STRUCTURE-PRESERVING FLOWS OF SYMPLECTIC MATRIX PAIRS

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Abstract. We construct a nonlinear differential equation of matrix pairs $(\mathcal{M}(t), \mathcal{L}(t))$ that are invariant (**Structure-Preserving Property**) in the class of symplectic matrix pairs

$$\left\{ (\mathcal{M}, \mathcal{L}) = \left(\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix} \mathcal{S}_2, \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix} \mathcal{S}_1 \right) \middle| \; X = [X_{ij}]_{1 \le i,j \le 2} \text{ is Hermitian} \right\},$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are two fixed symplectic matrices. Furthermore, its solution also preserves deflating subspaces on the whole orbit (**Eigenvector-Preserving Property**). Such a flow is called a *structure-preserving flow* and is governed by a Riccati differential equation (RDE) of the form $\dot{W}(t) = [-W(t), I]\mathscr{H}[I, W(t)^\top]^\top$, $W(0) = W_0$, for some suitable Hamiltonian matrix $\mathscr{H}$. We then utilize the Grassmann manifolds to extend the domain of the structure-preserving flow to the whole $\mathbb{R}$ except some isolated points. On the other hand, the Structure-Preserving Doubling Algorithm (SDA) is an efficient numerical method for solving algebraic Riccati equations and nonlinear matrix equations. In conjunction with the structure-preserving flow, we consider two special classes of symplectic pairs: $\mathcal{S}_1 = \mathcal{S}_2 = I_{2n}$ and $\mathcal{S}_1 = \mathcal{J}$, $\mathcal{S}_2 = -I_{2n}$ as well as the associated algorithms SDA-1 and SDA-2. It is shown that at $t = 2^{k-1}, k \in \mathbb{Z}$ this flow passes through the iterates generated by SDA-1 and SDA-2, respectively. Therefore, the SDA and its corresponding structure-preserving flow have identical asymptotic behaviors. Taking advantage of the special structure and properties of the Hamiltonian matrix, we apply a symplectically similarity transformation to reduce $\mathscr{H}$ to a Hamiltonian Jordan canonical form $\mathfrak{J}$. The asymptotic analysis of the structure-preserving flows and RDEs is studied by using $e^{\mathfrak{J}t}$. Some asymptotic dynamics of the SDA are investigated, including the linear and quadratic convergence.

1. Introduction.

The following three important types of algebraic Riccati equations appear in many different fields of applied mathematics, sciences and engineering.

- Continuous-time Algebraic Riccati Equation (CARE) [24, 28]:

$$-XGX + A^H X + XA + H = 0; \tag{1.1}$$

- Discrete-time Algebraic Riccati Equation (DARE) [24, 28]:

$$X = A^H X (I + GX)^{-1} A + H; \tag{1.2}$$

- Nonlinear Matrix Equation (NME) [11]:

$$X + A^H X^{-1} A = Q, \tag{1.3}$$

where $A, H, G, Q \in \mathbb{C}^{n \times n}$ with $G = G^H$, $H = H^H$ and $Q = Q^H$. The CARE (1.1) and DARE (1.2) with G and H being positive semi-definite arise in the linear quadratic (LQ) optimal control problems in continuous- and discrete-time, respectively (see [24, 28]). In the H_∞-control problems, G and H in (1.1) or (1.2) can possibly be indefinite [13, 28]. In addition, solutions of a CARE are the equilibria of Riccati differential equations (RDEs) that arise frequently from optimal controls [23, 25], or two-point boundary value problems [9, 10]. The NME (1.3) occurs in applications such as analysis of ladder networks, dynamic programming and stochastic filtering

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(see [2, 11, 12] and the references therein). Nonclassical Riccati-type equations for which the conjugate transpose in (1.3) is replaced by the complex transpose also have various applications; for example, in the vibration analysis of fast trains [16] and the Green's function calculation in nano research [15].

The Structure-Preserving Doubling Algorithms (SDAs) [3, 20, 27] are employed for solving the stabilizing solutions of the three Riccati-type equations (1.1)-(1.3). We now describe these SDAs for solving DARE/CARE and NME.

- For solving DAREs (1.2), the pairs $(\mathcal{M}_k, \mathcal{L}_k) = \left(\begin{bmatrix} A_k & 0 \\ -H_k & I \end{bmatrix}, \begin{bmatrix} I & G_k \\ 0 & A_k^H \end{bmatrix} \right)$ are generated by

  Algorithm SDA-1 [20, 27]. Let $A_1 = A$, $G_1 = G$, and $H_1 = H$.
  For $k = 1, 2, \ldots$, compute

$$\begin{aligned} A_{k+1} &= A_k(I + G_kH_k)^{-1}A_k, \\ G_{k+1} &= G_k + A_kG_k(I + H_kG_k)^{-1}A_k^H, \\ H_{k+1} &= H_k + A_k^H(I + H_kG_k)^{-1}H_kA_k. \end{aligned}$$

- For solving CAREs (1.1), one can transform it into a DARE (1.2) by using the Cayley transform and a left equivalence transform [4]. Then **Algorithm** SDA-1 can be employed to find the desired solutions of CAREs.
- For solving NMEs, the pairs $(\mathcal{M}_k, \mathcal{L}_k) = \left(\begin{bmatrix} A_k & 0 \\ Q_k & -I \end{bmatrix}, \begin{bmatrix} -P_k & I \\ A_k^H & 0 \end{bmatrix} \right)$ are generated by

  Algorithm SDA-2 [3, 27]. Let $A_1 = A$, $Q_1 = Q$, and $P_1 = 0$.
  For $k = 1, 2, \ldots$, compute

$$\begin{aligned} A_{k+1} &= A_k(Q_k - P_k)^{-1}A_k, \\ Q_{k+1} &= Q_k - A_k^H(Q_k - P_k)^{-1}A_k, \\ P_{k+1} &= P_k + A_k(Q_k - P_k)^{-1}A_k^H. \end{aligned}$$

Assume that the sequences $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ generated by the SDAs are well-defined. The sequences $\{(H_k, G_k, A_k)\}_{k=1}^{\infty}$ by SDA-1 and $\{(Q_k, P_k, A_k)\}_{k=1}^{\infty}$ by SDA-2 converge quadratically to $(H^*, G^*, 0)$ and $(Q^*, P^*, 0)$, respectively, if $(\mathcal{M}_1, \mathcal{L}_1)$ has no unimodular eigenvalues [3, 27]. The sequences by SDA-1 and SDA-2 converge linearly of the rate $1/2$ if the partial multiplicities of the unimodular eigenvalues of $(\mathcal{M}_1, \mathcal{L}_1)$ are all even [3, 20]. Here H^* and G^* are the stabilizing solutions of (1.2) and its dual equation: $Y = AY(I + HY)^{-1}A^H + G$, respectively, with the spectral radius $\rho((I + GH^*)^{-1}A) < 1$. Q^* and P^* are the maximal and the minimal solutions of (1.3), respectively, with $Q^* - X$ and $X - P^*$ being positive semi-definite, where X is any solution of (1.3).

Furthermore, the SDA has two preserving properties [22, 27]:

Eigenvector-Preserving Property: For each case above, if $\mathcal{M}_kU = \mathcal{L}_kUS$ or $\mathcal{M}_kVT = \mathcal{L}_kV$, where $U, V \in \mathbb{C}^{2n\times r}$ are of full column rank and $S, T \in \mathbb{C}^{r\times r}$, then $\mathcal{M}_{k+1}U = \mathcal{L}_{k+1}US^2$ or $\mathcal{M}_{k+1}VT^2 = \mathcal{L}_{k+1}V$, i.e., the SDA preserves the deflating subspaces for each k and squares the eigenvalue matrix;

Structure-Preserving Property: The sequences of pairs $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ generated by SDA-1 and SDA-2 are symplectic (see Definition 2.1) and are, respectively, contained in the sets

$$\mathbb{S}_1 = \left\{ \left(\begin{bmatrix} A & 0 \\ H & I \end{bmatrix}, \begin{bmatrix} I & G \\ 0 & A^H \end{bmatrix} \right) | A,\ H = H^H, G = G^H \in \mathbb{C}^{n\times n} \right\}, \quad (1.4a)$$

and

$$\mathbb{S}_2 = \left\{ \left(\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \begin{bmatrix} P & I \\ A^H & 0 \end{bmatrix} \right) | A, Q = Q^H, P = P^H \in \mathbb{C}^{n\times n} \right\}. \quad (1.4b)$$

To study the symplectic pairs, we first quote the following useful theorem.

THEOREM 1.1 (see [29]). *Let $(\mathcal{M}, \mathcal{L})$ be a regular symplectic pair with $\mathcal{M}$, $\mathcal{L} \in \mathbb{C}^{2n\times 2n}$. Then there exist symplectic matrices $\mathcal{S}_1$, $\mathcal{S}_2$ and a Hermitian matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ such that $(\mathcal{M}, \mathcal{L}) \overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix} \mathcal{S}_2, \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix} \mathcal{S}_1 \right)$. Here we say $(A_1, B_1) \overset{l.e.}{\sim} (A_2, B_2)$, if $A_1 = CA_2$, $B_1 = CB_2$ for some invertible matrix C.*

Theorem 1.1 provides us a classification for symplectic pairs. Specifically, let $\mathcal{S}_1, \mathcal{S}_2$ be symplectic and $\mathbb{H}(2n)$ be the set of all $2n \times 2n$ Hermitian matrices. We denote the class of symplectic pairs generated by $\mathcal{S}_1, \mathcal{S}_2$ as

$$\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2} = \left\{ \left(\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix} \mathcal{S}_2, \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix} \mathcal{S}_1 \right) | \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathbb{H}(2n) \right\}. \quad (1.5a)$$

It is easily seen that each pair $(\mathcal{M}, \mathcal{L}) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ is symplectic. We define a bijective transformation $T_{\mathcal{S}_1,\mathcal{S}_2} : \mathbb{H}(2n) \to \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ by

$$T_{\mathcal{S}_1,\mathcal{S}_2}(X) = \left(\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix} \mathcal{S}_2, \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix} \mathcal{S}_1 \right). \quad (1.5b)$$

Therefore, the invariant sets $\mathbb{S}_1$ and $\mathbb{S}_2$ for SDA-1 and SDA-2, respectively, can be rewritten as $\mathbb{S}_1 = \mathbb{S}_{I_{2n},I_{2n}}$ and $\mathbb{S}_2 = \mathbb{S}_{\mathcal{J},-I_{2n}}$. Note that $\mathbb{S}_1 \not\subseteq \mathbb{S}_2$, $\mathbb{S}_2 \not\subseteq \mathbb{S}_1$.

The purpose of this paper is to focus on the flows $(\mathcal{M}(t), \mathcal{L}(t))$ on a specified $\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ (i.e., the **Structure-Preserving Property**) that have **Eigenvector-Preserving Property**. Such a flow is called a *structure-preserving flow*. For two special classes $\mathbb{S}_1$ and $\mathbb{S}_2$ of symplectic pairs, the structure-preserving flow passes through a sequence of iterates generated by SDAs. Finding a smooth curve with a specific structure that passes through a sequence of iterates generated by some numerical algorithm is a popular topic studied by many researchers, especially in the study of the so-called Toda flow that links matrices/matrix pairs generated by QR/QZ-algorithm [5, 6, 7, 32]. In [22], a parameterized curve is constructed in $\mathbb{S}_2$ passing through the iterates generated by the fixed-point iteration, the SDA and the Newton's method with some additional conditions.

Our first main contribution in this paper is to construct a nonlinear differential equation associated with the structure-preserving flow satisfying $(\mathcal{M}(1), \mathcal{L}(1)) = (\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$. Before the description of this main result, we first quote the preliminary result in Theorem 2.7. Suppose that $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. There exist an invertible matrix $\mathbf{U} = [U_1 | U_0, U_\infty] \in \mathbb{C}^{2n\times 2\hat{n}} \times \mathbb{C}^{2n\times \ell} \times \mathbb{C}^{2n\times \ell}$ and a Hamiltonian $\widehat{\mathcal{H}} \in \mathbb{C}^{2\hat{n}\times 2\hat{n}}$ such that $\mathcal{M}_1 U_0 = 0$, $\mathcal{L}_1 U_\infty = 0$, and $\mathcal{M}_1 U_1 = \mathcal{L}_1 U_1 e^{\widehat{\mathcal{H}}}$. Here, the spaces spanned by U_0, U_∞, and U_1 represent the eigenspaces corresponding to zero eigenvalues, infinity eigenvalues, and finite nonzero eigenvalues of the pair, respectively.

Main Result I (Theorem 3.3). *Let $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ be a regular symplectic pair with $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$ and $X_1 = T^{-1}_{\mathcal{S}_1,\mathcal{S}_2}(\mathcal{M}_1, \mathcal{L}_1)$. There exists a Hamiltonian $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ such that the solution of the initial value problem (IVP):*

$$\dot{X}(t) = \mathcal{M}(t)\mathcal{H}\mathcal{J}\mathcal{M}(t)^H, \quad X(1) = X_1, \quad (1.6)$$

for $t \in (t_0, t_1)$, satisfies $\mathcal{M}(t)U_0 = 0$, $\mathcal{L}(t)U_\infty = 0$ and $\mathcal{M}(t)U_1 = \mathcal{L}(t)U_1 e^{\widehat{\mathcal{H}}t}$, where $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$. This implies that the pair $(\mathcal{M}(t), \mathcal{L}(t))$ preserves its deflating subspaces as t varies, i.e., the flow satisfies the so-called ***Eigenvector-Preserving Property***.

We further show that the IVP (1.6) is governed by an RDE. We then adopt the Grassmann manifold and Radon's lemma ([31] or see Theorem 3.14) to extend the domain of the structure-preserving flow to the whole $\mathbb{R}$ except some isolated points. For the $\mathbb{S}_2$ case, the phase portrait of the extended flow is the parameterized curve constructed in [22].

Our second main contribution shows that the extended flow passes through the k-th iterate generated by the SDA with initial pair $(\mathcal{M}_1, \mathcal{L}_1)$ at $t = 2^{k-1}$.

Main Result II (Theorem 3.19). *Let $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_1$ or $\mathbb{S}_2$ defined in* (1.4) *be regular with* $\operatorname{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. *Suppose $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ is the sequence generated by the SDAs. Then $(\mathcal{M}_k, \mathcal{L}_k) = (\mathcal{M}(2^{k-1}), \mathcal{L}(2^{k-1}))$, where $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$ and $X(t)$ is the extended solution of the IVP* (1.6).

Therefore, the SDA and its associated structure-preserving flow have identical asymptotic behaviors, including the stability, instability, and periodicity of the dynamics. By applying the asymptotic analysis of the flow to the Main Result II, our third main contribution concerns the convergence of the SDAs.

Main Result III (Theorems 4.2, 4.4 and 4.7). *Let $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_1$ or $\mathbb{S}_2$ be regular with* $\operatorname{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$ *and $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ be given in Main Result I. Let $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ be generated by the SDAs and $[X_{ij}^k]_{1\leqslant i,j\leqslant 2} = T_{\mathcal{S}_1,\mathcal{S}_2}^{-1}(\mathcal{M}_k, \mathcal{L}_k)$. Then*

- *(i) each X_{ij}^k converges quadratically if $\mathcal{H}$ has no pure imaginary eigenvalues;*
- *(ii) each X_{ij}^k converges linearly if each pure imaginary eigenvalue of $\mathcal{H}$ has only even partial multiplicities;*
- *(iii) suppose that $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ has only one eigenvalue $i\alpha$ with two odd partial multiplicities. Then X_{11}^k, X_{22}^k converge linearly and X_{12}^k, X_{21}^k approach closed curves in $\mathbb{C}^{n\times n}$. In the latter case, the closed curves consist of rank one matrices.*

Similar convergence analysis to assertions Main Result III (i) and (ii) have been carried out in [3, 16] and [20], respectively. In both cases, the proof of convergence for the matrix $\mathcal{H}$ is only applicable when the Jordan blocks, with pure imaginary eigenvalues, are of even sizes. We hereby present an asymptotic behavior of SDA in Main Result III (iii) for these Jordan blocks are of odd sizes.

This paper is organized as follows. In Section 2, we introduce some preliminary results of the eigenstructures of regular symplectic pairs. In Subsection 3.1, we prove Main Result I (Theorem 3.3). The connection between the structure-preserving flow and RDE is studied in Subsection 3.2. In Subsection 3.3, we then apply the Grassmann manifold to extend the flow to the whole $\mathbb{R}$. In Subsection 3.4, we prove Main Result II (Theorem 3.19). In Section 4, we give a brief description for the proof of Main Result III (Theorems 4.2, 4.4 and 4.7). Some numerical experiments are presented in Section 5.

2. Eigenstructures of regular symplectic pairs. We first introduce some notation, definitions and the algebraic structures that we consider in this paper. Let $\mathcal{J}_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix. For convenience, we use $\mathcal{J}$ for $\mathcal{J}_n$ by dropping the subscript "n" if the order of $\mathcal{J}_n$ is clear in the context. Two subspaces $\mathbb{U}$ and $\mathbb{V}$ of $\mathbb{C}^{2n}$ are called $\mathcal{J}$-orthogonal if $u^H \mathcal{J} v = 0$ for each $u \in \mathbb{U}$ and $v \in \mathbb{V}$. A subspace $\mathbb{U}$ of $\mathbb{C}^{2n}$ is called isotropic if $x^H \mathcal{J} y = 0$ for any $x, y \in \mathbb{U}$. An

n-dimensional isotropic subspace is called a Lagrangian subspace. The jth column of an identity matrix I is denoted by e_j and $\|\cdot\|$ denotes the Forbenious norm.

DEFINITION 2.1.

1. *A matrix $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ is Hamiltonian if $\mathcal{H}\mathcal{J} = (\mathcal{H}\mathcal{J})^H$.*
2. *A matrix $\mathcal{S} \in \mathbb{C}^{2n\times 2n}$ is symplectic if $\mathcal{S}\mathcal{J}\mathcal{S}^H = \mathcal{J}$.*
3. *A matrix pair $(\mathcal{M}, \mathcal{L}) \in \mathbb{C}^{2n\times 2n} \times \mathbb{C}^{2n\times 2n}$ is symplectic if $\mathcal{M}\mathcal{J}\mathcal{M}^H = \mathcal{L}\mathcal{J}\mathcal{L}^H$.*

Denote by $Sp(n)$ the multiplicative group of all $2n \times 2n$ symplectic matrices and by $\mathbb{H}(2n)$ the set of all $2n \times 2n$ Hermitian matrices. A matrix pair (A, B) with $A, B \in \mathbb{C}^{n\times n}$ is said to be *regular* if $\det(A - \lambda B) \neq 0$ for some $\lambda \in \mathbb{C}$. Note that the matrix pair (A, B) is said to have eigenvalues at infinity if B is singular. To see this, write the eigenvalue problem $Ax = \lambda Bx$ in the reciprocal form $Bx = \lambda^{-1}Ax$. In the case B is singular, we have $Bx = 0Ax$ whenever x is a null vector of B. This means that x is an eigenvector of the eigenvalue problem corresponding to eigenvalue $\lambda^{-1} = 0$, i.e., $\lambda = \infty$. The matrix pairs (A_1, B_1) and $(A_2, B_2) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}$ are said to be left equivalent, denoted by $(A_1, B_1) \overset{\text{l.e.}}{\sim} (A_2, B_2)$, if $A_1 = CA_2$, $B_1 = CB_2$ for some invertible matrix C. It is well-known that for a regular matrix pair (A, B) there are invertible matrices P and Q which transform (A, B) to the Kronecker canonical form [14] as

$$PAQ = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \quad PBQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \tag{2.1}$$

where J is a Jordan matrix corresponding to the finite eigenvalues of (A, B) and N is a nilpotent Jordan matrix corresponding to the infinity eigenvalues. The index of a matrix pair (A, B) is the index of nilpotency of N, i.e., the matrix pair (A, B) is of index ν, denoted by $\nu = \operatorname{ind}_\infty(A, B)$, if $N^{\nu-1} \neq 0$ and $N^\nu = 0$.

REMARK 2.1. *Let (A, B) be a regular pair. Then $\operatorname{ind}_\infty(A, B) = 0$ if B is invertible and $\operatorname{ind}_\infty(A, B) = 1$ if the infinity eigenvalues of (A, B) are semi-simple.*

2.1. Eigenvector-Preserving Property.

Motivated by the **Eigenvector-Preserving Property** of iterations of SDAs, we extend this property to a flow. Let $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ be regular and $\operatorname{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. A flow $\{(\mathcal{M}(t), \mathcal{L}(t)) \mid t \in \mathbb{R}\} \subseteq \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ with $(\mathcal{M}(1), \mathcal{L}(1)) = (\mathcal{M}_1, \mathcal{L}_1)$ having the **Eigenvector-Preserving Property** can be stated as follows:

> *Assume that $\mathcal{M}_1 U_0 = 0$, $\mathcal{L}_1 U_\infty = 0$, and $\mathcal{M}_1 U_1 = \mathcal{L}_1 U_1 S$, where $\mathbf{U} = [U_1, U_0, U_\infty] \in \mathbb{C}^{2n\times 2n}$ and S are invertible. Then*
>
> $$\mathcal{M}(t)U_0 = 0,\ \mathcal{L}(t)U_\infty = 0,\ \mathcal{M}(t)U_1 = \mathcal{L}(t)U_1 S^t \tag{2.2}$$
>
> *hold.*

We shall note that the flow $(\mathcal{M}(t), \mathcal{L}(t))$ in (2.2) preserves eigenvectors whenever S in (2.2) is diagonalizable, otherwise, it preserves the deflating subspaces only. Here in (2.2), S^t, for $t \in \mathbb{R}$, is defined in the sense of a matrix function. Because S invertible, it follows from [17, Definition 1.11 and Theorem 1.17] that the matrix function S^t is well-defined. We shall show in Theorem 2.2 that the invertibility of the matrix S can result from the condition $\operatorname{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. In this case, (2.2) can have an alternative form as (2.1) in which $P\mathcal{M}(t)Q = J^t \oplus I$ and $P\mathcal{L}(t)Q = I \oplus 0$.

2.2. Regular symplectic pairs with $\operatorname{ind}_\infty(\mathcal{M}, \mathcal{L}) \leqslant 1$.

The proof of the following theorem can be found in [21, 26]

THEOREM 2.2. *Suppose $(\mathcal{M}, \mathcal{L})$ is a regular symplectic pair with $\mathcal{M}, \mathcal{L} \in \mathbb{C}^{2n\times 2n}$ and $\operatorname{ind}_\infty(\mathcal{M}, \mathcal{L}) \leqslant 1$. Then there is $\hat{n} \leqslant n$ such that $\operatorname{rank}(\mathcal{M}) = \operatorname{rank}(\mathcal{L}) = n + \hat{n}$. In* *addition, there exist an invertible matrix* $\mathbf{U} = [U_1|U_0, U_\infty] \in \mathbb{C}^{2n\times 2\hat{n}} \times \mathbb{C}^{2n\times \ell} \times \mathbb{C}^{2n\times \ell}$ *with* $\ell = n - \hat{n}$ *and a symplectic matrix* $\widehat{\mathcal{S}} \in \mathbb{C}^{2\hat{n}\times 2\hat{n}}$ *such that*

$$\mathbf{U}^H \mathcal{J}_n \mathbf{U} = \mathcal{J}_{\hat{n}} \oplus \mathcal{J}_\ell \tag{2.3}$$

and

$$\mathcal{M}U_0 = 0, \quad \mathcal{L}U_\infty = 0, \quad \mathcal{M}U_1 = \mathcal{L}U_1\widehat{\mathcal{S}}. \tag{2.4}$$

REMARK 2.3. *Theorem 2.2 shows that the assumption of the* ***Eigenvector Preserving Property*** *holds if* $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$.

Note that the matrix $\widehat{\mathcal{S}}$ in Theorem 2.2 is symplectic. There is a Hamiltonian matrix $\widehat{\mathcal{H}}$ satisfying $e^{\widehat{\mathcal{H}}} = \widehat{\mathcal{S}}$ (see e.g., [19, 21, 30]). Using $\widehat{\mathcal{H}}$, we shall construct a Hamiltonian matrix $\mathcal{H}$ which has invariant subspaces spanned by U_0, U_∞, and U_1.

THEOREM 2.4. *Suppose* $(\mathcal{M}, \mathcal{L})$ *is a regular symplectic pair with* $\mathcal{M}, \mathcal{L} \in \mathbb{C}^{2n\times 2n}$ *and* $\mathrm{ind}_\infty(\mathcal{M}, \mathcal{L}) \leqslant 1$. *Let the matrices* $\mathbf{U}$ *and* $\widehat{\mathcal{S}}$ *be given as in Theorem 2.2, and* $\widehat{\mathcal{H}} \in \mathbb{C}^{2\hat{n}\times 2\hat{n}}$ *be the Hamiltonian matrix such that* $e^{\widehat{\mathcal{H}}} = \widehat{\mathcal{S}}$. *Then the matrix*

$$\mathcal{H} = \mathbf{U}\begin{bmatrix} \widehat{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix}(\mathcal{J}_{\hat{n}} \oplus \mathcal{J}_\ell)^H \mathbf{U}^H \mathcal{J} \tag{2.5}$$

is Hamiltonian.

Proof. Since $\widehat{\mathcal{H}}$ is Hamiltonian, we have $\mathcal{H}\mathcal{J} = -\mathbf{U}(\widehat{\mathcal{H}}\mathcal{J}_{\hat{n}}^H \oplus 0)\mathbf{U}^H = \mathcal{J}^H\mathcal{J}^H\mathbf{U}(\mathcal{J}_{\hat{n}}\widehat{\mathcal{H}}^H \oplus 0)\mathbf{U}^H = \mathcal{J}^H\mathcal{H}^H$. Hence, $\mathcal{H}$ is Hamiltonian. □

REMARK 2.5. *Suppose that* $(\mathcal{M}, \mathcal{L})$ *is a real regular symplectic pair. Then* $\mathbf{U}$ *and* $\widehat{\mathcal{S}}$ *can be chosen real. In [8], under the assumptions*

(i) $\widehat{\mathcal{S}}$ *has an even number of Jordan blocks of each size relative to every negative eigenvalue;*

(ii) the size of two identical Jordan blocks corresponding to eigenvalue -1 *is odd;*

it is shown that there is a real Hamiltonian matrix $\widehat{\mathcal{H}}$ *such that* $e^{\widehat{\mathcal{H}}} = \widehat{\mathcal{S}}$. *Hence, under these assumptions, the Hamiltonian* $\mathcal{H}$ *defined in* (2.5) *can be chosen real.*

Suppose that $\mathcal{L}$ is invertible. It follows from Theorem 2.2 that $\mathcal{M}$ is also invertible. Therefore, U_0 and U_∞ in (2.4) are absent. On the other hand, the matrix $\mathcal{L}^{-1}\mathcal{M}$ is symplectic, hence there exists a Hamiltonian matrix $\mathcal{H}$ such that $e^{\mathcal{H}} = \mathcal{L}^{-1}\mathcal{M}$, i.e., $\mathcal{M} = \mathcal{L}e^{\mathcal{H}}$. For the case that $\mathcal{L}$ is singular and $(\mathcal{M}, \mathcal{L})$ is a regular symplectic pair with $\mathrm{ind}_\infty(\mathcal{M}, \mathcal{L}) = 1$, it is natural to ask whether there is a Hamiltonian matrix $\mathcal{H}$ such that $\mathcal{M} = \mathcal{L}e^{\mathcal{H}}$. However, this is not true (see Lemma 2.6 for the necessary condition). The pair $(\mathcal{M}, \mathcal{L})$ satisfies the equality $\mathcal{M}\Pi_0 = \mathcal{L}\Pi_\infty e^{\mathcal{H}}$ for some suitable matrices Π_0 and Π_∞. To see this, we need the following lemma.

LEMMA 2.6. *Suppose that* $(\mathcal{M}, \mathcal{L})$ *is a regular symplectic pair. If* $\mathcal{M} = \mathcal{L}W$ *for some nonsingular* W, *then both* $\mathcal{M}$ *and* $\mathcal{L}$ *are invertible.*

Proof. From Theorem 1.1, there are matrices $\mathcal{S}_1, \mathcal{S}_2 \in Sp(n)$ and $X = [X_{ij}]_{1\le i,j\le 2} \in \mathbb{H}(2n)$ such that $\mathcal{M} = C\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}\mathcal{S}_2$, $\mathcal{L} = C\begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix}\mathcal{S}_1$, where C is nonsingular. Suppose that $\mathcal{M} = \mathcal{L}W$. Then we have $\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}\mathcal{S}_2 = \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix}\mathcal{S}_1 W$. Since $\mathcal{S}_1$, $\mathcal{S}_2$ and W are nonsingular, it is easily seen that X_{12} and X_{21} are nonsingular. Thus, $\mathcal{M}$ and $\mathcal{L}$ are invertible. □

THEOREM 2.7. *Suppose* $(\mathcal{M},\mathcal{L})$ *is a regular symplectic pair with* $\mathcal{M},\mathcal{L}\in\mathbb{C}^{2n\times 2n}$ *and* $\text{ind}_\infty(\mathcal{M},\mathcal{L})\leqslant 1$. *Let the matrices* $\mathbf{U}$ *and* $\mathcal{H}$ *be given as in Theorems 2.2 and 2.4, respectively. Let*

$$\Pi_0=\mathbf{U}(I_{2\hat{n}}\oplus I_\ell\oplus 0)\mathbf{U}^{-1}\ \text{and}\ \Pi_\infty=\mathbf{U}(I_{2\hat{n}}\oplus 0\oplus I_\ell)\mathbf{U}^{-1}, \tag{2.6}$$

where $\mathbf{U}^{-1}=(\mathcal{J}_{\hat{n}}\oplus\mathcal{J}_\ell)^H\mathbf{U}^H\mathcal{J}$. *Then we have* $\mathcal{M}\Pi_0=\mathcal{L}\Pi_\infty e^{\mathcal{H}}$.

REMARK 2.8. *Note that both* Π_0 *and* Π_∞ *are idempotent, i.e.,* $\Pi_0^2=\Pi_0$ *and* $\Pi_\infty^2=\Pi_\infty$. *Here* Π_0 *(*Π_∞*, respectively) is a projection onto the space spanned by the eigenspace corresponding to the finite eigenvalues (nonzero eigenvalues, respectively) along the space spanned by* U_∞ *(by* U_0*, respectively). In addition, if* $\text{ind}_\infty(\mathcal{M},\mathcal{L})=0$*, i.e.,* $\mathcal{M}$ *and* $\mathcal{L}$ *are invertible and* $\hat{n}=n$*, then* $\Pi_0=\Pi_\infty=I$. *Therefore,* $\mathcal{M}=\mathcal{L}e^{\mathcal{H}}$ *with some appropriate Hamiltonian matrix* $\mathcal{H}$. *This coincides with Lemma 2.6.*

Proof. [Proof of Theorem 2.7] From (2.4) and (2.6), we have

$$\mathcal{M}\Pi_0=[\mathcal{L}U_1\widehat{\mathcal{S}}|0,0]\mathbf{U}^{-1}=\mathcal{L}[U_1|U_0,U_\infty](\widehat{\mathcal{S}}\oplus 0\oplus I_\ell)\mathbf{U}^{-1}=\mathcal{L}\Pi_\infty\mathbf{U}(\widehat{\mathcal{S}}\oplus I_{2\ell})\mathbf{U}^{-1}.$$

It follows from (2.5) and the fact $e^{\widehat{\mathcal{H}}}=\widehat{\mathcal{S}}$ that $e^{\mathcal{H}}=\mathbf{U}(e^{\widehat{\mathcal{H}}}\oplus e^0)\mathbf{U}^{-1}=\mathbf{U}(\widehat{\mathcal{S}}\oplus I_{2\ell})\mathbf{U}^{-1}$. The assertion follows. ∎

To make the correspondence between the constructed matrices in the previous lemmas/theorems and the symplectic pairs $(\mathcal{M},\mathcal{L})$, we use the following notations throughout this paper.

DEFINITION 2.2. *For a given regular symplectic pair* $(\mathcal{M},\mathcal{L})$ *with* $\text{ind}_\infty(\mathcal{M},\mathcal{L})\leqslant 1$. *Let* $\hat{n}:=\text{rank}(\mathcal{M})-n$ *and* $\ell=n-\hat{n}$. *We define* $\mathbf{U}(\mathcal{M},\mathcal{L}):=[U_1|U_0,U_\infty]\in\mathbb{C}^{2n\times 2\hat{n}}\times\mathbb{C}^{2n\times\ell}\times\mathbb{C}^{2n\times\ell}$ *and* $\widehat{\mathcal{S}}(\mathcal{M},\mathcal{L})\in\mathbb{C}^{2\hat{n}\times 2\hat{n}}$ *that satisfy* (2.3) *and* (2.4). *Let* $\widehat{\mathcal{H}}(\mathcal{M},\mathcal{L})$ *be the matrix that satisfies* $e^{\widehat{\mathcal{H}}(\mathcal{M},\mathcal{L})}=\widehat{\mathcal{S}}(\mathcal{M},\mathcal{L})$*, and let* $\mathcal{H}(\mathcal{M},\mathcal{L})$ *and* $\Pi_0(\mathcal{M},\mathcal{L})$*,* $\Pi_\infty(\mathcal{M},\mathcal{L})$ *be the matrices defined in* (2.5) *and* (2.6)*, respectively.*

2.3. A perturbation result for symplectic pairs. We now provide a perturbation theory for a regular symplectic pair $(\mathcal{M},\mathcal{L})$ with $\text{ind}_\infty(\mathcal{M},\mathcal{L})=1$. In this case, the infinity eigenvalues of the pair $(\mathcal{M},\mathcal{L})$ exist and are semi-simple, and hence, so does the zero eigenvalues, by Theorem 2.2. After a small perturbation of order $O(\varepsilon)$ in a specific direction, the perturbed pair $(\mathcal{M}^\varepsilon,\mathcal{L}^\varepsilon)$ preserves the deflating subspaces spanned by U_0, U_∞, and U_1. Moreover, finite and nonzero eigenvalues of $(\mathcal{M}^\varepsilon,\mathcal{L}^\varepsilon)$ are remained the same as that of $(\mathcal{M},\mathcal{L})$. Zero and infinity eigenvalues of $(\mathcal{M},\mathcal{L})$ are perturbed to finite eigenvalues of $(\mathcal{M}^\varepsilon,\mathcal{L}^\varepsilon)$ of order $O(\varepsilon)$ and of order $O(1/\varepsilon)$, respectively.

THEOREM 2.9. *Suppose* $(\mathcal{M},\mathcal{L})$ *is a regular symplectic pair with* $\mathcal{M},\mathcal{L}\in\mathbb{C}^{2n\times 2n}$ *and* $\text{ind}_\infty(\mathcal{M},\mathcal{L})=1$. *Let* $\mathbf{U}=\mathbf{U}(\mathcal{M},\mathcal{L})$ *and* $\widehat{\mathcal{S}}=\widehat{\mathcal{S}}(\mathcal{M},\mathcal{L})$ *be given as in Definition 2.2 and let* $\Phi^\varepsilon\in\mathbb{C}^{\ell\times\ell}$ *be a family of nonsingular matrices with* $\|\Phi^\varepsilon\|\leqslant\varepsilon$ *for each* $\varepsilon>0$. *If*

$$\mathcal{M}^\varepsilon=\mathcal{M}+\Delta\mathcal{M}^\varepsilon,\quad \mathcal{L}^\varepsilon=\mathcal{L}+\Delta\mathcal{L}^\varepsilon, \tag{2.7}$$

where $\Delta\mathcal{M}^\varepsilon=-\mathcal{L}U_0\Phi^{\varepsilon H}U_\infty^H\mathcal{J}$*,* $\Delta\mathcal{L}^\varepsilon=\mathcal{M}U_\infty\Phi^\varepsilon U_0^H\mathcal{J}$*, then* $(\mathcal{M}^\varepsilon,\mathcal{L}^\varepsilon)$ *is a regular symplectic pair with* $\mathcal{L}^\varepsilon$ *being invertible. Moreover,* $\mathcal{M}^\varepsilon$ *and* $\mathcal{L}^\varepsilon$ *satisfy*

$$\mathcal{M}^\varepsilon U_0=\mathcal{L}^\varepsilon U_0\Phi^{\varepsilon H},\ \mathcal{M}^\varepsilon U_\infty\Phi^\varepsilon=\mathcal{L}^\varepsilon U_\infty,\ \mathcal{M}^\varepsilon U_1=\mathcal{L}^\varepsilon U_1\widehat{\mathcal{S}}, \tag{2.8}$$

and $(\mathcal{M}^\varepsilon,\mathcal{L}^\varepsilon)\to(\mathcal{M},\mathcal{L})$ *as* $\varepsilon\to 0$.

Proof. From (2.3), it holds that

$$U_0^H\mathcal{J}U_\infty=I,\ U_\infty^H\mathcal{J}U_0=-I,\ U_\infty^H\mathcal{J}U_\infty=U_0^H\mathcal{J}U_0=0. \tag{2.9}$$

For each $\varepsilon > 0$, from (2.7) and (2.9) it holds that

$$
\begin{aligned}
\mathcal{M}^\varepsilon \mathcal{J} \mathcal{M}^{\varepsilon H} &= (\mathcal{M} + \Delta\mathcal{M}^\varepsilon)\mathcal{J}(\mathcal{M} + \Delta\mathcal{M}^\varepsilon)^H \\
&= \mathcal{M}\mathcal{J}\mathcal{M}^H + \mathcal{M}\mathcal{J}\Delta\mathcal{M}^{\varepsilon H} + \Delta\mathcal{M}^\varepsilon\mathcal{J}\mathcal{M}^H + \Delta\mathcal{M}^\varepsilon\mathcal{J}\Delta\mathcal{M}^{\varepsilon H} \\
&= \mathcal{L}\mathcal{J}\mathcal{L}^H - \mathcal{M}U_\infty\Phi^\varepsilon U_0^H\mathcal{L}^H + \mathcal{L}U_0\Phi^{\varepsilon H}U_\infty^H\mathcal{M}^H \\
&= \mathcal{L}\mathcal{J}\mathcal{L}^H + \Delta\mathcal{L}^\varepsilon\mathcal{J}\mathcal{L}^H + \mathcal{L}\mathcal{J}\Delta\mathcal{L}^{\varepsilon H} + \Delta\mathcal{L}^\varepsilon\mathcal{J}\mathcal{L}^H + \Delta\mathcal{L}^\varepsilon\mathcal{J}\Delta\mathcal{L}^{\varepsilon H} \\
&= (\mathcal{L} + \Delta\mathcal{L}^\varepsilon)\mathcal{J}(\mathcal{L} + \Delta\mathcal{L}^\varepsilon)^H = \mathcal{L}^\varepsilon\mathcal{J}\mathcal{L}^{\varepsilon H}.
\end{aligned}
$$

That is, $(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon)$ forms a symplectic pair. Now, we show that $\mathcal{L}^\varepsilon$ is invertible. Since $(\mathcal{M}, \mathcal{L})$ is a regular symplectic pair, there exists a nonzero constant λ_0 such that $\mathcal{M} - \lambda_0\mathcal{L}$ is invertible. Using the fact that $\mathbf{U}$ is nonsingular, it follows from (2.4) that

$$
(\mathcal{M} - \lambda_0\mathcal{L})\mathbf{U} = [\mathcal{M}U_1 - \lambda_0\mathcal{L}U_1, -\lambda_0\mathcal{L}U_0, \mathcal{M}U_\infty] = \left[\mathcal{L}U_1(\widehat{\mathcal{S}} - \lambda_0 I), -\lambda_0\mathcal{L}U_0, \mathcal{M}U_\infty\right]
$$

is nonsingular, and hence, $\widehat{\mathcal{S}} - \lambda_0 I$ is also invertible. Since Φ^ε is nonsingular, from (2.7) and (2.9) together with the fact that $U_0^H\mathcal{J}U_1 = 0$, we have

$$
\mathcal{L}^\varepsilon\mathbf{U} = [\mathcal{L}U_1, \mathcal{L}U_0, \mathcal{M}U_\infty\Phi^\varepsilon] = (\mathcal{M} - \lambda_0\mathcal{L})\mathbf{U}\left((\widehat{\mathcal{S}} - \lambda_0 I)^{-1} \oplus (-\lambda_0)^{-1}I \oplus \Phi^\varepsilon\right)
$$

is invertible and then $\mathcal{L}^\varepsilon$ is invertible. Hence, $(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon)$ is a regular symplectic pair.

From (2.4) and (2.9), we have

$$
\begin{aligned}
\mathcal{M}^\varepsilon U_0 &= (\mathcal{M} + \Delta\mathcal{M}^\varepsilon)U_0 = -\mathcal{L}U_0\Phi^{\varepsilon H}U_\infty^H\mathcal{J}U_0 = \mathcal{L}U_0\Phi^{\varepsilon H} = \mathcal{L}^\varepsilon U_0\Phi^{\varepsilon H}, \\
\mathcal{L}^\varepsilon U_\infty &= (\mathcal{L} + \Delta\mathcal{L}^\varepsilon)U_\infty = \mathcal{M}U_\infty\Phi^\varepsilon U_0^H\mathcal{J}U_\infty = \mathcal{M}U_\infty\Phi^\varepsilon = \mathcal{M}^\varepsilon U_\infty\Phi^\varepsilon, \\
\mathcal{M}^\varepsilon U_1 &= (\mathcal{M} + \Delta\mathcal{M}^\varepsilon)U_1 = \mathcal{M}U_1 = \mathcal{L}U_1\widehat{\mathcal{S}} = (\mathcal{L} + \Delta\mathcal{L}^\varepsilon)U_1\widehat{\mathcal{S}} = \mathcal{L}^\varepsilon U_1\widehat{\mathcal{S}}.
\end{aligned}
$$

Thus, equations of (2.8) hold. Since $\|\Phi^\varepsilon\| \leqslant \varepsilon$, $(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon) \to (\mathcal{M}, \mathcal{L})$ as $\varepsilon \to 0$. ◻

COROLLARY 2.10. *Suppose $(\mathcal{M}, \mathcal{L}) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ is a regular symplectic pair with* $\mathrm{ind}_\infty(\mathcal{M}, \mathcal{L}) \leqslant 1$. *Let Φ^ε be nonsingular with $\|\Phi^\varepsilon\| \leqslant \varepsilon$ for each $\varepsilon > 0$ sufficiently small, and $\mathcal{M}^\varepsilon$, $\mathcal{L}^\varepsilon$ be given as in Theorem 2.9. Then there exists $(\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ for $\varepsilon \geqslant 0$ sufficiently small, such that $(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon) \overset{\text{l.e.}}{\sim} (\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon)$. Moreover, for each $\varepsilon > 0$ sufficiently small, $\widetilde{\mathcal{M}}^\varepsilon$ and $\widetilde{\mathcal{L}}^\varepsilon$ are invertible satisfying* (2.8) *and $(\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon) \to (\mathcal{M}, \mathcal{L})$ as $\varepsilon \to 0$.*

Proof. Since $(\mathcal{M}, \mathcal{L}) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$, it holds that $\mathcal{M} = \begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}\mathcal{S}_2$, $\mathcal{L} = \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix}\mathcal{S}_1$, where $[X_{ij}]_{1\leq i,j\leq 2}$ is Hermitian. Since $\|\Phi^\varepsilon\| \leqslant \varepsilon$ for $\varepsilon > 0$ sufficiently small, from (2.7) we have

$$
\mathcal{M}^\varepsilon = \begin{bmatrix} X_{12} + O(\varepsilon) & O(\varepsilon) \\ X_{22} + O(\varepsilon) & I + O(\varepsilon) \end{bmatrix}\mathcal{S}_2,\ \mathcal{L}^\varepsilon = \begin{bmatrix} I + O(\varepsilon) & X_{11} + O(\varepsilon) \\ O(\varepsilon) & X_{21} + O(\varepsilon) \end{bmatrix}\mathcal{S}_1.
$$

Applying row operations to $(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon)$ we have

$$
(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon) \overset{\text{l.e.}}{\sim} \left(\begin{bmatrix} \widetilde{X}_{12}(\varepsilon) & 0 \\ \widetilde{X}_{22}(\varepsilon) & I \end{bmatrix}\mathcal{S}_2, \begin{bmatrix} I & \widetilde{X}_{11}(\varepsilon) \\ 0 & \widetilde{X}_{21}(\varepsilon) \end{bmatrix}\mathcal{S}_1\right) \equiv \left(\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon\right),
$$

where $\widetilde{X}_{ij}(\varepsilon) = X_{ij} + O(\varepsilon)$ for $1 \leqslant i, j \leqslant 2$. Hence, $(\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon) \to (\mathcal{M}, \mathcal{L})$ as $\varepsilon \to 0$. Using the fact that $(\mathcal{M}^\varepsilon, \mathcal{L}^\varepsilon) \overset{\text{l.e.}}{\sim} (\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon)$, it follows from Theorem 2.9 that $\widetilde{\mathcal{M}}^\varepsilon$ and $\widetilde{\mathcal{L}}^\varepsilon$ are invertible, and satisfy the equalities of (2.8). Since $(\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon)$ is symplectic and $[\widetilde{X}_{ij}(\varepsilon)]_{1\leq ij\leq 2}$ is Hermitian, we have $(\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ for $\varepsilon \geqslant 0$ sufficiently small. ◻

3. Structure-preserving flows.

3.1. Construction of structure-preserving flows. Suppose that $(\mathcal{M}_1, \mathcal{L}_1)$ is a regular symplectic pair with $\operatorname{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. From Theorem 1.1, there exist two symplectic matrices $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$. In this subsection we shall construct a differential equation with $(\mathcal{M}_1, \mathcal{L}_1)$ as an initial matrix pair whose solution is the structure-preserving flow.

We first prove the case for which $\mathcal{L}_1$ is invertible.

THEOREM 3.1. *Let $\mathcal{S}_1$, $\mathcal{S}_2 \in Sp(n)$, $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ be Hamiltonian and $X_1 \in \mathbb{H}(2n)$. Suppose $X(t)$, for $t \in (t_0, t_1)$ and $t_0 < 1 < t_1$, is the solution of the IVP:*

$$\dot{X}(t) = \mathcal{M}(t)\mathcal{H}\mathcal{J}\mathcal{M}(t)^H, \quad X(1) = X_1, \tag{3.1}$$

where $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$. If the pair $(\mathcal{M}_1, \mathcal{L}_1) \equiv (\mathcal{M}(1), \mathcal{L}(1))$ satisfies $\mathcal{M}_1 = \mathcal{L}_1 e^{\mathcal{H}_1}$ for some Hamiltonian $\mathcal{H}_1 \in \mathbb{C}^{2n\times 2n}$, then

$$\mathcal{M}(t) = \mathcal{L}(t)e^{\mathcal{H}_1}e^{\mathcal{H}(t-1)} \tag{3.2}$$

for all $t \in (t_0, t_1)$.

Proof. Partition $X_1 = [X_{ij}^1]_{1\le i,j\le 2}$ and $X(t) = [X_{ij}(t)]_{1\le i,j\le 2}$. From the assumption $\mathcal{M}_1 = \mathcal{L}_1 e^{\mathcal{H}_1}$ and Lemma 2.6, we see that both $\mathcal{M}(1)$ and $\mathcal{L}(1)$ are invertible. On the other hand, the solution $X(t)$ of the IVP (3.1) is continuous. Therefore, we denote $(\tilde{t}_0, \tilde{t}_1)$ the *connected component* of the open set $\{t \in (t_0, t_1) | \det(\mathcal{M}(\mathrm{t})) \neq 0,\ \det(\mathcal{L}(\mathrm{t})) \neq 0\}$ that contains $t = 1$. We first show that assertion (3.2) holds for $t \in (\tilde{t}_0, \tilde{t}_1)$. By the fact that

$$\mathcal{M}(t) = \begin{bmatrix} X_{12}(t) & 0 \\ X_{22}(t) & I \end{bmatrix} \mathcal{S}_2, \quad \mathcal{L}(t) = \begin{bmatrix} I & X_{11}(t) \\ 0 & X_{21}(t) \end{bmatrix} \mathcal{S}_1,$$

we have

$$\dot{X} = \begin{bmatrix} \dot{X}_{12} & 0 & 0 & \dot{X}_{11} \\ \dot{X}_{22} & 0 & 0 & \dot{X}_{21} \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -X_{12}^H & -X_{22}^H \\ -X_{11}^H & -X_{21}^H \\ I_n & 0 \end{bmatrix} = [\dot{\mathcal{M}}, \dot{\mathcal{L}}] \begin{bmatrix} \mathcal{J}\mathcal{M}^H \\ -\mathcal{J}\mathcal{L}^H \end{bmatrix}. \tag{3.3}$$

Plugging (3.3) into the first equation of (3.1) and multiplying $\mathcal{M}^{-H}\mathcal{J}^H$ from the right to the resulting equation, we have

$$[\dot{\mathcal{M}}, \dot{\mathcal{L}}] \begin{bmatrix} I \\ \mathcal{J}\mathcal{L}^H\mathcal{M}^{-H}\mathcal{J} \end{bmatrix} = \mathcal{M}\mathcal{H}, \quad t \in (\tilde{t}_0, \tilde{t}_1). \tag{3.4}$$

Since $\mathcal{M}\mathcal{J}\mathcal{M}^H = \mathcal{L}\mathcal{J}\mathcal{L}^H$ and both $\mathcal{M}$ and $\mathcal{L}$ are invertible, (3.4) becomes $\dot{\mathcal{M}} - \dot{\mathcal{L}}(\mathcal{L}^{-1}\mathcal{M}) = \mathcal{M}\mathcal{H}$. Multiplying $\mathcal{L}^{-1}$ from the left of the last equation, we thus obtain $\mathcal{L}^{-1}\dot{\mathcal{M}} - (\mathcal{L}^{-1}\dot{\mathcal{L}}\mathcal{L}^{-1})\mathcal{M} = \mathcal{L}^{-1}\mathcal{M}\mathcal{H}$. This coincides with $\frac{d}{dt}(\mathcal{L}^{-1}\mathcal{M}) = (\mathcal{L}^{-1}\mathcal{M})\mathcal{H}$. Therefore, using the initial condition in (3.1) and the fact $\mathcal{M}_1 = \mathcal{L}_1 e^{\mathcal{H}_1}$, it follows that $\mathcal{L}(t)^{-1}\mathcal{M}(t) = e^{\mathcal{H}_1}e^{\mathcal{H}(t-1)}$ for $t \in (\tilde{t}_0, \tilde{t}_1)$. Hence, assertion (3.2) holds.

Now we claim that $\tilde{t}_0 = t_0$ and $\tilde{t}_1 = t_1$. We only prove the case $\tilde{t}_1 = t_1$. Suppose that $\tilde{t}_1 < t_1$. Then $\mathcal{M}(\tilde{t}_1)$ and $\mathcal{L}(\tilde{t}_1)$ exist. Since $(\tilde{t}_0, \tilde{t}_1)$ is a connected component of $\{t \in (t_0, t_1) | \det(\mathcal{M}(\mathrm{t})) \neq 0,\ \det(\mathcal{L}(\mathrm{t})) \neq 0\}$, $\mathcal{M}(\tilde{t}_1)$ and $\mathcal{L}(\tilde{t}_1)$ are singular. Using (3.2) and taking the limit $t \to \tilde{t}_1^-$, we have $\mathcal{M}(\tilde{t}_1) = \mathcal{L}(\tilde{t}_1)e^{\mathcal{H}_1}e^{\mathcal{H}(\tilde{t}_1-1)}$. Since $e^{\mathcal{H}_1}e^{\mathcal{H}(\tilde{t}_1-1)}$ is invertible, $\mathcal{M}(\tilde{t}_1)$ and $\mathcal{L}(\tilde{t}_1)$ are invertible by Lemma 2.6. This is a contradiction. Hence, $\tilde{t}_0 = t_0$ and $\tilde{t}_1 = t_1$. $\square$

REMARK 3.2. *(i) In Theorem 3.1, since X_1 and $\mathcal{HJ}$ are Hermitian, it is easily seen that $X(t)$ is also Hermitian for $t \in (t_0, t_1)$. From the definition that $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$, we have that the curve $\{(\mathcal{M}(t), \mathcal{L}(t)) | t \in (t_0, t_1)\} \subset \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$. (ii) Suppose $(\mathcal{M}_1, \mathcal{L}_1)$ is a real symplectic pair. Under the conditions in Remark 2.5, $\mathcal{H}_1$ can be chosen real. If $\mathcal{H}$ is also real, then the curve $\{(\mathcal{M}(t), \mathcal{L}(t)) | t \in (t_0, t_1)\} \subset \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ is real.*

The following theorem is the detail version of Main Result I.

THEOREM 3.3. *Let $\mathcal{S}_1$, $\mathcal{S}_2 \in Sp(n)$ and $X_1 \in \mathbb{H}(2n)$ be given such that the symplectic pair $(\mathcal{M}_1, \mathcal{L}_1) = T_{\mathcal{S}_1,\mathcal{S}_2}(X_1)$ is regular with $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. Let the idempotent matrices $\Pi_0 = \Pi_0(\mathcal{M}_1, \mathcal{L}_1)$, $\Pi_\infty = \Pi_\infty(\mathcal{M}_1, \mathcal{L}_1)$ and the Hamiltonian matrix $\mathcal{H} = \mathcal{H}(\mathcal{M}_1, \mathcal{L}_1)$ be defined in Definition 2.2 such that (from Theorem 2.7)*

$$\mathcal{M}_1 \Pi_0 = \mathcal{L}_1 \Pi_\infty e^{\mathcal{H}}. \tag{3.5}$$

If $X(t)$, for $t \in (t_0, t_1)$, $t_0 < 1 < t_1$, is the solution of the IVP (3.1)*, then*

$$\mathcal{M}(t)\Pi_0 = \mathcal{L}(t)\Pi_\infty e^{\mathcal{H}t}, \tag{3.6}$$

or equivalently,

$$\mathcal{M}(t)U_0 = 0, \quad \mathcal{L}(t)U_\infty = 0, \quad \mathcal{M}(t)U_1 = \mathcal{L}(t)U_1 e^{\widehat{\mathcal{H}}t}, \tag{3.7}$$

where $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$, for all $t \in (t_0, t_1)$.

REMARK 3.4. *Note that (i) if $\mathcal{M}_1$ and $\mathcal{L}_1$ are invertible, this implies $\Pi_0 = \Pi_\infty = I$, and hence the result of Theorem 3.3 is consistent with Theorem 3.1 in which $\mathcal{H}_1$ is replaced by $\mathcal{H}$; and (ii) from Definition 2.2 of $\mathcal{H}$, Π_0 and Π_∞, it is easily seen that the invariant properties* (3.6) *and* (3.7) *are equivalent. This shows that the flow $(\mathcal{M}(t), \mathcal{L}(t))$ satisfies **Eigenvector-Preserving Property**. Actually, this flow $(\mathcal{M}(t), \mathcal{L}(t))$ is the structure-preserving flow with the initial $(\mathcal{M}_1, \mathcal{L}_1)$.*

Proof. [Proof of Theorem 3.3] Applying Theorem 2.9 and Corollary 2.10 with $\Phi^\varepsilon = \varepsilon I$, we see that $(\mathcal{M}_1 + \Delta\mathcal{M}^\varepsilon, \mathcal{L}_1 + \Delta\mathcal{L}^\varepsilon)$ is left equivalent to the symplectic pair

$$(\mathcal{M}_1^\varepsilon, \mathcal{L}_1^\varepsilon) \equiv (\widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{L}}^\varepsilon) = \left(\begin{bmatrix} X_{12}^{1\varepsilon} & 0 \\ X_{22}^{1\varepsilon} & I \end{bmatrix} \mathcal{S}_2, \begin{bmatrix} I & X_{11}^{1\varepsilon} \\ 0 & X_{21}^{1\varepsilon} \end{bmatrix} \mathcal{S}_1 \right) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$$

for ε sufficiently small. In addition, $\mathcal{M}_1^\varepsilon$ and $\mathcal{L}_1^\varepsilon$ are invertible for $\varepsilon > 0$ and $(\mathcal{M}_1^\varepsilon, \mathcal{L}_1^\varepsilon) \to (\mathcal{M}_1, \mathcal{L}_1)$ as $\varepsilon \to 0$. Let $X^\varepsilon(t) = \begin{bmatrix} X_{11}^\varepsilon(t) & X_{12}^\varepsilon(t) \\ X_{21}^\varepsilon(t) & X_{22}^\varepsilon(t) \end{bmatrix}$ be the solution of the IVP

$$\dot{X}^\varepsilon(t) = \mathcal{M}^\varepsilon(t)\mathcal{HJ}\mathcal{M}^\varepsilon(t)^H, \quad X^\varepsilon(1) = X_1^\varepsilon,$$

where $X_1^\varepsilon = \begin{bmatrix} X_{11}^{1\varepsilon} & X_{12}^{1\varepsilon} \\ X_{21}^{1\varepsilon} & X_{22}^{1\varepsilon} \end{bmatrix}$ and $(\mathcal{M}^\varepsilon(t), \mathcal{L}^\varepsilon(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X^\varepsilon(t))$. By the continuous dependence of the solution on the initial condition of the IVP (see e.g. Section 8.4 in [18]), we have $(\mathcal{M}^\varepsilon(t), \mathcal{L}^\varepsilon(t)) \to (\mathcal{M}(t), \mathcal{L}(t))$ as $\varepsilon \to 0$. On the other hand, it follows from Theorem 3.1 that $\mathcal{M}^\varepsilon(t) = \mathcal{L}^\varepsilon(t)(\mathcal{L}_1^{\varepsilon -1}\mathcal{M}_1^\varepsilon)e^{\mathcal{H}(t-1)}$. Consequently,

$$\mathcal{M}^\varepsilon(t)e^{-\mathcal{H}t}e^{\mathcal{H}} = \mathcal{L}^\varepsilon(t)(\mathcal{L}_1^{\varepsilon -1}\mathcal{M}_1^\varepsilon). \tag{3.8}$$

Let $\mathbf{U} = \mathbf{U}(\mathcal{M}_1, \mathcal{L}_1) = [U_1 | U_0, U_\infty]$ satisfy (2.3) and (2.4) in which $(\mathcal{M}, \mathcal{L})$ is replaced by $(\mathcal{M}_1, \mathcal{L}_1)$. From (2.8), we have

$$\mathcal{M}_1^\varepsilon[U_1, U_0, U_\infty](I_{2\hat{n}} \oplus I_\ell \oplus \varepsilon I_\ell) = \mathcal{L}_1^\varepsilon[U_1, U_0, U_\infty](\widehat{\mathcal{S}} \oplus \varepsilon I_\ell \oplus I_\ell). \tag{3.9}$$

Plugging (3.9) into (3.8) and using the fact that $e^{\mathcal{H}} = \mathbf{U}(\widehat{\mathcal{S}} \oplus I_\ell \oplus I_\ell)\mathbf{U}^{-1}$, we have

$$\mathcal{M}^\varepsilon(t)e^{-\mathcal{H}t}\mathbf{U}(I_{2\hat{n}} \oplus I_\ell \oplus \varepsilon I_\ell)\mathbf{U}^{-1} = \mathcal{L}^\varepsilon(t)\mathbf{U}(I_{2\hat{n}} \oplus \varepsilon I_\ell \oplus I_\ell)\mathbf{U}^{-1}.$$

When ε approaches 0, it follows from (2.6) that $\mathcal{M}(t)e^{-\mathcal{H}t}\Pi_0 = \mathcal{L}(t)\Pi_\infty$. Since $e^{-\mathcal{H}t}$ commutes with Π_0, we obtain assertion (3.6), and then (3.7). $\square$

COROLLARY 3.5. *Theorem 3.3 holds true if Eq.* (3.1) *is replaced by*

$$\dot{X}(t) = \mathcal{L}(t)\mathcal{H}\mathcal{J}\mathcal{L}(t)^H, \quad X(1) = X_1.$$

Proof. It suffices to show that $\mathcal{M}(t)\mathcal{H}\mathcal{J}\mathcal{M}(t)^H = \mathcal{L}(t)\mathcal{H}\mathcal{J}\mathcal{L}(t)^H$. Using definitions of $\Pi_0 = \Pi_0(\mathcal{M}_1, \mathcal{L}_1)$ and $\Pi_\infty = \Pi_\infty(\mathcal{M}_1, \mathcal{L}_1)$, we have $\mathcal{M}(t) = \mathcal{M}(t)\Pi_0$, $\mathcal{L}(t) = \mathcal{L}(t)\Pi_\infty$. It follows from (3.6) and the symplecticity of $e^{\mathcal{H}t}$ that

$$\begin{aligned}\mathcal{M}(t)\mathcal{H}\mathcal{J}\mathcal{M}(t)^H &= \mathcal{M}(t)\Pi_0\mathcal{H}\mathcal{J}\Pi_0^H\mathcal{M}(t)^H = \mathcal{L}(t)\Pi_\infty e^{\mathcal{H}t}\mathcal{H}\mathcal{J}(e^{\mathcal{H}t})^H\Pi_\infty^H\mathcal{L}(t)^H \\ &= \mathcal{L}(t)\Pi_\infty\mathcal{H}\mathcal{J}\Pi_\infty^H\mathcal{L}(t)^H = \mathcal{L}(t)\mathcal{H}\mathcal{J}\mathcal{L}(t)^H.\end{aligned}$$

$\square$

Now, we study the invariance property (3.6). To this end, for given $\mathcal{S}_1$, $\mathcal{S}_2 \in Sp(n)$, we let $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ be regular with $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. Let the idempotent matrices $\Pi_0 = \Pi_0(\mathcal{M}_1, \mathcal{L}_1)$, $\Pi_\infty = \Pi_\infty(\mathcal{M}_1, \mathcal{L}_1)$ and $\mathcal{H} = \mathcal{H}(\mathcal{M}_1, \mathcal{L}_1)$ be defined as in Definition 2.2. Consider the linear system with unknowns $(\mathcal{M}, \mathcal{L})$:

$$\begin{cases}\mathcal{M}\Pi_0 = \mathcal{L}\Pi_\infty e^{\mathcal{H}t}, \\ (\mathcal{M}, \mathcal{L}) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2},\end{cases} \tag{3.10}$$

where $t \in \mathbb{R}$ plays the role as a parameter. It is clear from Theorem 3.3 that the solution $(\mathcal{M}(t), \mathcal{L}(t))$ of the IVP (3.1) is contained in the manifold described by (3.10). In the following, we shall show that the consistency of Eq. (3.10) at specified t implies the uniqueness of the solution $(\mathcal{M}, \mathcal{L})$, for which the pair $(\mathcal{M}, \mathcal{L})$ is regular. From the definition of $\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ in (1.5), the second equation of (3.10) implies $(\mathcal{M}, \mathcal{L}) = T_{\mathcal{S}_1,\mathcal{S}_2}(X)$ for some $X = [X_{ij}] \in \mathbb{H}(2n)$. The linear system (3.10) can be rewritten as

$$\begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix}\mathcal{S}_2\mathbf{U}(I_{2\hat{n}} \oplus I_\ell \oplus 0) = \begin{bmatrix} I & X_{11} \\ 0 & X_{21} \end{bmatrix}\mathcal{S}_1\mathbf{U}(e^{\widehat{\mathcal{H}}t} \oplus 0 \oplus I_\ell). \tag{3.11}$$

The following lemma can be obtained by direct calculations.

LEMMA 3.6. *Let*

$$E_{11} = (I_\ell \oplus 0), \quad E_{22} = (0 \oplus I_\ell), \tag{3.12a}$$

$$\mathbf{V}_1 \equiv \begin{bmatrix} \mathbf{V}_1^1 \\ \mathbf{V}_2^1 \end{bmatrix} = \mathcal{S}_1\mathbf{U}, \quad \mathbf{V}_2 \equiv \begin{bmatrix} \mathbf{V}_1^2 \\ \mathbf{V}_2^2 \end{bmatrix} = \mathcal{S}_2\mathbf{U}, \tag{3.12b}$$

where $\mathbf{V}_i^j \in \mathbb{C}^{n\times 2n}$ for each $1 \leqslant i,j \leqslant 2$. Then the linear system (3.11) *is equivalent to the alternative form:*

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} -\mathbf{V}_2^1(e^{\widehat{\mathcal{H}}t} \oplus E_{22}) \\ \mathbf{V}_1^2(I_{2\hat{n}} \oplus E_{11}) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1^1(e^{\widehat{\mathcal{H}}t} \oplus E_{22}) \\ -\mathbf{V}_2^2(I_{2\hat{n}} \oplus E_{11}) \end{bmatrix}. \tag{3.13}$$

LEMMA 3.7. *Let (A,B) be a regular pair with $A, B \in \mathbb{C}^{n\times n}$. Suppose that $CA+DB=0$ and $[C,D]\in\mathbb{C}^{n\times 2n}$ is of full row rank. Then (D,C) is a regular pair.*

Proof. Since (A,B) is regular, there exists $\lambda_0\in\mathbb{C}$ such that $A-\lambda_0 B$ is invertible and $[A^\top, B^\top]^\top$ is of full column rank. Therefore,

$$0 = [C,D]\begin{bmatrix} A \\ B \end{bmatrix}(A-\lambda_0 B)^{-1} = [C,D]\begin{bmatrix} I & \lambda_0 I \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -\lambda_0 I \\ 0 & I \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}(A-\lambda_0 B)^{-1}$$
$$= [C, D+\lambda_0 C]\begin{bmatrix} I \\ B(A-\lambda_0 B)^{-1} \end{bmatrix}. \tag{3.14}$$

It is easily seen that $\operatorname{rank}[C, D+\lambda_0 C] = \operatorname{rank}[C,D] = n$. Since the row vectors of $[-B(A-\lambda_0 B)^{-1}, I]$ form a basis of left null space of $\begin{bmatrix} I \\ B(A-\lambda_0 B)^{-1} \end{bmatrix}$, it follows from (3.14) that there is a nonsingular matrix W such that $[C, D+\lambda_0 C] = W[-B(A-\lambda_0 B)^{-1}, I]$. Then $D+\lambda_0 C$ is invertible and hence (D,C) is regular. □

THEOREM 3.8. *Let $(\mathcal{M}_1,\mathcal{L}_1)\in\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ be a regular pair with $\operatorname{ind}_\infty(\mathcal{M}_1,\mathcal{L}_1)\leqslant 1$ and $\mathbf{U}\equiv[U_1,U_0,U_\infty]=\mathbf{U}(\mathcal{M}_1,\mathcal{L}_1)$. Suppose $(\mathcal{M},\mathcal{L})$ is a solution of (3.10) at some $t\in\mathbb{R}$. Then*

(i) $(\mathcal{M},\mathcal{L})$ is regular;

(ii) $(\mathcal{M},\mathcal{L})$ is the unique solution of (3.10);

(iii) It holds that

$$\mathcal{M}U_0 = 0, \quad \mathcal{L}U_\infty = 0, \quad \mathcal{M}U_1 = \mathcal{L}U_1 e^{\widehat{\mathcal{H}}t}. \tag{3.15}$$

Conversely, if $(\mathcal{M},\mathcal{L})\in\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ satisfies (3.15), then $(\mathcal{M},\mathcal{L})$ is a solution of (3.10).

Proof. From (3.11), we have $[-\mathcal{L},\mathcal{M}]\begin{bmatrix} \mathbf{U}(e^{\widehat{\mathcal{H}}t}\oplus E_{22}) \\ \mathbf{U}(I_{2\hat{n}}\oplus E_{11}) \end{bmatrix} = 0$. Since $\operatorname{rank}([-\mathcal{L},\mathcal{M}]) = 2n$ and $\Big((e^{\widehat{\mathcal{H}}t}\oplus E_{22}), (I_{2\hat{n}}\oplus E_{11})\Big)$ is regular, it follows from Lemma 3.7 that $(\mathcal{M},\mathcal{L})$ is regular. Hence, assertion (i) holds.

Next, we show that the linear system (3.10) has a unique solution. From Lemma 3.6, it suffices to show that the matrix $\begin{bmatrix} -\mathbf{V}_2^1(e^{\widehat{\mathcal{H}}t}\oplus E_{22}) \\ \mathbf{V}_1^2(I_{2\hat{n}}\oplus E_{11}) \end{bmatrix}$ in (3.13) is invertible. Suppose that $y\in\mathbb{C}^{2n}$ satisfying $\begin{bmatrix} -\mathbf{V}_2^1(e^{\widehat{\mathcal{H}}t}\oplus E_{22}) \\ \mathbf{V}_1^2(I_{2\hat{n}}\oplus E_{11}) \end{bmatrix} y = 0$. Let $z_1 = (e^{\widehat{\mathcal{H}}t}\oplus E_{22})y$ and $z_2 = (I_{2\hat{n}}\oplus E_{11})y$. Then we have $\mathbf{V}_2^1 z_1 = 0$ and $\mathbf{V}_1^2 z_2 = 0$. Since the linear system (3.13) is consistent, we obtain that $\mathbf{V}_1^1 z_1 = 0$ and $\mathbf{V}_2^2 z_2 = 0$. Hence, $\mathbf{V}_1 z_1 = 0$ and $\mathbf{V}_2 z_2 = 0$. From (3.12b), we have $\mathbf{V}_1$ and $\mathbf{V}_2$ are invertible, and hence, $z_1 = z_2 = 0$. Therefore, $y = 0$. Assertion (ii) follows.

Since (3.11) is an alternative form of (3.10), and (3.11) is equivalent to (3.15) whenever $(\mathcal{M},\mathcal{L})\in\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$, assertion (iii) follows directly. □

REMARK 3.9. *Given two symplectic matrices $\mathcal{S}_1$ and $\mathcal{S}_2$, the linear system (3.10) may have no solution in $\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$. We consider a simple example. Let $\mathcal{S}_1=\mathcal{S}_2=I_2$, $\mathcal{H} = \begin{bmatrix} 0 & \pi/2 \\ -\pi/2 & 0 \end{bmatrix}$ and $t=1$. Then $e^{\mathcal{H}t} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. It is easily seen that (3.10) has no solution in $\mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$.*

From Theorem 3.8 (ii), the unique solution of (3.10) can be written as an implicit function $(\mathcal{M}(t),\mathcal{L}(t))$ for t on the set

$$\mathcal{T}_X = \{t\in\mathbb{R} \mid (3.10) \text{ has a solution at } t\}. \tag{3.16}$$

REMARK 3.10. *Let* $X(t) = T^{-1}_{\mathcal{S}_1,\mathcal{S}_2}(\mathcal{M}(t),\mathcal{L}(t))$, *where* $(\mathcal{M}(t),\mathcal{L}(t))$ *is the solution of* (3.10) *at* $t \in \mathcal{T}_X$. *We see that for each* $t \in \mathcal{T}_X$, $X(t)$ *satisfies* (3.13) *and* $\begin{bmatrix} -\mathbf{V}_2^1(e^{\widehat{\mathcal{H}}t} \oplus E_{22}) \\ \mathbf{V}_1^2(I_{2\hat{n}} \oplus E_{11}) \end{bmatrix}$ *is invertible, hence,* $X(t)$ *is continuously differentiable. Consequently,* $\mathcal{T}_X$ *is open.*

THEOREM 3.11. *Suppose that* $(\mathcal{M}(t),\mathcal{L}(t))$ *is the solution of* (3.10) *for* $t \in (\tilde{t}_0,\tilde{t}_1) \subseteq \mathcal{T}_X$, *where* $\tilde{t}_0 < 1 < \tilde{t}_1$. *Then* $X(t) = T^{-1}_{\mathcal{S}_1,\mathcal{S}_2}(\mathcal{M}(t),\mathcal{L}(t))$ *for* $t \in (\tilde{t}_0,\tilde{t}_1)$ *is the solution of the IVP* (3.1).

Proof. Let $Y(t) = T^{-1}_{\mathcal{S}_1,\mathcal{S}_2}(\mathcal{M}(t),\mathcal{L}(t))$ for $t \in (\tilde{t}_0,\tilde{t}_1)$. From Remark 3.10, $Y(t)$ is continuously differentiable. Suppose that $X(t)$ for $t \in (t_0,t_1)$ is the solution of the IVP (3.1), where (t_0,t_1) is the maximal interval. It follows from Theorem 3.3 that $T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$ is the solution of (3.10) at $t \in (t_0,t_1)$. If $(\tilde{t}_0,\tilde{t}_1) \subseteq (t_0,t_1)$, then the uniqueness of the solution of (3.10) implies that $Y(t) = X(t)$ for $t \in (\tilde{t}_0,\tilde{t}_1)$, and hence $X(t) = T^{-1}_{\mathcal{S}_1,\mathcal{S}_2}(\mathcal{M}(t),\mathcal{L}(t))$, for $t \in (\tilde{t}_0,\tilde{t}_1)$, is the solution of the IVP (3.1). Now we claim that $(\tilde{t}_0,\tilde{t}_1) \subseteq (t_0,t_1)$. We prove the case $\tilde{t}_1 \leqslant t_1$. On the contrary, suppose that $\tilde{t}_1 > t_1$. Then $t_1 \in (\tilde{t}_0,\tilde{t}_1) \subseteq \mathcal{T}_X$, and hence, $(\mathcal{M}(t_1),\mathcal{L}(t_1))$ is the solution of (3.10) at $t = t_1$. By the uniqueness of solution of (3.10), we have $X(t) = Y(t)$ for $t \in (t_0,t_1)$. We also note that $\dot{Y}(t)$ is continuous at $t_1 \in (\tilde{t}_0,\tilde{t}_1)$. Therefore,

$$\dot{Y}(t_1) - \mathcal{M}(t_1)\mathcal{H}\mathcal{J}\mathcal{M}(t_1)^H = \lim_{t \to t_1^-} \dot{Y}(t) - \mathcal{M}(t)\mathcal{H}\mathcal{J}\mathcal{M}(t)^H = 0.$$

Hence, the solution $X(t)$ of the IVP (3.1) can be extended to t_1. This is a contradiction because (t_0,t_1) is the maximal interval of the IVP (3.1). □

REMARK 3.12. *Theorem 3.11 shows that the connected component of* $\mathcal{T}_X$ *containing* $t = 1$ *coincides with the maximal interval of the IVP* (3.1). *Moreover, the phase portrait of the IVP* (3.1) *is the curve* $\{(\mathcal{M}(t),\mathcal{L}(t))$ *defined by* (3.10) $\mid t \in (\tilde{t}_0,\tilde{t}_1) \subseteq \mathcal{T}_X\}$. *The solution of the IVP* (3.1) *can be extended to whole* $\mathcal{T}_X$ *by using the so-called Grassmann manifold which will be studied in Subsection 3.3 for details.*

3.2. Structure-preserving flow vs. Riccati differential equation. In this subsection, we shall show that the IVP (3.1) is governed by a *Riccati differential equation (RDE)*. In addition, by adopting Radon's Lemma (see Theorem 3.14), the structure-preserving flow can be represented as an explicit form. Throughout this subsection, we suppose that the assumptions in Theorem 3.3 are satisfied. First, since $\mathcal{S}_2$ is symplectic and $\mathcal{H}$ is Hamiltonian, $\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}$ is also Hamiltonian, say

$$\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1} = \begin{bmatrix} A & S \\ D & -A^H \end{bmatrix}, \tag{3.17}$$

where $A, S, D \in \mathbb{C}^{n\times n}$ with $S^H = S$ and $D^H = D$. Suppose that $X(t) = [X_{ij}(t)]_{1\leqslant i,j\leqslant 2}$, for $t \in (t_0,t_1)$ and $t_0 < 1 < t_1$, is the solution of (3.1). We then have

$$\begin{aligned} \begin{bmatrix} \dot{X}_{11} & \dot{X}_{12} \\ \dot{X}_{21} & \dot{X}_{22} \end{bmatrix} &= \begin{bmatrix} X_{12} & 0 \\ X_{22} & I \end{bmatrix} \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}\mathcal{J} \begin{bmatrix} X_{12}^H & X_{22}^H \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -X_{12}SX_{12}^H & -X_{12}SX_{22}^H + X_{12}A \\ -X_{22}SX_{12}^H + A^H X_{12}^H & -X_{22}SX_{22}^H + X_{22}A + A^H X_{22}^H + D \end{bmatrix}, \\ X_{ij}(1) &= X_{ij}^1 \text{ for } 1 \leqslant i,j \leqslant 2. \end{aligned} \tag{3.18}$$

That is, $X_{ij}(t)$ for $1 \leqslant i, j \leqslant 2$ satisfy the coupled differential equations

$$\dot{X}_{11} = -X_{12} S X_{12}^H, \tag{3.19a}$$

$$\dot{X}_{12} = -X_{12} S X_{22}^H + X_{12} A, \tag{3.19b}$$

$$\dot{X}_{21} = -X_{22} S X_{12}^H + A^H X_{12}^H, \tag{3.19c}$$

$$\dot{X}_{22} = -X_{22} S X_{22}^H + X_{22} A + A^H X_{22}^H + D, \tag{3.19d}$$

with $X_{ij}(1) = X_{ij}^1$, where A, D and S are given in (3.17). Note that S, D and the initial matrix X_{22}^1 are Hermitian. From (3.19d), $X_{22}(t)$ is Hermitian for $t \in (t_0, t_1)$. Therefore, by taking a time shift, $W(t) = X_{22}(t+1)$, $t \in (t_0 - 1, t_1 - 1)$, is the solution of the RDE:

$$\dot{W}(t) = -W(t) S W(t) + W(t) A + A^H W(t) + D, \quad W(0) = W_0, \tag{3.20}$$

with $W_0 = X_{22}^1$.

Remark 3.13. *Suppose that $W(t)$, for $t \in (t_0 - 1, t_1 - 1)$ and $t_0 - 1 < 0 < t_1 - 1$, is a solution of the RDE* (3.20). *Using the fact $X_{22}(t) = W(t-1)$, $t \in (t_0, t_1)$, we can get $X_{12}(t)$ for $t \in (t_0, t_1)$ by solving the linear differential equation* (3.19b) *with $X_{12}(1) = X_{12}^1$. Since $X_{21}^1 = X_{12}^{1H}$, it follows from* (3.19b) *and* (3.19c) *that $X_{21}(t) = X_{12}(t)^H$, for $t \in (t_0, t_1)$. Finally, $X_{11}(t)$ for $t \in (t_0, t_1)$ can be obtained directly from* (3.19a). *So, solving the IVP* (3.1) *is governed by solving the RDE* (3.20).

Theorem 3.14. *[1, Radon's Lemma] Let A, S, $D \in \mathbb{C}^{n \times n}$ with $S^H = S$ and $D^H = D$, then the following statements hold.*

(i) Let $W(t)$ be a solution of the RDE (3.20) *in the interval $(t_0 - 1, t_1 - 1)$ containing zero. If $Q(t)$ is a solution of the IVP $\dot{Q}(t) = (SW(t) - A)Q(t)$, with initial $Q(0) = I_n$ and $P(t) := W(t)Q(t)$, then $Y(t) \equiv [Q(t)^\top, P(t)^\top]^\top$ is the solution of the linear IVP*

$$\dot{Y}(t) = \widetilde{\mathcal{H}} Y(t), \quad Y(0) = [I, W_0^\top]^\top, \tag{3.21a}$$

where

$$\widetilde{\mathcal{H}} = \begin{bmatrix} -A & S \\ D & A^H \end{bmatrix}. \tag{3.21b}$$

(ii) Let $Y(t) \equiv [Q(t)^\top, P(t)^\top]^\top$ be the solution of (3.21). *If $Q(t)$ is invertible for $t \in (t_0 - 1, t_1 - 1) \subset \mathbb{R}$, then $W(t) \equiv P(t)Q(t)^{-1}$ is a solution of RDE* (3.20).

Remark 3.15. *Using the definition of $\widetilde{\mathcal{H}}$ in* (3.21b), *it follows from* (3.17) *that*

$$\widetilde{\mathcal{H}} = -(I \oplus (-I)) \mathcal{S}_2 \mathcal{H} \mathcal{S}_2^{-1} (I \oplus (-I)). \tag{3.22}$$

Therefore, if $\mathcal{S}_2 \mathcal{H} \mathcal{S}_2^{-1} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \Lambda$, *then* $\widetilde{\mathcal{H}} \begin{bmatrix} U_1 \\ -U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ -U_2 \end{bmatrix} (-\Lambda)$.

Corollary 3.16. *Let $Y(t) \equiv [Q(t)^\top, P(t)^\top]^\top$ and $W(t)$ be the solution of* (3.21) *and* (3.20), *respectively, with $W_0 = X_{22}^1$. If $Q(t)$ is invertible, for $t \in (t_0 - 1, t_1 - 1)$ and $t_0 - 1 < 0 < t_1 - 1$, then the solutions of* (3.19d) *and* (3.19b) *are*

$$X_{22}(t) = W(t-1) = P(t-1)Q(t-1)^{-1} \text{ and } X_{12}(t) = X_{12}^1 Q(t-1)^{-1}, \tag{3.23}$$

respectively, for $t \in (t_0, t_1)$. In addition, $X_{21}(t) = X_{12}(t)^H = Q(t-1)^{-H} X_{21}^1$.

Proof. The detail of the proof can be found in [21]. $\square$

Let $\mathcal{S}_1\mathcal{H}\mathcal{S}_1^{-1} = \begin{bmatrix} A_\star & S_\star \\ D_\star & -A_\star^H \end{bmatrix}$. From Corollary 3.5 and a similar calculation as (3.18), we obtain that $X_{11}(t)$ and $X_{21}(t)$ satisfy

$$\begin{aligned} \dot{X}_{11} &= X_{11}D_\star X_{11}^H + X_{11}A_\star^H + A_\star X_{11}^H - S_\star, \\ \dot{X}_{21} &= X_{21}D_\star X_{11}^H + X_{21}A_\star^H, \end{aligned} \tag{3.24}$$

with $X_{11}(1) = X_{11}^1$ and $X_{21}(1) = X_{21}^1$. Similarly, by using the fact that the solution $X_{11}(t)$ is Hermitian and taking the time shift, $t \to t+1$, we see that $W_\star(t) = X_{11}(t+1)$ is the solution of the RDE

$$\dot{W}_\star(t) = W_\star(t)D_\star W_\star(t) + W_\star(t)A_\star^H + A_\star W_\star(t) - S_\star, \quad W_\star(0) = X_{11}^1. \tag{3.25}$$

Let $Y_\star(t) = [Q_\star(t)^\top, P_\star(t)^\top]^\top$ be the solution of the linear differential equation

$$\dot{Y}_\star(t) = \widetilde{\mathcal{H}}_\star Y_\star(t), \quad Y_\star(0) = [I, X_{11}^{1^\top}]^\top, \tag{3.26a}$$

where

$$\widetilde{\mathcal{H}}_\star \equiv \begin{bmatrix} -A_\star^H & -D_\star \\ -S_\star & A_\star \end{bmatrix} = \mathcal{J}^{-1}\mathcal{S}_1\mathcal{H}\mathcal{S}_1^{-1}\mathcal{J}. \tag{3.26b}$$

Suppose that $Q_\star(t)$ is invertible for $t \in (t_0^\star - 1, t_1^\star - 1)$ and $t_0^\star - 1 < 0 < t_1^\star - 1$. By Radon's Lemma and Corollary 3.16, the solution $X_{11}(t)$, $X_{21}(t)$ of (3.24) can be formulated by

$$X_{11}(t) = W_\star(t-1) = P_\star(t-1)Q_\star(t-1)^{-1}, \; X_{21}(t) = X_{21}^1 Q_\star(t-1)^{-1}, \tag{3.27}$$

respectively, for $t \in (t_0^\star, t_1^\star)$. Comparing (3.22) and (3.26b) yields that $\widetilde{\mathcal{H}}_\star$ and $-\widetilde{\mathcal{H}}$ are similar.

The nonsingularity of $Q(t)$ and $Q_\star(t)$ plays an important role to determine whether $X_{22}(t)$ and $X_{11}(t)$ exist, respectively. The following theorem claims that both $Q(t)$ and $Q_\star(t)$ are invertible simultaneously.

THEOREM 3.17. *Let $Q(t)$, $P(t)$, $Q_\star(t)$ and $P_\star(t)$ be the matrix functions given in* (3.21) *and* (3.26)*, respectively. Then we have*

$$\{t \in \mathbb{R} |\ \det(Q(t)) \neq 0\} = \{t \in \mathbb{R} |\ \det(Q_\star(t)) \neq 0\}. \tag{3.28}$$

In addition, if $\hat{t} \in \mathbb{R}$ such that $\det(Q(\hat{t})) = 0$, then $\|P(t)Q(t)^{-1}\|$, $\|X_{12}^1 Q(t)^{-1}\|$, $\|P_\star(t)Q_\star(t)^{-1}\|$, and $\|X_{21}^1 Q_\star(t)^{-1}\| \to \infty$ as $t \to \hat{t}$.

Proof. Let Π_0, Π_∞, $\mathbf{U} \equiv [U_1|U_0, U_\infty]$ and $\mathcal{H}$ be given in Definition 2.2 that satisfy (3.5). Using the facts that $\mathcal{S}_1$, $\mathcal{S}_2$ and $e^{\mathcal{H}t}$ are symplectic and applying (3.21), (3.22) and (3.26), we have

$$\begin{aligned} Q(t) = [I, 0]\begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} &= [I, 0]\mathcal{S}_2 e^{-\mathcal{H}t}\mathcal{S}_2^{-1}\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} I \\ X_{22}^1 \end{bmatrix} \\ &= [I, 0]\mathcal{S}_2 e^{-\mathcal{H}t}\mathcal{S}_2^{-1}\mathcal{J}\begin{bmatrix} X_{22}^1 \\ I \end{bmatrix} = [I, 0]\mathcal{S}_2\mathcal{J}(e^{\mathcal{H}t})^H\mathcal{S}_2^H\begin{bmatrix} X_{22}^1 \\ I \end{bmatrix}, \end{aligned} \tag{3.29}$$

$$Q_\star(t) = [I, 0]\begin{bmatrix} Q_\star(t) \\ P_\star(t) \end{bmatrix} = -[0, I]\mathcal{S}_1 e^{\mathcal{H}t}\mathcal{J}\mathcal{S}_1^H\begin{bmatrix} I \\ X_{11}^1 \end{bmatrix}. \tag{3.30}$$

Suppose that $\hat{t} \in \mathbb{R}$ such that $\det(Q(\hat{t})) = 0$. We first claim that $\lim_{t\to\hat{t}} \|P(t)Q(t)^{-1}\| = \lim_{t\to\hat{t}} \|X_{12}^1 Q(t)^{-1}\| = \infty$. Since $[Q(\hat{t})^\top, P(\hat{t})^\top]^\top = e^{\widetilde{\mathcal{H}}\hat{t}}[I, X_{22}^1]^\top$ is of full column rank and $Q(\hat{t})$ is singular, it is easily seen that $\lim_{t\to\hat{t}} \|P(t)Q(t)^{-1}\| = \infty$. Now, we show that $\lim_{t\to\hat{t}} \|X_{12}^1 Q(t)^{-1}\| = \infty$. Since $Q(t)$ is continuous and $Q(\hat{t})$ is singular, it suffices to show that $X_{12}^1 x_0 \neq 0$, where $Q(\hat{t})x_0 = 0$ with $x_0 \neq 0$. We prove it by contradiction. Suppose that $X_{12}^1 x_0 = 0$. Since $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$, Eq. (3.5) can be written in the form

$$\begin{bmatrix} X_{12}^1 & 0 \\ X_{22}^1 & I \end{bmatrix} \mathcal{S}_2 \Pi_0 = \begin{bmatrix} I & X_{11}^1 \\ 0 & X_{21}^1 \end{bmatrix} \mathcal{S}_1 \Pi_\infty e^{\mathcal{H}}. \tag{3.31}$$

Using the facts that $X_{12}^1 = {X_{21}^1}^H$ and $X_{12}^1 x_0 = 0$, it follows from the second row of (3.31) that $x_0^H [X_{22}^1, I]\mathcal{S}_2 \Pi_0 = 0$. Since $\Pi_0[U_1, U_0] = [U_1, U_0]$, we have $x_0^H [X_{22}^1, I]\mathcal{S}_2[U_1, U_0] = 0$. Using the definition of $\mathcal{H}$ in (2.5) yields

$$\begin{aligned} x_0^H [X_{22}^1, I]\mathcal{S}_2 e^{\mathcal{H}t} &= x_0^H [X_{22}^1, I]\mathcal{S}_2 [U_1|U_0, U_\infty] \left[\begin{array}{c|c} e^{\widehat{\mathcal{H}}t} & 0 \\ \hline 0 & I_{2\ell} \end{array}\right] [U_1|U_0, U_\infty]^{-1} \\ &= \left[0, 0, x_0^H [X_{22}^1, I]\mathcal{S}_2 U_\infty\right] [U_1|U_0, U_\infty]^{-1}, \end{aligned}$$

which is independent of the parameter t. Therefore, we may denote $z_0^H = x_0^H [X_{22}^1, I]\mathcal{S}_2 e^{\mathcal{H}t}$. Post-multiplying x_0 to (3.29), it holds that

$$Q(t)x_0 = [I, 0]\mathcal{S}_2 \mathcal{J} \left(e^{\mathcal{H}t^H} \mathcal{S}_2^H \begin{bmatrix} X_{22}^1 \\ I \end{bmatrix} x_0 \right) = [I, 0]\mathcal{S}_2 \mathcal{J} z_0$$

which is independent of the parameter t. Because $Q(0) = I$ and $x_0 \neq 0$, we have $Q(\hat{t})x_0 = Q(0)x_0 \neq 0$. This contradicts that $Q(\hat{t})x_0 = 0$.

Now, we show that $\lim_{t\to\hat{t}} \|P_\star(t)Q_\star(t)^{-1}\| = \lim_{t\to\hat{t}} \|X_{21}^1 Q_\star(t)^{-1}\| = \infty$. Using the fact that $X_{21}^1 Q_\star(t)^{-1} = X_{21}(t+1) = X_{12}(t+1)^H = (X_{12}^1 Q(t)^{-1})^H$, we have $\lim_{t\to\hat{t}} \|X_{21}^1 Q_\star(t)^{-1}\| = \infty$. Consequently, $Q_\star(\hat{t})$ is singular. Then $\lim_{t\to\hat{t}} \|P_\star(t)Q_\star(t)^{-1}\| = \infty$ can be proven by the similar argument for $\lim_{t\to\hat{t}} \|P(t)Q(t)^{-1}\| = \infty$. This proves the inclusion $\{t \in \mathbb{R} | \det(Q(t)) = 0\} \subseteq \{t \in \mathbb{R} | \det(Q_\star(t)) = 0\}$. The conclusion can be shown accordingly by (3.30). Hence, (3.28) holds true. □

Now, let

$$\mathcal{T}_W = \{t \in \mathbb{R} | \ Q(t) \text{ is invertible}\}. \tag{3.32}$$

Theorem 3.17 enables us to write the set $\mathcal{T}_W$ in an alternative form $\mathcal{T}_W = \{t \in \mathbb{R} | \ Q_\star(t) \text{ is invertible}\}$. From (3.29), $\det(Q(t))$ is a nonzero analytic function, and hence, the zeros of $\det(Q(t))$ are isolated. It follows $\mathcal{T}_W$ is the set that $\mathbb{R}$ subtracts some isolated points, and hence, $\mathcal{T}_W$ is a union of open intervals, say

$$\mathcal{T}_W = \bigcup_{k\in\mathbb{Z}} (\hat{t}_k, \hat{t}_{k+1}). \tag{3.33}$$

Here $\det(Q(\hat{t}_k)) = 0$ for each k and $\cdots < \hat{t}_{-1} < \hat{t}_0 < \hat{t}_1 < \cdots$. Since $Q(0) = I$, it implies that $0 \in \mathcal{T}_W$. For convenience, say $0 \in (\hat{t}_0, \hat{t}_1)$. Therefore, from Radon's Lemma it follows that $(\hat{t}_0, \hat{t}_1)$ is the maximal interval of the RDEs (3.20) and (3.25). In Subsection 3.3, we shall extend the domain of $W(t)$ and $W_\star(t)$ to whole $\mathcal{T}_W$.

3.3. The extension of structure-preserving flow: the phase portrait on Grassmann manifolds. Let $G^n(\mathbb{C}^{2n})$ be the Grassmann manifold that consists of n-dimensional subspaces of a $2n$-dimensional space, equipped with an appropriate topology (see e.g., [1]). Intrinsically, $G^n(\mathbb{C}^{2n})$ can be written as

$$G^n(\mathbb{C}^{2n}) = \left\{ \mathrm{Im}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \mid A, B \in \mathbb{C}^{n\times n} \text{ and rank}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = n \right\}.$$

Here $\mathrm{Im}\big([A^\top, B^\top]^\top\big)$ denotes the column space spanned by $[A^\top, B^\top]^\top$. It is easily seen that $\mathbb{C}^{n\times n}$ can be embedded into $G^n(\mathbb{C}^{2n})$ by $\psi(W) = \mathrm{Im}\left([I, W^\top]^\top\right)$. Let $G_0^n(\mathbb{C}^{2n}) = \left\{\mathrm{Im}\left([A^\top, B^\top]^\top\right) \in G^n(\mathbb{C}^{2n}) \mid A \in \mathbb{C}^{n\times n} \text{ is invertible}\right\}$. Then $G_0^n(\mathbb{C}^{2n}) = \psi(\mathbb{C}^{n\times n})$ is the image of ψ. Note that the Grassmann manifold $G^n(\mathbb{C}^{2n})$ is a compact analytic manifold of dimension n^2 and that $G_0^n(\mathbb{C}^{2n})$ is an open dense subset of $G^n(\mathbb{C}^{2n})$.

Radon's Lemma leads us to consider a natural extension of the solution of the RDE (3.20) in $\mathbb{C}^{n\times n}$ to a function on the Grassmann manifold $G^n(\mathbb{C}^{2n})$, via the process by the embedding $\psi(W(t)) = \mathrm{Im}([I, W(t)^\top]^\top) = \mathrm{Im}([Q(t)^\top, P(t)^\top]^\top)$. Hence, the solution of the RDE (3.20) on $G^n(\mathbb{C}^{2n})$ is just the solution of (3.21). Note that the maximal interval of the solution of (3.21) is $\mathbb{R}$. In addition, the representation of Theorem 3.14 (ii) holds not only for all $t \in (\hat{t}_0, \hat{t}_1)$ but also for $t \in \mathcal{T}_W$ defined in (3.32). Hence, the *extended solution* of the RDE (3.20) is

$$W(t) = P(t)Q(t)^{-1}, \quad \text{for } t \in \mathcal{T}_W$$

where $[Q(t)^\top, P(t)^\top]^\top$ is the solution of (3.21). Here, $\psi(W(t)) \in G_0^n(\mathbb{C}^{2n})$ for $t \in \mathcal{T}_W$. In the case $t \notin \mathcal{T}_W$, i.e., $t = \hat{t}_k$ for some $k \in \mathbb{Z}$, $W(t)$ does not exist but $\mathrm{Im}\left([Q(t)^\top, P(t)^\top]^\top\right) \in G^n(\mathbb{C}^{2n}) \setminus G_0^n(\mathbb{C}^{2n})$. Since $[Q(t)^\top, P(t)^\top]^\top = e^{\widetilde{\mathcal{H}}t}[I, W_0^\top]^\top$, both $Q(t)$ and $P(t)$ are analytic functions of t, and hence, $W(t)$ is meromorphic. We note that the unboundedness of $\mathcal{T}_W$ implies that the limit, $\lim_{t\to\infty} W(t)$, is meaningful. The asymptotic phenomena of the phase portrait of the RDE (3.20) can be investigated by using the extended solution of the RDE.

Theorem 3.17 shows that $Q(t)$ and $Q_\star(t)$ are simultaneously invertible, where $Y(t) = [Q(t)^\top, P(t)^\top]^\top$ and $Y_\star(t) = [Q_\star(t)^\top, P_\star(t)^\top]^\top$ are the solutions of (3.21) and (3.26), respectively. From Corollary 3.16 and (3.27), the *extended solution*, $X(t) = [X_{ij}(t)]_{1\leqslant i,j\leqslant 2}$, of the IVP (3.1) can be defined as

$$\begin{aligned} X_{11}(t) &= P_\star(t-1)Q_\star(t-1)^{-1}, & X_{12}(t) &= X_{12}^1 Q(t-1)^{-1}, \\ X_{21}(t) &= X_{21}^1 Q_\star(t-1)^{-1}, & X_{22}(t) &= P(t-1)Q(t-1)^{-1}, \end{aligned} \tag{3.34}$$

for $t \in \mathcal{T}_W + 1$, where $\mathcal{T}_W + 1$ denotes the set

$$\mathcal{T}_W + 1 \equiv \{t+1 | t \in \mathcal{T}_W\} = \{t \in \mathbb{R} |\ Q(t-1) \text{ is invertible}\}. \tag{3.35}$$

In Remark 3.12, we demonstrate that the maximal interval of the IVP (3.1), i.e., the maximal interval of $\mathcal{T}_W + 1$ containing 1, coincides with the connected component of $\mathcal{T}_X$ containing 1. In the following theorem we will show that $\mathcal{T}_W + 1 = \mathcal{T}_X$ and $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$ satisfies (3.10) for $t \in \mathcal{T}_W + 1$, where $X(t)$ is the extended solution of the IVP (3.1), and vice versa.

THEOREM 3.18. *Suppose the assumptions of Theorem 3.3 hold.*

(i) If $X(t)$, for $t \in \mathcal{T}_W + 1$, is the extended solution of the IVP (3.1)*, then $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$ satisfies* (3.10) *for $t \in \mathcal{T}_W + 1$;*

(ii) $\mathcal{T}_W + 1 = \mathcal{T}_X$ where $\mathcal{T}_X$ is defined in (3.16)*;*

(iii) if $(\mathcal{M}(t), \mathcal{L}(t))$ is the solution of (3.10) *for $t \in \mathcal{T}_X$, then $X(t) = T^{-1}_{\mathcal{S}_1,\mathcal{S}_2}(\mathcal{M}(t), \mathcal{L}(t))$ is the extended solution of the IVP* (3.1).

Proof. We first prove assertion (i). Suppose that $X(t) = [X_{ij}(t)]_{1\leqslant i,j\leqslant 2}$ for $t \in \mathcal{T}_W + 1$, defined in (3.34), is the extended solution of the IVP (3.1). Since $X_{22}(t)$ is Hermitian and $X_{21}(t) = X_{12}(t)^H$, it holds that

$$[X_{21}(t), X_{22}(t)] = Q(t-1)^{-H}[X_{12}^{1^H}, P(t-1)^H], \tag{3.36}$$

where $[Q(t)^\top, P(t)^\top]^\top$ is the solution of the IVP (3.21). Using the definitions of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ in (2.5) and (3.22), respectively, we have

$$\begin{bmatrix} Q(t-1) \\ P(t-1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \mathcal{S}_2 \mathbf{U}(e^{-\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell}) \mathbf{U}^{-1} \mathcal{S}_2^{-1} \begin{bmatrix} I \\ -X_{22}^1 \end{bmatrix}. \tag{3.37}$$

Since $\mathcal{S}_2$ and $e^{-\mathcal{H}(t-1)} = \mathbf{U}(e^{-\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell})\mathbf{U}^{-1}$ are symplectic, we have

$$\mathcal{J}\mathcal{S}_2 \mathbf{U}(e^{-\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell}) \mathbf{U}^{-1} \mathcal{S}_2^{-1} = \mathcal{S}_2^{-H} \mathbf{U}^{-H} (e^{\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell})^H \mathbf{U}^H \mathcal{S}_2^H \mathcal{J}.$$

Applying the last equation to (3.37) it follows that

$$\begin{aligned} \mathbf{U}^H \mathcal{S}_2^H \begin{bmatrix} P(t-1) \\ Q(t-1) \end{bmatrix} &= -(e^{\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell})^H \mathbf{U}^H \mathcal{S}_2^H \mathcal{J} \begin{bmatrix} I \\ -X_{22}^1 \end{bmatrix} \\ &= (e^{\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell})^H \mathbf{U}^H \mathcal{S}_2^H \begin{bmatrix} X_{22}^1 \\ I \end{bmatrix}. \end{aligned} \tag{3.38}$$

Using the fact that $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ satisfies (3.5), we have

$$\begin{bmatrix} X_{12}^1 & 0 \\ X_{22}^1 & I \end{bmatrix} \mathcal{S}_2 \mathbf{U}(I_{2\hat{n}} \oplus E_{11}) = \begin{bmatrix} I & X_{11}^1 \\ 0 & X_{21}^1 \end{bmatrix} \mathcal{S}_1 \mathbf{U}(e^{\widehat{\mathcal{H}}} \oplus E_{22}), \tag{3.39}$$

where E_{11} and E_{22} are defined in (3.12a) and $\hat{n} = n - \ell$. Since ${X_{22}^1}^H = X_{22}^1$ and ${X_{21}^1}^H = X_{12}^1$, it follows from (3.38) and (3.39) that

$$\begin{aligned} &(I_{2\hat{n}} \oplus E_{11})[{\mathbf{V}_2^2}^H, {\mathbf{V}_1^2}^H] \begin{bmatrix} Q(t-1) \\ P(t-1) \end{bmatrix} = (I_{2\hat{n}} \oplus E_{11}) \mathbf{U}^H \mathcal{S}_2^H \begin{bmatrix} P(t-1) \\ Q(t-1) \end{bmatrix} \\ &= (e^{\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell})^H (I_{2\hat{n}} \oplus E_{11})^H \mathbf{U}^H \mathcal{S}_2^H \begin{bmatrix} X_{22}^1 \\ I \end{bmatrix} \\ &= (e^{\widehat{\mathcal{H}}(t-1)} \oplus I_{2\ell})^H (e^{\widehat{\mathcal{H}}} \oplus E_{22})^H \mathbf{U}^H \mathcal{S}_1^H \begin{bmatrix} 0 \\ {X_{21}^1}^H \end{bmatrix} = (e^{\widehat{\mathcal{H}}t} \oplus E_{22})^H {\mathbf{V}_2^1}^H X_{12}^1, \end{aligned}$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ are defined in (3.12b). We then have

$$[-(e^{\widehat{\mathcal{H}}t} \oplus E_{22})^H {\mathbf{V}_2^1}^H, (I_{2\hat{n}} \oplus E_{11}){\mathbf{V}_1^2}^H] \begin{bmatrix} X_{12}^1 \\ P(t-1) \end{bmatrix} = -(I_{2\hat{n}} \oplus E_{11}) {\mathbf{V}_2^2}^H Q(t-1).$$

Combining the last equation and (3.36), we obtain $[X_{21}(t), X_{22}(t)] \begin{bmatrix} -\mathbf{V}_2^1(e^{\widehat{\mathcal{H}}t} \oplus E_{22}) \\ \mathbf{V}_1^2(I_{2\hat{n}} \oplus E_{11}) \end{bmatrix} = -\mathbf{V}_2^2(I_{2\hat{n}} \oplus E_{11})$. Therefore, the equality of the second row of (3.13) holds. The equality of the first row can be accordingly obtained by using the formulas for $X_{11}(t)$ and $X_{12}(t) = X_{21}(t)^H$ in (3.34) and the solution $Y_\star(t) = [Q_\star(t)^\top, P_\star(t)^\top]^\top$ of the linear differential equation (3.26). Since (3.13) is equivalent to (3.10) by Lemma 3.6, this shows assertion (i).

Now we prove assertion (ii). From (i), we have $\mathcal{T}_W + 1 \subseteq \mathcal{T}_X$. From (3.33) and (3.35), we obtain that $\mathcal{T}_W + 1 = \bigcup_{k\in\mathbb{Z}} (\hat{t}_k + 1, \hat{t}_{k+1} + 1) \subseteq \mathcal{T}_X$. For each $k \in \mathbb{Z}$, we have that $(\hat{t}_k + 1, \hat{t}_{k+1} + 1) \subseteq \mathcal{T}_X$. Choosing a point $t_{k+1/2} \in (\hat{t}_k + 1, \hat{t}_{k+1} + 1)$, it follows from (i) that $(\mathcal{M}(t_{k+1/2}), \mathcal{L}(t_{k+1/2})) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t_{k+1/2}))$ is the solution of (3.10) at $t = t_{k+1/2}$. A similar argument to Theorem 3.11 and Remark 3.12 shows that $(\hat{t}_k + 1, \hat{t}_{k+1} + 1)$ is the connected component of $\mathcal{T}_X$ containing $t_{k+1/2}$. Hence, $\mathcal{T}_W + 1 = \mathcal{T}_X$.

Now we prove assertion (iii). From Theorem 3.8 it follows that the solution $(\mathcal{M}(t), \mathcal{L}(t))$ of (3.10) is unique for each $t \in \mathcal{T}_X$. Therefore, assertions (i) and (ii) lead to the fact that $X(t) = T_{\mathcal{S}_1,\mathcal{S}_2}^{-1}(\mathcal{M}(t), \mathcal{L}(t))$ is the extended solution. This completes the proof. ∎

3.4. Structure-preserving flow vs. SDA. The structure-preserving flow $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t)) \in \mathbb{S}_{\mathcal{S}_1,\mathcal{S}_2}$ with the initial $(\mathcal{M}(1), \mathcal{L}(1)) = (\mathcal{M}_1, \mathcal{L}_1)$ has been constructed in Theorem 3.3, where $X(t)$ for $t \in \mathcal{T}_X$ is the extended solution of the IVP (3.1). In addition, Theorem 3.18 shows the phase portrait of this flow is actually the curve $\mathcal{C}_{\mathcal{M}_1,\mathcal{L}_1} = \{(\mathcal{M}(t), \mathcal{L}(t)) \mid (\mathcal{M}(t), \mathcal{L}(t)) \text{ is a solution of (3.10) at } t \in \mathcal{T}_X\}$. On the other hand, $\mathcal{C}_{\mathcal{M}_1,\mathcal{L}_1}$ is actually the so-called *Structuring-Preserving Curve* constructed in the work [22] for considering the case $\mathbb{S}_2$, in which $\mathcal{C}_{\mathcal{M}_1,\mathcal{L}_1}$ passes through the iterations of the SDA-2. We now show the Main Result II which is a generalization of the work [22].

THEOREM 3.19 (Main Result II). *Let $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ be the sequence generated by SDA-1 or SDA-2 with $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_1$ or $\mathbb{S}_2$, respectively, and $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$. Then $(\mathcal{M}_k, \mathcal{L}_k) = (\mathcal{M}(2^{k-1}), \mathcal{L}(2^{k-1}))$, where $(\mathcal{M}(t), \mathcal{L}(t)) = T_{\mathcal{S}_1,\mathcal{S}_2}(X(t))$ and $X(t)$ is the extended solution of the IVP* (3.1).

Proof. By the **Eigenvector-Preserving Property** for SDAs, we have

$$\mathcal{M}_k U_0 = 0, \quad \mathcal{L}_k U_\infty = 0 \quad \text{and } \mathcal{M}_k U_1 = \mathcal{L}_k U_1 e^{\widehat{\mathcal{H}} 2^{k-1}} \tag{3.40}$$

for each $k \in \mathbb{N}$, where the initial pair $(\mathcal{M}_1, \mathcal{L}_1)$ satisfies (2.4) with $\widehat{\mathcal{S}} = e^{\widehat{\mathcal{H}}}$. By applying Theorem 3.18 (*iii*) to (3.40), we have $(\mathcal{M}_k, \mathcal{L}_k) = (\mathcal{M}(2^{k-1}), \mathcal{L}(2^{k-1}))$, and hence, the assertion follows. ◻

4. Application to the convergence analysis of SDA. The structure-preserving flows are governed by the RDEs of the compact form:

$$\dot{W}(t) = [-W(t), I]\mathscr{H}[I, W(t)^\top]^\top, \; W(0) = W_0, \tag{4.1}$$

where $\mathscr{H}$ is a Hamiltonian matrix. By Theorem 3.14, the extended solution $W(t; \mathscr{H}, W_0) = P(t; \mathscr{H}, W_0)Q(t; \mathscr{H}, W_0)^{-1}$ where $Q(t; \mathscr{H}, W_0)$ and $P(t; \mathscr{H}, W_0)$ are of the form

$$Y(t; \mathscr{H}, W_0) \equiv \begin{bmatrix} Q(t; \mathscr{H}, W_0) \\ P(t; \mathscr{H}, W_0) \end{bmatrix} = e^{\mathscr{H} t} \begin{bmatrix} I \\ W_0 \end{bmatrix}. \tag{4.2}$$

Specifically, RDEs (3.20) and (3.25) are of the compact form as in (4.1) with $\mathscr{H} = \widetilde{\mathcal{H}} = -(I_n \oplus -I_n)\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}(I_n \oplus -I_n)$ and $\mathscr{H} = \widetilde{\mathcal{H}}_\star = \mathcal{J}^{-1}\mathcal{S}_1\mathcal{H}\mathcal{S}_1^{-1}\mathcal{J}$ given in (3.22) and (3.26b), respectively. Applying the relation between $\widetilde{\mathcal{H}}$ and $\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}$ together with (4.2), Eqs. (3.23) and (3.27) have the alternative form:

$$\begin{aligned} X_{22}(t) &= W(t-1; \widetilde{\mathcal{H}}, X_{22}^1) = -W(t-1; -\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1), \\ &= -P(t-1; -\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)Q(t-1; -\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)^{-1}, \\ &= -P(-t+1; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)Q(-t+1; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)^{-1}, \\ &= -W(-t+1; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1) \\ X_{12}(t) &= X_{12}^1 Q(-t+1; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)^{-1}, \end{aligned} \tag{4.3a}$$

$$\begin{aligned} X_{11}(t) &= W(t-1; \widetilde{\mathcal{H}}_\star, X_{11}^1) = P(t-1; \widetilde{\mathcal{H}}_\star, X_{11}^1)Q(t-1; \widetilde{\mathcal{H}}_\star, X_{11}^1)^{-1}, \\ X_{21}(t) &= X_{21}^1 Q(t-1; \widetilde{\mathcal{H}}_\star, X_{11}^1)^{-1}, \end{aligned} \tag{4.3b}$$

for $t \in \mathcal{T}_W + 1$. We conclude that

(*i*) the large time behaviors of $X_{22}(t)$, $X_{12}(t)$ as $t \to \infty$ are determined by $W(t; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)$ and $Q(t; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)^{-1}$ as $t \to -\infty$;

(*ii*) the large time behaviors of $X_{11}(t)$, $X_{21}(t)$ as $t \to \infty$ are determined by $W(t; \widetilde{\mathcal{H}}_\star, X_{11}^1)$ and $Q(t; \widetilde{\mathcal{H}}_\star, X_{11}^1)^{-1}$ as $t \to \infty$.

Note that Hamiltonian matrices $\mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}$ and $\widetilde{\mathcal{H}}_\star$ are symplectically similar. By assertions (i) and (ii) above, we see that the asymptotic behaviors of $X_{22}(t)$, $X_{12}(t)$ and $X_{11}(t)$, $X_{21}(t)$ as $t \to \infty$ are governed by

$$Y(t; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1) = \mathcal{S}_2 e^{\mathcal{H}t}\mathcal{S}_2^{-1}\begin{bmatrix} I \\ -X_{22}^1 \end{bmatrix} (\text{as } t \to -\infty), \tag{4.4a}$$

and

$$Y(t; \widetilde{\mathcal{H}}_\star, X_{11}^1) = \mathcal{J}^{-1}\mathcal{S}_1 e^{\mathcal{H}t}\mathcal{S}_1^{-1}\mathcal{J}\begin{bmatrix} I \\ X_{11}^1 \end{bmatrix} (\text{as } t \to \infty), \tag{4.4b}$$

respectively. For both cases in (4.4a) and (4.4b), $e^{\mathcal{H}t}$ is involved. Therefore, for a given Hamiltonian matrix $\mathscr{H}$, we are interested in the study of the asymptotic behavior of the solution, $W(t) = P(t)Q(t)^{-1}$, of the RDE (4.1) and $Q(t)^{-1}$ as $t \to \pm\infty$.

Due to the special structure of the Hamiltonian matrix $\mathscr{H}$, rather than applying a Jordan canonical form to $\mathscr{H}$, we shall adopt the Hamiltonian Jordan canonical form for studying the asymptotic behavior of RDEs. A canonical form of a Hamiltonian matrix under symplectic similarity transformations has been investigated in [26]. Let N_k be the $k \times k$ nilpotent matrix, and let $N_k(\lambda) = \lambda I_k + N_k$ be the Jordan block of the eigenvalue λ with size k.

In the following, we shall apply the asymptotic analysis of (4.3). Throughout this section, we fix $(\mathcal{S}_1, \mathcal{S}_2) = (I, I)$ (the $\mathbb{S}_1$ class) or $(\mathcal{J}, -I)$ (the $\mathbb{S}_2$ class) and let the pair $(\mathcal{M}_1, \mathcal{L}_1) \in \mathbb{S}_1$ or $\mathbb{S}_2$ be regular with $\mathrm{ind}_\infty(\mathcal{M}_1, \mathcal{L}_1) \leqslant 1$.

4.1. Proof of Main Result III (i)**.** To simplify the proof, we suppose that $\mathcal{H}$ has no other eigenvalues than λ and $-\bar{\lambda}$ and there is only one Jordan block for each of the two eigenvalues. Denote $r = \mathrm{Re}(\lambda) > 0$. Then from [26], there is a symplectic matrix $\mathcal{S}$ such that $\mathfrak{J} := \mathcal{S}^{-1}\mathcal{H}\mathcal{S} = N_n(\lambda) \oplus (-N_n(\lambda)^H)$. Let $\mathcal{S}_- = \mathcal{S}_2\mathcal{S}$, $\mathcal{S}_+ = \mathcal{J}^{-1}\mathcal{S}_1\mathcal{S}$. Partition $\mathcal{S}_\pm = \left[\begin{array}{c|c} U_1^\pm & V_1^\pm \\ \hline U_2^\pm & V_2^\pm \end{array}\right]$, where $U_1^\pm, U_2^\pm, V_1^\pm, V_2^\pm \in \mathbb{C}^{n\times n}$. Let

$$\begin{bmatrix} W_1^- \\ W_2^- \end{bmatrix} = \mathcal{S}_-^{-1}\begin{bmatrix} I \\ -X_{22}^1 \end{bmatrix}, \quad \begin{bmatrix} W_1^+ \\ W_2^+ \end{bmatrix} = \mathcal{S}_+^{-1}\begin{bmatrix} I \\ X_{11}^1 \end{bmatrix} \in \mathbb{C}^{2n\times n}. \tag{4.5}$$

Then we have following results.

Lemma 4.1. *If U_1^+, V_1^-, W_1^+ and W_2^- are invertible, then*

$$X_{22}(t) = -V_2^-(V_1^-)^{-1} + O(e^{-2rt}t^{2(n-1)}), \quad X_{12}(t) = O(e^{-rt}t^{n-1}), \tag{4.6a}$$
$$X_{11}(t) = U_2^+(U_1^+)^{-1} + O(e^{-2rt}t^{2(n-1)}), \quad X_{21}(t) = O(e^{-rt}t^{n-1}), \tag{4.6b}$$

as $t \to \infty$.

Proof. We only prove (4.6a). Eq. (4.6b) can be obtained similarly. Since $\mathfrak{J} = N_n(\lambda) \oplus (-N_n(\lambda)^H)$, we have $e^{\mathfrak{J}t} = (e^{\lambda t}\Phi_n) \oplus (e^{-\bar{\lambda}t}\Phi_n^{-H})$, where $\Phi_n \equiv \Phi_n(t) = e^{N_n t}$. From (4.4a), we have

$$\begin{bmatrix} Q(t) \\ P(t) \end{bmatrix} \equiv Y(t; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1) = \left[\begin{array}{c|c} U_1^- & V_1^- \\ \hline U_2^- & V_2^- \end{array}\right]\begin{bmatrix} e^{\lambda t}\Phi_n W_1^- \\ e^{-\bar{\lambda}t}\Phi_n^{-H}W_2^- \end{bmatrix}. \tag{4.7}$$

It is well known that $\Phi_n^{-1}(t) = e^{-N_n t}$ is a polynomial of order $n-1$ and hence, $\|\Phi_n^{-H}\| = O(t^{n-1})$. Since $Q(-t+1) = e^{\bar{\lambda}(t-1)}(V_1^-\Phi_n^{-H}W_2^- + e^{-2r(t-1)}U_1^-\Phi_n W_1^-)$, if V_1^- and W_2^- are invertible, then we have $X_{12}(t) = X_{12}^1 Q(-t+1)^{-1} = O(e^{-rt}t^{n-1})$

as $t \to \infty$. From (4.3a), the second assertion in (4.6a) follows. Using the facts that Φ_n, V_1^- and W_2^- are invertible, we obtain from (4.3a) and (4.7) that

$$\begin{aligned}
&X_{22}(t) = -P(-t+1)Q(-t+1)^{-1} \\
&= -(V_2^- + e^{-2r(t-1)}U_2^-\Phi_n W_1^-(W_2^-)^{-1}\Phi_n^H)(V_1^- + e^{-2r(t-1)}U_1^-\Phi_n W_1^-(W_2^-)^{-1}\Phi_n^H)^{-1} \\
&= -V_2^-(V_1^-)^{-1} + O(e^{-2rt}t^{2(n-1)}),
\end{aligned}$$

as $t \to \infty$. We complete the proof. □

Applying the results of Lemma 4.1 to Theorem 3.19 directly, the following theorem follows.

THEOREM 4.2. *Let $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ be the sequence generated by the SDA-1 or SDA-2. Suppose that $\mathcal{H}$ has only eigenvalues λ and $-\bar{\lambda}$ with $r = \mathrm{Re}(\lambda) > 0$. If U_1^+, V_1^-, W_1^+ and W_2^- are invertible, then*

$$\begin{aligned}
X_{22}^k &= -V_2^-(V_1^-)^{-1} + O(e^{-r2^k}2^{2nk}), \quad & X_{12}^k &= O(e^{-r2^{k-1}}2^{nk}), \\
X_{11}^k &= U_2^+(U_1^+)^{-1} + O(e^{-r2^k}2^{2nk}), \quad & X_{21}^k &= O(e^{-r2^{k-1}}2^{nk}),
\end{aligned}$$

as $k \to \infty$, where $[X_{ij}^k]_{1\le i,j\le 2} \equiv T_{\mathcal{S}_1,\mathcal{S}_2}^{-1}(\mathcal{M}_k, \mathcal{L}_k)$.

This theorem shows that the SDA exhibits a quadratic convergence whenever none of nonzero eigenvalues of $\mathcal{H}$ are pure imaginary. A similar convergence analysis has been carried out in [3, 16].

4.2. Proof of Main Result III *(ii)***.** We only consider that $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ has only one pure imaginary eigenvalue $i\alpha$ having one Jordan block of size $2n$. From [26], there is a symplectic matrix $\mathcal{S}$ such that $\mathfrak{J} := \mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \left[\begin{array}{c|c} N_n(i\alpha) & \beta e_n e_n^H \\ \hline 0 & -(N_n(i\alpha))^H \end{array}\right]$, where $\beta \in \{-1, 1\}$. Note that $N_{2n}(i\alpha) = \Theta\mathfrak{J}\Theta^{-1}$, where $\Theta = I_n \oplus (-\beta P_n) \in \mathbb{R}^{2n\times 2n}$ and $P_n(j, n+1-j) = (-1)^j$, $j = 1, \cdots, n$. We denote

$$\begin{aligned}
\Gamma_n \equiv \Gamma_n(t) &= \begin{bmatrix} \frac{t^n}{n!} & \frac{t^{(n+1)}}{(n+1)!} & \cdots & \frac{t^{2n-1}}{(2n-1)!} \\ \frac{t^{(n-1)}}{(n-1)!} & \frac{t^n}{n!} & \ddots & \frac{t^{(2n-2)}}{(2n-2)!} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{t^1}{1!} & \cdots & \cdots & \frac{t^n}{n!} \end{bmatrix}, \qquad (4.8) \\
\widehat{\Gamma}_n \equiv \widehat{\Gamma}_n(t) &= \Gamma_n P_n,
\end{aligned}$$

Then $e^{\mathfrak{J}t} = e^{i\alpha t}\Theta^{-1}e^{N_{2n}t}\Theta$ has the structured form $e^{\mathfrak{J}t} = e^{i\alpha t}\left[\begin{array}{c|c} \Phi_n(t) & -\beta\widehat{\Gamma}_n(t) \\ \hline 0 & (\Phi_n(t))^{-H} \end{array}\right]$, where $t \in \mathbb{R}$, $\Phi_n(t) = e^{N_n t}$ and $\widehat{\Gamma}_n(t)$ is defined in (4.8).

Let $\mathcal{S}_- = \mathcal{S}_2\mathcal{S}$, $\mathcal{S}_+ = \mathcal{J}^{-1}\mathcal{S}_1\mathcal{S}$. Partition $\mathcal{S}_\pm = \left[\begin{array}{c|c} U_1^\pm & V_1^\pm \\ \hline U_2^\pm & V_2^\pm \end{array}\right]$, where $U_j^\pm, V_j^\pm \in \mathbb{C}^{n\times n}$, $j = 1, 2$. Let $W_1^\pm$ and $W_2^\pm$ be defined as (4.5). Then we have following results.

LEMMA 4.3. *If $U_1^\pm$, $W_2^\pm$ are invertible, then*

$$X_{22}(t) = -U_2^-(U_1^-)^{-1} + O(t^{-1}), \quad X_{12}(t) = O(t^{-1}), \qquad (4.9a)$$

$$X_{11}(t) = U_2^+(U_1^+)^{-1} + O(t^{-1}), \quad X_{21}(t) = O(t^{-1}), \qquad (4.9b)$$

as $t \to \infty$.

Proof. We only prove (4.9a). Eq. (4.9b) can be obtained similarly. From the structure form of $e^{\mathfrak{J}t}$ above and (4.4a) we have

$$\left[\begin{array}{c} Q(t) \\ P(t) \end{array}\right] \equiv Y(t; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1) = e^{i\alpha t}\left[\begin{array}{c|c} U_1^- & V_1^- \\ \hline U_2^- & V_2^- \end{array}\right]\left[\begin{array}{c} \Phi_n W_1^- - \beta\widehat{\Gamma}_n^{2n-1}W_2^- \\ \Phi_n^{-H}W_2^- \end{array}\right].$$

Since W_2^- is invertible, we let $\Omega(t) = (\Phi_n W_1^-(W_2^-)^{-1} - \beta\widehat{\Gamma}_n^{2n-1})^{-1}$. Using the fact that $\Phi_n^{-H}\Omega(t)^{-1} = O(t^{-1})$ as $t \to -\infty$ (the detail can be found [21]), we have

$$\left[\begin{array}{c} Q(t) \\ P(t) \end{array}\right](W_2^-)^{-1}\Omega(t) = e^{i\alpha t}\left[\begin{array}{c|c} U_1^- & V_1^- \\ \hline U_2^- & V_2^- \end{array}\right]\left[\begin{array}{c} I \\ O(t^{-1}) \end{array}\right] = e^{i\alpha t}\left[\begin{array}{c} U_1^- + O(t^{-1}) \\ U_2^- + O(t^{-1}) \end{array}\right],$$

as $t \to -\infty$. Then $Q(-t+1)^{-1} = e^{i\alpha(t-1)}(W_2^-)^{-1}\Omega(-t+1)(U_1^- + O(t^{-1}))^{-1} = O(t^{-1})$ because $\Omega(-t+1) = O(t^{-1})$ as $t \to \infty$. Since U_1^- is invertible, from (4.3a) we have $X_{22}(t) = -P(-t+1)Q(-t+1)^{-1} = -U_2^-(U_1^-)^{-1} + O(t^{-1})$ and $X_{12}(t) = X_{12}^1 Q(-t+1)^{-1} = O(t^{-1})$, as $t \to \infty$. We complete the proof. □

Applying the results of Lemma 4.3 to Theorem 3.19, the Theorem 4.4 follows.

THEOREM 4.4. *Let $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^{\infty}$ be the sequence generated by the SDA-1 or SDA-2. Suppose that $\mathcal{H}$ has only one pure imaginary eigenvalue $i\alpha$ with partial multiplicity $2n$. If $U_1^{\pm}$, $W_2^{\pm}$ are invertible, then*

$$\begin{aligned} X_{22}^k &= -U_2^-(U_1^-)^{-1} + O(2^{-k}), \quad & X_{12}^k &= O(2^{-k}), \\ X_{11}^k &= U_2^+(U_1^+)^{-1} + O(2^{-k}), \quad & X_{21}^k &= O(2^{-k}), \end{aligned}$$

as $k \to \infty$, where $[X_{ij}^k]_{1\le i,j\le 2} \equiv T_{\mathcal{S}_1,\mathcal{S}_2}^{-1}(\mathcal{M}_k, \mathcal{L}_k)$.

This theorem shows that the SDA exhibits a linear convergence whenever the partial multiplicities corresponding to nonzero pure imaginary eigenvalues of $\mathcal{H}$ are even. A similar convergence analysis has been proven in [20].

4.3. Proof of Main Result III *(iii)***.** Suppose that $\mathcal{H}$ has only one pure imaginary eigenvalue $i\alpha$ with two odd partial multiplicities $2n_1 + 1$ and $2n_2 + 1$. Note that $n_1 + n_2 + 1 = n$. By [26], there is a symplectic matrix $\mathcal{S}$ such that $\mathfrak{J} := \mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \left[\begin{array}{c|c} R & D \\ \hline 0 & -R^H \end{array}\right]$, where

$$R = \left[\begin{array}{ccc} N_{n_1}(i\alpha) & 0 & -\frac{\sqrt{2}}{2}e_{n_1} \\ 0 & N_{n_2}(i\alpha) & -\frac{\sqrt{2}}{2}e_{n_2} \\ 0 & 0 & i\alpha \end{array}\right], D = \frac{\sqrt{2}}{2}i\beta\left[\begin{array}{ccc} 0 & 0 & e_{n_1} \\ 0 & 0 & -e_{n_2} \\ -e_{n_1}^H & e_{n_2}^H & 0 \end{array}\right], \tag{4.10}$$

$\beta \in \{-1, 1\}$. Let $\mathcal{S}_- = \mathcal{S}_2\mathcal{S}$, $\mathcal{S}_+ = \mathcal{J}^{-1}\mathcal{S}_1\mathcal{S}$. Partition $\mathcal{S}_{\pm} = \left[\begin{array}{cc|cc} U_1^{\pm} & u_1^{\pm} & V_1^{\pm} & v_1^{\pm} \\ \hline U_2^{\pm} & u_2^{\pm} & V_2^{\pm} & v_2^{\pm} \end{array}\right]$, where $U_j^{\pm}, V_j^{\pm} \in \mathbb{C}^{n\times(n-1)}$ and $u_j^{\pm}, v_j^{\pm} \in \mathbb{C}^n$ for $j = 1, 2$. Let $W_1^{\pm}$ and $W_2^{\pm}$ be defined as (4.5). The following lemma is useful for the reduction of $Y(t)$. The detail of proof can be found in [21].

LEMMA 4.5. *Let $W_2^{\pm}$ be invertible and $\mathbf{W}^{\pm} := W_1^{\pm}(W_2^{\pm})^{-1} = \left[\begin{array}{c|c} \mathbf{W}_{1,1}^{\pm} & \mathbf{w}_{1,2}^{\pm} \\ \hline \mathbf{w}_{2,1}^{\pm} & \mathbf{w}_{2,2}^{\pm} \end{array}\right]$, where $\mathbf{W}_{1,1}^{\pm} \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$, $\mathbf{w}_{1,2}^{\pm}, (\mathbf{w}_{2,1}^{\pm})^H \in \mathbb{C}^{n_1+n_2}$ and $\mathbf{w}_{2,2}^{\pm} \in \mathbb{C}$. Let*

$$\mathbf{U}_{\pm} \equiv \left[\begin{array}{c} \mathbf{U}_1^{\pm} \\ \hline \mathbf{U}_2^{\pm} \end{array}\right] = \left[\begin{array}{c|c} U_1^{\pm} & f_u^{\pm}u_1^{\pm} + f_v^{\pm}v_1^{\pm} \\ \hline U_2^{\pm} & f_u^{\pm}u_2^{\pm} + f_v^{\pm}v_2^{\pm} \end{array}\right], \tag{4.11}$$

where $(f_u^{\pm}, f_v^{\pm}) = (-1)^{n_1}(\mathbf{w}_{2,2}^{\pm}, 1)$ *if* n *is odd; otherwise* $(f_u^{\pm}, f_v^{\pm}) = (-1)^{n_1} i\beta(-1, \mathbf{w}_{2,2}^{\pm})$. *Then there exist nonsingular matrices* $\Omega_{\pm}(t)$ *of the form*

$$\Omega_{\pm}(t) = \left[\begin{array}{c|c} O(t^{-2}) & O(t^{-1}) \\ \hline 0 & e^{-i\alpha t} \end{array}\right], \quad \text{as } t \to \pm\infty, \tag{4.12}$$

satisfying

$$\begin{aligned} Y(t; \mathcal{S}_2 \mathcal{H} \mathcal{S}_2^{-1}, -X_{22}^1)(W_2^-)^{-1}\Omega_-(t) &= \mathbf{U}_- + O(t^{-1}), \quad \text{as } t \to -\infty \\ Y(t; \widetilde{\mathcal{H}}_\star, X_{11}^1)(W_2^+)^{-1}\Omega_+(t) &= \mathbf{U}_+ + O(t^{-1}), \quad \text{as } t \to \infty. \end{aligned} \tag{4.13}$$

Then we have following results.

LEMMA 4.6. *If* $\mathbf{U}_1^{\pm}$, $W_2^{\pm}$ *are invertible, then*

$$\begin{aligned} X_{22}(t) &= -\mathbf{U}_2^-(\mathbf{U}_1^-)^{-1} + O(t^{-1}), \\ X_{12}(t) &= e^{i\alpha(t-1)} X_{12}^1 (W_2^-)^{-1} e_n e_n^H (\mathbf{U}_1^-)^{-1} + O(t^{-1}), \end{aligned} \tag{4.14a}$$

$$\begin{aligned} X_{11}(t) &= \mathbf{U}_2^+(\mathbf{U}_1^+)^{-1} + O(t^{-1}), \\ X_{21}(t) &= e^{-i\alpha(t-1)} X_{21}^1 (W_2^+)^{-1} e_n e_n^H (\mathbf{U}_1^+)^{-1} + O(t^{-1}), \end{aligned} \tag{4.14b}$$

as $t \to \infty$, *where* $\mathbf{U}_j^{\pm}$ *for* $j = 1, 2$ *are defined in* (4.11).

Proof. We only prove (4.14a). Eq. (4.14b) can be obtained similarly. Denote $[Q(t)^\top, P(t)^\top]^\top = Y(t; \mathcal{S}_2\mathcal{H}\mathcal{S}_2^{-1}, -X_{22}^1)$. From Lemma 4.5 and (4.3a), (4.4a) we have $X_{22}(t) = -P(-t+1)Q(-t+1)^{-1} = -\mathbf{U}_2^-(\mathbf{U}_1^-)^{-1} + O(t^{-1})$, as $t \to \infty$. Using the structure of $\Omega_-(t)$ in (4.12), we see $\Omega_-(-t+1) = e^{i\alpha(t-1)} e_n e_n^H + O(t^{-1})$ as $t \to \infty$. From (4.13) it follows that

$$\begin{aligned} Q(-t+1)^{-1} &= (W_2^-)^{-1}\Omega_-(-t+1)(\mathbf{U}_1^- + O(t^{-1}))^{-1} \\ &= e^{i\alpha(t-1)}(W_2^-)^{-1} e_n e_n^H (\mathbf{U}_1^-)^{-1} + O(t^{-1}), \end{aligned}$$

as $t \to \infty$. Hence, $X_{12}(t) = X_{12}^1 Q(-t+1)^{-1} = e^{i\alpha(t-1)} X_{12}^1 (W_2^-)^{-1} e_n e_n^H (\mathbf{U}_1^-)^{-1} + O(t^{-1})$ as $t \to \infty$. ◻

Applying Lemma 4.6 to Theorem 3.19, the Theorem 4.7 follows.

THEOREM 4.7. *Let* $\{(\mathcal{M}_k, \mathcal{L}_k)\}_{k=1}^\infty$ *be the sequence generated by the SDA-1 or SDA-2. Suppose that* $\mathcal{H}$ *has only one eigenvalue* $i\alpha$ *with two odd partial multiplicities* $2n_1 + 1$ *and* $2n_2 + 1$. *If* $\mathbf{U}_1^{\pm}$, $W_2^{\pm}$ *are invertible, then*

$$\begin{aligned} X_{22}^k &= -\mathbf{U}_2^-(\mathbf{U}_1^-)^{-1} + O(2^{-k}), \;\; X_{12}^k = e^{i\alpha(2^{k-1}-1)} X_{12}^1 (W_2^-)^{-1} e_n e_n^H (\mathbf{U}_1^-)^{-1} + O(2^{-k}), \\ X_{11}^k &= \mathbf{U}_2^+(\mathbf{U}_1^+)^{-1} + O(2^{-k}), \;\; X_{21}^k = e^{-i\alpha(2^{k-1}-1)} X_{21}^1 (W_2^+)^{-1} e_n e_n^H (\mathbf{U}_1^+)^{-1} + O(2^{-k}), \end{aligned}$$

as $k \to \infty$, *where* $[X_{ij}^k]_{1 \le i,j \le 2} \equiv T_{\mathcal{S}_1,\mathcal{S}_2}^{-1}(\mathcal{M}_k, \mathcal{L}_k)$.

5. Numerical experiments. In this section, we show some numerical experiments to demonstrate the eigenvalue effects in Theorems 4.2 and 4.7. Here we fix $(\mathcal{S}_1, \mathcal{S}_2) = (I, I)$ (the $\mathbb{S}_1$ class).

EXAMPLE 5.1. Consider the Hamiltonian Jordan canonical form $\mathfrak{J} = N_2(\lambda) \oplus (-N_2(\lambda)^H)$, where $\lambda = r + i\alpha$ and $r > 0$. We construct the Hamiltonian matrix $\mathcal{H} = \mathcal{S}\mathfrak{J}\mathcal{S}^{-1} \in \mathbb{R}^{4\times 4}$, where $\mathcal{S}$ is a randomly generated symplectic matrix. Using the symplectic matrix $e^{\mathcal{H}}$, we can construct initial matrices $X_{11}^1, X_{12}^1, X_{21}^1, X_{22}^1 \in \mathbb{R}^{2\times 2}$ with $X_{12}^1 = (X_{21}^1)^\top$ and X_{11}^1, X_{22}^1 being symmetric such that $\mathcal{M}_1 = \mathcal{L}_1 e^{\mathcal{H}}$, where $(\mathcal{M}_1, \mathcal{L}_1) = T_{\mathcal{S}_1,\mathcal{S}_2}([X_{ij}^1])$. Because the $\mathbb{S}_1$ class is considered, $\mathcal{S}_- = \mathcal{S}$. Then $X_{12}(t)$

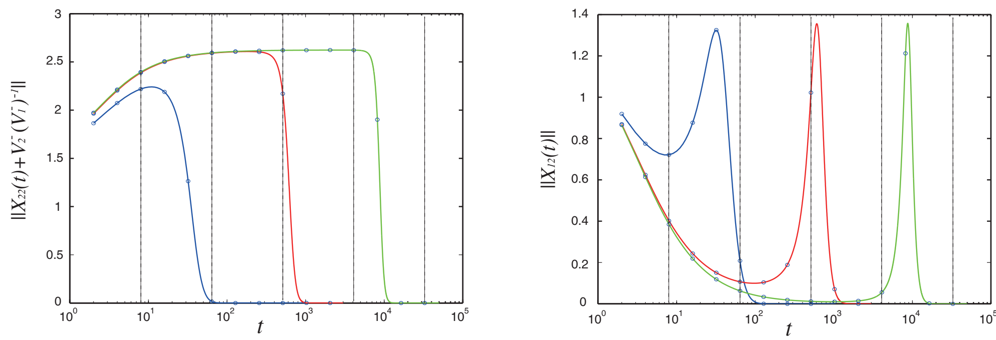

FIG. 5.1. $\|X_{22}(t) + V_2^-(V_1^-)^{-1}\|$ *and* $\|X_{12}(t)\|$ *are plotted for* $1 \leqslant t \leqslant 1.5 \times 10^4$ *with* $r = 0.1$ *(blue line),* 0.01 *(red line) and* 0.001 *(green line).*

and $X_{22}(t)$ can be computed by the formulas in (4.3a). Here, we let $\alpha = 0.801$ be fixed, and vary $r = 0.1$, 0.01, and 0.001. We then plot $\|X_{22}(t) + V_2^-(V_1^-)^{-1}\|$ and $\|X_{12}(t)\|$ for $1 \leqslant t \leqslant 1.5 \times 10^4$ in Figure 5.1. The blue, red and green lines in Figure 5.1 represents for the residuals with respect to $r = 0.1$, 0.01, and 0.001, respectively. The circles on the lines represent for the residuals $\|X_{22}^k + V_2^-(V_1^-)^{-1}\|$ and $\|X_{12}^k\|$ for the SDA at $k = 3$, 6, 9, 12 and 15. In each case, we can see that the convergence occurs, but the time for the residual bounded by the tolerance is postponed when r is smaller.

EXAMPLE 5.2. Consider the Hamiltonian Jordan canonical form $\mathfrak{J} = \left[\begin{array}{c|c} R & D \\ \hline 0 & -R^H \end{array}\right]$, where R and D have form in (4.10) with $\alpha = 1.771$, $n_1 = 2$, $n_2 = 1$, $n = n_1 + n_2 + 1 = 4$ and $\beta = 1$. We construct the Hamiltonian matrix $\mathcal{H} = \mathcal{S}\mathfrak{J}\mathcal{S}^{-1} \in \mathbb{R}^{8\times 8}$, where $\mathcal{S}$ is a randomly generated symplectic matrix. Using the symplectic matrix $e^{\mathcal{H}}$, we can construct initial matrices $X_{11}^1, X_{12}^1, X_{21}^1, X_{22}^1 \in \mathbb{R}^{4\times 4}$ with $X_{12}^1 = (X_{21}^1)^\top$ and X_{11}^1, X_{22}^1 being symmetric such that $\mathcal{M}_1 = \mathcal{L}_1 e^{\mathcal{H}}$, where $(\mathcal{M}_1, \mathcal{L}_1) = T_{\mathcal{S}_1,\mathcal{S}_2}([X_{ij}^1])$. Therefore, $\mathcal{S}_- = \mathcal{S}$. Then $X_{12}(t)$ and $X_{22}(t)$ can be computed by the formulas in (4.3a). We then plot $\|X_{22}(t) + \mathbf{U}_2^-(\mathbf{U}_1^-)^{-1}\|$ and $\|X_{12}(t) - e^{i\alpha(t-1)}X_{12}^1(W_2^-)^{-1}e_n e_n^H(\mathbf{U}_1^-)^{-1}\|$ for $1 \leqslant t \leqslant 10^3$ in Figure 5.2 (a). We can see that both of them approach 0 as $t \to \infty$. It is shown in Lemma 4.6 that $X_{12}(t)$ approaches a rank-one periodic function with period $2\pi/\alpha$. In Figure 5.2 (b), we plot the phase portrait of the $(1,1)$-entry of $X_{12}(t)$ that illustrates how it approaches a limit cycle.

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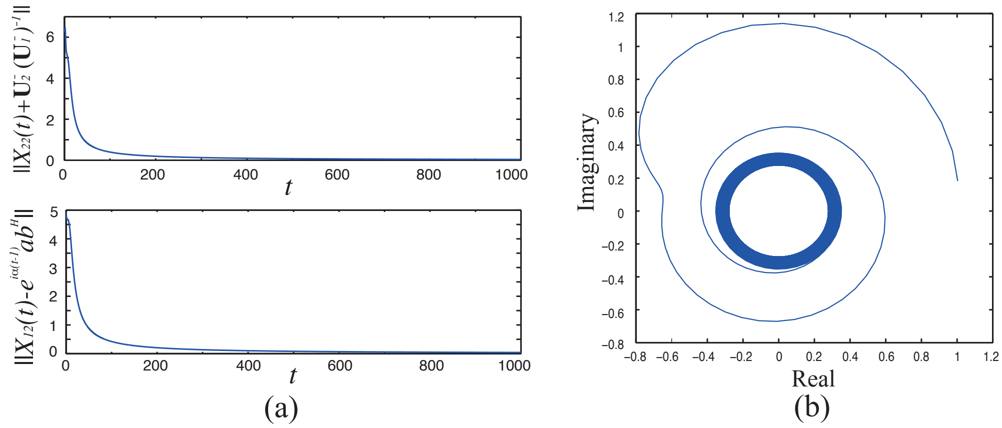

FIG. 5.2. *Set* $\alpha = 1.771$*,* $n_1 = 2$*,* $n_2 = 1$ *and* $\beta = 1$*. (a)* $\|X_{22}(t) + \mathbf{U}_2^-(\mathbf{U}_1^-)^{-1}\|$ *and* $\|X_{12}(t) - e^{i\alpha(t-1)}ab^H\|$ *with* $a = X_{12}^1(W_2^-)^{-1}e_n$ *and* $b = (\mathbf{U}_1^-)^{-H}e_n$ *are plotted for* $1 \leqslant t \leqslant 10^3$*. (b) Phase portrait of the* $(1,1)$*-entry of* $X_{12}(t)$ *is plotted for* $1 \leqslant t \leqslant 10^3$*.*

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