A STRUCTURE-PRESERVING CURVE FOR SYMPLECTIC PAIRS AND ITS APPLICATIONS

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Abstract. The main purpose of this paper is the study of numerical methods for the stabilizing solution of the matrix equation $X + A^*X^{-1}A = Q$, where Q is Hermitian positive definite. We construct a smooth curve parameterized by $t \geq 1$ of symplectic pairs with a special structure, in which the curve passes through all iteration points generated by the known numerical methods, including the fixed-point iteration, the structured preserving doubling algorithm (SDA), and Newton's method under some specified condition. In the theoretical section, we give a necessary and sufficient condition for the existence of this structured symplectic pairs for each parameter $t \geq 1$. We also characterize the behavior of this curve. In the application section, we use this curve to measure the convergence rates of those numerical methods. Numerical results illustrating these solutions are also presented.

1. Introduction.

The nonlinear matrix equations (NMEs)

$$X + A^*X^{-1}A = Q, \tag{1.1}$$

where $A,\ Q \in \mathbb{C}^{n\times n}$ and Q is Hermitian positive definite, arises in several applications. Various aspects of the NME, such as solvability, numerical solution, perturbation and applications, are discussed in [1, 8, 9, 10, 11, 12, 16, 18, 20, 21, 22] and the references therein. Under some conditions, such as the corresponding rational matrix-valued function

$$\psi(\lambda) = Q - \lambda^{-1}A - \lambda A^* \tag{1.2}$$

is regular and positive semi-definite for all λ on the unit circle $\mathbb{T}$, it has been established that the NME has a unique stabilizing solution X in [9]. It is further known that this unique solution X is Hermitian positive definite and the maximal solution. Moreover, the stabilizing solution X can be used to factor $\psi(\lambda)$ as

$$\psi(\lambda) = (\lambda^{-1}X - A^*)X^{-1}(\lambda X - A), \tag{1.3}$$

and satisfies $\det(\lambda X - A) \neq 0$ for $|\lambda| > 1$. Let

$$\mathcal{M} = \begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 0 & I \\ A^* & 0 \end{bmatrix}. \tag{1.4}$$

The pencil $\mathcal{M} - \lambda\mathcal{L}$ is a linearization of matrix polynomial $\phi(\lambda) \equiv \lambda\psi(\lambda)$. It follows from (1.2) or (1.3) that λ is an eigenvalue of $(\mathcal{M}, \mathcal{L})$ if and only if $1/\bar{\lambda}$ is an eigenvalue of $(\mathcal{M}, \mathcal{L})$ with the same multiplicity. Here λ can be 0 or ∞. Because the stabilizing solution X is positive definite, we obtain from (1.3) that the multiplicity of the unimodular eigenvalue, λ_0, of $\psi(\lambda)$ is even and the length of Jordan chain corresponding to λ_0 is at least 2. The stabilizing solution X of NMEs (1.1) can be formulated as

$$\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} 0 & I \\ A^* & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} S,$$

where $S = X^{-1}A \in \mathbb{C}^{n\times n}$ with $\rho(S) \leq 1$.

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The numerical methods for the stabilizing solution X of NMEs (1.1) originate from the fixed-point iteration [1, 9, 12]

$$X_{k+1} = Q - A^* X_k^{-1} A, \quad \text{with } X_0 = Q. \tag{1.5}$$

It is proven in [9] that the sequence $\{X_k\}$ generated by the fixed-point iteration converges to the stabilizing solution X. Newton's method has been studied by [12], in which the authors proved that the convergence is quadratic if $\rho(X^{-1}A) < 1$. If $\rho(X^{-1}A) = 1$ and all eigenvalues of $X^{-1}A$ on the unit circle are semisimple, then the convergence is at least linear with rate $1/2$. Recently, the structure preserving doubling algorithm (SDA) has been studied by [3, 4, 15, 16]. The convergence of SDA is quadratic if $\rho(X^{-1}A) < 1$. If $\rho(X^{-1}A) = 1$ (without any assumption on the unit eigenvalues), it is shown in [4] that the convergence of SDA is at least linear with rate $1/2$. The relation between fixed-point iteration and SDA has been studied in [3]. It is shown that if the sequence $\{X_k\}$ is generated by (1.5), then the sequence $\{Q_k\}$ generated by SDA is $\{X_{2^k-1}\}$.

Intrinsically, the SDA generates the sequence of matrix pairs of the form

$$\mathcal{M}_k = \begin{bmatrix} A_k & 0 \\ Q_k & -I \end{bmatrix}, \quad \mathcal{L}_k = \begin{bmatrix} -P_k & I \\ A_k^* & 0 \end{bmatrix},$$

satisfying $Q_k = Q_k^*$, $P_k = P_k^*$, $A_0 = A$, $Q_0 = Q$ and $P_0 = 0$. It is further shown by [15] that

(i) if x is an eigenvector of the matrix pencil $\mathcal{M} - \lambda\mathcal{L}$ corresponding to an eigenvalue λ_0, i.e., $\mathcal{M}x = \lambda_0 \mathcal{L}x$, then $\mathcal{M}_k x = \lambda_0^{2^k} \mathcal{L}_k x$. In other words, the sequence of matrix pairs $\{(\mathcal{M}_k, \mathcal{L}_k)\}$ is eigenvector-preserving and eigenvalue-doubling.

(ii) the pairs $\{(\mathcal{M}_k, \mathcal{L}_k)\}$ preserve the symplectic pairs structure, i.e.,

$$\mathcal{M}_k \mathcal{J} \mathcal{M}_k^* = \mathcal{L}_k \mathcal{J} \mathcal{L}_k^*, \quad \text{where } \mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Now, let

$$\mathcal{S} = \left\{ \left(\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \begin{bmatrix} -P & I \\ A^* & 0 \end{bmatrix} \right) \mid A,\ Q = Q^*,\ P = P^* \in \mathbb{C}^{n\times n} \right\} \tag{1.6}$$

be a subset of symplectic pairs with a special structure. Motivated by the eigenvector-preserving property of the SDA mentioned above, for any A, $Q = Q^* \in \mathbb{C}^{n\times n}$, it is natural to ask whether there is a unique curve $\mathcal{C} \equiv \{(\mathcal{M}(t), \mathcal{L}(t)) | t \geq 1\} \subseteq \mathcal{S}$ satisfying the eigenvector preserving condition:

EVP. If $\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} S$, where $U, V \in \mathbb{C}^{n\times m}$ and $S \in \mathbb{C}^{m\times m}$, then

$$\mathcal{M}(t) \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{L}(t) \begin{bmatrix} U \\ V \end{bmatrix} S^t. \tag{1.7}$$

Because the solution curve $\mathcal{C}$ is in $\mathcal{S}$, the curve $\mathcal{C}$ is called a *structure-preserving curve*. If the unique structure-preserving curve $\mathcal{C}$ exists, then the symplectic pairs $(\mathcal{M}_k, \mathcal{L}_k)$ generated by SDA are on the curve $\mathcal{C}$ and satisfy $\mathcal{M}(2^k) = \mathcal{M}_k$, $\mathcal{L}(2^k) = \mathcal{L}_k$. Therefore, the curve $\mathcal{C}$ passes through all iteration points $(\mathcal{M}_k, \mathcal{L}_k)$ generated by SDA. To find a smooth curve with a specific structure that passes through a sequence generated by some numerical algorithm is a topic studied by many researchers, especially in the study of the so-called Toda flow that connects matrices in each step of the QR-algorithm (see, e.g., [2, 5, 6, 7, 19] and the works cited therein). In the study of Toda flows, the curve is the solution of a nonlinear ordinary differential equation of matrices in which the eigenvalues are preserved and the eigenvectors vary in t. Rather than the invariance property of Toda flows, the curve we focus on in this paper that satisfies **EVP** shall preserve the eigenvectors.

To be more precise, in **EVP**, we use a matrix function operator S^t where $S \in \mathbb{C}^{m\times m}$ and $t > 1$. By assuming that the eigenvalues of S on $\mathbb{R}^- \cup \{0\}$ are semi-simple, S can be written as

$$S = W \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} W^{-1}. \tag{1.8}$$

Here, J is an invertible Jordan canonical form and all negative eigenvalues of J are semi-simple. Let $\log(z)$ denote the *principle logarithm* of nonzero $z \in \mathbb{C}$. Because $z^t = \exp(t\log(z))$ for each nonzero $z \in \mathbb{C}$, it follows from [13, Theorem 1.17] that S^t, for $t \geq 1$, can be defined as follows.

Definition 1.1. Suppose that eigenvalues of S on $\mathbb{R}^-\cup\{0\}$ are semi-simple and let S have the Jordan canonical form (1.8). Then, for each $t \geq 1$,

$$S^t = W \begin{bmatrix} J^t & 0 \\ 0 & 0 \end{bmatrix} W^{-1},$$

where $J^t = \exp(t\log(J))$ and $\log(J)$ is the principle logarithm of J defined in [13].

Because $\mathcal{M} - \lambda\mathcal{L}$ is the linearization of the matrix polynomial $\phi(\lambda) = \lambda\psi(\lambda)$, the matrix-valued function, $\psi(\lambda)$, plays an important role in the study of NMEs (1.1) (see, e.g., [9]). In this paper, we make two assumptions regarding $\psi(\lambda)$:

A1. Every eigenvalue of matrix polynomial $\phi(\lambda) = \lambda\psi(\lambda)$ on $\mathbb{R}^- \cup \{0\}$ is semi-simple.

A2. The function $\psi(\lambda)$ is positive definite for all $|\lambda| = 1$.

It follows from assumption **A1** that the matrix function operator S^t, $t \geq 1$, in (1.7) is well-defined. Assumption **A2** leads to the solvability of NMEs (1.1).

The structure-preserving curve problem can be formulated as follows.

Structure-preserving curve problem: *Given A, $Q = Q^* \in \mathbb{C}^{n\times n}$ satisfying assumptions* **A1** *and* **A2***, find a curve $\mathcal{C} = \{(\mathcal{M}(t), \mathcal{L}(t))|t \geq 1\} \subseteq \mathcal{S}$ such that* **EVP** *holds.*

For real case, if the given matrices A and $Q = Q^\top$ are real, then we may hope that the structure-preserving curve belongs to a set of real symplectic pairs

$$\mathcal{S}_R = \left\{ \left(\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \begin{bmatrix} -P & I \\ A^\top & 0 \end{bmatrix} \right) \mid A,\ Q = Q^\top,\ P = P^\top \in \mathbb{R}^{n\times n} \right\}.$$

The structure-preserving curve problem in the real case is also considered and can be formulated as follows.

Structure-preserving curve problem in the real case: *Given A, $Q = Q^\top \in \mathbb{R}^{n\times n}$ satisfying assumptions* **A1** *and* **A2***, find a curve $\mathcal{C} = \{(\mathcal{M}(t), \mathcal{L}(t))|t \geq 1\} \subseteq \mathcal{S}_R$ such that* **EVP** *holds.*

Our first main result in this paper is concerned with the solution curve of the structure-preserving curve problems in the specified structure of symplectic pairs in $\mathcal{S}$.

Theorem 1.1 (Main Result 1)**.** *Let A, $Q = Q^* \in \mathbb{C}^{n\times n}$ be given such that they satisfy assumptions* **A1** *and* **A2***. Suppose that X_L is the unique stabilizing solution of NME (1.1) and $S_1 = X_L^{-1}A$. Then, there exist S_2 and $X_S = X_S^* \in \mathbb{C}^{n\times n}$, with S_2 being similar to S_1, such that the solution of the structure-preserving curve problems can be characterized as*

$$\mathcal{M}(t) = \begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} \text{ and } \mathcal{L}(t) = \begin{bmatrix} -P(t) & I \\ A(t)^* & 0 \end{bmatrix}$$

if and only if $1 \notin \sigma(S_1^t S_2^{t^*})$, *where*

$$\begin{aligned} P(t) &= \left(X_S - X_L S_1^t S_2^{t^*}\right)\left(I - S_1^t S_2^{t^*}\right)^{-1}, \\ A(t) &= \left(-P(t) + X_L\right) S_1^t, \\ Q(t) &= X_L + A(t)^* S_1^t. \end{aligned} \tag{1.9}$$

In addition, eigenvalues of $S_1^t S_2^{t^*}$ *are real and non-negative for all* $t \geq 1$.

The matrices S_1, S_2, X_S and X_L are completely determined by A and Q. We will construct these matrices in Section 2. From Main Result 1, we see that $1 \notin \sigma(S_1^t S_2^{t^*})$ is the necessary and sufficient condition of the solvability. Because $\rho(S_1) = \rho(S_2) < 1$, $\rho(S_1^t S_2^{t^*}) \to 0$ as $t \to \infty$ is implied, and hence, it holds for the existence of the curve $\mathcal{C}$ in the structure-preserving curve problems for all sufficiently large t. Our second main result is concerned with the boundedness of the solution curve on certain intervals

$$E_{[a,b)} = \{t \in [a,b) |\ \rho(S_1^t S_2^{t^*}) < 1\} \tag{1.10}$$

as well as the nested set property for these $E_{[a,b)}$'s.

Theorem 1.2 (Main Result 2)**.** *(i) For each* $k \in \mathbb{N}$, $\{t+1 |\ t \in E_{[k,k+1)}\} \subseteq E_{[k+1,k+2)}$. *(ii) All positive integers are contained in* $E_{[1,\infty)}$. *(iii) If* $t \in E_{[1,\infty)}$, *then* $0 \leq P(t+1) \leq X_S$ *and* $X_L \leq Q(t)$.

In Main Result 2, we have that the solvability of the structure-preserving curve problems hold for all positive integers. Our third main result is concerned with the relationship between the solution curve defined on the positive integers and the known numerical schemes for solving NMEs: the fixed-point iteration, the SDA, and Newton's method.

Theorem 1.3 (Main Result 3)**.** *Let* A, $Q = Q^* \in \mathbb{C}^{n\times n}$ *be given such that they satisfy assumptions* **A1** *and* **A2**. *Let* $A(t)$, $Q(t)$ *and* $P(t)$ *be given in* (1.9). *Then, (i)* $\{Q(k+1)\}_{k=0}^{\infty}$ *is the sequence generated by the fixed-point iteration* (1.5)*; (ii) the pairs*

$$(\mathcal{M}_k, \mathcal{L}_k) = \left(\begin{bmatrix} A(2^k) & 0 \\ Q(2^k) & -I \end{bmatrix}, \begin{bmatrix} -P(2^k) & I \\ A(2^k)^* & 0 \end{bmatrix} \right), k = 0, 1, 2, \ldots,$$

are the sequence generated by the SDA; (iii) if we further assume that $A^*Q^{-1}A = AQ^{-1}A^*$, *then* $\{Q(2^{k+1}-1)\}_{k=0}^{\infty}$ *is the sequence generated by Newton's method.*

Indeed, the solution curve passes through the orbits of the three existing numerical methods, and hence, the parameterized curve forms a nature measurement of the convergence speeds of these methods. This paper is organized as follows. In Section 2, we introduce some preliminary results. The proofs of Main Results 1, 2 and 3 are given in Sections 3, 4, and 5, respectively. In Section 6, we give two numerical examples. The first example shows that there exists $t > 1$ such that $1 \in \sigma(S_1^t S_2^{t^*})$. The second example illustrates that the solution curve $Q(t)$ does not pass through the Newton iterations in which the condition $A^*Q^{-1}A = AQ^{-1}A^*$ is not satisfied.

2. Preliminaries. In this section, we shall introduce some notations and definitions and give some preliminary results related to symplectic matrix pairs. Finally, we shall redescribe those two structure-preserving curve problems. Throughout this paper, we denote the unit circle in complex plane by $\mathbb{T}$. For a matrix $A \in \mathbb{C}^{n\times n}$, we use $\sigma(A)$ and $\rho(A)$ to denote the spectrum and spectral radius of A, respectively. A^* and $A^\top$ denote the conjugate transpose and transpose of A, respectively. For Hermitian matrices A_1, $A_2 \in \mathbb{C}^{n\times n}$, we use $A_1 > A_2$ ($A_1 \geq A_2$) to denote that $A_1 - A_2$ is positive definitive (positive semi-definite).

Lemma 2.1 (Theorem 1.13 in [13])**.** *Suppose the eigenvalues of* $S \in \mathbb{C}^{n\times n}$ *on* $\mathbb{R}^- \cup \{0\}$ *are semi-simple. If* Y *commutes with* S, *then* Y *commutes with* S^t *for each* $t \geq 1$.

Definition 2.1. A matrix pair $(\mathcal{M}, \mathcal{L})$ is called a symplectic pair if

$$\mathcal{M}\mathcal{J}\mathcal{M}^* = \mathcal{L}\mathcal{J}\mathcal{L}^*, \tag{2.1}$$

where $\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Two symplectic pairs, $(\mathcal{M}_1, \mathcal{L}_1)$ and $(\mathcal{M}_2, \mathcal{L}_2)$, are called equivalent if there exists an invertible matrix $C \in \mathbb{C}^{2n\times 2n}$ such that $\mathcal{M}_1 = C\mathcal{M}_2$ and $\mathcal{L}_1 = C\mathcal{L}_2$.

Definition 2.2. A subspace $\mathcal{U}$ in $\mathbb{C}^{2n}$ is called isotropic if $x^*\mathcal{J}y = 0$ for all $x, y \in \mathcal{U}$. $\mathcal{U}$ is said to be a Lagrangian subspace if it is a maximal isotropic subspace.

Suppose the columns of $\begin{bmatrix} U \\ V \end{bmatrix} \in \mathbb{C}^{2n\times n}$ span a Lagrangian deflating subspace of $(\mathcal{M}, \mathcal{L})$ corresponding to (T, K), that is,

$$\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} T = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} K. \tag{2.2}$$

A sufficient condition for the invertibility of matrix U in (2.2) is given in the following.

Theorem 2.1. *Suppose the columns of* $\begin{bmatrix} U \\ V \end{bmatrix} \in \mathbb{C}^{2n\times n}$ *form a basis of the Lagrangian deflating subspace of* $(\mathcal{M}, \mathcal{L})$ *corresponding to* (T, K)*. If there exists* $\eta_0 \in \mathbb{T}$ *such that* $\psi(\eta_0)$ *is positive definite, then* U *is invertible.*

Proof. Because $\psi(\eta_0)$ is positive definite for some $\eta_0 \in \mathbb{T}$, it is easily seen from (1.2) and (1.4) that $\mathcal{M} - \eta_0\mathcal{L}$ is invertible. From (2.2), we also have that $K - \eta_0 T$ is invertible. Then,

$$(\mathcal{M} + \eta_0\mathcal{L}) \begin{bmatrix} U \\ V \end{bmatrix} T = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} K + \eta_0\mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} T = \mathcal{L} \begin{bmatrix} U \\ V \end{bmatrix} (K + \eta_0 T),$$

$$(\mathcal{M} + \eta_0\mathcal{L}) \begin{bmatrix} U \\ V \end{bmatrix} K = \mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} K + \eta_0\mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} T = \mathcal{M} \begin{bmatrix} U \\ V \end{bmatrix} (K + \eta_0 T).$$

It follows that $(\mathcal{M} + \eta_0\mathcal{L}) \begin{bmatrix} U \\ V \end{bmatrix} (K - \eta_0 T) = (\mathcal{M} - \eta_0\mathcal{L}) \begin{bmatrix} U \\ V \end{bmatrix} (K + \eta_0 T)$. Because $\mathcal{M} - \eta_0\mathcal{L}$ and $K - \eta_0 T$ are invertible, we have

$$(\mathcal{M} - \eta_0\mathcal{L})^{-1}(\mathcal{M} + \eta_0\mathcal{L}) \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} (K + \eta_0 T)(K - \eta_0 T)^{-1},$$

that is,

$$(\mathcal{M} - \eta_0\mathcal{L})^{-1}(\mathcal{M} + \eta_0\mathcal{L})\mathcal{U} \subseteq \mathcal{U}, \tag{2.3}$$

where $\mathcal{U} = \operatorname{span}\left\{\begin{bmatrix} U \\ V \end{bmatrix}\right\}$ is a Lagrangian deflating subspace of $(\mathcal{M}, \mathcal{L})$.

Computing $(\mathcal{M} - \eta_0\mathcal{L})^{-1}$ gives

$$(\mathcal{M} - \eta_0\mathcal{L})^{-1} = \begin{bmatrix} -\bar{\eta}_0\psi^{-1}(\eta_0) & \psi^{-1}(\eta_0) \\ -\bar{\eta}_0(Q - \eta_0 A^*)\psi^{-1}(\eta_0) & (Q - \eta_0 A^*)\psi^{-1}(\eta_0) \end{bmatrix},$$

then we have $(\mathcal{M} - \eta_0\mathcal{L})^{-1}(\mathcal{M} + \eta_0\mathcal{L}) = \begin{bmatrix} \star & -2\psi^{-1}(\eta_0) \\ \star & \star \end{bmatrix}$. Now, we suppose that there is a vector $x \in \mathbb{C}^n$ such that $Ux = 0$, i.e., $\begin{bmatrix} 0 \\ Vx \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} x \in \mathcal{U}$. From (2.3), we obtain

$$(\mathcal{M} - \eta_0\mathcal{L})^{-1}(\mathcal{M} + \eta_0\mathcal{L}) \begin{bmatrix} 0 \\ Vx \end{bmatrix} = \begin{bmatrix} -2\psi^{-1}(\eta_0)Vx \\ \star \end{bmatrix} \in \mathcal{U}.$$

Because $\mathcal{U}$ is isotropic, we have

$$
\begin{aligned}
0 &= [0,\ x^*V^*]\mathcal{J}(\mathcal{M}-\eta_0\mathcal{L})^{-1}(\mathcal{M}+\eta_0\mathcal{L})\begin{bmatrix} 0 \\ Vx \end{bmatrix} \\
&= 2x^*V^*\psi^{-1}(\eta_0)Vx.
\end{aligned}
$$

Because $\psi(\eta_0)$ is positive definite, this fact implies $Vx = 0$. Hence, $x = 0$. Therefore, U is invertible.

Under assumption **A2**, the symplectic pencil $\mathcal{M} - \lambda\mathcal{L}$ has no eigenvalue on $\mathbb{T}$. Because $\mathcal{M} - \lambda\mathcal{L}$ is a linearization of $\phi(\lambda) = \lambda\psi(\lambda)$, the identity $\psi(\lambda) = \psi^*(1/\bar{\lambda})$ implies the generalized eigenvalues of $\mathcal{M} - \lambda\mathcal{L}$ occur in pairs $(\lambda_0, 1/\bar{\lambda}_0)$ including $(0, \infty)$. From [14, 17], we know that λ and $1/\bar{\lambda}$ have the same size of Jordan blocks. Suppose that $\mathrm{span}\{Z_1\}$ and $\mathrm{span}\{Z_2\}$ form the stable and unstable deflating subspaces of $(\mathcal{M}^*, \mathcal{L}^*)$, i.e.,

$$
\mathcal{M}^*[Z_1, Z_2]\begin{bmatrix} I_n & 0 \\ 0 & J_s \end{bmatrix} = \mathcal{L}^*[Z_1, Z_2]\begin{bmatrix} J_s^* & 0 \\ 0 & I_n \end{bmatrix}, \tag{2.4}
$$

where $Z_1, Z_2 \in \mathbb{C}^{2n\times n}$ and $J_s \in \mathbb{C}^{n\times n}$ consists of stable Jordan blocks, i.e., $\rho(J_s) < 1$. Let

$$
\mathcal{W} = \mathcal{J}[\mathcal{M}^*Z_2, \mathcal{L}^*Z_1] \in \mathbb{C}^{2n\times 2n}. \tag{2.5}
$$

It follows from (2.4) that $\mathcal{W}$ is invertible. Partition $\mathcal{W}$ as

$$
\mathcal{W} = \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix}, \tag{2.6}
$$

where $W_i \in \mathbb{C}^{n\times n}$ for $i = 1, \ldots, 4$.

Lemma 2.2. *The invertible matrix $\mathcal{W}$ given in* (2.5) *satisfies*

$$
\mathcal{M}\mathcal{W}\begin{bmatrix} I_n & 0 \\ 0 & J_s^* \end{bmatrix} = \mathcal{L}\mathcal{W}\begin{bmatrix} J_s & 0 \\ 0 & I_n \end{bmatrix}. \tag{2.7}
$$

Proof. Premultiplying (2.4) by $\mathcal{L}\mathcal{J}$ and using (2.1), we have

$$
\mathcal{L}\mathcal{J}\mathcal{M}^*[Z_1, Z_2]\begin{bmatrix} I_n & 0 \\ 0 & J_s \end{bmatrix} = \mathcal{M}\mathcal{J}\mathcal{M}^*[Z_1, Z_2]\begin{bmatrix} J_s^* & 0 \\ 0 & I_n \end{bmatrix}.
$$

It follows that

$$
\mathcal{M}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \mathcal{L}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} J_s. \tag{2.8}
$$

Similarly, premultiplying (2.4) by $\mathcal{M}\mathcal{J}$ and using (2.1) yields

$$
\mathcal{L}\mathcal{J}\mathcal{L}^*[Z_1, Z_2]\begin{bmatrix} I_n & 0 \\ 0 & J_s \end{bmatrix} = \mathcal{M}\mathcal{J}\mathcal{L}^*[Z_1, Z_2]\begin{bmatrix} J_s^* & 0 \\ 0 & I_n \end{bmatrix}.
$$

It follows that

$$
\mathcal{M}\begin{bmatrix} W_3 \\ W_4 \end{bmatrix} J_s^* = \mathcal{L}\begin{bmatrix} W_3 \\ W_4 \end{bmatrix}. \tag{2.9}
$$

Equation (2.7) follows directly from (2.8), (2.9) and (2.6).

Lemma 2.2 shows that $\operatorname{span}\left\{\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\right\}$ and $\operatorname{span}\left\{\begin{bmatrix} W_3 \\ W_4 \end{bmatrix}\right\}$ form the stable and unstable deflating subspaces of $(\mathcal{M}, \mathcal{L})$, respectively.

Lemma 2.3. *Let $\mathcal{D} = \mathcal{W}^*\mathcal{J}\mathcal{W}$. Then, $\mathcal{D} \in \mathbb{C}^{2n\times 2n}$ is skew-Hermitian and has the form $\mathcal{D} = \begin{bmatrix} 0 & D_1 \\ -D_1^* & 0 \end{bmatrix}$, where $D_1 \in \mathbb{C}^{n\times n}$ is invertible that satisfies $D_1 J_s^* = J_s^* D_1$.*

Proof. It follows from (2.4) and (2.5) that

$$\mathcal{W}\begin{bmatrix} J_s & 0 \\ 0 & I_n \end{bmatrix} = \mathcal{J}[\mathcal{M}^* Z_2 J_s, \mathcal{L}^* Z_1] = \mathcal{J}\mathcal{L}^*[Z_2, Z_1].$$

Consequently, we obtain

$$\begin{aligned}\left[\begin{array}{c|c} J_s^* & 0 \\ \hline 0 & I_n \end{array}\right]\mathcal{W}^*\mathcal{J}\mathcal{W}\left[\begin{array}{c|c} J_s & 0 \\ \hline 0 & I_n \end{array}\right] &= \begin{bmatrix} Z_2^* \\ Z_1^* \end{bmatrix}\mathcal{L}\mathcal{J}\mathcal{L}^*[Z_2, Z_1] = \begin{bmatrix} Z_2^* \\ Z_1^* \end{bmatrix}\mathcal{M}\mathcal{J}\mathcal{M}^*[Z_2, Z_1] \\ &= \begin{bmatrix} Z_2^*\mathcal{M} \\ J_s Z_1^*\mathcal{L} \end{bmatrix}\mathcal{J}[\mathcal{M}^* Z_2, \mathcal{L}^* Z_1 J_s^*] = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & J_s \end{array}\right]\mathcal{W}^*\mathcal{J}\mathcal{W}\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & J_s^* \end{array}\right]. \end{aligned} \tag{2.10}$$

Comparing each sides of (2.10) and using the fact that $\rho(J_s) < 1$, we see $\mathcal{D} = \mathcal{W}^*\mathcal{J}\mathcal{W} = \begin{bmatrix} 0 & D_1 \\ -D_1^* & 0 \end{bmatrix}$ and $J_s^* D_1 = D_1 J_s^*$. Because both $\mathcal{W}$ and $\mathcal{J}$ are invertible, it follows that D_1 is also invertible. This completes the proof.

From (2.6) and Lemma 2.3, we have

$$W_1^* W_2 - W_2^* W_1 = 0, \tag{2.11a}$$
$$W_3^* W_4 - W_4^* W_3 = 0, \tag{2.11b}$$
$$W_1^* W_4 - W_2^* W_3 = D_1. \tag{2.11c}$$

Equations (2.11a) and (2.11b) show that the stable and unstable deflation subspaces, $\operatorname{span}\left\{\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\right\}$ and $\operatorname{span}\left\{\begin{bmatrix} W_3 \\ W_4 \end{bmatrix}\right\}$, are isotropic subspaces. That is, they are Lagrangian deflation subspaces. Under assumption **A2**, it follows from Theorem 2.1 that W_1 and W_3 are invertible. Let

$$\begin{aligned} X_L &= W_2 W_1^{-1}, \\ X_S &= W_4 W_3^{-1}. \end{aligned} \tag{2.12}$$

It is shown in [9] that X_L and X_S are positive definite and positive semi-definite, respectively. Note that 0 is not an eigenvalue of the pair $\mathcal{M} - \lambda\mathcal{L}$ only if X_S is positive definite. Furthermore, $X_L - X_S \geq 0$.

Remark 2.1. Because

$$\begin{bmatrix} I & I \\ X_L & X_S \end{bmatrix} = \mathcal{W}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & W_3^{-1} \end{bmatrix}$$

is invertible, the Hermitian matrix $X_L - X_S$ is actually positive definite.

Substituting (2.12) into (2.11c) yields

$$W_1^*(X_S - X_L)W_3 = D_1. \tag{2.13}$$

Let

$$S_1 = W_1 J_s W_1^{-1} \text{ and } S_2 = W_3^{*^{-1}} J_s W_3^{*}. \tag{2.14}$$

Then, (2.7) can be rewritten as

$$\begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix} \begin{bmatrix} I & I \\ X_L & X_S \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S_2^* \end{bmatrix} = \begin{bmatrix} 0 & I \\ A^* & 0 \end{bmatrix} \begin{bmatrix} I & I \\ X_L & X_S \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & I \end{bmatrix}. \tag{2.15}$$

The structure-preserving curve problem and structure-preserving curve problem in the real case can be redescribed by the following two problems.

Problem SPC. *Given A, $Q = Q^* \in \mathbb{C}^{n\times n}$ that satisfy the assumptions* **A1** *and* **A2**. *For each $t \geq 1$, find $A(t)$, $P(t)$, $Q(t) \in \mathbb{C}^{n\times n}$ such that*

$$\begin{cases} \begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} \begin{bmatrix} I & S_2^{t^*} \\ X_L & X_S S_2^{t^*} \end{bmatrix} = \begin{bmatrix} -P(t) & I \\ A(t)^* & 0 \end{bmatrix} \begin{bmatrix} S_1^t & I \\ X_L S_1^t & X_S \end{bmatrix}, \\ P(t)^* = P(t), \quad Q(t)^* = Q(t). \end{cases} \tag{2.16}$$

where X_L, X_S, S_1, $S_2 \in \mathbb{C}^{n\times n}$ are given in (2.12), (2.14) *and satisfy equation* (2.15).

Problem SPC-R. *Given A, $Q = Q^\top \in \mathbb{R}^{n\times n}$ that satisfy the assumptions* **A1** *and* **A2**. *For each $t \geq 1$, find $A(t)$, $P(t)$, $Q(t) \in \mathbb{R}^{n\times n}$ such that*

$$\begin{cases} \begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} \begin{bmatrix} I & S_2^{t^\top} \\ X_L & X_S S_2^{t^\top} \end{bmatrix} = \begin{bmatrix} -P(t) & I \\ A(t)^\top & 0 \end{bmatrix} \begin{bmatrix} S_1^t & I \\ X_L S_1^t & X_S \end{bmatrix}, \\ P(t)^\top = P(t), \quad Q(t)^\top = Q(t). \end{cases} \tag{2.17}$$

where X_L, X_S, S_1, $S_2 \in \mathbb{R}^{n\times n}$ are given in (2.12), (2.14) *and satisfy equation* (2.15).

Remark 2.2. (i) From (2.15), it is easily seen that when $t = 1$, equations (2.16)/(2.17) have solution $A(1) = A$, $Q(1) = Q$ and $P(1) = 0$. (ii) In (2.16), we mean $S_2^{t^*}$ by $(S_2^t)^*$. Here we note that $(S_2^t)^* \neq (S_2^*)^t$ if S_2 has negative eigenvalues.

3. Solving Problem SPC and Problem SPC-R. In this section, we give a necessary and sufficient condition for the solvability of equations (2.16) and (2.17). Let A, $Q = Q^* \in \mathbb{C}^{n\times n}$ satisfy assumptions **A1** and **A2**. Therefore, there exist Hermitian matrices X_L, X_S in (2.12) and stable matrices S_1, S_2 in (2.14). Here, X_L is positive definite, X_S is positive semi-definite with $X_L - X_S > 0$ and S_1, S_2 have the same spectrum (i.e., $\sigma(S_1) = \sigma(S_2)$). To solve equations (2.16)/(2.17), we first consider the matrix equation

$$\begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} \begin{bmatrix} I & S_2^{t^*} \\ X_L & X_S S_2^{t^*} \end{bmatrix} = \begin{bmatrix} -P(t) & I \\ B(t) & 0 \end{bmatrix} \begin{bmatrix} S_1^t & I \\ X_L S_1^t & X_S \end{bmatrix}, \tag{3.1}$$

where $A(t)$, $B(t)$, $Q(t)$ and $P(t) \in \mathbb{C}^{n\times n}$. Suppose $\{A(t), B(t), Q(t), P(t)\}$ is a solution of (3.1) that satisfies $A(t) = B(t)^*$, $P(t) = P(t)^*$ and $Q(t) = Q(t)^*$. It is naturally a solution of (2.16) and vice versa. In addition, if $A(t)$, $B(t)$, $Q(t)$ and $P(t)$ are real matrices, it is a solution of (2.17). First, we solve equation (3.1). To this end, we apply a column operation to (3.1) that yields

$$\begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ X_L & (X_S - X_L) S_2^{t^*} \end{bmatrix} = \begin{bmatrix} -P(t) & I \\ B(t) & 0 \end{bmatrix} \begin{bmatrix} S_1^t & I - S_1^t S_2^{t^*} \\ X_L S_1^t & X_S - X_L S_1^t S_2^{t^*} \end{bmatrix}.$$

It follows that

$$\begin{cases} P(t)\left(I - S_1^t S_2^{t^*}\right) = X_S - X_L S_1^t S_2^{t^*}, \\ A(t) = \left(-P(t) + X_L\right) S_1^t, \\ B(t)\left(I - S_1^t S_2^{t^*}\right) = \left(X_L - X_S\right) S_2^{t^*}, \\ Q(t) = X_L + B(t) S_1^t. \end{cases} \tag{3.2}$$

From (2.15), we have $A(1) = B(1)^* = A$, $Q(1) = Q$ and $P(1) = 0$ is solution of (3.1). It follows from (3.2) that $P(1) = 0 = X_S - X_L S_1 S_2{}^*$, which implies

$$X_S = X_L S_1 S_2{}^*. \tag{3.3}$$

Lemma 3.1. *Eigenvalues of $S_1 S_2^*$ are real, nonnegative and strictly inside the unit circle.*

Proof. From (3.3), we have $X_L^{-1} X_S$ and $S_1 S_2^*$ have the same eigenvalues. Suppose that λ is an eigenvalue of $X_L^{-1} X_S$, then $(X_S - \lambda X_L)x = 0$, where x is a the corresponding eigenvector. Then, $x^* X_S x - \lambda x^* X_L x = 0$. Because $X_L > 0$ and $X_S \geq 0$,

$$0 \leq \lambda = \frac{x^* X_S x}{x^* X_L x} \in \mathbb{R}.$$

Using the fact that $X_L > X_S$, we have $\lambda < 1$. The proof is complete.

From (2.14), we have

$$S_1^t S_2^{t^*} = W_1 J_s^t W_1^{-1} W_3 J_s^{t^*} W_3^{-1}.$$

Suppose that λ is an eigenvalue of $S_1^t S_2^{t^*}$, then $S_1^t S_2^{t^*} - \lambda I$ is singular and so is

$$J_s^t W_1^{-1} W_3 J_s^{t^*} - \lambda W_1^{-1} W_3. \tag{3.4}$$

Multiplying D_1^* from the left of (3.4) and using Lemmas 2.1 and 2.3, we have

$$J_s^t D_1^* W_1^{-1} W_3 J_s^{t^*} - \lambda D_1^* W_1^{-1} W_3 \tag{3.5}$$

is singular. From (2.13), we obtain

$$D_1^* W_1^{-1} W_3 = W_3^* (X_S - X_L) W_3 \tag{3.6}$$

is Hermitian negative definite. Hence, $-D_1^* W_1^{-1} W_3$ can be written in the Choleskey factorization $-D_1^* W_1^{-1} W_3 = L L^*$. Let

$$\Phi = L^{-1} J_s L. \tag{3.7}$$

From (3.5), we have $\Phi^t \Phi^{t^*} - \lambda I$ is singular. Hence, we obtain

$$\sigma(S_1^t S_2^{t^*}) = \sigma(\Phi^t \Phi^{t^*}), \tag{3.8}$$

where Φ is given in (3.7). We thus have the following consequence.

Theorem 3.1. *Eigenvalues of $S_1^t S_2^{t^*}$ are real and nonnegative for any $t \geq 1$.*

The following lemma is useful to prove the existence of solution curve for **Problem SPC**.

Lemma 3.2. *For each $t \geq 1$, we have*

(i) $(X_L - X_S) S_1^t = S_2^t (X_L - X_S)$;

(ii) $(X_L - X_S)S_1^t {S_2^t}^* = \left(S_1^t {S_2^t}^*\right)^* (X_L - X_S)$ *and* ${S_1^t}^* S_2^t (X_L - X_S) = (X_L - X_S)\left({S_1^t}^* S_2^t\right)^*$

Proof. (i) Because $A(1) = B(1)^*$ and $P(1) = 0$, it follows from (3.2) that

$$(I - S_1 {S_2}^*)^* X_L S_1 = S_2(X_L - X_S). \tag{3.9}$$

Applying (3.3) to (3.9) yields

$$(X_L - X_S)S_1 = S_2(X_L - X_S).$$

The first assertion of this lemma follows from Lemma 2.1 directly.

(ii) We only prove the first equality. The other assertion can be accordingly obtained. For each $t \geq 1$,

$$\begin{aligned}
(X_L - X_S)S_1^t {S_2^t}^* &= S_2^t (X_L - X_S) {S_2^t}^* \quad (\text{ by } (i)) \\
&= S_2^t [S_2^t (X_L - X_S)]^* \\
&= S_2^t [(X_L - X_S) S_1^t]^* \quad (\text{ by } (i)) \\
&= S_2^t {S_1^t}^* (X_L - X_S) = \left(S_1^t {S_2^t}^*\right)^* (X_L - X_S).
\end{aligned}$$

We thus complete the proof.

The following theorem gives a sufficient condition for the solvability of (2.16).

Theorem 3.2. *For each* $t \geq 1$, *if* $1 \notin \sigma(S_1^t {S_2^t}^*)$ *then* (3.1) *is uniquely solvable with* $P(t)^* = P(t)$, $B(t) = A(t)^*$ *and* $Q(t)^* = Q(t)$.

Proof. Because $1 \notin \sigma(S_1^t {S_2^t}^*)$, it follows from (3.2) that (3.1) is uniquely solvable. First, we claim $P(t)$ is Hermitian. Multiplying $(I - S_1^t {S_2^t}^*)^*$ from the left of the first equation of (3.2), we get

$$\begin{aligned}
(I - S_1^t {S_2^t}^*)^* P(t) \left(I - S_1^t {S_2^t}^*\right) &= (I - S_1^t {S_2^t}^*)^* \left(X_S - X_L S_1^t {S_2^t}^*\right) \\
&= X_S - (S_1^t {S_2^t}^*)^* X_S - X_L S_1^t {S_2^t}^* + (S_1^t {S_2^t}^*)^* X_L S_1^t {S_2^t}^*.
\end{aligned}$$

To see this claim, it suffices to show that $(S_1^t {S_2^t}^*)^* X_S + X_L S_1^t {S_2^t}^*$ is Hermitian. From Lemma 3.2, we have

$$\begin{aligned}
0 &= (X_L - X_S) S_1^t {S_2^t}^* - \left(S_1^t {S_2^t}^*\right)^* (X_L - X_S) \\
&= \left[(S_1^t {S_2^t}^*)^* X_S + X_L S_1^t {S_2^t}^*\right] - \left[X_S S_1^t {S_2^t}^* + (S_1^t {S_2^t}^*)^* X_L\right].
\end{aligned}$$

Hence, $(S_1^t {S_2^t}^*)^* X_S + X_L S_1^t {S_2^t}^*$ is Hermitian.

Next, we show that $B(t) = A(t)^*$. Taking the conjugate transpose of the second equation of (3.2) and using the fact that $P(t)$ is Hermitian, we have

$$\begin{aligned}
A(t)^* &= {S_1^t}^* (-P(t) + X_L) \\
&= {S_1^t}^* (-X_S + X_L S_1^t {S_2^t}^*)(I - S_1^t {S_2^t}^*)^{-1} + {S_1^t}^* X_L \\
&= {S_1^t}^* (-X_S + X_L S_1^t {S_2^t}^* + X_L - X_L S_1^t {S_2^t}^*)(I - S_1^t {S_2^t}^*)^{-1} \\
&= {S_1^t}^* (X_L - X_S)(I - S_1^t {S_2^t}^*)^{-1} \\
&= (X_L - X_S) {S_2^t}^* (I - S_1^t {S_2^t}^*)^{-1} \quad (\text{by Lemma 3.2 } (i)) \\
&= B(t).
\end{aligned}$$

Finally, we show that $Q(t)$ is Hermitian. Because $P(t) = P(t)^*$ and $B(t) = A(t)^*$, we have

$$B(t)S_1^t = A(t)^* S_1^t = (S_1^t)^*(-P(t) + X_L)S_1^t$$

is Hermitian. Hence, $Q(t) = X_L + B(t)S_1^t$ is Hermitian.

Theorem 3.2 shows that $1 \notin \sigma(S_1^t S_2^{t^*})$ is a sufficient condition for the solvability of (2.16). In the following theorem, we will see that it is also necessary.

Theorem 3.3. *Suppose that $1 \in \sigma(S_1^t S_2^{t^*})$ for some $t \geq 1$. Then,* (3.1) *has no solution.*

Proof. Suppose $1 \in \sigma(S_1^t S_2^{t^*})$ for some specified t. There is a nonzero vector $x \in \mathbb{C}^n$ such that $S_1^t S_2^{t^*} x = x$. From the first equation of (3.2), we obtain that $P(t)$ satisfies

$$P(t)0 = P(t)(I - S_1^t S_2^{t^*})x = (X_S - X_L S_1^t S_2^{t^*})x = (X_S - X_L)x.$$

However, from Remark 2.1 $X_S - X_L$ is invertible and x is nonzero, which is a contradiction. Hence (3.1) has no solution at t.

From Theorems 3.2 and 3.3, we have the following consequence for the solvability of **Problem SPC**.

Theorem 3.4. *The solution curve of* **Problem SPC** *can be uniquely characterized as*

$$\left\{ (A(t), Q(t), P(t)) |\ t \geq 1 \text{ and } 1 \notin \sigma(S_1^t S_2^{t^*}) \right\},$$

where $A(t)$, $Q(t)$ and $P(t)$ are given as in (1.9).

Next, we consider the real case. Suppose that A, $Q = Q^\top \in \mathbb{R}^{n\times n}$ satisfy assumptions **A1** and **A2**. Because $\mathcal{M} - \lambda\mathcal{L}$ has no eigenvalue on $\mathbb{T}$, it follows from (2.12) and (2.14) that X_L, X_S, S_1 and S_2 are real matrices. We first quote the important property that has been proven in [13, Theorem 1.31].

Theorem 3.5. *If $A \in \mathbb{R}^{n\times n}$ has no eigenvalues on $\mathbb{R}^-$, then the principal logarithm of A, $\log(A)$, is a real matrix.*

It is easily seen from Definition 1.1 and Theorem 3.5 that S_1^t and S_2^t are real matrices for each $t \geq 1$ provided that none of the eigenvalues of S_1 are in $\mathbb{R}^-$. If $1 \notin \sigma(S_1^t S_2^{t^*})$, it follows from (1.9) that $A(t)$, $Q(t)$ and $P(t)$ are real matrices. Then, we have the following result.

Theorem 3.6. *Suppose that A, $Q = Q^\top \in \mathbb{R}^{n\times n}$ and $\lambda\psi(\lambda)$ has no eigenvalue on $\mathbb{R}^-$. The solution curve of* **Problem SPC-R** *can be characterized as*

$$\left\{ (A(t), Q(t), P(t)) |\ t \geq 1 \text{ and } 1 \notin \sigma(S_1^t S_2^{t^\top}) \right\},$$

where $A(t)$, $Q(t)$ and $P(t)$ are real and defined in (1.9).

The solvability of (2.16) depends on whether $S_1^t S_2^{t*}$ has eigenvalue 1. In the following, we will see that it is solvable for t sufficiently large and a lower bound for t can be estimated.

Theorem 3.7. *Suppose that the eigenvalues of $\mathcal{M} - \lambda\mathcal{L}$ are semi-simple. Let*

$$T_0 = (\log \lambda_{\min} - \log \lambda_{\max}) / (2 \log \rho(J_s)),$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are, respectively, the maximal and minimal eigenvalues of positive definite matrix $W_3^*(X_L - X_S)W_3$. Then, (2.16) *has solution $(A(t), Q(t), P(t))$ for all $t > T_0$.*

Proof. It suffices to show that $\rho(S_1^t S_2^{t^*}) < 1$ for all $t > T_0$. Suppose that λ is an eigenvalue of $S_1^t S_2^{t^*}$. It follows from (3.4)-(3.6) that there is a nonzero vector x with $\|x\| = 1$ such that

$$J_s^t W_3^*(X_L - X_S)W_3 J_s^{t^*} x = \lambda W_3^*(X_L - X_S)W_3 x.$$

Because $t > T_0$ and J_s is diagonal, $\|J_s^{t^*}x\|^2 < (\rho(J_s))^{2T_0} = \lambda_{\min}/\lambda_{\max}$. Hence,

$$\lambda = \frac{x^* J_s^t W_3^* (X_L - X_S) W_3 J_s^{t^*} x}{x^* W_3^* (X_L - X_S) W_3 x} \leq \frac{\lambda_{\max} \|J_s^{t^*} x\|^2}{\lambda_{\min}} < \frac{\lambda_{\max} \frac{\lambda_{\min}}{\lambda_{\max}}}{\lambda_{\min}} = 1.$$

By Theorem 3.1, we have $\lambda \geq 0$. So, $\rho(S_1^t S_2^{t^*}) < 1$ for all $t > T_0$.

4. Behavior of the solution curve. From Theorems 3.2 and 3.3, we have $1 \notin \rho(S_1^t S_2^{t^*})$ as the necessary and sufficient condition for the solvability of equation (2.16). Because $\rho(S_1) = \rho(S_2) < 1$, we know that $\rho(S_1^t S_2^{t^*}) \to 0$ as $t \to \infty$. Therefore, for each $t \in E_{[a,b)}$, (2.16) is solvable, where $E_{[a,b)}$ is defined in (1.10). In this section, we first characterize the relation between intervals, $E_{[k,k+1)}$ and $E_{[k+1,k+2)}$, for each $k \in \mathbb{N}$. From this property, we consequently have that for each $k \in \mathbb{N}$, (2.16) has a unique solution $(A(k), Q(k), P(k))$. Furthermore, we also study the monotonicity of $\{Q(k)\}_{k=1}^{\infty}$ and $\{P(k)\}_{k=1}^{\infty}$. The following lemma is useful, and the proof is straightforward.

Lemma 4.1. *Suppose that $\Theta \in \mathbb{C}^{n\times n}$ is a Hermitian matrix and $\Phi \in \mathbb{C}^{n\times n}$ satisfies $\rho(\Phi\Phi^*) < 1$. Then, $\rho(\Phi\Theta\Phi^*) < \rho(\Theta)$.*

In the following, we shall see that these $E_{[k,k+1)}$'s have the "nested set property".

Theorem 4.1. *For each $k \in \mathbb{N}$, $\{t + 1|\ t \in E_{[k,k+1)}\} \subseteq E_{[k+1,k+2)}$.*

Proof. For any $t \in E_{[k,k+1)}$, we have $\rho(S_1^t S_2^{t^*}) < 1$. From (3.8) we have $\rho(S_1^{t+1} S_2^{t+1^*}) = \rho(\Phi^{t+1}\Phi^{t+1^*}) = \rho(\Phi(\Phi^t\Phi^{t^*})\Phi^*)$. By Lemma 4.1, we obtain

$$\rho(S_1^{t+1} S_2^{t+1^*}) < \rho(\Phi^t \Phi^{t^*}) = \rho(S_1^t S_2^{t^*}) < 1.$$

Hence, $t + 1 \in E_{[k+1,k+2)}$.

Remark 4.1. (i) Suppose that $E_{[k,k+1)} = [k, k + 1)$ for some $k \in \mathbb{N}$. Theorem 4.1 shows that $E_{[k,\infty)} = [k, \infty)$, which means that (2.16) is solvable for $t \in [k, \infty)$. (ii) From Lemma 3.1, we have $1 \in E_{[1,2)}$. By Theorem 4.1, it is easily seen that $k \in E_{[k,k+1)}$ for each $k \in \mathbb{N}$. That is, for any $k \in \mathbb{N}$, (2.16) has a unique solution $(A(k), Q(k), P(k))$.

Lemma 4.2. *For each $t \in E_{[1,\infty)}$, $(X_L - X_S)(I - S_1^t S_2^{t^*})$ and $(I - S_1^{t^*} S_2^t)(X_L - X_S)$ are Hermitian positive definite.*

Proof. By Lemma 3.2 (ii), we have $(X_L - X_S)(I - S_1^t S_2^{t^*})$ and $(I - S_1^{t^*} S_2^t)(X_L - X_S)$ are Hermitian. Now, we show that the eigenvalues of those two matrices are positive. Let $X_L - X_S = LL^*$ be the Cholesky factorization of $X_L - X_S$. Then, we have

$$\begin{aligned} L^{-1}(X_L - X_S)(I - S_1^t S_2^{t^*})L^{-1^*} &= L^*(I - S_1^t S_2^{t^*})L^{*^{-1}}, \\ L^{-1}(I - S_1^{t^*} S_2^t)(X_L - X_S)L^{-1^*} &= L^{-1}(I - S_1^{t^*} S_2^t)L. \end{aligned} \tag{4.1}$$

By Theorem 3.1, we have that

$$\sigma(S_1^{t^*} S_2^t) = \sigma((S_1^{t^*} S_2^t)^*) = \sigma(S_2^{t^*} S_1^t) = \sigma(S_1^t S_2^{t^*}).$$

Because $\rho(S_1^t S_2^{t^*}) < 1$, it follows from Theorem 3.1 that the eigenvalues of $L^*(I - S_1^t S_2^{t^*})L^{*^{-1}}$ and $L^{-1}(I - S_1^{t^*} S_2^t)L$ are positive. From (4.1), we obtain that $(X_L - X_S)(I - S_1^t S_2^{t^*})$ and $(I - S_1^{t^*} S_2^t)(X_L - X_S)$ are Hermitian positive definite.

In the following, we shall see the boundedness of $Q(t)$ and $P(t)$ for $t \in E_{[1,\infty)}$.

Theorem 4.2. *Let $t \in E_{[1,\infty)}$. Then, $P(t + 1) \leq X_S$ is positive semi-definite. Furthermore, if $A \in \mathbb{C}^{n\times n}$ is invertible then $P(t + 1) < X_S$ is positive definite.*

Proof. Because $t \in E_{[1,\infty)}$, from Theorem 4.1, we have $t+1 \in E_{[1,\infty)}$. Hence $P(t+1)^* = P(t+1)$ exists. Next, we show that $P(t+1)$ is positive semi-definite. Using the first equation of (3.2) and equation (3.3), we have

$$\begin{aligned}
(I - S_1^{t+1} {S_2^{t+1}}^*)^* P(t+1) \left(I - S_1^{t+1} {S_2^{t+1}}^*\right) &= (I - S_1^{t+1} {S_2^{t+1}}^*)^* X_L (S_1 {S_2}^* - S_1^{t+1} {S_2^{t+1}}^*) \\
&= \left[(I - S_1 {S_2}^*)^* + (S_1 {S_2}^* - S_1^{t+1} {S_2^{t+1}}^*)^*\right] X_L (S_1 {S_2}^* - S_1^{t+1} {S_2^{t+1}}^*) \\
&\geq (I - S_1 {S_2}^*)^* X_L (S_1 {S_2}^* - S_1^{t+1} {S_2^{t+1}}^*) \\
&= (X_L - X_S)(S_1 {S_2}^* - S_1^{t+1} {S_2^{t+1}}^*) \quad \text{(by (3.3))} \\
&= (X_L - X_S) S_1 (I - S_1^t {S_2^t}^*) {S_2}^* \\
&= S_2 (X_L - X_S)(I - S_1^t {S_2^t}^*) {S_2}^*. \quad \text{(by Lemma 3.2 } (i)) \qquad (4.2)
\end{aligned}$$

Applying Lemma 4.2 to (4.2), we have $P(t+1) \geq 0$. Next, we show that $P(t+1) \leq X_S$. From the first equation of (3.2) and (3.3), we also have

$$\begin{aligned}
X_S - P(t+1) &= X_S - [(X_S - X_L) S_1^{t+1} {S_2^{t+1}}^* + X_S (I - S_1^{t+1} {S_2^{t+1}}^*)](I - S_1^{t+1} {S_2^{t+1}}^*)^{-1} \\
&= (X_L - X_S) S_1^{t+1} {S_2^{t+1}}^* (I - S_1^{t+1} {S_2^{t+1}}^*)^{-1} \\
&= S_2^{t+1} (X_L - X_S) {S_2^{t+1}}^* (I - S_1^{t+1} {S_2^{t+1}}^*)^{-1}. \quad \text{(by Lemma 3.2 } (i))
\end{aligned}$$

This finding implies that

$$\begin{aligned}
(I - S_1^{t+1} {S_2^{t+1}}^*)^* (X_S - P(t+1))(I - S_1^{t+1} {S_2^{t+1}}^*) &= (I - S_2^{t+1} {S_1^{t+1}}^*)(S_2^{t+1} (X_L - X_S) {S_2^{t+1}}^*) \\
&= S_2^{t+1} [(I - {S_1^{t+1}}^* S_2^{t+1})(X_L - X_S)] {S_2^{t+1}}^*. \qquad (4.3)
\end{aligned}$$

By Lemma 4.2, we have $X_S - P(t+1) \geq 0$. Therefore, $P(t+1) \leq X_S$ is positive semi-definite.

Suppose that $A \in \mathbb{C}^{n \times n}$ is invertible, then $\mathcal{M} - \lambda \mathcal{L}$ has no zero eigenvalue, i.e., J_s is invertible. From (2.14), we have S_2 is invertible. By (4.2) and (4.3), we obtain that $0 < P(t+1) < X_S$.

Theorem 4.3. *Let $t \in E_{[1,\infty)}$. Then, $Q(t) \geq X_L$ is positive definite. Furthermore, if $A \in \mathbb{C}^{n \times n}$ is invertible, then $Q(t) > X_L$.*

Proof. Because $t \in E_{[1,\infty)}$, $(I - S_1^t {S_2^t}^*)$ is invertible. From the first equation of (3.2), we have

$$\begin{aligned}
X_L - P(t) &= X_L - [(X_S - X_L) + X_L (I - S_1^t {S_2^t}^*)](I - S_1^t {S_2^t}^*)^{-1} \\
&= (X_L - X_S)(I - S_1^t {S_2^t}^*)^{-1} \\
&= (I - S_1^t {S_2^t}^*)^{-*} [(I - S_1^t {S_2^t}^*)^* (X_L - X_S)](I - S_1^t {S_2^t}^*)^{-1}.
\end{aligned}$$

From Lemma 4.2, we obtain that $X_L - P(t) > 0$. Using the fact that $B(t) = A(t)^*$ and the second and last equations of (3.2), we have

$$Q(t) = X_L + A(t)^* S_1^t = X_L + {S_1^t}^* (X_L - P(t)) S_1^t.$$

Because $X_L - P(t)$ and X_L are positive definite, $Q(t) \geq X_L$ is positive definite. Furthermore, if $A \in \mathbb{C}^{n \times n}$ is invertible, then S_1 is invertible. Hence, $Q(t) > X_L$.

Remark 4.2. From Theorems 4.2 and 4.3, it is easily seen that the matrices X_L and X_S defined in (2.12) are the lower bound of $\{Q(k)\}_{k=1}^{\infty}$ and upper bound of $\{P(k)\}_{k=1}^{\infty}$, respectively.

In the following, we shall see that $\{Q(k)\}_{k=1}^{\infty}$ and $\{P(k)\}_{k=1}^{\infty}$ also have the monotonicity. The proof shall be straightforward from the further results in Section 5.

Theorem 4.4. *For each $k \in \mathbb{N}$, $P(k+1) \geq P(k)$ and $Q(k) \geq Q(k+1)$.*

5. Applications. The solution curve of **Problem SPC** is a union of piecewise smooth curves parameterized by $t \in I_{[1,\infty)}$. Here

$$I_{[1,\infty)} \equiv \{t \in [1,\infty)|\ 1 \notin \sigma(S_1^t S_2^{t*})\}.$$

In Section 4, we study the case for $k \in \mathbb{N} \subset I_{[1,\infty)}$. In this section, we shall propose two applications. First, we develop a new algorithm in which every iteration step reaches the solution curve of **Problem SPC** with certain parameters $k \in \mathbb{N}$. Second, we use this solution curve to measure the convergence rates of fixed-point iteration, SDA and Newton's method for solving NMEs (1.1).

We give a useful result in the following theorem.

Theorem 5.1. *Let $A,\ Q \in \mathbb{C}^{n\times n}$ satisfy assumptions* **A1** *and* **A2**. *Assume that $(A(t_1), Q(t_1), P(t_1))$ and $(A(t_2), Q(t_2), P(t_2))$ are on the solution curve of* **Problem SPC** *with $t_1, t_2 \in I_{[1,\infty)}$. If $Q(t_1) - P(t_2)$ is invertible, then $t \equiv t_1 + t_2 \in I_{[1,\infty)}$ and*

$$\begin{cases} A(t) = A(t_1)[Q(t_1) - P(t_2)]^{-1}A(t_2), \\ Q(t) = Q(t_2) - A(t_2)^*[Q(t_1) - P(t_2)]^{-1}A(t_2), \\ P(t) = P(t_1) + A(t_1)[Q(t_1) - P(t_2)]^{-1}A(t_1)^*. \end{cases} \tag{5.1}$$

Proof. Writing

$$\mathcal{M}(t_i) = \begin{bmatrix} A(t_i) & 0 \\ Q(t_i) & -I \end{bmatrix},\quad \mathcal{L}(t_i) = \begin{bmatrix} -P(t_i) & I \\ A(t_i)^* & 0 \end{bmatrix},\quad i = 1, 2,$$

we have

$$\mathcal{M}(t_i)\begin{bmatrix} I \\ X_L \end{bmatrix} = \mathcal{L}(t_i)\begin{bmatrix} I \\ X_L \end{bmatrix} S_1^{t_i} \text{ and } \mathcal{M}(t_i)\begin{bmatrix} I \\ X_S \end{bmatrix} S_2^{t_i*} = \mathcal{L}(t_i)\begin{bmatrix} I \\ X_S \end{bmatrix},$$

for $i = 1, 2$. Because $Q(t_1) - P(t_2)$ is invertible, denote

$$\mathcal{M}_* = \begin{bmatrix} A(t_1)[Q(t_1) - P(t_2)]^{-1} & 0 \\ -A(t_2)^*[Q(t_1) - P(t_2)]^{-1} & I \end{bmatrix},\quad \mathcal{L}_* = \begin{bmatrix} I & -A(t_1)[Q(t_1) - P(t_2)]^{-1} \\ 0 & A(t_2)^*[Q(t_1) - P(t_2)]^{-1} \end{bmatrix}.$$

It is easy to verify that $\mathcal{M}_*\mathcal{L}(t_2) = \mathcal{L}_*\mathcal{M}(t_1)$. Let

$$\begin{aligned} \widehat{\mathcal{M}} \equiv \mathcal{M}_*\mathcal{M}(t_2) &= \begin{bmatrix} A(t_1)[Q(t_1) - P(t_2)]^{-1}A(t_2) & 0 \\ Q(t_2) - A(t_2)^*[Q(t_1) - P(t_2)]^{-1}A(t_2) & -I \end{bmatrix}, \\ \widehat{\mathcal{L}} \equiv \mathcal{L}_*\mathcal{L}(t_1) &= \begin{bmatrix} -P(t_1) - A(t_1)[Q(t_1) - P(t_2)]^{-1}A(t_1)^* & I \\ A(t_2)^*[Q(t_1) - P(t_2)]^{-1}A(t_1)^* & 0 \end{bmatrix}. \end{aligned}$$

Thus, it is easily seen that

$$\widehat{\mathcal{M}}\begin{bmatrix} I \\ X_L \end{bmatrix} = \widehat{\mathcal{L}}\begin{bmatrix} I \\ X_L \end{bmatrix} S_1^{t_1+t_2} \text{ and } \widehat{\mathcal{M}}\begin{bmatrix} I \\ X_S \end{bmatrix} (S_2^{t_1+t_2})^* = \widehat{\mathcal{L}}\begin{bmatrix} I \\ X_S \end{bmatrix}.$$

By the uniqueness of the solution of (2.16), we obtain $t = t_1 + t_2 \in I_{[1,\infty)}$ and

$$\mathcal{M}(t) \equiv \begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} = \widehat{\mathcal{M}},\quad \mathcal{L}(t) \equiv \begin{bmatrix} -P(t) & I \\ A(t)^* & 0 \end{bmatrix} = \widehat{\mathcal{L}},$$

where $A(t)$, $Q(t)$ and $P(t)$ are given in (5.1).

Remark 5.1. (i) Suppose that $t_1, t_2 \in \mathbb{N}$. From Theorems 4.2 and 4.3, we have that $Q(t_1) - P(t_2) \geq X_L - X_S$ is positive definite. Hence, the condition of Theorem 5.1 holds. (ii) Suppose that $t_1 = t_2 \in \mathbb{N}$. The iteration (5.1) is SDA and has been derived in [3, 4, 15, 16].

Suppose $(A_k, Q_k, P_k) = (A(t_1), Q(t_1), P(t_1))$ and $(A_\ell, Q_\ell, P_\ell) = (A(t_2), Q(t_2), P(t_2))$ are on the solution curve for **Problem SPC** at some positive integers t_1 and t_2, respectively. Note that $(A(1), Q(1), P(1)) = (A, Q, 0)$ are given initially. By applying (A_k, Q_k, P_k) and (A_ℓ, Q_ℓ, P_ℓ) to equation (5.1) and setting $(A_{k+1}, Q_{k+1}, P_{k+1}) = (A(t), Q(t), P(t))$ with $t = t_1 + t_2$, the following algorithm for solving NME (1.1) can be inductively derived.

Algorithm 5.1. *Given A, $Q \in \mathbb{C}^{n\times n}$ satisfying the assumptions* **A1** *and* **A2**. *Let $A_0 = A, Q_0 = Q, P_0 = 0$.*
For $k = 0, 1, \ldots$, choose $\ell \in \{0, 1, \ldots, k\}$ and compute

$$\begin{aligned} A_{k+1} &= A_\ell (Q_\ell - P_k)^{-1} A_k, \\ Q_{k+1} &= Q_k - A_k^* (Q_\ell - P_k)^{-1} A_k, \\ P_{k+1} &= P_\ell + A_\ell (Q_\ell - P_k)^{-1} A_\ell^*. \end{aligned}$$

Similarly, if we apply $(A_\ell, Q_\ell, P_\ell) = (A(t_1), Q(t_1), P(t_1))$ and $(A_k, Q_k, P_k) = (A(t_2), Q(t_2), P(t_2))$ to (5.1), we have an alternate form of Algorithm 5.1.

Algorithm 5.2. *Given A, $Q \in \mathbb{C}^{n\times n}$ satisfying the assumptions* **A1** *and* **A2**. *Let $A_0 = A, Q_0 = Q, P_0 = 0$.*
For $k = 0, 1, \ldots$, choose $\ell \in \{0, 1, \ldots, k\}$ and compute

$$\begin{aligned} A_{k+1} &= A_k (Q_k - P_\ell)^{-1} A_\ell, \\ Q_{k+1} &= Q_\ell - A_\ell^* (Q_k - P_\ell)^{-1} A_\ell, \\ P_{k+1} &= P_k + A_k (Q_k - P_\ell)^{-1} A_k^*. \end{aligned}$$

Here we note that, at each step k, if the ℓ's are chosen to be the same for both Algorithms 5.1 and 5.2, then these two algorithms generate the same sequences $\{(A_k, Q_k, P_k)\}_{k=0}^\infty$. By setting $\ell = k$, Algorithms 5.1 and 5.2 are the well-known SDA. In addition,

$$A_k = A(2^k),\ Q_k = Q(2^k),\ P_k = P(2^k), \tag{5.2}$$

where $A(\cdot)$, $Q(\cdot)$ and $P(\cdot)$ are given in (1.9). If we set $\ell = 0$, then Algorithms 5.1 and 5.2 generate the sequence

$$A_{k+1} = A(Q - P_k)^{-1} A_k = A_k Q_k^{-1} A, \tag{5.3a}$$

$$Q_{k+1} = Q_k - A_k^* (Q - P_k)^{-1} A_k = Q - A^* Q_k^{-1} A, \tag{5.3b}$$

$$P_{k+1} = A(Q - P_k)^{-1} A^* = P_k + A_k Q_k^{-1} A_k^*. \tag{5.3c}$$

Here,

$$A_k = A(k+1),\ Q_k = Q(k+1),\ P_k = P(k+1). \tag{5.4}$$

It is easily seen that (5.3b) is the fixed-point iteration (1.5). From (5.2) and (5.4), for a sequence $\{X_k\}$ generated by the fixed-point iteration, then the sequence $\{Q_k\}$ generated by SDA is $\{X_{2^k-1}\}$. From (5.3c), we have iteration

$$P_{k+1} = A(Q - P_k)^{-1} A^*, \quad \text{with } P_0 = 0.$$

On the other hand, from (2.16), we see that the sequence P_k converges to the positive semi-definite matrix X_S. This finding coincides with the iteration derived in [9]. Therefore, if we let $Y_k = Q - P_k$, then Y_k converges to the maximal solution of dual equation

$$Y + AY^{-1}A^* = Q.$$

Now, we are in a position to prove Theorem 4.4.

Proof. [Proof of Theorem 4.4] First, let $Q_k = Q(k+1)$ and $Y_k = Q - P(k+1)$. Hence, $\{Q_k\}$ and $\{Y_k\}$ are the sequences generated by the fixed-point iteration for $X + A^*X^{-1}A = Q$ and $Y + AY^{-1}A^* = Q$, respectively. It is reported in [9] that both Q_k and Y_k are monotonically decreasing. Accordingly, $P(k+1) \geq P(k)$ and $Q(k) \geq Q(k+1)$ are positive semi-definite.

Next, we consider Newton's method [12].

Algorithm 5.3. *Given A, $Q \in \mathbb{C}^{n\times n}$ satisfying the assumptions* **A1** *and* **A2**.
Let $X_0 = Q$. For $k = 1, 2, \ldots$,
Compute $L_k = X_{k-1}^{-1}A$.
Solve $X_k - L_k^* X_k L_k = Q - 2L_k^* A$.

Analogically to the relationship between the fixed-point iteration and the SDA mentioned above, we may also hope that the iterations $\{X_k\}_{k=0}^{\infty}$ generated by Newton's method are the same as $\{Q(t_k)\}_{k=0}^{\infty}$ for some $t_k \in \mathbb{N}$. Unfortunately, this result is not true in the general case. We need an additional assumption:

A3. The equation $A^*Q^{-1}A = AQ^{-1}A^*$ holds.

Lemma 5.1. *Suppose that A, $Q \in \mathbb{C}^{n\times n}$ satisfy assumption* **A3** *and Q is positive definite. Then there exists an invertible matrix $\Gamma \in \mathbb{C}^{n\times n}$ such that $\Gamma^*Q\Gamma = I$ and $\Gamma^*A\Gamma$ is a diagonal matrix.*

Proof. Because Q is positive definite, there exists a lower-triangular matrix L such that $Q^{-1} = LL^*$. From assumption **A3**, we have

$$L^*A^*LL^*AL = L^*A^*Q^{-1}AL = L^*AQ^{-1}A^*L = L^*ALL^*A^*L.$$

Hence, L^*AL is normal. Then, a unitary matrix Υ exists such that $\Upsilon^*(L^*AL)\Upsilon$ is diagonal. Let $\Gamma = L\Upsilon$. We have that $\Gamma^*Q\Gamma = I$ and $\Gamma^*A\Gamma$ is a diagonal matrix.

Suppose that A, $Q \in \mathbb{C}^{n\times n}$ satisfy assumptions **A1**, **A2** and **A3**. From Lemma 5.1, there is an invertible matrix Γ such that $\Gamma^*Q\Gamma = I$ and $\Gamma^*A\Gamma$ is a diagonal matrix. Suppose that $\{A_k\}$, $\{Q_k\}$ and $\{P_k\}$ are generated by (5.3) with $A_0 = A$, $Q_0 = Q$ and $P_0 = 0$. Let

$$\widehat{A}_k = \Gamma^*A_k\Gamma, \quad \widehat{Q}_k = \Gamma^*Q_k\Gamma, \quad \widehat{P}_k = \Gamma^*P_k\Gamma, \tag{5.5}$$

for $k = 0, 1, \ldots$. It follows that $\{\widehat{A}_k\}$, $\{\widehat{Q}_k\}$ and $\{\widehat{P}_k\}$ are diagonal and satisfy

$$\begin{aligned}
\widehat{A}_{k+1} &= \widehat{A}(\widehat{Q} - \widehat{P}_k)^{-1}\widehat{A}_k = \widehat{A}_k\widehat{Q}_k^{-1}\widehat{A}, \\
\widehat{Q}_{k+1} &= \widehat{Q}_k - \widehat{A}_k^*(\widehat{Q} - \widehat{P}_k)^{-1}\widehat{A}_k = \widehat{Q} - \widehat{A}^*\widehat{Q}_k^{-1}\widehat{A}, \\
\widehat{P}_{k+1} &= \widehat{A}(\widehat{Q} - \widehat{P}_k)^{-1}\widehat{A}^* = \widehat{P}_k + \widehat{A}_k\widehat{Q}_k^{-1}\widehat{A}_k^*.
\end{aligned} \tag{5.6}$$

Lemma 5.2. *Suppose that A, $Q \in \mathbb{C}^{n\times n}$ satisfy assumptions* **A1**, **A2** *and* **A3**. *Then, the sequences $\{A_k\}$, $\{Q_k\}$ and $\{P_k\}$ generated by (5.3) with $A_0 = A$, $Q_0 = Q$ and $P_0 = 0$ satisfy (i) $A_\ell^*Q_k^{-1}A_\ell = A_\ell Q_k^{-1}A_\ell^*$, (ii) $P_k + Q_k = Q$ for all $\ell, k \in \mathbb{N}$.*

Proof. From (5.5)-(5.6) and using the fact that $\widehat{A}_k$, $\widehat{Q}_k$ and $\widehat{P}_k$ are diagonal, we have

$$\Gamma^*A_\ell^*Q_k^{-1}A_\ell\Gamma = \Gamma^*A_\ell^*\Gamma(\Gamma^*Q_k\Gamma)^{-1}\Gamma^*A_\ell\Gamma = \widehat{A}_\ell^*\widehat{Q}_k^{-1}\widehat{A}_\ell = \widehat{A}_\ell\widehat{Q}_k^{-1}\widehat{A}_\ell^* = \Gamma^*A_\ell Q_k^{-1}A_\ell^*\Gamma,$$

for all $\ell, k \in \mathbb{N}$. This finding proves assertion (i). Next, we show that assertion (ii) holds. Clearly, it holds true for $k = 0$. Suppose that (ii) holds for $k = m$, then for $k = m + 1$, we have

$$\begin{aligned} P_k + Q_k &= P_{m+1} + Q_{m+1} = A(Q - P_m)^{-1} A^* + Q - A^* Q_m^{-1} A \\ &= Q + A(Q - P_m)^{-1} A^* - A Q_m^{-1} A^* \quad (\text{by } (i)) \\ &= Q + A[(Q - P_m)^{-1} - Q_m^{-1}] A \quad (\text{by induction hypothesis}) \\ &= Q. \end{aligned}$$

Therefore, an induction argument leads to assertion (ii).

Theorem 5.2. *Suppose that A, $Q \in \mathbb{C}^{n\times n}$ satisfy assumptions* **A1**, **A2** *and* **A3**. *If X_k is generated by Newton's method, then $X_k = Q(2^{k+1} - 1)$ where $Q(\cdot)$ is given in* (1.9).

Proof. The proof is given by induction on the number k. Let

$$\tau_k = 2^{k+1} - 1.$$

Clearly, when $k = 0$, the equalities $X_0 = Q = Q(1) = Q(\tau_0)$ hold. Suppose that $X_k = Q(\tau_k)$ when $k = m \in \mathbb{N}$. For $k = m + 1$, it follows from (5.3) and Lemma 5.2 that

$$\begin{aligned} Q(\tau_k) =& Q(\tau_{m+1}) = Q(2\tau_m + 1) = Q(2\tau_m) - A(2\tau_m)^* [Q - P(2\tau_m)]^{-1} A(2\tau_m) \\ =& Q(2\tau_m) - A(2\tau_m)^* Q(2\tau_m)^{-1} A(2\tau_m). \end{aligned} \tag{5.7}$$

Applying $t_1 = t_2 = \tau_m$ to (5.1), we have

$$\begin{aligned} A(2\tau_m) &= A(\tau_m) \left[Q(\tau_m) - P(\tau_m)\right]^{-1} A(\tau_m), \\ Q(2\tau_m) &= Q(\tau_m) - A(\tau_m)^* \left[Q(\tau_m) - P(\tau_m)\right]^{-1} A(\tau_m). \end{aligned} \tag{5.8}$$

Then, it follows from Lemma 5.2 again that

$$\begin{aligned} & A(2\tau_m)^* Q(2\tau_m)^{-1} A(2\tau_m) \\ &= A(\tau_m)^* \left[Q(\tau_m) - P(\tau_m)\right]^{-1} A(\tau_m)^* Q(2\tau_m)^{-1} A(\tau_m) \left[Q(\tau_m) - P(\tau_m)\right]^{-1} A(\tau_m) \\ &= A(\tau_m)^* \left[Q(\tau_m) - P(\tau_m)\right]^{-1} A(\tau_m) Q(2\tau_m)^{-1} A(\tau_m)^* \left[Q(\tau_m) - P(\tau_m)\right]^{-1} A(\tau_m). \end{aligned} \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7) and using the Sherman-Morrison-Woodbury formula, we obtain

$$Q(\tau_m) - Q(\tau_{m+1}) = A(\tau_m)^* \left[Q(\tau_m) - P(\tau_m) - A(\tau_m) Q(\tau_m)^{-1} A(\tau_m)^*\right]^{-1} A(\tau_m). \tag{5.10}$$

By induction hypothesis, we have $X_m = Q(\tau_m)$. Multiplying X_m^{-1} from the right and left of (5.10), gives

$$\begin{aligned} & X_m^{-1} - X_m^{-1} Q(\tau_{m+1}) X_m^{-1} \\ &\quad = Q(\tau_m)^{-1} A(\tau_m)^* \left[Q(\tau_m) - P(\tau_m) - A(\tau_m) Q(\tau_m)^{-1} A(\tau_m)^*\right]^{-1} A(\tau_m) Q(\tau_m)^{-1}. \end{aligned}$$

Applying the Sherman-Morrison-Woodbury formula and using (5.8), it follows that

$$\begin{aligned} X_m^{-1} - X_m^{-1} Q(\tau_{m+1}) X_m^{-1} &= -Q(\tau_m)^{-1} + \left[Q(\tau_m) - A(\tau_m)^* [Q(\tau_m) - P(\tau_m)]^{-1} A(\tau_m)\right]^{-1} \\ &= -X_m^{-1} + Q(2\tau_m)^{-1}. \end{aligned} \tag{5.11}$$

From (5.3b) and (5.11), we have

$$\begin{aligned}Q(\tau_{m+1}) &= Q(2\tau_m+1) = Q - A^*Q(2\tau_m)^{-1}A \\ &= Q - 2A^*X_m^{-1}A + A^*X_m^{-1}Q(\tau_{m+1})X_m^{-1}A.\end{aligned}$$

This finding implies that $Q(\tau_k) = Q(\tau_{m+1})$ satisfies

$$Q(\tau_{m+1}) - A^*X_m^{-1}Q(\tau_{m+1})X_m^{-1}A = Q - 2A^*X_m^{-1}A.$$

Hence, we obtain that $X_k = X_{m+1} = Q(\tau_{m+1}) = Q(\tau_k)$. By the mathematical induction, for each $k \in \mathbb{N}$, we have $X_k = Q(2^{k+1}-1)$, where X_k is generated by Newton's method.

6. Numerical results. In this section, we present some numerical results. All numerical experiments are conducted using MATLAB R2010a with double-precision floating-point arithmetic ($eps \approx 2.22 \times 10^{-16}$).

Example 6.1. We randomly generate two complex matrices A and $Q = Q^*$ of dimension 11. Choose a suitable $\varrho \in \mathbb{R}$ and set $Q := Q + \varrho I_{11}$ such that A, Q satisfy assumptions **A1** and **A2**. We compute X_L, X_S, S_1 and S_2 by using the QZ algorithm (see (2.15)). Thus, the solution curve $\{(A(t), Q(t), P(t)) \mid t \geq 1\}$ can be parameterized by (1.9). In this experiment, we only compute the solution curves, $A(t)$, $Q(t)$ and $P(t)$, for $t \in [1,5]$. To measure the accuracy of computed solution curve of (2.16), we use the absolute residual

$$\mathrm{Res}(t) = \left\| \begin{bmatrix} A(t) & 0 \\ Q(t) & -I \end{bmatrix} \begin{bmatrix} I & {S_2^t}^* \\ X_L & X_S {S_2^t}^* \end{bmatrix} - \begin{bmatrix} -P(t) & I \\ A(t)^* & 0 \end{bmatrix} \begin{bmatrix} S_1^t & I \\ X_L S_1^t & X_S \end{bmatrix} \right\|,$$

where $\|\cdot\|$ is spectral norm. In Figure 6.1, we plot the residual $\mathrm{Res}(t)$. From Theorem 3.2, we have that $Q(t)$ and $P(t)$ are Hermitian matrices. To show this point, we also illustrate $\|Q(t) - Q(t)^*\|$ and $\|P(t) - P(t)^*\|$ in Figure 6.1. From Theorems 3.2 and 3.3, we know that $1 \notin \sigma(S_1^t {S_2^t}^*)$ is the necessary and sufficient condition for the solvability of (2.16). On the other hand, it follows from Theorem 3.1 that $\rho(S_1^t {S_2^t}^*)$ is the maximal eigenvalue of $S_1^t {S_2^t}^*$. In Figure 6.2, we plot $\|A(t)\|$, $\|Q(t)\|$ and $\|P(t)\|$ and $\rho(S_1^t {S_2^t}^*)$. For this example, we see that there are four points, $t_1 = 1.1339$, $t_2 = 1.8682$, $t_3 = 2.3709$ and $t_4 = 2.7123$, at which $\rho(S_1^{t_i} {S_2^{t_i}}^*) = 1$. This result implies that the equation (2.16) has no solution for $t = t_1, \ldots, t_4$. We also can see that $\|A(t)\|$, $\|Q(t)\|$ and $\|P(t)\|$ in Figure 6.2 blow up at $t = t_1, \ldots, t_4$. The intervals $E_{[k,k+1)}$ defined in (1.10) can be observed as

$$\begin{aligned}&E_{[1,2)} = [1,t_1) \cup (t_2,2), \quad E_{[2,3)} = [2,t_3) \cup (t_4,3), \\ &E_{[k,k+1)} = [k,k+1), \text{ for } k = 3,4\ldots.\end{aligned}$$

It is easily to see that these $E_{[k,k+1)}$'s satisfy $\{t+1 | t \in E_{[k,k+1)}\} \subseteq E_{[k+1,k+2)}$ for each $k \in \mathbb{N}$, which has been shown in Theorem 4.1. Therefore, equation (2.16) is solvable for $t \geq 3$.

Example 6.2. We randomly generate two complex matrices A and $Q = Q^*$ of dimension 11. Choose a suitable $\varrho \in \mathbb{R}$ and set $Q := Q + \varrho I_{11}$ such that A, Q satisfy assumptions **A1** and **A2**. In this example, $\rho(S_1^t {S_2^t}^*) < 1$ for each $t \geq 1$, and hence, $E_{[1,\infty)} = [1,\infty)$. However, assumption **A3** does not hold. We first compute X_1, X_2 and X_3 by Newton's method and construct the solution curves, $A(t)$, $Q(t)$ and $P(t)$, for $t \in [1,16]$ by (1.9). To see the relationship between the solution curve of (2.16) and Newton's iterations X_1, X_2 and X_3, we compute

$$R_{X_i}(t) = \frac{\|X_i - Q(t)\|}{\|X_i\|}, \quad \text{for } i = 1,2,3.$$

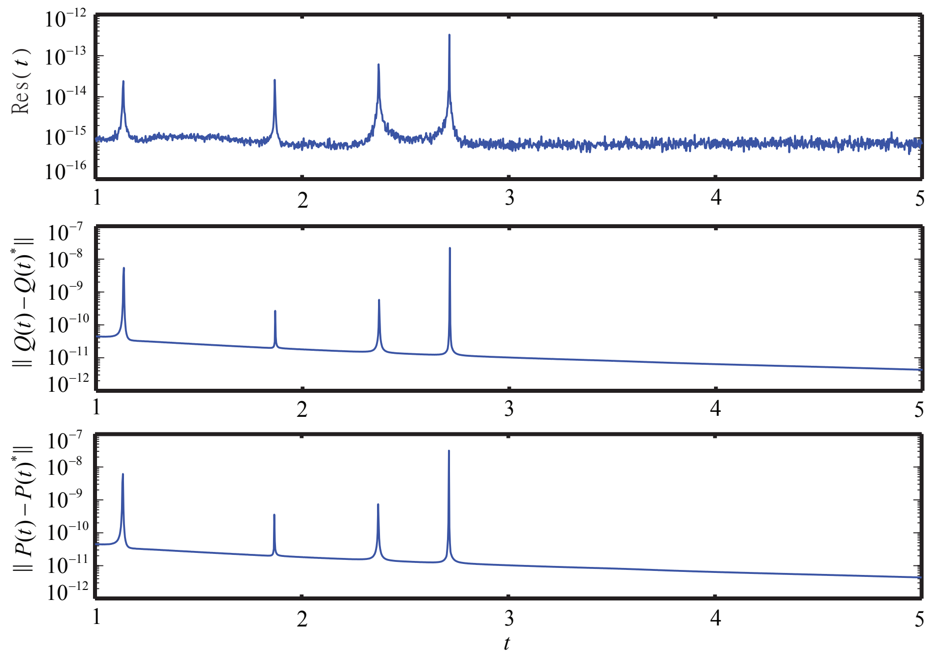

FIG. 6.1. *Res*(t), $\|Q(t) - Q(t)^*\|$ *and* $\|P(t) - P(t)^*\|$ *for* $t \in [1, 5]$.

Because assumption **A3** does not hold, it is not guaranteed that $\{Q(2^{k+1} - 1)\}_{k=0}^{\infty}$ is the sequence generated by Newton's method as mentioned in Main Result 3. That is, we cannot guarantee that $R_{X_1}(3)$, $R_{X_2}(7)$ and $R_{X_3}(15)$ are zero. In Figure 6.3, we plot $R_{X_1}(t)$, $R_{X_2}(t)$ and $R_{X_3}(t)$. We see that the values $R_{X_1}(t_1)$, $R_{X_2}(t_2)$ and $R_{X_3}(t_3)$ are between 10^{-3} and 10^{-2} when t_1, t_2 and t_3 are near 3, 6 and 13, respectively.

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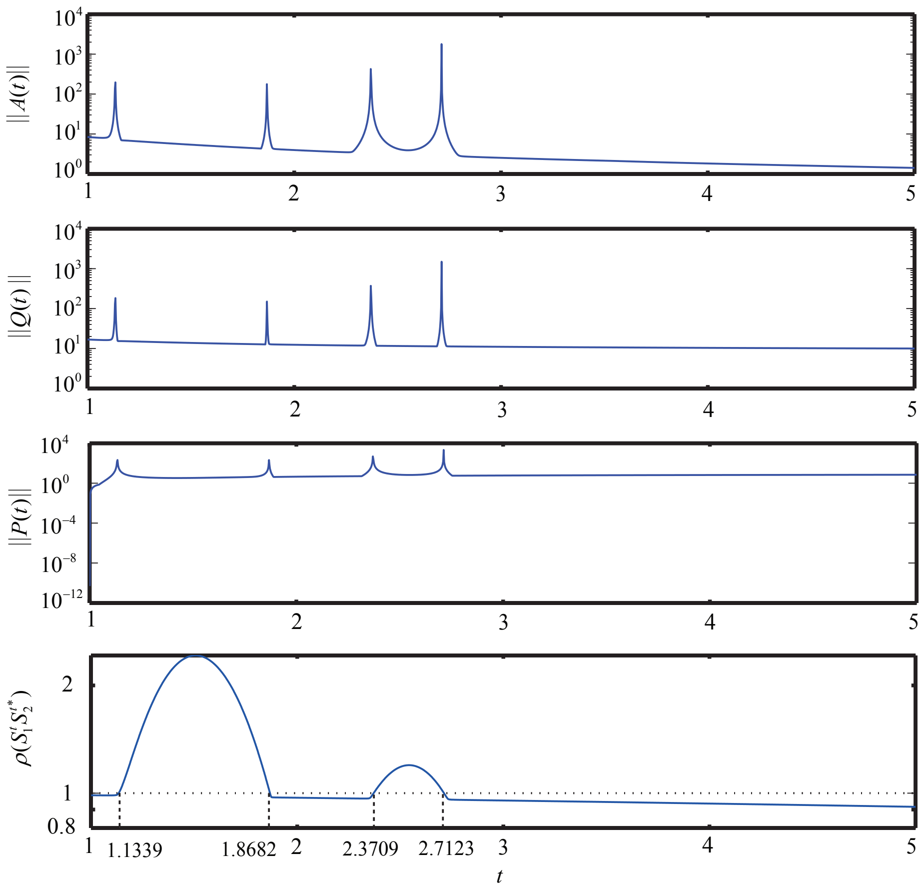

FIG. 6.2. $\|A(t)\|$, $\|Q(t)\|$, $\|P(t)\|$ *and* $\rho(S_1^t S_2^{t\,*})$ *for* $t \in [1, 5]$.

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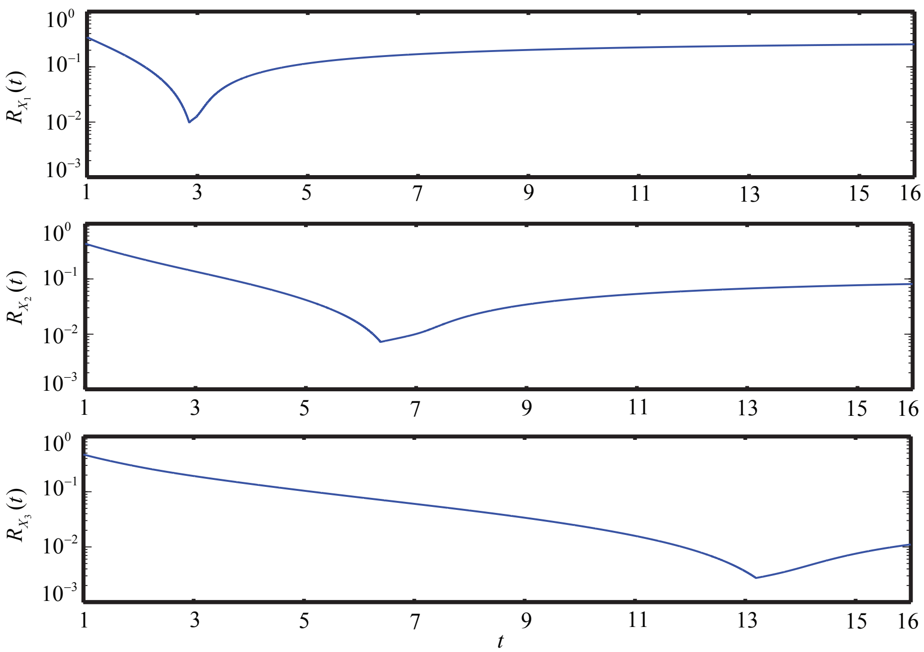

FIG. 6.3. *$R_{X_1}(t)$, $R_{X_2}(t)$ and $R_{X_3}(t)$ for $t \in [1, 16]$.*

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