grave{Y}_L(x)\, I_\uparrow}\, N && (7.2.34)\\
&\quad -\mu_8\, \overline{\nu}\nu_8\, I_\uparrow \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow\, \overline{N} \;-\; \overline{\mu_8}\, \nu\overline{\nu_8}\, \overline{I_\uparrow \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow}\, N\,, && (7.2.35)
\end{aligned}$$

where the complex functions a and b are invariant under (7.2.25), and N is the monomial

$$N \;=\; T^{(-1)}_{[0]}\, T^{(1)}_{[2]}\, \overline{T^{(-1)}_{[0]}\, T^{(0)}_{[0]}}\,. \tag{7.2.36}$$

We split up the analysis of the EL equations corresponding to (7.2.32–7.2.35) into several lemmas. We say that the summands (7.2.34) or (7.2.35) are *non-trivial* if there are admissible dynamical mass matrices such that this summand or one of the

expressions obtained by applying the replacements (7.2.18) and/or (7.2.19) are non-zero. Furthermore, we refer to two phase functions $\alpha, \beta \in S^1$ as being *independent* if for α fixed, β can take any value in S^1 and vice versa[1]

LEMMA 7.2.6. *Under the assumptions (I)–(III), ν is independent of the phase functions μ, μ_8, ν_8, and $\overline{\mu}\mu_8$. The term (7.2.35) is non-trivial.*

Proof. Suppose that the dynamical mass matrices were zero in the neutrino sector, i.e.

$$Y_L\, I_8 \;\equiv\; 0 \;\equiv\; Y_R\, I_8 \tag{7.2.37}$$

Then the Dirac operator, and thus also the fermionic projector, would be invariant on the neutrino sector. As a consequence, the chiral Dirac particles would drop out of all composite expressions due to chiral cancellations, in contradiction to (II). We conclude that (7.2.37) is false. Since we are free to choose a convenient representation of the dynamical mass matrices, we can assume that the matrices

$$\acute{Y}_L\, I_8\,, \qquad \acute{Y}_R\, I_8\,, \qquad \grave{Y}_L\, I_8\,, \qquad \grave{Y}_R\, I_8 \tag{7.2.38}$$

do not all vanish identically. The contributions to the fermionic projector which involve the matrix products $I_8 \acute{Y}_{L/R}$ or $\grave{Y}_{L/R} I_8$ enter only the perturbation calculation for the kernel of $P(x,y)\, P(y,x)$, and according to Theorem G.5.1 they drop out of the EL equations. Thus (II) is satisfied only if

$$(I_\uparrow + I_\downarrow)\, \acute{Y}_L\, I_8 \;\not\equiv\; 0 \qquad \text{or} \qquad (I_\uparrow + I_\downarrow)\, \acute{Y}_R\, I_8 \;\not\equiv\; 0\,.$$

This shows that (7.2.35) is non-trivial.

According to (III), there is a free dynamical gauge field which couples differently to the Dirac particles in the sectors $n = 1$ and $n = 8$. The corresponding free gauge potentials describe relative phase shifts of the fermionic projector on $\mathrm{Im}\, I_1$ and $\mathrm{Im}\, I_8$. These relative phases are captured by μ_8 and $\mu_8 \nu_8$ (see (7.2.9)). Since the free gauge potentials on the line segment $\overline{xy}$ can be chosen arbitrarily, it follows that ν is independent of μ_8 and $\mu_8 \nu_8$. A similar argument for I_7 instead of I_1 shows that ν and $\overline{\mu}\mu_8$ are independent. ∎

LEMMA 7.2.7. *Imposing at most one additional relation between the basic fractions (besides those of Theorem 6.2.2), ν and μ are independent. The term (7.2.34) is non-trivial.*

Proof. Assume to the contrary that ν and μ are dependent or that (7.2.34) is trivial. Then the phases in (7.2.32–7.2.34) are all dependent on ν. The independence of the phases established in Lemma 7.2.6 yields that the EL equations must be satisfied separately for (7.2.35). Imposing at most one additional relation between the basic fractions, we cannot arrange that (7.2.35) drops out of the EL equations. We thus obtain that for a suitable complex κ,

$$I_\uparrow\, \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow \;=\; \kappa(x,y)\, I_\uparrow\,, \tag{7.2.39}$$

and this condition must also be satisfied after the replacements (7.2.18) and/or (7.2.19) for the same κ. Since the rank of I_8 is smaller than that of $I_\uparrow$, the lhs of (7.2.39) is a

[1] Online version: The arguments in the following lemmas need to be modified if we allow for a local chiral transformation as considered in the book [**5**] (listed in the references in the preface to the second online edition).

singular matrix, and thus κ vanishes identically. This implies that the lhs of (7.2.39) is trivial (i.e. vanishes also after the replacements (7.2.18, 7.2.19)), in contradiction to Lemma 7.2.6. ∎

Having established that the phases in (7.2.32) and (7.2.33) are independent of those in (7.2.34) and (7.2.35), we can now apply the uniform splitting lemma to (7.2.33).

LEMMA 7.2.8. *Imposing at most one additional relation between the basic fractions, we can arrange by a constant unitary transformation that for all $a, b = 1, \ldots, p$ and $c, d = p+1, \ldots, 7$,*

$$(I_\uparrow \grave{Y}_{L/R}\, I_\uparrow)_b^{(a\alpha)} \;=\; \delta_b^a\, u_{L/R}^\alpha\,, \qquad (I_\downarrow \grave{Y}_{R/L}\, I_\downarrow)_d^{(c\alpha)} \;=\; \delta_d^c\, u_{L/R}^\alpha \tag{7.2.40}$$

with $u_{L/R}(x) \in \mathbb{C}^3$.

Proof. It clearly suffices to consider one chirality. Since ν is independent of μ and $\mu_8\nu_8$, the EL equations imply that

$$I_\uparrow \acute{Y}_R(y)\, I_\uparrow \grave{Y}_L(x)\, I_\uparrow \;=\; \lambda(x,y)\, I_\uparrow\,. \tag{7.2.41}$$

The dynamical mass matrices can be chosen independently at x and y. Denoting the class of admissible matrices $I_\uparrow \grave{Y}_L I_\uparrow$ by $\mathcal{B}$, we are in the setting of Lemma 7.1.3 with $p_1 = p$ and $p_2 = 3p$. Since p_2 is divisible by p_1, we can, possibly after increasing r, assume that $p_2 - r p_1 = 0$, and thus

$$I_\uparrow \grave{Y}_L\, I_\uparrow \;=\; U\, (\underbrace{u_L \oplus \cdots \oplus u_L}_{p \text{ summands}})$$

with $u_L(x) \in \mathbb{C}^3$. Omitting the constant unitary transformation and writing out the components, this is just the lhs of (7.2.40). Under the replacement (7.2.25), ν as well as α and β are unchanged. As a consequence, also the function λ in (7.2.41) is invariant under (7.2.25), and this implies that the mappings π_i of Lemma 7.1.3 obtained for $B = I_\uparrow \grave{Y}_L I_\uparrow$ and $B = I_\downarrow \grave{Y}_R I_\downarrow$ are all unitarily equivalent. This proves the rightmost equation of (7.2.40). ∎

It remains to analyze (7.2.34) and (7.2.35).

LEMMA 7.2.9. *The EL equations to degree 4 can be satisfied only if we impose at least one additional relation between the basic fractions.*

Proof. In the limit $y \to x$, the matrices $I_\uparrow \acute{Y}_R(y)\, I_\cdot\, \grave{Y}_L(x)\, I_\uparrow$ can be written in the form $B^* B$ with $B = I_\cdot\, \grave{Y}_L\, I_\uparrow$ and are therefore Hermitian and positive semidefinite. This shows that (7.2.34) and (7.2.35) cannot cancel each other identically. According to Lemma 7.2.7, (7.2.34) is non-trivial. It suffices to consider the case that (7.2.34) does not vanish identically (in the other cases when (7.2.34) is non-zero after applying (7.2.18, 7.2.19) the argument is analogous). Then we can arrange a contribution to (7.2.33–7.2.35) of the form $(\mu A N + \overline{\mu A N})$ with a matrix $A \neq 0$. The same contribution must be present after performing the replacements (7.2.25). Since these replacements transform μ into $\overline{\mu}$ (see (7.2.18) and (7.2.19)), we obtain a condition of the form

$$\mu A N + \overline{\mu A N} \;=\; \overline{\mu} B N + \mu \overline{B N} \qquad \text{for all } \mu \in S^1 \tag{7.2.42}$$

with B a matrix. Without introducing an additional relation between the basic fractions, we must treat N and $\overline{N}$ as being independent, and thus (7.2.42) has no solution. ∎

Using that A and B go over to positive matrices as $y \to x$, one sees that in order to arrange that (7.2.42) has a solution, we need to impose that N and $\overline{N}$ coincide in the EL equations, i.e.[2]

$$(N - \overline{N}) \left(\overline{T^{(0)}_{[0]}} \right)^{-1} = 0 \,. \tag{7.2.43}$$

The next lemma is again an application of the uniform splitting lemma and uses the non-degeneracy assumption of Def. 7.1.2.

LEMMA 7.2.10. *Suppose that the basic fractions satisfy (in addition to the conditions of Theorem 6.2.2) the relation (7.2.43) with N according to (7.2.36). Then the parameter p in (6.2.8) is equal to 4. The phase factors in the neutrino sector are determined by*

$$\nu_8 = \nu \qquad \text{and} \qquad \mu_8 = \mu \text{ or } \overline{\mu} \,. \tag{7.2.44}$$

We can arrange by constant unitary transformations that for $a, b = 1, 2, 3$,

$$(I_\downarrow \grave{Y}_{L/R}\, I_\uparrow)^{(a+4\,\alpha)}_b = \delta^a_b\, v^\alpha_{L/R} \,, \qquad (I_\uparrow \grave{Y}_{R/L}\, I_\downarrow)^{(a\alpha)}_{b+4} = \delta^a_b\, \overline{v^\alpha_{L/R}} \tag{7.2.45}$$

with $v_{L/R}(x) \in \mathbb{C}^3$. In the two cases for μ_8 in (7.2.44),

$$(I_8\, \grave{Y}_{L/R}\, I_\uparrow)^{(8\alpha)}_4 = v^\alpha_{L/R} \text{ or } \overline{v^\alpha_{L/R}} \,, \tag{7.2.46}$$

respectively. Furthermore,

$$I_8\, \grave{Y}_{R/L}\, I_\downarrow = 0 \,. \tag{7.2.47}$$

Proof. Imposing (7.2.43) and using (7.2.40), the EL equations (7.2.20) reduce to the conditions

$$\begin{aligned} \lambda(x,y)\, I_\uparrow &= \mu\, I_\uparrow \acute{Y}_R(y)\, I_\downarrow \grave{Y}_L(x)\, I_\uparrow + \overline{\mu\, I_\uparrow \acute{Y}_R(y)\, I_\downarrow \grave{Y}_L(x)\, I_\uparrow} \\ &\quad + \mu_8\, \overline{\nu}\nu_8\, I_\uparrow \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow + \overline{\mu_8\, \nu\overline{\nu_8}\, I_\uparrow \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow} \,. \end{aligned} \tag{7.2.48}$$

We first prove that the phase factors must be dependent in the sense that

$$\mu_8\, \overline{\nu}\nu_8 = \mu \text{ or } \overline{\mu} \,. \tag{7.2.49}$$

Assuming the contrary, we must treat the four summands in (7.2.48) as being independent, and thus

$$I_\uparrow \acute{Y}_R(y)\, I_\downarrow \grave{Y}_L(x)\, I_\uparrow = \kappa(x,y)\, I_\uparrow \,. \tag{7.2.50}$$

Performing the replacement (7.2.25) and using that μ transforms to $\overline{\mu}$, we obtain furthermore that

$$I_\downarrow \acute{Y}_L(y)\, I_\uparrow \grave{Y}_R(x)\, I_\downarrow = \kappa(x,y)\, I_\downarrow \tag{7.2.51}$$

[2] Online version: For a difficulty to realize this relation between the basic fractions by a suitable regularization see the proof of Lemma 3.10.3 in the book [**5**] (listed in the references in the preface to the second online edition). One should keep in mind that, following the consideration after [**5**, eq. (3.7.13)], the relation must still hold if we replace the factors $T^{(1)}_{[2]}$ and $\overline{T^{(1)}_{[2]}}$ by a non-zero real constant.

with the same κ as in (7.2.50). We apply Lemma 7.1.3 to (7.2.50) (with $p_1 = p$ and $p_2 = 3(7-p)$) and to (7.2.51) (with $p_1 = 7-p$ and $p_2 = p$). Leaving out the constant unitary transformations, we obtain the representations

$$I_\downarrow \grave{Y}_L \, I_\uparrow \;=\; \begin{pmatrix} \overbrace{b \oplus \cdots \oplus b}^{p \text{ summands}} \\ 0 \end{pmatrix}, \qquad I_\uparrow \grave{Y}_R \, I_\downarrow \;=\; \begin{pmatrix} \overbrace{\overline{b} \oplus \cdots \oplus \overline{b}}^{7-p \text{ summands}} \\ 0 \end{pmatrix}, \tag{7.2.52}$$

where $\overline{b}$ is the complex conjugate of the vector $b \in \mathbb{C}^3$. According to Lemma 7.2.7, (7.2.34) is non-trivial. Since the contributions to the EL equations involving μ are unchanged when applying the replacements (7.2.18) and (7.2.19), we can arrange that (7.2.50) does not vanish, and thus $b \neq 0$. On the lhs of (7.2.52), the inequality $rp_1 \leq p_2$ implies that $r < 3$. Thus on the rhs of (7.2.52), the number of zero rows is $3p - r(7-p) > 3$. Therefore, $I_p \grave{Y}_R I_\downarrow = 0$, or, equivalently, by taking the adjoint and in components,

$$(\acute{Y}_L)^a_{(d.)} \;=\; 0 \qquad \text{for } d = p \text{ and } a = p+1, \ldots, 7.$$

On the other hand, the lhs of (7.2.52) implies that

$$(\grave{Y}_L)^a_{(d.)} \;\neq\; 0 \qquad \text{for } d = p \text{ and } a = p+1, \ldots, 7.$$

The non-degeneracy assumption of Def. 7.1.2 allows us to conclude that

$$(\acute{Y}_L)^a_{(d.)} \;=\; 0 \qquad \text{for all } a = p+1, \ldots, 7 \text{ and } d \neq a.$$

This implies that $I_\downarrow \acute{Y}_L I_\uparrow = 0$, in contradiction to the rhs of (7.2.52) and the fact that $b \neq 0$.

Repeating the above argument for the opposite chirality gives in analogy to (7.2.49) that

$$\mu_8 \;=\; \mu \text{ or } \overline{\mu}\,. \tag{7.2.53}$$

Using that μ and ν are independent according to Lemma 7.2.7, (7.2.49) and (7.2.53) are equivalent to (7.2.44).

In the case $\mu_8 = \mu$, the EL equations (7.2.20) reduce to the conditions

$$I_\uparrow \acute{Y}_R(y) \, (I_\downarrow + I_8) \, \grave{Y}_L(x) \, I_\uparrow \;=\; \kappa(x,y) \, I_\uparrow\,. \tag{7.2.54}$$

After the replacement (7.2.25), the phase factors in (7.2.48) are no longer dependent (cf. (7.2.18) and (7.2.19)), and thus we get the conditions

$$I_\downarrow \acute{Y}_L(y) \, I_\uparrow \grave{Y}_R(x) \, I_\downarrow \;=\; \overline{\kappa(x,y)} \, I_\downarrow \tag{7.2.55}$$

$$I_\downarrow \acute{Y}_L(y) \, I_8 \grave{Y}_R(x) \, I_\downarrow \;=\; 0\,. \tag{7.2.56}$$

The last relation implies (7.2.47). Applying the above argument for (7.2.50) and (7.2.51) to (7.2.54) and (7.2.55), we again get a contradiction unless $\mathrm{Rg} I_\uparrow = \mathrm{Rg}(I_\downarrow + I_8)$. This shows that $p = 4$. Possibly after increasing r, we obtain in analogy to (7.2.52) the representations

$$(I_\downarrow + I_8) \, \grave{Y}_L \, I_\uparrow \;=\; b \oplus b \oplus b \oplus b\,, \qquad I_\uparrow \grave{Y}_R \, I_\downarrow \;=\; \overline{b} \oplus \overline{b} \oplus \overline{b}\,. \tag{7.2.57}$$

Writing these relations in components gives (7.2.45, 7.2.46).

In the case $\mu_8 = \overline{\mu}$, we obtain in analogy to (7.2.54) the condition

$$I_\uparrow \acute{Y}_R(y) \, I_\downarrow \grave{Y}_L(x) \, I_\uparrow \;+\; \overline{I_\uparrow \acute{Y}_R(y) \, I_8 \grave{Y}_L(x) \, I_\uparrow} \;=\; \kappa(x,y) \, I_\uparrow\,, \tag{7.2.58}$$

and after the replacement (7.2.25) again the conditions (7.2.55, 7.2.56). The lhs of (7.2.55) can be split into a product of matrices of the form $A(y)\,B(x)$. Since the equations (7.2.55) and (7.2.58) involve the same function $\kappa(x,y)$, the matrices on the lhs of (7.2.58) must split in the same way. To this end, the matrix $I_\uparrow \acute{Y}_R(y)\, I_8\, \grave{Y}_L(x)\, I_\uparrow$ must (possibly after a constant unitary transformation) be diagonal for all x and y, so that the spectral adjoint reduces to the complex conjugate (i.e. to taking the complex conjugate of all matrix entries). After taking this complex conjugate, we can proceed exactly as in the case $\mu_8 = \mu$ above. The only difference is that we obtain a representation not for the matrix $I_8 \grave{Y}_L I_\uparrow$ but for its complex conjugate, and this leads to the complex conjugate in (7.2.46). ∎

We remark that the fact that the partial trace is non-cyclic, (7.1.2), is essential for the above construction to work. Namely, according to the lhs of (7.2.45),

$$I_\uparrow \acute{Y}_L\, I_8\, \grave{Y}_R\, I_\uparrow \quad \overset{\text{in general}}{\neq} \quad 0\,. \tag{7.2.59}$$

On the other hand, the weak causality compatibility condition, Def. 7.1.1, implies that

$$I_8\, \acute{Y}_R\, I_\uparrow\, \grave{Y}_L\, I_8 \;=\; X_R\, I_8\, \acute{Y}_R\, I_\uparrow\, \grave{Y}_L\, I_8 \;=\; 0\,. \tag{7.2.60}$$

If the partial trace were cyclic, (7.2.59) and (7.2.60) would be inconsistent.

Combining the previous lemmas and choosing a convenient representation for the dynamical mass matrices gives the main result of this section.

THEOREM 7.2.11. **(spontaneous block formation)** *We consider the EL equations corresponding to the Lagrangian (5.5.15) in the presence of chiral and scalar potentials (6.0.1–6.0.5) to the degree 4 on the light cone. We assume that the Dirac operator is weakly causality compatible and that the dynamical mass matrices are non-degenerate (see Defs. 7.1.1 and 7.1.2). Then, following (IV), we need to introduce two relations between the basic fractions. Imposing that*

$$(M - \overline{M})\, \overline{T^{(0)}_{[0]}}^{-1} \;=\; 0 \;=\; (N - \overline{N})\, \overline{T^{(0)}_{[0]}}^{-1} \tag{7.2.61}$$

with M and N according to (6.2.6, 7.2.36), we can arrange by constant unitary transformations that the Dirac operator is of the following form,

$$i\partial\!\!\!/ - m\,\chi_L \left(Y^q_R \oplus Y^q_R \oplus Y^q_R \oplus Y^l_R\right) \;-\; m\,\chi_R \left(Y^q_L \oplus Y^q_L \oplus Y^q_L \oplus Y^l_L\right) \tag{7.2.62}$$

$$+ (\chi_R\, A\!\!\!/_L + A\!\!\!/_V) \left(\sigma^3 \oplus \sigma^3 \oplus \sigma^3 \oplus \sigma^3\right) \tag{7.2.63}$$

$$+ (A\!\!\!/^q\, \mathbb{1}) \oplus 0_{\mathbf{C}^2} \;+\; 0_{\mathbf{C}^6} \oplus (A\!\!\!/^l\, \mathbb{1} + A\!\!\!/^s\, \sigma^3)\,. \tag{7.2.64}$$

Here $Y^{q/l}_{L/R}$ are 2×2 matrices on the sectors which depend also on the generations, i.e. in components

$$Y^{q/l}_c \;=\; (Y^{q/l}_c)^{(a\alpha)}_{(b\beta)} \qquad \text{with } a,b = 1,2,\ \alpha,\beta = 1,2,3, c = L/R.$$

The chiral and vector potentials are trivial on the generations and depend only on the sector index. A_L, A_V and A^l are vector fields, and A^q is a 3×3 matrix potential ($\mathbb{1}$ and σ^3 are Pauli matrices). The vector field A^s is a function of A_L and A_R; the two possible choices are

$$A^s \;\equiv\; 0 \qquad \text{or} \qquad A^s \;\equiv\; -A_L - 2A_V\,. \tag{7.2.65}$$

The dynamical gauge groups (see Def. 6.1.1) are given by

$$\mathcal{G} \;=\; U(1)_L \times \mathcal{F}\,, \qquad\qquad \mathcal{F} \;=\; U(1)_V \times U(3)^q \times U(1)^l\,, \tag{7.2.66}$$

where the indices clarify to which potentials in the Dirac operator the groups correspond.

Proof. Lemmas 7.2.4 and 7.2.5 do not immediately apply here because they are based on the assumption that we have only one relation between the basic fractions. But it is straightforward to check that if in these lemmas we allowed for an additional relation between the basic fractions, the argument of Lemma 7.2.9 would still go through, thus making it necessary to introduce a third relation between the basic fractions.

Collecting the results of Lemmas 7.2.4–7.2.10 and choosing a convenient representation for the dynamical mass matrices, we obtain that the dynamical mass matrices are block diagonal as in (7.2.62). Thus it remains to derive the dynamical gauge group and the form of the corresponding gauge potentials. Possibly after reordering the sectors, the $U(1)_L$ is precisely the group $\mathcal{B}_4$ in Def. 6.2.1. The free gauge group is obtained by taking the maximal subgroup of $\mathcal{F}_4$ for which the gauge potentials respect the phase conditions (7.2.44). In the two cases in (7.2.65), the $U(1)_V$ shifts the phases of μ and μ_8 by the same or the opposite amount, respectively. This corresponds to the two cases in (7.2.44). The other potentials must leave the phase functions unchanged, and thus they must coincide on the sectors which are mapped into each other by the dynamical mass matrices (7.2.45). This gives the group $U(3)^q \times U(1)^l$. ∎

We point out that, except for the potentials A^q, the Dirac operator splits into four direct summands. The first three summands are identical and involve massive Dirac particles, whereas the chiral Dirac particles are contained in the last summand. The gauge potentials A^q describe an interaction between the Dirac particles in the three identical summands. In analogy to the standard model, it is natural to identify the fermions in the first three and the last summands with the quarks and leptons, respectively. In order to make these notions precise, we first observe that for the fermionic projector, the above splitting means that for all contributions considered so far[3],

$$P(x,y) \;=\; U(x,y) \Big(P^q \oplus P^q \oplus P^q \oplus P^l \Big)\,, \tag{7.2.67}$$

where U is the generalized phase transformation by the potentials A^q,

$$U(x,y) \;=\; \text{Pexp}\left(-i \int_0^1 d\tau\; A^q_j(\tau y + (1-\tau)x)\, (y-x)^j \right). \tag{7.2.68}$$

The unitary transformation (7.2.68) clearly commutes with the direct sum and thus drops out of the closed chain,

$$A_{xy} \;=\; A^q_{xy} \oplus A^q_{xy} \oplus A^q_{xy} \oplus A^l_{xy} \qquad \text{with} \qquad A^{q/l}_{xy} \;\equiv\; P^{q/l}(x,y)\, P^{q/l}(y,x)\,. \tag{7.2.69}$$

DEF. 7.2.12. *The first three direct summands in (7.2.67) and (7.2.69) are referred to as the* **quark blocks**. *The last direct summand is the* **lepton block**.

[3]We remark for clarity that that the contributions to the fermionic projector which involve the $U(3)^q$ gauge fields or currents, which have not been considered so far, do not split in the form (7.2.67).

7.3. The Dynamical Mass Matrices in the Quark and Neutrino Blocks

We now specify the dynamical mass matrices in the quark and neutrino blocks.

THEOREM 7.3.1. *Under the assumptions of Theorem 7.2.11, the EL equations are satisfied to degree 4 on the light cone if and only if the matrices $Y^q_{L/R}$ and $Y^l_{L/R}$ in (7.2.62) have (after suitable constant unitary transformations) the following properties at all space-time points,*

$$\hat{Y}^q_L = (\hat{Y}^q_R)^* = \begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix} \tag{7.3.1}$$

$$\hat{Y}^l_L = (\hat{Y}^l_R)^* = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \tag{7.3.2}$$

$$\grave{Y}^q_L = \begin{pmatrix} a & \mathcal{V}_L\, \overline{b} \\ \mathcal{U}_L\, b & \overline{a} \end{pmatrix}, \qquad \grave{Y}^q_R = \begin{pmatrix} \overline{a} & \mathcal{V}_R\, \overline{b} \\ \mathcal{U}_R\, b & a \end{pmatrix} \tag{7.3.3}$$

and in the two cases in (7.2.65),

$$\grave{Y}^l_L = \begin{pmatrix} a & 0 \\ \mathcal{W}_L\, b & 0 \end{pmatrix}, \qquad \grave{Y}^l_R = \begin{pmatrix} \overline{a} & * \\ \mathcal{W}_R\, b & * \end{pmatrix} \tag{7.3.4}$$

and

$$\grave{Y}^l_L = \begin{pmatrix} a & 0 \\ \mathcal{W}_L\, \overline{b} & 0 \end{pmatrix}, \qquad \grave{Y}^l_R = \begin{pmatrix} \overline{a} & * \\ \mathcal{W}_R\, \overline{b} & * \end{pmatrix}, \tag{7.3.5}$$

respectively. Here we use a matrix notation in the sector index. In (7.3.3–7.3.5), the matrix entries are vectors in $\mathbb{C}^3$ (and this takes into account the dependence on the generations). The parameter c is complex, $a, b \in \mathbb{C}^3$, and the stars stand for any vectors in $\mathbb{C}^3$. The off-diagonal elements are non-trivial in the sense that there is a space-time point where $b \neq 0$. The matrices $\mathcal{U}_{L/R}$, $\mathcal{V}_{L/R}$, $\mathcal{W}_{L/R} \in U(3)$ are constant unitary transformations.

Proof. We only consider the first case in (7.2.65); the second is obtained in the same way keeping track of the complex conjugates. The weak causality condition of Def. 7.1.1 implies that $\grave{Y}_L I_8 = 0$. On the other hand, we already observed after (7.2.38) that the matrix product $\grave{Y}_R I_8$ enters only the perturbation calculation for the kernel, which is trivial according to Theorem G.5.1. This explains the zeros and stars in (7.3.2, 7.3.4). Then (7.3.1, 7.3.2) follow immediately from Lemma 7.2.4 and Lemma 7.2.5. A short calculation using (7.3.1, 7.2.61, 7.2.44) yields that the EL equations to degree 4 reduce to the conditions

$$\int_x^z dz\; I_\uparrow \acute{Y}_L \grave{Y}_R\, I_\uparrow = \alpha(x,y)\, I_\uparrow \tag{7.3.6}$$

$$\overline{\nu}\, I_\uparrow \acute{Y}_R(y)\, I_\uparrow \grave{Y}_L(x)\, I_\uparrow + \overline{\nu\, I_\uparrow \acute{Y}_R(y)\, I_\uparrow \grave{Y}_L(x)\, I_\uparrow} = \beta(x,y)\, I_\uparrow \tag{7.3.7}$$

$$\mu\, I_\uparrow \acute{Y}_R(y)\, (I_\downarrow + I_8)\, \grave{Y}_L(x)\, I_\uparrow + \overline{\mu}\, \overline{I_\uparrow \acute{Y}_R(y)\, (I_\downarrow + I_8)\, \grave{Y}_L(x)\, I_\uparrow} = \gamma(x,y)\, I_\uparrow \tag{7.3.8}$$

as well as to the conditions obtained by the replacements

$$L \longleftrightarrow R\,, \qquad \nu \longleftrightarrow \overline{\nu} \tag{7.3.9}$$

and/or

$$I_\uparrow + I_8 \longrightarrow I_\downarrow\,, \qquad I_\downarrow \longrightarrow I_\uparrow\,, \qquad \mu \longleftrightarrow \overline{\mu}\,, \qquad \nu \longleftrightarrow \overline{\nu} \tag{7.3.10}$$

with the complex functions α, β, and γ unchanged. We first substitute (7.2.40) into (7.3.7). Comparing with the relation obtained by applying (7.3.9), one sees that $\langle u_L, u_L \rangle = \overline{\langle u_R, u_R \rangle}$, and thus we can arrange with a constant unitary transformation that $u_L = \overline{u_R}$. This explains the diagonal entries in (7.3.3, 7.3.4). Substituting (7.2.45) and (7.2.46) into (7.3.8) and comparing with the relation obtained by applying (7.3.10), we obtain similarly that $\langle u_R, u_R \rangle = \overline{\langle u_L, u_L \rangle}$ and thus, up to a constant unitary transformation, $v_L = \overline{v_L}$. Since we have already used the freedom in choosing orthogonal bases in order to arrange that $u_L = \overline{u_R}$, we now need to take into account these unitary transformations. This gives the off-diagonal elements in (7.3.3, 7.3.4). We conclude that (7.3.1–7.3.4) are necessary conditions. Substituting (7.3.1–7.3.4) into (7.3.6–7.3.8) and applying (7.3.9, 7.3.10), one verifies immediately that these conditions are also sufficient. The last statement in Lemma 7.2.7 implies that b is non-trivial. ∎

In Chapters 6 and 7 we always restricted attention to our model variational principle (5.5.14). We now make a few general comments on how our methods could be extended to other two-point actions, and which features of the Lagrangian are important for getting a physically interesting continuum limit.

The methods of Chapter 6 immediately apply to other two-point actions; the only obstruction is that the gauge terms in the operator $Q(x, y)$ must be simple fractions in $T_\circ^{(n)}$ and $\overline{T_\circ^{(n)}}$. The general mechanism is that the eigenvalues of A_{xy} are influenced by the gauge terms (cf. (6.1.10, 6.2.12, 6.2.13)). When analyzed in the EL equations, this leads to conditions for the eigenvalues of the "phase matrices" W_c (see (6.1.15) or (6.2.22)), and these conditions can finally be translated into constraints for the dynamical gauge fields. In this last step one uses crucially that the EL equations are nonlocal in the sense that they yield relations between the chiral potentials even at distant points (see e.g. (6.2.27)). This gives rise to *global constraints*, i.e. conditions which must hold in all of space-time. For example, Theorem 6.2.2 states that the dynamical gauge group in case (2) must be contained in one of the groups $(\mathcal{G}_p)_{p=0,\ldots,3}$ in the whole space-time, but it cannot be the group $\mathcal{G}_{p_1}$ in one region of space-time and a different group $\mathcal{G}_{p_2}$ in another region (as one sees by considering line integrals which join the two regions).

For the spontaneous block formation, it is essential that the EL equations are satisfied only if the eigenvalues of A_{xy} are highly degenerate. The requirement that these degeneracies should be respected by the scalar potentials can then be used to show that the potentials must split globally into a direct sum.

While this general mechanism should occur similarly for most other Lagrangians, the details depend sensitively on the particular form of the action. Our model Lagrangian has the special feature that it involves only the absolute squares of the eigenvalues of A_{xy}. This is the reason why Theorem 6.1.2 involves only the absolute squares of ν_{nc}, (6.1.15), leading to the relatively weak constraint for the dynamical gauge group (6.1.16) (if we had, for example, considered instead the polynomial Lagrangian (5.5.4), the gauge terms to highest degree would have led to conditions also for the phases of ν_{nc}, giving rise to much stronger conditions). To the next lower degree on the light cone, the phases of ν_{nc} do enter the analysis. But since perturbing the absolute square gives rise to a real part, $\Delta|\lambda_{ncs}|^2 = 2\mathrm{Re}(\overline{\lambda_{ncs}}\, \Delta\lambda_{ncs})$, we can easily arrange that only the real part of ν_{ncs} comes into play, and so the phases are fixed only up to signs. This is a major advantage of our action over e.g. polynomial actions,

where the same flexibility for the phases can be arranged only for a large degree of the polynomial. Another action which has the nice property that it depends only on the absolute squares of the eigenvalues is the determinant action (5.5.12). Working with the spectral trace leads to the specific problem that one must handle spectral adjoints. This is clearly a technical complication, but we do not consider it to be essential for the spontaneous block formation.

REMARK 7.3.2. *(Massive neutrinos)* In the analysis of Chapters 6 and 7, the structure of the neutrino sector was used several times: In the vacuum, the chiral cancellations were useful because, as a consequence, the EL equations were trivially satisfied in the neutrino sector (see (5.3.2) and (5.4.11)). In Theorem 6.1.2, the chiral cancellations in the neutrino sector are the reason why the dynamical gauge fields are not allowed to describe a mixing between the neutrinos and the massive fermions (see the argument leading to (6.1.22)). In the proof of Theorem 6.2.2, it was essential that the number of massive sectors is odd (see (6.2.30)). Finally, in the analysis of the degeneracies we always treated the neutrino sector separately.

Generally speaking, the chiral fermions lead to complications in the case with interaction, because the dynamical gauge fields were not allowed to describe a mixing between the massive and the chiral fermions, and this made it necessary to take scalar potentials into account. Also in view of recent experimental observations, it thus seems tempting to consider a neutrino sector which is built up of massive chiral Dirac seas. This is indeed possible, although we see the following difficulties. First, it is not clear how chiral fermions should be described in Minkowski space (for details see §C.1). Furthermore, building in massive chiral fermions is certainly not easy. Namely, if the resulting neutrino sector does not give rise to chiral cancellations, we must extend the Lagrangian in order to arrange that the EL equations are satisfied in the vacuum. The analysis of the interaction would also be considerably different. Finally, one should keep in mind that the recent experiments do not measure the mass of the neutrinos directly, but merely observe neutrino oscillations, i.e. a mixing of the neutrinos in different generations. This mixing could also be explained for massless neutrinos if the interaction of the neutrinos were suitably modified. For these reasons, we feel that before moving on to massive neutrinos, one should first get a better understanding of variational principles for a massless neutrino sector.

CHAPTER 8

The Effective Gauge Group

In this chapter we will reformulate the interaction of the Dirac particles with chiral and scalar fields as specified in Theorems 7.2.11 and 7.3.1 as an interaction via "effective" non-Abelian gauge fields. Before working out the details in §8.1 and §8.2, we begin by explaining the general construction. Consider the Dirac equation in the presence of chiral and scalar potentials (6.0.5). Since the dynamical mass matrix $Y_L = (Y_L)^{(a\alpha)}_{(b\beta)}$ (with $a, b = 1, \ldots, 8$ and $\alpha, \beta = 1, 2, 3$) need not be Hermitian, we cannot diagonalize it by a unitary transformation. But using the polar decomposition, we can at least represent Y_L in the form

$$Y_L \;=\; U_L\, Y^{\text{eff}}\, U_R^{-1} \tag{8.0.1}$$

with two unitary matrices $U_{L/R} \in U(3 \times 8)$ and Y^{eff} a diagonal matrix with real non-negative entries[1]. We introduce the so-called *chiral transformation* V by

$$V \;=\; \chi_L\, U_R \;+\; \chi_R\, U_R\,. \tag{8.0.2}$$

Note that the adjoint of V,

$$V^* \;=\; \chi_R\, U_L^{-1} \;+\; \chi_L\, U_R^{-1}\,,$$

is in general different from its inverse, which we denote by a bar,

$$\overline{V} \;\equiv\; V^{-1} \;=\; \chi_L\, U_L^{-1} \;+\; \chi_R\, U_R^{-1}\,.$$

Thus the chiral transformation need not be unitary. The chiral transformation of the Dirac operator is defined by and computed to be

$$\overline{V}^*\; (i\partial\!\!\!/ + \chi_L(A\!\!\!/_R - mY_R) + \chi_R(A\!\!\!/_L - mY_L))\, \overline{V} \;=\; i\partial\!\!\!/ + \chi_L\, A\!\!\!/_R^{\text{eff}} + \chi_R\, A\!\!\!/_L^{\text{eff}} - mY^{\text{eff}}$$

with Y^{eff} as in (8.0.1) and

$$A_c^{\text{eff}} \;=\; U_c^{-1} A_c U_c \;+\; iU_c^{-1}\,(\partial U_c)\,, \qquad\qquad c \in \{L, R\}. \tag{8.0.3}$$

Finally, the effective fermionic projector is obtained from the auxiliary fermionic projector by the chiral transformation

$$P^{\text{eff}} \;=\; V\, P\, V^*\,. \tag{8.0.4}$$

[1]For the reader not familiar with the polar decomposition we outline the construction. For a matrix $A \in \text{Mat}(\mathbf{C}^n)$ we introduce the Hermitian and positive semidefinite matrix $R = \sqrt{A^* A}$. A short calculation shows that the matrix V defined by

$$V\, u \;=\; \begin{cases} u & \text{for } u \in \text{Ker}\, R \\ A\, R^{-1}\, u & \text{for } u \in (\text{Ker}\, R)^\perp \end{cases}$$

is unitary and satisfies the relation $A = VR$. Diagonalizing R by a unitary transformation, i.e. $R = WDW^{-1}$ with D diagonal and W unitary, we obtain the desired representation $A = U_1 D U_2^{-1}$ with $U_1 \equiv VW$ and $U_2 \equiv W$.

It satisfies the effective Dirac equation

$$\left(i\partial\!\!\!/ + \chi_L \,A\!\!\!/_R^{\text{eff}} + \chi_R \,A\!\!\!/_L^{\text{eff}} - mY^{\text{eff}}\right) P \;=\; 0\,. \tag{8.0.5}$$

Since the chiral transformation is one-to-one, the effective fermionic projector gives an equivalent formulation of the physical system. The advantage of the effective description is that the effective mass matrix Y^{eff} is diagonal. This means that if we interpret the sector index after the chiral transformation as labeling the different types of Dirac particles (like u, d, e, ν_e, etc.), the effective scalar potentials describe a dynamical shift of the mass of each type of fermion, whereas the interaction between different types of fermions is described only by the effective chiral potentials. Thus, apart from the fact that we allow for dynamical mass shifts, the Dirac particles interact as in the standard model via chiral fields.

In general, the effective potentials have locally the form of non-Abelian gauge potentials. But they cannot be chosen at every point independently, because it must be possible to represent them in the form (8.0.3) with A_c the Abelian gauge potentials of Theorem 7.2.11. We refer to (8.0.3) as the *gauge condition*.

For clarity, we point out that the unitary transformations in the polar decomposition (8.0.1) are not uniquely determined. Thus, similar to the freedom of choosing different gauges, there is a certain arbitrariness in the choice of U_L and U_R. Since at infinity the dynamical mass matrices go over to the mass matrix Y of the vacuum, we can and will always choose $U_{L/R}$ such that

$$\lim_{x\to\infty} U_{L/R}(x) \;=\; \mathbb{1}\,. \tag{8.0.6}$$

8.1. The Chiral Transformation in the Quark Blocks

Using the splitting (7.2.69), we may disregard the $U(3)^q$ potentials and can analyze the chiral transformation in the quark and lepton blocks separately. In this section, we consider a quark block and for ease in notation omit the superscript q. According to Theorem 7.3.1, the EL equations to degree 4 give information only on the partial traces of the dynamical mass matrices. Therefore, the dynamical mass matrices, and as a consequence also the chiral transformation and the effective potentials, are not completely determined. This means that we have a certain freedom to arbitrarily change these objects, and we shall use this freedom to make the following assumption on the form of the effective chiral gauge potentials.

DEF. 8.1.1. *The effective chiral potential A_c, $c \in \{L, R\}$, has* **unitary mixing** *if for every space-time point x there is a unitary matrix $W_c \in U(3)$ and a $U(2)$ potential a_c such that at x,*

$$A_c^{\text{eff}} \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W_c \end{pmatrix} a_c \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W_c^{-1} \end{pmatrix} \;=\; \begin{pmatrix} a_c^{11} & a_c^{12}\, W_c^{-1} \\ a_c^{21}\, W_c & a_c^{22} \end{pmatrix} \tag{8.1.1}$$

(here as in Theorem 7.3.1 we use a matrix notation in the sectors). The matrix W_c is referred to as the **mixing matrix**.

Thus we impose that the effective chiral potentials be trivial on the generations except for a unitary mixing of the generations in the off-diagonal matrix elements. This assumption is clearly satisfied for the gauge potentials in the standard model if we choose $W_R \equiv \mathbb{1}$ and W_L equal to the CKM mixing matrix. Our ansatz is more general in that we allow for both left- and right-handed mixing matrices and that

$W_c = W_c(x)$ need not be a constant matrix. Ultimately, the assumption of unitary mixing should be justified from the EL equations. But this makes it necessary to consider the EL equations to the degree 3 on the light cone, and we do not want to enter this analysis here. Therefore, we simply take Def. 8.1.1 as a physically reasonable technical simplification.

Our first lemma characterizes those chiral transformations which respect the condition of Def. 8.1.1.

LEMMA 8.1.2. *The effective chiral potential has unitary mixing if and only if the unitary transformation U_c in (8.0.2) is for all x of the form*

$$U_c \;=\; \begin{pmatrix} u_c^{11} & u_c^{12}\, W_c^{-1} \\ u_c^{21}\, W_c & u_c^{22} \end{pmatrix} \qquad \text{with } u_c \in U(2). \tag{8.1.2}$$

Furthermore, the mixing matrix is constant.

If U_c is of the form (8.1.2) with W_c a constant matrix, it is obvious that the corresponding effective chiral potential (8.0.3) has unitary mixing. In order to show that the converse is also true, we must analyze the differential equation for U_c and use the boundary conditions at infinity (8.0.6).

Proof of Lemma 8.1.2. It suffices to prove the "only if" part. Thus we assume that A_c^{eff} has unitary mixing and shall derive that U_c is of the form (8.1.2). For ease in notation we omit the subscript c. According to Theorem 7.2.11, A is diagonal and can thus be written as $A = \alpha \mathbb{1} + \beta \sigma^3$ with real functions α and β. When substituting into (8.0.3), α yields a contribution to A^{eff} with unitary mixing, independent of the form of U. Thus α is irrelevant for the following argument, and we can assume that A is a multiple of σ^3.

Let Ω be the set where the field tensor $F = dA - iA \wedge A$ is non-zero,

$$\Omega \;=\; \{x \,|\, F(x) \neq 0\}\,.$$

We shall first prove that on each connected component Ω_C of Ω, U is for all $x \in \overline{\Omega_C}$ of the form

$$U(x) \;=\; \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} u(x) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \tag{8.1.3}$$

with $u \in U(2)$ and constant unitary matrices $V_1, V_2, W \in U(3)$. To this end, we differentiate (8.0.3) and (8.1.1) to obtain

$$U^{-1}\, F\, U \;=\; F^{\text{eff}} \;=\; \begin{pmatrix} f^{11} & f^{12}\, W^{-1} \\ f^{21}\, W & f^{22} \end{pmatrix} \tag{8.1.4}$$

with $f = da + a \wedge a$ (these relations can be understood immediately from the behavior of the field tensor under gauge transformations). At $x \in \Omega_c$, $0 \neq F \sim \sigma^3$. Using this fact in (8.1.4) shows that $U(x)$ must be of the form

$$U \;=\; \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \tag{8.1.5}$$

with $B_1, B_2 \in U(3)$ and $\varphi \in \mathbb{R}$. Hence at x, the first summand in (8.0.3) is of the required form (8.1.1), and thus the second summand must also be of this form. Computing $iU^{-1}(\partial U)$ for U according to (8.1.5), one sees that this term is of the form (8.1.1) only if at x,

$$\partial W \;=\; 0 \qquad \text{and} \qquad \partial B_{1/2} \;\sim\; B_{1/2} \tag{8.1.6}$$

(in the special case $\sin\varphi = 0$, we merely obtain that $\partial(B_2 W^{-1}) \sim B_2 W^{-1}$, but since in this case U only involves the product $B_2 W^{-1}$, we can arrange that $\partial W = 0$). Integrating (8.1.6) gives (8.1.3).

Let $\Lambda = \mathbb{R}^4 \setminus \Omega$ be the set where F vanishes. We next prove that on each connected component Λ_C of Λ, U is of the form

$$U(x) \;=\; \begin{pmatrix} e^{i\phi(x)} & 0 \\ 0 & e^{-i\phi(x)} \end{pmatrix} V\, u_{\text{eff}}(x) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \tag{8.1.7}$$

with real ϕ, $u_{\text{eff}} \in U(2)$ and constant matrices $V \in U(6)$ and $W \in U(3)$. In order to derive this formula, we first use that $F = 0$ on Λ_C to represent A as a pure gauge potential, i.e.

$$A \;=\; iB^{-1}\,(\partial B) \qquad \text{with} \qquad B \;=\; \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \tag{8.1.8}$$

and a real function ϕ. According to the first part of (8.1.4), F^{eff} also vanishes. Let us consider what this tells us about the terms in (8.1.1). Using that the phase factors can be absorbed into a^{21}, we can arrange that $W^{-1}(\partial W)$ is trace-free. Then the contributions to F^{eff} involving ∂W and ∂a are linearly independent. From this we conclude that W is constant on Λ_C and that A^{eff} can be represented as

$$A^{\text{eff}} \;=\; i \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W \end{pmatrix} u_{\text{eff}}^{-1}\,(\partial u_{\text{eff}}) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W^{-1} \end{pmatrix} \tag{8.1.9}$$

with $u_{\text{eff}} \in U(2)$. On the other hand, substituting (8.1.8) into (8.0.3) gives

$$A^{\text{eff}} \;=\; i(BU)^{-1}\,\partial(BU)\,. \tag{8.1.10}$$

Differentiating the unitary matrix

$$B\,U \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W \end{pmatrix} u_{\text{eff}}^{-1}$$

and using (8.1.9) and (8.1.10), one sees that this matrix is constant on Λ_C, proving (8.1.7).

Note that the representation (8.1.7) poses a weaker constraint on U than equation (8.1.3). We shall now prove that on Λ_C even (8.1.3) holds. If Λ_C extends to infinity, we can according to (8.0.6) assume that

$$\lim_{\Lambda_C \ni x \to \infty} \phi \;=\; 0\,, \qquad \lim_{\Lambda_C \ni x \to \infty} u_{\text{eff}} \;=\; \mathbb{1} \qquad \text{and} \qquad V \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W \end{pmatrix}.$$

Then (8.1.7) indeed goes over to (8.1.3). If conversely Λ_C is compact, we choose $y \in \partial\Lambda_C$. Then at y both (8.1.3) and (8.1.7) hold, and comparing these formulas one sees that V must be a diagonal matrix. This implies that on Λ_C, (8.1.7) reduces to (8.1.3).

We just showed that for all X, U can be represented in the form (8.1.3), where V is constant on each connected component of Ω and Λ. Possibly after multiplying U by piecewise constant unitary transformations and/or absorbing a constant unitary transformation from u into V_1, V_2, or W, we can assume that all matrices in (8.1.3) are continuous. The asymptotics at infinity (8.0.6) finally yields that $V_1 = \mathbb{1}$ and $V_2 = W$. ∎

Using the result of the previous lemma in (8.0.1), we can now compute the dynamical mass matrices and analyze the conditions of Theorem 7.3.1. We restrict attention to the special case, which will be of relevance later, that the right-handed chiral transformation is trivial.

LEMMA 8.1.3. *Assume that $U_R \equiv \mathbb{1}$. If (7.3.1) and (7.3.3) are satisfied, the mixing matrix and the potential u_L in (8.1.1) has the following properties,*

$$\acute{Y}\,\grave{W}_L \;=\; 0 \;=\; \acute{W}_L\,\grave{Y} \tag{8.1.11}$$

$$|Y\,\grave{W}_L| \;=\; |\grave{Y}| \;=\; |\acute{W}_L\,Y| \tag{8.1.12}$$

$$u_L \;\in\; SU(2)\,. \tag{8.1.13}$$

Furthermore, (7.3.1) and (7.3.3) are also satisfied if we leave U_L unchanged and set the effective scalar potentials to zero,

$$Y^{\mathit{eff}} \;\equiv\; Y\,. \tag{8.1.14}$$

Conversely, if $U_R \equiv \mathbb{1}$ and (8.1.11–8.1.14) are satisfied, then (7.3.1) and (7.3.3) hold.

Proof. Evaluating (8.0.1) for U_L according to (8.1.1) and $U_R \equiv \mathbb{1}$ gives

$$Y_L \;=\; \begin{pmatrix} u_L^{11}\,Y_1^{\text{eff}} & u_L^{12}\,W_L^{-1}\,Y_2^{\text{eff}} \\ u_L^{21}\,W_L\,Y_1^{\text{eff}} & u_L^{22}\,Y_2^{\text{eff}} \end{pmatrix}$$

with $Y^{\text{eff}} = \text{diag}(Y_1^{\text{eff}}, Y_2^{\text{eff}})$. Assume that (7.3.1) and (7.3.3) are satisfied. Let us evaluate these relations for the off-diagonal elements of Y_L. Since b in (7.3.3) is non-trivial, the function $u_L^{21}(x)$ does not vanish identically, but clearly it is zero at infinity. As usual, we implicitly assume that u_L^{21} decays asymptotically at infinity, without necessarily being zero outside a compact set. Then we can apply a perturbation argument to the lower left matrix element of Y_L. Namely, (7.3.1) yields that $\acute{W}_L \grave{Y}_1^{\text{eff}} = 0$, and taking the asymptotic limit gives the rhs of (8.1.11). The rhs of (8.1.11) is obtained similarly from the upper right matrix element of Y_L. Applying the above perturbation argument to the off-diagonal terms in (7.3.3) yields (8.1.12).

We next evaluate (7.3.1) for the diagonal elements of Y_L. Since Y^{eff} is a positive matrix, $\hat{Y}_1^{\text{eff}}$ and $\hat{Y}_2^{\text{eff}}$ are real and ≥ 0. Furthermore, u_L satisfies as a $U(2)$ matrix the relation $|u_L^{11}| = |u_L^{22}|$. From (7.3.1) we conclude that $u_L^{11} = \overline{u_L^{22}}$, and thus $u \in SU(2)$.

Finally, it is straightforward to check that (8.1.11–8.1.14) imply the (7.3.1) as well as (7.3.3). ∎

In the remainder of this section we shall analyze and discuss the gauge condition (8.0.3). First, we substitute (8.1.2) and pull the constant mixing matrix outside,

$$A_c^{\text{eff}} \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W_c \end{pmatrix} \left(u_c^{-1} A_c u_c \;+\; i u_c^{-1}\,(\partial u_c)\right) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W_c^{-1} \end{pmatrix}.$$

Next, we decompose the potential and the unitary transformation into the $U(1)$ and $SU(2)$ parts, i.e.

$$A_c \;=\; \alpha\,\mathbb{1} + a\,\sigma^3 \qquad \text{and} \qquad u_c \;=\; e^{-i\phi}\,v$$

with real functions α, a, ϕ and $v \in SU(2)$. This gives

$$A_c^{\text{eff}} \;=\; (\alpha + \partial\phi)\,\mathbb{1} \;+\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W_c \end{pmatrix} \left[a\,v^{-1}\sigma^3 v \;+\; i\,v^{-1}(\partial v)\right] \begin{pmatrix} \mathbb{1} & 0 \\ 0 & W_c^{-1} \end{pmatrix}. \tag{8.1.15}$$

Thus ϕ describe a usual $U(1)$ gauge transformation. The square bracket can be regarded as an $SU(2)$ potential, and the matrix W_c introduces a unitary mixing in the off-diagonal elements. The remaining question is in which way the expression in the square brackets gives a constraint for the $SU(2)$ potential.

DEF. 8.1.4. *The field tensor $F = dA - iA \wedge A$ of an $SU(2)$ potential A is* **simple** *if for every x there is a real-valued 2-form Λ and $s \in su(2)$ such that*

$$F(x) \;=\; \Lambda\, s\,. \tag{8.1.16}$$

LEMMA 8.1.5. *An $SU(2)$ potential A can be represented in the form*

$$A \;=\; a\, v^{-1} \sigma^3 v \;+\; i\, v^{-1} (\partial v) \tag{8.1.17}$$

with $a(x) \in \mathbb{R}$ and $v(x) \in SU(2)$ if and only if its field tensor is simple.

Proof. If A is of the form (8.1.17), its field tensor is given by

$$F \;=\; f\, v^{-1} \sigma^3 v$$

with $f = da$, and this is obviously simple.

Assume conversely that F is simple. We choose $v_1 \in SU(2)$ such that

$$v_1\, s\, v_1^{-1} \;=\; \lambda\, \sigma^3$$

with $\lambda \in \mathbb{R}$ and introduce the gauge potential $\tilde{A}$ by

$$\tilde{A} \;=\; v_1\, A\, v_1^{-1} \;-\; i\, v_1 (\partial v_1^{-1})\,. \tag{8.1.18}$$

From (8.1.16) one sees that the corresponding field tensor is

$$\tilde{F} \;=\; f\, \sigma^3$$

with the real-valued 2-form $f = \lambda\Lambda$. From the fact that $\tilde{F}$ is closed we conclude that $df = 0$, and thus there is a 1-form a with $f = da$ (note that we are working in Minkowski space, which is clearly simply connected). By construction, the $SU(2)$ potentials $\tilde{A}$ and $a\sigma^3$ have the same field tensor $\tilde{F}$. As a consequence, they are related to each other by an $SU(2)$ transformation, i.e.

$$\tilde{A} \;=\; a\, v_2^{-1} \sigma^3 v_2 \;+\; i\, v_2^{-1} (\partial v_2) \tag{8.1.19}$$

with $v_2 \in SU(2)$. Substituting (8.1.19) into (8.1.18) and solving for A gives (8.1.17) with $v = v_2 v_1$. ∎

With this lemma we have reformulated the gauge condition (8.1.15) as a structure condition for the effective field tensor. This makes it possible to regard the effective chiral potentials as locally defined objects. More precisely, we shall treat the effective chiral potentials as local gauge potentials, which are constrained only by local conditions like e.g. that the effective field tensor be simple, but we shall not consider the corresponding chiral transformation (which involves integrating the effective potentials and is therefore defined in a nonlocal way). In particular, when we have conditions between the effective potentials in the quark and neutrino sectors, we shall always satisfy them by *local relations*, i.e. by algebraic or differential equations involving the effective potentials. This procedure corresponds to the usual requirement of locality in physics. It could be further justified later by the fact that the EL equations yield differential equations for the effective potentials (the "field equations"), and it seems impossible to satisfy such differential equations if the effective potentials obey nonlocal constraints.

8.2. The Chiral Transformation in the Lepton Block

We come to the analysis in the lepton block; for ease in notation the superscript l will be omitted. As a consequence of the chiral massless fermions, the dynamical matrices are different in the lepton and quark blocks. More precisely, Y_L and Y_R must now be of the form (7.3.2) and (7.3.4, 7.3.5). We shall first show that these conditions are incompatible with a unitary mixing and then resolve this problem by modifying the mixing in the right-handed component. Let us assume that (7.3.4) or (7.3.5) are satisfied for dynamical mass matrices of the form (8.0.1) with $U_{L/R}$ according to Lemma 8.1.2. Then, choosing a space-time point where $b \neq 0$, we have

$$(U_L\, Y^{\text{eff}}\, \grave{U}_R^{-1})\, I_2 \;=\; 0 \qquad \text{but} \qquad I_2\, (U_R\, Y^{\text{eff}}\, \grave{U}_L^{-1}) \;\neq\; 0\,, \tag{8.2.1}$$

where $I_{1/2}$ are again the projectors on the two sectors. Introducing the unit vectors $n = 3^{-\frac{1}{2}}\,(1,1,1) \in \mathbb{C}^3$ and $u = (0, n) \in \mathbb{C}^6$, we can write the first condition in (8.2.1) without a partial trace as

$$U_L\, Y^{\text{eff}}\, U_R^{-1}\, u \;=\; 0\,. \tag{8.2.2}$$

In the vacuum, the mass matrix $Y^{\text{eff}} = Y$ is strictly positive in the first sector. A perturbation argument yields that, at least for weak fields, the effective mass matrix is of the form $Y^{\text{eff}} = \text{diag}(Y_1^{\text{eff}}, Y_2^{\text{eff}})$ with $Y_1^{\text{eff}} > 0$. Therefore, the condition (8.2.1) can only be satisfied if $U_R^{-1}u$ vanishes in the first sector. Thus, using (8.1.2),

$$\begin{pmatrix} v^{11} & v^{12}\, W_c^{-1} \\ v^{21}\, W_c & v^{22} \end{pmatrix} \begin{pmatrix} 0 \\ n \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ * \end{pmatrix}$$

with $v = u_R^{-1}$. This implies that $v^{12} = 0$ and thus $U_R \equiv 1$. Using this result in (8.2.2), we obtain that $Y^{\text{eff}}u = 0$, and since Y^{eff} is a diagonal matrix with non-negative entries, we conclude that $Y_2^{\text{eff}} = 0$. Finally, the relations $U_R = \mathbb{1}$ and $Y_2^{\text{eff}} = 0$ imply that $I_2 U_R Y^{\text{eff}} = 0$, in contradiction to the second equation in (8.2.1).

In order to avoid the above contradiction, the vector $U_R^{-1}u$ must vanish identically in the first sector without U_R being trivial. The natural way to arrange this is to replace the unitary matrix W_R in (8.1.1) by a matrix which is zero on $<n>$ and is unitary on $<n>^\perp$. In analogy to the procedure in the previous section, we first introduce the corresponding effective potentials and determine U_R afterwards. We let Π be the projector

$$\Pi \;=\; |n><n| \qquad \text{with} \qquad n \;=\; \frac{1}{\sqrt{3}}\,(1,1,1)\,. \tag{8.2.3}$$

DEF. 8.2.1. *The effective potential A_R^{eff} has* **projected mixing** *if for every space-time point there is a unitary matrix $W_R \in U(3)$ with*

$$W_R\, n \;=\; n\,,$$

as well as real functions b_R^1 and b_R^2 and a $U(2)$ potential a_R such that at x,

$$A_R^{\text{eff}} \;=\; \begin{pmatrix} b_R^1 & 0 \\ 0 & b_R^2 \end{pmatrix} \;+\; (1-\Pi) \begin{pmatrix} a_R^{11} & a_R^{12}\, W_R^{-1} \\ A_R^{21}\, W_R & a_R^{22} \end{pmatrix}. \tag{8.2.4}$$

W_R is called the **mixing matrix**.

LEMMA 8.2.2. *The effective potential A_R^{eff} has projected mixing if and only if the unitary transformation U_R in (8.0.2) is for all x of the form*

$$U_R \;=\; \Pi \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} + (\mathbb{1} - \Pi) \begin{pmatrix} u_{11} & u^{12}\, W_R^{-1} \\ u^{21}\, W_R & u^{22} \end{pmatrix} \tag{8.2.5}$$

with $v_{1/2} \in U(1)$ and $u \in U(2)$. Furthermore, the matrix W_R is constant.

Proof. We consider the effective potential on $<n>$ and $<n>^\perp$ separately. On $<n>^\perp$, A_R^{eff} is of the form as in Def. 8.1.1, and so Lemma 8.1.3 applies. On $<n>$, on the other hand, A_R^{eff} is a diagonal potential, and integrating the differential equation for U_R as in the proof of Lemma 8.1.2 shows that U_R is also diagonal. ∎

The effective potentials in the second summand in (8.2.4) have different properties than usual gauge fields. First, one should keep in mind that the right-handed potential A_{eff}^R does not couple to the left-handed massless fermions, and therefore the off-diagonal elements in (8.2.4) cannot be regarded as describing an interaction between the massive leptons and the neutrinos. Indeed, one must be careful about associating any physical interaction to the second summand in (8.2.4), because the factor $(\mathbb{1} - \Pi)$ gives zero when the partial trace is taken, and also because some degrees of freedom of the corresponding potentials drop out of the fermionic projector when $\tilde{t}$ is multiplied by the chiral asymmetry matrix (7.1.3). For these reasons, we regard the off-diagonal elements in (8.2.4) as describing a new type of interaction whose physical significance is not clear at the moment. We refer to an effective potential which involves a factor $(\mathbb{1} - \Pi)$ as a *nil potential*.

The next lemma is very useful because it allows us to compute the vectors a and b in Theorem 7.3.1 without specifying U_R. In this way, we can get around the detailed analysis of the nil potential.

LEMMA 8.2.3. *Suppose that A_L^{eff} and A_R^{eff} have unitary and projected mixing, respectively. Then U_L and U_R can be chosen such that*

$$\grave{Y}_L \;=\; U_L\, \grave{Y}^{eff}\,.$$

Proof. Since $(\mathbb{1} - \Pi)n = 0$, the partial trace of the second summand in (8.0.4) is zero, whereas in the first summand the factor Π drops out. Thus $\grave{Y}_L = U_L Y^{\text{eff}} \grave{V}$ with V a diagonal $U(2)$ matrix. This matrix commutes with Y^{eff} and can thus be absorbed into U_L. ∎

8.3. Derivation of the Effective Gauge Group

We are now ready to prove the main result of this chapter.

THEOREM 8.3.1. *We consider the EL equations corresponding to the Lagrangian (5.5.13) under the assumptions of Theorem 7.2.11. Assume furthermore that the right-handed effective potentials in the lepton block have projected mixing and that all other effective potentials have unitary mixing (see Defs. 8.1.1 and 8.2.1). Imposing local relations between the effective potentials (as explained on page 194), the right-handed*

chiral transformation is trivial in the quark blocks, $U_R^q \equiv \mathbb{1}$. The mixing matrices are constant and satisfy the relations

$$\acute{Z}\, \grave{W}_L^q \;=\; \acute{W}_L^q\, \grave{Z} \;=\; \acute{Z}\, \grave{W}_L^l \;=\; 0 \tag{8.3.1}$$

$$|Z\, \grave{W}_L^q| \;=\; |\acute{W}_L^q\, Z| \;=\; |Z\, \grave{W}_L^l| \;=\; |\grave{Z}|\,, \tag{8.3.2}$$

where $Z = \frac{1}{m}\, diag(m_1, m_2, m_3)$ is the mass matrix of the massive fermions. The effective Dirac operator is of the following form,

$$i\partial\!\!\!/ - m \left(Y_q^{\mathit{eff}} \oplus Y_q^{\mathit{eff}} \oplus Y_q^{\mathit{eff}} \oplus Y_l^{\mathit{eff}}\right) \tag{8.3.3}$$

$$+ \chi_R \left(A\!\!\!/_L^{\mathit{eff}} \oplus A\!\!\!/_L^{\mathit{eff}} \oplus A\!\!\!/_L^{\mathit{eff}} \oplus A\!\!\!/_L^{\mathit{eff}}\right) \;+\; \chi_L\, A\!\!\!/_R \left(\sigma^3 \oplus \sigma^3 \oplus \sigma^3 \oplus \sigma^3\right) \tag{8.3.4}$$

$$+ (A\!\!\!/^q\, \mathbb{1}) \oplus \left(A\!\!\!/^l\, \mathbb{1} \;+\; (\mathbb{1} - \Pi)\, A\!\!\!/_R^{nil}\right)\,. \tag{8.3.5}$$

Here A_L^{eff} is a 2×2 matrix potential, A^q is a 3×3 matrix potential, A_R and A^l are vector fields, and A^{nil} is a nil potential. The effective gauge group is

$$\mathcal{G}^{\mathit{eff}} \;=\; SU(2)_L^{\mathit{eff}} \times U(1)_R \times U(3)^q \times U(1)^l\,. \tag{8.3.6}$$

The only constraint for the chiral potentials is that the field tensor corresponding to A_L^{eff} must be simple (see Def. 8.1.4). The EL equations to degree 4 are satisfied for the same effective potentials if the effective scalar potentials are set to zero,

$$Y^{\mathit{eff}} \;=\; Y\,. \tag{8.3.7}$$

Proof. We rewrite (7.3.7) and (7.3.8) in terms of the effective potentials. According to Lemma 8.2.3, $\grave{Y}_L^l$ is independent of U_R, and thus β and γ can be expressed in terms of Y_l^{eff}, the Abelian potentials A_V, A_s in (7.2.63), and the non-Abelian effective potential A_L^{eff} in the lepton block. Furthermore, (7.3.7) and (7.3.8) can be satisfied by local relations between the effective potentials only if the non-Abelian effective gauge fields coincide in the quark and lepton blocks. This yields the effective chiral gauge group (8.3.6) and the form of the corresponding potentials in (8.3.4) and (8.3.5). Lemma 8.1.5 shows that the gauge conditions (8.0.3) are satisfied if and only if the field tensor corresponding to A_L^{eff} is simple.

According to Lemmas 8.1.2 and 8.2.2, the mixing matrices are constant. Using Lemma 8.1.3, we obtain (8.3.1) and (8.3.2) in the quark blocks. The corresponding relations in the lepton block are obtained similarly from (7.3.2) and (7.3.4). Finally, (8.3.7) follows immediately from Lemma 8.1.3 and an analogous perturbation argument in the lepton block. ∎

Note that (8.3.1) and (8.3.2) are not satisfied if W_L is equal to the identity matrix. Thus the EL equations imply that the off-diagonal components of the effective gauge fields involve a non-trivial mixing of the generations. The fact that we may set Y^{eff} equal to Y, (8.3.7), means that the effective scalar potentials are irrelevant for the derivation of the effective gauge group. But this does not answer the question whether effective scalar potentials may occur in the system or not; to this end one must analyze the EL equations to lower degree on the light cone.

Finally we point out that Theorem 8.3.1 only gives necessary conditions for the effective potentials. But it is to be expected the analysis of the EL equations to degree 3 will give further constraints for the effective potentials. Taking this into account,

the results of Theorem 8.3.1 are in perfect agreement with physics: The $SU(3)^q$ and $SU(2)_L^{\text{eff}}$ can be identified with the strong and weak gauge groups, respectively. The coupling of the corresponding gauge potentials to the fermions is exactly as in the standard model. The $SU(3)^q$ is a free gauge group (see the discussion on page 152), and this implies that the corresponding gauge fields are necessarily massless. However, the $SU(2)_L^{\text{eff}}$ is spontaneously broken. The electromagnetic potential corresponds to a linear combination of the potentials of the subgroup $SU(2)_L^{\text{eff}} \times U(1)_R \times U(1)^q \times U(1)^l \subset \mathcal{G}^{\text{eff}}$, characterized by the property that it is a traceless vector potential. In order to make the connection to the standard model more precise, it remains to explain why only this particular linear combination occurs, and furthermore one must analyze the masses of the spontaneously broken gauge fields in the resulting field equations. To answer these questions, one needs to analyze the EL equations to degree 3 on the light cone; this is an interesting project for the future.

APPENDIX A

Connection to the Fock Space Formalism

In this appendix it is shown that for an observer who is making measurements only in a subsystem of the whole physical system, the description of a many-fermion system with the fermionic projector is equivalent to the fermionic Fock space formalism, provided that the number of fermions of the whole system (including the particles of the sea) is infinite. The following consideration applies in the same way to either a space-time continuum or to discrete space-time. Before beginning we point out that the action principle, from which the fundamental physical equations can be deduced, involves the fermions only via the Dirac action $<\Psi, (i\partial\!\!\!/ + \mathcal{B} - m)\Psi>$. For the formulation of the Dirac action one only needs on the fermionic Fock space the time/position operators and the operator $\partial\!\!\!/$, which are all one-particle operators. Therefore, we can say that many-particle operators (like for example in the four-fermion coupling of the Fermi model) are not essential for the formulation of the quantum field theory of the standard model. Having this in mind, we may here restrict attention to one-particle operators[1].

Let P be a fermionic projector acting on the vector space H. The one-particle observables correspond to operators $\mathcal{O}$ on H. Our subsystem is described by a non-degenerate subspace $K \subset H$; we decompose H as a direct sum $H = K \oplus L$ with $L = K^\perp$. We assume that the observables are localized in N; i.e. they are trivial on L,

$$\mathcal{O}_{|L} \;=\; 0_{|L}\,. \tag{A.1}$$

We choose a (properly normalized) basis $\Psi_1, \ldots, \Psi_n$ of the subspace $P(H) \subset H$, and decompose the states Ψ_j in the form

$$\Psi_j \;=\; \Psi_j^K + \Psi_j^L \qquad \text{with } \Psi_j^K \in K, \Psi_j^L \in L.$$

Substituting into (3.2.1), we obtain for the many-particle wave function the expression

$$\Psi \;=\; \sum_{\pi \in \mathcal{P}(n)} (-1)^{|\pi|} \left(\bigwedge_{j \in \pi} \Psi_j^K \right) \wedge \left(\bigwedge_{j \notin \pi} \Psi_j^L \right), \tag{A.2}$$

where $\mathcal{P}(n)$ denotes the set of all subsets of $\{1, \ldots, n\}$. For measurements in our subsystem, we must calculate the expectation value $<\Psi|\mathcal{O}|\Psi>_F$[2], where the operators

[1] Online version: For the description of *entanglement*, it is indeed necessary to consider two-particle observables; see the paper "Entanglement and second quantization in the framework of the fermionic projector" (arXiv:0911.0076 [math-ph]).

[2] We remark for clarity that this expectation value does not coincide with that of a measurement in nonrelativistic quantum mechanics. Namely, in the continuum, the scalar product $<.|.>$ involves a time integration. But one can get a connection to nonrelativistic measurements by considering operators $\mathcal{O}$ with a special time dependence (which, for example, act on the wave functions only in a short time interval $[t, t + \Delta t]$).

$\mathcal{O}$ act on the Fock space according to

$$\mathcal{O}(\Psi_1 \wedge \cdots \wedge \Psi_n) \;=\; (\mathcal{O}\Psi_1) \wedge \cdots \wedge \Psi_n + \Psi_1 \wedge (\mathcal{O}\Psi_2) \cdots \wedge \Psi_n + \cdots + \Psi_1 \wedge \cdots \wedge (\mathcal{O}\Psi_n)\,,$$

and where $<.|.>_F$ is the scalar product on the Fock space, induced by the scalar product $<.|.>$ on H. It is useful to rewrite the expectation value with the statistical operator S, i.e.

$$<\Psi|\mathcal{O}|\Psi>_F \;=\; \mathrm{tr}_F(S\,\mathcal{O}) \quad \text{with} \quad S \;=\; |\Psi><\Psi|_F\,,$$

where tr_F denotes the trace in the Fock space. Using (A.1), we can take the partial trace over L and obtain, applying (A.2),

$$<\Psi|\mathcal{O}|\Psi>_F \;=\; \mathrm{tr}_{F_K}(S^K\,\mathcal{O}) \qquad \text{with} \tag{A.3}$$

$$S^K \;=\; \sum_{k=0}^{n} \sum_{\substack{\pi,\,\pi' \in \mathcal{P}(n),\\ \#\pi = \#\pi' = k}} c_{\pi,\pi'} \;|\wedge_{i\in\pi}\, \Psi_i^K><\wedge_{j\in\pi'}\, \Psi_j^K|_{F_K} \tag{A.4}$$

$$c_{\pi,\pi'} \;=\; (-1)^{|\pi|+|\pi'|} <\wedge_{i\notin\pi}\, \Psi_i^L|\wedge_{j\notin\pi'}\, \Psi_j^L>_F\,,$$

where tr_{F_K} is the trace in the Fock space $F_K = \oplus_{k=0}^{\infty} \wedge^k K$ generated by K. Thus our subsystem is described by a statistical operator S^K on F_K, which is composed of mixed states consisting of different numbers of particles. Since the constants $c_{\pi,\pi'}$ depend on the wave functions Ψ^L outside our subsystem, we can consider them as arbitrary numbers.

In the limit when the number n of particles of the whole system tends to infinity, (A.4) goes over to a statistical operator of the form

$$S^K \;=\; \sum_{k=0}^{\infty} \sum_{\alpha,\beta=0}^{\infty} c_{\alpha\beta}^{(k)}\; |\Psi_\alpha^{(k)}><\Psi_\beta^{(k)}|_{F_K} \tag{A.5}$$

with arbitrary complex coefficients $c_{\alpha\beta}^{(k)}$ and k-particle states $\Psi_\alpha^{(k)} \in F_K^k$. This statistical operator differs from a general statistical operator S_{gen}^K in that it is diagonal on the k-particle subspaces (i.e. that the wave functions in the "bra" and in the "ket" of (A.5) are both k-particle states); more precisely, S_{gen}^K has, compared to (A.5), the more general form

$$S_{\mathrm{gen}}^K \;=\; \sum_{k,l=0}^{\infty} \sum_{\alpha,\beta=0}^{\infty} c_{\alpha\beta}^{(k,l)}\; |\Psi_\alpha^{(k)}><\Psi_\beta^{(l)}|_{F_K}\,. \tag{A.6}$$

We remark for clarity that a pure state of the Fock space $\Psi \in F_K$ has a decomposition $\Psi = \sum_{k=0}^{\infty} \lambda_k \Psi^{(k)}$, and thus the corresponding statistical operator is

$$S \;=\; |\Psi><\Psi|_{F_K} \;=\; \sum_{k,l=0}^{\infty} \lambda_k\, \overline{\lambda_l}\; |\Psi^{(k)}><\Psi^{(l)}|_{F_K}\,.$$

This statistical operator is a special case of (A.6), but it is *not* of the form (A.5).

The difference between (A.5) and (A.6) becomes irrelevant if we keep in mind that all physically relevant observables commute with the particle number operator. Namely in this case, every expectation value reduces to the sum of the expectation

values in the k-particle Fock spaces,

$$\begin{aligned} \mathrm{tr}_{F_K}(S^K_{\mathrm{gen}}\,\mathcal{O}) &= \sum_{k,l=0}^{\infty}\sum_{\alpha,\beta=0}^{\infty} c^{(k,l)}_{\alpha\beta} <\Psi^{(l)}_{\beta} \mid \mathcal{O} \mid \Psi^{(k)}_{\alpha}>_{F_K} \\ &= \sum_{k=0}^{\infty}\sum_{\alpha,\beta=0}^{\infty} c^{(k,k)}_{\alpha\beta} <\Psi^{(k)}_{\beta} \mid \mathcal{O} \mid \Psi^{(k)}_{\alpha}>_{F_K} . \end{aligned}$$

If we choose the coefficients $c^{(k)}_{\alpha\beta}$ in (A.5) to be $c^{(k)}_{\alpha\beta} = c^{(k,k)}_{\alpha\beta}$, these expectation values are also obtained from the statistical operator S^K,

$$\mathrm{tr}_{F_K}(S^K_{\mathrm{gen}}\,\mathcal{O}) \;=\; \mathrm{tr}_{F_K}(S^K\,\mathcal{O})\,.$$

We conclude that it is no loss of generality to describe the subsystem by the statistical operator S^K.

APPENDIX B

Some Formulas of the Light-Cone Expansion

This appendix is a compilation of some formulas of the light-cone expansion. More precisely, we list the phase-free contribution to the light-cone expansion of the Green's functions (cf. Def. 2.5.5). According to Def. 2.5.5 and Theorem 2.5.6, the light-cone expansion of the Green's functions is immediately obtained by inserting ordered exponentials into the line integrals. Furthermore, as explained after (2.5.45), the formulas can be applied directly to the fermionic projector; they then describe the singularities of $\tilde{P}(x,y)$ on the light cone. Without loss of generality, we restrict attention to the left handed component of the Green's functions. We compute precisely those contributions which will be of relevance in Appendix G and in Chapters 6–8. The following formulas were all generated by a computer program, see [**F6**] for details.

We begin with the perturbation by a chiral perturbation to first order. The phase-free contribution (denoted by a corresponding superscript on the equal sign) is

$$\chi_L \left(-s \left(\chi_L \A_R + \chi_R \A_L\right) s\right)(x,y) \overset{\text{phase-free}}{=} \not\xi\, \mathcal{O}((y-x)^0) + \mathcal{O}((y-x)^2)$$

$$+\chi_L\, S^{(0)}(x,y)\, \xi^i \int_x^y dz\, [0,1 \,|\, 0]\, (\partial\!\!\!/ A_{Li}) \tag{B.1}$$

$$-\chi_L\, S^{(0)}(x,y) \int_x^y dz\, [0,0 \,|\, 0]\, \A_L \tag{B.2}$$

$$+\chi_L\, S^{(0)}(x,y)\, \A_L(x) \tag{B.3}$$

$$+\frac{1}{2}\, \chi_L\, S^{(0)}(x,y)\, \not\xi \int_x^y dz\, [0,0 \,|\, 0]\, (\partial\!\!\!/ \A_L) \tag{B.4}$$

$$-\chi_L\, S^{(0)}(x,y)\, \not\xi \int_x^y dz\, [1,0 \,|\, 0]\, (\partial^i A_{Li}) \tag{B.5}$$

$$+\frac{1}{2}\chi_L\, S^{(0)}(x,y)\, \not\xi\, \xi^i \int_x^y dz\, [0,0 \,|\, 1]\, (\Box A_{Li}) \tag{B.6}$$

$$+\chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz\, [0,1 \,|\, 1]\, (\partial\!\!\!/ \Box A_{Li}) \tag{B.7}$$

$$+\chi_L\, S^{(1)}(x,y) \int_x^y dz\, [0,2 \,|\, 0]\, (\Box \A_L) \tag{B.8}$$

$$-2\chi_L\, S^{(1)}(x,y) \int_x^y dz\, [0,0 \,|\, 1]\, (\partial\!\!\!/ \partial^i A_{Li})\,, \tag{B.9}$$

where again $\xi \equiv (y-x)$. The notation $\not\xi\, \mathcal{O}((y-x)^2)$ means that we leave out all contributions which are of the order $\mathcal{O}((y-x)^0)$ and have a leading factor $\not\xi$. This formula has the disadvantage that it contains partial derivatives of the chiral potential; it would be better for physical applications to work instead with the Yang-Mills field tensor and the Yang-Mills current. Therefore, we introduce left and right handed

gauge-covariant derivatives $D^{L/R}$,

$$D^L_j \;=\; \frac{\partial}{\partial x^j} \;-\; iA_{Lj}\,, \qquad\qquad D^R_j \;=\; \frac{\partial}{\partial x^j} \;-\; iA_{Rj}\,,$$

and define the corresponding field tensor and current as usual by the commutators

$$F^c_{jk} \;=\; i\left[D^c_j,\, D^c_k\right]\,, \qquad\qquad j^c_l \;=\; \Big[D^{c\,k},\, F^c_{lk}\Big] \qquad\qquad (c = L \text{ or } R). \tag{B.10}$$

In the case of an Abelian gauge field, this formula reduces to the familiar formulas for the electromagnetic field tensor and current,

$$F^c_{jk} \;=\; \partial_j A_{c\,k} - \partial_k A_{c\,j}\,, \qquad\qquad j^c_l \;=\; \partial_{lk} A^k_c - \Box A_{c\,l}\,.$$

Notice, however, that in the general case of a system of Dirac seas, (B.10) involves quadratic and cubic terms in the potential.

By substituting (B.10) into (B.1–B.9) and manipulating the line integrals with integrations by parts, one can rewrite the phase-free contribution in a way where the linear terms in the potential are gauge invariant. For example, we can combine (B.1–B.3) by transforming the line integrals as

$$\begin{aligned}
&\xi^k \int_x^y dz\, [0,1\,|\,0]\, (\not\partial A_{Lk}) \;=\; \xi^k \int_x^y dz\, [0,1\,|\,0]\, (\gamma^j F^L_{jk} \;+\; \partial_k \not A_L) \;+\; \mathcal{O}(A_L^2) \\
&\quad=\;\; \xi^k \int_x^y dz\, [0,1\,|\,0]\, \gamma^j F^L_{jk} \;-\; \not A_L(x) \;+\; \int_x^y dz\, [0,0\,|\,0]\, \not A_L \;+\; \mathcal{O}(A_L^2)\,.
\end{aligned} \tag{B.11}$$

This procedure yields (in the non-Abelian case) quadratic and cubic terms in the potential which are *not* gauge invariant. Fortunately, these gauge-dependent terms are all compensated by corresponding contributions to the higher order Feynman diagrams. We thus obtain

$$\begin{aligned}
&\chi_L \sum_{k=0}^{\infty} ((-s\,(\chi_L \not A_R + \chi_R \not A_L))^k\, s)(x,y) \;\overset{\text{phase-free}}{=}\; \not\xi\, \mathcal{O}((y-x)^0) + \mathcal{O}((y-x)^2) \\
&\quad +\chi_L\, S^{(0)}(x,y)\, \xi^i \int_x^y dz\, [0,1\,|\,0]\, \gamma^l F^L_{li} \\
&\quad +\frac{1}{4}\, \chi_L\, S^{(0)}(x,y)\, \not\xi \int_x^y dz\, [0,0\,|\,0]\, \gamma^j \gamma^k\, F^L_{jk} \\
&\quad -\frac{1}{2}\, \chi_L\, S^{(0)}(x,y)\, \not\xi\, \xi^i \int_x^y dz\, [0,0\,|\,1]\, j^L_i \\
&\quad -i\chi_L\, S^{(0)}(x,y)\, \not\xi\, \xi_i \xi^j \int_x^y dz_1\, [0,1\,|\,1]\, F^L_{kj} \int_{z_1}^y dz_2\, [0,1\,|\,0]\, F^{ki}_L \\
&\quad +\chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz\, [0,1\,|\,1]\, (\not\partial j^L_i) \\
&\quad +\chi_L\, S^{(1)}(x,y) \int_x^y dz\, [0,2\,|\,0]\, j^L_k\, \gamma^k \\
&\quad -i\chi_L\, S^{(0)}(x,y)\, \not\xi\, \xi_i \xi^j \int_x^y dz_1\, [0,1\,|\,1]\, F^L_{kj} \int_{z_1}^y dz_2\, [0,1\,|\,0]\, F^{ki}_L \\
&\quad +i\chi_L\, S^{(1)}(x,y)\, \xi^i \xi^j \int_x^y dz_1\, [0,3\,|\,0]\, \gamma^k\, F^L_{kj} \int_{z_1}^y dz_2\, [0,0\,|\,1]\, j^L_i
\end{aligned}$$

$$
\begin{aligned}
&+i\chi_L\, S^{(1)}(x,y)\, \xi^i \xi^j \int_x^y dz_1\, [0,2 \,|\, 1]\, j_j^L \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, \gamma^l\, F_{li}^L \\
&-2i\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \int_x^y dz_1\, [0,2 \,|\, 1]\, F_{mj}^L \int_{z_1}^y dz_2\, [0,2 \,|\, 0]\, (\partial\!\!\!/ F_L^{mi}) \\
&-2i\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \int_x^y dz_1\, [0,2 \,|\, 1]\, (\partial\!\!\!/ F_{kj}^L) \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F_L^{ki} \\
&+i\chi_L\, S^{(1)}(x,y)\, \xi^i \xi^j \int_x^y dz_1\, [0,2 \,|\, 1]\, \gamma^k F_{kj}^L \int_{z_1}^y dz_2\, [0,2 \,|\, 0]\, j_i^L \\
&-\frac{i}{2}\, \chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \,|\, 0]\, \gamma^j F_{ji}^L \int_{z_1}^y dz_2\, [0,0 \,|\, 0]\, \gamma^k \gamma^l F_{kl}^L \\
&-\frac{i}{2}\, \chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \,|\, 0]\, \gamma^j \gamma^k F_{jk}^L \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, \gamma^l F_{li}^L \\
&+2i\chi_L\, S^{(1)}(x,y)\, \xi_i \int_x^y dz_1\, [0,3 \,|\, 0]\, \gamma^j F_{jk}^L \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F_L^{ki} \\
&-2i\chi_L\, S^{(1)}(x,y)\, \xi^j \int_x^y dz_1\, [0,1 \,|\, 1]\, F_{ij}^L \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, \gamma_k F_L^{ki} \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \xi^k \\
&\quad \times \int_x^y dz_1\, [0,4 \,|\, 0]\, \gamma^l F_{lk}^L \int_{z_1}^y dz_2\, [0,1 \,|\, 1]\, F_{mj}^L \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, F_L^{mi} \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \xi^k \\
&\quad \times \int_x^y dz_1\, [0,3 \,|\, 1]\, \gamma^l F_{lk}^L \int_{z_1}^y dz_2\, [0,3 \,|\, 0]\, F_{mj}^L \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, F_L^{mi} \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \xi^k \\
&\quad \times \int_x^y dz_1\, [0,3 \,|\, 1]\, F_{mk}^L \int_{z_1}^y dz_2\, [0,3 \,|\, 0]\, \gamma^l F_{lj}^L \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, F_L^{mi} \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi^i \xi_j \xi^k \\
&\quad \times \int_x^y dz_1\, [0,3 \,|\, 1]\, F_{mk}^L \int_{z_1}^y dz_2\, [0,3 \,|\, 0]\, F_L^{mj} \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, \gamma^l F_{li}^L\, .
\end{aligned}
$$

We call this formulation of the phase-free contributions purely in terms of the Yang-Mills field tensor and the Yang-Mills current the *gauge invariant form* of the light-cone expansion.

It remains to consider the scalar/pseudoscalar perturbation; i.e. we must study how the dynamic mass matrices $Y_{L/R}(x)$ show up in the light-cone expansion. We begin with the case of a single mass matrix. To first order in the external potential, the corresponding Feynman diagram has the light-cone expansion

$$
\begin{aligned}
&\chi_L\, m\, (-s\, (-\chi_L Y_R - \chi_R Y_L)\, s)(x,y) \\
&\quad \overset{\text{phase-free}}{=} \quad \frac{1}{2}\, \chi_L\, m\, S^{(0)}(x,y)\, \not{\xi} \int_x^y dz\, [0,0 \,|\, 0]\, (\partial\!\!\!/ Y_L) \\
&\qquad\qquad +\chi_L\, m\, S^{(0)}(x,y)\, Y_L(x) \;+\; \mathcal{O}((y-x)^0)\, . \qquad \text{(B.12)}
\end{aligned}
$$

The higher orders in the chiral potentials yield no phase-free contributions. The next orders in the mass parameter are treated similarly. The contributions quadratic in m

are

$$
\begin{aligned}
&\chi_L\, m^2 \sum_{n_1,n_2,n_3=0}^{\infty} ((-s\, \not{A}_L)^{n_1}\, s\, Y_L\, s\, (-\not{A}_R\, s)^{n_2}\, Y_R\, s\, (-\not{A}_R\, s)^{n_3})(x,y) \\
&\overset{\text{phase-free}}{=} \frac{i}{2}\, \chi_L\, m^2\, S^{(0)}(x,y)\, \not{\xi} \int_x^y dz\, [0,0 \mid 0]\, Y_L\, Y_R \\
&\quad + i\chi_L\, m^2\, S^{(1)}(x,y) \int_x^y dz\, [0,1 \mid 0]\, Y_L\, \gamma^j (D_j Y_R) \\
&\quad + i\chi_L\, m^2\, S^{(1)}(x,y) \int_x^y dz\, [0,1 \mid 0]\, \gamma^j (D_j Y_L)\, Y_R \\
&\quad - i\chi_L\, m^2\, S^{(1)}(x,y)\, Y_L \int_x^y dz\, [0,0 \mid 0]\, \gamma^j (D_j Y_R) \\
&\quad + \chi_L\, m^2\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^j F^L_{ji} \int_{z_1}^y dz_2\, [0,0 \mid 0]\, Y_L\, Y_R \\
&\quad + \chi_L\, m^2\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma^j F^L_{ji} \\
&\quad + \not{\xi}\, \mathcal{O}((y-x)^0) + \mathcal{O}((y-x)^2)\,,
\end{aligned}
$$

whereas there is only one terms cubic in m,

$$
\begin{aligned}
&\chi_L\, m^3 \sum_{n_1,n_2,n_3,n_4=0}^{\infty} ((-s\, \not{A}_L)^{n_1}\, s\, Y_L\, s\, (-\not{A}_R\, s)^{n_2}\, Y_R\, s\, (-\not{A}_L\, s)^{n_3}\, Y_L\, s\, (-\not{A}_R\, s)^{n_4})(x,y) \\
&\overset{\text{phase-free}}{=} \chi_L\, m^3\, S^{(1)}(x,y)\, Y_L \int_x^y dz\, [0,0 \mid 0]\, Y_R\, Y_L \\
&\qquad + \not{\xi}\, \mathcal{O}((y-x)^0) + \mathcal{O}((y-x)^2)\,.
\end{aligned}
$$

To the order $\sim m^4$ and higher all contributions are on the light cone of the order $\not{\xi}\mathcal{O}((y-x)^0) + \mathcal{O}((y-x)^2)$.

The above Feynman diagrams completely characterize the Green's functions to the order $\mathcal{O}((y-x)^0)$ on the light cone. Notice that in agreement with Theorem 2.5.6 we get only a finite number of phase-free contributions.

APPENDIX C

Normalization of Chiral Fermions

In this appendix we describe a method for normalizing chiral fermions. The main difficulty is that for a proper normalization one needs to give the chiral fermions a small rest mass; this will be discussed in Section C.1 for a single Dirac sea in Minkowski space. In Section C.2 we develop a method for analyzing the normalization of chiral fermions with a small generalized "mass," whereas Section C.3 gives the general construction including the infrared regularization and the interaction.

C.1. Massive Chiral Fermions – Preparatory Discussion

Before introducing the infrared regularization, we need to understand how a chiral Dirac sea can be normalized in infinite volume using some kind of "δ-normalization." To this end we consider a non-interacting left-handed fermionic projector in Minkowski space,

$$P(x,y) \;=\; \chi_L\, t_m(x,y)|_{m=0}\;, \tag{C.1.1}$$

where we set

$$t_m \;=\; \frac{1}{2}\,(p_m - k_m)$$

with p_m and k_m according to (2.2.4, 2.2.5). Naively, products of this fermionic projector vanish due to chiral cancellations,

$$\begin{aligned} P^2(x,y) \;&=\; \int d^4z\, P(x,z)\, P(z,y) \;=\; \int d^4z\, \chi_L\, t_0(x,z)\, \chi_L\, t_0(z,y) \\ &=\; \int d^4z\, \chi_L\, \chi_R\, t_0(x,z)\, t_0(z,y) \;\overset{\text{formally}}{=}\; 0\,. \end{aligned} \tag{C.1.2}$$

However, this formal calculation has no meaning in the formalism of causal perturbation theory §2.2 because in this formalism we are not allowed to multiply Dirac seas of the same fixed mass. Instead, we must treat the masses as variable parameters. Thus before we can give a mathematical meaning to products of chiral Dirac seas, we must extend the definition of a chiral Dirac sea to non-zero rest mass.

Giving chiral Dirac particles a mass is a delicate issue which often leads to confusion and misunderstandings. Therefore, we discuss the situation in the example (C.1.1) in detail. In momentum space, the distribution t_m, $m \geq 0$, takes the form

$$t_m(k) \;=\; (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0)\,.$$

On the mass shell, the range of the (4×4)-matrix $\not{k} + m$ is two-dimensional; this corresponds to a twofold degeneracy of the eigenspaces of the Dirac operator $(\not{k} - m)$ for any fixed k. If $m = 0$, the Dirac equation splits into two separate equations for the left- and right-handed component of the spinor, and this makes it possible to project out half of the eigenvectors simply by multiplying by χ_L,

$$P(k) \;=\; \chi_L\, \not{k}\, \delta(k^2)\, \Theta(-k^0)\,. \tag{C.1.3}$$

If $m > 0$, this method cannot be applied because the left- and right-handed subspaces are no longer invariant. In particular, the product $\chi_L t_m$ for $m > 0$ is not Hermitian and is not a solution of the Dirac equation. Nevertheless, we can project out one of the degenerate eigenvectors as follows. For given k on the lower mass shell we choose a vector q with

$$kq \;=\; 0 \qquad \text{and} \qquad q^2 \;=\; -1\,. \tag{C.1.4}$$

A short calculation shows that

$$[t_m(k),\, \rho \not{q}] \;=\; 0 \qquad \text{and} \qquad (\rho \not{q})^2 \;=\; \mathbb{1}$$

(where ρ is again the pseudoscalar matrix (1.2.13)). This means that the matrix $\rho \not{q}$ has eigenvalues ± 1, and that the Dirac equation is invariant on the corresponding eigenspaces. Projecting for example onto the eigenspace corresponding to the eigenvalue -1 gives

$$P_m(k) \;:=\; \frac{1}{2}\,(\mathbb{1} - \rho \not{q})\,(\not{k} + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)\,. \tag{C.1.5}$$

Thus, similar to the procedure in the massless case (C.1.3), P_m is obtained from t_m by projecting out half of the Dirac eigenstates on the lower mass shell. But in contrast to (C.1.3), the construction of P_m depends on the vector field q, which apart from the conditions (C.1.4) can be chosen arbitrarily. A short calculation shows that P_m is idempotent in the sense that

$$P_m\, P_{m'} \;=\; \delta(m - m')\, P_m\,. \tag{C.1.6}$$

The distribution (C.1.5) can be regarded as a generalization of the chiral Dirac sea (C.1.3) to the massive case. In order to make this connection clearer, we now show that (C.1.5) reduces to (C.1.3) in the limit $m \searrow 0$. Thus, for fixed $\vec{k} \neq 0$ and variable $m > 0$, we let k be on the lower mass shell, $k(m) = (-\sqrt{|\vec{k}|^2 + m^2}, \vec{k})$, and choose $q(m)$ such that (C.1.4) is satisfied. A simple example for q is

$$q(m) \;=\; \frac{1}{m}\left(-|\vec{k}|,\, \sqrt{|\vec{k}|^2 + m^2}\, \frac{\vec{k}}{|\vec{k}|}\right). \tag{C.1.7}$$

In this example, k and mq coincide as $m \searrow 0$; more precisely,

$$k - mq = \mathcal{O}(m^2)\,.$$

This relation holds for a large class of functions $q(m)$. Therefore, it seems general enough to concentrate on the situation where

$$k - mq \;=\; m^2\, v \qquad \text{with } v(m) = \mathcal{O}(m^0)\,. \tag{C.1.8}$$

Solving this relation for q and substituting into (C.1.5) gives

$$P_m(k) \;=\; \frac{1}{2}\left(\mathbb{1} - \rho\, \frac{\not{k}}{m} + m\rho \not{v}\right)(\not{k} + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)\,. \tag{C.1.9}$$

Using that on the mass shell $\not{k}(\not{k} + m) = m(\not{k} + m)$, we get

$$P_m(k) \;=\; \frac{1}{2}\,(\mathbb{1} - \rho + m\rho \not{v})\,(\not{k} + m)\,\delta(k^2 - m^2)\,\Theta(-k^0)\,. \tag{C.1.10}$$

If now we take the limit $m \searrow 0$, we obtain precisely (C.1.3), i.e.

$$\lim_{m \searrow 0} P_m \;=\; P \tag{C.1.11}$$

with convergence in the sense of distributions. This calculation shows that (C.1.5) indeed includes (C.1.3) as a limiting case and that the dependence on q drops out as $m \searrow 0$.

The distribution (C.1.5) gives a possible definition of a massive chiral Dirac sea. However, it would be too restrictive to use only (C.1.5) as the basis of our construction, because there are other common ways to give chiral Dirac particles a rest mass. These alternatives are more general than (C.1.5) in that the wave functions are no longer solutions of the Dirac equation. To give a simple example, one could describe a massive left-handed Dirac sea for $m > 0$ by

$$P_m(k) \;=\; \Big(\chi_L \,\not{k} + \frac{m}{4}\Big)\; \delta\Big(k^2 - \frac{m^2}{4}\Big)\; \Theta(-k^0)\,. \tag{C.1.12}$$

This distribution has the advantage over (C.1.5) that it is Lorentz invariant, but it is clearly not a solution of the Dirac equation. As $m \searrow 0$, we again recover the massless chiral Dirac sea (C.1.1). We compute the operator product $P_m P_{m'}$ in momentum space,

$$\begin{aligned}
(P_m\, P_{m'})(k) &= \Big(\chi_L \,\not{k} + \frac{m}{4}\Big) \Big(\chi_L \,\not{k} + \frac{m'}{4}\Big)\, \delta\Big(k^2 - \frac{m^2}{4}\Big)\, \delta\Big(k^2 - \frac{m'^2}{4}\Big)\, \Theta(-k^0) \\
&= \delta\Big(\frac{m^2}{4} - \frac{m'^2}{4}\Big) \Big(\frac{m+m'}{4}\, \chi_L \,\not{k} + \frac{mm'}{16}\Big)\; \delta\Big(k^2 - \frac{m^2}{4}\Big)\, \Theta(-k^0) \\
&= \delta(m-m') \Big(\chi_L \,\not{k} + \frac{m}{8}\Big)\; \delta\Big(k^2 - \frac{m^2}{4}\Big)\, \Theta(-k^0)\,,
\end{aligned}$$

where in the last step we used that $m, m' > 0$. Note that in the last line the summand $m/8$ appears (instead of the summand $m/4$ in (C.1.12)), and therefore P_m is not idempotent in the sense (C.1.6). On the other hand, one can argue that (C.1.6) is a too strong normalization condition, because we are interested in the situation when the masses of the chiral particles are arbitrarily small, and thus it seems sufficient that (C.1.6) should hold in the limit $m, m' \searrow 0$. In this limit, the problematic summands $m/4$ and $m/8$ both drop out, and thus we can state the idempotence of P_m as follows,

$$\lim_{m,m' \searrow 0} \big(P_m\, P_{m'} - \delta(m-m')\, P_m\big) \;=\; 0\,. \tag{C.1.13}$$

The above example shows that, in order to have more flexibility to give the chiral Dirac particles a mass, it is preferable to work with the weaker normalization condition (C.1.13) instead of (C.1.6). Comparing with the naive calculation (C.1.2), one sees that introducing the mass changes the behavior of the operator products completely, even if the masses are arbitrarily small. Therefore, we refer to the limit $m, m' \searrow 0$ in (C.1.13) as the *singular mass limit*.

For the correct understanding of the singular mass limit, it is important to observe that, in contrast to operator products as considered in (C.1.13), the formalism of the continuum limit is well-behaved as $m \searrow 0$. Namely, in the continuum limit we consider an expansion in powers of m. The different orders in m have a different singular behavior on the light cone. In particular, to every order on the light cone only a finite number of orders in m contribute. Thus to every order on the light cone, the m-dependence is polynomial and therefore smooth. Expressed in terms of the kernel, the limit $m \searrow 0$ is singular when we form the product $P(x,z)\, P(z,y)$ and integrate over z (as in (C.1.2)). But if we take the closed chain $P(x,y)\, P(y,x)$ and consider

the singularities on the light cone, the limit $m \searrow 0$ is regular and well-behaved. This justifies why in Chapters 6–8 it was unnecessary to give the neutrinos a mass and take the limit $m \searrow 0$ afterwards. We could treat the neutrino sector simply as being composed of massless chiral particles. In particular, the chiral cancellations in the formalism of the continuum limit are consistent with the singular mass limit.

Our next goal is to develop the mathematical framework for analyzing the singular mass limit for a fermionic projector with interaction. Clearly, this framework should be general enough to include the examples (C.1.5) and (C.1.12). Thus we first return to (C.1.5). After writing P_m in the form (C.1.10), it seems natural to interpret the leading factor as a generalization of the chiral asymmetry matrix X. This is indeed convenient in the vacuum, because introducing the operator X_m by

$$X_m(k) \;=\; \frac{1}{2}\,(\mathbb{1} - \rho - m\rho \not{v}(k))\,, \tag{C.1.14}$$

we obtain in analogy to the corresponding formulas for massless chiral particles that

$$P_m \;=\; X_m\, t_m \;=\; t_m\, X_m^*\,.$$

Unfortunately, the operator X_m does not seem to be useful in the case with interaction. The reason is that X_m depends on the momentum k, and this leads to the following serious difficulties. First, the k-dependence of X_m makes it very difficult to satisfy the analogue of the causality compatibility condition

$$X_m^*\,(i\partial\!\!\!/ + \mathcal{B} - m) \;=\; (i\partial\!\!\!/ + \mathcal{B} - m)\,X_m\,.$$

As a consequence, it is in general not possible to commute the chiral asymmetry matrix through the operator products of the causal perturbation expansion; in particular $X_m\,\tilde{t}_m$ and $\tilde{t}_m\,X_m$ do in general not coincide (where $\tilde{t}_m$ is the interacting Dirac sea as defined via the causal perturbation expansion). Even if we assume that there is a canonical definition of the fermionic projector P_m obtained by suitably inserting factors of X_m and X_m^* into the operator product expansion for $\tilde{t}_m$, we cannot expect that the correspondence to the massless Dirac sea is respected; i.e. in the case with interaction,(C.1.11) will in general be violated. In order to explain how this comes about, we point out that our argument leading to (C.1.8) was based on the assumption that k converges to the mass cone as $m \searrow 0$. More precisely, if $\lim_{m\searrow 0} k(m)$ is not on the mass cone, the function v will diverge like $v(m) \sim m^{-2}$, and so $X_m(k)$ will not converge to X as $m \searrow 0$. Thus $\lim_{m\searrow 0} X_m = X$ only if in this limit all the momenta are on the mass cone. But in the causal perturbation expansion off-shell momenta also appear (note that the Green's functions are non-zero away from the mass cone). This means that in the limit $m \searrow 0$, the momenta are in general not on the lower mass cone, and so X_m will not converge to X. Because of these problems, we conclude that it is not admissible to first perform the perturbation expansion for t_m and to multiply by X_m afterwards. Instead, the k-dependence of X_m must be taken into account in the perturbation expansion.

At this point it is very helpful that we stated the normalization condition for a chiral Dirac sea in the form (C.1.13). The key observation is that if we substitute (C.1.10) into (C.1.13), compute the operator product and take the limits $m, m' \searrow 0$, all contributions to (C.1.10) which are at least quadratic in m drop out. More precisely, if we expand P_m in the form

$$P_m \;=\; \Big(\chi_L \not{k} \;+\; \frac{m}{2}\,(\mathbb{1} - \rho) \;+\; \frac{m}{2}\,\rho \not{v} \not{k} + \mathcal{O}(m^2)\Big)\;\delta(k^2 - m^2)\,\Theta(-k^0)\,, \tag{C.1.15}$$

the error term is of no relevance for the normalization condition (C.1.13). Taking the inner product of (C.1.8) with k and using the first part of (C.1.4) together with the relation $k^2 = m^2$, one sees that $vk = 1$. We use this identity in (C.1.15) to obtain

$$P_m \;=\; \Big(\chi_L \,\not{k} \;+\; \frac{m}{2} \;+\; \frac{m}{4}\,\rho\,[\not{v}, \not{k}]\Big)\;\delta(k^2 - m^2)\,\Theta(-k^0)\,. \tag{C.1.16}$$

Writing P_m in this form has the advantage that we can pull out the chiral projectors by setting

$$P_m \;=\; \frac{1}{2}\,(X\,\tilde{t}_m + \tilde{t}_m\,X^*) \tag{C.1.17}$$

with $X = \chi_L$ and

$$\tilde{t}_m \;=\; \Big(\not{k} + m + \frac{m}{2}\,\rho\,[\not{v}, \not{k}]\Big)\;\delta(k^2 - m^2)\,\Theta(-k^0)\,.$$

Again neglecting terms quadratic in m, $\tilde{t}_m$ is a solution of the Dirac equation,

$$(i\partial\!\!\!/ + \mathcal{B}_0 - m)\,\tilde{t}_m \;=\; 0\,, \tag{C.1.18}$$

where

$$\mathcal{B}_0(k) \;=\; -\frac{m}{2}\,\rho\,[\not{v}, \not{k}]\,. \tag{C.1.19}$$

The formulation of the vacuum (C.1.17) and (C.1.18, C.1.19) has the advantage that the interaction can easily be introduced. Namely, in order to describe the interaction we simply insert the external potentials into the Dirac equation (C.1.18). In this way, the problems mentioned after (C.1.14) get resolved. Instead of working with a k-dependent chiral asymmetry matrix X_m, the k-dependent vector field v in (C.1.10) is now taken into account by a perturbation $\mathcal{B}_0$ of the Dirac equation, making it possible to apply perturbative methods in the spirit of §2.2.

An obvious technical problem in this approach is that the perturbation operator $\mathcal{B}_0$, (C.1.19), is not of a form previously considered in that it is nonlocal, is not causality compatible and does not decay at infinity. This problem will be analyzed in detail in Section C.2. What makes the problem tractible is that $\mathcal{B}_0$ tends to zero as $m \searrow 0$ and is *homogeneous*, meaning that its kernel $\mathcal{B}_0(x, y)$ depends only on the difference $x - y$.

Let us verify in which generality the above method (C.1.17, C.1.18) applies. In the example (C.1.12), we can write the chiral Dirac sea in the form (C.1.17) with

$$\tilde{t}_m \;=\; \Big(\not{k} + \frac{m}{2}\Big)\;\delta\Big(k^2 - \frac{m^2}{4}\Big)\;\Theta(-k^0)\,, \tag{C.1.20}$$

and $\tilde{t}_m$ is a solution of the Dirac equation (C.1.18) with $\mathcal{B}_0 = m/2$. Thus in this case, $\mathcal{B}_0$ is a homogeneous local operator. More generally, the method of pulling out the chiral asymmetry (C.1.17) applies to any distribution P_m of the form

$$P_m(k) \;=\; \big(\chi_L\,(\text{odd}) + (\text{even}) + \mathcal{O}(m^2)\big)\;\delta(k^2 - c\,m^2)\,\Theta(-k^0)\,,$$

where "(odd)" and "(even)" refer to a product of an odd and even number of Dirac matrices, respectively (and c is a constant). Namely, the corresponding $\tilde{t}_m$ is

$$\tilde{t}_m(k) \;=\; \big((\text{odd}) + 2\,(\text{even}) + \mathcal{O}(m^2)\big)\;\delta(k^2 - c\,m^2)\,\Theta(-k^0)\,.$$

Hence the only restriction of the method (C.1.17, C.1.18) is that the *right-handed odd contribution to* P_m should be *of the order* $\mathcal{O}(m^2)$. For example, our method does *not* apply to

$$P_m(k) \;=\; \big(\chi_L\,\not{k} + m\,\chi_R\,f\,\not{k} + m + \mathcal{O}(m^2)\big)\;\delta(k^2 - m^2)\,\Theta(-k^0)$$

with a scalar function $f(k)$, although in this case the normalization condition (C.1.13) is satisfied. Dropping this restriction would make it necessary to give up (C.1.17) and thus to treat the trace compatibility on a level which goes far beyond what we can accomplish here. It is our view that assuming that the right-handed odd contribution to P_m is of the order $\mathcal{O}(m^2)$ is a reasonable technical simplification.

We close our discussion with a comment on the example (C.1.12). We saw above that P_m can be written in the form (C.1.17) with $\tilde{t}_m$ according to (C.1.20), and that $\tilde{t}_m$ is a solution of the Dirac equation (C.1.18) with the perturbation $\mathcal{B}_0 = m/2$. An alternative point of view is that $\tilde{t}_m$ is a solution of the free Dirac equation of half the mass,

$$(i\partial\!\!\!/ - M)\, \tilde{t}_m \;=\; 0 \qquad \text{with} \qquad M = \frac{m}{2}\,. \tag{C.1.21}$$

We refer to the method of considering a Dirac equation in which the mass parameter is multiplied by a constant as the *modified mass scaling*. The modified mass scaling has the advantage that one can satisfy the normalization conditions for chiral Dirac seas (C.1.13) with P_m according to (C.1.17) and $\tilde{t}_m$ a solution of the free Dirac equation.

C.2. The Homogeneous Perturbation Expansion

In the above examples we saw that there are different methods for giving a chiral Dirac sea a rest mass, which all correspond to inserting a suitable homogeneous operator $\mathcal{B}_0$ into the Dirac equation. Furthermore, we found that the terms quadratic in the mass were irrelevant for the normalization of the Dirac sea, and this suggests that it should be possible to treat $\mathcal{B}_0$ perturbatively. This is indeed possible, as we shall now show for a general class of operators $\mathcal{B}_0$.

For simplicity, we again consider a single Dirac sea. We let $\mathcal{B}_0$ be a *homogeneous operator*, whose further properties will be specified below. In order to keep track of the different orders in perturbation theory, we multiply $\mathcal{B}_0$ by a small parameter $\varepsilon > 0$. Exactly as in (2.6.10), we insert a parameter μ into the Dirac equation, which then reads

$$(i\partial\!\!\!/ + \varepsilon\, \mathcal{B}_0 - \mu\, \mathbb{1})\, \Psi \;=\; 0\,.$$

Here the Dirac operator is homogeneous and is therefore diagonal in momentum space. Thus for given momentum k, the Dirac equation reduces to the 4×4 matrix equation

$$(k\!\!\!/ + \varepsilon\, \mathcal{B}_0(k) - \mu)\, \Psi(k) \;=\; 0\,. \tag{C.2.1}$$

Our aim is to introduce and analyze the spectral projectors and Green's functions of the Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0$, where we regard μ as the eigenvalue. In preparation, we shall now analyze the matrix equation (C.2.1) for fixed k in a perturbation expansion to first order in ε. If $k^2 \neq 0$, the matrix $k\!\!\!/$ is diagonalizable with eigenvalues and spectral projectors

$$\mu_\pm \;=\; \pm\mu_k\,, \qquad E_\pm \;=\; \frac{1}{2}\left(\mathbb{1} \pm \frac{k\!\!\!/}{\mu_k}\right), \tag{C.2.2}$$

where we set $\mu_k = \sqrt{k^2}$ (if $k^2 < 0$, our sign convention is such that μ_k lies in the upper complex half plane). The eigenspaces $\text{Im}\, E_\pm$ are two-dimensional. The spectral projectors $E_\pm$ become singular as $k^2 \to 0$. The reason is that on the mass cone $\mathcal{C} := \{k \,|\, k^2 = 0\}$, the matrix $k\!\!\!/$ is not diagonalizable. We will address this problem later and for the moment simply assume that $k^2 \neq 0$. We next consider the Dirac

operator $\not{k} + \varepsilon \mathcal{B}_0$ for small ε. Perturbing the eigenspaces $\operatorname{Im} E_\pm$ gives rise to two-dimensional invariant subspaces, and a standard calculation shows that the projectors $E^\varepsilon_\pm$ onto these subspaces are given by

$$E^\varepsilon_s \;=\; E_s \;+\; s\,\frac{\varepsilon}{2\mu_k}\,(E_s\,\mathcal{B}_0\,E_{\bar s} + E_{\bar s}\,\mathcal{B}_0\,E_s) \;+\; \mathcal{O}(\varepsilon^2) \tag{C.2.3}$$

with $s = \pm$ and $\bar s = -s = \mp$. It remains to diagonalize the operator $\not{k} + \varepsilon\mathcal{B}_0$ on the invariant subspaces $\operatorname{Im} E^\varepsilon_s$. This is carried out in the next lemma. We choose three (possibly complex) Lorentz vectors $(q_i)_{i=1,2,3}$ such that

$$\langle q_i, k\rangle = 0 \qquad \text{and} \qquad \langle q_i, q_j\rangle = -\delta_{ij}\,. \tag{C.2.4}$$

More precisely, if k is time-like, we choose the (q_i) as a real orthonormal basis of the space-like hypersurface $<k>^\perp$. If on the other hand q is space-like, we choose q_1 and q_2 real and space-like, whereas q_3 is time-like and imaginary. We use the vector notation $\vec q = (q_1, q_2, q_3)$ and introduce the matrices $\Sigma_{1,2,3}$ by

$$\vec\Sigma \;=\; \rho\,\not{\vec q}\,. \tag{C.2.5}$$

LEMMA C.2.1. *Suppose that $k^2 \neq 0$ and that for small ε, the matrix $\not{k} + \varepsilon\mathcal{B}_0$ is diagonalizable. Then its eigenvalues $(\mu^a_s)_{s=\pm,a=1/2}$ are given by*

$$\mu^{1/2}_+ \;=\; \mu_k \;+\; \varepsilon\,(\nu_+ \pm \tau_+) \;+\; \mathcal{O}(\varepsilon^2) \tag{C.2.6}$$

$$\mu^{1/2}_- \;=\; -\mu_k \;+\; \varepsilon\,(\nu_- \mp \tau_-) \;+\; \mathcal{O}(\varepsilon^2)\,, \tag{C.2.7}$$

where

$$\nu_s \;=\; \frac{1}{2}\operatorname{Tr}(E_s\,\mathcal{B}_0) \tag{C.2.8}$$

$$\vec\tau_s \;=\; \frac{1}{2}\operatorname{Tr}(\vec\Sigma\,E_s\,\mathcal{B}_0) \tag{C.2.9}$$

$$\tau_s \;=\; \sqrt{(\tau^1_s)^2 + (\tau^2_s)^2 + (\tau^3_s)^2}\,. \tag{C.2.10}$$

The corresponding spectral projectors can be written as

$$E^a_s \;=\; \Pi^a\,E_s \;+\; s\,\frac{\varepsilon}{2\mu_k}\,(\Pi^a\,E_s\,\mathcal{B}_0\,E_{\bar s} + E_{\bar s}\,\mathcal{B}_0\,\Pi^a\,E_s) \;+\; \mathcal{O}(\varepsilon^2) \tag{C.2.11}$$

with

$$\Pi^{1/2} \;=\; \frac{1}{2}\left(\mathbb{1} \pm \frac{1}{\tau_s}\,\vec\tau_s\vec\Sigma\right). \tag{C.2.12}$$

If $\vec\tau_s = 0$, the invariant subspace $Im\,E^\varepsilon_s$ is an eigenspace to first order in ε; i.e.

$$(\not{k} + \varepsilon\mathcal{B}_0)|_{Im\,E^\varepsilon_s} \;=\; (s\,\mu_k + \varepsilon\,\nu_s)\;\mathbb{1}|_{Im\,E^\varepsilon_s} \;+\; \mathcal{O}(\varepsilon^2)\,.$$

Proof. We restrict attention to the invariant subspace $\operatorname{Im} E^\varepsilon_+$; for E^ε_- the proof is similar. A short calculation using (C.2.5) and (C.2.2, C.2.4) shows that

$$[\Sigma_i,\,E_+] \;=\; 0\,, \qquad \Sigma_i^2 \;=\; \mathbb{1}\,, \qquad \operatorname{Tr}(\Sigma_i\,\Sigma_j\,E_+) \;=\; 2\,\delta_{ij}\,.$$

This means that the matrices Σ_i are invariant on $\operatorname{Im} E_+$, have the eigenvalues ±1 on this subspace and are orthogonal. Thus by choosing a suitable basis (and possibly after changing the orientation of $\vec\Sigma$ by exchanging Σ_1 with Σ_2), we can arrange that the matrices $\vec\Sigma|_{\operatorname{Im} E_+}$ coincide with the Pauli matrices $\vec\sigma$. To first order in ε, the eigenvalues are obtained by diagonalizing $\mathcal{B}_0$ on the unperturbed invariant subspace $\operatorname{Im} E_+$. A short calculation shows that the 2×2 matrix $\nu\mathbb{1} + \vec\tau\vec\sigma$ has the eigenvalues $\nu \pm \tau$ and

corresponding spectral projectors $\Pi_{1/2} = \frac{1}{2}(\mathbb{1} + \frac{1}{\tau}\,\vec{\tau}\vec{\sigma})$ with $\tau = \sqrt{(\tau_1)^2 + (\tau_2)^2 + (\tau_3)^2}$. This gives (C.2.6, C.2.7). Finally, (C.2.11) follows from standard perturbation theory without degeneracies. ∎

To avoid confusion, we point out that in general $\tau_s \neq |\vec{\tau}_s|$ because (C.2.10) involves ordinary squares instead of absolute squares. In particular, it is possible that $\tau_s = 0$ although $\vec{\tau}_s \neq 0$. However, in this case the 2×2 matrix $\varepsilon \mathcal{B}_0|_{\text{Im } E_s}$ is not diagonalizable, and thus the above lemma does not apply.

REMARK C.2.2. In the proof of the previous lemma we used that the three matrices $\Sigma_i|_{\text{Im } E_+}$ can be represented as the Pauli matrices σ_i. It is instructive to verify explicitly that these matrices satisfy the correct commutation relations, for example

$$\frac{i}{2}\,[\Sigma_1,\, \Sigma_2]|_{\text{Im } E_+} \;=\; \Sigma_3|_{\text{Im } E_+}\,.$$

We now give this calculation in detail. By a choice of coordinates, we can arrange that $k = (\omega, \vec{p})$ and $q_{1/2} = (0, \vec{q}_{1/2})$. The standard identity between the Dirac matrices $i\sigma_{jk} = \frac{\rho}{2}\,\epsilon_{jklm}\,\sigma^{lm}$ yields that (possibly after changing the orientation of $\vec{\Sigma}$),

$$i \not{q}_1 \not{q}_2 \;=\; \frac{\rho}{2|\vec{p}|}\,[\not{k}, \gamma^0]\,. \tag{C.2.13}$$

From the definition of $\vec{\Sigma}$, (C.2.5), one sees that $[\Sigma_1, \Sigma_2] = -2 \not{q}_1 \not{q}_2$, and using (C.2.13) as well as the identity $[\mu_k, \gamma^0] = 0$, we conclude that

$$\frac{i}{2}\,[\Sigma_1,\, \Sigma_2] \;=\; -\frac{\rho}{2|\vec{p}|}\,[\not{k} - \mu_k,\, \gamma^0]\,. \tag{C.2.14}$$

In order to simplify the rhs of (C.2.14) on Im E_+, we use that E_+ satisfies the Dirac equation

$$(\not{k} - \mu_k)\, E_+ \;=\; 0\,. \tag{C.2.15}$$

This identity allows us to replace the commutator with $\not{k} - \mu_k$ by an anti-commutator,

$$[\not{k} - \mu_k,\, \gamma^0]\, E_+ \;=\; \{\not{k} - \mu_k,\, \gamma^0\}\, E_+ \;=\; (2\omega - 2\mu_k \gamma^0)\, E_+\,. \tag{C.2.16}$$

Multiplying (C.2.15) by $2\omega/\mu_k$ and adding (C.2.16) gives

$$[\not{k} - \mu_k,\, \gamma^0]\, E_+ \;=\; \frac{2\omega}{\mu_k}\left(\not{k} - \frac{\mu^2}{\omega}\,\gamma^0\right) E_+\,.$$

Using this identity in (C.2.14) gives

$$\frac{i}{2}\,[\Sigma_1,\, \Sigma_2]|_{\text{Im } E_+} \;=\; \rho\, \not{q}_3|_{\text{Im } E_+}$$

with

$$q_3 \;=\; -\frac{\omega}{\mu_k\, |\vec{p}|}\left(\not{k} - \frac{\mu_k^2}{\omega}\,\gamma^0\right),$$

and a short calculation shows that this vector q_3 has indeed all the properties listed after (C.2.4).

We shall now define the spectral projectors and Green's functions corresponding to the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$. We denote the spectrum of the matrix in (C.2.1) by $\sigma^\varepsilon(k)$,

$$\sigma^\varepsilon(k) \;=\; \sigma(\not{k} + \varepsilon \mathcal{B}_0(k))\,.$$

It is natural to define the spectrum σ^ε of the Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0$ as the union of the $\sigma^\varepsilon(k)$s,

$$\sigma^\varepsilon \;=\; \bigcup_{k\in\mathbb{R}^4} \sigma^\varepsilon(k)\,.$$

As we saw above, the matrix $k\!\!\!/ + \varepsilon\mathcal{B}_0(k)$ in general is not diagonalizable, and thus we cannot introduce the spectral projectors for all k pointwise. But since the diagonalizable matrices are dense in $\mathrm{Gl}(\mathbb{C}^4)$, it is reasonable to assume that the matrix $k\!\!\!/ + \varepsilon\mathcal{B}_0(k)$ is *diagonalizable for almost all (a.a.) k*. Our formalism will involve momentum integrals where sets of measure zero are irrelevant. Therefore, we may in what follows restrict attention to those k for which the matrix $k\!\!\!/ + \varepsilon\mathcal{B}_0(k)$ is diagonalizable. Moreover, we shall assume that $\mathcal{B}_0$ is *smooth* and *bounded*. According to (C.2.2), the spectrum of the unperturbed Dirac operator is $\sigma^{\varepsilon=0} = \mathbb{R}\cup i\mathbb{R}$. The next lemma shows that the real part of the spectrum is stable under perturbations.

LEMMA C.2.3. *Suppose that $k^2 > 0$. Then for ε sufficiently small, $\sigma^\varepsilon(k) \subset \mathbb{R}$.*

Proof. Choosing coordinates such that $k = (\omega, \vec{0})$, it is obvious that the eigenspaces of $k\!\!\!/$ are definite, i.e.

$$\prec\Psi \mid \Psi\succ \neq 0 \qquad \text{for all eigenvectors } \Psi.$$

By continuity, the eigenspaces of $k\!\!\!/ + \varepsilon\mathcal{B}_0(k)$ will also be definite for sufficiently small ε. As a consequence, the corresponding eigenvalues are real, because

$$\lambda \prec\Psi \mid \Psi\succ \;=\; \prec\Psi \mid (k\!\!\!/ + \varepsilon\mathcal{B}_0)\,\Psi\succ \;=\; \prec(k\!\!\!/ + \varepsilon\mathcal{B}_0)\,\Psi \mid \Psi\succ \;=\; \overline{\lambda}\prec\Psi \mid \Psi\succ\,. \qquad \blacksquare$$

Unfortunately, we have a-priori no control of how the imaginary part of the spectrum changes with ε. For this reason, it is most convenient to introduce the spectral projectors for all $\mu \in \mathbb{C}$, such that they vanish identically for $\mu \notin \sigma^\varepsilon$. For the normalization, we work with δ-distributions supported at one point in the complex plane. More precisely, we set

$$\begin{aligned} \delta^2(z) &= \delta(\mathrm{Re}\, z)\,\delta(\mathrm{Im}\, z) \\ \int_{\mathbf{C}} d^2z\,\cdots &= \int_{\mathbb{R}^2} d(\mathrm{Re}\, z)\,d(\mathrm{Im}\, z)\,\cdots\,. \end{aligned}$$

DEF. C.2.4. *For $\mu \in \mathbb{C}$ and $k \in \mathbb{R}^4$ we set*

$$p_\mu^\varepsilon(k) \;=\; \sum_{s=\pm,\, a=1/2} E_s^a(k)\,\delta^2(\mu - \mu_s^a(k)) \tag{C.2.17}$$

$$k_\mu^\varepsilon(k) \;=\; \epsilon(k^0)\,p_\mu^\varepsilon(k) \tag{C.2.18}$$

$$s_\mu^\varepsilon(k) \;=\; \int_{\mathbf{C}} d^2\nu\,\frac{PP}{\mu-\nu}\,p_\nu^\varepsilon(k)\,. \tag{C.2.19}$$

We also consider p_μ^ε, k_μ^ε, and s_μ^ε as multiplication operators in momentum space.

In formal calculations, the operators p_μ^ε and k_μ^ε are solutions of the Dirac equation,

$$(i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - \mu)\,p_\mu^\varepsilon \;=\; 0 \;=\; (i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - \mu)\,k_\mu^\varepsilon\,,$$

and satisfy in analogy to (2.2.26–2.2.29) the multiplication rules

$$p^\varepsilon_\mu \, p^\varepsilon_{\mu'} \;=\; k^\varepsilon_\mu \, k^\varepsilon_{\mu'} \;=\; \delta^2(\mu-\mu')\, p^\varepsilon_\mu \tag{C.2.20}$$

$$p^\varepsilon_\mu \, k^\varepsilon_{\mu'} \;=\; k^\varepsilon_\mu \, p^\varepsilon_{\mu'} \;=\; \delta^2(\mu-\mu')\, k^\varepsilon_\mu \tag{C.2.21}$$

as well as the "completeness relation"

$$\int_{\mathbf{C}} p^\varepsilon_\mu \, d^2\mu \;=\; \mathbb{1}\,.$$

Using these identities in (C.2.19) yields that

$$(i\partial\!\!\!/ + \varepsilon \mathcal{B}_0 - \mu)\, s^\varepsilon_\mu \;=\; \mathbb{1}\,.$$

Thus on a formal level, the operators p^ε_μ, k^ε_μ and s^ε_μ are the spectral projectors and Green's functions of the Dirac operator, respectively. In order to give these operators a mathematical meaning, we can proceed as follows. Let k be such that the matrix $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$ can be diagonalized. Then the functional calculus for finite matrices (as defined e.g. via the approximation by polynomials) allows us to introduce for $f \in C^1(\mathbb{C})$ the matrix $f(k\!\!\!/ + \varepsilon \mathcal{B}_0(k))$. Formally, we can write the functional calculus with the spectral projectors,

$$\int_{\mathbf{C}} f(\mu)\, p^\varepsilon_\mu(k)\, d^2\mu \;=\; f(k\!\!\!/ + \varepsilon \mathcal{B}_0(k))\,. \tag{C.2.22}$$

We can use this relation to give the integral in (C.2.22) a rigorous sense for a.a. k. The same argument applies to k^ε_μ. For s^ε_μ, we can similarly use the formal identity

$$\int_{\mathbf{C}} f(\mu)\, s^\varepsilon_\mu(k)\, d^2\mu \;\overset{(C.2.19)}{=}\; \int_{\mathbf{C}} g(\mu)\, p^\varepsilon_\nu(k)\, d^2\mu \tag{C.2.23}$$

with

$$g(\nu) \;=\; \int_{\mathbf{C}} \frac{\mathrm{PP}}{\mu-\nu}\, f(\mu)\, d^2\mu\,.$$

In this way, one sees that the operators p^ε_μ, k^ε_μ and s^ε_μ are well-defined when evaluated weakly in μ and k.

Under additional assumptions, we can make sense of the operators in Def. C.2.4 even for fixed real μ. We first justify the δ-distribution and the principal part.

LEMMA C.2.5. *Suppose that for a given interval $I \subset \mathbb{R}$, the spectral projectors E^a_s in (C.2.17) are bounded uniformly in $\mu \in I$ and $k \in \mathbb{R}^4$. Then for a.a. $\mu \in I$, the operators p^ε_μ, k^ε_μ, and s^ε_μ are well-defined distributions in momentum space.*

Proof. We write the Dirac equation $(k\!\!\!/ + \varepsilon \mathcal{B}_0(k))\Psi = 0$ in the Hamiltonian form

$$\omega\, \Psi \;=\; H(\omega,\mu)\, \Psi \qquad \text{with} \qquad H(\omega,\mu) \;=\; -\gamma^0\, (\vec{k\!\!\!/} + \varepsilon \mathcal{B}_0(\omega,\vec{k}) - \mu \mathbb{1})$$

and $k = (\omega, \vec{k})$. In what follows we keep $\vec{k}$ fixed and consider this equation for variable parameters $\omega, \mu \in \mathbb{R}$. The matrix $H(\omega,\mu)$ is Hermitian with respect to the positive scalar product $(.|.) = \prec.|\gamma^0|.\succ$. Thus it can be diagonalized; we denote its eigenvalues (counting multiplicities) by $\Omega_1 \leq \cdots \leq \Omega_4$. The min-max principle (see [**RS**]) allows us to write Ω_n as

$$\Omega_n \;=\; \min_{U,\, \dim U = n} \; \max_{u \in U,\, \|u\|=1} \|Hu\|\,,$$

where $\|.\|$ is the norm induced by $(.|.)$ and U denotes a subspace of $\mathbb{C}^4$. It follows from this representation that the Ω_n depend Lipschitz-continuously on ω and μ. Namely,

$$\begin{aligned}
\Omega_n(\omega) &= \min_{U,\,\dim U=n} \max_{u\in U,\,\|u\|=1} \|H(\omega)\,u\| \\
&= \min_{U,\,\dim U=n} \max_{u\in U,\,\|u\|=1} \|H(\omega')\,u + (H(\omega)-H(\omega'))\,u\| \\
&\le \min_{U,\,\dim U=n} \max_{u\in U,\,\|u\|=1} \big(\|H(\omega')\,u\| + \|H(\omega)-H(\omega')\|\,\|u\|\big) \\
&= \Omega_n(\omega') + \|H(\omega)-H(\omega')\|\,.
\end{aligned}$$

Using that $\mathcal{B}_0(k)$ is C^1 with bounded derivatives, we obtain the estimate

$$\|H(\omega)-H(\omega')\| \;\le\; \|\varepsilon\gamma^0\,(\mathcal{B}_0(\omega)-\mathcal{B}_0(\omega'))\| \;\le\; \varepsilon c\,|\omega-\omega'|$$

and thus $\Omega_n(\omega)-\Omega_n(\omega')\le\varepsilon c|\omega-\omega'|$. Exchanging the roles of ω and ω' gives the bound

$$|\Omega_n(\omega)-\Omega_n(\omega')| \;\le\; \varepsilon c\,|\omega-\omega'|\,. \tag{C.2.24}$$

A similar calculation shows that

$$|\Omega_n(\mu)-\Omega_n(\mu')| \;\le\; |\mu-\mu'|\,. \tag{C.2.25}$$

We next consider for given n the equation

$$\omega \;=\; \Omega_n(\omega,\mu)\,. \tag{C.2.26}$$

The following argument shows that for sufficiently small ε, this equation has a unique solution ω_n, which depends Lipschitz-continuously on μ. Let ϕ (for fixed μ and n) be the mapping

$$\phi \;:\; \mathbb{R}\to\mathbb{R} \;:\; \omega\mapsto\Omega_n(\omega,\mu)\,.$$

According to (C.2.24),

$$|\phi(\omega)-\phi(\omega')| \;=\; |\Omega_n(\omega)-\Omega_n(\omega')| \;\le\; \varepsilon c\,|\omega-\omega'|\,.$$

Thus if we choose ε small enough, ϕ is a contraction. The Banach fixed point theorem yields a unique fixed point ω_n. The dependence on the parameter μ is controlled by (C.2.24) and (C.2.25). Namely,

$$\begin{aligned}
|\omega_n(\mu)-\omega_n(\mu')| &= |\Omega_n(\omega_n(\mu),\mu)-\Omega_n(\omega_n(\mu'),\mu')| \\
&\le \varepsilon c\,|\omega_n(\mu)-\omega_n(\mu')| \;+\; |\mu-\mu'|
\end{aligned}$$

and thus

$$|\omega_n(\mu)-\omega_n(\mu')| \;\le\; (1-\varepsilon c)^{-1}\,|\mu-\mu'|\,. \tag{C.2.27}$$

If we regard the spectral projector (C.2.17) as a distribution in ω, it is supported at those ω for which the Dirac equation $(\not{k}+\varepsilon\mathcal{B}_0-\mu)\Psi=0$ has a non-trivial solution. These are precisely the solutions ω_n of the equation (C.2.26). Thus we can write p^ε_μ as

$$p^\varepsilon_\mu \;=\; \sum_{n=1}^{4} E^a_s(\omega_n)\;\delta(\omega-\omega_n)\;\delta(\mathrm{Im}\,\mu)\left|\frac{\partial\omega(\mu)}{\partial\mu}\right|\,, \tag{C.2.28}$$

where the parameters $a=a(n)$ and $s=s(n)$ must be chosen such that $\mu^a_s(\omega_n)=\mu$. Since $\omega_n(\mu)$ is Lipschitz (C.2.27), the factor $|\partial_\mu\omega_n(\mu)|$ in (C.2.28) is well-defined for a.a. μ and is uniformly bounded. Thus $p^\varepsilon_\mu(\omega)$ is a well-defined distribution for a.a. μ. The same argument applies to k^ε_μ.

It remains to justify the Green's function s_μ^ε. We can write it in the Hamiltonian framework as

$$s_\mu^\varepsilon \;=\; \frac{\mathrm{PP}}{\not{k} + \varepsilon \mathcal{B}_0 - \mu\, \mathbb{1}} \;=\; \frac{\mathrm{PP}}{\omega - H(\omega,\mu)}\, \gamma^0 \,.$$

Thus denoting the spectral projectors of H by $(F_n)_{n=1,\ldots,4}$, we have

$$s_\mu^\varepsilon(\omega) \;=\; \sum_{n=1}^4 \frac{\mathrm{PP}}{\omega - \Omega_n(\omega,\mu)}\, F_n(\omega,\mu)\, \gamma^0 \,. \tag{C.2.29}$$

According to (C.2.24), $\Omega_n(\omega)$ is Lipschitz and thus differentiable almost everywhere with $|\partial_\omega \Omega_n| \leq \varepsilon c$. The spectral projectors $F_n(\omega)$ can also be chosen to be Lipschitz. As a consequence, the principal part in (C.2.29) is well-defined for a.a. μ. ∎

This lemma involves the strong assumption that the spectral projectors E_s^a must be uniformly bounded. We shall now analyze this assumption in detail. As one sees from (C.2.2) in the limit $\mu \to 0$, the spectral projectors can have poles and thus in general are *not* uniformly bounded. Thus we need to impose an extra condition, which we will formulate using the following notion.

Def. C.2.6. *Let A be a 4×4 matrix, which is Hermitian (with respect to $\prec.|.\succ$). A point $\mu \in \sigma(A)$ is called ε-**definite** if there is a subset $\sigma_+ \subset \sigma(A)$ such that*

(i) *The invariant subspace I_+ corresponding to σ_+ is definite.*
(ii) *dist$(\sigma_+,\, \sigma(A) \setminus \sigma_+) > \varepsilon$.*

Lemma C.2.7. *If $\mu \in \sigma(A)$ is ε-definite, the matrix A is diagonalizable on I_+, and its spectral projectors E_a are bounded by*

$$\|E_a\| \;\leq\; c \left(\frac{\|A\|}{\varepsilon} \right)^3 , \tag{C.2.30}$$

where $\|.\|$ is a matrix norm and c is a constant which depends only on the choice of the norm $\|.\|$.

Proof. It clearly suffices to consider a particular matrix norm. We introduce the positive scalar product $(.|.) = \prec.|\gamma^0|.\succ$, let $\|.\| = (.|.)^{\frac{1}{2}}$ be the corresponding norm, and set

$$\|A\| \;=\; \sup_{\Psi \text{ with } \|\Psi\|=1} \|A\Psi\| \,.$$

We denote the projector onto I_+ by E. E can be constructed with a functional calculus. Namely, let $\mathcal{P}(z)$ be a complex polynomial satisfying the conditions

$$\mathcal{P}|_{\sigma_+} \;=\; 1 \qquad \text{and} \qquad \mathcal{P}|_{\sigma_-} \;=\; 0 \,.$$

Since these are at most four conditions, $\mathcal{P}$ can be chosen to be of degree three,

$$\mathcal{P}(z) \;=\; \sum_{n=0}^3 c_n\, z^n \,.$$

Furthermore, the fact that A is ε-definite can be used to bound the coefficients c_n by

$$|c_n| \;\leq\; \frac{C}{\varepsilon^n} \tag{C.2.31}$$

with a suitable constant C (this is easily seen from a scaling argument). The projector E is given by $E = \mathcal{P}(A)$, and (C.2.31) gives the estimate

$$\|E\| \;\leq\; \sum_{n=0}^{3} \frac{C}{\varepsilon^n}\, \|A\|^n \;\leq\; C \left(\frac{\|A\|}{\varepsilon} \right)^3 , \tag{C.2.32}$$

where we used $\varepsilon < \|A\|$ in the last step.

By definition, $\text{Im}\, E = I_+$ is a definite subspace. We can assume without loss of generality that it is positive, i.e.

$$\prec \Psi \,|\, E\, \Psi \succ \;\geq\; 0 \qquad \text{for all } \Psi.$$

The matrix $A|_{I_+}$ is Hermitian with respect to the positive scalar product $\prec .|.\succ|_{I_+}$. Thus it has a spectral decomposition with eigenvalues μ_a and corresponding spectral projectors E_a, $a = 1, \ldots, N$,

$$A|_{I_+} \;=\; \sum_{a=1}^{n} \mu_a\, E_a|_{I_+} \,.$$

Extending the E_a by zero to the invariant subspace corresponding to $\sigma(A) \setminus \sigma_+$, the spectral projectors satisfy the relations

$$E_a^* = E_a = E_a^2 \,, \qquad \sum_{a=1}^{N} E_a = E \,, \qquad \prec \Psi \,|\, E_a\, \Psi \succ \geq 0 \;\text{ for all } \Psi,$$

where the star denotes the adjoint with respect to $\prec .|.\succ$.

We introduce the operators F and F_a by

$$F \;=\; \gamma^0\, E \,, \qquad F_a \;=\; \gamma^0\, E_a \,.$$

It is straightforward to check that these operators have the following properties,

$$F_a^+ \;=\; F_a \,, \qquad (\Psi \,|\, F_a\, \Psi) \;\geq\; 0 \tag{C.2.33}$$

$$\sum_a F_a \;=\; F \,, \tag{C.2.34}$$

where "$+$" denotes the adjoint with respect to $(.|.)$. The relations (C.2.33) mean that the F_a are positive self-adjoint operators on a Hilbert space. This makes it possible to estimate the norm of the spectral projectors as follows,

$$\begin{aligned} \|E_a\| \;&=\; \|\gamma^0\, F_a\| \;\leq\; \|\gamma^0\|\, \|F_a\| \;\leq\; \|F_a\| \;=\; \sup_{\Psi \text{ with } \|\Psi\|=1} (\Psi \,|\, F_a\, \Psi) \\ &\leq\; \sup_{\Psi \text{ with } \|\Psi\|=1} \sum_{b=1}^{N} (\Psi \,|\, F_b\, \Psi) \;=\; \sup_{\Psi \text{ with } \|\Psi\|=1} (\Psi \,|\, F\, \Psi) \\ &=\; \|F\| \;=\; \|\gamma^0\, E\| \;\leq\; \|E\| \,. \end{aligned}$$

We now apply (C.2.32) to obtain (C.2.30). ∎

DEF. C.2.8. *The Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an* **ε-definite kernel** *if for all $\mu \in (-\varepsilon, \varepsilon)$ and all k with $\mu \in \sigma^\varepsilon(k)$, μ is in the ε-definite spectrum of the matrix $k\!\!\!/ + \varepsilon \mathcal{B}_0(k)$.*

Combining Lemma C.2.5 and Lemma C.2.7 gives the following result.

THEOREM C.2.9. *If the Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0$ has an ε-definite kernel, then its spectral projectors and Green's functions (as given in Def. C.2.4) are well-defined distributions in momentum space for a.a. $\mu \in (-\varepsilon, \varepsilon)$.*

It remains to specify under which assumptions on $\mathcal{B}_0$ the Dirac operator has an ε-definite kernel. We decompose $\mathcal{B}_0$ as

$$\mathcal{B}_0(k) \;=\; \alpha\, \mathbb{1} + i\beta\, \rho + v\!\!\!/ + \rho\, a\!\!\!/ + \frac{i\rho}{2}\, w_{ij}\, \sigma^{ij} \,. \tag{C.2.35}$$

Here α, β, v, a, and w are real potentials (namely the scalar, pseudoscalar, vector, axial, and bilinear potentials, respectively; clearly we assume w to be anti-symmetric). We introduce the function $\Delta(k)$ as the following combination of the axial and bilinear potentials,

$$\Delta^2 \;=\; -k^2\, \langle a, a\rangle + \langle a, k\rangle^2 - w_{ij} k^j\, w^{il} k_l \,. \tag{C.2.36}$$

The first two summands can also be written as

$$-\, k^2\, \langle a, a\rangle + \langle a, k\rangle^2 \;=\; -k^2 \left(a - \frac{1}{k^2}\, \langle a, k\rangle\, k \right)^2 \,. \tag{C.2.37}$$

For timelike k, the vector inside the round brackets is spacelike, and thus $(C.2.37) \geq 0$. Similarly, the vector $w_{ij}k^j$ is spacelike for k timelike. We conclude that

$$\Delta(k) \;\geq\; 0 \qquad \text{if } k^2 > 0. \tag{C.2.38}$$

Furthermore, $\Delta(q)$ vanishes on the mass cone $\mathcal{C} = \{q^2 = 0\}$ if and only if q is collinear to the vector a and is an eigenvector of w,

$$a \;=\; \nu q \qquad \text{and} \qquad w_{ij} q^j \;=\; \lambda\, q_i \qquad (\nu, \lambda \in \mathbb{R},\, q \in \mathcal{C}). \tag{C.2.39}$$

Expanding (C.2.36), one sees that in this case, Δ is finite to the next order on the light cone, i.e.

$$\Delta(q) = 0 \quad \Longrightarrow \quad l \equiv \lim_{k \to q} \frac{1}{k^2}\, \Delta(k) \text{ exists.} \tag{C.2.40}$$

Qualitatively speaking, the next theorem states that the Dirac operator has an ε-definite kernel if and only if the scalar potential is non-zero and dominates the axial and bilinear potentials.

THEOREM C.2.10. *Suppose that for all $q \in \mathcal{C}$,*

$$|\alpha(q)| \;>\; \frac{3}{2} + \begin{cases} \left| \dfrac{w_{ij}(q) a^i q^j}{\Delta(q)} \right| & \text{if } \Delta(q) \neq 0 \\[2ex] \Big(1 + \Theta(1 - 2\sqrt{|l(q)|}\Big)\, \sqrt{|l(q)|} & \text{if } \Delta(q) = 0. \end{cases} \tag{C.2.41}$$

Then for sufficiently small ε, the Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0$ has an ε-definite kernel. If conversely there is $q \in \mathcal{C}$ for which the opposite inequality holds (i.e. (C.2.41) with "$>$" replaced by "$<$"), then the Dirac operator has no ε-definite kernel.

Proof. A short calculation using (C.2.8, C.2.2, C.2.35) gives

$$\nu_\pm \;=\; \alpha \pm \frac{1}{\mu_k}\, \langle v, k\rangle \,. \tag{C.2.42}$$

In the special case $k\!\!\!/ = \mu_k\, \gamma^0$ and $\vec{q} = \vec{\gamma}$, we obtain furthermore from (C.2.9) that

$$(\tau_\pm)_r \;=\; a_r \pm w_{r0} \qquad (r = 1, 2, 3).$$

Thus, according to (C.2.10),

$$(\tau_\pm)^2 \;=\; \sum_{r=1}^3 (a_r)^2 \pm 2\, a_r\, w_{r0} + (w_{r0})^2\,,$$

and this can be written covariantly as

$$(\tau_\pm)^2 \;=\; -\langle a,a\rangle + \frac{1}{\mu_k^2}\,\langle a,k\rangle^2 - \frac{1}{\mu_k^2}\, w_{ij} k^i\, w^{il} k_l \mp \frac{2}{\mu_k}\, w_{ij} a^i k^j\,. \tag{C.2.43}$$

This tensor equation is valid for any time-like k, and it is easy to check that it holds for spacelike k as well.

Let $q \in \mathcal{C}$. We first consider the case $\Delta(q) \neq 0$. By continuity, $\Delta \neq 0$ in a neighborhood U of q, and according to (C.2.38), Δ is positive in U. We substitute (C.2.42) and (C.2.43) into (C.2.6) and (C.2.7). In order to remove the singularities at $\mu_k = 0$, we write the eigenvalues μ_s^a in the form

$$\left.\begin{aligned} \mu_+^{1/2} &= \sqrt{k^2 + 2\varepsilon\delta_{1/2}} + \varepsilon\,(\alpha \pm \kappa_+) + \mathcal{O}(\varepsilon^2) \\ \mu_-^{1/2} &= -\sqrt{k^2 + 2\varepsilon\delta_{1/2}} + \varepsilon\,(\alpha \mp \kappa_-) + \mathcal{O}(\varepsilon^2) \end{aligned}\right\}, \tag{C.2.44}$$

where we set

$$\delta_{1/2} \;=\; \langle v,k\rangle \pm \Delta\,, \qquad\qquad \kappa_\pm \;=\; \tau_\pm - \frac{1}{\mu_k}\,\Delta\,.$$

The functions $\kappa_\pm$ have the following expansion,

$$\kappa_\pm \;=\; \frac{1}{\mu_k}\left(\sqrt{\Delta^2 \mp 2\mu_k\, w_{ij} a^i k^j} - \Delta\right) \;=\; \mp \frac{w_{ij} a^i k^j}{\Delta} \;+\; \mathcal{O}(\mu_k)\,. \tag{C.2.45}$$

In particular, one sees that these functions are bounded locally uniformly in μ_k. Let us verify under which conditions the Dirac operator restricted to U has an ε-definite kernel. Suppose that $\mu_+^1 \in (-\varepsilon, \varepsilon)$. Then, due to the square root in (C.2.44),

$$k^2 + 2\varepsilon\delta_1 \;=\; \mathcal{O}(\varepsilon^2)\,.$$

It follows from (C.2.44) that

$$\begin{aligned} \mu_\pm^2 &= \sqrt{k^2 + 2\varepsilon\delta_2} + \mathcal{O}(\varepsilon) \;=\; \sqrt{\mathcal{O}(\varepsilon^2) + 2\varepsilon(\delta_2 - \delta_1)} + \mathcal{O}(\varepsilon) \\ &= \sqrt{-4\varepsilon\,\Delta} + \mathcal{O}(\varepsilon) \;\sim\; \sqrt{\varepsilon} \end{aligned}$$

and therefore

$$|\mu_+^1 - \mu_+^2| \;\sim\; \sqrt{\varepsilon} \;\gg\; \varepsilon\,.$$

Moreover, we obtain from (C.2.44) and (C.2.45) that

$$\begin{aligned} \mu_+^1 - \mu_-^1 &= 2\mu_+^1 + 2\varepsilon\alpha - \varepsilon(\kappa_+ - \kappa_-) + \mathcal{O}(\varepsilon^2) \\ &= 2\mu_+^1 + 2\varepsilon\left(\alpha + \frac{w_{ij}\, a^i k^j}{\Delta} + \mathcal{O}(\mu_k)\right) + \mathcal{O}(\varepsilon^2)\,. \end{aligned}$$

Thus the condition $|\mu_+^1 - \mu_-^1| > \varepsilon$ is satisfied if

$$\left|\alpha + \frac{w_{ij} a^i k^j}{\Delta}\right| \;>\; \frac{3}{2}\,.$$

As is proved in Lemma C.2.11 below, the eigenspace corresponding to μ_+^1 is definite. We conclude that μ_+^1 is an ε-definite eigenvalue of A. Repeating the above argument in the three other cases $\mu_-^2, \mu_\pm^2 \in (-\varepsilon, \varepsilon)$, one obtains that for sufficiently small ε, the

kernel of the Dirac operator is ε-definite in U. If conversely (C.2.41) holds with ">" replaced by "<", it is straightforward to check that the Dirac operator for small ε has no ε-definite kernel.

It remains to consider the case $\Delta(q) = 0$. We write the eigenvalues μ_s^a as

$$\left.\begin{aligned} \mu_+^{1/2} &= \sqrt{k^2 + 2\varepsilon \langle v, k\rangle} + \varepsilon\,(\alpha \pm \tau_+) \\ \mu_-^{1/2} &= -\sqrt{k^2 + 2\varepsilon \langle v, k\rangle} + \varepsilon\,(\alpha \mp \tau_-)\,. \end{aligned}\right\} \tag{C.2.46}$$

According to (C.2.40), the first three summands in (C.2.43) have a finite limit at q. Furthermore, (C.2.39) yields that

$$\frac{2}{\mu_k}\, w_{ij} a^i k^j \;=\; \mathcal{O}(\mu_k)\,.$$

We conclude that the functions $\tau_\pm$ in a neighborhood of q have the expansion

$$\tau_\pm \;=\; \sqrt{|l|} + \mathcal{O}(\sqrt{|\mu_k|})\,. \tag{C.2.47}$$

For small ε, $\mu_k \sim \sqrt{\varepsilon}$, and so the term $\mathcal{O}(\sqrt{\mu_k})$ is of higher order in ε and can be omitted. Furthermore, the following continuity argument varying l shows that the eigenvalues μ_+^a and μ_-^a correspond to positive and negative eigenvectors, respectively: If $l = 0$, only the scalar and vector potentials enter the perturbation calculation to first order in ε (see (C.2.46, C.2.47)). If only scalar and vector potentials are present, the spectral decomposition of the matrix $\not{k} + \varepsilon \mathcal{B}_0$ is easily obtained from the identity

$$[(\not{k} + \varepsilon \not{v} + \varepsilon\alpha) - \varepsilon\alpha]^2 \;=\; (k + \varepsilon v)^2\, \mathbb{1}\,.$$

One sees that the eigenvalues are twofold degenerate, $\sigma^\varepsilon(k) = \{\mu_+, \mu_-\}$, and that if they are real, the corresponding eigenspaces are definite. The parameter l removes the degeneracy of these eigenspaces, but the resulting invariant subspaces remain definite.

Suppose that $\mu_+^1 \in (-\varepsilon, \varepsilon)$. We consider the two subcases $2\sqrt{|l|} > \varepsilon$ and $2\sqrt{|l|} < \varepsilon$ separately. In the first case, $|\mu_s^1 - \mu_s^2| > \varepsilon$, and thus we must arrange that

$$|\mu_+^1 - \mu_\mp^2| > \varepsilon \qquad (2\sqrt{l} > \varepsilon). \tag{C.2.48}$$

In the second case, $|\mu_s^1 - \mu_s^2| < \varepsilon$. Thus we must combine the eigenvalues to pairs and consider the definite eigenspaces corresponding to the sets $\sigma_s = \{\mu_s^1, \mu_s^2\}$, $s = \pm$, and must satisfy the condition

$$\text{dist}\,(\sigma_+, \sigma_-) \;>\; \varepsilon \qquad (2\sqrt{l} < \varepsilon). \tag{C.2.49}$$

Evaluating (C.2.48) and (C.2.49) using (C.2.46, C.2.47) and analyzing similarly the three other cases $\mu_-^1, \mu_\pm^2 \in (-\varepsilon, \varepsilon)$ gives the condition (C.2.41). ∎

LEMMA C.2.11. *Let A be a Hermitian matrix (with respect to $\prec .|. \succ$). If $\mu \in \sigma(A)$ is real and the corresponding invariant eigenspace I is one-dimensional, then I is a definite eigenspace.*

Proof. Since each invariant subspace contains at least one eigenvector, I is clearly an eigenspace. We must show that I is definite. Assume to the contrary that $I = <\Psi>$ is null, i.e.

$$A\Psi = \lambda\Psi \qquad \text{with} \qquad \lambda \in \mathbb{R} \text{ and } \prec \Psi | \Psi \succ = 0.$$

We denote the invariant subspaces of A by $(I_\mu)_{\mu \in \sigma(A)}$. Since $I_\lambda = <\Psi>$ is one-dimensional and null, there must be an invariant subspace I_μ, $\mu \neq \lambda$, which is not orthogonal to Ψ,

$$I_\mu \cap <\Psi>^\perp \;\neq\; \emptyset\,.$$

We choose on I_μ a basis $(e_1, \ldots, e_n)$ such that A is in the Jordan form, i.e.

$$A|_{I_\mu} \;=\; \begin{pmatrix} \mu & 1 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \mu \end{pmatrix}.$$

Let $k \in \{1, \ldots, n\}$ be the smallest index such that $\prec e_k | \Psi \succ \neq 0$. Then

$$\begin{aligned} \lambda \prec e_k | \Psi \succ &= \prec e_k \,|\, A\Psi \succ \;=\; \prec A e_k \,|\, \Psi \succ \\ &= \mu \prec e_k \,|\, \Psi \succ \;+\; \prec e_{k-1} \,|\, \Psi \succ \;=\; \mu \prec e_k \,|\, \Psi \succ . \end{aligned}$$

This is a contradiction. ∎

Suppose that the homogeneous operator $\mathcal{B}_0$ satisfies the condition (C.2.41) in Theorem C.2.10. Then the Dirac operator has an ε-definite kernel. As a consequence, the distributions $t^\varepsilon_\mu = \frac{1}{2}(p^\varepsilon_\mu - k^\varepsilon_\mu)$ are well-defined (see Def. C.2.4 and Theorem C.2.9). Following (C.1.17), we introduce the fermionic projector by

$$P^\varepsilon_\mu \;=\; \frac{1}{2}\left(X\, t^\varepsilon_\mu + t^\varepsilon_\mu\, X^*\right) \tag{C.2.50}$$

with $X = \chi_L$. In order to analyze the normalization of P^ε_μ, we consider the product

$$P^\varepsilon_\mu\, P^\varepsilon_{\mu'} \;=\; \frac{1}{4}\left(X\, t^\varepsilon_\mu\, t^\varepsilon_{\mu'}\, X^* + X\, t^\varepsilon_\mu\, X\, t^\varepsilon_{\mu'} + t^\varepsilon_\mu\, X^*\, t^\varepsilon_{\mu'}\, X^*\right). \tag{C.2.51}$$

According to (C.2.20) and (C.2.21),

$$t^\varepsilon_\mu\, t^\varepsilon_{\mu'} \;=\; \delta^2(\mu - \mu')\, t^\varepsilon_\mu\,. \tag{C.2.52}$$

Thus the only problem is to compute the products $t^\varepsilon_\mu X t^\varepsilon_{\mu'}$ and $t^\varepsilon_\mu X^* t^\varepsilon_{\mu'}$. Using the relations $\chi_{L\,/\,R} = \frac{1}{2}(\mathbb{1} \mp \rho)$ together with (C.2.52), this problem reduces to making mathematical sense of the operator product

$$t^\varepsilon_\mu\, \rho\, t^\varepsilon_{\mu'}\,.$$

It seems impossible to give this expression a meaning without making additional assumptions on $\mathcal{B}_0$. For simplicity, we shall impose a quite strong condition, which is motivated as follows. The spectral projectors p_μ corresponding to the unperturbed Dirac operator $i\partial\!\!\!/ - \mu$ satisfy the relations $\rho\, p_\mu\, \rho = p_{-\mu}$ and thus $p_\mu\, \rho\, p_\mu = 0$ ($\mu > 0$). It is natural to demand that the last identity should also hold in the presence of the homogeneous perturbation for small ε.

DEF. C.2.12. *The kernel of the homogeneous Dirac operator $i\partial\!\!\!/ + \mathcal{B}(\varepsilon, k)$ is* **ε-orthogonal to** *ρ if for all $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, the corresponding spectral projectors $E_\mu(k)$ and $E_{\mu'}(k)$ satisfy the condition*

$$E_\mu\, \rho\, E_{\mu'} \;=\; 0\,. \tag{C.2.53}$$

If the kernel of the Dirac operator is ε-definite and ε-orthogonal to ρ, it follows immediately that for all $\mu, \mu' \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$t^\varepsilon_\mu \,\rho\, t^\varepsilon_{\mu'} \;=\; 0\,. \tag{C.2.54}$$

Using (C.2.52) and (C.2.54) in (C.2.51), we see that

$$P^\varepsilon_\mu \,P^\varepsilon_{\mu'} \;=\; \delta^2(\mu-\mu')\,\frac{1}{8}\left(X\,P^\varepsilon_\mu + P^\varepsilon_\mu\,X^* + 2\,X\,P^\varepsilon_\mu\,X^*\right)\,.$$

Now we can take the limits $\varepsilon, \mu \searrow 0$ to obtain

$$\lim_{\varepsilon\searrow 0}\lim_{\mu,\mu'\searrow 0}\left(P^\varepsilon_\mu\,P^\varepsilon_{\mu'} - \frac{1}{2}\,\delta^2(\mu-\mu')\,P^\varepsilon_\mu\right) \;=\; 0\,. \tag{C.2.55}$$

In analogy to (C.1.13), this relation states that the fermionic projector is idempotent (apart from the factor $\frac{1}{2}$ which will be treated in Section C.3 using the modified mass scaling).

In the remainder of this section, we analyze under which assumptions on $\mathcal{B}_0$ the kernel of the Dirac operator is ε-orthogonal to ρ. We begin with a simple calculation in first order perturbation theory.

LEMMA C.2.13. *Suppose that the Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0$ has an ε-definite kernel and that the homogeneous potentials in (C.2.35) satisfy for all $k \in \mathbb{R}^4$ the relations*

$$\beta(k) \;=\; 0 \qquad \text{and} \qquad \epsilon_{ijlm}\,w^{ij}(k)\,k^l \;=\; 0\,. \tag{C.2.56}$$

Then for all k and $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$E_\mu(k)\,\rho\,E_\mu(k) \;=\; \mathcal{O}(\varepsilon^2)\,. \tag{C.2.57}$$

Proof. Choose k and $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Since the Dirac operator has an ε-definite kernel, the invariant subspace I corresponding to the set $\{\mu, \mu'\} \subset \sigma^\varepsilon(k)$ is definite (notice that $\mu \in (-\varepsilon, \varepsilon)$ and $|\mu - \mu'| < \varepsilon$). We saw in the proof of Theorem C.2.10 that the invariant subspaces $\text{Im}\,E^\varepsilon_+$ and $\text{Im}\,E^\varepsilon_+$ (with $E^\varepsilon_\pm$ according to (C.2.3)) are definite. Thus $I \subset \text{Im}\,E^\varepsilon_+$ or $I \subset \text{Im}\,E^\varepsilon_-$. Therefore, it suffices to show that for all $s = \pm$,

$$E^\varepsilon_s\,\rho\,E^\varepsilon_s \;=\; \mathcal{O}(\varepsilon^2)\,. \tag{C.2.58}$$

Substituting (C.2.3) and using the relations $\rho E_\pm \rho = E_\mp$, we obtain the equivalent condition

$$E_s\,\{\mathcal{B}_0, \rho\}\,E_s \;=\; 0\,. \tag{C.2.59}$$

This equation means that the matrix $\{\mathcal{B}_0, \rho\}$ must vanish on the two-dimensional subspace $\text{Im}\,E_s$. Since on this subspace, the matrices $\vec{\Sigma}$, (C.2.5), have a representation as the Pauli matrices, we can restate (C.2.59) as the four conditions

$$\text{Tr}\left(E_s\,\{\mathcal{B}_0, \rho\}\right) \;=\; 0 \;=\; \text{Tr}\left(\vec{\Sigma}\,E_s\,\{\mathcal{B}_0, \rho\}\right)\,.$$

Evaluating these relations using (C.2.2, C.2.5, C.2.35) gives (C.2.56). ∎

This lemma is not satisfactory because it gives no information on how the error term in (C.2.57) depends on k. More specifically, the error term may have poles on the mass cone (and explicit calculations show that such poles $\sim k^{-2n}$ indeed occur for $n = 1$ and $n = 2$). Since in the limit $\varepsilon \searrow 0$ the kernel of the Dirac operator is the mass cone, it is far from obvious how to control the error term in this limit. In other words,

(C.2.57) cannot be interpreted as "the kernel of the Dirac operator is ε-orthogonal to ρ up to a small error term."

In order to resolve this difficulty, we must proceed non-perturbatively. In generalization of our previous ansatz $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$, we shall consider the Dirac operator $i\partial\!\!\!/ + \mathcal{B}^\varepsilon$, where we assume that $\mathcal{B}^\varepsilon(k)$ is a homogeneous potential which is smooth in both arguments and has the power expansion

$$\mathcal{B}^\varepsilon(k) \;=\; \varepsilon\, \mathcal{B}_0(k) + \varepsilon^2\, \mathcal{B}_1(k) + \varepsilon^3\, \mathcal{B}_2(k) + \cdots \,. \tag{C.2.60}$$

The higher order potentials $\mathcal{B}_1, \mathcal{B}_2, \ldots$ are irrelevant for Def. C.2.8 because they are negligible for small ε. In particular, the statement of Theorem C.2.10 remains valid without changes. Furthermore, the potentials $\mathcal{B}_1, \mathcal{B}_2, \ldots$ should be irrelevant for the statement of idempotence (C.2.55) because (C.2.55) involves a limit $\varepsilon \searrow 0$. Therefore, it seems unnecessary to enter a detailed study of these potentials. The only point of interest is under which assumptions on $\mathcal{B}_0$ there exist smooth potentials $\mathcal{B}_1$, $\mathcal{B}_2$,... such that the spectral projectors corresponding to the Dirac operator $i\partial\!\!\!/ + \mathcal{B}^\varepsilon$ satisfy the conditions (C.2.53) exactly.

THEOREM C.2.14. *Suppose that the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0$ has an ε-definite kernel and that the homogeneous potentials in (C.2.35) satisfy for all k the relations (C.2.56). Then there is an $\varepsilon > 0$ and a smooth potential $\mathcal{B}^\varepsilon(k)$ having the expansion (C.2.60) such that the kernel of the Dirac operator $i\partial\!\!\!/ + \mathcal{B}^\varepsilon$ is ε-orthogonal to ρ.*

Proof. Choose k and $\mu, \mu' \in \sigma^\varepsilon(k) \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Similar to what was described before (C.2.58), we know from the proof of Theorem C.2.10 that the matrix $A \equiv k\!\!\!/ + \mathcal{B}^\varepsilon(k)$ has a positive and a negative definite invariant subspace, one of which contains $\operatorname{Im} E_\mu \cup \operatorname{Im} E_{\mu'}$. Again denoting the projectors onto these subspaces by E^ε_+ and E^ε_-, respectively, it thus suffices to show that for $s = \pm$,

$$E^\varepsilon_s\, \rho\, E^\varepsilon_s \;=\; 0\,. \tag{C.2.61}$$

We first evaluate these conditions in a special spinor basis. Namely, we let e_1 and e_2 be an orthonormal basis of $\operatorname{Im} E^\varepsilon_+$ and set $e_3 = \rho e_1$, $e_4 = \rho e_2$. The conditions (C.2.61) imply that e_3 and e_4 span $\operatorname{Im} E^\varepsilon_-$. Using the relation $\rho^2 = \mathbb{1}$ as well as that the subspaces $<\{e_1, e_2\}>$ and $<\{e_3, e_4\}>$ are invariant under A, we conclude that the matrices ρ and A are of the form

$$\rho \;=\; \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{C.2.62}$$

where we used a block matrix notation corresponding to the splitting

$$\mathbb{C}^4 \;=\; <\{e_1, e_2\}> \oplus <\{e_3, e_4\}>\,,$$

and "$*$" denotes an arbitrary block matrix entry. Furthermore, the relation $\rho^* = -\mathbb{1}$ yields that

$$\prec e_3 \,|\, e_3 \succ \;=\; -1 \;=\; \prec e_4 \,|\, e_4 \succ\,,$$

and thus the basis (e_α) is pseudo-orthonormal,

$$\prec \Psi \,|\, \Phi \succ \;=\; \sum_{\alpha=1}^4 s_\alpha\, \overline{\Psi^\alpha}\, \Phi^\alpha \qquad \text{with} \qquad s_1 = s_2 = 1,\; s_3 = s_4 = -1. \tag{C.2.63}$$

We see that the matrix ρ and the spin scalar product are in the usual Dirac representation. In this representation, the fact that A is block diagonal (C.2.62) can be

expressed by saying that A must be a real linear combination of the 8 matrices

$$\mathbb{1}, \quad \gamma^0, \quad \rho\vec{\gamma}, \quad \rho\gamma^0\vec{\gamma}\,. \tag{C.2.64}$$

We next express this result in a general basis, but again in the Dirac representation. Since the representations of the matrix ρ, (C.2.62), and of the scalar product, (C.2.63), are fixed, the freedom in choosing the basis is described by even $U(2,2)$ transformations. This group, which we denote by $U(2,2)^{\text{even}}$, contains the normal Abelian subgroup $U = \{\exp(\vartheta\rho/2) : \vartheta \in \mathbb{R}\}$. Acting by U on (C.2.64) gives the matrices

$$\mathbb{1}, \quad ((\cosh\vartheta + \rho\,\sinh\vartheta)\,\gamma^0, \quad ((\cosh\vartheta + \rho\,\sinh\vartheta)\,\rho\vec{\gamma}, \quad \rho\gamma^0\vec{\gamma}\,. \tag{C.2.65}$$

When the factor group $U(2,2)^{\text{even}}/U$ acts on (C.2.65), the resulting transformations correspond precisely to Lorentz transformations of the tensor indices (see Lemma 1.2.1 for details). Thus the conditions (C.2.61) are satisfied if and only if A is of the form

$$A \;=\; \alpha\,\mathbb{1} + ((\cosh\vartheta + \rho\,\sinh\vartheta)\,\not{v} + ((\rho\,\cosh\vartheta + \sinh\vartheta)\,\not{a} + \rho\not{v}\not{b} \tag{C.2.66}$$

with a time-like vector field u and two vector fields a and b, which are orthogonal to u,

$$\langle u, a\rangle \;=\; 0 \;=\; \langle u, b\rangle\,. \tag{C.2.67}$$

We substitute the identity $A = \not{k} + \mathcal{B}^\varepsilon(k)$ into (C.2.66) and solve for $\mathcal{B}^\varepsilon(k)$. Expanding in powers of ε gives the result. ∎

C.3. The General Construction, Proof of Idempotence

In this section we shall make precise what "idempotence" means for a fermionic projector with chiral asymmetry in the presence of a general interaction. We proceed in several steps. We begin with a straightforward extension of the results of Section C.2 to systems of Dirac seas. Then we introduce the interaction and perform the causal perturbation expansion. After putting in an infrared regularization, we can define the fermionic projector. Finally, idempotence is established via a singular mass limit.

We begin with a system of Dirac seas in the vacuum, described by the mass matrix Y and the chiral asymmetry matrix X (see §2.3). In order to give the chiral fermions a "small generalized mass," we introduce a homogeneous operator $\mathcal{B}_0$ and consider for $\varepsilon > 0$ the Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - mY$. For simplicity, we assume that $\mathcal{B}_0$ is diagonal on the sectors and is non-trivial only in the chiral blocks, i.e.

$$(\mathcal{B}_0)^{(a\alpha)}_{(b\beta)} \;=\; \delta^a_b\,\delta^\alpha_\beta\,\mathcal{B}_0^{(a\alpha)} \qquad \text{with} \qquad \mathcal{B}_0^{(a\alpha)} = 0 \text{ if } X_a = \mathbb{1}.$$

Then on each sector the methods of Section C.2 apply; let us collect the assumptions on $\mathcal{B}_0$ and the main results: For every index $(a\alpha)$ with $X^{(a\alpha)} \neq \mathbb{1}$ we assume that

(1) $\mathcal{B}_0^{(a\alpha)}(k)$ depends smoothly on $k \in \mathbb{R}^4$ and grows at most polynomially at infinity.
(2) The (4×4)-matrix $\not{k} + \varepsilon\mathcal{B}_0^{(a\alpha)}(k)$ is diagonalizable for a.a. k.
(3) $\mathcal{B}_0^{(a\alpha)}$ has the decomposition into scalar, vector, axial, and bilinear potentials,

$$\mathcal{B}_0^{(a\alpha)}(k) \;=\; \alpha\,\mathbb{1} + \not{v} + \rho\,\not{a} + \frac{i\rho}{2}\,w_{ij}\,\sigma^{ij}\,,$$

such that for all $k \in \mathbb{R}^4$ and $q \in \mathcal{C}$ the following conditions are satisfied,

$$\epsilon_{ijlm}\, w^{ij}(k)\, k^l \;=\; 0$$

$$|\alpha(q)| \;>\; \frac{3}{2} + \begin{cases} \left| \dfrac{w_{ij}(q) a^i q^j}{\Delta(q)} \right| & \text{if } \Delta(q) \neq 0 \\ \Big(1 + \Theta(1 - 2\sqrt{|l(q)|}\Big)\, \sqrt{|l(q)|} & \text{if } \Delta(q) = 0. \end{cases}$$

(with Δ and l defined by (C.2.36) and (C.2.40), $\mathcal{C} = \{k \,|\, k^2 = 0\}$ is the mass cone).

Then for sufficiently small ε, the Dirac operator $i\partial\!\!\!/ + \varepsilon \mathcal{B}_0^{(a\alpha)}$ has an ε-definite kernel (see Def. C.2.8 and Theorem C.2.10). Thus for a.a. $\mu \in (-\varepsilon, \varepsilon)$, the spectral projectors $p_\mu^{\varepsilon,(a\alpha)}$, $k_\mu^{\varepsilon,(a\alpha)}$ and the Green's functions $s_\mu^{\varepsilon,(a\alpha)}$ are well-defined distributions in momentum space (see Def. C.2.4 and Theorem C.2.9). Furthermore, the kernel of the Dirac operator is ε-orthogonal to ρ (see Def. C.2.12 and Theorem C.2.14; for simplicity we here omit the higher order potentials $\mathcal{B}_1, \mathcal{B}_2, \ldots$ in (C.2.60), this is justified because these potentials obviously drop out in the singular mass limit), and this can be stated in the form (cf. (C.2.17))

$$p_\mu^{\varepsilon,(a\alpha)}\, \rho\, p_{\mu'}^{\varepsilon,(a\alpha)} \;=\; 0 \qquad \text{for all } \mu, \mu' \in (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2})\,.$$

We build up the spectral projectors $p_{+\mu}^\varepsilon$, $k_{+\mu}^\varepsilon$ and the Green's function $s_{+\mu}^\varepsilon$ of the whole system by taking direct sums; namely,

$$A_{+\mu}^\varepsilon \;=\; \bigoplus_{a,\alpha} \begin{cases} A_{m_{a\alpha}+\mu} & \text{if } X_a = \mathbb{1} \\ A_{\frac{\mu}{2}}^{\varepsilon,(a\alpha)} & \text{if } X_a \neq \mathbb{1}, \end{cases} \tag{C.3.1}$$

where A stands for p, k, or s. Note that in the chiral blocks the mass parameter $\frac{\mu}{2}$ (and not μ) is used. The purpose of this *modified mass scaling* is to get rid of the factor $\frac{1}{2}$ in the normalization of a chiral Dirac sea (C.2.55) (also see the paragraph after (C.1.21)). The corresponding Dirac operator is

$$i\partial\!\!\!/ + \varepsilon \mathcal{B}_0 - mY - \mu Z\,,$$

where the matrix $Z \equiv \frac{1}{2}(X + X^*)$ takes into account the modified mass scaling. The spectral projectors satisfy the multiplication rules

$$\left.\begin{aligned} p_{+\mu}^\varepsilon\, p_{+\mu'}^\varepsilon \;&=\; k_{+\mu}^\varepsilon\, k_{+\mu'}^\varepsilon \;=\; \delta^2(\mu - \mu')\, Z^{-1}\, p_{+\mu}^\varepsilon \\ p_{+\mu}^\varepsilon\, k_{+\mu'}^\varepsilon \;&=\; k_{+\mu}^\varepsilon\, p_{+\mu'}^\varepsilon \;=\; \delta^2(\mu - \mu')\, Z^{-1}\, k_{+\mu}^\varepsilon \end{aligned}\right\} \tag{C.3.2}$$

$$C_{+\mu}^\varepsilon\, \rho\, C_{+\mu'}^\varepsilon \;=\; 0 \qquad \text{for } \mu, \mu' \in (-\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2})\,, \tag{C.3.3}$$

where C stands for k or p. The Green's functions satisfy the relations

$$\left.\begin{aligned} C_{+\mu}^\varepsilon\, s_{+\mu'}^\varepsilon \;&=\; s_{+\mu}^\varepsilon\, C_{+\mu'}^\varepsilon \;=\; \frac{\mathrm{PP}}{\mu - \mu'}\, Z^{-1}\, C_{+\mu}^\varepsilon \\ s_{+\mu}^\varepsilon\, s_{+\mu'}^\varepsilon \;&=\; \frac{\mathrm{PP}}{\mu - \mu'}\, Z^{-1}\, (s_{+\mu}^\varepsilon - s_{+\mu'}^\varepsilon)\,. \end{aligned}\right\} \tag{C.3.4}$$

These multiplication rules differ from those in §2.2 only by the additional factor Z^{-1}.

To describe the interaction, we insert a potential $\mathcal{B}$ into the Dirac operator, which then reads

$$i\partial\!\!\!/ + \mathcal{B} + \varepsilon \mathcal{B}_0 - mY - \mu Z\,. \tag{C.3.5}$$

We assume that Y and $\mathcal{B}$ have the following properties:

(a) Only the chiral particles are massless, i.e.

$$Y^{(a\alpha)} > 0 \qquad \text{if } X_a = \mathbb{1}.$$

(b) $\mathcal{B}$ is the operator of multiplication with the Schwartz function $\mathcal{B}(x)$.
(c) Y and $\mathcal{B}$ are causality compatible, i.e.

$$X^* \,(i\partial\!\!\!/ + \mathcal{B} - mY) \;=\; (i\partial\!\!\!/ + \mathcal{B} - mY)\, X\,. \tag{C.3.6}$$

In order to introduce the spectral projectors with interaction $\tilde{p}^\varepsilon_{+\mu}$ and $\tilde{k}^\varepsilon_{+\mu}$, we take the operator expansion of causal perturbation theory §2.3 and replace the operators according to $A \to A^\varepsilon_{+\mu}$ (with $A = p$, k, or s). All the operator products of the resulting expansion are well-defined for a.a. μ (note that $\tilde{\mathcal{B}}(k)$ has rapid decay and $A^\varepsilon_{+\mu}(k)$ grows at most polynomially at infinity).

For the infrared regularization, we proceed exactly as in §2.6 and replace space by the three-dimensional torus (2.6.2). Furthermore, we "average" the mass parameter μ. More precisely, combining (2.6.13) with (C.2.50), the auxiliary fermionic projector is defined by

$$P^{\varepsilon,\delta} \;=\; \frac{1}{2} \int_{(0,\delta)\times(-\delta,\delta)} (X\, \tilde{t}^\varepsilon_{+\mu} + \tilde{t}^\varepsilon_{+\mu}\, X^*)\, d^2\mu\,, \tag{C.3.7}$$

where as usual $\tilde{t}^\varepsilon_{+\mu} = \frac{1}{2}(\tilde{p}^\varepsilon_{+\mu} - \tilde{k}^\varepsilon_{+\mu})$. Finally, the *regularized fermionic projector* is obtained by taking the partial trace,

$$(P^{\varepsilon,\delta})^a_b \;=\; \sum_{\alpha=1}^{g(a)} \sum_{\beta=1}^{g(b)} (P^{\varepsilon,\delta})^{(a\alpha)}_{(b\beta)}\,. \tag{C.3.8}$$

Before we can prove idempotence, we need to impose the following extension of the non-degeneracy assumption (2.6.1). We set

$$\sigma^\varepsilon_{(a\alpha)} \;=\; \sigma(k\!\!\!/ + \varepsilon \mathcal{B}^{(a\alpha)}_0)\,.$$

DEF. C.3.1. *The Dirac operator $i\partial\!\!\!/ + \varepsilon\mathcal{B}_0 - mY$ has* **ε-non-degenerate masses** *if for all a and $\beta \neq \gamma$,*

$$\sigma^\varepsilon_{(b\beta)} \cap (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \neq \emptyset \quad\Longrightarrow\quad \sigma^\varepsilon_{(b\gamma)} \cap (-\varepsilon, \varepsilon) = \emptyset\,. \tag{C.3.9}$$

Roughly speaking, the next theorem states that the masses are ε-non-degenerate if they are non-degenerate in the massive sectors and if the homogeneous potentials in the chiral sectors are sufficiently different from each other.

THEOREM C.3.2. *Suppose that Y and $\mathcal{B}_0$ have the following properties:*

(i) *In the massive blocks (i.e. $X_a = \mathbb{1}$), the masses are non-degenerate,*

$$Y^{(b\beta)} \neq Y^{(b\gamma)} \qquad \text{if } \beta \neq \gamma.$$

(ii) *In the chiral blocks (i.e. $X_a \neq \mathbb{1}$), for all $\beta \neq \gamma$ and all $q \in \mathcal{C}$ either*

$$\langle v^{(b\beta)}, q\rangle + s\, \Delta^{(b\beta)}(q) \;\neq\; \langle v^{(b\gamma)}, q\rangle + s'\, \Delta^{(b\gamma)}(q) \qquad \text{for all } s, s' \in \{\pm 1\} \tag{C.3.10}$$

or else

$$|\alpha^{(b\beta)}(q) - \alpha^{(b\gamma)}(q)| \;>\; 2 \;+\; 2\,|d^{(b\beta)}(q) + d^{(b\gamma)}(q)| \tag{C.3.11}$$

with

$$d(q) \;=\; \begin{cases} \left| \dfrac{w_{ij}(q) a^i q^j}{\Delta(q)} \right| & \text{if } \Delta(q) \neq 0 \\ \sqrt{|l(q)|} & \text{if } \Delta(q) = 0 \end{cases}$$

(and Δ, l according to (C.2.36) and (C.2.40)).

Then for sufficiently small ε, the Dirac operator has ε-non-degenerate masses.

Proof. The condition in (i) follows immediately from the fact that the eigenvalues μ_s^a in the two sectors differ precisely by $m(Y^{(b\beta)} - Y^{(b\gamma)})$. For part (ii) we consider the formulas for the eigenvalues (C.2.44) and (C.2.46). If (C.3.10) holds, the eigenvalues in the two sectors all differ by contributions of the order $\sqrt{\varepsilon}$, and so (C.3.9) is satisfied for small ε. If on the other hand (C.3.10) is violated, there are eigenvalues in two different sectors such that the square roots in (C.2.44) and/or (C.2.46) coincide. Thus these eigenvalues differ by $(\alpha+\sigma)^{(b\beta)} - (\alpha+\sigma)^{(b\gamma)}$, where each σ is an element of the set $\{\pm\kappa_+, \pm\kappa_-, \pm\tau_+, \pm\tau_-\}$. The condition (C.3.11) guarantees that this difference is greater than 2ε, and so (C.3.9) is again satisfied. ∎

We can now state the main result of this chapter.

THEOREM C.3.3. **(Idempotence)** *Consider the Dirac operator (C.3.5) under the above assumptions (1)–(3) and (a)–(c). Assume furthermore that the masses are ε-non-degenerate (see Def. C.3.1 and Theorem C.3.2). Then the corresponding fermionic projector (C.3.7, C.3.8) satisfies the identity*

$$\lim_{\varepsilon \searrow 0} \lim_{\delta \searrow 0} \delta \left(\int_{\mathbb{R} \times T^3} \sum_{b=1}^{N} P_b^a(x,z)\, P_c^b(z,y)\, d^4z \;-\; P_c^a(x,y) \right) \;=\; 0 \tag{C.3.12}$$

with convergence as a distribution to every order in perturbation theory.

Proof. Similar to (C.2.19), the Green's function $s^\varepsilon_{+\mu}$ has a spectral representation in a mass parameter ν. We want to decompose $s^\varepsilon_{+\mu}$ into contributions $\dot{s}^\varepsilon_{+\mu}$ and $\breve{s}^\varepsilon_{+\mu}$ where $|\nu - \mu|$ is small and large, respectively. To this end, we introduce in each sector the operator

$$\begin{cases} \dot{s}_\mu^{\varepsilon,(a\alpha)} \;=\; \displaystyle\int_{B_{\varepsilon/4}(\mu)} \frac{\mathrm{PP}}{\nu - \mu}\, p_\mu^\varepsilon \, d^2\nu & \text{if } X_1 \neq \mathbb{1} \\ \dot{s}_{m_{a\alpha}+\mu} \;=\; \displaystyle\int_{B_{\varepsilon/2}(\mu)} \frac{\mathrm{PP}}{\nu - \mu}\, p_{m_{a\alpha}+\nu} \, d^2\nu & \text{if } X_1 = \mathbb{1} \end{cases}$$

and define $\dot{s}^\varepsilon_{+\mu}$ by taking as in (C.3.1) the direct sum. Setting $\breve{s}^\varepsilon_{+\mu} = s^\varepsilon_{+\mu} - \dot{s}^\varepsilon_{+\mu}$, we obtain the decomposition

$$s^\varepsilon_{+\mu} \;=\; \dot{s}^\varepsilon_{+\mu} + \breve{s}^\varepsilon_{+\mu}\,. \tag{C.3.13}$$

Our first step is to show that for small μ, the matrix $\breve{s}^\varepsilon_{+\mu}(k)$ is bounded, more precisely that

$$\|s^\varepsilon_{+\mu}(k)\| \;\leq\; \frac{C(k)}{\varepsilon^7} \qquad \text{for } \mu \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \tag{C.3.14}$$

with $C(k)$ a smooth function with at most polynomial growth at infinity (the exponent 7 is probably not optimal, but (C.3.14) is sufficient for our purpose). It clearly suffices to prove (C.3.14) in a given sector $(a\alpha)$; for simplicity the sector index will be omitted

(i.e. $\mathcal{B}_0 \equiv \mathcal{B}_0^{(a\alpha)}$). Furthermore, we only consider the case $X_a \neq \mathbb{1}$; the other case is analogous (and even simpler, because in the massive sectors no homogeneous potentials are present). We introduce the projector $E(k)$ by

$$E \;=\; \sum_{(a,s)\in\mathcal{S}} E_s^a \qquad \text{with} \qquad \mathcal{S} = \Big\{(a,s) \text{ with } |\mu_s^a - \mu| < \frac{\varepsilon}{4}\Big\}$$

According to Lemma C.2.7,

$$\|E\| \;\leq\; \frac{C_1(k)}{\varepsilon^3} \tag{C.3.15}$$

with $C_1(k)$ smooth with at most polynomial growth at infinity. The matrix $\breve{s}^\varepsilon_{+\mu}$ has a simple spectral representation,

$$\breve{s}^\varepsilon_{+\mu} \;=\; \sum_{(a,s)\notin\mathcal{S}} \frac{1}{\mu_s^a - \mu}\, E_s^a\,.$$

Unfortunately, this representation is not suitable for estimates, because we have no control of $\|E_s^a\|$ for $(a,s) \notin \mathcal{S}$. To avoid this problem, we rewrite $\breve{s}^\varepsilon_{+\mu}$ as follows,

$$\begin{aligned}\breve{s}^\varepsilon_{+\mu} &= \left(\sum_{(a,s)\notin\mathcal{S}} \frac{1}{\mu_s^a - \mu}\, E_s^a \;+\; \sum_{(a,s)\in\mathcal{S}} \frac{1}{\mu_s^a - \mu + \varepsilon}\, E_s^a\right)(\mathbb{1} - E) \\ &= (\not{k} + \varepsilon\mathcal{B}_0 - \mu + \varepsilon E)^{-1}\,(\mathbb{1} - E)\,.\end{aligned}$$

Introducing the "Hamiltonian" $H = -\gamma^0\,(\vec{\not{k}} + \varepsilon\mathcal{B}_0(\omega,\vec{k}) - \mu + \varepsilon E)$, we obtain

$$\breve{s}^\varepsilon_{+\mu} \;=\; (\omega - H)^{-1}\,\gamma^0\,(\mathbb{1} - E)\,. \tag{C.3.16}$$

The matrix $H(k)$ is Hermitian with respect to the positive scalar product $(.|.) = \prec.|\gamma^0|.\succ$ and can thus be diagonalized, i.e.

$$H \;=\; \sum_{n=1}^4 \Omega_n\, F_n$$

with real eigenvalues Ω_n and spectral projectors F_n. Substituting into (C.3.16) gives

$$\breve{s}^\varepsilon_{+\mu} \;=\; \sum_{n=1}^4 \frac{1}{\omega - \Omega_n}\, F_n\,\gamma^0\,(\mathbb{1} - E)\,,$$

and (C.3.15) yields the bound

$$\|\breve{s}^\varepsilon_{+\mu}\| \;\leq\; 2 \max_n \frac{1}{|\omega - \Omega_n|}\,\|\mathbb{1} - E\| \;\leq\; \frac{C_2(k)}{\varepsilon^3}\,\max_n \frac{1}{|\omega - \Omega_n|}\,. \tag{C.3.17}$$

It remains to estimate the factors $|\omega - \Omega_n|$ from below. We use that the determinant is multiplicative to obtain

$$\begin{aligned}\prod_{n=1}^4 (\omega - \Omega_n) &= \det(\omega - H) \;=\; \det(\gamma^0\,(\omega - H)) \\ &= \det(\not{k} + \varepsilon\mathcal{B}_0 - \mu + \varepsilon E) \;=\; \prod_{(a,c)\notin\mathcal{S}} (\mu_c^a - \mu) \prod_{(a,c)\in\mathcal{S}} (\mu_c^a - \mu + \varepsilon)\,.\end{aligned}$$

Taking the absolute value, the factors $|\mu_c^a - \mu|$ and $|\mu_c^a - \mu + \varepsilon|$ are all greater than $\frac{\varepsilon}{4}$, and thus

$$\prod_{n=1}^{4} |\omega - \Omega_n| \;\geq\; \left(\frac{\varepsilon}{4}\right)^4 . \tag{C.3.18}$$

Since the eigenvalues of the Hermitian matrix $\omega - H$ can be estimated by the sup-norm of the matrix,

$$|\omega - \Omega_n| \;\leq\; \|\omega - H\| ,$$

we can deduce from (C.3.18) that each factor $|\omega - \Omega_n|$ is bounded by

$$|\omega - \Omega_n| \;\geq\; \|\omega - H\|^{-3} \left(\frac{\varepsilon}{4}\right)^4 .$$

Substituting this inequality into (C.3.17) gives (C.3.14).

The causal perturbation expansion expresses $\tilde{t}^\varepsilon_{+\mu}$ as a sum of operator products of the form

$$\tilde{t}^\varepsilon_{+\mu} \;\asymp\; A^\varepsilon_{+\mu}\, \mathcal{B}_0\, A^\varepsilon_{+\mu} \cdots A^\varepsilon_{+\mu}\, \mathcal{B}_0\, A^\varepsilon_{+\mu} , \tag{C.3.19}$$

where each factor A stands for p, k, or s. Since $\mathcal{B}_0(k)$ has rapid decay and $A^\varepsilon_\mu(k)$ grows at most polynomially, these operator products are well-defined. According to (C.3.7) and (C.3.8), the first summand inside the brackets in (C.3.12) can be written as

$$\begin{aligned}\frac{1}{4}\int_0^\delta d\mu \int_0^\delta d\mu' \sum_b \sum_{\beta,\gamma} \Big(& X_a\, (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)}\, X_c^* \;+\; X_a\, (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, X_b\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)} \\ & + (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, X_b^*\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)}\, X_c^* \;+\; (t^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, X_b^*\, X_b\, (t^\varepsilon_{+\mu'})^{(b\gamma)}_{(c\delta)} \Big) .\end{aligned} \tag{C.3.20}$$

When we substitute (C.3.19) into (C.3.20), the difficult point is to multiply the rightmost factor A of the first factor t to the leftmost factor A of the second factor t. More precisely, we must analyze the following operator products,

$$(\cdots A^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, (A^\varepsilon_{+\mu'} \cdots)^{(b\gamma)}_{(c\delta)} \tag{C.3.21}$$

$$(\cdots A^\varepsilon_{+\mu})^{(a\alpha)}_{(b\beta)}\, \rho\, (A^\varepsilon_{+\mu'} \cdots)^{(b\gamma)}_{(c\delta)} \tag{C.3.22}$$

with $A = p$, k, or s.

If one of the factors A in (C.3.21) or (C.3.22) is the Green's function, we substitute (C.3.13) and expand. Since $\breve{s}^\varepsilon_{+\mu}$ is bounded (C.3.14), the products involving $\breve{s}^\varepsilon_{+\mu}$ have a finite limit as $\delta \searrow 0$. Since the two integrals in (C.3.20) give a factor δ^2, these products all drop out when the limit $\delta \searrow 0$ is taken in (C.3.12). Thus it suffices to consider the case when the factors A in (C.3.21) and (C.3.22) stand for p, k, or $\dot{s}$.

Since the Dirac operator has ε-non-degenerate masses, the distributions $A^\varepsilon_{+\mu}(k)$ have disjoint supports in different sectors. More precisely, for all $\mu, \mu' \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$\text{supp}\, (A^{\varepsilon,(b\beta)}_{\mu}) \cap \text{supp}\, (A^{\varepsilon,(b\gamma)}_{\mu'}) \;=\; \emptyset \qquad \text{if } \beta \neq \gamma \text{ and } X_b \neq \mathbb{1},$$

where each factor A stands for p, k, or $\dot{s}$. A similar relation holds in the massive blocks. Therefore, (C.3.21) and (C.3.22) vanish if $\beta \neq \gamma$.

In the case $\beta = \gamma$, (C.3.22) is zero because the Dirac operator is ε-orthogonal to ρ (C.3.3). Thus, using a matrix notation in the sectors, we only need to take into account the operator products

$$(\cdots A^\varepsilon_{+\mu})(A^\varepsilon_{+\mu'} \cdots)$$

with $A = p$, k, or s (here we may again consider s instead of $\dot{s}$ because, as we saw above, all factors $\breve{s}$ drop out in the limit $\delta \searrow 0$). Now we can apply the multiplication rules (C.3.2) and (C.3.4). Applying (C.3.2) gives a factor $\delta^2(\mu - \mu')$, and we can carry out the μ'-integral. After dividing by δ, we can take the limits $\delta \searrow 0$ and $\varepsilon \searrow 0$. Using that in this limit the Dirac operator is causality compatible (C.3.6), we can "commute X through" the resulting operator products (see §2.3). In this way, one recovers precisely the unregularized fermionic projector $P = \lim_{\varepsilon,\delta \searrow 0} P^{\varepsilon,\delta}$. If (C.3.4) is applied, the resulting principal part is bounded after the integrals over μ and μ' are carried out, and we can take the limits $\delta \searrow 0$ and $\varepsilon \searrow 0$. After commuting X through the resulting operator products we find that all terms cancel. ∎

For understanding better what the above results mean physically, it is instructive to consider a cosmological situation where the 4-volume of space-time is finite. In this case, the limits $\varepsilon, \delta \searrow 0$ in (C.3.12) are a merely mathematical idealization corresponding to the fact that the size of the universe is very large compared to the usual length scales on earth. We can extrapolate from (C.3.7, C.3.8) to get some information on how the properly normalized physical fermionic projector should look like: The parameter δ is to be chosen of the order T^{-1} with T the lifetime of the universe (also see §2.6). Then, due to the μ-integral in (C.3.7), the Dirac seas are built up from those fermionic states whose momenta lie in a thin strip around the mass cone. Naively, the modified mass scaling implies that for neutrinos this strip must be thinner. However, this naive picture is misleading because the detailed form of the chiral Dirac seas depend strongly on the homogeneous operator $\mathcal{B}_0$, which is unknown. We point out that in (C.3.12) the order of limits is essential: we must first take the infinite volume limit and then the limit $\varepsilon \searrow 0$. This means for our cosmology in finite 4-volume that the homogeneous perturbation $\varepsilon \mathcal{B}_0$ must be large compared to T^{-1}. One possibility for realizing this is to give the neutrinos a small rest mass. But, as shown above, the same can be achieved by more general, possibly nonlocal potentials which do not decay at infinity.

APPENDIX D

The Regularized Causal Perturbation Theory

In §4.5 we gave a procedure for regularizing the formulas of the light-cone expansion (4.5.5–4.5.10). We shall now derive this regularization procedure. The basic idea is to extend the causal perturbation expansion of §2.2 to the case with regularization, in such a way that the causality and gauge symmetry are preserved for macroscopic perturbations. Using the methods of §2.5 one can then analyze the behavior of the so-regularized Feynman diagrams near the light cone. For simplicity, we will restrict attention to the first order in perturbation theory. But our methods could be applied also to higher order Feynman diagrams, and the required gauge symmetry suggests that our main result, Theorem D.2, should hold to higher order in perturbation theory as well.

We first state our assumptions on the fermionic projector of the vacuum. As in Chapter 4 we describe the vacuum by a fermionic projector $P(x,y)$ of the form (4.1.3) with vector-scalar structure (4.1.5). For small energy-momentum, $\hat{P}$ should coincide with the unregularized fermionic projector of the vacuum, i.e.

$$\hat{P}(k) \;=\; (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0) \qquad \text{if } |k^0| \ll E_P \text{ and } |\vec{k}| \ll E_P. \tag{D.1}$$

Furthermore, we assume that the vector component is null on the light cone (i.e. that (4.4.21) holds with $\varepsilon_{\text{shear}} \ll 1$), and that P satisfies all the regularity assumptions considered in §4.3 and §4.4. For simplicity, we finally assume that $\hat{P}$ is supported inside the lower mass cone,

$$\text{supp } \hat{P} \;\subset\; \overline{\mathcal{C}^\wedge} \tag{D.2}$$

(with $\mathcal{C}^\wedge$ according to (5.6.15)). This last condition is quite strong, but nevertheless reasonable. In particular, it is satisfied when P is composed of one-particle states which are small perturbations of the Dirac eigenstates on the lower mass shell.

In this appendix we shall address the question of how one can introduce a classical external field into the system. For clarity, we will develop our methods mainly in the example of an electromagnetic field. As described in §4.1, we consider the regularized fermionic projector as a model for the fermionic projector of discrete space-time. In this sense, the regularization specifies the microscopic structure of space-time. Following the concept of macroscopic potentials and wave functions introduced in §4.1, the electromagnetic field should modify the fermionic projector only on length scales which are large compared to the Planck length, but should leave the microscopic structure of space-time unchanged. In order to fulfill this requirement, we impose the following conditions. First of all, we assume that the electromagnetic field be "macroscopic" in the sense that it can be described by an electromagnetic potential A which vanishes outside the low-energy region, i.e.

$$\hat{A}(k) \;=\; 0 \qquad \text{unless } |k^0| \ll E_P \text{ and } |\vec{k}| \ll E_P, \tag{D.3}$$

where $\hat{A}$ is the Fourier transform of A. We denote the fermionic projector in the presence of the electromagnetic field by $P[\A]$. In order to prevent that the electromagnetic potential might influence the microscopic structure of space-time locally, we demand that A can locally be made to zero by a gauge transformation. Thus we impose that the usual behavior under $U(1)$ gauge transformations

$$P[\A + (\partial\!\!\!/ \Lambda)](x,y) \;=\; e^{i\Lambda(x)}\, P[\A](x,y)\, e^{-i\Lambda(y)} \tag{D.4}$$

(with a real function Λ) should hold also for the regularized fermionic projector, assuming that the involved potentials A and $(A + \partial\Lambda)$ are both macroscopic (D.3). We point out that, because of the gauge symmetry in discrete space-time (following from the freedom in choosing the gauge (3.3.8)), the local phase transformations in (D.4) are irrelevant in the equations of discrete space-time, and thus the transformation law (D.4) implies the freedom to transform the electromagnetic potential according to $\A \to \A + \partial\!\!\!/ \Lambda$. Finally, we must rule out the possibility that the electromagnetic potential might influence the microscopic structure of space-time in a nonlocal way. For this purpose, we impose that the perturbation expansion for the regularized fermionic projector be causal, in the sense introduced in §2.2.

Let us consider how these conditions can be implemented in the perturbation theory to first order. We first recall the standard perturbation theory for Dirac eigenstates. For a solution Ψ of the free Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$, the perturbation to first order in A, which we denote by $\Delta\Psi[\A]$, is given by

$$\Delta\Psi[\A](x) \;=\; -\int d^4y\, s_m(x,y)\, \A(y)\, \Psi(y)\,, \tag{D.5}$$

where $s_m(x,y)$ is the Dirac Green's function (2.2.18),

$$s_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \frac{\mathrm{PP}}{k^2 - m^2}\, (k\!\!\!/ + m)\, e^{-ik(x-y)}\,. \tag{D.6}$$

If we consider $s_m(x,y)$ as the integral kernel of an operator s_m and the potentials as multiplication operators, we can calculate $\Delta\Psi$ in the case $\A = \partial\!\!\!/ \Lambda$ to be

$$\begin{aligned} \Delta\Psi[\partial\!\!\!/\Lambda] \;&=\; -s_m\, (\partial\!\!\!/ \Lambda)\, \Psi \;=\; i s_m\, [i\partial\!\!\!/ - m,\, \Lambda]\, \Psi \\ &=\; i((i\partial\!\!\!/ - m) s_m)\, \Lambda\, \Psi \;-\; i s_m\, \Lambda\, ((i\partial\!\!\!/ - m)\Psi) \;=\; i\Lambda\, \Psi\,. \end{aligned} \tag{D.7}$$

Thus in this case, $\Delta\Psi(x) = i\Lambda(x)\, \Psi(x)$ is simply the contribution linear in Λ to the phase transformed wave function $\exp(i\Lambda(x))\, \Psi(x)$; this shows explicitly that the perturbation calculation is gauge invariant.

As a consequence of the regularization, the fermionic projector $P(x,y)$ is in general not composed of Dirac eigenstates. Therefore, we next consider a wave function Ψ which is not necessarily a solution of the free Dirac equation. But according to (D.2), we may assume that its Fourier transform $\hat{\Psi}$ has its support in the interior of the mass cone,

$$\text{supp } \hat{\Psi} \;\subset\; \{k \,|\, k^2 \geq 0\}\,. \tag{D.8}$$

In this case we can introduce $\Delta\Psi[\A]$ as follows. The spectral projector p_μ of the free Dirac operator $i\partial\!\!\!/$ corresponding to the eigenvalue $\mu \in \mathbb{R}$ has the form

$$p_\mu(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \epsilon(\mu)\, (k\!\!\!/ + \mu)\, \delta(k^2 - \mu^2)\, e^{-ik(x-y)} \tag{D.9}$$

(see (2.2.4); notice that we added the step function $\epsilon(\mu)$ to allow for the case $\mu < 0$). Since the real axis is only part of the spectrum of the free Dirac operator (namely, the

free Dirac operator has also an imaginary spectrum), the spectral projectors $(p_\mu)_{\mu \in \mathbb{R}}$ are clearly not complete, i.e. $\int_{-\infty}^{\infty} p_\mu d\mu \neq \mathbb{1}$. By integrating (D.9) over μ,

$$\int_{-\infty}^{\infty} p_\mu(x,y)\, d\mu \;=\; \int \frac{d^4k}{(2\pi)^4}\, \Theta(k^2)\, e^{-ik(x-y)}\,, \tag{D.10}$$

one sees more precisely that the operator $\int_{-\infty}^{\infty} p_\mu d\mu$ is the projector on all the momenta in the mass cone. But according to (D.8), Ψ lies in the image of this projector, and we can thus use the spectral projectors p_μ to decompose Ψ into eigenstates of the free Dirac operator. Each eigenstate can then be perturbed using (D.5). This leads us to introduce $\Delta\Psi[\mathcal{A}]$ according to

$$\Delta\Psi[\mathcal{A}] \;=\; -\int_{-\infty}^{\infty} d\mu\; s_\mu\, \mathcal{A} p_\mu\, \Psi\,. \tag{D.11}$$

This definition of $\Delta\Psi$ shows the correct behavior under gauge transformations; namely, similar to (D.7),

$$\begin{aligned} \Delta\Psi[\partial\!\!\!/\Lambda] &= i \int_{-\infty}^{\infty} d\mu\; s_\mu\, [i\partial\!\!\!/ - \mu,\, \Lambda]\, p_\mu\, \Psi \\ &= i\Lambda \left(\int_{-\infty}^{\infty} p_\mu\, d\mu \right) \Psi \;\overset{(D.10, D.8)}{=}\; i\Lambda\, \Psi\,. \end{aligned} \tag{D.12}$$

Thinking in terms of the decomposition (2.2.1) of the fermionic projector into the one-particle states, it seems natural to introduce the perturbation of the fermionic projector $\Delta P[\mathcal{A}]$ by perturbing each one-particle state according to (D.11). This leads to the formula

$$\Delta P[\mathcal{A}] \;=\; -\int_{-\infty}^{\infty} d\mu\, (s_\mu\, \mathcal{A} p_\mu\, P \;+\; P\, p_\mu\, \mathcal{A} s_\mu)\,. \tag{D.13}$$

The gauge symmetry can again be verified explicitly. Namely, a calculation similar to (D.12) using (D.2) yields that

$$\Delta P[\partial\!\!\!/\Lambda](x,y) \;=\; i\Lambda(x)\, P(x,y) - iP(x,y)\, \Lambda(y)\,,$$

and this is the contribution linear in Λ to (D.4). The perturbation calculation (D.13) is immediately extended to a general perturbation operator $\mathcal{B}$ by setting

$$\Delta P[\mathcal{B}] \;=\; -\int_{-\infty}^{\infty} d\mu\, (s_\mu\, \mathcal{B} p_\mu\, P \;+\; P\, p_\mu\, \mathcal{B} s_\mu)\,. \tag{D.14}$$

Let us verify if the perturbation calculation (D.14) is causal in the sense of §2.2. Since it seems impossible to write (D.14) in a manifestly causal form, we apply here a different method, which allows us to analyze the causality of the perturbation expansion in momentum space. As mentioned in §2.5, the causality of the perturbation expansion can be understood via the causality of the line integrals over the external potentials and fields which appear in the light cone expansion. More precisely, causality means that the light-cone expansion of $\Delta P(x,y)$ should involve only line integrals along the line segment $\overline{xy}$, but no unbounded line integrals like for example $\int_0^\infty d\lambda\, \mathcal{B}(\lambda y + (1-\lambda)x)$. This way of understanding the causality of the perturbation expansion yields a simple condition in momentum space. Namely, if $\mathcal{B}$ has the form of a plane wave of momentum q, i.e. $\mathcal{B}(x) = \mathcal{B}_q \exp(-iqx)$, then the unbounded line integrals become infinite when q goes to zero (for $\mathcal{B}_q$ fixed), whereas integrals along the line segment $\overline{xy}$ are clearly bounded in this limit. Hence we can say that the perturbation calculation (D.14) is causal only if it is regular in the limit $q \to 0$. In order

to analyze this condition, we substitute the explicit formulas (D.6, D.9) into (D.14) and obtain

$$\begin{aligned}\Delta P[\mathcal{B}](x,y) \;&=\; -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu) \int \frac{d^4k}{(2\pi)^4} \\ &\times \Big(\frac{\mathrm{PP}}{(k+q)^2-\mu^2}\, (\not k + \not q + \mu)\, \mathcal{B}_q\, (\not k + \mu)\, \delta(k^2-\mu^2)\, \hat{P}(k)\, e^{-i(k+q)x+iky} \\ &\quad + \hat{P}(k)\, \delta(k^2-\mu^2)\, (\not k + \mu)\, \mathcal{B}_q\, (\not k - \not q + \mu)\, \frac{\mathrm{PP}}{(k-q)^2-\mu^2}\, e^{-ikx+i(k-q)y} \Big) .\end{aligned}$$

We set $q = \varepsilon \check{q}$ with a fixed vector $\check{q}$ and consider the behavior for $\epsilon \searrow 0$. Taking only the leading order in ε, one can easily carry out the μ-integration and gets

$$\begin{aligned}\Delta P[\mathcal{B}](x,y) \;&=\; -\frac{1}{\varepsilon} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \\ &\times \Big(\frac{\mathrm{PP}}{2k\check{q} + \varepsilon \check{q}^2}\, (\not k\, \mathcal{B}_q + \mathcal{B}_q\, \not k)\, \hat{P}(k) \\ &\qquad + \hat{P}(k)\, (\not k\, \mathcal{B}_q + \mathcal{B}_q\, \not k)\, \frac{\mathrm{PP}}{-2k\check{q} + \varepsilon \check{q}^2} \Big) + \mathcal{O}(\varepsilon^0) . \qquad \text{(D.15)}\end{aligned}$$

Since

$$\lim_{\epsilon \searrow 0} \frac{\mathrm{PP}}{2k\check{q} + \varepsilon \check{q}^2} \;=\; \lim_{\epsilon \searrow 0} \frac{\mathrm{PP}}{2k\check{q} - \varepsilon \check{q}^2} \;=\; \frac{\mathrm{PP}}{2k\check{q}}$$

in the sense of distributions in the argument $k\check{q}$ (notice that this kind of convergence is sufficient using the regularity of $\hat{P}$), the leading singularity of (D.15) for $\varepsilon \searrow 0$ has the form

$$-\frac{1}{\varepsilon} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{\mathrm{PP}}{2k\check{q}}\, \Big[\{\mathcal{B}_q,\, \not k\},\, \hat{P}(k) \Big] . \qquad \text{(D.16)}$$

Taking the Fourier transform in the variable $(x-y)$, it is clear that (D.16) vanishes only if the commutator/anti-commutator combination $[\{\mathcal{B}_q, \not k\}, \hat{P}(k)]$ is zero for all k. Since the perturbation $\mathcal{B}_q$ can be arbitrary, one sees (for example by considering a scalar perturbation, $\mathcal{B}_q \sim \mathbb{1}$) that it is a necessary condition for the perturbation calculation (D.14) to be regular in the limit $q \to 0$ that

$$[\not k,\, \hat{P}(k)] \;=\; 0 \qquad \text{for all } k. \qquad \text{(D.17)}$$

This commutator vanishes only if the vector field $v(k)$ in (4.1.5) is a multiple of k, or, using the notation of §4.4, if the surface states have no shear. We conclude that the perturbation calculation (D.14) is in general not causal.

Before resolving this causality problem, we briefly discuss how this problem comes about. The condition (D.17) can be stated equivalently that the operator P must commute with the free Dirac operator. In other words, the perturbation calculation (D.14) is causal only if the fermionic projector of the vacuum is composed of eigenstates of the free Dirac operator. In this formulation, our causality problem can be understood directly. Namely, since our perturbation method is based on the perturbation calculation (D.5) for Dirac eigenstates, it is not astonishing that the method is inappropriate for non-eigenstates, because the perturbation expansion is then performed around the wrong unperturbed states. It is interesting to see that this shortcoming leads to a breakdown of causality in the perturbation expansion.

In order to comply with causality, we must modify the perturbation calculation (D.14). Our idea is to deduce the perturbation calculation for the fermionic projector from that for a modified fermionic projector, which satisfies the causality condition (D.17). The simplest idea for modifying the fermionic projector would be to introduce a unitary transformation $\hat{U}(k) \in U(2,2)$ which makes the vector $v(k)$ in (4.1.5) parallel to k, more precisely

$$\hat{U}(k)^{-1}\, v_j(k)\, \gamma^j\, \hat{U}(k) \;=\; \lambda(k)\, \not{k} \qquad \text{with } \lambda(k) \in \mathbb{R}.$$

However, a unitary transformation is too restrictive because it keeps the Lorentzian scalar product $v(k)^2$ invariant, and thus it cannot be used for example in the case when $v(k)$ is space-like, but k is time-like. Therefore, we shall consider here a linear combination of unitary transformations. More precisely, we introduce for $L > 1$ and $l = 1, \ldots, L$ unitary operators $\hat{U}_l(k) \in U(2,2)$ and real coefficients c_l such that[1]

$$\sum_{l=1}^{L} c_l(k) = 1 \qquad \text{and} \qquad v_j(k)\, \gamma^j \;=\; \sum_{l=1}^{L} c_l(k)\, \hat{U}_l(k)\, \lambda(k)\, \not{k}\, \hat{U}_l(k)^{-1} \tag{D.18}$$

with $\lambda(k) \in \mathbb{R}$. The existence of $(\hat{U}_l, c_l)$ is guaranteed by the fact that the $U(2,2)$ transformations comprise Lorentzian transformations §1.5. Clearly, the representation (D.18) is not unique. According to (D.1), we can choose the transformation (D.18) to be the identity in the low-energy region, and can thus assume that

$$\hat{U}_l(k) \;=\; \mathbb{1} \qquad \text{if } |k^0| \ll E_P \text{ and } |\vec{k}| \ll E_P. \tag{D.19}$$

Furthermore, the regularity assumptions and the particular properties of the fermionic projector mentioned before (D.2) give rise to corresponding properties of the operators $\hat{U}_l$; this will be specified below (see (D.31, D.52)). The operators obtained by multiplication with $\hat{U}_l(k)$ in momentum space are denoted by U_l; they have in position space the kernels

$$U_l(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{U}_l(k)\, e^{-ik(x-y)}\,. \tag{D.20}$$

Finally, we introduce the "modified fermionic projector" Q by replacing the vector field $v(k)$ in (4.1.5) by $\lambda(k)\, \not{k}$, i.e.

$$\hat{Q}(k) \;=\; (\lambda(k)\, \not{k} \;+\; \phi(k)\, \mathbb{1})\, f(k)\,. \tag{D.21}$$

According to (D.18), the fermionic projector P is obtained from Q by the transformation

$$P \;=\; \sum_{l=1}^{L} c_l\, U_l\, Q\, U_l^{-1}\,. \tag{D.22}$$

The modified fermionic projector (D.21) satisfies the condition $[\hat{Q}(k),\, \not{k}] = 0$. Hence the perturbation calculation for Q does not suffer from our above causality problem, and we can introduce $\Delta Q[\mathcal{B}]$ in analogy to (D.14) by

$$\Delta Q[\mathcal{B}] \;:=\; -\int_{-\infty}^{\infty} d\mu\, (s_\mu\, \mathcal{B}\, p_\mu\, Q \;+\; Q\, p_\mu\, \mathcal{B}\, s_\mu)\,. \tag{D.23}$$

[1] Online version: Taking such linear combinations has the disadvantage that normalization and definiteness properties are not preserved. Therefore, it is preferable to use instead the construction in the book [**5**, Appendix F] (listed in the references in the preface to the second online edition).

We now deduce the perturbation of P by applying to (D.23) a transformation analogous to that in (D.22), namely

$$\Delta P[\mathcal{B}] \;:=\; \sum_{l=1}^{L} c_l\, U_l\, \Delta Q[\mathcal{B}]\, U_l^{-1} \tag{D.24}$$

$$\;=\; -\sum_{l=1}^{L} c_l \int_{-\infty}^{\infty} d\mu\, U_l \,(s_\mu\, \mathcal{B}\, p_\mu\, Q \;+\; Q\, p_\mu\, \mathcal{B}\, s_\mu)\, U_l^{-1}\,. \tag{D.25}$$

This last transformation should not affect the causality (in the sense of §2.2) because if (D.23) is regular when the momentum q of the bosonic potential goes to zero, then the transformed operator (D.24) will clearly also be regular in this limit. We call (D.25) the *regularized causal perturbation* of the fermionic projector to first order.

The perturbation calculation (D.25) requires a detailed explanation. Qualitatively speaking, the difference between (D.14) and (D.25) is that the spectral projectors p_μ, the Green's functions s_μ, and the perturbation operator $\mathcal{B}$ have been replaced by the unitarily transformed operators

$$U_l\, p_\mu\, U_l^{-1}\,, \qquad U_l\, s_\mu\, U_l^{-1} \quad \text{and} \qquad U_l\, \mathcal{B}\, U_l^{-1}\,, \tag{D.26}$$

and that a linear combination is taken. According to (D.19), the unitary transformations in (D.26) have no influence on the macroscopic properties of these operators, i.e. on the behavior when these operators are applied to wave functions with support in the low-energy region. But the transformation (D.26) changes the operators on the microscopic scale, in such a way that causality is fulfilled in the perturbation expansion. We point out that in the case where $\mathcal{B}$ is the usual operator of multiplication with the external potentials, the transformed operator $U_l \mathcal{B} U_l^{-1}$ is in general no longer a multiplication operator in position space; thus one can say that the classical potentials have become nonlocal on the microscopic scale. In order to better understand why the causality problem of (D.14) has disappeared in (D.25), it is useful to observe that Q commutes with the spectral projectors p_μ. This means that Q is composed of eigenstates of the Dirac operator, so that the perturbation expansion is now performed around the correct unperturbed states.

Let us consider a gauge transformation. In the case $\mathcal{B} = \not{\partial}\Lambda$, the perturbation (D.25) is computed to be

$$\Delta P[\not{\partial}\Lambda] \;=\; i \sum_{l=1}^{L} c_l \int_{-\infty}^{\infty} d\mu\, U_l \,(s_\mu\, [i\not{\partial} - \mu,\, \Lambda]\, p_\mu\, Q \;+\; Q\, p_\mu\, [i\not{\partial} - \mu,\, \Lambda]\, s_\mu)\, U_l^{-1}$$

$$\;=\; i \sum_{l=1}^{L} c_l \int_{-\infty}^{\infty} d\mu\, U_l \,(\Lambda\, p_\mu\, Q \;-\; Q\, p_\mu\, \Lambda)\, U_l^{-1}$$

$$\;=\; \sum_{l=1}^{L} c_l \left(i U_l \Lambda \left(\int_{-\infty}^{\infty} p_\mu\, d\mu \right) Q U_l^{-1} \;-\; i U_l Q \left(\int_{-\infty}^{\infty} p_\mu\, d\mu \right) \Lambda U_l^{-1} \right). \tag{D.27}$$

By construction of $\hat{Q}$, we can assume that the distributions $\hat{P}$ and $\hat{Q}$ have the same support, and thus (D.2) holds for $\hat{Q}$ as well,

$$\text{supp}\; \hat{Q} \;\subset\; \overline{\mathcal{C}^\wedge}\,. \tag{D.28}$$

Hence, according to (D.10), the projectors $\int_{-\infty}^{\infty} p_\mu d\mu$ in (D.27) can be omitted, and we conclude that

$$\Delta P[\partial\!\!\!/\Lambda] \;=\; \sum_{l=1}^{L} c_l \left(i U_l \Lambda U_l^{-1}\, U_l Q U_l^{-1} \;-\; i U_l Q U_l^{-1}\, U_l \Lambda U_l^{-1} \right) . \tag{D.29}$$

If in this formula we were allowed to replace the factors $U_l \Lambda U_l^{-1}$ by Λ, we could substitute in (D.22) and would obtain the contribution linear in Λ to the required transformation law (D.4). Indeed, the difference between Λ and $U_l \Lambda U_l^{-1}$ is irrelevant, as one sees in detail as follows. We consider one summand in (D.29) and set for ease in notation $U = U_l$. According to (D.19), the operators Λ and $U \Lambda U^{-1}$ coincide macroscopically (i.e. when applied to functions with support in the low-energy region), and thus (D.29) yields gauge symmetry on the macroscopic scale. However, such a macroscopic gauge symmetry is not sufficient for us: to ensure that the microscopic structure of space-time is not influenced by the electromagnetic field, it is essential that (D.4) holds even on the Planck scale. In order to show microscopic gauge invariance, we consider the operator $U \Lambda U^{-1}$ in momentum space,

$$(U \Lambda U^{-1}\, f)(q) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{U}(q)\, \hat{\Lambda}(q-p)\, \hat{U}(p)^{-1}\, f(p) \,, \tag{D.30}$$

where $\hat{\Lambda}$ is the Fourier transform of Λ, and f is a test function in momentum space. Since we assume that the electromagnetic potential $A\!\!\!/ = \partial\!\!\!/ \Lambda$ is macroscopic (D.3), the integrand in (D.30) vanishes unless $q - p$ is in the low-energy region. More precisely, we can say that

$$|q^0 - p^0|,\, |\vec{q} - \vec{p}| \;\sim\; l_{\text{macro}}^{-1} \,,$$

where l_{macro} denotes a typical length scale of macroscopic physics. Since the vector $q - p$ is in this sense small, it is reasonable to expand the factor $\hat{U}(q)$ in (D.30) in a Taylor series around p. As the operators $\hat{U}_l$ are characterized via (D.18), we can assume that they have similar regularity properties as P. In particular, we may assume that the partial derivatives of $\hat{U}_l(p)$ scale in powers of E_P^{-1}, in the sense that there should be a constant $c \ll l_{\text{macro}} E_P$ such that

$$|\partial^\kappa \hat{U}_l(p)| \;\leq\; \left(\frac{c}{E_P} \right)^{|\kappa|} \qquad \text{for any multi-index } \kappa. \tag{D.31}$$

From this we conclude that the Taylor expansion of $\hat{U}(q)$ around p is an expansion in powers of $(l_{\text{macro}} E_P)^{-1}$, and thus

$$\begin{aligned} (U \Lambda U^{-1}\, f)(q) \;&=\; \int \frac{d^4p}{(2\pi)^4}\, \hat{U}(p)\, \hat{\Lambda}(q-p)\, \hat{U}(p)^{-1}\, f(p) \\ &\quad + (\text{higher orders in } (l_{\text{macro}} E_P)^{-1}). \end{aligned} \tag{D.32}$$

Using that $\hat{\Lambda}(q-p)$ is a multiple of the identity matrix, the factors $\hat{U}(p)$ and $\hat{U}(p)^{-1}$ in (D.32) cancel each other. We conclude that the operators $U \Lambda U^{-1}$ and Λ coincide up to higher order in $(l_{\text{macro}} E_P)^{-1}$. For the integral kernels in position space, we thus have

$$(U \Lambda U^{-1})(x,y) \;=\; \Lambda(x)\, \delta^4(x-y) \;+\; (\text{higher orders in } (l_{\text{macro}} E_P)^{-1}). \tag{D.33}$$

We point out that this statement is much stronger than the equality of the operators $U \Lambda U^{-1}$ and Λ on the macroscopic scale that was mentioned at the beginning of this

paragraph. Namely, (D.33) shows that these operators coincide even microscopically, up to a very small error term. Notice that it was essential for the derivation that Λ is a scalar function (for example, (D.33) would in general be false if we replaced Λ by $\not{A}$). Using (D.33) in each summand of (D.29) and applying (D.22), we conclude that

$$\begin{aligned} \Delta P[\not{\partial}\Lambda](x,y) &= i\Lambda(x)\, P(x,y) \;-\; iP(x,y)\, \Lambda(y) \\ &\quad + (\text{higher orders in } (l_{\text{macro}} E_P)^{-1}). \end{aligned} \tag{D.34}$$

This shows gauge symmetry of the perturbation calculation (D.25).

It is interesting that, according to (D.34), gauge symmetry holds only up to an error term. This is unproblematic as long as the length scales of macroscopic physics are large compared to the Planck length. But (D.34) indicates that the regularized causal perturbation theory fails when energy or momentum of the perturbation $\mathcal{B}$ are of the order of the Planck energy. In this case, the distinction between the "macroscopic" and "microscopic" length scales, on which our constructions relied from the very beginning (cf. (D.3)), can no longer be made, and it becomes impossible to introduce a causal and gauge invariant perturbation theory.

We conclude the discussion of the regularized causal perturbation expansion by pointing out that our construction was based on condition (D.17), which is only a necessary condition for causality. Hence the causality of (D.25) has not yet been proved. We shall now perform the light-cone expansion of (D.25). This will show explicitly that the light-cone expansion involves, to leading orders in $(l_{\text{macro}} E_P)^{-1}$ and $(l E_P)^{-1}$, no unbounded line integrals, thereby establishing causality in the sense of §2.2.

In the remainder of this appendix, we will analyze the regularized causal perturbation calculation (D.25) near the light cone. Our method is to first perform the light-cone expansion of ΔQ, and then to transform the resulting formulas according to (D.24) to finally obtain the light-cone expansion of ΔP. In preparation, we describe how a decomposition into Dirac eigenstates can be used for an analysis of the operator Q near the light cone. A short computation using (D.21, D.28) yields that $\hat{Q}$ can be represented in the form

$$\hat{Q}(k) \;=\; \int_{-\infty}^{\infty} d\mu\; w_\mu(\vec{k})\; \epsilon(\mu)\; (\not{k} + \mu)\; \delta(k^2 - \mu^2)\; \Theta(-k^0) \tag{D.35}$$

with the real-valued distribution

$$w_\mu(\vec{k}) = (\phi(k) + \mu\, \lambda(k))\, f(k) \qquad \text{and} \qquad k(\vec{k}) = (-\sqrt{|\vec{k}|^2 + \mu^2}, \vec{k}). \tag{D.36}$$

This representation can be understood as follows. According to (D.9), the distributions $\epsilon(\mu)\; (\not{k} + \mu)\; \delta(k^2 - \mu^2)$ in the integrand of (D.35) are the spectral projectors of the free Dirac operator in momentum space. The factor $\Theta(-k^0)$ projects out all states on the upper mass cone, and the function $w_\mu(\vec{k})$ multiplies the states on the lower mass shell $k = (-\sqrt{|\vec{k}|^2 + \mu^2}, \vec{k})$ with a scalar weight factor. In this sense, (D.35) can be regarded as the spectral decomposition of the operator Q into Dirac eigenstates. Notice that the factor $\delta(k^2 - \mu^2)\; \Theta(-k^0)$ in (D.35) is the Fourier transform of the distribution T_a, (2.5.40). Exactly as described for the scalar component in §4.3, we are here interested only in the regularization effects for large energy or momentum and may thus disregard the logarithmic mass problem (see §2.5 for details). Therefore, we

"regularize" T_a according to (2.5.42) and consider instead of (D.35) the operator

$$\hat{Q}^{\text{reg}}(k) \;:=\; \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, (\not{k} + \mu)\, T^{\text{reg}}_{\mu^2}(k)\,,$$

where $T^{\text{reg}}_a(k)$ is the Fourier transform of (2.5.42). We expand the distribution $T^{\text{reg}}_{\mu^2}$ in a power series in μ^2,

$$\hat{Q}^{\text{reg}}(k) \;=\; \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, (\not{k} + \mu) \sum_{n=0}^{\infty} \frac{1}{n!}\, T^{(n)}(k)\, \mu^{2n}$$

with $T^{(n)}$ according to (2.5.43). Commuting the integral and the sum, we obtain

$$\hat{Q}^{\text{reg}}(k) \;=\; 32\pi^3 \sum_{n=0}^{\infty} \frac{1}{n!} \Big(g_{[n]}(\vec{k})\, \not{k} \;+\; h_{[n]}(\vec{k}) \Big)\, T^{(n)}(k) \tag{D.37}$$

with

$$g_{[n]}(\vec{k}) \;=\; \frac{1}{32\pi^3} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, \mu^{2n} \tag{D.38}$$

$$h_{[n]}(\vec{k}) \;=\; \frac{1}{32\pi^3} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, \mu^{2n+1}\,. \tag{D.39}$$

The representation (D.37) is very useful because it reveals the behavior of the operator Q near the light cone. To see this, we consider the Fourier transform of (D.37) in light-cone coordinates (s, l, x_2, x_3). For the Fourier transform of the factor $T^{(n)}(k)$, we have the representation (4.5.2). This representation can immediately be extended to the Fourier transform of $\not{k}\, T^{(n)}(k)$ by acting on (4.5.2) with the differential operator $i\not{\partial}$; more precisely in light-cone coordinates $y - x = (s, l, x_2, x_3)$,

$$\begin{aligned} &\int \frac{d^4k}{(2\pi)^4}\, \not{k}\, T^{(n)}(k)\, e^{-ik(x-y)} \\ &\quad= \;-\frac{1}{32\pi^3}\, (-il)^{n-2} \int_0^{\infty} \left[il\, \gamma^s \left(\frac{1}{u^{n-1}} \right)^{\text{reg}} - (n-1)\, \gamma^l \left(\frac{1}{u^n} \right)^{\text{reg}} \right] e^{-ius}\,. \end{aligned} \tag{D.40}$$

In order to treat the factors $g_{[n]}$ and $h_{[n]}$ in (D.37), we note that the Fourier transform of (D.37) can be computed similar as described in §4.3 by integrating out the transversal momenta according to (4.3.4) and analyzing the remaining two-dimensional Fourier integral (4.3.7) with the integration-by-parts method (4.3.14). If this is done, the functions $g_{[n]}$ and $h_{[n]}$ appear in the integrand of (4.3.7). Our regularity assumption on the fermionic projector of the vacuum (see §4.3 and §4.4) imply that $g_{[n]}$ and $h_{[n]}$ are smooth functions, whose partial derivatives scale in powers of E_P^{-1}. Hence all derivative terms of the functions $g_{[n]}$ and $h_{[n]}$ which arise in the integration-by-parts procedure (4.3.14) are of higher order in $(lE_P)^{-1}$. Taking into account only the leading order in $(lE_P)^{-1}$, we thus obtain a representation of the fermionic projector of the vacuum involving only $g_{[n]}$ and $h_{[n]}$ at the boundary $v = \alpha_u$. Comparing this representation with (4.5.2) and (D.40), we conclude that the Fourier transform of (D.37) is obtained, to leading order in $(lE_P)^{-1}$, simply by inserting the functions $g_{[n]}$ and $h_{[n]}$ into the integrands of (4.5.2) and (D.40), evaluated along the line $\vec{k} = (k_x = 2u, k_y = 0, k_z = 0)$.

Thus

$$\begin{aligned} Q^{\mathrm{reg}}(s,l) \;=\; & -\sum_{n=0}^{\infty} \frac{1}{n!}\,(-il)^{n-1} \int_0^\infty \left(\frac{1}{u^n}\right)^{\mathrm{reg}} e^{-ius}\, h_{[n]}(u)\, du \\ & -\sum_{n=0}^{\infty} \frac{1}{n!}\,(-il)^{n-2} \int_0^\infty \left[il\,\gamma^s \left(\frac{1}{u^{n-1}}\right)^{\mathrm{reg}} - (n-1)\,\gamma^l \left(\frac{1}{u^n}\right)^{\mathrm{reg}} \right] e^{-ius}\, g_{[n]}(u)\, du \\ & + (\text{higher orders in } (lE_P)^{-1}), \end{aligned} \tag{D.41}$$

where $h_{[n]}(u)$ and $g_{[n]}(u)$ are the functions (D.38, D.39) with $\vec{k} = (-2u, 0, 0)$.

The decomposition of the operator Q into Dirac eigenstates (D.35) is also useful for analyzing its perturbation ΔQ.

LEMMA D.1. *Let $B(x) \in C^2(\mathbb{R}^4) \cap L^1(\mathbb{R}^4)$ be a matrix potential which decays so fast at infinity that the functions $x_i \mathcal{B}(x)$ and $x_i x_j \mathcal{B}(x)$ are also L^1. Then the light-cone expansion of the operator $\Delta Q[\mathcal{B}]$, (D.23), is obtained by regularizing the light-cone expansion of the Dirac sea to first order in the external potential as follows. A summand of the light-cone expansion of the Dirac sea which is proportional to m^p,*

$$m^p \textit{ (iterated line integrals in bosonic potentials and fields) } T^{(n)}(s,l)\,,$$

must be replaced by

$$\begin{aligned} & (-1) \textit{ (iterated line integrals in bosonic potentials and fields)} \\ & \quad \times (-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{reg} e^{-ius} \times \begin{cases} h_{[\frac{p-1}{2}]} & \textit{for } p \textit{ odd} \\ g_{[\frac{p}{2}]} & \textit{for } p \textit{ even} \end{cases} \\ & \quad + \textit{(rapid decay in } l) + \textit{(higher orders in } (lE_P)^{-1}, (l_{macro} E_P)^{-1}). \end{aligned} \tag{D.42}$$

A contribution $\sim m^p$ which contains a factor $(y-x)_j \gamma^j$,

$$m^p \textit{ (iterated line integrals in bosonic potentials and fields) } (y-x)_j \gamma^j\, T^{(n)}(s,l)\,,$$

is to be replaced by

$$\begin{aligned} & (-1) \textit{ (iterated line integrals in bosonic potentials and fields)} \\ & \quad \times (-il)^{n-1} \int_{-\infty}^{\infty} du \left[2l\,\gamma^s \left(\frac{1}{u^n}\right)^{reg} + 2in\,\gamma^l \left(\frac{1}{u^{n+1}}\right)^{reg} \right] \\ & \quad \times e^{-ius} \times \begin{cases} h_{[\frac{p-1}{2}]} & \textit{for } p \textit{ odd} \\ g_{[\frac{p}{2}]} & \textit{for } p \textit{ even} \end{cases} + \textit{(contributions } \sim \gamma^2, \gamma^3) \\ & \quad + \textit{(rapid decay in } l) + \textit{(higher orders in } (lE_P)^{-1}, (l_{macro} E_P)^{-1}). \end{aligned} \tag{D.43}$$

In these formulas, $g_{[n]}$ and $h_{[n]}$ are the functions (D.38, D.39) with $\vec{k} = (-2u, 0, 0)$.

Proof. By substituting (D.6) and (D.35) into (D.23), we obtain the following representation for ΔQ in momentum space,

$$\begin{aligned}\Delta Q[\mathcal{B}]\left(k+\frac{q}{2},\, k-\frac{q}{2}\right) \;&=\; -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not{k}+\frac{\not{q}}{2}+\mu)\, \mathcal{B}_q\, (\not{k}-\frac{\not{q}}{2}+\mu) \\ &\quad\times \left(\frac{\mathrm{PP}}{(k+\frac{q}{2})^2-\mu^2}\, w_\mu(\vec{k}-\frac{\vec{q}}{2})\, T_{\mu^2}(k-\frac{q}{2}) \right. \\ &\qquad\qquad \left. + w_\mu(\vec{k}+\frac{\vec{q}}{2})\, T_{\mu^2}(k+\frac{q}{2})\, \frac{\mathrm{PP}}{(k-\frac{q}{2})^2-\mu^2} \right) \\ &=\; \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not{k}+\frac{\not{q}}{2}+\mu)\, \mathcal{B}_q\, (\not{k}-\frac{\not{q}}{2}+\mu) \\ &\qquad\times \frac{\mathrm{PP}}{2kq} \left(w_\mu(\vec{k}+\frac{\vec{q}}{2})\, T_{\mu^2}(k+\frac{q}{2}) \;-\; w_\mu(\vec{k}-\frac{\vec{q}}{2})\, T_{\mu^2}(k-\frac{q}{2}) \right). \end{aligned} \tag{D.44}$$

Using the methods developed in [**F5**], we now perform the light-cone expansion in momentum space and then transform back to position space. Since we are here interested in the regularization effects for large energy or momentum, we may disregard the logarithmic mass problem and work on the level of the formal light-cone expansion of [**F5**, Section 3] (our constructions could be made rigorous using the resummation method of [**F5**, Section 4]). As in [**F5**, Section 3], we expand the distributions T_{μ^2} in a Taylor series in q and rewrite the resulting k-derivatives as derivatives with respect to μ^2. This gives

$$T_{\mu^2}(k \pm \frac{q}{2}) \;=\; \sum_{j,r=0}^{\infty} c_{jr}\, (\pm kq)^j \left(\frac{q^2}{4}\right)^r T_{\mu^2}^{(j+r)}(k) \tag{D.45}$$

with combinatorial factors c_{jr} whose detailed form is not needed in what follows. Next, we expand (D.45) in a Taylor series in μ^2 and obtain

$$T_{\mu^2}(k \pm \frac{q}{2}) \;=\; \sum_{n,j,r=0}^{\infty} c_{njr}\, \mu^{2n}\, (\pm kq)^j \left(\frac{q^2}{4}\right)^r T^{(n+j+r)}(k) \tag{D.46}$$

with new combinatorial factors c_{njr}. We substitute the expansions (D.46) into (D.44) and write the even and odd terms in kq together,

$$\begin{aligned}\Delta Q[\mathcal{B}]\left(k+\frac{q}{2},\, k-\frac{q}{2}\right) \;&=\; -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not{k}+\frac{\not{q}}{2}+\mu)\, \mathcal{B}_q\, (\not{k}-\frac{\not{q}}{2}+\mu) \\ &\times \left(\frac{\mathrm{PP}}{2kq} \sum_{n,j,r=0,\, j \text{ even}}^{\infty} c_{njr}\, \mu^{2n}\, (kq)^j \left(\frac{q^2}{4}\right)^r T^{(n+j+r)}(k) \left(w_\mu(\vec{k}+\frac{\vec{q}}{2}) - w_\mu(\vec{k}-\frac{\vec{q}}{2}) \right) \right. \\ &\left. + \frac{\mathrm{PP}}{2kq} \sum_{n,j,r=0,\, j \text{ odd}}^{\infty} c_{njr}\, \mu^{2n}\, (kq)^j \left(\frac{q^2}{4}\right)^r T^{(n+j+r)}(k) \left(w_\mu(\vec{k}+\frac{\vec{q}}{2}) + w_\mu(\vec{k}-\frac{\vec{q}}{2}) \right) \right). \end{aligned} \tag{D.47}$$

We first consider the contributions to (D.47) for even j. These terms contain the factor $(w_\mu(\vec{k}+\frac{\vec{q}}{2}) - w_\mu(\vec{k}-\frac{\vec{q}}{2}))$. If the distribution w_μ were a smooth function and its derivatives had the natural scaling behavior in powers of the Planck length, we could immediately conclude that $|w_\mu(\vec{k}+\frac{\vec{q}}{2}) - w_\mu(\vec{k}-\frac{\vec{q}}{2})| \sim |\vec{q}|\, |\partial w_\mu| \sim (l_{\text{macro}} E_P)^{-1}$, and

thus all the terms for even j would be negligible. Unfortunately, the situation is more difficult because w_μ is in general not a smooth function (cf. (D.36)), and we obtain the desired regularity in $\vec{k}$ only after the μ-integration has been carried out. This makes it necessary to use the following argument. Consider one summand in (D.47) for even j. After carrying out the μ-integration, this summand yields a finite number of contributions to $\Delta Q(k+\frac{q}{2}, k-\frac{q}{2})$ of the following form,

$$\frac{\text{PP}}{kq}\,(kq)^j \left(\frac{q^2}{4}\right)^r \cdots \mathcal{B}_q \cdots T^{(n+j+r)}(k) \left[g(\vec{k}+\frac{\vec{q}}{2}) - g(\vec{k}+\frac{\vec{q}}{2})\right], \tag{D.48}$$

where each symbol "$\cdots$" stands for a possible factor $\not{k}$ or $\not{q}$, and where g is a scalar function, which coincides with one of the functions $g_{[n]}$ or $h_{[n]}$ (see (D.38) and (D.39)). As already mentioned after (D.40), our regularity assumptions on the fermionic projector of the vacuum imply that the functions $g_{[n]}$ and $h_{[n]}$, and thus also g, are smooth, and that their derivatives scale in powers of the Planck length. Applying the fundamental theorem of calculus, we rewrite the square bracket in (D.48) as a line integral,

$$(D.48) \;=\; \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda\, \frac{\text{PP}}{kq}\,(kq)^j \left(\frac{q^2}{4}\right)^r \cdots \mathcal{B}_q \cdots T^{(n+j+r)}(k)\, (\vec{q}\,\vec{\nabla}) g(\vec{k}+\lambda\vec{q})\,. \tag{D.49}$$

We now transform (D.49) to position space. Our regularity assumptions on $\mathcal{B}$ mean in momentum space that $\mathcal{B}(q) \in C^2 \cap L^\infty$. Using furthermore the regularity of $\vec{\nabla} g$, we can carry out the q-integration in the Fourier integral. Carrying out also the integral over λ, we end up with a contribution to $\Delta Q(x,y)$ of the form

$$\int \frac{d^4k}{(2\pi)^4}\, T^{(n+j+r)}(k)\; F(k, x+y)\, e^{-ik(x-y)} \tag{D.50}$$

with a (matrix-valued) function F which is differentiable in k and is of the order $(l_{\text{macro}} E_P)^{-1}$. In the low-energy region, the function g in (D.48) is constant and thus F is homogeneous in k of degree at most $j+1$. After transforming to light-cone coordinates, this implies that (D.50) is close to the light cone dominated by the fermionic projector of the vacuum, in the sense that in light-cone coordinates, $|(D.50)| \leq \text{const}(l)\, |P(s,l)|$. We conclude that all summands in (D.47) for even j are of higher order in $(l_{\text{macro}} E_P)^{-1}$.

It remains to consider the summands in (D.47) for odd j. In this case, one factor kq cancels the principal value, and we obtain

$$\begin{aligned}
\Delta Q[\mathcal{B}]\left(k+\frac{q}{2},\, k-\frac{q}{2}\right) \;&=\; -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not{k}+\frac{\not{q}}{2}+\mu)\, \mathcal{B}_q\, (\not{k}-\frac{\not{q}}{2}+\mu) \\
&\quad \times \sum_{n,j,r=0}^{\infty} C_{njr}\, \mu^{2n}\, (kq)^{2j} \left(\frac{q^2}{4}\right)^r T^{(n+2j+1+r)}(k) \left(w_\mu(\vec{k}+\frac{\vec{q}}{2}) + w_\mu(\vec{k}-\frac{\vec{q}}{2})\right) \\
&\quad + (\text{higher orders in } (l_{\text{macro}} E_P)^{-1})
\end{aligned} \tag{D.51}$$

with some combinatorial factors C_{njr}. This formula has similarities to the light-cone expansion of the Dirac sea in momentum space [**F5**, equation (3.15)]. In [**F5**, Section 3], we proceeded by rewriting the factors kq as k-derivatives acting on $T^{(.)}$. When taking the Fourier transform, these k-derivatives were integrated by parts onto the exponential factor $\exp(-ik(x-y))$ to yield factors $(y-x)$. After collecting and rearranging all resulting terms, we obtained the line-integrals of the light-cone expansion. This method can be applied also to the integrand of (D.51), and we can carry out

the μ-integration afterwards. We shall not go through all these constructions steps in detail here, but merely consider what happens in principle. Whenever a k-derivative ∂_{k^i} acts on the factors w_μ in the integration-by-parts procedure, we get instead of a factor $(y-x)_i\, w_\mu$ (which is obtained when the k-derivative acts on the exponential $\exp(-ik(x-y))$) a factor $\partial_i w_\mu$. After carrying out the μ-integration, one sees that the resulting term is of higher order in $(lE_P)^{-1}$. Thus we can, to leading order in $(lE_P)^{-1}$, neglect all derivatives of the factors w_μ. But then, the integration-by-parts procedure reduces to the construction in [**F5**, Section 3], and we thus obtain precisely the line integrals of the light-cone expansion [**F5**]. Furthermore, we can replace the factor $(w_\mu(\vec{k}+\frac{\vec{q}}{2})+w_\mu(\vec{k}-\frac{\vec{q}}{2}))$ in (D.51) by $2w_\mu(\vec{k})$, because a Taylor expansion of this factor around $\vec{q}=0$ amounts, again after carrying out the μ-integration, to an expansion in powers of $(l_{\text{macro}}E_P)^{-1}$, and it thus suffices to take into account the leading term of this expansion. These considerations show that the light-cone expansion of (D.51) differs from that in [**F5**] merely by the additional μ-integration and the factor $w_\mu(\vec{k})$. Hence the light-cone expansion of (D.51) is obtained from that of the Dirac sea by the following replacements,

$$\begin{aligned} m^p\, T^{(n)}(x,y) \quad &\to \quad \int \frac{d^4k}{(2\pi)^4} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, \mu^p\, T^{(n)}(k)\, e^{-ik(x-y)}\, w_\mu(\vec{k}) \\ m^p\, (y-x)_j \gamma^j\, T^{(n)}(x,y) & \\ &\to \quad \int \frac{d^4k}{(2\pi)^4} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, \mu^p\, (-2ik\!\!\!/)\, T^{(n+1)}(k)\, e^{-ik(x-y)}\, w_\mu(\vec{k}) \end{aligned}$$

(where we used the identity $(y-x)^i T^{(n)}(x,y) = 2\partial_{x^i} T^{(n+1)}(x,y)$; see [**F5**, equation (3.5)]). The lemma follows by carrying out the μ-integrals applying (D.38, D.39) and by analyzing the behavior near the light cone as explained before (D.41).

∎

From this lemma we can deduce the light-cone expansion of the regularized fermionic projector.

THEOREM D.2. *The light-cone expansion of the regularized causal perturbation (D.25) is obtained by regularizing the light-cone expansion of the Dirac sea to first order in the external potential as follows. A summand of the light-cone expansion of the Dirac sea which is proportional to m^p,*

$$m^p \text{ (iterated line integrals in bosonic potentials and fields) } T^{(n)}(s,l)\,,$$

must be replaced by (4.5.7). A contribution $\sim m^p$ which contains a factor $(y-x)_j\gamma^j$,

$$m^p \text{ (iterated line integrals in bosonic potentials and fields) } (y-x)_j\gamma^j\, T^{(n)}(s,l)\,,$$

is to be replaced by (4.5.9). In these formulas, g, h, a and b are the regularization functions introduced in §4.3 and §4.4 (see (4.4.27, 4.3.25, 4.5.10, 4.4.19)).

Proof. As mentioned at the beginning of this appendix, we assume here that the vector component is null on the light cone (4.4.21). Let us consider what this condition tells us about the operators U_l. According to (D.19), the operators $\hat{U}_l$ are trivial in the low-energy region. Conversely, for large energy or momentum, (4.4.21) yields that the vector field $v(k)$ is parallel to k, up to a perturbation of the order $\varepsilon_{\text{shear}}$. Hence

we can assume that the transformation (D.22) is a small perturbation of the identity, in the sense that

$$c_l\, |\hat{U}_l(k) - \mathbb{1}| \;\sim\; \varepsilon_{\text{shear}} \qquad \text{for all } k. \tag{D.52}$$

We next derive the light-cone expansion of ΔP by transforming the result of Lemma D.1 according to (D.24). Since the transformation (D.24) is small in the sense of (D.52), it leaves the iterated line integrals in (D.42) and (D.43) unchanged to leading order in $\varepsilon_{\text{shear}}$. Hence it suffices to consider the transformation of the u-integrals in (D.42) and (D.43). The u-integral in (D.42) is as a homogeneous scalar operator invariant under the unitary transformations. In the u-integral in (D.43), on the other hand, only the Dirac matrices γ^l and γ^s are modified. More precisely, we have to leading order in $\varepsilon_{\text{shear}}$,

$$\begin{aligned}
\sum_{l=1}^{L} c_l\, (\hat{U}_l \gamma^s \hat{U}_l^{-1})(u, v=\alpha_u) &= \gamma^s + \frac{b_1(u)}{u^2}\, \gamma^l + (\text{contributions} \sim \gamma^2, \gamma^3) \\
\sum_{l=1}^{L} c_l\, (\hat{U}_l \gamma^l \hat{U}_l^{-1})(u, v=\alpha_u) &= \gamma^l + \frac{b_2(u)}{u^2}\, \gamma^s + (\text{contributions} \sim \gamma^2, \gamma^3)
\end{aligned}$$

with suitable regularization functions b_s and b_l which are small in the following sense,

$$\frac{b_{1/2}(u)}{u^2} \;\sim\; \varepsilon_{\text{shear}}\,.$$

Notice that in the high-energy region $u \sim E_P$, the contribution $\sim \gamma^l$ in the integrand of (D.43) is smaller than the contribution $\sim \gamma^s$ by a relative factor of $(lE_P)^{-1}$. Hence we can neglect b_2, whereas b_1 must be taken into account. We conclude that the transformation (D.24) of the contributions (D.42) and (D.43) is carried out simply by the replacement

$$\gamma^s \;\to\; \gamma^s \;+\; \frac{b_1(u)}{u^2}\, \gamma^l\,. \tag{D.53}$$

It remains to derive relations between the regularization functions $g_{[n]}$, $h_{[n]}$, and b_s, which appear in the transformed contributions (D.42) and (D.43), and the regularization functions g, h, a, and b in (4.5.7) and (4.5.9). For this, we apply the transformation (D.22) to Q^{reg}, (D.41). Exactly as described above, this transformation reduces to the replacement (D.53), and we obtain the following expansion of the fermionic projector near the light cone,

$$\begin{aligned}
P^{\text{reg}}(s,l) \;=\; & -\sum_{n=0}^{\infty} \frac{1}{n!}\, (-il)^{n-1} \int_0^\infty \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\, h_{[n]}(u)\, du \\
& -\sum_{n=0}^{\infty} \frac{1}{n!}\, (-il)^{n-2} \int_0^\infty e^{-ius}\, g_{[n]}(u) \\
& \qquad \times \left[il\, \gamma^s \left(\frac{1}{u^{n-1}}\right)^{\text{reg}} - (n-1)\, \gamma^l \left(\frac{1}{u^n}\right)^{\text{reg}} + il\, \gamma^s\, b(u) \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} \right] du \\
& + (\text{higher orders in } \varepsilon_{\text{shear}},\, (lE_P)^{-1}).
\end{aligned}$$

Comparing this result with the formulas for the fermionic projector derived in §4.3 and §4.4 (see (4.3.27, 4.3.28) and (4.4.6, 4.4.7)), one gets the following identities between the regularization functions,

$$g_{[n]}(u) \;=\; g(u)\, a(u)^n\,, \qquad h_{[n]}(u) \;=\; h(u)\, a(u)^n\,, \qquad b_1(u) \;=\; b(u)\,. \qquad \blacksquare$$

We finally explain in which sense the regularized causal perturbation theory is unique. In order to ensure regularity of the perturbation theory in the limit when the momentum q of the external field goes to zero, one must satisfy a causality condition similar to (D.17), and to this end one has to work with a modified fermionic projector Q. Since we must modify the direction of the vector field v, it is natural to describe the transformation from Q to P by linear combinations of unitary transformations (D.22). Nevertheless, we remark that one could just as well work with a different or more general transformation $Q \to P$. The reason is that the particular form of this transformation enters only in the proof of Theorem D.2, and we use merely that this transformation is close to the identity, in the sense similar to (D.52). Hence the restriction to transformations of type (D.22) is no loss in generality. Furthermore, we point out that the gauge symmetry (D.34) uniquely determines the precise form of how the potential $\mathcal{B}$ enters into the perturbation calculation (e.g. one may not replace $\mathcal{B}$ in (D.25) by $U_l^{-1} \mathcal{B} U_l$). We conclude that our construction of the regularized causal perturbation theory is canonical up to the freedom in choosing the coefficients $c_l(k)$ and the unitary transformations $\hat{U}_l(k)$. By assuming that the unitary transformations are regular (D.31) and small (D.52), the arbitrariness in choosing $(c_l, \hat{U}_l)$ was constrained so much that it has no influence on the regularization of the light-cone expansion. Indeed, the c_l and $\hat{U}_l$ do not enter the statement of Theorem D.2. Thus we can say that the regularized causal perturbation expansion is unique up to contributions of higher order in $(lE_P)^{-1}$, $(l_{\text{macro}} E_P)^{-1}$ and $\varepsilon_{\text{shear}}$.

APPENDIX E

Linear Independence of the Basic Fractions

In this appendix we consider simple fractions of degree $L \geq 2$ of the form

$$\frac{T_{\circ}^{(a_1)} \cdots T_{\circ}^{(a_\alpha)}\, \overline{T_{\circ}^{(b_1)} \cdots T_{\circ}^{(b_\beta)}}}{T_{[0]}^{(c_1)} \cdots T_{[0]}^{(c_\gamma)}\, \overline{T_{[0]}^{(d_1)} \cdots T_{[0]}^{(d_\delta)}}} \tag{E.1}$$

with integer parameters $\alpha, \beta \geq 1$, $\gamma, \delta \geq 0$ which satisfy the additional conditions

$$c_j, d_j \;\in\; \{-1, 0\} \tag{E.2}$$

$$\alpha - \gamma \;>\; \beta - \delta \;\geq\; 1\,. \tag{E.3}$$

We prove the following theorem which makes precise that the only relations between the simple fractions are given by the integration-by-parts rules.

THEOREM E.1. *Assume that a linear combination of simple fractions (E.1–E.3) vanishes when evaluated weakly on the light cone (4.5.29) to leading order in $(lE_P)^{-1}$ and $(l_{macro}E_P)^{-1}$, for any choice of η and the regularization functions. Then the linear combination is trivial after suitably applying the integration-by-parts rules.*

The condition (E.3) ensures that the simple fractions are asymmetric under complex conjugations. Such an asymmetry is essential for our proof. However, (E.3) could easily be weakened or replaced by other asymmetry conditions. Also, (E.2) and the fact that the denominator involves only the square indices [0] is mainly a matter of convenience. The reason why we are content with (E.1–E.3) is that all EL equations in this book can be expressed in terms of simple fractions of this form.

We point out that the above theorem does not imply that the basic monomials are independent in the sense that, by choosing suitable regularization functions, the basic regularization parameters can be given arbitrary values. Theorem E.1 states that there are no identities between the basic fractions, but the basic regularization parameters might nevertheless be constrained by inequalities between them (e.g. certain regularization parameters might be always positive). Furthermore, we remind that the assumptions of positivity of the scalar component and of half occupied surface states (see the last paragraph of §4.4) yield relations between the regularization functions which might give additional constraints for the regularization parameters. For these reasons, one should in applications always verify that the values for the basic regularization parameters obtained in the effective continuum theory can actually be realized by suitable regularization functions.

In the proof we will work with a class of regularization functions for which the Fourier integrals and the weak evaluation integral can be computed explicitly; then we will analyze in detail how the resulting formulas depend on the regularization. More

precisely, we choose the regularization functions in (4.5.12, 4.5.14) as follows,

$$g(u) \;=\; u^{\sigma-1}\,(1+\varepsilon\,u^\nu)\,e^{-\frac{u}{2E_P}}\,\Theta(u)\,, \quad h(u) \;=\; u^{4\nu}\,g(u) \tag{E.4}$$

$$a(u) \;=\; u^{8\nu}\,, \qquad\qquad\qquad\qquad b(u) \;=\; u^{2\nu} \tag{E.5}$$

with real parameters $\varepsilon, \sigma, \nu, E_P > 0$. These regularization functions have all the properties required in §2.5 if $\sigma \approx 1$ and $\varepsilon \ll \nu \ll 1$; note that the factor $e^{-\frac{u}{2E_P}}$ gives the desired decay on the scale of the Planck energy. Using the decay of the integrand for large $\operatorname{Re} u$, we can deform the integration contours to obtain for any $\rho > 0$,

$$\int_0^\infty u^{\rho-1}\,e^{-\frac{u}{2E_P}}\,e^{-ius}\,du \;=\; \left(is+\frac{1}{2E_P}\right)^{-\rho}\int_0^\infty v^{\rho-1}\,e^{-v}\,dv \;=\; \frac{\Gamma(\rho)}{z^\rho}\,, \tag{E.6}$$

where in the last step we set

$$z \;=\; is+\frac{1}{2E_P}$$

and used the definition of the gamma function. Here the power $z^{-\rho}$ is understood as $\exp(-\rho\log z)$ with the logarithm defined on the complex plane cut along the negative real axis. By analytic continuation we we can extend (E.6) to ρ in the complex plane with the exception of the poles of the gamma function,

$$\int_0^\infty u^{\rho-1}\,e^{-\frac{u}{2E_P}}\,e^{-ius}\,du \;=\; \frac{\Gamma(\rho)}{z^\rho}\,, \qquad \text{for } \rho\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}. \tag{E.7}$$

This Fourier integral is also useful for computing the L^2-scalar product of the Fourier transform via Plancherel. Namely, under the conditions $\rho, \rho' > \frac{1}{2}$ we obtain

$$\int_0^\infty u^{\rho+\rho'-2}e^{-E_P u}du \;=\; \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\Gamma(\rho)}{z^\rho}\,\frac{\Gamma(\rho')}{\overline{z}^{\rho'}}\,ds\,,$$

and computing the integral on the left gives

$$\int_{-\infty}^\infty \frac{1}{z^\rho\,\overline{z}^{\rho'}}\,ds \;=\; 2\pi\,E_P^{\rho+\rho'-1}\,\frac{\Gamma(\rho+\rho'-1)}{\Gamma(\rho)\,\Gamma(\rho')} \qquad \text{for } \rho,\rho' > \tfrac{1}{2}. \tag{E.8}$$

The fact that the integral (E.7) diverges when ρ tends to a negative integer corresponds precisely to the logarithmic mass problem as discussed after (4.5.1). A short calculation shows that the infrared regularization can be introduced here simply by subtracting the pole, i.e. for $n\in\mathbb{N}_0$

$$\int_0^\infty \left(u^{-n-1}\right)^{\text{reg}}\,e^{-\frac{u}{2E_P}}\,e^{-ius}\,du \;=\; \lim_{\rho\to -n}\left(\frac{\Gamma(\rho)}{z^\rho}-\frac{(-1)^n}{n!}\,\frac{z^n}{\rho+n}\right). \tag{E.9}$$

Using this formula in (4.5.12, 4.5.14), we obtain

$$T^{(n)}_{[p]} \;=\; -(-il)^{n-1}\sum_{k=0,1}\varepsilon^k\,\frac{\Gamma(\sigma-n+(4p+k)\nu)}{z^{\sigma-n+(4p+k)\nu}} \;-\; \text{(IR-reg)} \tag{E.10}$$

$$T^{(n)}_{\{p\}} \;=\; -(-il)^{n-1}\sum_{k=0,1}\varepsilon^k\,\frac{\Gamma(\sigma-n+(4p+k+2)\nu)}{z^{\sigma-n+(4p+k+2)\nu}} \;-\; \text{(IR-reg)}\,, \tag{E.11}$$

where "(IR-reg)" means that we subtract a counter term as in (E.9).

For clarity we disregard the infrared regularization for the moment and consider only the zeroth order in ε. Substituting the obtained formulas for $T_\circ^{(n)}$ and their complex conjugates into (E.1), we obtain

$$\begin{aligned}(E.1) \;=\; & \frac{(-1)^{A-C+\beta-\delta}}{(-il)^L}\,\frac{\Gamma(\sigma-a_1+\circ\nu)\cdots\Gamma(\sigma-a_\alpha+\circ\nu)}{\Gamma(\sigma-c_1)\cdots\Gamma(\sigma-c_\gamma)}\,\frac{1}{z^{(\alpha-\gamma)\sigma-A+C+\bullet\nu}} \\ & \times\frac{\Gamma(\sigma-b_1+\circ\nu)\cdots\Gamma(\sigma-b_\beta+\circ\nu)}{\Gamma(\sigma-d_1)\cdots\Gamma(\sigma-d_\delta)}\,\frac{1}{\overline{z}^{(\beta-\delta)\sigma-B+D+\bullet\nu}}\,, \qquad (E.12)\end{aligned}$$

where $A=\sum_{j=1}^{\alpha}a_j$, $B=\sum_{j=1}^{\beta}b_j$, $C=\sum_{j=1}^{\gamma}c_j$, $D=\sum_{j=1}^{\delta}d_j$. Here the parameter $\circ$ takes into account the lower indices of the corresponding factors $T_\circ^{(a_j)}$ or $\overline{T_\circ^{(b_j)}}$; more precisely for an index $[p]$ and $\{p\}$ it is equal to $4p$ and $4p+2$, respectively. The parameter $\bullet$ stands for the sum of the parameters $\circ$ in the same line. Integrating over s,

$$\int_{-\infty}^{\infty}\frac{T_\circ^{(a_1)}\cdots T_\circ^{(a_\alpha)}\,\overline{T_\circ^{(b_1)}\cdots T_\circ^{(b_\beta)}}}{T_{[0]}^{(c_1)}\cdots T_{[0]}^{(c_\gamma)}\,\overline{T_{[0]}^{(d_1)}\cdots T_{[0]}^{(d_\delta)}}}\,ds\,, \qquad (E.13)$$

using (E.8) and leaving out irrelevant prefactors, we obtain the expression

$$\begin{aligned}&\frac{\Gamma(\sigma-a_1+\circ\nu)\cdots\Gamma(\sigma-a_\alpha+\circ\nu)\,\Gamma(\sigma-b_1+\circ\nu)\cdots\Gamma(\sigma-b_\beta+\circ\nu)}{\Gamma(\sigma-c_1)\cdots\Gamma(\sigma-c_\gamma)\,\Gamma(\sigma-d_1)\cdots\Gamma(\sigma-d_\delta)} \\ &\times\frac{E_P^{\lambda-1}\,\Gamma(\lambda-1)}{\Gamma((\alpha-\gamma)\sigma-A+C+\bullet\nu)\,\Gamma((\beta-\delta)\sigma-B+D+\bullet\nu)}\,, \qquad (E.14)\end{aligned}$$

where λ is the sum of the arguments of the two gamma functions in the denominator of the second line. Since the E_P-dependence tells us about λ, we are led to a combination of gamma functions as considered in the next lemma. Although the statement of the lemma is not surprising, the proof is a bit delicate, and we give it in detail.

LEMMA E.2. *Consider for given parameters $N,M\in\mathbb{N}$ quotients of gamma functions of the form*

$$\frac{\Gamma(\sigma-a_1+\nu b_1)\cdots\Gamma(\sigma-a_J+\nu b_J)}{\Gamma(\sigma-c_1+\nu d_1)\cdots\Gamma(\sigma-c_K+\nu d_K)}\,\frac{1}{\Gamma(n_1\sigma-l_1+\nu m_1)\,\Gamma(n_2\sigma-l_2+\nu m_2)} \qquad (E.15)$$

with integers $J,K\geq 0$ and $a_j,b_j,c_j,d_j,n_j,l_j,m_j\in\mathbb{Z}$, which satisfy the relations

$$n_1+n_2 \;=\; N\,, \qquad n_1\;>\;n_2\;\geq\;1 \qquad (E.16)$$

$$m_1+m_2 \;=\; M\,. \qquad (E.17)$$

If a linear combination of expressions of the form (E.15–E.17) vanishes for all (σ,ν) in an open set of $\mathbb{R}^2$, then the linear combination is trivial after suitably applying the identity

$$x\,\Gamma(x) \;=\; \Gamma(x+1)\,. \qquad (E.18)$$

Proof. Assume that a linear combination of terms of the form (E.15–E.17) vanishes for all (σ,ν) in an open set of $\mathbb{R}^2$. By analytic continuation we can assume that the linear combination vanishes for σ and μ in the whole complex plane with the exception of the poles of the gamma functions.

We first consider the asymptotics for large σ and ν. If we fix ν and choose σ large, we can approximate the gamma functions with the Stirling formula

$$\Gamma(x) \;=\; \sqrt{2\pi}\,x^{x+\frac{1}{2}}\,e^{-x}\,\bigl(1+\mathcal{O}(x^{-1})\bigr) \qquad (E.19)$$

to obtain

$$\log(E.15) \;=\; (J-K-n_1-n_2)\,\sigma\,(\log(\sigma)-1) - \sum_{i=1,2} n_i \log(n_i)\,\sigma \;+\; \mathcal{O}(\log\sigma)\,.$$

Terms with a different asymptotics cannot compensate each other in the linear combination and must therefore vanish separately. Thus we can restrict attention to a linear combination with fixed values of the parameters

$$J-K-n_1-n_2 \qquad \text{and} \qquad \sum_{i=1,2} n_i \log(n_i)\,. \tag{E.20}$$

More generally, we can choose $\nu=\varepsilon\sigma$ for small fixed $\varepsilon\geq 0$. Then for large σ,

$$\begin{aligned}\log(E.15) \;=\; & \left(\sum_{j=1}^{J}(1+\varepsilon b_i) - \sum_{k=1}^{K}(1+\varepsilon d_k) \;-\; \sum_{i=1,2}(n_i+\varepsilon m_i)\right)\sigma\,(\log(\sigma)-1)\\ & + \sum_{j=1}^{J}(1+\varepsilon b_j)\;\log(1+\varepsilon b_j)\,\sigma - \sum_{k=1}^{K}(1+\varepsilon d_k)\;\log(1+\varepsilon d_k)\,\sigma\\ & - \sum_{i=1,2}(n_i+\varepsilon m_i)\;\log(n_i+\varepsilon m_i)\,\sigma \;+\; \mathcal{O}(\log\sigma)\,.\end{aligned}$$

Expanding in powers of ε, we see that the asymptotics also determines the parameter

$$\sum_{i=1,2} m_i\;\log(n_i)\,. \tag{E.21}$$

We can assume that the parameters (E.20, E.21) are the same for all summands of our linear combination. Combining (E.16) with the right of (E.20), we can compute n_1 and n_2. Furthermore, (E.17) and (E.21) uniquely determine m_1 and m_2.

By iteratively applying (E.18) we can write each term (E.15) as

$$\frac{\mathcal{P}(\sigma,\nu)}{\mathcal{Q}(\sigma,\nu)}\;\frac{\Gamma(\sigma+\nu b_1)\cdots\Gamma(\sigma+\nu b_J)}{\Gamma(\sigma+\nu d_1)\cdots\Gamma(\sigma+\nu d_K)}\;\frac{1}{\Gamma(n_1\sigma+\nu m_1)\,\Gamma(n_2\sigma+\nu m_2)}\,,$$

where $\mathcal{P}$ and $\mathcal{Q}$ are polynomials in σ and ν (which clearly depend also on all the integer parameters). After bringing the summands of the linear combination on a common denominator, the numerator must vanish identically. Thus it suffices to consider a sum of expressions of the form

$$\mathcal{P}(\sigma,\nu)\;\Gamma(n_1\sigma+\nu m_1)\,\Gamma(n_2\sigma+\nu m_2)\prod_{l=1}^{L-2}\Gamma(\sigma+\nu b_l) \tag{E.22}$$

with new polynomials $\mathcal{P}$ and parameters L, b_l. According to the left of (E.20), the parameter L is the same for all summands. We denote the number of such summands by P. For notational convenience we also write (E.22) in the more general form

$$\mathcal{P}(\sigma,\nu)\;\prod_{l=1}^{L}\Gamma(n_l\sigma+\nu a_l) \tag{E.23}$$

with new parameters n_l and a_l. We evaluate (E.23) at successive points $\sigma+p$ with $p = 0,\dots,P-1$. Again using (E.18), we obtain the expression

$$\mathcal{P}(\sigma+p,\nu)\prod_{l=1}^{L}\left(\prod_{q=0}^{n_l p-1}(n_l\sigma+q+\nu a_l)\right)\Gamma(n_l\sigma+\nu a_l)\,.$$

We now consider the asymptotic regime

$$\sigma \;\gg\; \nu \;\gg\; P\,.$$

Then (E.23) simplifies to

$$\mathcal{P}(\sigma,\nu)\prod_{l=1}^{L}(n_l\sigma+\nu a_l)^{n_l p}\,\Gamma(n_l\sigma+\nu a_l)\,\big(1+\mathcal{O}(p\,\sigma^{-1})+\mathcal{O}(p\,\nu^{-1})\big)\,.$$

It is convenient to divide by $\prod_{l=1}^{L}(n_l\sigma)^{n_l p}$ (this is possible in view of the fact that the parameters n_i and L in (E.22) are known). This gives

$$\mathcal{P}(\sigma,\nu)\prod_{l=1}^{L}\left[1+\frac{\nu a_l}{\sigma n_l}\right]^{n_l p}\,\Gamma(n_l\sigma+\nu a_l)\,\big(1+\mathcal{O}(p\,\sigma^{-1})+\mathcal{O}(p\,\nu^{-1})\big)\,.$$

Since the parameters n_i and m_i in (E.22) are also known, we can finally divide by the factors for $l=1,2$ to obtain the expressions

$$F_p(\sigma,\nu)\;:=\;\mathcal{P}(\sigma,\nu)\prod_{l=1}^{L-2}\left[1+b_l\,\frac{\nu}{\sigma}\right]^{p}\,\Gamma(n_l\sigma+\nu b_l)\,.$$

These functions satisfy the simple relations

$$F_p(\sigma,\nu)\;=\;F_0(\sigma,\nu)\,G\Big(\frac{\nu}{\sigma}\Big)^{p} \tag{E.24}$$

with

$$G(\lambda)\;:=\;\prod_{l=1}^{L-2}\left[1+\lambda\,b_l\right]\,. \tag{E.25}$$

Let us verify that the function $G(\lambda)$ determines all the parameters b_l in (E.22): Assume that the functions G and $\tilde{G}$ corresponding to two choices of the parameters b_l are equal. Collecting and counting common factors, their quotient $G(\lambda)/\tilde{G}(\lambda)$ can be written as

$$\frac{G}{\tilde{G}}\;=\;\prod_{i=1}^{I}[1+\lambda\,b_i]^{q_i}$$

for some parameters $q_i\in\mathbb{Z}$ satisfying the conditions

$$\sum_{i=1}^{I}q_i\;=\;0\,. \tag{E.26}$$

Here the b_i are (with a slight abuse of notation) a selection of the parameters b_l and $\tilde{b}_l$, which are all different from each other, and I denotes the number of such parameters. We must show that the powers q_i are all zero. To this end we take the logarithm,

$$\log G-\log\tilde{G}\;=\;\sum_{i=1}^{I}q_i\log\left(1+\lambda\,b_i\right).$$

Expanding in powers of λ up to the order I, we obtain the conditions

$$\sum_{i=1}^{I} q_i\, b_i^l \;=\; 0 \qquad\quad \text{for all } l = 1, \ldots, I.$$

We write these equations in the matrix form

$$A\Psi = 0 \tag{E.27}$$

with

$$A \;=\; \begin{pmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_I \\ b_1^2 & b_2^2 & \cdots & b_I^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{I-1} & b_2^{I-1} & \cdots & b_I^{I-1} \end{pmatrix}, \qquad \Psi \;=\; \begin{pmatrix} q_1 b_1 \\ q_2 b_2 \\ \vdots \\ q_I b_I \end{pmatrix}.$$

An elementary consideration shows that

$$\det A \;=\; \prod_{1 \le i < j \le I} (b_j - b_i) \;\neq\; 0 \tag{E.28}$$

because the b_j are all different. We conclude that the matrix A is invertible and thus $\Psi = 0$. Hence all the powers q_i vanish whenever $b_i \neq 0$. In the remaining case $b_i = 0$ the corresponding power q_i is zero because of (E.26).

We now return to (E.24). By assumption the sum of the functions $F_p(\sigma, \nu)$ vanishes,

$$0 \;=\; \sum_{\alpha=1}^{P} F_p^{(\alpha)}(\sigma, \nu) \qquad\quad \text{for } p = 0, \ldots, P-1,$$

where the index (α) labels the summands of the linear combination. Using (E.24) and keeping σ fixed, we can again write these equations in matrix form (E.27) with

$$A(\nu) \;=\; \begin{pmatrix} 1 & 1 & \cdots & 1 \\ G_{(1)} & G_{(2)} & \cdots & G_{(P)} \\ G_{(1)}^2 & G_{(2)}^2 & \cdots & G_{(P)}^2 \\ \vdots & \vdots & \ddots & \vdots \\ G_{(1)}^{P-1} & G_{(2)}^{P-1} & \cdots & G_{(P)}^{P-1} \end{pmatrix}, \qquad \Psi(\nu) \;=\; \begin{pmatrix} F_0^{(1)} \\ F_0^{(2)} \\ \vdots \\ F_0^{(P)} \end{pmatrix}.$$

Suppose that $\Psi(\nu_0) \neq 0$. Then there is $\varepsilon > 0$ such that $\Psi(\nu) \neq 0$ for all $\nu \in B_\varepsilon(\nu_0)$. Computing the determinant of A again using the formula (E.28) we conclude that for each $\nu \in B_\varepsilon(\nu_0)$, at least two of the functions $G_{(\alpha)}$ coincide. Since there is only a finite number of combinations to choose the indices, there must be two indices $(\alpha) \neq (\beta)$ such that the function $G^{(\alpha)} - G^{(\beta)}$ has an infinite number of zeros on $B_\varepsilon(\nu_0) B_\varepsilon(\nu_0)$. Due to analyticity, it follows that $G^{(\alpha)} \equiv G^{(\beta)}$, in contradiction to our above result that the functions $G^{(\alpha)}$ are all different. We conclude that $\Psi(\nu_0) = 0$ and thus

$$F_0^{(\alpha)}(\sigma, \nu) \;=\; 0 \qquad\quad \text{for all } \alpha = 1, \ldots P\,.$$

This means that the terms (E.22) all vanish identically. ∎

Proof of Theorem E.1. Consider a linear combination $\mathfrak{L}$ of simple fractions which satisfies the assumptions of the theorem. We regularize according to (E.4, E.5, E.10, E.11), expand in powers of ε and evaluate weakly on the light cone (E.13). This gives a series of terms of the general form (E.14) (note that due to (E.2) no infrared

regularization is necessary in the denominator, and so the counter terms appear only in the numerator). The scaling in E_P distinguishes between the contributions with different values of λ. For every basic fractions (E.1) we introduce the parameter

$$N = \alpha - \gamma + \beta - \delta \tag{E.29}$$

and let $N_{\max}$ be the maximum which this parameter attains for the basic fractions in $\mathfrak{L}$. We restrict attention to those contributions (E.14) where λ can be written as

$$\lambda \;=\; N\sigma - l + m\nu \tag{E.30}$$

with integer parameters l, m. These contributions all come from those simple fractions for which the parameter (E.29) is equal to N. Furthermore, since the counter terms in (E.10, E.11) involve no factor $z^{-\sigma}$ (see (E.9)), they contribute only for $\lambda = n\sigma + \cdots$ with $n < N_{\max}$ and thus do not show up in our analysis. By considering the contributions (E.14) with λ of the form (E.30), we will show that all simple fractions with $N = N_{\max}$ vanish after suitably applying the integration-by-parts rules. Then the corresponding counter terms also drop out, because the infrared regularization is compatible with the integration-by-parts rules. Hence these simple fractions completely drop out of $\mathfrak{L}$, and we can proceed inductively to the analysis of the simple fractions with $N < N_{\max}$. This argument allows us to completely ignore the infrared regularization in what follows.

Since the summands with different value of λ scale differently in E_P, we can assume that the parameter λ is the same for all simple fractions in $\mathfrak{L}$. Dividing (E.14) through $E_P^{\lambda-1}\,\Gamma(\lambda-1)$ and using (E.3), we obtain terms which are precisely of the form as considered in Lemma E.2. Therefore, our representation of $\mathfrak{L}$ as a linear combination of quotients of gamma functions (E.14) is unique up to applying (E.18). We will consider this arbitrariness later and for the moment consider a fixed choice of summands of the form (E.14).

Our goal is to get a one-to-one connection between quotients of gamma functions of the form (E.14) and our original simple fractions. Unfortunately, to zeroth order in ε one cannot reconstruct the simple fraction from the expression (E.14), because (E.14) is symmetric in the parameters a_j and b_k, and thus it is impossible to tell which of these parameters came from a factor $T_\circ^{(.)}$ or $\overline{T_\circ^{(.)}}$. This is the reason why we need to consider the higher orders in ε as well. Out of the many terms of the general form (E.13) we select a few terms according to the following rules, which we apply one after the other:

(i) No factors $\Gamma(\sigma+\nu)$ or $\Gamma(\sigma+1+\nu)$ appear.
(ii) For the factor $\Gamma((\alpha-\gamma)\sigma - n + m\nu)$ the parameter m is maximal.
(iii) The number of factors $\Gamma(\sigma - n + (2m+1)\nu)$ in the numerator is minimal.

By "maximal" (and similarly "minimal") we mean that there is no summand for which the corresponding parameter is larger (no matter how all other parameters look like). Note that the factor $\Gamma((\alpha-\gamma)\sigma - n + m\nu)$ in (E.14) is uniquely determined because of (E.3).

Assuming that $\mathfrak{L}$ is non-trivial, the above procedure gives us at least one term of the form (E.13) (note that the zero-order term in ε clearly satisfies (i)). The point is that we can uniquely construct from this term a corresponding simple fraction from $\mathfrak{L}$, as the following consideration shows. According to (i), the factors $T_{[0]}^{(-1)}$ and $T_{[0]}^{(0)}$ are taken into account only to lowest order in ε, because otherwise a factor of the form $\Gamma(\sigma - n + \nu)$, $n = -1, 0$, would appear. In particular, one sees from (E.2) that we

do not get ε-terms of the denominator. Hence the higher orders in ε are obtained simply by expanding the numerator in (E.1) in powers of ε. With the rule (ii) we arranged that *all* factors $T_\circ^{(n)}$ with $n > 0$ or $\circ \neq [0]$ are taken into account linearly in ε. On the other hand, (iii) ensures that *none* of the factors $\overline{T_\circ^{(n)}}$ is taken into account linearly in ε. Therefore, all gamma functions in the numerator whose argument contains an odd number times ν belong to a factor $T_\circ^{(n)}$. Conversely, the gamma functions of the form $\Gamma(\sigma - n + 2m\nu)$ belong to a factor $\overline{T_\circ^{(n)}}$, at least when $n > 0$ or $\circ \neq [0]$. In this way, the gamma functions determine the simple fraction up to factors of $T_{[0]}^{(-1)}$, $T_{[0]}^{(0)}$ and $\overline{T_{[0]}^{(-1)}}$, $\overline{T_{[0]}^{(0)}}$. But the factors $T_{[0]}^{(-1)}$ and $T_{[0]}^{(0)}$ can easily be determined from the argument of the factor $\Gamma((\alpha - \gamma)\sigma - n + m\nu)$, because $\alpha - \gamma$ gives us how many factors $T_{[0]}^{(.)}$ we muse use, whereas n tells us about how many factors $T_{[0]}^{(-1)}$ we must use. Since λ is known, we also know the arguments of the factor $\Gamma((\beta - \delta)\sigma - B + D + \bullet\nu)$ in (E.13), and this determines in turn the factors $\overline{T_{[0]}^{(-1)}}$ and $\overline{T_{[0]}^{(0)}}$.

We conclude that the above construction allows us to determine one summand of $\mathfrak{L}$. Subtracting this summand, we can proceed iteratively to determine all other summands of $\mathfrak{L}$. This construction is unique up to the transformation of the gamma functions with (E.18).

We conclude the proof by establishing a one-to-one correspondence between the transformation (E.18) of the gamma functions and the integration-by-parts rule for the simple fraction. To every simple fraction (E.1) we can associate a contribution of the form (E.14) which satisfies the rules (i)–(iii) with the following symbolic replacements,

$$\begin{aligned}
T_{[0]}^{(-1)}, \overline{T_{[0]}^{(-1)}} &\longrightarrow \Gamma(\sigma + 1)\,, & T_{[0]}^{(0)}, \overline{T_{[0]}^{(0)}} &\longrightarrow \Gamma(\sigma) \\
T_{[p]}^{(n)} &\longrightarrow \Gamma(\sigma - n + (4p+1)\nu) \\
\overline{T_{[p]}^{(n)}} &\longrightarrow \Gamma(\sigma - n + 4p\nu) \qquad \text{(if } n > 0 \text{ or } p > 0\text{)} \\
T_{\{p\}}^{(n)} &\longrightarrow \Gamma(\sigma - n + (4p+3)\nu)\,, & \overline{T_{\{p\}}^{(n)}} &\longrightarrow \Gamma(\sigma - n + (4p+2)\nu)\,.
\end{aligned}$$

These replacement rules determine the first line in (E.14), whereas the arguments of the gamma functions in the second line are obtained as explained above by adding the arguments of the gamma functions in the first line. If (E.18) is applied to the gamma functions in the first line of (E.14),

$$\Gamma(\sigma - n + \circ\nu) \;\longrightarrow\; (\sigma - (n+1) + \circ\nu)\,\Gamma(\sigma - (n+1) + \circ\nu)\,,$$

we take this into account with the following symbolic transformation inside the simple fraction,

$$T_\circ^{(n)} \;\longrightarrow\; \nabla T_\circ^{(n+1)}\,.$$

Here ∇ is the derivation as introduced in (4.5.34). Using the Leibniz rule this correspondence can be extended to composite expressions; for example,

$$\begin{aligned}
&\nabla\left(\frac{1}{T_\circ^{(n)}}\right) \;=\; -\frac{\nabla T_\circ^{(n)}}{(T_\circ^{(n)})^2} \\
&\quad\longleftrightarrow\quad -\frac{(\sigma - n + \circ\sigma)\,\Gamma(\sigma - n + \circ\sigma)}{\Gamma(\sigma - n + \circ\sigma)^2} \;=\; -\frac{(\sigma - n + \circ\sigma)}{\Gamma(\sigma - n + \circ\sigma)}
\end{aligned}$$

$$\begin{aligned}\nabla(T_{\circ_1}^{(n_1)}T_{\circ_2}^{(n_2)}) \;&=\; \nabla(T_{\circ_1}^{(n_1)})\,T_{\circ_2}^{(n_2)} + T_{\circ_1}^{(n_1)}\,\nabla(T_{\circ_2}^{(n_2)})\\ \longleftrightarrow\quad &(2\sigma-(n_1+n_2-2)+(\circ_1+\circ_2)\nu)\,\Gamma(\sigma-n_1+\circ_1\nu)\,\Gamma(\sigma-n_1+\circ_1\nu)\end{aligned}$$

and similarly for other composite expressions. It remains to consider the transformations of the gamma functions in the second line of (E.14). Since λ is fixed, we can only increment the argument of one gamma function in the denominator and at the same time decrement the argument of the other, for example

$$\begin{aligned}&\frac{(\alpha-\gamma)\sigma-(n+1)+\bullet_1\nu}{\Gamma((\alpha-\gamma)\sigma-n+\bullet_1\nu)\,\Gamma((\beta-\delta)\sigma-m+\bullet_2\nu)}\\ \longrightarrow\quad &\frac{(\beta-\delta)\sigma-m+\bullet_2\nu)}{\Gamma((\alpha-\gamma)\sigma-(n+1)+\bullet_1\nu)\,\Gamma((\beta-\delta)\sigma-(m-1)+\bullet_2\nu)}\,.\end{aligned}$$

This transformation can be related to the integration by parts rule

$$\int_{-\infty}^{\infty}\nabla(\cdots)\,\overline{(\cdots)}\,ds \;\longrightarrow\; -\int_{-\infty}^{\infty}(\cdots)\,\nabla\overline{(\cdots)}\,ds\,,$$

where $(\cdots)$ and $\overline{(\cdots)}$ stand for simple fractions composed of $T_\circ^{(n)}$ and $\overline{T_\circ^{(n)}}$, respectively. As is easily verified, these replacements rules are all compatible with each other and with the Leibniz rule. They allow us to identify the transformation (E.18) with the integration-by-parts rules. ∎

APPENDIX F

The Commutator $[P, Q]$

The Euler-Lagrange equations corresponding to our variational principles involve the commutator $[P, Q]$ (5.2.9), where Q is a composite expression in the fermionic projector. In Chapter 4 we developed a method with which composite expressions in the fermionic projector can be evaluated weakly on the light cone. In this appendix we shall describe in detail how these methods can be used to evaluate the commutator $[P, Q]$ in the continuum limit. We begin by collecting a few formulas from §4.5 and bring them into a form convenient for what follows. The kernel $Q(x, y)$ can be written as a linear combination of terms of the form (4.5.22)

$$f(x,y)\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}}\,, \tag{F.1}$$

where f is a smooth function composed of the bosonic fields and fermionic wave functions. Here the factors $T_\circ^{(a_j)}$ and $\overline{T_\circ^{(b_k)}}$ are the regularized distributions of the light-cone expansion. The quotient of monomials is called a simple fraction, and its degree L is defined by (4.5.27). If $L > 1$, the monomial becomes singular on the light cone when the regularization is removed by letting $E_P \to \infty$. In light-cone coordinates (s, l, x_2, x_3), this singular behavior on the light cone is quantified by a weak integral over s for fixed x_2, x_3, and $l \gg E_P^{-1}$. More precisely (4.5.29),

$$\begin{aligned}\int_{-\infty}^{\infty} ds\, (\eta f)(s)\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \;&=\; \frac{c_{\text{reg}}}{(il)^L}\, (\eta f)(s=0)\, \log^g(E_P)\, E_P^{L-1} \\ &\quad + (\text{higher orders in } (lE_P)^{-1} \text{ and } (l_{\text{macro}} E_P)^{-1})\,,\end{aligned} \tag{F.2}$$

where g is an integer, c_{reg} is the so-called regularization parameter, and η is a test function, which must be macroscopic in the sense that its derivatives scale in powers of l^{-1} or l_{macro}^{-1}. The asymptotic formula (F.2) applies on the upper light cone $s = 0$, but by taking the adjoint and using that Q is Hermitian, $Q(x, y)^* = Q(y, x)$, it is immediately extended to the lower light cone. Furthermore, we can integrate (F.2) over l, x_2, and x_3, provided that $l \gg E_P^{-1}$. In polar coordinates $(y - x) = (t, r, \Omega = (\vartheta, \varphi))$, we thus have

$$\begin{aligned}&\int_{-\infty}^{\infty} dt \int_{r_0}^{\infty} r^2\, dr \int_{S^2} d\Omega\, \eta(t, r, \Omega)\, f(x,y)\, \frac{T_\circ^{(a_1)} \cdots T_\circ^{(a_\alpha)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_\beta)}}}{T_\circ^{(c_1)} \cdots T_\circ^{(c_\gamma)}\, \overline{T_\circ^{(d_1)} \cdots T_\circ^{(d_\delta)}}} \\ &= \log^g(E_P)\, E_P^{L-1} \int_{\mathbb{R}\setminus[-r_0, r_0]} t^2\, dt \int_{S^2} d\Omega\, (\eta f)_{|r=|t|}\, \frac{c_{\text{reg}}(\Omega)}{(it)^L} \\ &\quad + (\text{higher orders in } (rE_P)^{-1} \text{ and } (l_{\text{macro}} E_P)^{-1})\,,\end{aligned}$$

valid for every $r_0 \gg E_P^{-1}$. Next we expand the function f in a Taylor series around $(y-x) \equiv \xi = 0$,

$$f(x,y) \;=\; \sum_J f_J(x)\, \xi^J \tag{F.3}$$

with J a multi-index, and write the Taylor coefficients together with the regularization parameter. Collecting all contributions, we obtain for Q the weak evaluation formula

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \int_{r_0}^{\infty} r^2\, dr \int_{S^2} d\Omega\, \eta(t,r,\Omega)\, Q(x,y) \\ &\quad= \sum_{L=2}^{L_{\max}} \sum_{g=0}^{g_{\max}} \log^g(E_P)\, E_P^{L-1} \sum_J \int_{\mathbb{R}\setminus[-r_0,r_0]} dt\, t^{2-L} \int_{S^2} d\Omega\, \eta\, h_J(\Omega)\, \xi^J \\ &\qquad + (\text{higher orders in } ((r+|t|)\, E_P)^{-1} \text{ and } (l_{\text{macro}} E_P)^{-1}) \;+\; o(E_P) \end{aligned} \tag{F.4}$$

with suitable functions $h_J(\Omega)$, which depend on L and g. The integrand on the right side of (F.4) is evaluated on the light cone $r = |t|$. The maximal degree of the monomials $L_{\max}$ as well as $g_{\max}$ are clearly finite parameters. Notice that the monomials of degree $L < 2$ are omitted in (F.4); this is justified as follows. For $L < 2$, the integral (F.2) diverges at most logarithmically as $E_P \to \infty$, and furthermore has a pole in l of order at most one. Thus the corresponding contribution to (F.4) is at most logarithmically divergent as $E_P \to \infty$, with bounds uniform in r_0. This is what we mean by $o(E_P)$.

We point out that the asymptotic expansion near the light cone (F.4) does not give any information on the behavior of $Q(x,y)$ near the origin, i.e. when x and y are so close that $r, |t| \sim E_P^{-1}$. Namely, due to the restriction $r_0 \gg E_P^{-1}$, the region near the origin is excluded from the integration domain. Also, near the origin the terms of higher order in $((r+|t|)E_P)^{-1}$, which are left out in (F.4), cannot be neglected. As explained in detail in Appendix D, the reason for this limitation is that near the origin, Q depends essentially on the detailed form of the fermionic projector on the Planck scale and thus remains undetermined within the method of variable regularization.

Our aim is to evaluate the commutator $[P,Q]$ using the expansion (F.4). The main difficulty is that products of the operators P and Q, like for example

$$(Q\, P)(x,y) \;=\; \int d^4z\, Q(x,z)\, P(z,y)\,, \tag{F.5}$$

involve $Q(x,z)$ near the origin $x = z$, where (F.4) does not apply. In order to explain our strategy for dealing with this so-called *problem at the origin*, we briefly discuss a simple one-dimensional example. Assume that we are given a function $f(x)$, $x \in \mathbb{R}$, and a positive integer n such that for all $x_0 \gg E_P^{-1}$ and test functions η,

$$\begin{aligned} &\int_{\mathbb{R}\setminus[-x_0,x_0]} f(x)\, \eta(x)\, dx \\ &\quad= \int_{\mathbb{R}\setminus[-x_0,x_0]} \frac{\eta(x)}{x^n}\, dx \;+\; (\text{higher orders in } (x E_P)^{-1}). \end{aligned} \tag{F.6}$$

In analogy to (F.4), this formula does not give any information on the behavior of $f(x)$ near the origin $x = 0$. Thus there are many different functions satisfying (F.6), a typical example is

$$f(x) \;=\; \frac{1}{(x - i E_P^{-1})^n}\,. \tag{F.7}$$

The question is if (F.6) is useful for analyzing the weak integral

$$\int_{-\infty}^{\infty} f(x)\, \eta(x)\, dx\,. \tag{F.8}$$

The answer to this question depends very much on the properties of η. If η is an arbitrary test function with compact support, we can restrict attention to test functions with support away from the origin, $\text{supp}\, \eta \subset \mathbb{R} \setminus [-x_0, x_0]$. Then (F.6) applies, and we find that $f(x) \sim x^{-n}$. Thus by evaluating (F.8) for suitable test functions, we can find out that, as long as $|x| \gg E_P^{-1}$, $f(x)$ behaves like the function x^{-n}, which has a pole of order n at the origin. We refer to this statement as we can *detect the pole of f by testing with η*. Unfortunately, the situation becomes more difficult if we assume that η belongs to a more restricted class of functions. Assume for example that $\eta(x)$ is rational, goes to zero at infinity and has all its poles in the upper half plane $\{\text{Im}\; x > 0\}$. Then for f as in (F.7), the integral (F.8) can be closed to a contour integral in the lower complex plane, and we get zero, independent of η. This shows that when testing only with rational functions with poles in the upper half plane, the formula (F.6) is of no use, and we cannot detect the pole of f. Indeed, the problem in (F.5) can be understood in a similar way. If we apply the operator product QP to a test function η and write the result as $(QP)\eta = Q(P\eta)$, the problem of making sense out of the integral (F.5) can be restated by saying that Q may be tested only with the functions $P\eta$. In other words, the test functions must lie in the image of P, i.e. they must be negative-energy solutions of the Dirac equation. Thus the question is if by evaluating only with such special functions, can we nevertheless detect the poles of Q, and if yes, how can this be done? Once these questions are settled, we can compute the operator products PQ, QP and take their difference.

For clarity we begin the analysis with the simplified situation where both P and Q are homogeneous, i.e.

$$P(x,y) \;=\; P(y-x)\,, \qquad\qquad Q(x,y) \;=\; Q(y-x)\,. \tag{F.9}$$

Under this assumption, the operators P and Q are diagonal in momentum space,

$$P(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{P}(k)\, e^{-ik(x-y)}\,, \qquad Q(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{Q}(k)\, e^{-ik(x-y)}\,,$$

and their products can be taken "pointwise" in k, i.e. in the example (F.5),

$$(QP)(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \hat{Q}(k)\, \hat{P}(k)\, e^{-ik(x-y)}\,. \tag{F.10}$$

Due to this simplification, it is preferable to work in momentum space. Now the problem at the origin becomes apparent in the Fourier integral

$$\hat{Q}(k) \;=\; \int d^4\xi\; Q(\xi)\, e^{-ik\xi} \qquad\qquad (\xi \equiv y - x), \tag{F.11}$$

where we must integrate over a neighborhood of $\xi = 0$. In order to handle this problem, we must carefully keep track of how the unknown behavior of Q near the origin effects the Fourier integral: If we consider the integral in position space

$$\int d^4\xi\; \eta(\xi)\, Q(\xi) \tag{F.12}$$

with a smooth, macroscopic function η, we can make the unknown contribution near the origin to the integral small by assuming that η goes to zero sufficiently fast near

$\xi = 0$. Thus we expect that (F.12) is well-defined provided that the partial derivatives of $\eta(\xi)$ at $\xi = 0$ vanish up to the order n,

$$\partial^I \eta(0) \;=\; 0 \qquad \text{for all } I \text{ with } |I| \leq n, \tag{F.13}$$

where n is a sufficiently large parameter (which we shall specify below). In momentum space, the conditions (F.13) take the form

$$\int d^4k \; k^I \, \hat{\eta}(k) \;=\; 0\,, \qquad\qquad |I| \leq n.$$

This means that the Fourier transform of Q is well-defined by (F.11), as long as it is evaluated weakly only with test functions $\hat{\eta}(k)$ which are orthogonal to the polynomials k^I. Equivalently, we can say that $\hat{Q}(k)$ is defined only up to the polynomials k^I,

$$\hat{Q}(k) \;=\; \int d^4\xi \, Q(\xi)\, e^{-ik\xi} \bmod \; \mathcal{P}^n(k)\,, \tag{F.14}$$

where $\mathcal{P}^n(k)$ denotes the polynomials in k of degree at most n. Let us compute the Fourier integral (F.14) using the expansion (F.4), and at the same time determine the parameter n. Our method is to consider (F.4) for $\eta = \exp(-ik\xi)$ and to choose n so large that we can take the limit $r_0 \searrow 0$ to obtain the Fourier integral (F.14). For each summand in (F.4), the resulting t-integral is of the form

$$\lim_{r_0 \searrow 0} \int_{\mathbb{R} \setminus [-r_0, r_0]} dt \, t^{2-L+|J|} \, e^{-i\omega t} \bmod \; \mathcal{P}^n(\omega)\,.$$

Here in the integrand one may distinguish between the two regions $|t| > \omega^{-1}$, where the factor $t^{2-L+|J|}$ is regular and $e^{-i\omega t}$ is oscillating, and $|t| < \omega^{-1}$, where the pole of $t^{2-L+|J|}$ must be taken into account and the exponential is well-approximated by a Taylor polynomial. Since we are interested in the scaling behavior of the integral over the pole, it suffices to consider the region $|t| < \omega^{-1}$, and calculating modulo $\mathcal{P}^n(\omega)$, the leading contribution to the integral is

$$\lim_{r_0 \searrow 0} \int_{[-\omega^{-1}, \omega^{-1}] \setminus [-r_0, r_0]} dt \, t^{2-L+|J|} \, \frac{(-i\omega t)^{n+1}}{(n+1)!}\,. \tag{F.15}$$

If $n \geq L - |J| - 3$, the integrand is bounded near $t = 0$. In the case $n = L - |J| - 4$, the limit in (F.15) may be defined as a principal value, whereas for $n < L - |J| - 4$, (F.15) is ill-defined. Thus we need to assume that $n \geq L - |J| - 4$. Moreover, we must ensure that the terms of higher order in $((r + |t|)E_P)^{-1}$, which are omitted in (F.4), are negligible in the Fourier integral. Since these terms are regularized on the Planck scale, the scaling behavior of these higher order terms is in analogy to (F.15) given by the integrals

$$\int_{[-\omega^{-1}, \omega^{-1}] \setminus [-E_P^{-1}, E_P^{-1}]} dt \, \frac{t^{3-L+|J|+n}}{(tE_P)^n}\,, \qquad\qquad n \geq 1.$$

A simple calculation shows that these integrals are negligible compared to (F.15) if and only if $n \geq L - |J| - 3$, and under this assumption, they are of higher order in

ω/E_P. We conclude that the Fourier transform of Q has the expansion

$$\begin{aligned}\hat{Q}(k) \;&=\; \sum_{L=2}^{L_{\max}} \sum_{g=0}^{g_{\max}} \log^g(E_P)\, E_P^{L-1} \sum_J \\ &\quad\times \int_{-\infty}^{\infty} dt\, t^{-L+|J|+2} \int_{S^2} d\Omega\, h_J(\check{\xi})\, \check{\xi}^J\, e^{-ik\check{\xi}t} \bmod\; \mathcal{P}^{L-|J|-3}(k) \\ &\quad + (\text{higher orders in } k/E_P \text{ and } (l_{\text{macro}} E_P)^{-1}) \;+\; o(E_P)\,, \end{aligned} \tag{F.16}$$

where $\check{\xi}$ is the "unit null vector" $\check{\xi} = (1,\, \Omega \in S^2 \subset \mathbb{R}^3)$ and $h_J(\check{\xi} = (1,\Omega)) \equiv h_J(\Omega)$. Carrying out the t-integration gives the following result.

LEMMA F.1. *Suppose that the operator Q is homogeneous. Then its Fourier transform $\hat{Q}$ is of the form*

$$\begin{aligned}\hat{Q}(k) \;&=\; \sum_{L=2}^{L_{max}} \sum_{g=0}^{g_{max}} \log^g(E_P)\, E_P^{L-1} \sum_J \hat{Q}_J^{Lg}(k) \;mod\; \mathcal{P}^{L-|J|-3}(k) \\ &\quad + \textit{(higher orders in } k/E_P \textit{ and } (l_{macro} E_P)^{-1}\textit{)} \;+\; o(E_P) \end{aligned} \tag{F.17}$$

with

$$\begin{aligned}\hat{Q}_J^{Lg}(k) \;&=\; -2\pi i\, (-i\omega)^{L-|J|-3} \\ &\quad\times \int_{S^2} d\Omega\, h_J(\check{\xi})\, \check{\xi}^J \;\times \begin{cases} \dfrac{(\check{k}\check{\xi})^{L-|J|-3}}{(L-|J|-3)!}\, \Theta(\omega \check{k}\check{\xi}) & \textit{if } |J| < L-2 \\ \epsilon(\omega)^{L-|J|-3}\, \delta^{(2+|J|-L)}(\check{k}\check{\xi}) & \textit{if } |J| \geq L-2\,, \end{cases} \end{aligned} \tag{F.18}$$

where $\omega = k^0$ is the energy and $\check{k} \equiv k/\omega$ (ϵ is again the step function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise).

Proof. The t-integral in (F.16) is of the form

$$\int_{-\infty}^{\infty} t^{-n}\, e^{-i\lambda t}\, dt \bmod\; \mathcal{P}^{n-1}(k)$$

with $n = L - |J| - 2$ and $\lambda = k\check{\xi}$. For $n = 0$, we have

$$\int_{-\infty}^{\infty} e^{-i\lambda t}\, dt \;=\; 2\pi\, \delta(\lambda)\,. \tag{F.19}$$

The case $n < 0$ follows by differentiating this equation $(-n)$ times with respect to λ. In order to treat the case $n > 0$, we integrate (F.19) n times in the variable λ. The integration constant is a polynomial in λ of degree $n-1$ and can thus be omitted. ∎

Let us briefly discuss the above expansion. The parameters L and g give the scaling behavior in the Planck energy. The multi-index J enters at two different points: it determines via the factor $\omega^{-|J|}$ the dependence on the energy, and it also influences the S^2-integral. This integral gives detailed information on the behavior of $\hat{Q}(k)$ in $\check{k}$, but it is independent of $|\omega|$. Integrating over S^2 takes into account the angular dependence of the regularization functions and of the macroscopic physical objects and tells about how the different angles contribute to $\hat{Q}$. In the case $|J| \geq L-2$, the integrand has a δ-like singularity localized at $\check{k}\check{\xi} = 0$, and so the S^2-integral reduces to integrating over the intersection of the hyperplane $\{\xi \mid k\xi = 0\}$ with the two-sphere $t = 1 = r$.

This intersection is empty for time-like k, and is a one-sphere for space-like k. As a consequence, $\hat{Q}_J^{Lg}(k)$ is zero inside the mass cone $\mathcal{C}$ and in general non-zero outside, without being regular on its boundary $\{k^2 = 0\}$. If on the other hand $|J| < L - 2$, the factor $\Theta(\omega \check{k} \check{\xi})$ is essential, because without this factor, we would simply have a polynomial in k of degree $L - |J| - 3$, being zero modulo $\mathcal{P}^{L-|J|-3}(k)$. Note that for any $\check{\xi}$, the factor $\Theta(\omega \check{k} \check{\xi})$ vanishes inside the lower mass cone $\{\check{k}^2 > 0, \omega < 0\}$, whereas it is in general non-zero otherwise. This means that $\hat{Q}_J^{Lg}(k)$ again vanishes in the interior of the lower mass cone and is not regular on the mass cone $\{k^2 = 0\}$. The singular behavior on the lower mass cone is made more explicit in the following lemma.

LEMMA F.2. *The operators $\hat{Q}_J^{Lg}(k)$, (F.18), vanish inside the lower mass cone $\{\check{k}^2 > 0,\ \omega < 0\}$. Near the lower mass cone, they have the asymptotic form*

$$\begin{aligned} \hat{Q}_J^{Lg}(k) &= 2\pi^2\, i \left(-\frac{i\omega}{2}\right)^{L-|J|-3} h_J(\check{k})\, \check{k}^J\, (1 + O(\check{k}^2)) \\ &\quad \times \begin{cases} \dfrac{\check{k}^{2(L-|J|-2)}}{(L-|J|-2)!}\, \Theta(-\check{k}^2) & \text{if } |J| \le L-2 \\ (-1)^{1+|J|-L}\, \delta^{(1+|J|-L)}(\check{k}^2) & \text{if } |J| > L-2\,. \end{cases} \end{aligned} \tag{F.20}$$

Proof. Without loss of generality, we can assume that $\check{k}$ points in the tx-direction of our Cartesian coordinate system, i.e. $\check{k} = (1, \lambda, 0, 0)$ with $\lambda \ge 0$. Then $\check{k}\check{\xi} = 1 - \lambda \cos\vartheta$, and we can write (F.18) in the region $\omega < 0$ as

$$\begin{aligned} \hat{Q}_J^{Lg}(k) &= -2\pi i\, (-i\omega)^{L-|J|-3} \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi\; h_J(\check{\xi})\, \check{\xi}^J \\ &\quad \times \begin{cases} \dfrac{(1-\lambda\cos\vartheta)^{L-|J|-3}}{(L-|J|-3)!}\, \Theta(\lambda\cos\vartheta - 1) & \text{if } |J| < L-2 \\ (-1)^{L-|J|-3}\, \delta^{(2+|J|-L)}(1-\lambda\cos\vartheta) & \text{if } |J| \ge L-2\,. \end{cases} \end{aligned}$$

Inside the lower mass cone, the parameter $\lambda < 1$, and the integrand is identically equal to zero. Outside and near the lower mass cone, $1 \le \lambda \approx 1$, and the integrand vanishes unless $\cos\vartheta \approx 1$. Hence to leading order in $\check{k}^2$, we may replace $h_J(\check{\xi})\, \check{\xi}^J$ by its value at the coordinate pole $\vartheta = 0$ and carry out the φ-integration,

$$\begin{aligned} \hat{Q}_J^{Lg}(k) &= -4\pi^2\, i\, (-i\omega)^{L-|J|-3}\, h_J(\check{k})\, \check{k}^J\, (1 + O(\check{k}^2)) \\ &\quad \times \int_{-1}^{1} du\; \times \begin{cases} \dfrac{(1-\lambda u)^{L-|J|-3}}{(L-|J|-3)!}\, \Theta(\lambda u - 1) & \text{if } |J| < L-2 \\ (-1)^{L-|J|-3}\, \delta^{(2+|J|-L)}(1-\lambda u) & \text{if } |J| \ge L-2\,. \end{cases} \end{aligned}$$

In the case $|J| < L - 2$, the remaining integral is of the form

$$\int_{-1}^{1} (1-\lambda u)^n\, \Theta(\lambda u - 1)\, du \;=\; -\frac{1}{\lambda(n+1)}\, (1-\lambda)^{n+1}\, \Theta(\lambda - 1)\,,$$

whereas in the case $|J| \ge L - 2$,

$$\begin{aligned} \int_{-1}^{1} \delta^{(n)}(1-\lambda u)\, du &= \left(-\frac{d}{d\lambda}\right)^n \int_{-1}^{1} u^{-n}\, \delta(1-\lambda u)\, du \\ &= \left(-\frac{d}{d\lambda}\right)^n (\lambda^{n-1}\, \Theta(\lambda - 1))\,. \end{aligned}$$

Finally, we use that $\lambda = 1 + O(\check{k}^2)$ and $\check{k}^2 = 1 - \lambda^2 = 2(1-\lambda) + O(\check{k}^4)$. ∎

According to Lemma F.1 and Lemma F.2, all the information on the behavior of Q on the light cone contained in (F.4) is encoded in momentum space in a neighborhood of the lower mass cone. More precisely, this information can be retrieved as follows. Due to the factor $\omega^{-|J|}$ in (F.18) and (F.20), the sum over the multi-index J in (F.17) is an expansion in powers of ω^{-1}. Thus by considering $\hat{Q}$ for large energies, more precisely for ω in the range

$$m^2\, l_{\text{macro}},\, l_{\text{macro}}^{-1} \;\ll\; |\omega| \;\ll\; E_P\,, \tag{F.21}$$

we can make the contributions for large $|J|$ small (the restriction $|\omega| \ll E_P$ is clearly necessary because the terms of higher order in k/E_P are omitted in (F.17)). Hence in this energy range, the series in (F.17) converges fast, and the scaling behavior in E_P and ω, as well as the dependence on $\check{k}$ given explicitly in (F.20), allow us to determine the functions $h_J(\Omega)$ completely.

Having computed the Fourier transform of Q, we can now take the product with the operator P according to (F.10). We begin with the simplest case where we take for P one massive Dirac sea in the vacuum (2.6.13) with $m > 0$. In this case, $\hat{P}(k)$ is supported inside the lower mass cone $\mathcal{C}^\wedge$. According to Lemma F.2, the operators $\hat{Q}_J^{Lg}$ vanish identically inside the lower mass cone. Hence the supports of $\hat{Q}_J^{Lg}$ and $\hat{P}$ do not intersect, and it follows immediately that

$$\hat{Q}_J^{Lg}\, \hat{P} \;=\; 0\,. \tag{F.22}$$

This means that after multiplying by $\hat{P}$, all the information contained in the expansion of Lemma F.1 is lost. We refer to this difficulty as the *problem of disjoint supports*. Using the notion introduced after (F.8), it is impossible to detect the poles of Q by testing with the negative-energy solutions of the free Dirac equation. This situation is indeed quite similar to the example (F.6, F.8) for f according to (F.7) and rational test functions with support in the upper half plane, in particular since the Fourier transform $\eta(k) = \int_{-\infty}^{\infty} \eta(x)\, \exp(-ikx)$ of such a test function is supported in the half line $k < 0$, and can thus be regarded as a one-dimensional analogue of the negative-energy solutions of the free Dirac equation.

It is an instructive cross-check to see how the problem of disjoint supports comes about if instead of analyzing the behavior of Q in position space (F.4) and then transforming to momentum space, we work exclusively in momentum space. For simplicity we give this qualitative argument, which will not enter the subsequent analysis, only in the special case of a monomial, i.e. instead of (F.1) for an expression of the form

$$f(x,y)\, T_\circ^{(a_1)} \cdots T_\circ^{(a_p)}\, \overline{T_\circ^{(b_1)} \cdots T_\circ^{(b_q)}}\,. \tag{F.23}$$

In this case, we can, instead of taking the product of the factors $T_\circ^{(a)}$ and $\overline{T_\circ^{(b)}}$ in position space, also compute their convolution in momentum space. As explained in §4.2, the singular behavior of the fermionic projector on the light cone is determined by states near the lower mass cone. More precisely, the main contribution to $P(x,y)$ comes from states close to the hypersurface $\mathcal{H} = \{k \,|\, k\xi = 0\}$, which for ξ on the light cone is tangential to the mass cone $C = \{k^2 = 0\}$, so that the singularity on the light cone can be associated to the states in a neighborhood of the straight line $\mathcal{H} \cap C$. For objects

derived from the fermionic projector like the regularized distributions $T_\circ^{(a_j)}$ and $\overline{T_\circ^{(b_k)}}$, this qualitative picture applies just as well. The Fourier transforms of the factors $T_\circ^{(a_j)}$ and $T_\circ^{(b_k)}$ are supported in the the interior of the lower mass cone. Thus when forming their convolution,

$$\hat{g}_1 := \frac{1}{(2\pi)^{4p}}\, \hat{T}_\circ^{(a_1)} * \cdots * \hat{T}_\circ^{(a_p)}\,, \qquad \hat{g}_2 := \frac{1}{(2\pi)^{4q}}\, \hat{T}_\circ^{(b_1)} * \cdots * \hat{T}_\circ^{(b_q)}\,, \tag{F.24}$$

the resulting convolution integrals are all finite integrals over a compact domain (e.g., the integrand in $(\hat{T}_\circ^{(a_1)} * \hat{T}_\circ^{(a_2)})(k) = \int d^4q\, \hat{T}_\circ^{(a_1)}(q)\, \hat{T}_\circ^{(a_2)}(k-q)$ vanishes unless q lies in the "diamond" $\{q^2 \geq 0,\, q^0 > 0\} \cap \{(q-k)^2 \geq 0,\, (q-k)^0 < 0\}$). Moreover, the supports of $\hat{g}_1$ and $\hat{g}_2$ are again inside the lower mass cone. Exactly as described for the fermionic projector in Section C.3, the behavior of $\hat{g}_1$ and $\hat{g}_2$ near the lower mass cone determines the well known singularities of g_1 and g_2 on the light cone, whereas the form of $\hat{g}_1$ and $\hat{g}_2$ in the high-energy region away from the mass cone depends essentially on the details of the regularization and is thus unknown. Using $\hat{g}_1$ and $\hat{g}_2$, we can write the Fourier transform of the monomial as

$$\hat{M}^{Lg}(k) := \frac{1}{(2\pi)^4} \int d^4\xi\, \hat{T}_\circ^{(a_1)} * \cdots * \hat{T}_\circ^{(a_p)}\, \overline{\hat{T}_\circ^{(b_1)} * \cdots * \hat{T}_\circ^{(b_q)}}\, e^{-ik\xi} \tag{F.25}$$

$$= \int d^4q\, \hat{g}_1(q)\, \hat{g}_2(q-k)\,. \tag{F.26}$$

In (F.1) the monomial is multiplied by the smooth function f. Thus the corresponding contribution to $\hat{Q}$ is obtained by taking the convolution of $\hat{f}$ with $\hat{M}^{Lg}$,

$$\hat{Q} \asymp \hat{f} * \hat{M}^{Lg}\,. \tag{F.27}$$

Since f is a macroscopic function, its Fourier transform $\hat{f}(q)$ is localized in a neighborhood of the origin, i.e. in the region $|q^0|, |\vec{q}| \sim l_{\text{macro}}^{-1}$. The Taylor expansion (F.3) corresponds to expanding $\hat{f}$ in terms of distributions supported at the origin, more precisely

$$\hat{f} = \sum_J \hat{f}_J \qquad \text{with} \qquad \hat{f}_J(k) = (2\pi)^4\, f_J\, (i\partial_k)^J \delta^4(k)\,, \tag{F.28}$$

and substituting this expansion into (F.27) yields the expansion of Lemma F.1,

$$\hat{Q}_J^{Lg} = \hat{f}_J * \hat{M}^{Lg}\,. \tag{F.29}$$

Since the distributions $\hat{f}_J(q)$ are supported at $q = 0$, the support of $\hat{Q}_J^{Lg}$ coincides with that of $\hat{M}^{Lg}$. Hence to discuss the problem of disjoint supports, we must consider $\hat{M}^{Lg}$ as given by the integral in (F.26). Note that this integral differs from a convolution in that the argument of $\hat{g}_2$ has the opposite sign; this accounts for the complex conjugation in (F.25). As a consequence, the integration domain is now not compact, and the integral is finite only due to the regularization. More precisely, the integration range is the intersection of two cones, as shown in Figure F.1 in a typical example. We have information on the integrand only when both q and $q-k$ are close to the lower mass cone, i.e. when q lies in the intersection of the dark shaded regions in Figure 4.1. Outside of this so-called *intersection region*, however, the integrand depends on the unknown high-energy behavior of $\hat{g}_1$ or $\hat{g}_2$. Since the intersection region becomes large when k comes close to the mass cone and does not depend smoothly in k for k on the mass cone, the contribution of the intersection region to the integral is localized in a

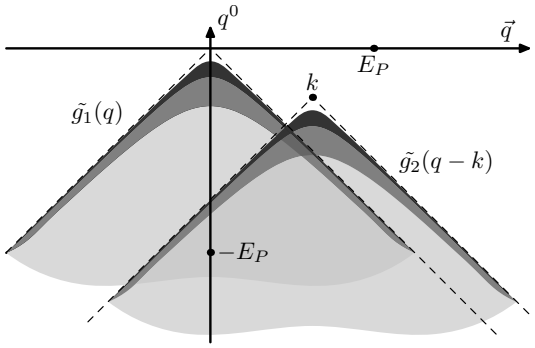

FIGURE F.1. Example for the integrand of $\hat{M}^{Lg}$.

neighborhood of and is not regular on the mass cone $\{k^2 = 0\}$. The contribution of the high-energy regions to the integral, on the other hand, is regular in k and thus well-approximated by a polynomial in k. This qualitative argument illustrates why in Lemma F.1, $\hat{Q}$ is determined only modulo a polynomial, and why the singular behavior on the light cone (F.4) is in momentum space encoded near the lower mass cone (see Lemma F.2 and the discussion thereafter). An important conclusion from Figure F.1 is that $\hat{Q}(k)$ is in general not zero in the interior of the lower mass cone, and even the intersection region gives a contribution there. Thus for generic regularizations or for simple regularizations like mollifying with a smooth function, the supports of $\hat{Q}$ and $\hat{P}$ will have a non-empty intersection, and even the singularities of Q on the light cone will contribute to the product QP. On the other hand, it seems possible that there are special regularizations for which the contributions from the high energy regions and the intersection region compensate each other in the integral (F.26) in such a way that $\hat{M}^{Lg}$ indeed vanishes inside the lower mass cone. We call such a regularization an *optimal regularization*. According to the method of variable regularization (see §4.1), we want to keep the regularization as general as possible. Therefore, we must allow for the possibility that the regularization is optimal, and this leads to the problem of disjoint supports.

The above consideration in momentum space gives a hint on how to resolve the problem of disjoint supports. Namely, let us assume for a moment that the macroscopic function f has nice decay properties at infinity. Then its Fourier transform $\hat{f}$ is a regular function. As a consequence, the convolution (F.27) mollifies $\hat{M}^{Lg}$ on the scale l_{macro}^{-1}, and the support of $\hat{Q}$ will be larger than that of $\hat{M}^{Lg}$. Clearly, l_{macro}^{-1} is very small on the Planck scale, but since the mass shell $\{k^2 = m^2\}$ and the mass cone $\{k^2 = 0\}$ come asymptotically close as the energy $|k^0|$ gets large, mollifying even on a small scale leads to an overlap of the supports of $\hat{Q}$ and $\hat{P}$. This is an effect which is not apparent in the expansion of Lemma F.1 because by expanding f in a Taylor series around $\xi = 0$, we did not use the decay properties of f at infinity, and thus we did not see the smoothing in momentum space (cf. also (F.28) and (F.29)). More generally, the above "mollifying argument" shows that the supports of $\hat{Q}$ and $\hat{P}$ should overlap if we *take into account the macroscopic perturbations* of P and Q more carefully. Thus in order to solve the problem of disjoint supports, we shall now compute the product QP in the case of general interaction, without assuming that P or Q are homogeneous.

Our key result will be an expansion of the operator product around the light cone (see Theorem F.5 below).

Let us specify our assumptions on P and Q. For Q we merely assume that the weak evaluation formula (F.4) holds. For P, on the other hand, we work with the formulas of the light-cone expansion, which are of the general form

$$P(x,y) \;=\; \sum_{p=-1}^{\infty} g_p(x,y)\, T^{(p)}(x,y) \;+\; (\text{smooth contributions})\,. \tag{F.30}$$

Here the g_p are smooth functions involving the bosonic potentials and fields, and the unspecified smooth contributions are composed of the fermionic wave functions as well as the non-causal contributions to the Dirac seas (see (2.3.19, 2.5.45) and Appendix B). For clarity, we shall consider the product of Q with each of the summands in (F.30) separately, i.e. we will for given $p \geq -1$ compute the product

$$Q\,R \qquad \text{with} \qquad R(x,y) \;=\; g(x,y)\, T^{(p)}(x,y) \tag{F.31}$$

and a smooth function g. To avoid confusion, we recall that in the case $p = -1$, $T^{(p)}$ is defined via a distributional derivative; more precisely, we assume in this case that $g(x,y)$ has the form $g = \xi_j\, f^j$ with smooth functions $f^j(x,y)$ and set similar to (2.5.21)

$$g(x,y)\, T^{(-1)}(x,y) \;=\; -2\, f^j(x,y)\, \frac{\partial}{\partial y^j} T^{(0)}(x,y)\,. \tag{F.32}$$

For technical convenience, we assume furthermore that g is a Schwartz function, $g \in \mathcal{S}(\mathbb{R}^4 \times \mathbb{R}^4)$, but this assumption is not essential and could be relaxed by approximation (see the discussion after (F.81) below).

The contributions to Q in (F.4) are supported on the light cone. Thus we can write them in the form

$$Q(x,y) \;=\; h(x,y)\, K_{a=0}(x,y) \tag{F.33}$$

with

$$K_{a=0}(x,y) \;=\; \frac{i}{4\pi^2}\, \delta(\xi^2)\, \epsilon(\xi^0) \tag{F.34}$$

and a function $h(x,y)$, which in general will have a pole at the origin $x = y$. This representation is useful because K_a is a solution of the Klein-Gordon equation, namely in momentum space

$$K_a(k) \;=\; \delta(k^2 - a)\, \epsilon(k^0)\,, \qquad a \in [0, \infty). \tag{F.35}$$

In what follows, we will also need the Green's function S_a of the Klein-Gordon equation defined by

$$S_a(k) \;=\; \frac{\mathrm{PP}}{k^2 - a} \;\equiv\; \frac{1}{2}\, \lim_{\varepsilon \searrow 0} \sum_{\pm} \frac{1}{k^2 - a \pm i\varepsilon}\,, \qquad a \in \mathbb{R}. \tag{F.36}$$

As is immediately verified with contour integrals, this Green's function is for $a > 0$ causal in the sense that $S_a(x,y)$ vanishes for space-like ξ. On the contrary if $a < 0$, $S_a(x,y)$ vanishes for time-like ξ. More precisely, the Green's function can be written as

$$S_a(x,y) \;=\; -\frac{1}{4\pi}\, \delta(\xi^2) \;+\; \Theta(a\xi^2)\, \epsilon(a)\, H_a(x,y)\,, \tag{F.37}$$

where H_a is a smooth solution of the Klein-Gordon equation with power expansion

$$H_a(x,y) \;=\; \frac{a}{16\pi} \sum_{j=0}^\infty \frac{(-1)^j}{j!\,(j+1)!}\, \frac{a^j\, \xi^{2j}}{4^j}\,. \tag{F.38}$$

It is convenient to also introduce the Green's function

$$S_a^{\bowtie} \;=\; S_a - \Theta(a)\, H_a\,, \tag{F.39}$$

which for all $a \in \mathbb{R}$ vanishes in time-like directions. As one sees explicitly using (F.38) and (F.37), both $H_a(x,y)$ and $S_a^{\bowtie}(x,y)$ are analytic in a for all $a \in \mathbb{R}$. Similarly, a short explicit calculation shows that $K_a(x,y)$ is analytic for $a \in [0,\infty)$. We set

$$K^{(n)} = \lim_{a \searrow 0} \left(\frac{d}{da}\right)^n K_a\,, \quad S_{\bowtie}^{(n)} = \left(\frac{d}{da}\right)^n S_{a\,|\,a=0}^{\bowtie}\,, \quad H^{(n)} = \left(\frac{d}{da}\right)^n H_{a\,|\,a=0}\,.$$

The following lemma gives the light-cone expansion for an operator product involving two factors $K^{(.)}$. A major difference to Lemma 2.5.2 is that the expansion now contains unbounded line integrals; this also requires a different method of the proof.

LEMMA F.3. *The operator product $K^{(l)} V K^{(r)}$ with $l, r \geq 0$ and a scalar function $V \in \mathcal{S}$ has the light-cone expansion*

$$\begin{aligned}
&(K^{(l)}\, V\, K^{(r)})(x,y) \\
=&-\frac{1}{2\pi^2} \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, H^{(l+r+n+1)}(x,y) \\
&-\frac{1}{2\pi^2} \sum_{n=0}^\infty \frac{1}{n!} \int_{-\infty}^\infty d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, S_{\bowtie}^{(l+r+n+1)}(x,y) \\
&+(\textit{non-causal contributions, smooth for } x \neq y).
\end{aligned} \tag{F.40}$$

We point out that, exactly as in §2.5, we do not study the convergence of the infinite series in (F.40), which are merely a convenient notation for the approximation by the partial sums.

Proof of Lemma F.3. We first consider the operator product $K_a V K_b$ for $a, b > 0$ in the case when V is a plane wave, $V(x) = \exp(-iqx)$. Then in momentum space (similar to [**F6**, eqn (3.9)]), the operator product takes the form

$$(K_a\, V\, K_b)\Big(p+\frac{q}{2},\, p-\frac{q}{2}\Big) \;=\; K_a\Big(p+\frac{q}{2}\Big)\; K_b\Big(p-\frac{q}{2}\Big)\;. \tag{F.41}$$

If $q^2 < 0$, we get contributions when either the two upper mass shells of the factors K_a and K_b intersect, or the two lower mass shells. Conversely if $q^2 > 0$, we only get cross terms between the upper and lower mass shells. Thus setting

$$u \;=\; \frac{a+b}{2}\,, \qquad v \;=\; \frac{a-b}{2}\,, \tag{F.42}$$

we have

$$\begin{aligned}(K_a\,V\,K_b)\Big(p+\frac{q}{2},\,p-\frac{q}{2}\Big) \;&=\; -\epsilon(q^2)\,\delta\Big((p+\frac{q}{2})^2-a\Big)\;\delta\Big((p-\frac{q}{2})^2-b\Big)\\ &=\; -\epsilon(q^2)\,\delta\Big(((p-\frac{q}{2})^2-b)\;+\;(2pq-2v)\Big)\;\delta\Big((p-\frac{q}{2})^2-b\Big)\\ &=\; -\epsilon(q^2)\,\delta(2pq-2v)\,\delta\bigg(p^2-pq+\frac{q^2}{4}-b\bigg)\\ &=\; -\frac{1}{2}\,\epsilon(q^2)\,\delta(pq-v)\,\delta\bigg(p^2+\frac{q^2}{4}-u\bigg)\,.\end{aligned}$$

Hence we can write our operator product as

$$K_a\,V\,K_b \;=\; \frac{d}{du}A_{uv} \tag{F.43}$$

where A_{uv} is the operator

$$A_{uv}\Big(p+\frac{q}{2},\,p-\frac{q}{2}\Big) \;=\; \frac{1}{2}\,\delta(pq-v)\,\Theta\bigg(q^2\,(p^2+\frac{q^2}{4}-u)\bigg)\,. \tag{F.44}$$

Our strategy is to first derive an expansion for A_{uv}. Then we will differentiate this expansion with respect to u and v and take the limits $a,b\searrow 0$ to get the desired expansion for $K^{(l)}VK^{(r)}$.

The right side of (F.44) involves a product of the form $\delta(\alpha)\,\Theta(\beta)$. This product can be transformed into a line integral as follows. Consider for $\varepsilon>0$ the function

$$f_\varepsilon(\alpha,\beta) \;=\; \frac{1}{\pi}\,\Theta(\varepsilon\beta-\alpha^2)\,(\varepsilon\beta-\alpha^2)^{-\frac{1}{2}}\,. \tag{F.45}$$

This function is zero unless $\beta>0$ and $\alpha\in[-\sqrt{\varepsilon\beta},\sqrt{\varepsilon\beta}]$. As $\varepsilon\searrow 0$, the size of this last interval tends to zero, and so α is confined to a smaller and smaller neighborhood of the origin. On the other hand, the integral over α stays bounded in this limit; namely,

$$\int_{-\infty}^{\infty} f_\varepsilon(\alpha,\beta)\,d\alpha \;=\; \Theta(\beta) \qquad\qquad \text{for all } \varepsilon>0.$$

From this we conclude that

$$\lim_{\varepsilon\searrow 0} f_\varepsilon(\alpha,\beta) \;=\; \delta(\alpha)\,\Theta(\beta) \tag{F.46}$$

with convergence as a distribution. Moreover, the formula

$$\int_{-\infty}^{\infty} \frac{\mathrm{PP}}{\tau^2+\gamma}\,d\tau \;=\; \pi\,\Theta(\gamma)\,\gamma^{-\frac{1}{2}} \tag{F.47}$$

allows one to write (F.45) as a contour integral. Putting these relations together, we obtain for A_{uv},

$$\begin{aligned}A_{uv}\Big(p+\frac{q}{2},\,p-\frac{q}{2}\Big) \;&\overset{(F.44,F.46)}{=}\; \frac{1}{2}\,\lim_{\varepsilon\searrow 0} f_\varepsilon\bigg(pq-v,\;q^2\,(p^2+\frac{q^2}{4}-u)\bigg)\\ &\overset{(F.45)}{=} \frac{1}{2\pi}\,\lim_{\varepsilon\searrow 0}\,\frac{1}{|\varepsilon q^2|}\,\Theta\bigg(\frac{p^2+\frac{q^2}{4}-u}{\varepsilon q^2}-\frac{(pq-v)^2}{\varepsilon^2 q^4}\bigg)\bigg(\frac{p^2+\frac{q^2}{4}-u}{\varepsilon q^2}-\frac{(pq-v)^2}{\varepsilon^2 q^4}\bigg)^{-\frac{1}{2}}\\ &\overset{(F.47)}{=} \frac{\epsilon(q^2)}{2\pi^2}\,\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}\frac{\mathrm{PP}}{\varepsilon q^2\tau^2+p^2+\dfrac{q^2}{4}-u-\dfrac{(pq-v)^2}{\varepsilon q^2}}\,d\tau\,.\end{aligned} \tag{F.48}$$

After shifting the integration variable according to $\tau \to \tau + (pq - v)/(\varepsilon q^2)$, we can identify the integrand with the Green's function (F.36),

$$\begin{aligned} & A_{uv}\Big(p + \frac{q}{2},\, p - \frac{q}{2}\Big) \\ &= \frac{\epsilon(q^2)}{2\pi^2} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \frac{\mathrm{PP}}{p^2 + 2\tau(pq - v) + \varepsilon^2 q^2 \tau^2 + \dfrac{q^2}{4} - u}\, d\tau \\ &= \frac{\epsilon(q^2)}{2\pi^2} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} S_{z(\varepsilon,\tau)}(p + \tau q)\,, \end{aligned} \tag{F.49}$$

where z is the "mass function"

$$z(\varepsilon, \tau) \;=\; u \;+\; 2\tau v \;+\; (1 - \varepsilon^2)\, \tau^2\, q^2 \;-\; \frac{q^2}{4}\,.$$

If we solve (F.39) for S_a and substitute into (F.49), we can take the limit $\varepsilon \searrow 0$ to obtain

$$A_{uv}\Big(p + \frac{q}{2},\, p - \frac{q}{2}\Big) \;=\; \frac{\epsilon(q^2)}{2\pi^2} \int_{-\infty}^{\infty} (S_z^{\bowtie} + \Theta(z)\, H_z)(p + \tau q)\, d\tau \tag{F.50}$$

with

$$z \;\equiv\; u \;+\; 2\tau v \;+\; (\tau^2 - \frac{1}{4})\, q^2\,. \tag{F.51}$$

The calculation so far was carried out for fixed momentum q of the potential. In order to describe the case of general $V \in \mathcal{S}$, we must integrate over q. Furthermore, we transform to position space by integrating over p (similar to [**F5**, eqn (3.10)]) and obtain

$$A_{uv}(x, y) \;=\; \int \frac{d^4q}{(2\pi)^4}\, \hat{V}(q) \int \frac{d^4p}{(2\pi)^4}\, A_{uv}(p + \frac{q}{2}, p - \frac{q}{2})\, e^{-ip(x-y)}\, e^{-i\,\frac{q}{2}\,(x+y)}\,,$$

where $\hat{V}$ is the Fourier transform of V. Substituting in (F.50) and pulling out the τ-integral gives

$$\begin{aligned} A_{uv}(x, y) \;&=\; \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\tau \int \frac{d^4q}{(2\pi)^4}\, \hat{V}(q)\, \epsilon(q^2)\, e^{-i\,\frac{q}{2}\,(x+y)} \\ &\qquad \times \int \frac{d^4p}{(2\pi)^4}\, (S_z^{\bowtie} + \Theta(z)\, H_z)(p + \tau q)\, e^{-ip(x-y)}\,, \end{aligned}$$

and, after shifting the integration variable p according to $p + \tau q \to p$, we can carry out the Fourier integral,

$$\int \frac{d^4p}{(2\pi)^4}\, (S_z^{\bowtie} + \Theta(z)\, H_z)(p + \tau q)\, e^{-ip(x-y)} \;=\; e^{iq\tau(x-y)}\, (S_z^{\bowtie} + \Theta(z)\, H_z)(x, y)\,,$$

and thus obtain

$$\begin{aligned} A_{uv}(x, y) \;&=\; \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\tau \int \frac{d^4q}{(2\pi)^4}\, \hat{V}(q)\, \epsilon(q^2) \\ &\qquad \times e^{-iq((\frac{1}{2} - \tau)x + (\frac{1}{2} + \tau)y)}\, (S_z^{\bowtie} + \Theta(z)\, H_z)(x, y)\,. \end{aligned} \tag{F.52}$$

In this way, we have transformed the line integral, which appeared in (F.48) as a contour integral in momentum space, into an integral along the straight line $(\frac{1}{2} - \tau)x + (\frac{1}{2} + \tau)y$ through the space-time points x and y.

The operator product $K^{(l)}VK^{(r)}$ is obtained from A_{uv} by differentiating with respect to u, v and setting $u = 0 = v$. More precisely, using (F.42) and (F.43),

$$(K^{(l)}\, V\, K^{(r)})(x,y) \;=\; \frac{1}{2^{l+r}} \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)^l \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^r \frac{\partial}{\partial u}\, A_{uv}|_{u=0=v}\;. \tag{F.53}$$

According to (F.51), the factors $S_z^{\bowtie}$, $\Theta(z)$, and H_z in (F.52) depend implicitly on u and v. Thus when substituting (F.52) into (F.53), we can carry out the partial derivatives with the sum, product, and chain rules. Let us first collect the terms for which all the derivatives act on the factors $S_z^{\bowtie}$ or H_z. This gives the contributions to $(K^{(l)}VK^{(r)})(x,y)$

$$\begin{aligned}&\frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\tau\, (\tfrac{1}{2}+\tau)^l\, (\tfrac{1}{2}-\tau)^r \int \frac{d^4q}{(2\pi)^4}\, \hat{V}(q)\, \epsilon(q^2)\, e^{-i\,\frac{q}{2}\,(x+y)} \\ &\qquad \times e^{iq\tau\,(x-y)}\, (S_z^{\bowtie\,(l+r+1)} + \Theta(z)\, H_z^{(l+r+1)})(x,y)\end{aligned}$$

with $z = (\tau^2 - \frac{1}{4})q^2$. After expanding in powers of z and introducing the new integration variable $\alpha = \tau + \frac{1}{2}$, we obtain

$$\begin{aligned}&\frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n \int \frac{d^4q}{(2\pi)^4}\, \epsilon(q^2)\, (-q^2)^n\, \hat{V}(q) \\ &\qquad \times e^{-iq(\alpha y + (1-\alpha)x)}\, (S_z^{\bowtie\,(l+r+n+1)} + \Theta((\alpha^2-\alpha)\, q^2)\, H_z^{(l+r+n+1)})(x,y)\,.\end{aligned} \tag{F.54}$$

If $\hat{V}$ is supported outside the mass cone, $\operatorname{supp} \hat{V} \subset \{q^2 < 0\}$, we can carry out the q-integral in (F.54) and obtain precisely the two series in (F.40). Conversely if $\hat{V}$ is supported inside the mass cone $\{q^2 > 0\}$, we get

$$\begin{aligned}&\frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}\setminus[0,1]} d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, H^{(l+r+n+1)}(x,y) \\ &+\frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, S_{\bowtie}^{(l+r+n+1)}(x,y)\,.\end{aligned}$$

This does not coincide with the series in (F.40). Using (F.39) and (F.37), the difference can be written as

$$\begin{aligned}&-\frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n \\ &\qquad \times (\Box^n V)_{|\alpha y+(1-\alpha)x}\, \epsilon(\xi^2)\, H^{(l+r+n+1)}(x,y)\,.\end{aligned} \tag{F.55}$$

Since unbounded line integrals are involved, this expression is clearly non-causal. We shall now prove that (F.55) is smooth for $x \neq y$. Notice that the line integrals in (F.55) are supported on the hyperplane $\{q \mid q\xi = 0\}$, e.g.

$$\int_{-\infty}^{\infty} V(\alpha y + (1-\alpha)x)\, d\alpha \;=\; \int \frac{d^4q}{(2\pi)^4}\, \hat{V}(q)\, 2\pi\delta(q\xi)\, e^{-i\,\frac{q}{2}\,(x+y)}\,. \tag{F.56}$$

For time-like ξ, this hyperplane does not intersect the support of $\hat{V}$, and thus (F.55) vanishes identically inside the light cone. Furthermore, (F.55) is clearly smooth in the region $\{\xi^2 < 0\}$ away from the light cone. Thus it remains to show that all the partial

derivatives of (F.55) vanish on the light cone. The boundary values of these partial derivatives on the light cone $\{\xi^2 = 0\}$ involve integrals of the form

$$\int_{-\infty}^{\infty} d\alpha\, \alpha^{l+k_1}\, (1-\alpha)^{r+k_2}\, (\alpha-\alpha^2)^n\, (\partial^K \Box^n V)_{|\alpha y+(1-\alpha)x}\, \xi^L$$

with parameters $k_1, k_2 \geq 0$ and multi-indices K, L. Similar to (F.56), these integrals are supported on the hypersurface $\{q \mid q\xi\} = 0$, and for ξ on the light cone (and $\xi \neq 0$), this hypersurface does not intersect the support of $\hat{V}$. We conclude that (F.54) coincides with (F.40), both in the case when supp $\hat{V} \subset \{q^2 < 0\}$ and when supp $\hat{V} \subset \{q^2 > 0\}$. Linearity and an approximation argument near the light cone yield that (F.54) coincides with (F.40) for general $V \in \mathcal{S}$.

It remains to consider the contributions when some of the derivatives in (F.53) act on the factor $\Theta(z)$ in (F.52). The resulting expressions are of the form

$$\int \frac{d^4q}{(2\pi)^4}\, \hat{V}(q)\, e^{-i\,\frac{q}{2}\,(x+y)} \int_{-\infty}^{\infty} d\tau\, \epsilon(q^2)\, \delta^{(\alpha)}(z)\, H_z^{(\beta)}(x,y)\, \mathcal{P}(\tau)\, e^{-iq\tau(y-x)} \tag{F.57}$$

with integers $\alpha, \beta \geq 0$ and a polynomial $\mathcal{P}(\tau)$. Using the formula

$$\delta(z) \;=\; \frac{1}{2\pi i}\, \lim_{\varepsilon \searrow 0} \left(\frac{1}{z - i\varepsilon} - \frac{1}{z + i\varepsilon} \right),$$

we can write the τ-integral in terms of the complex integrals

$$\lim_{\varepsilon \searrow 0} \epsilon(q^2) \int_{-\infty}^{\infty} d\tau\, \frac{1}{((\tau^2 - \frac{1}{4})\, q^2 \pm i\varepsilon)^{\alpha+1}}\, H_z^{(\beta)}\, \mathcal{P}(\tau)\, e^{-iq\tau(y-x)}\,.$$

Depending on the sign of $q(y-x)$, the integration contour can be closed either in the upper or in the lower half plane, and the residue theorem yields expressions of the form

$$\lim_{\varepsilon \searrow 0} \frac{1}{(q^2 \pm i\varepsilon)^\gamma}\, H^{(\kappa)}(x-y)\, \mathcal{P}(\tau)\, e^{-iq\tau(y-x)} \qquad \text{with} \qquad \tau = \pm\frac{1}{2} \tag{F.58}$$

and $\gamma \leq 2\alpha + 1$, $\beta \leq \kappa \leq \beta + \alpha$. These expressions are well-defined distributions, and thus the q-integral in (F.57) is finite. Due to the powers of $1/q^2$ in (F.58), the resulting contributions to the operator product $K^{(l)} V K^{(r)}$ are non-causal. Since the factor $H^{(\kappa)}$ in (F.58) is a polynomial in ξ and $\hat{V}(q)$ in (F.57) has rapid decay, these contributions are also smooth. ∎

The above lemma can be used to derive the light-cone expansion for the operator product $K^{(l)} V T^{(r)}$.

LEMMA F.4. *For $l, r \geq 0$ and $V \in \mathcal{S}$,*

$$\begin{aligned}
&(K^{(l)}\, V\, T^{(r)})(x,y) \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x} \\
&\quad \times \epsilon(y^0 - x^0)\, T^{(l+r+n+1)}(x,y) \;+\; (\textit{contributions smooth for } x \neq y)\,.
\end{aligned} \tag{F.59}$$

Proof. Using (F.39) and the fact that $H_a(x,y)$ is smooth in x and y according to (F.38), the light-cone expansion (F.40) yields that for $p, q \geq 0$,

$$\begin{aligned}
&(K^{(l)}\, V\, K^{(r)})(x,y) \\
&= -\frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\, S^{(l+r+n+1)}(x,y) \\
&\quad +(\text{contributions smooth for } x \neq y)\,, \qquad \text{(F.60)}
\end{aligned}$$

where $S^{(n)} = \lim_{a \searrow 0} S_a^{(n)}$. The main difference between (F.59) and (F.60) is that the factors $K^{(r)}$ and $S^{(l+r+n+1)}$ are replaced by corresponding factors $T^{(.)}$. The method of the proof is to realize these replacements by multiplying (F.60) with a suitable operator from the right.

In preparation, we rewrite the operators $S^{(.)}$ in (F.60) in terms of $K^{(.)}$ as follows. Using that multiplication in position space corresponds to convolution in momentum space, we have for $a > 0$,

$$\begin{aligned}
&\int K_a(x,y)\, \epsilon(y^0-x^0)\, e^{-ik\,\xi}\, d^4\xi \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \delta(\omega^2 - |\vec{k}|^2 - a)\, \epsilon(\omega)\, (-2i)\, \frac{\mathrm{PP}}{k^0-\omega} \\
&= \frac{1}{i\pi}\, \frac{1}{2\,|\omega|}\, \frac{\mathrm{PP}}{k^0-\omega}\Big|_{\omega=-\sqrt{|\vec{k}|^2+a}}^{\omega=\sqrt{|\vec{k}|^2+a}} = \frac{1}{i\pi}\, \frac{\mathrm{PP}}{k^2-a} = \frac{1}{i\pi}\, S_a(k)\,,
\end{aligned}$$

and thus

$$S_a(x,y) = i\pi\, K_a(x,y)\, \epsilon(y^0-x^0)\,.$$

We differentiate with respect to a and let $a \searrow 0$ to obtain

$$S^{(n)}(x,y) = i\pi\, K^{(n)}(x,y)\, \epsilon(y^0-x^0)\,.$$

Substituting into (F.60) gives

$$\begin{aligned}
(K^{(l)}\, V\, K^{(r)})(x,y) &= (\text{contributions smooth for } x \neq y) \\
&\quad + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\alpha\, \epsilon(y^0-x^0)\, \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n \\
&\qquad \times (\Box^n V)_{|\alpha y+(1-\alpha)x}\, K^{(l+r+n+1)}(x,y)\,. \qquad \text{(F.61)}
\end{aligned}$$

The operator T_a, $a \geq 0$, is obtained from K_a by projecting on the negative-energy states, more precisely

$$T_a = K_a\, \chi\,, \qquad \text{(F.62)}$$

where χ is the multiplication operator in momentum space

$$\chi(k) = -\Theta(-k^0)\,.$$

In position space, χ has the kernel

$$\chi(x,y) = \int \frac{d^4k}{(2\pi)^4}\, \chi(k)\, e^{-ik(x-y)} = -\frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \frac{1}{y^0-x^0-i\varepsilon}\, \delta^3(\vec{y}-\vec{x})\,. \qquad \text{(F.63)}$$

If a is positive, the mass shell $\{k^2 = a\}$ does not intersect the hyperplane $\{k^0 = 0\}$ where χ is not smooth, and thus we may differentiate (F.62) with respect to a to obtain

$$T_a^{(n)} = K_a^{(n)}\, \chi\,, \qquad a > 0. \qquad \text{(F.64)}$$

However, difficulties arise in (F.64) in the limit $a \searrow 0$. Namely, the limit of the left side exist only after "regularizing" $T_a^{(n)}$ by subtracting a polynomial in $(y-x)$ (see (2.5.42, 2.5.43). On the right side, the problem is that $K^{(n)}(x,y)$ behaves polynomially at timelike infinity, whereas $\chi(x,y)$ decays for large (y^0-x^0) only as $(y^0-x^0)^{-1}$, and so the product $K^{(n)}\chi$ does not exist. To cure this problem, we insert into (F.63) an exponentially decaying factor by introducing for $\kappa > 0$ the kernel

$$\chi_\kappa(x,y) \;=\; -\frac{1}{2\pi i} \lim_{\varepsilon \searrow 0} \frac{e^{-\kappa\,|y^0-x^0|}}{y^0-x^0-i\varepsilon}\, \delta^3(\vec{y}-\vec{x})\,. \tag{F.65}$$

The exponential factor changes the product with $K_a^{(n)}$ only by a contribution smooth in $y-x$, and thus

$$\begin{aligned} &T_a^{(n)}(x,y) + (\text{smooth contributions}) \\ &\quad = \;\; T_a^{(n)}(x,y) \;=\; (K_a^{(n)}\,\chi)(x,y) \;=\; (K_a^{(n)}\,\chi_\kappa)(x,y) + (\text{smooth contributions})\,. \end{aligned}$$

The very left and right of this equation converge for $a \searrow 0$, and we conclude that

$$(K^{(n)}\,\chi_\kappa)(x,y) \;=\; T^{(n)}(x,y) \;+\; (\text{smooth contributions}). \tag{F.66}$$

We multiply the operator product on the left of (F.61) by the operator χ_κ. Applying (F.66) and using that multiplying by a smooth operator gives something smooth, we get

$$(K^{(l)}\,V\,K^{(r)}\,\chi_\kappa)(x,y) \;=\; (K^{(l)}\,V\,T^{(r)})(x,y) \;+\; (\text{smooth contributions}). \tag{F.67}$$

It remains to show that multiplying the right side of (F.61) by the operator χ_κ gives the right side of (F.59). When we multiply the summands on the right side of (F.61) by χ_κ, we get according to (F.65) a convolution in the time coordinate of the form

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tau\, \frac{e^{-\kappa\,|\tau|}}{\tau - i\varepsilon}\, F(x; y^0-\tau, \vec{y})\, K^{(l+r+n+1)}(x; y^0-\tau, \vec{y})\,. \tag{F.68}$$

Here the function F stands for the line integral in (F.61); it is smooth unless $y^0-\tau = x^0$. We write the convolution integral (F.67) as

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tau\, e^{-\kappa\,|\tau|}\, \frac{F(x; y^0-\tau, \vec{y}) - F(x,y)}{\tau - i\varepsilon}\, K^{(l+r+n+1)}(x; y^0-\tau, \vec{y}) \tag{F.69}$$

$$+F(x,y) \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\tau\, \frac{e^{-\kappa\,|\tau|}}{\tau - i\varepsilon}\, K^{(l+r+n+1)}(x; y^0-\tau, \vec{y})\,. \tag{F.70}$$

The term (F.70) can be written as $F(x,y)\,(K^{(l+r+n+1)}\,\chi_\kappa)(x,y)$, and applying (F.66) gives precisely the summands on the right side of (F.59). Let us prove that (F.69) is smooth for $x \neq y$. Thus assume that $x \neq y$. Our above construction is Lorentz invariant in the sense that we may introduce the operator χ_κ in any reference frame, without influence on our operator products. Thus we can choose the reference frame such that $\vec{x} \neq \vec{y}$. The τ-integral in (F.69) can be regarded as an integral along the straight line $\{(y^0-\tau, \vec{y})\}$. Since $\vec{x} \neq \vec{y}$, this straight line does not intersect the point x. Due to causality of $K^{(.)}$, the integrand vanishes unless $|(y^0-\tau)-x^0| \geq |\vec{y}-\vec{x}|$. Thus if ξ is spacelike, i.e. $|y^0-x^0| < |\vec{y}-\vec{x}|$, the integrand vanishes in a neighborhood of $\tau = 0$, and as a consequence the integral (F.69) is smooth in x and y. On the other hand if ξ lies inside the cone $|y^0-x^0| > \frac{1}{2}\,|\vec{y}-\vec{x}|$, the straight line $\{(y^0-\tau, \vec{y})\}$ does for small enough τ not intersect the hyperplane $\{z \,|\, z^0 = x^0\}$ where F is not smooth.

Thus the function $F(x; y^0 - \tau, \vec{y})$ is smooth in a neighborhood of $\tau = 0$, and the mean value theorem yields that the bracket in (F.69) is smooth. This implies that (F.69) is again smooth in x and y.

It remains to show that multiplying the contributions smooth for $x \neq y$, which are not specified in (F.61), by the operator χ_κ, gives terms which are again smooth for $x \neq y$. Again using Lorentz invariance, we can choose a reference frame such that $\vec{x} \neq \vec{y}$. Then multiplying by χ_κ yields a convolution along the straight line $\{(y^0 - \tau, \vec{y})\}$, and this line integral does not intersect the point x. In this way, we can avoid integrating across the origin where the contributions in (F.61) may be singular. We get a convolution of χ_κ with a smooth function, and this is clearly finite and depends smoothly on x and y. ∎

After these preliminaries, we are ready to compute the operator product QR in an expansion around the light cone. For the statement of the result, we need to analytically extend $h_J(\Omega)$ in (F.4) from a function on S^2 to a function on Minkowski space and also regularize it at the origin: As a smooth function on S^2, h_J can be expanded in spherical harmonics. Since the spherical harmonics are the boundary values on S^2 of the harmonic polynomials on $\mathbb{R}^3$, we have the unique expansion

$$h_J(\vec{x}) \;=\; \sum_{n=0}^{\infty} \mathcal{P}_n(\vec{x})|_{\vec{x}\in S^2} \;, \tag{F.71}$$

where $\mathcal{P}_n(\vec{x})$ are suitable harmonic polynomials of degree n. The smoothness of h_J implies that the summands in (F.71) decay in n faster than any polynomial. As a consequence, the series in (F.71) converges absolutely for any $\vec{x} \in \mathbb{R}^3$, and we can even extend $h_J(\vec{x})$ to a unique function on $\mathbb{C}^3$. For $\varepsilon > 0$, we define the regularization h_J^ε of h_J by

$$h_J^\varepsilon(\xi) \;=\; h_J\left(\frac{\vec{\xi}}{\xi^0 - i\varepsilon}\right) . \tag{F.72}$$

THEOREM F.5. *Assume that the operator Q satisfies the weak evaluation formula (F.4). Then the operator product QR with R according to (F.31) with $p \geq -1$ and $g \in \mathcal{S}(\mathbb{R}^4 \times \mathbb{R}^4)$ has the expansion*

$$(Q\,R)(x,y) \;=\; \sum_{L=2}^{L_{max}} \sum_{g=0}^{g_{max}} \log^g(E_P)\, E_P^{L-1} \sum_J \tag{F.73}$$

$$\times\, (-2\pi) \sum_{n=0}^{\infty} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} d\alpha\, (1-\alpha)^p\, (\alpha - \alpha^2)^n\, \Box_z^n \left(\frac{h_J^\varepsilon(\zeta)\, \zeta^J}{(\zeta^0 - i\varepsilon)^{L-1}}\, g(z,y) \right)_{|z=\alpha y + (1-\alpha)x}$$

$$\times\, \epsilon(y^0 - x^0)\, T^{(p+n+1)}(x,y) \;\; \textit{mod}\;\; \mathcal{P}^{L-|J|-3}(\partial_x)\, R(x,y) \;+\; o(E_P) \tag{F.74}$$

$$+ \textit{ (contributions smooth for } x \neq y) \;+\; \textit{(higher orders in } (l_{macro} E_P)^{-1}) \tag{F.75}$$

with $\zeta \equiv z - x$.

Proof. We first consider the case $p \geq 0$. We fix two space-time points x_0, y_0 and set

$$Q(\xi) \;:=\; Q(x_0, x_0 + \xi)\,, \qquad R(\xi) \;:=\; R(y_0 - \xi, y_0)\,.$$

We regard $Q(\xi)$ and $R(\xi)$ as the integral kernels of corresponding homogeneous operators, which with slight abuse of notation we denote again by Q and R. Then

in momentum space (F.11), the operator Q has the expansion of Lemma F.1 and Lemma F.2. Let us compute the product $Q_J^{Lg}R$. According to its construction in Lemma F.1, $Q_J^{Lg}(\xi)$ behaves for $\xi \neq 0$ like

$$Q_J^{Lg}(\xi) \;=\; \delta(\xi^2)\,\epsilon(\xi^0)\,t^{-L+1}\,h_J(\check{\xi})\,\xi^J\,,$$

and furthermore its Fourier transform $\hat{Q}_J^{Lg}(k)$ vanishes inside the lower mass cone (cf. Lemma F.2). The distribution

$$\lim_{\varepsilon\searrow 0} K^{(0)}(x,y)\left[-4\pi^2 i\,(y^0-x^0-i\varepsilon)^{-L+1}\,h_J(\check{\xi})\,\xi^J\right] \tag{F.76}$$

has these two properties, as one sees immediately from (F.34) and when computing the Fourier transform of (F.76) with contour integrals. Indeed, a short calculation shows that (F.76) even coincides with $Q_J^{Lg}(\xi)$.

We introduce for $\varepsilon > 0$ the potential

$$V_\varepsilon(z) \;=\; -4\pi^2 i\,(z^0-x_0^0-i\varepsilon)^{-L+1}\,h_J^\varepsilon(z-x_0)\,(z-x_0)^J\,g(z,y_0)\,,$$

where h_J^ε is the regularization of h_J (F.72). This potential is a Schwartz function, and thus Lemma F.4 yields that

$$\begin{aligned}
&(K^{(0)}\,V^\varepsilon\,T^{(p)})(x,y)\\
&= \frac{1}{2\pi i}\sum_{n=0}^\infty \frac{1}{n!}\int_{-\infty}^\infty d\alpha\,(1-\alpha)^p\,(\alpha-\alpha^2)^n\,(\Box^n V_\varepsilon)_{|\alpha y+(1-\alpha)x}\\
&\qquad\times \epsilon(y^0-x^0)\,T^{(p+n+1)}(x,y) \;+\; (\text{contributions smooth for } x\neq y)\,.
\end{aligned} \tag{F.77}$$

We now set $x = x_0$, $y = y_0$ and take the limit $\varepsilon \searrow 0$. On the left side of (F.77), we can use that $(F.76) = Q_J^{Lg}(\xi)$ to obtain the operator product $(Q_J^{Lg}R)(x_0,y_0)$. Furthermore, due to our regularization of h_J^ε, the factors $(\Box^n V_\varepsilon)(z)$ on the right side of (F.77) are of the form smooth function times $(z^0-x^0-i\varepsilon)^{-l}$, and thus the limit $\varepsilon\searrow 0$ exists in each line integral in (F.77). We conclude that $(Q_J^{Lg}R)(x,y)$ coincides precisely with the series (F.74). Since in (F.77) we integrate across the origin, the higher orders in $((r+|t|)E_P)^{-1}$ in (F.4) (or equivalently the higher orders in k/E_P in (F.16)) yield contributions of higher order in $(l_{\text{macro}}E_P)^{-1}$. Finally, calculating modulo polynomials in (F.17) means in position space that $Q(x,y)$ is determined only modulo partial derivatives of $\delta^4(x-y)$, and this gives rise to the term $\mod \mathcal{P}^{L-|J|-3}(\partial_x)\,R(x,y)$ in (F.74). This concludes the derivation of (F.73–F.75) in the case $p \geq 0$.

In the above derivation we neglected the contributions smooth for $x \neq y$, which are not specified in (F.77), implicitly assuming that they remain smooth in the limit $\varepsilon \searrow 0$. This is justified as follows. As $\varepsilon \searrow 0$, the potential $V_\varepsilon(z)$ becomes singular only at $z = x_0$. Thus if the contributions smooth for $x \neq y$ in (F.77) had a non-smooth limit, the resulting non-smooth contributions to $(Q_J^{Lg}R)(x_0,y_0)$ would depend on $g(z,y_0)$ and its partial derivatives only at $z = x_0$; i.e. they would be of the form

$$\partial_{x_0}^J g(x_0,y_0)\,W_J(x_0,y_0)\,, \tag{F.78}$$

where W_J are distributions independent of g. Suppose that the Fourier transform $\hat{g}$ of $g(x_0+.,y_0)$ is supported inside the upper mass shell $\{k\,|\,k^2>0, k^0>0\}$. Then due to our $(-i\varepsilon)$-regularization, also the support of $\hat{V}_\varepsilon$ is inside the upper mass shell. As a consequence, the Fourier transforms of the distributions $K^{(0)}(x_0,.)$ and $(V_\varepsilon T^{\text{reg}\,(p)})(.,y_0)$ have disjoint supports, and the left side of (F.77) is zero. Furthermore for ξ close to

the light cone, the unbounded line integrals in (F.77) vanish (notice that they are supported on the hypersurface $\{k \,|\, k\xi = 0\}$). We conclude that if $\hat{g}$ is supported inside the upper mass shell, then the contributions smooth for $x \neq y$ in (F.77) are zero. Taking the limit $\varepsilon \searrow 0$ yields that the contributions (F.78) vanish if $\hat{g}$ is supported inside the upper mass shell, i.e.

$$\int d^4k \; k^J \; \hat{g}(k) \; W_J(x_0, y_0) \;=\; 0 \tag{F.79}$$

for all $\hat{g}$ with $\text{supp}\, \hat{g} \subset \{k \,|\, k^2 = 0, k^0 > 0\}$. Since the polynomials restricted to the upper mass cone are linearly independent, (F.79) implies that the contributions (F.78) are all identically zero.

The extension to the case $p = -1$ follows exactly as in [**F6**, Lemma 2.2]: We pull one derivative out of the operator product,

$$\begin{aligned} &\int d^4z \; Q(x_0, z) \; g(z, y_0) \; T^{(-1)}(z, y) \\ &\overset{(F.32)}{=} \; -2 \, \frac{\partial}{\partial y^j} \int d^4z \; Q(x_0, z) \; f^j(z, y_0) \; T^{(0)}(z, y)_{|y=y_0} \,, \end{aligned}$$

substitute for the integral the expansion (F.73–F.75) for $p = 0$, and differentiate through. ∎

The above theorem gives the clue for understanding the operator product QP as well as the commutator $[P, Q]$, as we shall now explain. Using that the product of Q with a smooth operator is smooth, we can write the operator product QP according to (F.30) in the form

$$(Q \, P)(x, y) \;=\; \sum_{p=-1}^{\infty} (Q \, P^p)(x, y) \;+\; (\text{smooth contributions}) \tag{F.80}$$

with

$$P^p(x, y) \;=\; g_p(x, y) \, T^{(p)}(x, y) \,. \tag{F.81}$$

The summands in (F.80) are precisely of the form considered in Theorem F.5, with the only exception that the functions g_p in (F.81) in general have no rapid decay at infinity. Fortunately, the behavior of the functions g_p at infinity is of no relevance to us, and we can apply Theorem F.5 to each summand in (F.80) using the following approximation argument. For fixed x and y, we choose a Schwartz function η which is identically equal to one on a compact set $K \ni x, y$. Then the function $(g_p \eta)$ has rapid decay at infinity, and Theorem F.5 applies to the operator product (F.31) with $g := g_p \eta$. In the discussion below leading to Corollary F.6, the function $g(z)$ enters only for z in a neighborhood of x and y. In this neighborhood, g and g_p coincide, and thus the dependence on η drops out. This shows that the behavior of the function g_p at infinity is indeed of no importance for Corollary F.6 below.

Let us briefly discuss the expansion (F.73–F.75). First of all, we point out that we calculate modulo terms of the form $\partial_x^K R(x, y)$ with $|K| \leq L - |J| - 3$. This corresponds to the fact that we have no information on the behavior of Q near the origin. For generic regularizations or simple regularizations like a cutoff in momentum space, the terms $\partial_x^K R(x, y)$ will not be zero. Thus in this case, the operator product QR does not vanish, even when we take for R a Dirac sea of the vacuum; this is in agreement with our consideration in momentum space after (F.29). However, the

situation is much different if we assume that we have a regularization where the terms $\partial_x^K R(x,y)$ all vanish. Namely, if we then take for R a Dirac sea of the vacuum, for example

$$R(x,y) \;=\; \frac{i}{2}\, \not{g}\, T^{(-1)}(x,y)\,,$$

the integrand of the line integral in (F.74) is a rational function with poles in the upper half plane, and QR is zero (up to the contributions not specified in Theorem F.5). This corresponds to our observation made after (F.22) that the poles of Q cannot be detected when testing with solutions of the free Dirac equation. The regularizations for which the terms $\partial_x^K R(x,y)$ vanish are just the optimal regularizations introduced after (F.29). The main advantage compared to our earlier consideration in momentum space after (F.29) is that the expansion (F.73–F.75) tells how the macroscopic perturbations of P and Q effect the operator product. In particular one sees that, when taking into account the macroscopic perturbations, the operator product QP does in general not vanish (even for optimal regularizations), and thus the problem of disjoint supports disappears.

In an interacting system, the factor $g(z,y)$ in (F.74) is composed of the bosonic potentials and fields. Thus in the generic situation, the line integrals in (F.74) will vanish only if the operator Q is identically zero. In order to make this argument clearer, it is useful not to think of P as a fixed object, but to consider small dynamic perturbations of P. More precisely, we consider perturbations of P induced by perturbations of the bosonic potentials of our physical system. In order not to disturb the Euler-Lagrange equations, these perturbations must not be arbitrary, but should satisfy the physical equations; a typical example are perturbations by an electromagnetic wave. Thus we consider variations of our system by small, physically admissible perturbations of P and study the effect on the operator product QP. We refer to this procedure for analyzing the operator product that we *test with physical perturbations of* P. Clearly, the requirement that the perturbation should satisfy the physical equations is a strong restriction (in particular, such perturbations are not dense in the L^2 topology). The reason why it is nevertheless a reasonable concept to use physical perturbations for testing is that these perturbations enter into (F.74) only along the one-dimensional line $\{\lambda y + (1-\lambda)x\}$. In the example of the perturbation by an electromagnetic wave, the electromagnetic field appears in the function g in (F.74), and by changing the location and amplitude of the wave, we can completely determine the function $h_J(\Omega)$ as well as the order $L - |J| - 1$ of the pole of the integrand at the origin. Notice furthermore that the summands in (F.73) scale in the Planck energy exactly as the corresponding summands in the expansion (F.4). We conclude that by testing the operator product QP with physical perturbations of P, we can reconstruct the weak evaluation formula (F.4) completely.

Next, it is instructive to consider the behavior of the summands in (F.74) as $y - x$ gets small. If $y - x$ is scaled like $(y-x) \to \lambda\,(y-x)$ with $\lambda > 0$, the variable transformation $\alpha \to \lambda^{-1}\alpha$ shows that as $\lambda \searrow 0$, the line integral blows up like $\lambda^{-(2n+p+1)}$. On the other hand, the factor $T^{(p+n+1)}$ goes to zero like λ^{2n+2p} in this limit. Thus each summand in (F.74) scales like λ^{-1+p}. It is remarkable that, no matter how large the order of the pole of Q at the origin is, the operator product QP has at the origin a pole of at most the order one. The reason is that in (F.74), we integrate over the pole of Q, and this regularizes the singularity at the origin. Since a pole of order one is integrable in three space dimensions, we do not need to study the operator product

$(QP)(x, y)$ at the origin $x = y$. According to the light-cone expansion of Theorem F.5, the information contained in the weak evaluation formula (F.4) is retrieved in the operator product QP by considering the singularities on the light cone away from the origin.

The expansion of Theorem F.5 immediately allows us to study also the commutator $[P, Q]$. Namely, by taking the adjoint, $(QR)(x, y)^* = (RQ)(y, x)$, the formula (F.73–F.75) applies to the operator product PQ as well, and we can take the difference $[P, Q] = PQ - QP$. The key observation is that in the product $(QR)(x, y)$, the pole of Q at the origin appears in (F.74) together with the factor $g(z, y)$ and $z \approx x$, whereas in the product $(RQ)(x, y)$, this pole is multiplied by $g(x, z)$ and $z \approx y$. Thus when testing $[P, Q](x, y)$ with perturbations of P, one can distinguish between the contributions from PQ and QP by considering perturbations which are localized near y and x, respectively.

These results are summarized as follows[1].

COROLLARY F.6. *To every order in E_P, the poles of $Q(x, y)$ at the origin $x = y$ can be detected in the commutator $[P, Q]$ by testing with physical perturbations of P.*

We close with a general comment on the significance of the constructions in this appendix. Due to the problem of disjoint supports, we could make sense out of the commutator $[P, Q]$ only after taking into account the macroscopic perturbations of P and Q. As a consequence, the relevant contributions to the commutator $[P, Q]$ are by several orders of $(l_{\text{macro}} E_P)^{-1}$ smaller than expected from a simple scaling argument. This can be interpreted that the causal structure of Minkowski space and the structure of the Dirac seas, which are the underlying reason for the problem of disjoint supports, have a tendency to making the commutator $[P, Q]$ small. In this way, the causal structure and the structure of the Dirac seas seem to correspond nicely to Euler-Lagrange equations of the form $[P, Q] = 0$.

[1] Online version: A simpler and cleaner method to obtain this result is to use the so-called method of *testing on null lines* as worked out in §3.5.2 in the book [**5**] (listed in the references in the preface to the second online edition).

APPENDIX G

Perturbation Calculation for the Spectral Decomposition of $P(x,y)\,P(y,x)$

In this appendix we shall develop a convenient method for analyzing the eigenvalues and spectral projectors of the matrix $A_{xy} \equiv P(x,y)\,P(y,x)$ and compute all contributions to the eigenvalues needed for the derivation of the effective gauge group in Chapter 7. Our strategy is as follows. We decompose the fermionic projector as

$$P \;=\; P_0 \;+\; \Delta P$$

with P_0 according to (6.1.1). This gives rise to the decomposition of A

$$A \;=\; A_0 \;+\; \Delta A \tag{G.1}$$

with

$$A_0 \;=\; P_0(x,y)\,P_0(y,x) \tag{G.2}$$

$$\Delta A \;=\; \Delta P(x,y)\,P_0(y,x) \;+\; P_0(x,y)\,\Delta P(y,x) \;+\; \Delta P(x,y)\,\Delta P(y,x)\,. \tag{G.3}$$

The eigenvalues and spectral projectors of A_0 were computed explicitly in Chapter 6, see (6.1.9, 6.1.10). On the light cone, $P_0(x,y)$ has singularities of order $\mathcal{O}((y-x)^{-4})$, whereas $\Delta P(x,y) = \mathcal{O}((y-x)^{-2})$. Likewise, ΔA is compared to A_0 of lower degree on the light cone. For this reason, ΔA can be treated perturbatively in the sense that the eigenvalues and spectral projectors of A can be expressed to any given degree on the light cone by a finite order perturbation calculation. Apart from the purely computational aspects, the main difficulty is that A_0 may have degenerate eigenvalues, and in this case we need to carefully analyze whether the degeneracy is removed by the perturbation. Our method is to first compute projectors on invariant subspaces of A (§G.1). Analyzing the perturbation on these invariant subspaces will then give the spectral decomposition of A (§G.4).

G.1. Perturbation of Invariant Subspaces

We write the spectral decomposition of A_0 as

$$A_0 \;=\; \sum_{k=1}^{K} \lambda_k\, F_k$$

with distinct eigenvalues λ_k and corresponding spectral projectors F_k. As in §5.3 we use the convention $\lambda_1 = 0$. Clearly, the F_k are the sum of the spectral projectors counting multiplicities,

$$F_k \;=\; \sum_{n,c,s \text{ with } \lambda_{ncs}=\lambda_k} F_{ncs} \tag{G.4}$$

with λ_{ncs} and F_{ncs} according to (6.1.10). Since the perturbation ΔA will in general split up the degenerate eigenvalues, we cannot expect that by perturbing F_k we obtain spectral projectors of the matrix A. But we can form projectors G_k on the space

spanned by all eigenvectors of A whose eigenvalues are sufficiently close to λ_k. The G_k are most conveniently introduced using contour integrals. We choose $\varepsilon > 0$ such that

$$|\lambda_i - \lambda_j| \;<\; 2\varepsilon \qquad \text{for all } i,j = 1,\ldots,K \text{ and } i \neq j.$$

Then we set

$$G_k \;=\; \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} (z-A)^{-1}\, dz\,, \tag{G.5}$$

The Cauchy integral formula shows that G_k is indeed a projector on the desired subspace.

The integral formula (G.5) is very useful for a perturbation expansion. To this end, we substitute (G.1) into (G.5) and compute the inverse with the Neumann series,

$$\begin{aligned} G_k &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} (z - A_0 - \Delta A)^{-1}\, dz \\ &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \left(\mathbb{1} - (z-A_0)^{-1}\,\Delta A\right)^{-1}\, (z-A_0)^{-1}\, dz \\ &= \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \sum_{n=0}^{\infty} \left((z-A_0)^{-1}\,\Delta A\right)^n\, (z-A_0)^{-1}\, dz\,. \end{aligned}$$

Interchanging the integral with the infinite sum gives the perturbation expansion,

$$G_k \;=\; \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|z-\lambda_k|=\varepsilon} \left((z-A_0)^{-1}\,\Delta A\right)^n\, (z-A_0)^{-1}\, dz\,, \tag{G.6}$$

where n is the order in perturbation theory. After substituting in the spectral representation for $(z-A_0)^{-1}$,

$$(z-A_0)^{-1} \;=\; \sum_{l=1}^{K} \frac{F_l}{z-\lambda_l}\,, \tag{G.7}$$

the contour integral in (G.6) can be carried out with residues. For example, we obtain to second order,

$$\begin{aligned} G_k \;=\; & F_k \;+\; \sum_{l\neq k} \frac{1}{\lambda_k - \lambda_l} \left(F_k\,\Delta A\,F_l \;+\; F_l\,\Delta A\,F_k\right) \;+\; \mathcal{O}((\Delta A)^3) \\ & + \sum_{l,m\neq k} \frac{1}{(\lambda_k-\lambda_l)(\lambda_k-\lambda_m)} \\ & \qquad \times \left(F_k\,\Delta A\,F_l\,\Delta A\,F_m + F_l\,\Delta A\,F_k\,\Delta A\,F_m + F_l\,\Delta A\,F_m\,\Delta A\,F_k\right) \\ & - \sum_{l\neq k} \frac{1}{(\lambda_k-\lambda_l)^2} \\ & \qquad \times \left(F_k\,\Delta A\,F_k\,\Delta A\,F_l + F_k\,\Delta A\,F_l\,\Delta A\,F_k + F_l\,\Delta A\,F_k\,\Delta A\,F_k\right). \end{aligned} \tag{G.8}$$

To order $n > 2$, the corresponding formulas are clearly more complicated, but even then they involve matrix products which are all of the form

$$F_{k_1}\,\Delta A\,F_{k_2}\,\Delta A\,\cdots\,F_{k_n}\,\Delta A\,F_{k_{n+1}}\,. \tag{G.9}$$

Substituting in (G.4) and expanding, we can just as well consider matrix products of the form (G.9) with the factors F_k replaced by F_{ncs}. Furthermore, for the computation

of the eigenvalues we need to take the expectation values of G_k with certain matrix elements of ΔA. This leads us to traces of matrix products of the form

$$\mathrm{Tr}\,(F_{n_1 c_1 s_1}\,\Delta A_1\,F_{n_2 c_2 s_2}\,\Delta A_2\,\cdots\,F_{n_l c_l s_l}\,\Delta A_l) \tag{G.10}$$

with $l = n+1$. We refer to a trace of the form (G.10) as a *matrix trace*. Our first task is to develop an efficient method for computing matrix traces (§G.2 and §G.3); after that we will proceed with the calculation of the eigenvalues of A (§G.4).

G.2. Factorization of Matrix Traces

If one attempts to calculate a matrix trace (G.10) directly by substituting in the formulas of the light-cone expansion, the resulting expressions become so complicated and involve so many Dirac matrices that they are almost impossible to handle. We shall now simplify the situation by giving a procedure which allows us to factor matrix traces into a product of so-called elementary traces, which are much easier to compute[1]. According to (G.3), we can assume that each factor ΔA_j in (G.10) is the product of a contribution to $P(x,y)$ with a contribution to $P(y,x)$. Denoting the contributions to $P(x,y)$ by B_j and using that the corresponding contributions to $P(y,x)$ are obtained by taking the adjoint with respect to the spin scalar product, we can write each ΔA_j in the form

$$\Delta A_j \;=\; B_{j_1}\,B_{j_2}^* \,.$$

Inserting the completeness relation

$$\sum_{ncs} F_{ncs} \;=\; \mathbb{1}$$

and expanding gives for (G.10) a sum of terms of the form

$$\mathrm{Tr}\left(F_{n_1 c_1 s_1}\,B_1\,F_{n_2 c_2 s_2}\,B_2^*\,\cdots\,F_{n_{k-1} c_{k-1} s_{k-1}}\,B_{k-1}\,F_{n_k c_k s_k}\,B_k^*\right) \tag{G.11}$$

with indices (n_j, s_j, c_j) (which are in general different from those in (G.10)) and $k = 2l$.

In order to handle the sector indices in (G.11), we introduce operators K_{n_1,n_2} which act on the sector index and map sector n_2 to sector n_1, i.e. in components

$$(K_{n_1 n_2})^n_{n'} \;=\; \delta^n_{n_1}\,\delta_{n' n_2}\,. \tag{G.12}$$

Then

$$F_{ncs} \;=\; K_{n1}\,F_{1cs}\,K_{1n}\,. \tag{G.13}$$

If we substitute this relation into (G.11) and combine the operators $K_\cdot$ and B_j to "new" operators B_j, we obtain a matrix trace again of the form (G.11), but with all indices n_j equal to one. Therefore in what follows we can restrict attention to the case of one sector and omit the sector indices. The generalization to several sectors will be straightforward by inserting operators $K_\cdot$ into the end formulas.

We choose a space-like unit vector u which is orthogonal to ξ and $\overline{\xi}$. Then the imaginary vector $v = iu$ satisfies the relations

$$v_j\,\xi^j \;=\; 0 \;=\; v_j\,\overline{\xi^j}\,, \qquad v^2 \;=\; 1\,, \qquad \overline{v} \;=\; -v\,. \tag{G.14}$$

An explicit calculation using (6.1.10) yields that

$$F_{R+} \;=\; \not{v}\,F_{L+}\,\not{v}\,, \qquad F_{L-} \;=\; \frac{1}{z}\,\not{\xi}\,\not{v}\,F_{L+}\,\not{v}\,\not{\xi}\,, \qquad F_{R-} \;=\; \frac{1}{z}\,\not{\xi}\,F_{L+}\,\not{\xi}\,. \tag{G.15}$$

[1] Online version: This factorization method is obtained in a somewhat easier way by computing the matrix elements in a double null spinor frame as explained in [**5**, §2.6.2] (see the references in the preface to the second online edition).

Substituting these formulas into (G.11), we obtain an expression involving only the spectral projector F_{L+}, namely

$$(G.11) \;=\; \mathrm{Tr}\,(F_{L+}\, C_1\, F_{L+}\, C_2 \cdots F_{L+}\, C_k) \tag{G.16}$$

with suitable matrices C_j. Since the F_{L+} are projectors on one-dimensional subspaces,

$$F_{L+}\, C\, F_{L+} \;=\; \mathrm{Tr}(F_{L+}\, C)\, F_{L+}\,.$$

By iteratively applying this relation in (G.16), we get the product of traces

$$\mathrm{Tr}\,(F_{L+}\, C_1)\;\mathrm{Tr}\,(F_{L+}\, C_2)\;\cdots\;\mathrm{Tr}\,(F_{L+}\, C_k)\;.$$

If we express the matrices C_j explicitly in terms of B_j and B_j^*, we obtain the following factorization formula,

$$\begin{aligned}&\mathrm{Tr}\,(F_{c_1 s_1}\, B_1\, F_{c_2 s_2}\, F_2^*\;\cdots\; B_{k-1}\, F_{c_k s_k}\, B_k^*)\\ &\quad=\;\; F_{s_1 s_2}^{c_1 c_2}(B_1)\, F_{s_2 s_3}^{c_2 c_3}(B_2^*)\;\cdots\; F_{s_{k-1} s_k}^{c_{k-1} c_k}(B_{k-1})\, F_{s_k s_1}^{c_k c_1}(B_k^*)\,,\end{aligned} \tag{G.17}$$

where $F_{s_i s_j}^{c_i c_j}$ are the so-called *elementary traces* defined by

$$\left.\begin{aligned}
F_{++}^{LL}(B) &= \mathrm{Tr}(F_+\, \chi_L\, B) \qquad, & F_{++}^{LR}(B) &= \mathrm{Tr}(F_+ \not{\psi}\, \chi_L\, B)\\
F_{+-}^{LL}(B) &= \mathrm{Tr}(\not{\xi}\, F_+ \not{\psi}\, \chi_L\, B) \quad, & F_{+-}^{LR}(B) &= \mathrm{Tr}(\not{\xi}\, F_+\, \chi_L\, B)\\
F_{-+}^{LL}(B) &= \frac{1}{z}\,\mathrm{Tr}(F_+ \not{\psi}\, \not{\xi} \chi_L\, B)\,, & F_{-+}^{LR}(B) &= \frac{1}{z}\,\mathrm{Tr}(F_+\, \not{\xi}\, \chi_L\, B)\\
F_{--}^{LL}(B) &= \frac{1}{z}\,\mathrm{Tr}(\not{\xi}\, F_+\, \not{\xi}\, \chi_L\, B)\,, & F_{--}^{LR}(B) &= \frac{1}{z}\,\mathrm{Tr}(\not{\xi}\, F_+ \not{\psi}\, \not{\xi}\, \chi_L\, B)\,.
\end{aligned}\right\} \tag{G.18}$$

These formulas are also valid for the opposite chirality after the replacements $L \leftrightarrow R$. The elementary traces of B^* are obtained by taking the complex conjugate,

$$\left.\begin{aligned}
F_{++}^{LL}(B^*) &= \overline{F_{--}^{RR}(B)} \quad, & F_{++}^{LR}(B^*) &= \overline{F_{--}^{LR}(B)}\\
F_{+-}^{LL}(B^*) &= \overline{F_{+-}^{RR}(B)} \quad, & F_{+-}^{LR}(B^*) &= \overline{F_{+-}^{LR}(B)}\\
F_{-+}^{RR}(B^*) &= \overline{F_{-+}^{LL}(B)} \quad, & F_{-+}^{LR}(B^*) &= \overline{F_{-+}^{LR}(B)}\\
F_{--}^{LL}(B^*) &= \overline{F_{++}^{RR}(B)} \quad, & F_{--}^{LR}(B^*) &= \overline{F_{++}^{LR}(B)}\,.
\end{aligned}\right\} \tag{G.19}$$

The relations (G.17–G.19) are verified by a straightforward calculation using (6.1.10, 6.1.8, G.14).

To summarize, the above procedure reduces the calculation of the matrix trace (G.10) to the computation of the elementary traces (G.18) for the contributions B to the light-cone expansion of $P(x,y)$. Taking the complex conjugate (G.19), one obtains the elementary traces of the corresponding contributions to $P(y,x)$. By applying (G.17) and, in the case of several sectors, by suitably inserting the operators $K_\cdot$, every matrix trace can be written as a linear combination of products of elementary traces.

G.3. Calculation of the Matrix Traces

We decompose $\Delta P(x,y)$ into its odd and even parts, denoted by B_o and B_e,

$$\Delta P(x,y) \;=\; B_o(x,y) \;+\; B_e(x,y)\,.$$

Explicit formulas for the fermionic projector in the presence of chiral and scalar potentials are listed in Appendix B. For the purpose of this paper, only the contributions involving the mass matrices $Y_{L/R}$ and their derivatives are of importance. But for

completeness and for later use, we will also compute the contributions which contain the chiral field strength and the chiral currents. However, we will omit all contributions quadratic in the field strength. Namely, these contributions are related to the energy-momentum tensor of the chiral fields, and it is therefore reasonable to postpone their analysis until gravity is also taken into consideration. Thus the phase-free contributions relevant here are

$$\begin{aligned}
\chi_L\, B_e &= \frac{1}{2}\,\chi_L\, m\, T^{(0)}(x,y)\, \not\xi \int_x^y dz\, \gamma^j\, (D_j Y_L) \\
&\quad + \chi_L\, m\, T^{(0)}(x,y)\, Y_L(x) \;+\; \mathcal{O}(\log|\xi^2|\,\xi^0) \\
\chi_L\, B_o &= \frac{i}{2}\,\chi_L\, m^2\, T^{(0)}(x,y)\, \not\xi \int_x^y dz\, Y_L\, Y_R \\
&\quad + i\chi_L\, m^2\, T^{(1)}(x,y) \int_x^y dz\, [0,1\,|\,0]\, Y_L\, \gamma^j (D_j Y_R) \\
&\quad + i\chi_L\, m^2\, T^{(1)}(x,y) \int_x^y dz\, [0,1\,|\,0]\, \gamma^j (D_j Y_L)\, Y_R \\
&\quad - i\chi_L\, m^2\, T^{(1)}(x,y)\, Y_L(x) \int_x^y dz\, \gamma^j (D_j Y_R) \\
&\quad + \chi_L\, T^{(0)}(x,y)\, \xi^i \int_x^y dz\, [0,1\,|\,0]\, \gamma^l F^L_{li} \\
&\quad + \frac{1}{4}\,\chi_L\, T^{(0)}(x,y)\, \not\xi \int_x^y dz\, \gamma^j \gamma^k\, F^L_{jk} \\
&\quad - \frac{1}{2}\,\chi_L\, T^{(0)}(x,y)\, \not\xi\, \xi^i \int_x^y dz\, [0,0\,|\,1]\, j^L_i \\
&\quad - \chi_L\, T^{(1)}(x,y)\, \xi^i \int_x^y dz\, [0,1\,|\,1]\, (\not\partial j^L_i) \\
&\quad - \chi_L\, T^{(1)}(x,y) \int_x^y dz\, [0,2\,|\,0]\, j^L_k\, \gamma^k \\
&\quad + \not\xi\, \mathcal{O}(\xi^{-2}) \;+\; \gamma^j\, F^L_{jk} \xi^k\, \mathcal{O}(\xi^{-2}) \;+\; \mathcal{O}(F_L^2) \;+\; \mathcal{O}(\log|\xi^2|\,\xi^0)\,.
\end{aligned}$$

A straightforward calculation yields for the elementary traces

$$F^{LR}_{+-}(P_0) \;=\; (\deg \le 1) \;=\; \frac{i}{2}\, X_L\, (z\, T^{(-1)}_{[0]}) \tag{G.20}$$

$$F^{LR}_{-+}(P_0) \;=\; (\deg \le 2) \;=\; \frac{i}{2}\, X_L\, T^{(-1)}_{[0]} \tag{G.21}$$

$$F^{LL}_{++}(B_e) \;=\; (\deg \le 1) \;=\; Y_L(x)\, T^{(0)}_{[1]} \;+\; (\deg < 1) \tag{G.22}$$

$$F^{LL}_{+-}(B_e) \;=\; (\deg \le 0) \tag{G.23}$$

$$F^{LL}_{-+}(B_e) \;=\; (\deg \le 1) \tag{G.24}$$

$$F^{LL}_{--}(B_e) \;=\; (\deg \le 1) \;=\; Y_L(y)\, T^{(0)}_{[1]} \;+\; (\deg < 1) \tag{G.25}$$

$$F^{LR}_{++}(B_o) \;=\; (\deg \le 1) \tag{G.26}$$

$$\;=\; v^j \xi^k \int_x^y dz\, [0,1\,|\,0]\, F^L_{jk}\, T^{(0)}_{[0]} \;+\; (\deg < 1) \tag{G.27}$$

$$+\frac{2i}{z-\overline{z}}\,\epsilon_{ijkl}\,\xi^i\,\overline{\xi}^j\,v^k\int_x^y dz\,[0,1\,|\,0]\,F_L^{lm}\,(\xi_m\,T_{[0]}^{(0)}) \tag{G.28}$$

$$+\frac{i}{z-\overline{z}}\,\epsilon^{ijkl}\,(\xi_i\overline{\xi}_j+\overline{\xi}_i\xi_j^{(0)}-\xi_i\xi_j^{(0)})\,v_k\int_x^y dz\,\xi^n F_{nl}^L\,T_{[0]}^{(0)} \tag{G.29}$$

$$\begin{aligned} F_{+-}^{LR}(B_o) &= (\deg\le 0)\\ &= \frac{i}{2}\int_x^y dz\,Y_L\,Y_R\,((z\,T_{[2]}^{(0)})\,+\,4\,T_{[2]}^{(1)})\,+\,(\deg<0) && \text{(G.30)}\\ &\quad -2i\,Y_L(x)\,Y_R(y)\,T_{[2]}^{(1)} && \text{(G.31)}\\ &\quad -\frac{1}{2}\,\xi_i\int_x^y dz\,[0,0\,|\,1]\,j_L^i\,((z\,T_{[0]}^{(0)})\,+\,8\,T_{[0]}^{(1)}) && \text{(G.32)}\\ &\quad +\frac{i}{2}\,\epsilon_{ijkl}\,\frac{z\,\overline{\xi}^i-\overline{z}\,\xi^i}{z-\overline{z}}\int_x^y F_L^{jk}\,(\xi^l\,T_{[0]}^{(0)}) && \text{(G.33)} \end{aligned}$$

$$\begin{aligned} F_{-+}^{LR}(B_o) &= (\deg\le 1)\\ &= \frac{i}{2}\int_x^y dz\,Y_L\,Y_R\,T_{[2]}^{(0)}\,+\,(\deg<1) && \text{(G.34)}\\ &\quad -\frac{1}{2}\,\xi_i\int_x^y dz\,[0,0\,|\,1]\,j_L^i\,T_{[0]}^{(0)} && \text{(G.35)}\\ &\quad -\frac{i}{2}\,\epsilon_{ijkl}\,\frac{\overline{\xi}^i-\xi^i}{z-\overline{z}}\int_x^y F_L^{jk}\,(\xi^l\,T_{[0]}^{(0)}) && \text{(G.36)} \end{aligned}$$

$$\begin{aligned} F_{--}^{LR}(B_o) &= (\deg\le 1)\\ &= v^j\xi^k\int_x^y dz\,[1,0\,|\,0]\,F_{jk}^L\,T_{[0]}^{(0)}\,+\,(\deg<1) && \text{(G.37)}\\ &\quad +\frac{i}{2}\,\epsilon^{ijkl}\,\xi_i\,v_j\int_x^y F_{kl}^L\,T_{[0]}^{(0)} && \text{(G.38)}\\ &\quad +\frac{2i}{z-\overline{z}}\,\epsilon_{ijkl}\,\xi^i\,\overline{\xi}^j\,v^k\int_x^y dz\,[0,1\,|\,0]\,F_L^{lm}\,(\xi_m\,T_{[0]}^{(0)}) && \text{(G.39)}\\ &\quad +\frac{i}{z-\overline{z}}\,\epsilon^{ijkl}\,(\xi_i\overline{\xi}_j+\overline{\xi}_i\xi_j^{(0)}-\xi_i\xi_j^{(0)})\,v_k\int_x^y dz\,\xi^n F_{nl}\,T_{[0]}^{(0)}\,. && \text{(G.40)} \end{aligned}$$

Here the totally anti-symmetric symbol ϵ_{ijkl} appears because we applied the identity

$$\mathrm{Tr}(\chi_{L/R}\,\not{a}\,\not{b}\,\not{c}\,\not{d})\;=\;2\,((ab)(cd)+(da)(bc)-(ac)(bd))\;\mp\;2i\,\epsilon_{ijkl}\,a^ib^jc^kd^l\,.$$

Therefore, the corresponding formulas for the opposite chirality are now obtained by the replacements

$$L\;\longleftrightarrow\;R\,,\qquad\qquad \epsilon_{ijkl}\;\longrightarrow\;-\,\epsilon_{ijkl}\,. \tag{G.41}$$

The elementary traces of the adjoints are computed via (G.19). All other elementary traces vanish.

Applying (G.17) and the degree estimates for the elementary traces and omitting all terms quadratic in the field strength, we can factor and estimate the following

matrix traces,

$$\begin{aligned}
\mathrm{Tr}(F_{L+}\,\Delta A) &= F^{LR}_{+-}(P_0)\,F^{RL}_{-+}(B_o^*) \;+\; F^{LR}_{+-}(B_o)\,F^{RL}_{-+}(P_0^*) \\
&\quad + F^{LL}_{++}(B_e)\,F^{LL}_{++}(B_e^*) \;+\; (\deg < 2) && \text{(G.42)}\\
\mathrm{Tr}(F_{L-}\,\Delta A) &= F^{LR}_{-+}(P_0)\,F^{RL}_{+-}(B_o^*) \;+\; F^{LR}_{-+}(B_o)\,F^{RL}_{+-}(P_0^*) \\
&\quad + F^{LL}_{--}(B_e)\,F^{LL}_{--}(B_e^*) \;+\; (\deg < 2) && \text{(G.43)}\\
\mathrm{Tr}(F_{Ls}\,\Delta A\,F_{Ls}\,\Delta A) &= \mathrm{Tr}(F_{Ls}\,\Delta A)\,\mathrm{Tr}(F_{Ls}\,\Delta A) \;=\; (\deg < 5) && \text{(G.44)}\\
\mathrm{Tr}(F_{Ls}\,\Delta A\,F_{Rs}\,\Delta A) &= (\deg < 5) && \text{(G.45)}\\
\mathrm{Tr}(F_{L+}\,\Delta A\,F_{R-}\,\Delta A) &= (F^{LL}_{++}(B_e)\,F^{LR}_{+-}(P_0^*) \;+\; F^{LR}_{+-}(P_0)\,F^{RR}_{--}(B_e^*)) \\
&\quad \times(F^{RR}_{--}(B_e)\,F^{RL}_{-+}(P_0^*) \;+\; F^{RL}_{-+}(P_0)\,F^{LL}_{++}(B_e^*)) + (\deg < 5) && \text{(G.46)}\\
\mathrm{Tr}(F_{L-}\,\Delta A\,F_{R+}\,\Delta A) &= (F^{LL}_{--}(B_e)\,F^{LR}_{-+}(P_0^*) \;+\; F^{LR}_{-+}(P_0)\,F^{RR}_{++}(B_e^*)) \\
&\quad \times(F^{RR}_{++}(B_e)\,F^{RL}_{+-}(P_0^*) \;+\; F^{RL}_{+-}(P_0)\,F^{LL}_{--}(B_e^*)) + (\deg < 5) && \text{(G.47)}\\
\mathrm{Tr}(F_{L+}\,\Delta A\,F_{L-}\,\Delta A) &= 0 \;=\; \mathrm{Tr}(F_{L-}\,\Delta A\,F_{L+}\,\Delta A)\,. && \text{(G.48)}
\end{aligned}$$

If we consider more generally the matrix trace of order l, factorization gives a linear combination of products of elementary traces as in (G.17) (with $k = 2l$). Let us estimate the degree of each of these products. Clearly, the number of factors $F^{\cdot}_{+-}$ equals the number of factors $F^{\cdot}_{-+}$, we denote the number of such pairs by p. Furthermore, let q be the number of factors $F^{\cdot}_{\cdot}(\Delta P^{\cdot})$ (where $\Delta P^{\cdot}$ stands for either ΔP or ΔP^*). According to (G.3), each ΔA contains at least one factor $\Delta P^{\cdot}$, hence $q \geq l$. The number of factors $F^{\cdot}_{++}$ and $F^{\cdot}_{--}$ is $2(l-p)$, and we saw above that each of these factors must involve $\Delta P^{\cdot}$, thus $q \geq 2(l-p)$. Adding our two upper bounds for q gives the inequality $q+p \geq 3l/2$. To estimate the degrees we first note that the degree of the pair $F^{\cdot}_{+-}(P^{\cdot}_0)\,F^{\cdot}_{-+}(P^{\cdot}_0)$ is three, and is decreased at least by one each time a $P^{\cdot}_0$ is replaced by $\Delta P^{\cdot}$. The total number of factors $F^{\cdot}_{+-}(\Delta P^{\cdot})$ and $F^{\cdot}_{-+}(\Delta P^{\cdot})$ is $q - 2(l-p)$. On the other hand, the degree of each factor $F^{\cdot}_{++}$ and $F^{\cdot}_{--}$ is at most one. Hence the degree of the matrix is bounded from below by $3p - (q - 2(l-p)) + 2(l-p) = 4l - (q+p)$. Substituting in our above lower bound for $q+p$ gives the degree estimate

$$\mathrm{Tr}(F_{c_1 s_1}\,\Delta A \,\cdots\, F_{c_l s_l}\,\Delta A) \;=\; \left(\deg < \frac{5}{2}\,l\right) \;=\; (\deg < 3l-1) \quad \text{for } l \geq 3. \qquad \text{(G.49)}$$

The above formulas are valid in the case $N = 1$ of one sector. The generalization to several sectors is done by inserting suitable operators $K_{\cdot}$ into the traces. This has no effect on the degree on the light cone, and thus the estimates of the matrix traces in (G.42–G.49) hold in the general case as well. We substitute the above results for the elementary traces (G.20–G.40) into (G.42–G.47) and insert the operators $K_{\cdot}$ to obtain the following explicit formulas:

$$\begin{aligned}
&\mathrm{Tr}(F_{nL+}\,\Delta A) \;=\; (\deg < 2) \\
&\quad +\mathrm{Tr}_S\left\{I_n\,\hat{Y}_L(x)\,\hat{Y}_L(y)\right\} T^{(0)}_{[1]}\,\overline{T^{(0)}_{[1]}} && \text{(G.50)}\\
&\quad +\frac{1}{4}\int_x^y dz\,\mathrm{Tr}_S\left\{I_n\,\acute{Y}_L\,\grave{Y}_R\,X_R\right\}((z\,T^{(0)}_{[2]}) \;+\; 4\,T^{(1)}_{[2]})\,\overline{T^{(-1)}_{[0]}} && \text{(G.51)}\\
&\quad -\mathrm{Tr}_S\left\{I_n\,\acute{Y}_L(x)\,\grave{Y}_R(y)\,X_R\right\} T^{(1)}_{[2]}\,\overline{T^{(-1)}_{[0]}} && \text{(G.52)}\\
&\quad +\frac{1}{4}\int_y^x dz\,\mathrm{Tr}_S\left\{I_n\,X_L\,\acute{Y}_R\,\grave{Y}_L\right\}(z\,T^{(-1)}_{[0]})\,\overline{T^{(0)}_{[2]}} && \text{(G.53)}
\end{aligned}$$

$$+\frac{i}{4}\,\xi_i\,\int_x^y dz\,[0,0\,|\,1]\,\mathrm{Tr}_S\left\{I_n\,j_L^i\,X_R\right\}((z\,T_{[0]}^{(0)})\,+\,8\,T_{[0]}^{(1)})\,\overline{T_{[0]}^{(-1)}} \tag{G.54}$$

$$-\frac{i}{4}\,\xi_i\int_y^x dz\,[0,0\,|\,1]\,\mathrm{Tr}_S\left\{I_n\,X_L\,j_R^i\right\}(z\,T_{[0]}^{(-1)})\,\overline{T_{[0]}^{(0)}} \tag{G.55}$$

$$+\frac{1}{4}\,\epsilon_{ijkl}\,\frac{z\,\overline{\xi}^i-\overline{z}\,\xi^i}{z-\overline{z}}\,\xi^l\,\int_x^y \mathrm{Tr}_S\left\{I_n\,F_L^{jk}\,X_R\right\}\,T_{[0]}^{(0)}\,\overline{T_{[0]}^{(-1)}} \tag{G.56}$$

$$+\frac{1}{4}\,\epsilon_{ijkl}\,\frac{\overline{\xi}^i-\xi^i}{z-\overline{z}}\,\xi_l\,\int_y^x \mathrm{Tr}_x\left\{I_n\,X_L\,F_R^{jk}\right\}\,(z\,T_{[0]}^{(-1)})\,(\overline{T_{[0]}^{(0)}}) \tag{G.57}$$

$$\mathrm{Tr}(F_{nL-}\,\Delta A)\;=\;(\deg<2)$$

$$+\mathrm{Tr}_S\left\{I_n\,\hat{Y}_L(y)\,\hat{Y}_L(x)\right\}T_{[1]}^{(0)}\,\overline{T_{[1]}^{(0)}} \tag{G.58}$$

$$+\frac{1}{4}\,\int_x^y dz\,\mathrm{Tr}_S\left\{I_n\,\acute{Y}_L\,\grave{Y}_R\,X_R\right\}T_{[2]}^{(0)}\,(\overline{z\,T_{[0]}^{(-1)}}) \tag{G.59}$$

$$+\frac{1}{4}\,\int_y^x dz\,\mathrm{Tr}_S\left\{I_n\,X_L\,\acute{Y}_R\,\grave{Y}_L\right\}T_{[0]}^{(-1)}\,((\overline{z\,T_{[2]}^{(0)}})\,+\,4\,\overline{T_{[2]}^{(1)}}) \tag{G.60}$$

$$-\mathrm{Tr}_S\left\{I_n\,X_L\,\acute{Y}_R(y)\,\grave{Y}_L(x)\right\}T_{[0]}^{(-1)}\,\overline{T_{[2]}^{(1)}} \tag{G.61}$$

$$-\frac{i}{4}\,\xi_i\int_y^x dz\,[0,0\,|\,1]\,\mathrm{Tr}_S\left\{I_n\,X_L\,j_R^i\right\}T_{[0]}^{(-1)}\,((\overline{z\,T_{[0]}^{(0)}})\,+\,8\,\overline{T_{[0]}^{(1)}}) \tag{G.62}$$

$$+\frac{i}{4}\,\xi_i\,\int_x^y dz\,[0,0\,|\,1]\,\mathrm{Tr}_S\left\{I_n\,j_L^i\,X_R\right\}T_{[0]}^{(0)}\,(\overline{z\,T_{[0]}^{(-1)}}) \tag{G.63}$$

$$-\frac{1}{4}\,\epsilon_{ijkl}\,\frac{\overline{\xi}^i-\xi^i}{z-\overline{z}}\,\xi^l\,\int_x^y \mathrm{Tr}_S\left\{I_n\,F_L^{jk}\,X_R\right\}\,T_{[0]}^{(0)}\,(\overline{z\,T_{[0]}^{(-1)}}) \tag{G.64}$$

$$-\frac{1}{4}\,\epsilon_{ijkl}\,\frac{z\,\overline{\xi}^i-\overline{z}\,\xi^i}{z-\overline{z}}\,\xi^l\int_y^x \mathrm{Tr}_S\left\{I_n\,X_L\,F_R^{jk}\right\}\,T_{[0]}^{(-1)}\,\overline{T_{[0]}^{(0)}} \tag{G.65}$$

$$\mathrm{Tr}(F_{nL+}\,\Delta A\,F_{n'R-}\,\Delta A)\;=\;(\deg<5)$$

$$-\frac{1}{4}\,\mathrm{Tr}_S\left\{I_n\left(\hat{Y}_L(x)\,X_L\,T_{[1]}^{(0)}\,(\overline{z\,T_{[0]}^{(-1)}})\,-\,X_L\,\hat{Y}_R(x)(z\,T_{[0]}^{(-1)})\,\overline{T_{[1]}^{(0)}}\right)\right.$$
$$\left.\times\,I_{n'}\left(\hat{Y}_R(y)\,X_R\,T_{[1]}^{(0)}\,\overline{T_{[0]}^{(-1)}}\,-\,X_R\,\hat{Y}_L(y)\,T_{[0]}^{(-1)}\,\overline{T_{[1]}^{(0)}}\right)\right\} \tag{G.66}$$

$$\mathrm{Tr}(F_{nL-}\,\Delta A\,F_{n'R+}\,\Delta A)\;=\;(\deg<5)$$

$$-\frac{1}{4}\,\mathrm{Tr}_S\left\{I_n\left(\hat{Y}_L(y)\,X_L\,T_{[1]}^{(0)}\,\overline{T_{[0]}^{(-1)}}\,-\,X_L\,\hat{Y}_R(y)\,T_{[0]}^{(-1)}\,\overline{T_{[1]}^{(0)}}\right)\right.$$
$$\left.\times\,I_{n'}\left(\hat{Y}_R(x)\,X_R\,T_{[1]}^{(0)}\,(\overline{z\,T_{[0]}^{(-1)}})\,-\,X_R\,\hat{Y}_L(x)\,(z\,T_{[0]}^{(-1)})\,\overline{T_{[1]}^{(0)}}\right)\right\} \tag{G.67}$$

G.4. Perturbation of the Non-Zero Eigenvalues

In §7 we calculated the eigenvalues λ_{ncs} of A in the presence of chiral and scalar potentials to the leading degree 3, (6.1.10). Now we shall compute the contributions to the non-zero eigenvalues of degree two, denoted by $\Delta\lambda_{ncs}$, $n=1,\ldots,7$ (the kernel of A will be considered in §G.5). To this end, we need to analyze the matrix A on the invariant subspaces $\mathrm{Im}G_k$. First, we choose for fixed $k>1$ a convenient basis of $\mathrm{Im}G_k$

as follows. The degeneracy of the unperturbed eigenspace $\mathrm{Im}F_k$ can be described by the index set I,

$$I \;=\; \{(ncs) \text{ with } \lambda_{ncs} = \lambda_k\}\,. \tag{G.68}$$

Note that, according to (6.1.10), s is the same for all elements $(ncs) \in I$, provided that the eigenvalue is non-zero. The index c, however, may take both values L and R, giving rise to the partition of I into I_L and I_R,

$$I_{L/R} \;=\; \{(ncs) \in I \text{ with } c = L/R\}\,.$$

The set I can be used to index a basis of F_k; namely we choose

$$(\phi_{ncs})_{(ncs)\in I} \qquad \text{with} \qquad 0 \neq \phi_{ncs} \;\in\; \mathrm{Im}\, F_{ncs}\,. \tag{G.69}$$

It is convenient to assume that the basis vectors are related to each other by

$$\phi_{n'cs} \;=\; K_{n'n}\, \phi_{ncs}\,, \qquad \phi_{n'\bar{c}s} \;=\; K_{n'n}\, \not{\psi}\, \phi_{ncs}\,; \tag{G.70}$$

this can clearly arranged according to (G.13–G.15). Since F_k projects onto a null space, the inner product of any two basis vectors ϕ_{ncs} vanishes. Thus in order to be able to evaluate vectors in $\mathrm{Im}F_k$ using the scalar product, we choose a "dual basis" $(\phi^{ncs})_{(ncs)\in I}$ of $\mathrm{Im}F_k^*$ given by

$$\phi^{ncs} \;\in\; \mathrm{Im}\, F^*_{ncs}\,, \qquad \phi^{n'cs} \;=\; K_{n'n}\, \phi^{ncs}\,, \qquad \phi^{n'\bar{c}s} \;=\; K_{n'n}\, \not{\psi}\, \phi^{ncs}\,. \tag{G.71}$$

The basis vectors and their duals are orthogonal in the sense that for $(ncs) \neq (n'c's)$,

$$<\phi^{ncs} \,|\, \phi_{n'c's}> \;=\; <F^*_{ncs}\, \phi^{ncs} \,|\, F_{n'c's}\, \phi_{n'c's}> \;=\; <\phi^{ncs} \,|\, F_{ncs}\, F_{n'c's}\, \phi_{n'c's}> \;=\; 0\,.$$

We normalize the basis vectors such that

$$<\phi^{ncs} \,|\, \phi_{n'c's}> \;=\; \delta^n_{n'}\, \delta^c_{c'} \qquad \text{for all } (ncs), (n'c's) \in I. \tag{G.72}$$

Next we introduce a basis $(\psi_{ncs})_{(ncs)\in I}$ of the invariant subspace $\mathrm{Im}G_k$ by applying the projector P_k to the ϕ_{ncs},

$$\psi_{ncs} \;=\; G_k\, \phi_{ncs}\,. \tag{G.73}$$

Finally, we introduce a basis $(\psi^{ncs})_{(ncs)\in I}$ which is dual to (ψ_{ncs}). We must be careful because projecting on $\mathrm{Im}(G_k)$ and $\mathrm{Im}(G_k)$, respectively, does not preserve the orthonormality; more precisely,

$$\begin{aligned} S^{ncs}_{n'c's} \;&\equiv\; < G^*_k\, \phi^{ncs} \,|\, \psi_{n'c's}> \;=\; < G^*_k\, \phi^{ncs} \,|\, G^k\, \phi_{n'c's}> \\ &=\; < \phi^{ncs} \,|\, G^k \,|\, \phi_{n'c's}> \overset{\text{in general}}{\neq} \delta^n_{n'}\, \delta^c_{c'}\,. \end{aligned} \tag{G.74}$$

But S is a perturbation of the identity, and thus it can be inverted within the perturbation expansion by a Neumann series. This makes it possible to introduce $(\psi^{ncs})_{(ncs)\in I}$ by

$$\psi^{ncs} \;=\; \sum_{(n'c's)\in I} (S^{-1})^{ncs}_{n'c's}\, G^*_k\, \phi^{n'c's}\,. \tag{G.75}$$

A short calculation shows that this basis of $\mathrm{Im}G^*_k$ is indeed dual to (ψ_{ncs}) in the sense that

$$<\psi^{ncs} \,|\, \psi_{n'c's}> \;=\; \delta^n_{n'}\, \delta^c_{c'} \qquad \text{for all } (ncs), (n'c's) \in I. \tag{G.76}$$

Using the basis (ψ_{ncs}) and its dual (ψ^{ncs}), we can write down matrix elements of A,

$$A^{ncs}_{n'c's} \;=\; <\psi^{ncs} \,|\, A \,|\, \psi_{n'c's}> \qquad \text{for } (ncs), (n'c's) \in I. \tag{G.77}$$

From the orthonormality (G.76) one sees that $A^{ncs}_{n'c's}$ is indeed a matrix representation for A in the basis (ψ_{ncs}), and thus the eigenvalues of A on the invariant subspace

$\mathrm{Im}G_k$ are obtained simply by diagonalizing this matrix. In the unperturbed case (i.e. if $\Delta A = 0$), the matrix $A^{ncs}_{n'c's}$ simplifies to

$$\begin{aligned} A^{ncs}_{n'c's} &= <\phi^{ncs} \,|\, A_0 \,|\, \phi_{n'c's}> \;=\; <\phi^{ncs} \,|\, A_0\, F_{n'c's}\, \phi_{n'c's}> \\ &= \lambda_k <\phi^{ncs} \,|\, \phi_{n'c's}> \;=\; \lambda_k\, \delta^n_{n'}\, \delta^c_{c'}\,, \end{aligned}$$

in agreement with the fact that $\mathrm{Im}F_k$ is an eigenspace of A_0 corresponding to the eigenvalue λ_k. Thus we see that the matrix elements $A^{ncs}_{n'c's}$ are to leading order on the light cone of degree 3. In the following theorem we compute the matrix elements up to contributions of degree < 2.

THEOREM G.4.1. *We consider the fermionic projector in the presence of chiral and scalar potentials (6.0.5) and in composite expressions disregard all terms quadratic in the field strength. Then for all $k = 2, \ldots, K$ and $(ncs), (n'c's) \in I$,*

$$\begin{aligned} A^{ncs}_{n'c's} &= \lambda_k\, \delta^n_{n'}\, \delta^c_{c'} \;+\; \delta^c_{c'}\, \mathrm{Tr}\,(F_{ncs}\, \Delta A\, K_{n'n}) \\ &\quad + \delta^c_{c'} \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l}\, \mathrm{Tr}\,(F_{ncs}\, \Delta A\, F_l\, \Delta A\, K_{n'n}) \;+\; (\deg < 2). \qquad \text{(G.78)} \end{aligned}$$

Proof. We begin by computing the matrix S, (G.74), and its inverse. This calculation will also illustrate how the relations (G.70) and (G.71) make it possible to rewrite expectation values as matrix traces and thus to apply the results of §G.2 and §G.3. In the case $c = c'$, we obtain from (G.74) and (G.70),

$$\begin{aligned} S^{ncs}_{n'cs} &= <\phi^{ncs} \,|\, G_k \,|\, \phi_{n'cs}> \;=\; <\phi^{ncs} \,|\, G_k \,|\, K_{n'n}\, \phi_{ncs}> \\ &\overset{(G.69,G.71)}{=} <F^*_{ncs}\, \phi^{ncs} \,|\, G_k\, K_{n'n} \,|\, F_{ncs}\, \phi_{ncs}> \\ &= <\phi^{ncs} \,|\, F_{ncs}\, G_k\, K_{n'n}\, F_{ncs} \,|\, \phi_{ncs}> \\ &\overset{(*)}{=} \mathrm{Tr}\,(F_{ncs}\, G_k\, K_{n'n})\; <\phi^{ncs} \,|\, F_{ncs} \,|\, \phi_{ncs}> \\ &= \mathrm{Tr}\,(F_{ncs}\, G_k\, K_{n'n})\; <\phi^{ncs} \,|\, \phi_{ncs}> \;\overset{(G.72)}{=}\; \mathrm{Tr}\,(F_{ncs}\, G_k\, K_{n'n})\,, \end{aligned}$$

where in (*) we used that F_{ncs} projects on a one-dimensional subspace. If we substitute the perturbation expansion for G_k, (G.6), into the obtained matrix trace, the estimate (G.49) shows that the orders $n > 2$ yield contributions to S of degree < -1. Thus it suffices to consider for G_k the second order expansion (G.8). This gives

$$S^{ncs}_{n'cs} \;=\; \delta^n_{n'} \;-\; \sum_{l \neq k} \frac{1}{(\lambda_k - \lambda_l)^2}\, \mathrm{Tr}\,(F_{ncs}\, \Delta A\, F_l\, \Delta A\, K_{n'n}) + (\deg < -1)\,. \qquad \text{(G.79)}$$

Note that of the matrix trace appearing here we need to take into account only the leading contributions of degree 5; these are easily obtained from (G.66) and (G.67). In the case $c \neq c'$, we obtain similarly

$$S^{ncs}_{n'\bar{c}s} \;=\; <\phi^{ncs} \,|\, G_k\, K_{n'n}\, \psi\!\!\!/\, \phi_{ncs}> \;=\; \mathrm{Tr}\,(F_{ncs}\, G_k\, K_{n'n}\, \psi\!\!\!/)\,.$$

We again substitute in the expansion for G_k (G.8). As a consequence of the additional factor $\psi\!\!\!/$, the contribution to zeroth order in ΔA now drops out. The first order contribution to $S^{ncs}_{n'\bar{c}s}$ is

$$\begin{aligned} &\sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l}\; (F_{ncs}\, \Delta A\, F_l\, K_{n'n}\, \psi\!\!\!/) \;=\; \frac{1}{\lambda_{ncs} - \lambda_{n'\bar{c}s}}\; (F_{ncs}\, \Delta A\, F_{n'\bar{c}s}\, K_{n'n}\, \psi\!\!\!/) \\ &= \frac{1}{\lambda_{ncs} - \lambda_{n'\bar{c}s}}\; (F_{ncs}\, \Delta A\, K_{n'n}\, \psi\!\!\!/) \;=\; (\deg < -1)\,, \end{aligned}$$

because according to (G.18) and (G.26) the last matrix trace has degree ≤ 1. Here we implicitly assumed that $\lambda_{ncs} \neq \lambda_{n'\bar{c}s}$, because otherwise we clearly get zero. A straightforward calculation using the factorization formula (G.17) as well as the estimates for the elementary traces following (G.20) shows that the second order contribution to $S^{ncs}_{n'\bar{c}s}$ also is of degree < -1. We conclude that

$$S^{ncs}_{n'\bar{c}s} \;=\; (\deg < -1)\,. \tag{G.80}$$

Now we can take the inverse of the expansions (G.79) and (G.80). This gives

$$(S^{-1})^{ncs}_{n'c's} \;=\; \delta^n_{n'}\delta^c_{c'} + \delta^c_{c'} \sum_{l \neq k} \frac{1}{(\lambda_k - \lambda_l)^2} \mathrm{Tr}\,(F_{ncs}\, \Delta A\, F_l\, \Delta A\, K_{n'n}) + (\deg < -1). \tag{G.81}$$

We next compute the expectation values

$$<\phi^{ncs} \,|\, A\, G_k \,|\, \phi_{n'c's}>$$

up to contributions of degree < 2. The method is the same as for the above calculation of the matrix S. In the case $c = c'$, we obtain the following matrix trace,

$$\begin{aligned} <\phi^{ncs} \,|\, A\, G_k \,|\, \phi_{n'cs}> \;&=\; <\phi^{ncs} \,|\, A\, G_k\, K_{n'n} \,|\, \phi_{ncs}> \\ &=\; <\phi^{ncs} \,|\, F_{ncs}\, A\, G_k\, K_{n'n}\, F_{ncs} \,|\, \phi_{ncs}> \;=\; \mathrm{Tr}\,(F_{ncs}\, A\, G_k\, K_{n'n})\,. \end{aligned}$$

Substituting in (G.1) and (G.6), the estimate (G.49) shows that it suffices to take into account G_k to second order (G.8). We get

$$\begin{aligned} <\phi^{ncs} \,|\, A\, G_k \,|\, \phi_{n'cs}> \;&=\; \lambda_k\, \delta^n_{n'}\, \delta^c_{c'} \;+\; \mathrm{Tr}\,(F_{ncs}\, \Delta A\, K_{n'n}) \\ &\quad + \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l}\, \mathrm{Tr}\,(F_{ncs}\, \Delta A\, F_l\, \Delta A\, K_{n'n}) \\ &\quad - \sum_{l \neq k} \frac{\lambda_k}{(\lambda_k - \lambda_l)^2}\, \mathrm{Tr}\,(F_{ncs}\, \Delta A\, F_l\, \Delta A\, K_{n'n}) \;+\; (\deg < 2). \end{aligned} \tag{G.82}$$

In the case $c \neq c'$, we can rewrite the expectation value as follows,

$$<\phi^{ncs} \,|\, A\, G_k \,|\, \phi_{n'\bar{c}s}> \;=\; <\phi^{ncs} \,|\, A\, G_k\, K_{n'n}\, \not{\psi} \,|\, \phi_{ncs}> \;=\; \mathrm{Tr}\,(F_{ncs}\, A\, G_k\, K_{n'n}\, \not{\psi})\,.$$

If we substitute in (G.1) and (G.8), factor the resulting matrix traces and use the estimates of the elementary traces of §G.3, we obtain that

$$<\phi^{ncs} \,|\, A\, G_k \,|\, \phi_{n'\bar{c}s}> \;=\; (\deg < 2)\,. \tag{G.83}$$

In order to bring the matrix elements (G.77) into a suitable form, we substitute the definitions (G.73) and (G.75) into (G.77) to obtain

$$\begin{aligned} A^{ncs}_{n'c's} \;&=\; \sum_{(\tilde{n}\tilde{c}s) \in I} (S^{-1})^{ncs}_{\tilde{n}\tilde{c}s}\, <G^*_k\, \phi^{\tilde{n}\tilde{c}x} \,|\, A \,|\, G_k\, \phi_{n'c's}> \\ &=\; \sum_{(\tilde{n}\tilde{c}s) \in I} (S^{-1})^{ncs}_{\tilde{n}\tilde{c}s}\, <\phi^{\tilde{n}\tilde{c}x} \,|\, A\, G_k \,|\, \phi_{n'c's}>\,, \end{aligned}$$

where in the last step we used that G_k commutes with A (as the projector on an invariant subspace). Putting in the expansions (G.81) and (G.82, G.83) gives the result. ∎

If there are no degeneracies, the above theorem reduces to the well-known formula of second order perturbation theory. The important result is that to the considered degree on the light cone, the matrix elements $A^{ncs}_{n'c's}$ are all zero if $c \neq c'$. In other

words, the left- and right-handed components are invariant subspaces of A. This fact immediately gives the following corollary.

COROLLARY G.4.2. *Consider the fermionic projector in the presence of chiral and scalar potentials (6.0.5), were in composite expressions we disregard all terms quadratic in the field strength. Suppose that the matrix $A^{ncs}_{n'c's}$, (G.78), is diagonal in the sector indices n, n' for all $k = 2, \ldots, K$. Then for $n = 1, \ldots, 7$, the contributions to the eigenvalues of degree two are*

$$\Delta\lambda_{nL+} = \mathrm{Tr}(F_{nL+}\,\Delta A) + \sum_{n'=1}^{8} \frac{1}{\lambda_{nL+} - \lambda_{n'R-}}\,\mathrm{Tr}(F_{nL+}\,\Delta A\,F_{n'R-}\,\Delta A) \tag{G.84}$$

$$\Delta\lambda_{nL-} = \mathrm{Tr}(F_{nL-}\,\Delta A) + \sum_{n'=1}^{8} \frac{1}{\lambda_{nL-} - \lambda_{n'R+}}\,\mathrm{Tr}(F_{nL-}\,\Delta A\,F_{n'R+}\,\Delta A)\,. \tag{G.85}$$

The traces appearing here are given explicitly by (G.50–G.67), where the line integrals are in phase-free form. The corresponding formulas for the opposite chirality are obtained by the replacements (G.41).

Proof. The result is an immediate consequence of Theorem G.4.1 and the estimates (G.42–G.48). ∎

G.5. Perturbation of the Kernel

The results of the previous section do not apply to the kernel of A. The reason is that for $k = 1$, the index set I, (G.68), is

$$I = \{(ncs) \text{ with } n = 8, c = L/R, s = \pm\}\,,$$

and this index set contains both elements with $s = +$ and $s = -$, giving rise to different types of matrix elements. On the other hand, the situation for the kernel is easier because the unperturbed spectral projector on the kernel satisfies the relations

$$X^*\,F_1\,X = 0 \tag{G.86}$$

$$\chi_R\,F_1\,X = 0 = X^*\,F_1\,\chi_L\,, \tag{G.87}$$

and furthermore we can simplify the calculations using that $\lambda_1 = 0$. Using these relations, it follows that, neglecting all contributions of degree < 2, the dimension of the kernel is not affected by the perturbation.

THEOREM G.5.1. *Consider the fermionic projector in the presence of chiral and scalar potentials (6.0.5) and assume that the fermionic projector is weakly causality compatible (see Def. 7.1.1). Suppose that in composite expressions all terms quadratic in the field strength are discarded. Then*

$$A\,G_1 = (\deg < 2)\,.$$

Proof. Using the definition (G.5),

$$\begin{aligned} A\,G_1 &= \frac{1}{2\pi i}\oint_{|z|=\varepsilon} A\,(z-A)^{-1}\,dz = \frac{1}{2\pi i}\oint_{|z|=\varepsilon} \big(z\,(z-A)^{-1} - \mathbb{1}\big)\,dz \\ &= \frac{1}{2\pi i}\oint_{|z|=\varepsilon} z\,(z-A)^{-1}\,dz\,. \end{aligned}$$

Performing the perturbation expansion gives, similar to (G.6),

$$A\,G_1 \;=\; \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z \left((z-A_0)^{-1}\,\Delta A\right)^n \,(z-A_0)^{-1}\,dz\,. \tag{G.88}$$

When we substitute in (G.7) and carry out the contour integral with residues, we get zero unless the factor z is differentiated. For this to occur, the pole at $z=0$ must be at least of order two, and thus we need to take into account only the orders in perturbation theory $n \geq 2$. If $n > 2$, we can as in the previous section transform the matrix products into matrix traces, and the estimate (G.49) yields that the resulting contributions to AG_1 are of degree < 2. Thus it suffices to consider the second order in perturbation theory,

$$\begin{aligned} A\,G_1 &= \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z\,(z-A_0)^{-1}\,\Delta A\,(z-A_0)^{-1}\,\Delta A\,(z-A_0)^{-1}\,dz \;+\; (\deg < 2) \\ &= -\sum_{l=2}^{K} \frac{1}{\lambda_l} \left(F_l\,\Delta A\,F_1\,\Delta A\,F_1 \;+\; F_1\,\Delta A\,F_1\,\Delta A\,F_l \;+\; F_1\,\Delta A\,F_l\,\Delta A\,F_1\right) \\ &\quad + (\deg < 2) \end{aligned} \tag{G.89}$$

The weak causality compatibility condition implies that

$$X\,P(x,y) \;=\; P(x,y) \;=\; P(x,y)\,X^*\,, \tag{G.90}$$

and similarly for composite expressions in the fermionic projector. As a consequence, the first two matrix products in (G.89) vanish; namely,

$$\Delta A\,F_1\,\Delta A \;=\; (\Delta A\,X^*)\,F_1\,(X\,\Delta A) \;=\; \Delta A\,(X^*\,F_1\,X)\,\Delta A \;\overset{(G.86)}{=}\; 0\,.$$

In the last matrix product in (G.89) we can apply (G.87),

$$F_1\,\Delta A\,F_l\,\Delta A\,F_1 \;=\; F_1\,(X\,\Delta A)\,F_l\,(\Delta A\,X^*)\,F_1 \;=\; \chi_L\,F_1\,\Delta A\,F_l\,\Delta A\,F_1\,\chi_R\,. \tag{G.91}$$

Next we substitute (G.4), rewrite the resulting operator products as matrix traces, factor these matrix traces into elementary traces, and apply the estimates of §G.3. This straightforward calculation shows that the matrix product (G.91) is of degree < 5 on the light cone. From (G.89) we conclude that AG_1 is of degree < 2. ∎

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